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QUANTUM ERGODICITY OF EIGENFUNCTIONS ON $PSL_2(\mathbf{Z})\backslash\mathbf{H}^2$

WENZHI LUO *and* PETER SARNAK[1]

To Wolfgang Schmidt on the Occasion of His 60th Birthday

1. Introduction

Schnirelman [SH], Colin de Verdière [CD] and Zelditch [Z1] have proven the quantum analogue of the geodesic flow on a compact Riemannian manifold Y being ergodic. Let Δ denote the Laplacian on Y and φ_j an orthonormal basis of L^2-eigenfunctions of Δ. The corresponding eigenvalues are denoted by λ_j. One forms the probability measures $d\mu_j(z) := |\varphi_j(z)|^2 \, dV(z)$, dV being the volume element (actually they consider a microlocalization of these measures to $S_1^*(Y)$, the unit cotangent bundle). If the geodesic flow on $S_1^*(Y)$ is ergodic they show the existence of a full density subsequence λ_{j_k} (i.e. one satisfying $\Sigma_{\lambda_{j_k}\leqslant\lambda} 1 \sim \Sigma_{\lambda_j \leqslant \lambda} 1$) for which $\mu_{j_k}(A) \to \mathrm{Vol}(A)/\mathrm{Vol}(Y)$ for all nice sets A (e.g. geodesic balls). Zelditch [Z2] has extended this result to noncompact surfaces such as the modular surface $X = PSL_2(\mathbf{Z})\backslash\mathbf{H}^2$. He shows that if $h \in C_{00}^\infty(X)$, the space of smooth functions on X with compact supports, and $\int_X h(x)\, dV(x) = 0$, then

$$(1) \qquad \sum_{\lambda_j \leqslant \lambda} |\langle h, \mu_j \rangle|^2 \ll_h \frac{\lambda}{\log \lambda}.$$

Here $A \ll_\tau B$ means $|A| \leqslant CB$ where C depends only on τ. Selberg [SE] has shown that in this case, $\Sigma_{\lambda_j \leqslant \lambda} 1 \sim \lambda/12$ and from this one can easily deduce the quantum ergodicity using (1).

Recent works of Hejhal-Rackner [H-R] and Rudnick-Sarnak [R-S] suggest that at least for this X much more is true. Namely, that there are no exceptional subsequences, that is $\mu_j \to dV$ as $j \to \infty$. This phenomenon might be called Quantum Unique Ergodicity. Hejhal-Rackner confirm this numerically while Rudnick-Sarnak show that certain natural candidates for such singular limits (that is, measures concentrated on closed geodesics) do not occur.

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This paper is concerned with this individual equidistribution conjecture for X. While we fall short of proving it we obtain a number of results in that direction. Firstly we prove the conjecture for the continuous part of the spectrum of X—that is we show that the Eisenstein series become individually equidistributed. Secondly for the discrete spectrum (cusp forms) we show that if exceptional subsequences occur they must be very sparse. We also introduce the discrepancy—a well-known measure of equidistribution for sequences—to quantify the measure of equidistribution of the μ_j's. This enables us to show that except for a sparse set of j's the μ_j's become equidistributed at a certain rate. For more background on this problem see the Lectures [SA]. Along the way we establish a conjecture of Iwaniec concerning the average size of Rankin-Selberg L-functions on their critical lines. The latter may be used to obtain new bounds for the remainder term in the Prime Geodesic Theorem (see below). We turn to a precise description of our results.

The spectrum of Δ on $L^2(X)$ consists of three types, see Hejhal [H2]:

(A) $\varphi_0(z) = \sqrt{3/\pi}$, the constant function;

(B) $\varphi_1(z), \varphi_2(z), \ldots$, an orthonormal basis of cusp forms, $\Delta\varphi_j + \lambda_j \varphi_j = 0$;

(C) $E\left(z, \frac{1}{2} + it\right)$, $t \geqslant 0$, the unitary Eisenstein series which furnish the continuous spectrum, $\Delta E + \left(\frac{1}{4} + t^2\right) E = 0$.

We will assume that the basis φ_j is chosen to be simultaneously eigenfunctions of the Hecke algebra. This choice is possible and in fact determines the φ_j's up to a scalar and hence determines μ_j uniquely. It is quite likely that there is only one o.n.b. of φ_j's anyway, since the numerical evidence points to the spectrum being simple [H3, ST]. We define $\mu_t = \left|E\left(z, \frac{1}{2} + it\right)\right|^2 dV(z)$. Note that $\mu_t(X) = \infty$, so that there is no canonical normalization of μ_t. Our first result is that μ_t become individually equidistributed.

Theorem **1.1.** — *Let* A, B *be compact Jordan measurable subsets of* X, *then*

$$\lim_{t\to\infty} \frac{\mu_t(A)}{\mu_t(B)} = \frac{\text{Vol}(A)}{\text{Vol}(B)}.$$

The renormalization is actually needed since we in fact show that as $t \to \infty$,

$$\mu_t(A) \sim \frac{48}{\pi} \text{Vol}(A) \log t. \tag{2}$$

Jakobson [J] has recently extended Theorem 1.1 to the microlocalizations $\tilde{\mu}_t$ of μ_t to $S_1^*(X)$.

To describe our main result concerning μ_j we introduce some norms on functions on X. For $H \in C^\infty(X)$ let $\| H \|_{k,\nu}$ be defined by:

$$\text{(3)} \qquad \| H \|_{k,\nu} = \max_{\nu_1+\nu_2 \leqslant \nu} \sup_{z \in F} \left| y^k \frac{\partial^{\nu_1+\nu_2} H}{(\partial x)^{\nu_1} (\partial y)^{\nu_2}} (z) \right|,$$

where F is the usual fundamental domain for X in **H**. For H an integrable function on X we denote by $\overline{H}$ the mean value $\frac{1}{\text{Vol}(X)} \int_X H(z)\, dV(z)$.

Theorem **1.2.** — *For* $\varepsilon > 0$ *and* $H \in C^\infty(X)$,

$$\sum_{\lambda_j \leqslant \lambda} \left| \int_X H(z)\, d\mu_j(z) - \overline{H} \right|^2 \ll_\varepsilon \| H \|_{8,8}^2\, \lambda^{(1/2)+\varepsilon},$$

the implied constant depending on ε *only.*

The upper bound here is essentially the square root of that in (1) and in fact is sharp (i.e. it cannot be replaced by any exponent less than 1/2). Theorem 1.2 asserts that on average $\left| \int_X H(z)\, d\mu_j(z) - \overline{H} \right|$ is of size $\lambda_j^{-1/4}$. We expect that this is true individually (see [SA]). As a corollary to Theorem 1.2 we can address a question of Zelditch [Z], as to the size of an exceptional subsequence. He showed in general that such a subsequence must be of zero density. The following asserts that an exceptional subsequence must be very thin.

Corollary **1.3.** — *Let* j_k *be a subsequence of* j*'s corresponding to a subset* $S \subset N$ *and for which* $\mu_{j_k} \to \nu \neq 3\, dV/\pi$. *Then for any* $\alpha > 1/2$, $| S \cap [1, N] | = O_\alpha(N^\alpha)$.

We also establish Theorem 1.2 for H an individual Eisenstein series, that is

$$\text{(4)} \qquad \sum_{\lambda_j \leqslant \lambda} \left| \left\langle E\left(\cdot, \frac{1}{2} + it\right), \mu_j \right\rangle \right|^2 \ll_\varepsilon (|t| + 1)^6\, \lambda^{(1/2)+\varepsilon}.$$

By Cauchy's inequality and in view of the integral representation

$$L(u_j \otimes u_j, s) = \frac{4\pi^s\, \Gamma(s)}{\Gamma^2(s/2)\, \Gamma(s/2 + it_j)\, \Gamma(s/2 - it_j)} \int_F E(z, s)\, |\varphi_j(z)|^2\, dV(z),$$

the last implies the following "mean Lindelöf" conjecture of Iwaniec [I1]:

$$\text{(5)} \qquad \sum_{\lambda_j \leqslant \lambda} \frac{\left| L\left(u_j \otimes u_j, \frac{1}{2} + it\right) \right|}{\cosh(\pi t_j)} \ll_\varepsilon (|t| + 1)^4\, \lambda^{1+\varepsilon},$$

where $L(u_j \otimes u_j, s)$ is the Rankin-Selberg L-function (see § 2 and § 3 below) and $\lambda_j = (1/4) + t_j^2$. We apply (5) to counting prime geodesics on X. Let

$$\pi_\Gamma(x) = |\{ P \mid N(P) \leqslant x \}|,$$

where P runs over the primitive conjugacy classes of $\Gamma = \mathrm{PSL}_2(\mathbf{Z})$ and $\mathrm{N(P)}$ is the corresponding norm [H1]; $\pi_\Gamma(x)$ counts the number of prime geodesics on X of length at most $\log x$.

Theorem **1.4.**

$$\pi_\Gamma(x) = \mathrm{li}(x) + \mathrm{O}(x^{(7/10)+\varepsilon}), \quad \text{for } \varepsilon > 0.$$

Here as usual, $\mathrm{li}(x) = \int_2^x dt/\log t$. For a general discrete cofinite $\Gamma \subset \mathrm{PSL}_2(\mathbf{R})$ the best-known bound for the remainder term is $\mathrm{O}(x^{3/4}/\log x)$ [RAN, SE], while for $\Gamma = \mathrm{PSL}_2(\mathbf{Z})$, Iwaniec in the paper quoted above established the bound $\mathrm{O}_\varepsilon(x^{(35/48)+\varepsilon})$.

In § 6 we will give an outline of a proof of Theorem 1.4. It is based on (5) and Iwaniec's method in [I1]. It should be pointed out that the expected remainder term here is $\mathrm{O}_\varepsilon(x^{(1/2)+\varepsilon})$. In the analogy with the Riemann zeta function and primes, the analogue of the Riemann Hypothesis for Selberg zeta function is true. Even so the abundance of eigenvalues puts the $\mathrm{O}_\varepsilon(x^{(1/2)+\varepsilon})$ bound completely out of reach. Indeed the $\mathrm{O}(x^{3/4})$ bound is reasonably straightforward but anything beyond that involves capturing cancellation in the sums over the eigenvalues.

To quantify equidistribution of the μ_j's it seems best to avoid issues of subsequences (as has been traditional in this problem) and to investigate directly the discrepancy. Various notions of discrepancy have been introduced in connection wich equidistribution of sequences [K-N]. The spherical cap discrepancy $\mathrm{D}(\mu)$ is defined by

$$\mathrm{D}(\mu) = \sup_{\mathrm{B}\subset \mathrm{X}} \left| \mu(\mathrm{B}) - \frac{3\mathrm{Vol(B)}}{\pi} \right| \tag{6}$$

where the supremum is over all injective geodesic balls in X. It is clear that if $\mathrm{D}(\mu_j) \to 0$ then μ_j become equidistributed and the size of $\mathrm{D}(\mu_j)$ gives the rate. The choice of balls in this geometry seems natural enough. A somewhat bold conjecture that emerges from Theorem 1.2 (see also [SA] and the question of Colin de Verdière [CD]) is that for $\varepsilon > 0$, $\mathrm{D}(\mu_j) \ll_\varepsilon \lambda_j^{-(1/4)+\varepsilon}$. As pointed out in [SA] this equidistribution rate if true would be optimal.

Theorem **1.5.**

$$\sum_{\lambda_j \leqslant \lambda} |\,\mathrm{D}(\mu_j)\,|^2 \ll \lambda^{(20/21)+\varepsilon}.$$

The main achievement in Theorem 1.5 is that the exponent of λ is less than 1. We have made no effort to reduce it further which is certainly possible by these methods. On the other hand the optimal exponent 1/2 seems out of reach by these methods. From Theorem 1.5 it follows that with exception of a very sparse set, the μ_j's become equidistributed at a rate which is a negative power of λ_j.

We end the introduction with some comments about the proofs. As pointed out in [SA] these questions of equidistribution are in part related to obtaining non-trivial bounds for Ranking-Selberg L-functions on their critical lines. Concerning Theorem 1 an involved but pleasing computation which exploits each factor of $E\left(z, \frac{1}{2} + it\right) \overline{E\left(z, \frac{1}{2} + it\right)}$ differently, leads to two features: (1) The entire problem in this case reduces to estimating L-functions. (2) The Rankin-Selberg L-functions in question conveniently factor into Euler products of degree at most two. We can therefore appeal to known estimates on the Riemann zeta function and L-functions of cusp forms (Meurman [MEU]) to prove the Theorem. For the case of cusp forms, viz Theorem 2 we no longer have any direct relation to L-functions. Our method is to first establish Theorem 2 for certain families of h's called incomplete Poincaré series. This allows us to exploit that φ_j is a Hecke eigenform and to represent $|\varphi_j(z)|^2$ with expressions involving the Fourier coefficients in quadratic polynomials. Eventually this allows us to use the Fourier coefficient—Kloosterman sum connection, that is the Petersson—Kuznetsov trace formula [KU], to convert the problem to estimating exponential sums. To do so effectively Weil's bound on Kloosterman sums is used as a key arithmetical ingredient. Also crucial in our analysis are the recent bounds of Iwaniec [I2] and Hoffstein-Lockhart [HN-L] for Fourier coefficients of cusp forms in the j aspect. Theorem 1.5 is derived from Theorem 1.2 using only Fourier analysis and geometry. The key point here is the structure of the spectral development of the characteristic function of a geodesic ball.

All of the above results may be proven for congruence subgroups of $PSL_2(\mathbf{Z})$ (save for the possibility of small eigenvalues $\lambda_j < 1/4$ intervening in Theorem 1.5). However inasmuch as we use heavily Poincaré and Eisenstein series (and related Fourier coefficients) we do not know how to establish these results for compact arithmetic surfaces.

2. Eisenstein Series

This section is devoted to the proof of Theorem 1.1. The Eisenstein series $E(z, s)$ for X are defined by

$$E(z, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} y(\gamma z)^s = \frac{1}{2} \sum_{(c,d)=1} \frac{y^s}{|cz + d|^{2s}}, \tag{7}$$

where $\Re(s) > 1$, $z = x + iy$, $\Gamma = PSL_2(\mathbf{Z})$ and

$$\Gamma_\infty = \{ z \mapsto z + n, n \in \mathbf{Z} \}.$$

The Fourier development of $E(z, s)$ is well known [SA2]:

$$E(z, s) = y^s + \varphi(s) y^{1-s} + \frac{2y^{1/2}}{\xi(2s)} \sum_{n=1}^{\infty} n^{s-1/2} \sigma_{1-2s}(n) K_{s-1/2}(2\pi n y) \cos(2\pi n x), \tag{8}$$

where $$\xi(s) = \pi^{-s/2}\,\Gamma(s/2)\,\zeta(s), \qquad \varphi(s) = \frac{\xi(2s-1)}{\xi(2s)},$$
$$\sigma_\nu(n) = \sum_{d\mid n} d^\nu$$

and K is the Bessel function.

In order to prove the equidistribution of μ_t we consider its inner products with various functions spanning $L^2(X)$. We begin with inner products with Maass cusp forms φ_j. Set

$$J_j(t) = \int_X \varphi_j\, d\mu_t = \int_X \varphi_j(z)\, E\left(z, \frac{1}{2} + it\right) E\left(z, \frac{1}{2} - it\right) \frac{dx\,dy}{y^2}. \tag{9}$$

To investigate this we first consider

$$I_j(s) = \int_X \varphi_j(z)\, E\left(z, \frac{1}{2} + it\right) E(z, s)\, \frac{dx\,dy}{y^2}. \tag{10}$$

Note that all of the above integrals converge rapidly since φ_j is a cusp form. Now for $\Re(s) > 1$ we can "unfold" the integral in (10) using the definition (7) to get

$$I_j(s) = \int_0^\infty \int_0^1 \varphi_j(z)\, E\left(z, \frac{1}{2} + it\right) y^s\, \frac{dx\,dy}{y^2}. \tag{11}$$

Since $E(z, s) = E(-\bar{z}, s)$ it follows that $I_j(s) \equiv 0$ if φ_j is an odd cusp form (the cusp forms are of two types $\varphi_j(-\bar{z}) = \varepsilon\varphi_j(z)$, $\varepsilon = 1, -1$) so we may assume that φ_j is even. In this case it has a Fourier development

$$\varphi_j(z) = y^{1/2} \sum_{n=1}^{\infty} \rho_j(1)\, \lambda_j(n)\, K_{it_j}(2\pi n y) \cos(2\pi n x). \tag{12}$$

Here $(1/4) + t_j^2 = \lambda_j$ and the coefficients $\lambda_j(n)$ satisfy the multiplicative relations which are a consequence of $\varphi_j(z)$ being a Hecke eigenform. (We will ignore the normalization $\rho_j(1)$ since in the discussion of this section j is fixed.) These amount to

$$L(\varphi_j, s) := \sum_{n=1}^{\infty} \frac{\lambda_j(n)}{n^s} = \prod_p \left(1 - \lambda_j(p)\, p^{-s} + p^{-2s}\right)^{-1}. \tag{13}$$

So

$$\begin{aligned} I_j(s) = \int_0^\infty \int_0^1 & \Big(y^{1/2} \sum_{n=1}^{\infty} \lambda_j(n)\, K_{it_j}(2\pi n y) \cos(2\pi n x)\Big) \\ & \left(y^{\frac{1}{2}+it} + \varphi\left(\frac{1}{2} + it\right) y^{\frac{1}{2}-it} + \frac{2y^{1/2}}{\xi(1+2it)} \sum_{n=1}^{\infty} n^{it}\, \sigma_{-2it}(n)\, K_{it}(2\pi n y) \cos(2\pi n x)\right) y^s\, \frac{dx\,dy}{y^2} \\ = \frac{1}{\xi(1+2it)} & \int_0^\infty \Big(\sum_{n=1}^{\infty} \lambda_j(n)\, K_{it_j}(2\pi n y)\, n^{it}\, \sigma_{-2it}(n)\, K_{it}(2\pi n y)\Big) y^s\, \frac{dy}{y} \\ = \frac{1}{\xi(1+2it)} & \left(\sum_{n=1}^{\infty} \frac{\lambda_j(n)\, n^{it}\, \sigma_{-2it}(n)}{n^s}\right) \int_0^\infty K_{it_j}(2\pi y)\, K_{it}(2\pi y)\, y^s\, \frac{dy}{y} \end{aligned}$$

which on evaluation of the integral ([G-R]) yields

$$(14)\qquad \frac{2\pi^{-s}}{\xi(1+2it)}\,\frac{\Gamma\left(\frac{s+it_j+it}{2}\right)\Gamma\left(\frac{s-it_j+it}{2}\right)\Gamma\left(\frac{s+it_j-it}{2}\right)\Gamma\left(\frac{s-it_j-it}{2}\right)}{\Gamma(s)}\,R(s)$$

where $\qquad R(s) = \sum_{n=1}^{\infty} \frac{\lambda_j(n)\, n^{it}\, \sigma_{-2it}(n)}{n^s}.$

Now $R(s)$ may be expressed in terms of the L-function $L(\varphi_j, s)$ in (13):

$$(15)\qquad R(s) = \frac{L(\varphi_j, s-it)\, L(\varphi_j, s+it)}{\zeta(2s)}.$$

To prove (15) first we factor $R(s)$ (we write $\lambda(n)$ for $\lambda_j(n)$ in short),

$$R(s) = \prod_p R_p(s)$$

where

$$(16)\qquad R_p(s) = \sum_{j=0}^{\infty} \lambda(p^j)\, p^{ijt}\, \sigma_{-2it}(p^j)\, p^{-js}$$

Now

$$\sigma_{-2it}(p^j) = \sum_{k=0}^{j} p^{-2itk} = \frac{1-p^{-2it(j+1)}}{1-p^{-2it}},$$

so

$$\begin{aligned} R_p(s) &= \sum_{j=0}^{\infty} \lambda(p^j)\, p^{ijt}\, \frac{1-p^{-2it(j+1)}}{1-p^{-2it}}\, p^{-js} \\ &= \frac{1}{1-p^{-2it}}\left(\sum_{j=0}^{\infty} \lambda(p^j)\, p^{-j(s-it)} - p^{-2it}\sum_{j=0}^{\infty} \lambda(p^j)\, p^{-j(s+it)}\right) \\ &= \frac{1}{1-p^{-2it}}\left(\frac{1}{1-\lambda(p)\,p^{-(s-it)}+p^{-2(s-it)}} - \frac{p^{-2it}}{1-\lambda(p)\,p^{-(s+it)}+p^{-2(s+it)}}\right) \\ &= \frac{1-p^{-2s}}{(1-\lambda(p)\,p^{-(s-it)}+p^{-2(s-it)})\,(1-\lambda(p)\,p^{-(s+it)}+p^{-2(s+it)})}. \end{aligned}$$

This proves (15). Using this in combination with (14), (11) and (9) we get

$$(17)\qquad J_j(t) = I_j\left(\frac{1}{2}-it\right)$$

$$= 2\pi^{-2it}\,\frac{\left|\Gamma\left(\frac{1}{4}+\frac{it_j}{2}\right)\right|^2}{|\zeta(1+2it)|^2}\,\frac{\Gamma\left(\frac{1}{4}-\frac{it_j}{2}-it\right)\Gamma\left(\frac{1}{4}+\frac{it_j}{2}-it\right)}{\left|\Gamma\left(\frac{1}{2}+it\right)\right|^2}\,L\left(\varphi_j, \frac{1}{2}\right)L\left(\varphi_j, \frac{1}{2}-2it\right).$$

One can check using the functional equation for $L(\varphi_j, s)$ that the RHS of (17) is real as it has to be in view of (9). An interesting point here is that $J_j(t) \equiv 0$ if $L(\varphi_j, 1/2) = 0$.

We are now ready to investigate the behavior of $J_j(t)$ as $t \to \infty$. Using Stirling's formula ($|\Gamma(\sigma + it)| \sim e^{-\pi|t|/2} |t|^{\sigma-(1/2)}$), we see that the Γ factors in (17) will yield a factor $\sim c\,|t|^{-1/2}$ as $t \to \infty$ (here t_j is fixed). This together with the arithmetic estimates:

(A) $(\log t)^{-1} \ll |\zeta(1 + it)| \ll \log t$,

(B) $\left|L\left(\varphi_j, \frac{1}{2} + it\right)\right| \ll_{j,\varepsilon} |t|^{(1/3)+\varepsilon}$, $\varepsilon > 0$,

yields

Proposition **2.1.**

$$|J_j(t)| \ll_{j,\varepsilon} |t|^{-(1/6)+\varepsilon}, \quad \text{for all } \varepsilon > 0.$$

The estimate (A) above is well-known in the theory of the Riemann zeta function ([T]). Actually later we will need an improvement of (A) due to Weyl. Indeed this method of Weyl of estimating exponential sums is used crucially by Meurman [MEU] in his proof of the estimate (B) above. The standard convexity bound for $L\left(\varphi, \frac{1}{2} + it\right)$ is $\left|L\left(\varphi_j, \frac{1}{2} + it\right)\right| \ll_{j,\varepsilon} |t|^{(1/2)+\varepsilon}$. Clearly this would not suffice to show $J_j(t) \to 0$ as $t \to \infty$. However any improvement (in the exponent) of the standard bound would suffice for our purpose. This completes the analysis of the inner products of μ_t with cusp forms. We turn now to inner products of μ_t with incomplete Eisenstein series.

Let $h(y) \in C^\infty(\mathbf{R}^+)$ be a rapidly decreasing function at 0 and ∞, that is, for any positive integer N, $h(y) = O_N(y^N)$ when $0 < y \leqslant 1$, and $h(y) = O_N(y^{-N})$ when $y > 1$. Let $H(s)$ be its Mellin transform

$$H(s) = \int_0^\infty h(y)\, y^{-s} \frac{dy}{y}. \tag{18}$$

Clearly $H(s)$ is entire in s and is of Schwartz class in t for each vertical line $\sigma + it$. The inversion formula gives

$$h(y) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} H(s)\, y^s\, ds \tag{19}$$

for any $\sigma \in \mathbf{R}$. For such an h we form the convergent series

$$F_h(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} h(y(\gamma z)) = \frac{1}{2\pi i} \int_{\Re(s) = 2} H(s)\, E(z, s)\, ds; \tag{20}$$

F_h belongs to $C^\infty(X)$ and is rapidly decreasing in the cusp (i.e. as $y \to \infty$). Hence we may form

$$\int_X F_h(z)\, d\mu_t(z) = \int_X F_h(z) \left| E\left(z, \frac{1}{2} + it\right)\right|^2 \frac{dx\,dy}{y^2}$$

$$= \frac{1}{2\pi i} \int_X \int_{\Re(s)=2} H(s)\, E(z,s)\, ds \left| E\left(z, \frac{1}{2} + it\right)\right|^2 \frac{dx\,dy}{y^2}$$

$$= \frac{1}{2\pi i} \int_0^\infty \int_{\Re(s)=2} H(s)\, y^s\, ds \int_0^1 \left| E\left(z, \frac{1}{2} + it\right)\right|^2 \frac{dx\,dy}{y^2}$$

$$= \frac{1}{2\pi i} \int_0^\infty \int_{\Re(s)=2} H(s)\, y^s\, ds \left(\left| y^{1/2+it} + \varphi\left(\frac{1}{2} + it\right) y^{1/2-it}\right|^2 \right.$$

$$\left. + \frac{2}{|\xi(1+2it)|^2} \sum_{n=1}^\infty y\, |\sigma_{-2it}(n)|^2\, |K_{it}(2\pi n y)|^2 \right) \frac{dy}{y^2}.$$

Now $\left|\varphi\left(\frac{1}{2} + it\right)\right|^2 = 1$, so that the first term above contributes (as $t \to \infty$)

$$(21) \qquad 2 \int_0^\infty h(y)\, \frac{dy}{y} + \text{a rapidly decreasing function of } t \text{ depending on } h.$$

The second term which we denote by $I_2(t)$ is

$$I_2(t) = \frac{1}{\pi i\, |\xi(1+2it)|^2} \int_{\Re(s)=2} H(s) \left(\sum_{n=1}^\infty \frac{|\sigma_{-2it}(n)|^2}{n^s}\right) \int_0^\infty |K_{it}(2\pi y)|^2 y^s \frac{dy}{y}\, ds.$$

The series can be evaluated as was first done by Ramanujan [RA]

$$\sum_{n=1}^\infty \frac{|\sigma_{-2it}(n)|^2}{n^s} = \frac{\zeta^2(s)\, \zeta(s-2it)\, \zeta(s+2it)}{\zeta(2s)}.$$

The y-integral is evaluated in terms of Γ-functions as before. We obtain (note that t is fixed while s is the variable for the integral)

$$I_2(t) = \frac{2}{i\pi\, |\xi(1+2it)|^2} \int_{\Re(s)=2} \frac{\pi^{-s} H(s)\, \zeta^2(s)\, \zeta(s-2it)\, \zeta(s+2it)\, \Gamma^2(s/2)\, \Gamma\left(\frac{s}{2} - it\right) \Gamma\left(\frac{s}{2} + it\right)}{\zeta(2s)\, \Gamma(s)}\, ds$$

$$= \frac{2}{i\pi\, |\xi(1+2it)|^2} \int_{\Re(s)=2} B(s)\, ds, \text{ say.}$$

Now shift the integral to $\Re(s) = 1/2$,

$$\mathrm{I}_2(t) = \frac{4\pi i}{i\pi\,|\,\xi(1+2it)|^2}\,\mathrm{Res}_{s=1}\,\mathrm{B}(s) + \frac{2}{i\pi\,|\,\xi(1+2it)|^2}\int_{\Re(s)=1/2}\mathrm{B}(s)\,ds + \mathrm{O}(t^{-10}).$$

The O-term comes from the contribution of poles at $s = 1 \pm 2it$. To justify shifting the contour we use Stirling formula to estimate the Γ-factors and the fact that $\mathrm{H}(\sigma + it)$ is rapidly decreasing in t. In fact using this and Weyl's bound

$$\zeta\left(\frac{1}{2} + it\right) \ll_\varepsilon t^{(1/6)+\varepsilon}, \tag{22}$$

we find that

$$\frac{2}{i\pi\,|\,\xi(1+2it)|^2}\int_{\Re(s)=1/2}\mathrm{B}(s)\,ds \ll_\varepsilon t^{(1/3)+\varepsilon}\,t^{-1/2} = t^{-(1/6)+\varepsilon}. \tag{23}$$

This corresponds to the bound in Proposition 2.1. The residue term is more complicated. Write $\mathrm{B}(s)$ as $\zeta^2(s)\,\mathrm{G}(s)$ where $\mathrm{G}(s)$ is holomorphic at $s = 1$. Then $\mathrm{Res}_{s=1}\,\mathrm{B}(s) = \mathrm{G}(1)\left(2\gamma + \frac{\mathrm{G}'}{\mathrm{G}}(1)\right)$, where γ is Euler's constant. Now a simple calculation gives

$$\mathrm{G}(1) = \frac{24}{\pi}\,\mathrm{H}(1)$$

and

$$\frac{\mathrm{G}'}{\mathrm{G}}(1) = \frac{\mathrm{H}'}{\mathrm{H}}(1) + \mathrm{C} + \frac{\zeta'}{\zeta}(1-2it) + \frac{\zeta'}{\zeta}(1+2it) + \frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+it\right) + \frac{\Gamma'}{\Gamma}\left(\frac{1}{2}-it\right),$$

C being independent of t. According to the Weyl-Hadamard-de la Vallée Poussin bound [T]

$$\frac{\zeta'}{\zeta}(1+it) \ll \frac{\log t}{\log\log t} \tag{24}$$

and Stirling's approximation

$$\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+it\right) = \log t + \mathrm{O}(1),$$

we have

$$\mathrm{Res}_{s=1}\,\mathrm{B}(s) = \frac{48}{\pi}\,\mathrm{H}(1)\log t + \mathrm{O}\left(\frac{\log t}{\log\log t}\right) \tag{25}$$

as $t \to \infty$. Note that

$$H(1) = \int_0^\infty h(y)\, \frac{dy}{y^2} = \int_X F_h(z)\, \frac{dx\, dy}{y^2}.$$

This leads to

Proposition **2.2.** — *Let* $F \in C^\infty(X)$ *be of the form* F_h *as in* (20). *Then*

$$\int_X F(z)\, d\mu_t(z) \sim \frac{48}{\pi} \left(\int_X F(z)\, \frac{dx\, dy}{y^2} \right) \log t$$

as $t \to \infty$.

With Proposition 2.1 and 2.2 we are ready to establish the following Proposition which by standard approximation arguments implies Theorem 1.1:

Proposition **2.3.** — *Let* $F \in C_{00}(X)$ (*i.e.* F *is a continuous function of compact support in* X); *then*

$$\int_X F(z)\, d\mu_t(z) \sim \frac{48}{\pi} \left(\int_X F(z)\, \frac{dx\, dy}{y^2} \right) \log t$$

as $t \to \infty$.

Proof. — It is easy to see that the functions of the form F_h as above together with the cusp forms φ_j are dense in $C_0(X)$ (the space of continuous functions vanishing in the cusp). Let $F \in C_{00}(X)$ and $\varepsilon > 0$; then we can find $G = G_1 + G_2$ with G_1 a finite sum of cusp forms and G_2 in the space of incomplete Eisenstein series with corresponding $h \in C_{00}^\infty(\mathbf{R}^+)$, such that $\| G - F \|_\infty < \varepsilon$. If $H = G - F$ then H is rapidly decreasing in the cusp and so we can find an $h_1 \geqslant 0$ which is rapidly decreasing and for which

$$H_1(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} h_1(y(\gamma z)) \geqslant | H(z) |,$$

and

$$\int_X H_1(z)\, dV(z) < 5\varepsilon.$$

Hence by positivity of $d\mu_t$ we have

$$\overline{\lim_{t \to \infty}} \frac{1}{\log t} \left| \int_X H(z)\, d\mu_t(z) \right| \leqslant \frac{240}{\pi}\, \varepsilon.$$

From this the Proposition follows easily.

To end this section we remark that while $| \langle F, \mu_t \rangle |$ is expected to be of size $t^{-1/2}$ for F in the space of cusp forms (and smooth), the above shows for F an incomplete Eisenstein series and of mean zero, that $\langle F, \mu_t \rangle / \log t \to 0$, but this convergence is very slow.

3. Incomplete Eisenstein Series and Poincaré Series

In this section we will establish Theorem 1.2 in a very special but important case, i.e., $H(z)$ is either an incomplete Eisenstein series or an incomplete Poincaré series. For these functions, we have the advantage of being able to "unfold" the integral $\int_X H(z)\, d\mu_j(z)$ and then appeal to automorphic L-function theory and the Petersson-Kuznetsov formula. In the next section we will see that Theorem 1.2 holds in general by an approximation argument.

Proposition **3.1.** — *Let* $h(x)$ *be a smooth function on* $(0, \infty)$, *supported in* (x_0, ∞), $x_0 > 0$, *such that for some* $C_{i,k} \geqslant 1$,

$$| h^{(i)}(x) | \leqslant C_{i,k}\, x^{-k}, \quad i, k \geqslant 0.$$

Let

$$P_{h,0}(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} h(y(\gamma z)).$$

Then for any $\varepsilon > 0$, $T \geqslant 1$, *we have*

$$\sum_{t_j \leqslant T} | \langle P_{h,0}, | u_j(z) |^2 \rangle - \overline{P_{h,0}} |^2 \ll_\varepsilon C_{1,1}\, C_{8,8}\, T^{1+\varepsilon},$$

where

$$\overline{P_{h,0}} = \frac{1}{\mathrm{Vol}(\Gamma \backslash \mathbf{H})} \int_{\Gamma \backslash \mathbf{H}} P_{h,0}(z)\, dV(z).$$

Proof. — By unfolding the integral and using (12) we have, with $\rho_j(n) = \rho_j(1)\, \lambda_j(n)$,

$$\int_{\Gamma \backslash \mathbf{H}} | u_j(z) |^2 \Big(\sum_{\gamma \in \Gamma_\infty \backslash \Gamma} h(y(\gamma z)) \Big)\, dV(z) = \sum_{k \neq 0} | \rho_j(k) |^2 \int_0^\infty K_{it_j}^2(r)\, h\left(\frac{r}{2\pi |k|} \right) \frac{dr}{r}.$$

Set $h_1(x) = h(x^{-1})$, and

$$G(s) = \int_0^\infty h_1(x)\, x^{s-1}\, dx = \int_0^\infty h(x)\, x^{-s-1}\, dx. \tag{26}$$

Then $G(s)$ is entire and by Mellin inversion

$$h_1(x) = \frac{1}{2\pi i} \int_{(\sigma)} G(s)\, x^{-s}\, ds, \quad \sigma > 0. \tag{27}$$

It is clear from the definition of $G(s)$, by partial integration, that

$$G(s) \ll_{\sigma_0} \frac{C_{l,l}}{| s(s-1) \dots (s-l+1) |} \tag{28}$$

for $l \geqslant 0$, $s \notin \mathbf{Z}$, and $1 \gg \sigma = \Re(s) \geqslant \sigma_0 > 0$. By Mellin transform and a well-known formula (see (35)), it follows that

$$\int_0^\infty K_{it_j}^2(r)\, h\left(\frac{r}{2\pi\,|\,k\,|}\right)\frac{dr}{r}$$
$$= \frac{1}{2\pi i}\int_{(\sigma)} \frac{G(s)}{(2\pi\,|\,k\,|)^s}\, 2^{-3+s}\,\Gamma\left(\frac{s+2it_j}{2}\right)\Gamma\left(\frac{s-2it_j}{2}\right)\frac{\Gamma^2\left(\frac{s}{2}\right)}{\Gamma(s)}\, ds.$$

where $\sigma > 1$. Then we have

$$(29)\qquad \int_{\Gamma\backslash\mathbf{H}} |\,u_j(z)\,|^2\, P_{h,0}(z)\, dV(z)$$
$$= \frac{1}{8\pi i}\int_{(\sigma)} \frac{G(s)\, L(u_j\otimes u_j, s)}{\pi^s}\,\Gamma\left(\frac{s+2it_j}{2}\right)\Gamma\left(\frac{s-2it_j}{2}\right)\frac{\Gamma^2\left(\frac{s}{2}\right)}{\Gamma(s)}\, ds,$$

where $$L(u_j\otimes u_j, s) = \sum_{n=1}^\infty |\,\rho_j(n)\,|^2\, n^{-s} = |\,\rho_j(1)\,|^2 \sum_{n=1}^\infty \lambda_j^2(n)\, n^{-s}.$$

We move the line of integration to $\Re(s) = 1/2$ and pass the simple pole of the integrand at $s=1$ with residue $\overline{P_{h,0}} = \frac{1}{\mathrm{Vol}(\Gamma\backslash\mathbf{H})}\int_{\Gamma\backslash\mathbf{H}} P_{h,0}(z)\, dV(z)$, in view of $\mathrm{Res}_{s=1} L(u_j\otimes u_j, s) = 12\pi^{-2}\cosh(\pi t_j)$ and $\overline{P_{h,0}} = 3\pi^{-1}\int_0^\infty h(y)\frac{dy}{y^2} = 3\pi^{-1}\, G(1)$, which follows from the unfolding method. In order to finish the proof, we need to understand the behavior of $L(u_j\otimes u_j, s)$ uniformly in j and s, which is of independent interest.

Let $L^{(2)}(u_j, s)$ stand for the second symmetric power L-function [SHI] attached to the Maass-Hecke form $u_j(z)$, and

$$R_j(s) = \zeta(2s)\sum_{n=1}^\infty \lambda_j^2(n)\, n^{-s},$$

the Rankin-Selberg convolution L-function. We have

$$R_j(s) = \zeta(s)\, L^{(2)}(u_j, s),\quad \sum_{n=1}^\infty \lambda_j^2(n)\, n^{-s} = \zeta(s)\sum_{n=1}^\infty \lambda_j(n^2)\, n^{-s}.$$

Hence

$$L^{(2)}(u_j, s) = \zeta(2s)\sum_{n=1}^\infty \lambda_j(n^2)\, n^{-s} = \sum_{n=1}^\infty c_j(n)\, n^{-s},$$
$$c_j(n) = \sum_{l^2 k = n} \lambda_j(k^2),$$

for $\Re(s) > 1$. It is well known from the work of Shimura [SHI] that $L^{(2)}(u_j, s)$ is entire and

$$\pi^{-3/2s}\,\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s}{2}+it_j\right)\Gamma\left(\frac{s}{2}-it_j\right)L^{(2)}(u_j, s)$$

is invariant under the change of variable $s \to 1 - s$. Define

$$\theta_j(s) = \pi^{-3/2s}\,\Gamma\left(\frac{s}{2} + it_j\right)\Gamma\left(\frac{s}{2} - it_j\right)$$

and let $w = (1/2) + it_0$. Consider the integral $(x > 0)$

$$\frac{1}{2\pi i}\int_{(\sigma)} L^{(2)}(u_j, s + w)\,\Gamma(s + l)\,\frac{x^{-s}}{s}\,ds, \tag{30}$$

where l is a positive integer, and $\sigma = (1/2) + 1/\log t_j$. Clearly (30) equals

$$\sum_{n=1}^{\infty} \frac{c_j(n)}{n^w}\,F(nx),$$

where
$$F(t) = \frac{1}{2\pi i}\int_{(\sigma)} \Gamma(s + l)\,\frac{t^{-s}}{s}\,ds = \int_t^{\infty} e^{-\xi}\,\xi^{l-1}\,d\xi.$$

Moving the line of integration in (30) to $-\sigma$, we pass the simple pole of the integrand at $s = 0$ with residue $\Gamma(l)\,L^{(2)}(u_j, w)$, and get

$$\Gamma(l)\,L^{(2)}(u_j, w) + \frac{1}{2\pi i}\int_{(-\sigma)} L^{(2)}(u_j, s + w)\,\Gamma(s + l)\,\frac{x^{-s}}{s}\,ds.$$

Now the integral equals

$$\frac{-1}{2\pi i}\int_{(\sigma)} L^{(2)}(u_j, -s + w)\,\Gamma(-s + l)\,\frac{x^{s}}{s}\,ds$$
$$= \frac{-1}{2\pi i}\int_{(\sigma)} L^{(2)}(u_j, s + \bar{w})\,\frac{\theta_j(s + \bar{w})}{\theta_j(-s + w)}\,\frac{\Gamma\left(\frac{s + \bar{w}}{2}\right)}{\Gamma\left(\frac{-s + w}{2}\right)}\,\Gamma(-s + l)\,\frac{x^{s}}{s}\,ds.$$

Moreover

$$\frac{\theta_j(s + \bar{w})}{\theta_j(-s + w)} = t_j^{2(s + \bar{w}) - 1}(\pi^{2s + \bar{w} - w})^{-3/2}\,(1 + O(|s + \bar{w}|^{2+\delta})\,t_j^{-1})$$
$$= \pi^{3it_0}\,t_j^{-2it_0}\left(\frac{t_j^2}{\pi^3}\right)^{s}(1 + O(|s + \bar{w}|^{2+\delta})\,t_j^{-1}).$$

In [I3] it is shown that $\Sigma_{n \leqslant N}\lambda_j^2(n) \ll t_j^{\varepsilon}N$, hence $L^{(2)}(u_j, s + \bar{w}) = R_j(s + \bar{w})/\zeta(s + \bar{w}) \ll t_j^{\delta}$, where δ is an arbitrarily small positive number. We conclude that the above integral equals

$$-\pi^{3it_0}\,t_j^{-2it_0}\sum_{n=1}^{\infty}\frac{c_j(n)}{n^{\bar{w}}}\,F\left(w, \frac{n\pi^3}{xt_j^2}\right) + O(|w|^{(5/2)+\delta}\,x^{\sigma}\,t_j^{\delta}), \tag{31}$$

where

$$(32)\qquad F(w, t) = \int_{(\sigma)} \Gamma(-s+l)\, \frac{\Gamma\left(\frac{s+\bar{w}}{2}\right)}{\Gamma\left(\frac{-s+w}{2}\right)}\, \frac{t^{-s}}{s}\, ds.$$

Hence, we have derived the "approximate functional equation"

$$\Gamma(l)\, L^{(2)}(u_j, w) = \sum_{n=1}^{\infty} \frac{c_j(n)}{n^w} F(nx) + \pi^{3it_0} t_j^{-2it_0} \sum_{n=1}^{\infty} \frac{c_j(n)}{n^{\bar{w}}} F\left(w, \frac{n\pi^3}{xt_j^2}\right) + O(|w|^{(5/2)+\delta} x^\sigma t_j^\delta).$$

Now we integrate both side of the above expression from $\pi^{3/2}/t_j$ to $e\pi^{3/2}/t_j$ with respect to the measure dx/x, which yields, for $t_j \sim T$ (we denote $T < t_j \leqslant 2T$ by $t_j \sim T$),

$$\Gamma(l)\, L^{(2)}(u_j, w) = \int_{T/t_j}^{eT/t_j} \sum_{n=1}^{\infty} \frac{c_j(n)}{n^w} F\left(\frac{n\pi^{3/2} y}{T}\right) \frac{dy}{y} + \pi^{3it_0} t_j^{-2it_0} \int_{T/et}^{T/t_j} \sum_{n=1}^{\infty} \frac{c_j(n)}{n^{\bar{w}}} F\left(w, \frac{n\pi^{3/2} y}{T}\right) \frac{dy}{y} + O(|w|^{(5/2)+\delta}\, T^{-(1/2)+\delta}).$$

We observe that:

1. $F(w, t) \ll_\varepsilon t^{-\varepsilon} |w|^\varepsilon$, $t > 0$. This is easily seen by moving the line (σ) in (32) to (ε).

2. $F(w, t) \ll_l (t/|w|)^{-l+(1/2)}$, $t > 0$. This is also easily seen by moving the line (σ) in (32), but this time to $(l - 1/2)$. Thus, if $t/|w| \gg T^\varepsilon$, then for any N, $F(w, t) \ll T^{-N}(|w|/t)^2$ by choosing $l = [N/\varepsilon] + 4$.

Hence, by Cauchy's inequality, we have

$$\sum_{T < t_j \leqslant 2T} |L^{(2)}(u_j, w)|^2 \ll \int_{1/2}^{e} \sum_{T < t_j \leqslant 2T} \left| \sum_{n=1}^{\infty} \frac{c_j(n)}{n^w} F\left(\frac{n\pi^{3/2} y}{T}\right) \right|^2 \frac{dy}{y} + \int_{1/2e}^{1} \sum_{T < t_j \leqslant 2T} \left| \sum_{n=1}^{\infty} \frac{c_j(n)}{n^{\bar{w}}} F\left(w, \frac{n\pi^{3/2} y}{T}\right) \right|^2 \frac{dy}{y} + O(T^{1+\delta} |w|^{5+\delta}).$$

Now if we choose $l = [10/\varepsilon] + 4$, and note that $|c_j(n)/n^{\bar{w}}| \ll n^{(1/2)+\delta}$ since $|\lambda_j(k)| \ll k^{1/2}$ [SA2], we have

$$\sum_{n \gg (T|w|)^{1+\varepsilon}} \frac{c_j(n)}{n^{\bar{w}}} F\left(w, \frac{n\pi^{3/2} y}{T}\right) \ll |w|^2.$$

Also

$$\sum_{n \gg T^{1+\varepsilon}} \frac{c_j(n)}{n^w} F\left(\frac{n\pi^{3/2} y}{T}\right) \ll 1.$$

Using [D-I1]

$$\sum_{t_j \leqslant T} \Big| \sum_{n \leqslant N} a_n\, v_j(n) \Big|^2 \ll (T^2 + N)\, T^{\varepsilon} \sum_{n \leqslant N} |\, a_n \,|^2,$$

the lower bound [I2], $v_j(1) \gg t_j^{-\varepsilon}$ (where $\rho_j(n) = \cosh(\pi t_j/2)\, v_j(n)$, $v_j(n) = v_j(1)\, \lambda_j(n)$), the definition of $c_j(n)$, the above remarks on F, and Cauchy's inequality, we deduce that

$$\sum_{T < t_j \leqslant 2T} |\, L^{(2)}(u_j, w) |^2 \ll T^{2+\varepsilon} \,|\, w \,|^{5+\varepsilon}. \tag{33}$$

Finally, by the upper bound $v_j(1) \ll t_j^{\varepsilon}$ [HN-L], and the crude bound $\zeta(w) = O(|\, w \,|^{1/2})$, we have proven

Theorem **3.2.**

$$\sum_{t_j \leqslant T} \frac{1}{\cosh 2\pi t_j} \,|\, L(u_j \otimes u_j, w) |^2 \ll T^{2+\varepsilon} \,|\, w \,|^{(11/2)+\varepsilon} \tag{34}$$

This establishes (4) and the conjecture of Iwaniec (5).

We return to (29). By Cauchy's inequality,

$$\left(\int_{\Gamma \backslash H} |\, u_j(z) |^2\, P_{h,0}(z)\, d\mu(z) - \overline{P_{h,0}} \right)^2$$

$$\ll \left(\int_{(1/2)} |\, G(s) |\, \frac{\left| \Gamma\left(\frac{s}{2}\right) \right|^4}{|\, \Gamma(s) |^2} \,|\, ds \,| \right) \left(\int_{(1/2)} |\, G(s) |\, |\, L(u_j \otimes u_j, s) |^2 \left| \Gamma\left(\frac{s + 2it_j}{2}\right) \Gamma\left(\frac{s - 2it_j}{2}\right) \right|^2 |\, ds \,| \right)$$

$$\ll C_{1,1} \int_{(1/2)} |\, G(s) |\, |\, L(u_j \otimes u_j, s) |^2 \left| \Gamma\left(\frac{s + 2it_j}{2}\right) \Gamma\left(\frac{s - 2it_j}{2}\right) \right|^2 |\, ds \,|.$$

But according to Theorem 3.2 and Stirling's formula,

$$\sum_{t_j \sim T} \int_{|\Im s| \leqslant T/10} |\, G(s) |\, |\, L(u_j \otimes u_j, s) |^2 \left| \Gamma\left(\frac{s + 2it_j}{2}\right) \Gamma\left(\frac{s - 2it_j}{2}\right) \right|^2 |\, ds \,| \ll C_{8,8}\, T^{1+\varepsilon}.$$

Also, we have trivially, again using Theorem 3.2 and (28),

$$\sum_{t_j \sim T} \int_{|\Im s| \geqslant T/10} |\, G(s) |\, |\, L(u_j \otimes u_j, s) |^2 \left| \Gamma\left(\frac{s + 2it_j}{2}\right) \Gamma\left(\frac{s - 2it_j}{2}\right) \right|^2 |\, ds \,| \ll C_{8,8}\, T^{1+\varepsilon}.$$

This completes the proof of Proposition 3.1.

Proposition **3.3.** — *Let $h(x)$ be a smooth function on $(0, \infty)$, supported in (x_0, ∞), $x_0 > 0$, such that for some $C_{i,k} \geqslant 1$,*

$$|\, h^{(i)}(x) \,| \leqslant C_{i,k}\, x^{-k}, \quad i, k \geqslant 0.$$

Let

$$P_{h,m}(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} h(y(\gamma z))\, e(mx(\gamma z)).$$

Then for any $\varepsilon > 0$, $m \neq 0$, $T \geqslant 1$, *we have*

$$\sum_{t_j \leqslant T} | \langle P_{h,m}, | u_j(z) |^2 \rangle |^2 \ll_\varepsilon m^2 (C_{2,2}^2 + C_{0,0}^2)\, T^{1+\varepsilon}.$$

Proof. — Without loss of generality we may assume that the first Fourier coefficient $\rho_j(1)$ of $u_j(z)$ is real. It follows from the unfolding method that

$$\int_{\Gamma\backslash H} | u_j(z) |^2 \Big(\sum_{\gamma \in \Gamma_\infty\backslash\Gamma} h(y(\gamma z))\, e(mx(\gamma z))\Big)\, dV(z)$$

$$= \sum_{k \neq 0, -m} \rho_j(k)\, \rho_j(k+m) \int_0^\infty K_{it_j}(r)\, K_{it_j}\left(\left| 1 + \frac{m}{k} \right| r\right) h\left(\frac{r}{2\pi\, |k|}\right) \frac{dr}{r}$$

$$= \rho_j(1) \sum_{d \mid m} \sum_{k \neq 0, -m/d} \rho_j(k^2 + km/d) \int_0^\infty K_{it_j}(r)\, K_{it_j}\left(\left| 1 + \frac{m}{dk} \right| r\right) h\left(\frac{r}{2\pi\, d\, |k|}\right) \frac{dr}{r}.$$

Here we use, for $k > 0$, $\rho_j(k) = \rho_j(1)\, \lambda_j(k)$, $\rho_j(-k) = \varepsilon_j\, \rho_j(k)$, $\varepsilon_j = \pm 1$, and the multiplicativity of Hecke eigenvalues

$$\lambda_j(n)\, \lambda_j(n) = \sum_{d \mid (n, m)} \lambda_j\left(\frac{nm}{d^2}\right).$$

By a standard dyadic partition it suffices to estimate the sums with $t_j \sim T$, $k \sim K$, $K > 0$. The cases where $dK \geqslant AT$ (A is sufficiently large) can be ignored, since from [G-R]

$$K_{it}(x) = \int_0^\infty e^{-x \cosh \tau} \cos t\tau\, d\tau,$$

we have

$$K_{it}(x) \ll x^{-1} e^{-x/2}, \quad x \geqslant 0.$$

So the contribution from these terms is exponentially small, in view of $\rho_j(1) \ll \cosh(\pi t_j/2)\, t_j^\varepsilon$, $\lambda_j(k) \ll k^{1/2}$, and $h(y)$ is supported in $y \geqslant y_0 > 0$. Henceforth, we assume $dK \leqslant AT$. Similarly, we can assume $m \ll T$, because otherwise $r \mid 1 + m/dk \mid > AT$. Recall [G-R]

(35)
$$\int_0^\infty x^{-\lambda} K_\mu(ax)\, K_\nu(bx)\, dx$$

$$= \frac{2^{-2-\lambda} a^{-\nu+\lambda-1} b^\nu}{\Gamma(1-\lambda)}\, \Gamma\left(\frac{1-\lambda+\mu+\nu}{2}\right) \Gamma\left(\frac{1-\lambda-\mu+\nu}{2}\right)$$

$$\Gamma\left(\frac{1-\lambda+\mu-\nu}{2}\right) \Gamma\left(\frac{1-\lambda-\mu-\nu}{2}\right) F\left(\frac{1-\lambda+\mu+\nu}{2}, \frac{1-\lambda-\mu+\nu}{2}, 1-\lambda, 1-\frac{b^2}{a^2}\right),$$

$$\Re(a+b) > 0, \quad \Re\lambda < 1 - |\Re\mu| - |\Re\nu|,$$

where

$$F(\alpha, \beta, \gamma, z) = \sum_{n=0}^{\infty} \frac{\alpha(\alpha+1)\dots(\alpha+n-1)\,\beta(\beta+1)\dots(\beta+n-1)}{\gamma(\gamma+1)\dots(\gamma+n-1)\,n!} z^n$$

is the hypergeometric series. Taking $\lambda = 1 - \varepsilon$, $\mu = \nu = it_j$, $a = b = 1$ or $a = b = |1 + m/dk|$, and using $ab \leqslant (a^2 + b^2)/2$ and Stirling's formula we obtain

$$\sum_{t_j \sim T} \left| \sum_{k \sim K} \rho_j(1)\, \rho_j(k^2 + km/d) \int_0^\infty K_{it_j}(r)\, K_{it_j}\left(\left|1 + \frac{m}{dk}\right| r\right) h\left(\frac{r}{2\pi\, d\,|k|}\right) \frac{dr}{r} \right|^2 \ll C_{0,0}^2\, K^2\, T^\varepsilon.$$

Here we have applied the following result due to Kuznetsov [KU, D-I2]:

$$\sum_{t_j \leqslant T} |v_j(l)|^2 e^{-t_j/T} = 2\pi^{-2}\, T^2 + O((T + l^{1/2})\, d(l)). \tag{36}$$

Hence we can assume $K \gg T^{1/2}$, and $m \ll K^{2/3}$. With the above reduction, we proceed to prove Proposition 3.3.

As before, set $h_1(x) = h(x^{-1})$, and define $G(s)$ as in (26). We use the Mellin transform to obtain (with $\sigma = \varepsilon$)

$$\int_0^\infty K_{it_j}(r)\, K_{it_j}(|1 + m/dk|\, r)\, h\left(\frac{r}{2\pi kd}\right) \frac{dr}{r}$$

$$= \frac{1}{2\pi i} \int_{(\sigma)} \frac{G(s)}{(2\pi\, dk)^s} \int_0^\infty r^{s-1}\, K_{it_j}(r)\, K_{it_j}(|1 + m/dk|\, r)\, dr\, ds.$$

To deal with the inner integral we apply the formula (35) and [G-R p. 1040]

$$\int_0^\infty r^{s-1}\, K_{it_j}(r)\, K_{it_j}((1 + m/dk)\, r)\, dr$$

$$= 2^{-3+s}\, \Gamma\left(\frac{s + 2it_j}{2}\right) \Gamma\left(\frac{s - 2it_j}{2}\right) (1 + m/dk)^{it}$$

$$\int_0^1 \tau^{s/2-1}(1-\tau)^{s/2-1} \left(1 + \frac{2\tau m}{dk} + \tau\left(\frac{m}{dk}\right)^2\right)^{-s/2-it_j} d\tau.$$

If we can show

$$\sum_{t_j \sim T} \left| \sum_{k \sim K} a_k\, v_j(k^2 + km/d)\, f(k, t_j) \right|^2 \ll m T^{2+\varepsilon} K, \tag{37}$$

where

$$a_k = a_k(m, d, \tau, s) = \left(k^2\left(1 + \frac{2\tau m}{dk} + \tau\left(\frac{m}{dk}\right)^2\right)\right)^{-s/2}$$

$$f(k, t_j) = f(m, d, k, \tau, t_j) = \left(\frac{1 + \dfrac{m}{dk}}{1 + \dfrac{2\tau m}{dk} + \tau\left(\dfrac{m}{dk}\right)^2}\right)^{it_j},$$

then from Cauchy's inequality, (28), Stirling formula, and by considering $|\Im s| \leqslant T/10$ and $|\Im s| > T/10$ separately, we obtain Proposition 3.3.

In the following we will prove (37). It suffices to treat the case where

$$|t_j - T| \leqslant T^{1-\varepsilon}.$$

We infer that

$$\sum_j e^{-((t_j - T)/T^{1-\varepsilon})^2} \,|\sum_{k\sim K} a_k\, v_j(k^2 + km/d) f(k, t_j)|^2$$
$$= \sum_{k, l\sim K} a_k \bar{a}_l \sum_j v_j(k^2 + km/d)\, \bar{v}_j(l^2 + lm/d)\, h(t_j) + O(1),$$

where $\quad h(t) = e^{-((t-T)/T^{1-\varepsilon})^2} f(k, t) f(l, -t) + e^{-((t+T)/T^{1-\varepsilon})^2} f(k, -t) f(l, t).$

Applying Kuznetsov's formula [KU] to the inner sum, we deduce that

$$\sum_j v_j(k^2 + km/d)\, \bar{v}_j(l^2 + lm/d)\, h(t_j)$$
$$= \frac{\delta_{k,l}}{\pi^2}\int_{-\infty}^{\infty} t\tanh(\pi t)\, h(t)\, dt - \frac{2}{\pi}\int_0^{\infty} \frac{h(t)}{|\zeta(1+2it)|^2}\, d_{it}(k^2 + km/d)\, d_{it}(l^2 + lm/d)\, dt$$
$$+ \frac{2i}{\pi}\sum_{c=1}^{\infty} c^{-1}\, S(k^2 + km/d, l^2 + lm/d; c)\int_{-\infty}^{\infty} J_{2it}\left(\frac{4\pi\sqrt{(k^2 + km/d)(l^2 + lm/d)}}{c}\right) t\, \frac{h(t)}{\cosh(\pi t)}\, dt.$$

Here $S(m, n; c)$ is the Kloosterman sum and

$$d_{it}(n) = \sum_{d_1 d_2 = n}\left(\frac{d_1}{d_2}\right)^{it}.$$

Since

$$\int_{-\infty}^{\infty} t\tanh(\pi t)\, h(t)\, dt \ll T^{2+\varepsilon},$$

$$\int_0^{\infty} \frac{h(t)}{|\zeta(1+2it)|^2}\, d_{it}(k^2 + km/d)\, d_{it}(l^2 + lm/d)\, dt \ll m^{\varepsilon}\, T^{1+\varepsilon},$$

their contribution is at most $m^\varepsilon T^{2+\varepsilon} K$. It remains to estimate the sum of Kloosterman sums which we shall do for each modulus c separately. If $c > K^2 T^{-1+\varepsilon}$, we estimate the integral transform of Bessel's function

$$\frac{2i}{\pi}\int_{-\infty}^{\infty} J_{2it}\left(\frac{4\pi\sqrt{(k^2+km/d)\,(l^2+lm/d)}}{c}\right) t\,\frac{h(t)}{\cosh(\pi t)}\,dt$$

by moving the line of integration to $\Im t = -B$, where B is a sufficiently large integer, depending upon ε.

From Poisson integral formula [G-R]

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_0^\pi \cos(z\cos\theta)\sin^{2\nu}\theta\,d\theta$$

and Stirling formula for $\Gamma(s)$, it is easy to see that the resulting integral is very small. The residues of the integrand at the relevant poles are also very small. Therefore it remains to consider terms with $c \leqslant K^2 T^{-1+\varepsilon}$. We have

$$\int_{-\infty}^{\infty} J_{2it}\left(\frac{4\pi\sqrt{(k^2+km/d)\,(l^2+lm/d)}}{c}\right) t\,\frac{h(t)}{\cosh(\pi t)}\,dt$$

$$= \sum_{u=\pm 1} u \int_{T-T^{1-\varepsilon}\log^2 T}^{T+T^{1-\varepsilon}\log^2 T} J_{2it}\left(\frac{4\pi\sqrt{(k^2+km/d)\,(l^2+lm/d)}}{c}\right) t\,\frac{h(t)}{\cosh(\pi t)}\,dt + O(T^{-1}).$$

We need the following Van der Corput's lemma [T]:

Lemma **3.4.** — (1) *If* $f'(x) \geqslant \mu > 0$, *or* $f'(x) \leqslant -\mu < 0$, $x \in [a,b]$, *then*

$$\int_a^b e^{if(x)}\,dx \ll \frac{1}{\mu}.$$

(2) *If* $f''(x) \geqslant r > 0$, *or* $f''(x) \leqslant -r < 0$, $x \in [a,b]$, *then*

$$\int_a^b e^{if(x)}\,dx \ll \frac{1}{\sqrt{r}}.$$

Now (see [ER])

$$J_{ir}(x) = \frac{1}{\sqrt{2\pi}}\,\frac{1}{(r^2+x^2)^{1/4}}\,e^{\frac{\pi r}{2}-\frac{\pi i}{4}+i\omega_r(x)}\,(1+O(r^{-1})),$$

$$\omega_r(x) = \sqrt{r^2+x^2} + r\log\frac{\sqrt{r^2+x^2}-r}{x},$$

and

$$\frac{\partial}{\partial r}\,\omega_r(x) = -\log\frac{\sqrt{r^2+x^2}+r}{x},$$

$$\frac{\partial^2}{\partial r^2}\,\omega_r(x) = -\frac{1}{\sqrt{r^2+x^2}}.$$

If $T/(K^2/c) > A^{-1}$, using part (1) of Lemma 3.4 and Weil's bound for Kloosterman sum [W]

$$|\,S(m, n; c)| \leqslant d(c)\,(m, n, c)^{\frac{1}{2}} c^{\frac{1}{2}},$$

we deduce that the corresponding contribution is

$$\ll \max_{c \ll K^2 T^{-1+\varepsilon}} \sqrt{c}\,\frac{1}{\sqrt{K^2/c}}\,T^{1+\varepsilon} \ll \frac{K^2}{T}\,\frac{T^{1+\varepsilon}}{K} \ll KT^{\varepsilon}.$$

If $T/(K^2/c) \leqslant A^{-1}$, and $c \gg m$, we use (1) of the Lemma to deduce that the corresponding contribution is

$$\ll \max_{c \ll K^2 T^{-1+\varepsilon}} \sqrt{c}\,\frac{1}{\sqrt{K^2/c}}\,T^{1+\varepsilon}\,\frac{K^2/c}{T} \ll KT^{\varepsilon}.$$

If $T/(K^2/c) \leqslant A^{-1}$, and $c \ll m$, we use (2) of the Lemma to deduce that the corresponding contribution is

$$\ll \max_{c \ll m} \sqrt{c}\,\frac{1}{\sqrt{K^2/c}}\,T^{1+\varepsilon}\,\sqrt{\frac{K^2}{c}} \ll \sqrt{m}\,T^{1+\varepsilon}.$$

Hence, summing over k, l, we obtain (37).

4. Approximation

The estimates for $\langle P, \mu_j \rangle$ which were established in the previous section for incomplete Poincaré series may be used to obtain similar bounds for a general smooth function F. To do this we need to approximate such functions by P's. Let $F \in C^{\infty}(X)$. Let $C_1, C_2, \ldots, C_L$ be a decomposition of X into neighbourhoods, C_1 of the cusp, C_2 of the point i, C_3 of the point $\rho = e^{\pi i/3}$ and C_j, $j = 4, \ldots, L$ containing no elliptic fixed points. If we choose a partition of unity subordinate to this decomposition, we can write $F = \sum_{j=1}^{L} F_j$ where F_j has the same smoothness properties as F and each F_j is supported in C_j. So for our purposes here we can assume to begin with that F is supported in such a neighbourhood C_{j_0}. Let $\widetilde{C}_j$ be the lift to $\mathbf{H}$ of C_j into a fundamental domain. Let $\widetilde{F}$ be the Γ_∞ periodic function on $\mathbf{H}$ which is equal to F in $\widetilde{C}_{j_0}$. Clearly $\widetilde{F}(x,y)$ is smooth and is supported in $y \geqslant 1/2$. Also

$$F(z) = \frac{1}{W_{j_0}} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \widetilde{F}(\gamma z), \tag{38}$$

where $W_{j_0} = 1$ if $j_0 \neq 2, 3$ and otherwise is the order of the stabilizer. Expanding $\widetilde{F}(z)$ in a Fourier series in x gives

$$\widetilde{F}(z) = \sum_{m=-\infty}^{\infty} h_m(y)\, e(mx), \tag{39}$$

hence

$$F(z) = \frac{1}{W_{j_0}} \sum_{m=-\infty}^{\infty} P_{h_m, m}(z). \tag{40}$$

This will serve as our means of approximating F by the P's. From partial integration we see that, for $\mu \geqslant 0$ and $k \geqslant 0$,

$$y^k \,|\, h_m^{(j)}(y) \,| \leqslant (|\, m \,| + 1)^{-\mu} \sup_{y \geqslant 1/2} y^k \left| \frac{\partial^{j+\mu}}{(\partial y)^j\, (\partial x)^\mu} F(z) \right|, \tag{41}$$

hence, for $\mu \geqslant 0$, and $L \geqslant 0$,

$$\max_{i,k \leqslant L} C_{i,k}(h_m) \leqslant (|\, m + 1 \,|)^{-\mu} \,||\, F \,||_{L, L+\mu}. \tag{42}$$

The norm is the one introduced in (3). Now

$$W_{j_0} \,|\, \langle F, \mu_j \rangle - \overline{F} \,| = |(\langle P_{h_0,0}\, \mu_j \rangle - \overline{P}_{h_0,0}) + \sum_{m \neq 0} \langle P_{h_m,m}\, \mu_j \rangle \,|.$$

Applying Cauchy's inequality to this with weight $\alpha_m > 0$, $\alpha_0 = 1$, gives

$$|\, \langle F, \mu_j \rangle - \overline{F} \,|^2 \\ \leqslant (\sum_m \alpha_m^{-1}) \, (|\, \langle P_{h_0,0}\, \mu_j \rangle - \overline{P}_{h_0,0} \,|^2 + \sum_{m \neq 0} \alpha_m \,|\, \langle P_{h_m,m}\, \mu_j \rangle \,|^2).$$

Summing this for $|\, t_j \,| \leqslant T$ yields

$$I = \sum_{t_j \leqslant T} |\, \langle F, \mu_j \rangle - \overline{F} \,|^2 \\ \leqslant (\sum_m \alpha_m^{-1}) \, (\sum_{t_j \leqslant T} |\, \langle P_{h_0,0}\, \mu_j \rangle - \overline{P}_{h_0,0} \,|^2 + \sum_{m \neq 0} \alpha_m \sum_{t_j \leqslant T} |\, \langle P_{h_m,m}\, \mu_j \rangle \,|^2).$$

We are ready to apply the main result of section 3. Using (42) with $\mu = 3$, $\alpha_m = (|\, m \,| + 1)^{3/2}$ and the bounds of section 3, we get

$$I \ll_\varepsilon \,||\, F \,||_{8,8}^2 \, T^{1+\varepsilon}.$$

This established Theorem 1.2.

Finally we establish the above quantity for $F = \varphi_{\lambda_k}$. We can estimate $\varphi_{\lambda_k}(z)$ from its Fourier expansion. It is easily seen that $\varphi_{\lambda_j}(z)$ is exponentially small for $y \gg t_j$ and also that $\| \varphi_{\lambda_j} \|_\infty \ll \lambda_j^{1/4}$ [I-S]. In fact in [I-S] a stronger estimate has been established. However we use the crude bound which together with the Fourier expansion yields

$$\| \varphi_{\lambda_k} \|_{8,8} \ll | t_k |^{17/2}.$$

Corollary **3.1.** — *Let* $\lambda_k \neq 0$; *then*

$$\sum_{t_j \leqslant T} | \langle \varphi_{\lambda_k}, \mu_j \rangle |^2 \ll_\varepsilon | t_k |^{17} T^{1+\varepsilon}.$$

Note that this bound is only of use when T is much larger than t_k since the trivial bound for the above sum is $T^2 | t_k |$. This Corollary is the analogue of Theorem 3.2 which is equivalent to

$$\sum_{t_j \leqslant T} \left| \left\langle E\left(\cdot, \frac{1}{2} + it\right), \mu_j \right\rangle \right|^2 \ll_\varepsilon (| t | + 1)^6 T^{1+\varepsilon}. \tag{43}$$

5. Discrepancy

This section is devoted to proving the upper bounds for the discrepancy $D(\mu_j)$ as claimed in Theorem 1.5. The noncompactness of X leads to some technical complications. To deal with these, choose $y_0 > 1$ and set

$$D^{y_0}(\mu_j) = \sup_{B(\zeta, r) \subset X,\, \zeta \in F,\, \Im(\zeta) \geqslant y_0} \left| \mu_j(B) - \frac{\mathrm{Vol}(B)}{\mathrm{Vol}(X)} \right|$$

and

$$D_{y_0}(\mu_j) = \sup_{B(\zeta, r) \subset X,\, \zeta \in F,\, \Im(\zeta) \leqslant y_0} \left| \mu_j(B) - \frac{\mathrm{Vol}(B)}{\mathrm{Vol}(X)} \right|,$$

so that

$$D(\mu_j) = \max(D^{y_0}(\mu_j), D_{y_0}(\mu_j)),$$

and in particular

$$D(\mu_j) \leqslant D^{y_0}(\mu_j) + D_{y_0}(\mu_j). \tag{44}$$

We will estimate D^{y_0} and D_{y_0} differently and we begin with $D_{y_0}(\mu_j)$. Let $k(z, \zeta)$ be a point-pair invariant on **H** (see [SE]) and $K(z, \zeta) = \sum_{\gamma \in \Gamma} k(\gamma z, \zeta)$. If $\chi_{B(\zeta, r)}(z)$ is the characteristic function of the ball B (which we are assuming is injective in X) and if we define $k_r(z, \zeta) = 1$ when $d(z, \zeta) < r$, and $= 0$ otherwise, then we have

$$\chi_B(z) = K_r(z, \zeta). \tag{45}$$

According to the spectral expansion [SE] we have, at least in the L^2-sense,

$$K_r(z, \zeta) = \sum_{k=0}^{\infty} h_r(t_k)\, \varphi_k(z)\, \overline{\varphi_k(\zeta)} + \frac{1}{4\pi}\int_{-\infty}^{\infty} h_r(t)\, E\left(z, \frac{1}{2}+it\right) \overline{E\left(\zeta, \frac{1}{2}+it\right)}\, dt. \tag{46}$$

Here $h_r(t)$ is the Harish-Chandra-Selberg transform [SEL] of k_r. In order to use the expansion (46) to estimate the discrepancy we must smooth K_r so as to deal with absolutely convergent series. For $\varepsilon > 0$ let ψ_ε be an approximate identity, that is, $\psi_\varepsilon(z, \zeta) \geqslant 0$ is supported in a ball of radius ε, and

$$\int_{\mathbf{H}} \psi_\varepsilon(z, \zeta)\, dV(z) = 1.$$

We can and will also choose $\psi_\varepsilon(z, \zeta)$ so that $\psi_\varepsilon(z, \zeta) \ll \varepsilon^{-2}$ and its Harish-Chandra-Selberg transform $h^{(\varepsilon)}$ satisfies $|h^{(\varepsilon)}(t)| \ll 1$ for $|t| \leqslant 1/\varepsilon$ and is rapidly decreasing for $|t| \gg 1/\varepsilon$. Given the ball $B = B(\zeta, r)$ as above, let $A_1 = B(\zeta, r_1)$, $A_2 = B(\zeta, r_2)$ where $r_1 = r - 2\varepsilon$ and $r_2 = r + 2\varepsilon$ (if $r_1 < 0$ then A_1 is taken to be the empty set and $\chi_{A_1} \equiv 0$). For a function $F(z)$ defined on $\Gamma\backslash\mathbf{H}$ we set

$$(F * \overline{\psi}_\varepsilon)(z) := \int_X F(\zeta)\, \overline{\psi}_\varepsilon(\zeta, z)\, dV(\zeta),$$

where $$\overline{\psi}_\varepsilon(z, \zeta) = \sum_{\gamma\in\Gamma} \psi_\varepsilon(\gamma z, \zeta).$$

It is easily seen that with these choices

$$k_{r_1} * \overline{\psi}_\varepsilon(z) \leqslant \chi_A(z) \leqslant k_{r_2} * \overline{\psi}_\varepsilon(z). \tag{47}$$

For $l = 1, 2$ the expansions of these functions take the form

$$k_{r_l} * \overline{\psi}_\varepsilon(z) = \sum_{k=0}^{\infty} h_{r_l}(t_k)\, h^{(\varepsilon)}(t_k)\, \varphi_k(z)\, \overline{\varphi_k(\zeta)} + \frac{1}{4\pi}\int_{-\infty}^{\infty} h_{r_l}(t)\, h^{(\varepsilon)}(t)\, E\left(z, \frac{1}{2}+it\right) \overline{E\left(\zeta, \frac{1}{2}+it\right)}\, dt. \tag{48}$$

The mean value over X of the functions in (47) differ by a quantity which is $O(\varepsilon)$ and we may therefore conclude that

$$\left|\mu_j(B) - \frac{\mathrm{Vol}(B)}{\mathrm{Vol}(X)}\right| \ll \varepsilon + \sum_{l=1}^{2} \Big| \sum_{k\neq 0} h_{r_l}(t_k)\, h^{(\varepsilon)}(t_k) \langle \mu_j, \varphi_k \rangle \overline{\varphi_k(\zeta)} + \frac{1}{4\pi}\int_{-\infty}^{\infty} h_{r_l}(t)\, h^{(\varepsilon)}(t) \left\langle \mu_j, E\left(\cdot, \frac{1}{2}+it\right)\right\rangle \overline{E\left(\zeta, \frac{1}{2}+it\right)}\, dt \Big|. \tag{49}$$

Now

$$| \sum_{k \neq 0} h_{r_l}(t_k)\, h^{(\varepsilon)}(t_k) \langle \mu_j, \varphi_k \rangle \overline{\varphi_k(\zeta)} |^2$$
$$\leqslant \Big(\sum_{k \neq 0} | h_{r_l}(t_k) |^2 \, | h^{(\varepsilon)}(t_k) | \, | \varphi_k(\zeta) |^2 \Big) \Big(\sum_{k \neq 0} | \langle \mu_j, \varphi_k \rangle |^2 \, | h^{(\varepsilon)}(t_k) | \Big),$$

which, according to the estimate on $h^{(\varepsilon)}$, is

$$\ll \Big(\sum_{|t_k| \ll 1/\varepsilon,\, k \neq 0} | h_{r_l}(t_k) |^2 \, | \varphi_k(\zeta) |^2 \Big) \Big(\sum_{|t_k| \ll 1/\varepsilon,\, k \neq 0} | \langle \mu_j, \varphi_k \rangle |^2 \Big).$$

Using (46) we see that

$$\sum_{|t_k| \ll 1/\varepsilon,\, k \neq 0} | h_{r_l}(t_k) |^2 \, | \varphi_k(\zeta) |^2 \ll \int_X | K_{A_2}(z, \zeta) |^2 \, dV(z). \tag{50}$$

The right hand side can easily be estimated uniformly in ζ, r for $\Im(\zeta) \leqslant y_0$ and one finds it to be $O(1 + \varepsilon^3 y_0)$. We have shown that

$$| \sum_{k \neq 0} h_{r_l}(t_k)\, h^{\varepsilon}(t_k) \langle \mu_j, \varphi_k \rangle \overline{\varphi_k(\zeta)} |^2 \ll (1 + \varepsilon^3 y_0) \sum_{|t_k| \ll 1/\varepsilon,\, k \neq 0} | \langle \mu_j, \varphi_k \rangle |^2.$$

The same considerations hold verbatim for the Eisenstein series contribution. On taking supremum over all balls with $\Im(\zeta) \leqslant y_0$ we are led to

$$| D_{y_0}(\mu_j) |^2 \ll \varepsilon^2 + (1 + \varepsilon^3 y_0) \Big(\sum_{|t_k| \ll 1/\varepsilon,\, k \neq 0} | \langle \mu_j, \varphi_k \rangle |^2 + \int_{|t| \leqslant 1/\varepsilon} \left| \left\langle \mu_j, E\left(\cdot, \frac{1}{2} + it\right) \right\rangle \right|^2 dt \Big). \tag{51}$$

Next we turn to estimating $D^{y_0}(\mu_j)$ which we do in a crude fashion. First we need an upper bound for $\sum_{t_j \leqslant T} | \varphi_j(z) |^2$. If $k_\varepsilon(z, \zeta)$, $\varepsilon = 1/T$ is an approximation to the identity as before (and for which $h(t) \geqslant 0$, which can be arranged), then

$$K_\varepsilon(z, z) = \sum_{\gamma \in \Gamma} k_\varepsilon(z, \gamma z) \gg \sum_{t_j \leqslant T} | \varphi_j(z) |^2.$$

On the other hand the middle term above is easily estimated as being $\ll T^2 + Ty$, where $y = \Im(z)$. It follows that

$$\sum_{t_j \leqslant T} | \varphi_j(z) |^2 \ll T^2 + Ty. \tag{52}$$

From the Fourier expansion of $\varphi_j(z)$, it is easily seen that $| \varphi_j(z) |^2$ is exponentially small for $y \gg | t_j |$. Hence in considering balls $B(\zeta, r)$ when computing $D^{y_0}(\mu_j)$, with $| t_j | \leqslant T$, we can ignore those balls with center ζ satisfying $\Im(\zeta) \gg T$. (Note also that the volume of such a ball is $O(T^{-2})$.) For $B = B(\zeta, r)$ with $y_0 \leqslant \Im(\zeta) \leqslant T$ we have

$$\left| \langle \mu_j, B \rangle - \frac{\mathrm{Vol}(B)}{\mathrm{Vol}(X)} \right| \leqslant \frac{\mathrm{Vol}(B)}{\mathrm{Vol}(X)} + | \langle \mu_j, B \rangle | \leqslant \frac{1}{y_0} + \int_0^1 \int_{y_0}^T | \varphi_j(z) |^2 \frac{dx\, dy}{y^2}. \tag{53}$$

Taking supremum over all of these balls and using the fact that $D(\mu_j) \ll 1$ we conclude that

$$(54) \qquad |D^{y_0}(\mu_j)|^2 \ll \frac{1}{y_0} + \int_0^1 \int_{y_0}^T |\varphi_j(z)|^2 \frac{dx\,dy}{y^2}.$$

Gathering the bounds, we have

$$(55) \qquad |D(\mu_j)|^2 \ll \varepsilon^2 + (1 + \varepsilon^3 y_0) \Big(\sum_{|t_k| \ll 1/\varepsilon,\, k \neq 0} |\langle \mu_j, \varphi_k \rangle|^2 + \int_{|t| \leqslant 1/\varepsilon} \left| \left\langle \mu_j, E\left(\cdot, \frac{1}{2} + it\right) \right\rangle \right|^2 dt \Big) + \frac{1}{y_0} + \int_0^1 \int_{y_0}^T |\varphi_j(z)|^2 \frac{dx\,dy}{y^2}.$$

We now sum these inequalities for $|t_j| \leqslant T$ and apply Corollary 3.1, (43) and (52), where in the latter we use $y \leqslant T$, to get

$$\sum_{t_j \leqslant T} D(\mu_j)^2 \ll_\eta T^2 \varepsilon^2 + (1 + y_0 \varepsilon^3) \left(\sum_{t_k \leqslant 1/\varepsilon} t_k^{17} + \int_{|t| \leqslant 1/\varepsilon} |t|^6 \, dt \right) T^{1+\eta} + \frac{T^2}{y_0}$$

for $\eta > 0$. That is

$$\sum_{t_j \leqslant T} D(\mu_j)^2 \ll_\eta T^2 \varepsilon^2 + (1 + y_0 \varepsilon^3)\, T^{1+\eta} \varepsilon^{-19} + \frac{T^2}{y_0}.$$

Now optimize the choices of $\varepsilon = T^{-\beta}$, $y_0 = T^\alpha$. This yields the bound $T^{2-(2/21)}$ for the RHS. This concludes the proof of Theorem 1.5.

6. Appendix

In this section we will give an outline of the proof of Theorem 1.4 which follows closely the method in [I1]. Indeed using the argument below together with the analysis in § 9 of [I1], one can improve the exponent 7/10 slightly. However the bound $O_\varepsilon(x^{(2/3)+\varepsilon})$ which is mentioned following equation (12) of [I1] appears to remain out of reach. On the other hand, the exponent 2/3 can be deduced conditionally on the Lindelöf Hypothesis for the usual Dirichlet L-functions (see [I4], p. 189). We first show that

$$(56) \qquad \sum_{t_j} X^{it_j} \exp(-t_j/T) \ll T^{5/4} X^{1/8} (\log T)^2.$$

Let $h(\xi)$ be a smooth function supported in [N, 2N] whose derivatives satisfy

$$|h^{(p)}(\xi)| \ll N^{-p}, \quad \text{for } p = 0, 1, 2, \ldots$$

and

$$\int_{-\infty}^{+\infty} h(\xi)\, d\xi = N.$$

Thus, in review of [I1 lemma 8] and (5), we have

$$\sum_n h(n) \mid v_j(n) \mid^2 = \frac{12}{\pi^2} N + r(t_j, N),$$

$$\sum_{t_j \leqslant T} \mid r(t_j, N) \mid \ll T^2 N^{1/2} (\log T)^2.$$

We deduce that

$$\frac{1}{N} \sum_n h(n) \left(\sum_t \mid v_j(n)\mid^2 X^{it_j} \exp(-t_j/T)\right)$$

$$= \frac{1}{N} \sum_{t_j} \left(\sum_n \mid v_j(n)\mid^2 h(n)\right) X^{it_j} \exp(-t_j/T)$$

$$= \frac{12}{\pi^2} \sum_{t_j} X^{it_j} \exp(-t_j/T) + O(T^2 N^{-1/2} (\log T)^2).$$

Therefore, we only need to treat

$$\sum_{t_j} \mid v_j(n)\mid^2 X^{it_j} \exp(-t_j/T) \tag{57}$$

for $n \in [N, 2N]$. Let $\varphi(x)$ be a smooth function on $[0, \infty]$ such that

$$\mid \varphi(x) \mid \ll x, \quad x \to 0,$$

$$\mid \varphi^{(p)}(x) \mid \ll x^{-3}, \quad x \to \infty,$$

for $p = 0, 1, 2, 3$. Define

$$\varphi_0 = \frac{1}{2\pi} \int_0^\infty J_0(y)\, \varphi(y)\, dy,$$

$$\varphi_B(x) = \int_0^1 \int_0^\infty \xi x J_0(\xi x)\, J_0(\xi y)\, \varphi(y)\, dy\, d\xi,$$

$$\varphi_H(x) = \int_1^\infty \int_0^\infty \xi x J_0(\xi x)\, J_0(\xi y)\, \varphi(y)\, dy\, d\xi,$$

$$\hat{\varphi}(t) = \frac{\pi}{2i \sinh \pi t} \int_0^\infty [J_{2it}(x) - J_{-2it}(x)]\, \varphi(x)\, \frac{dx}{x}.$$

With these definitions we have

$$\varphi(x) = \varphi_B(x) + \varphi_H(x)$$

and the Kuznetsov formula [KU] reads. For $l_1, l_2 \geqslant 1$,

$$\sum_t \hat{\varphi}(t_j)\, v_j(l_1)\, \overline{v_j(l_2)} + \frac{2}{\pi}\int_0^\infty \frac{\hat{\varphi}(t)}{|\zeta(1+2it)|^2}\, d_{it}(l_1)\, d_{it}(l_2)\, dt$$

$$= \delta_{l_1, l_2}\, \varphi_0 + \sum_{c=1}^{\infty} c^{-1}\, S(l_1, l_2; c)\, \varphi_H\big(4\pi \sqrt{l_1 l_2}/c\big).$$

For φ we choose

$$\varphi(x) = \frac{-\sinh\beta}{\pi}\, x \exp(ix\cosh\beta),$$

$$2\beta = \log X + \frac{i}{T}.$$

Then [D-I2 lemma 7]

$$\hat{\varphi}(t) = \frac{\sinh(\pi + 2i\beta)\, t}{\sinh \pi t},$$

$$\varphi_0 = \frac{-\cosh\beta}{2\pi^2 \sinh^2\beta},$$

$$\varphi_B(x) = \frac{-\sinh 2\beta}{2\pi}\int_0^1 \xi x J_0(\xi x)\, (\cosh^2\beta - \xi^2)^{-3/2}\, d\xi,$$

$$\varphi_H(x) = \frac{-\sinh 2\beta}{2\pi}\int_1^\infty \xi x J_0(\xi x)\, (\cosh^2\beta - \xi^2)^{-3/2}\, d\xi.$$

It is easy to see that

$$\hat{\varphi}(t_j) = X^{it_j} e^{-t_j/T} + O(e^{-\pi t_j}),$$

and $\varphi_0 \ll X^{-1/2}$,

$$\frac{-2}{\pi}\int_0^\infty \frac{\hat{\varphi}(t)}{|\zeta(1+2it)|^2}\, (d_{it}(n))^2\, dt \ll T(\log T)^2 d^2(n).$$

Furthermore, $J_0(y) \ll \min(1, y^{-1/2})$ and $|S(n, n; c)| \leqslant (n, c)^{1/2} c^{1/2} d(c)$, hence

$$S_n(\varphi_B) \ll N^{1/2} X^{-1/2} (\log N)^2 \quad \text{and} \quad S_n(\varphi) \ll N^{1/2} T^{1/2} X^{1/4} \log T,$$

where $$S_n(\psi) = \sum_{c=1}^{\infty} c^{-1}\, S(n, n; c)\, \psi\left(\frac{4\pi n}{c}\right).$$

So we conclude that

$$\begin{aligned}&\sum_{t_j} \exp(-t_j/T)\, X^{it_j}\\ &\ll N^{1/2}\, T^{1/2}\, X^{1/4} \log T + N^{1/2}\, X^{-1/2} \log^2 N + N^{-1/2}\, T^2 \log^2 T + T \log^2 T\\ &\ll T^{5/4}\, X^{1/8} \log^2 T,\end{aligned}$$

on taking $N = T^{3/2}\, X^{-1/4}$.

Next we show that the above estimate is still valid if we replace the smooth weight function $\exp(-t/T)$ by the characteristic function $\chi(t)$ of $[1, T]$. Take a smooth function $g(\xi)$ such that $\operatorname{supp}(g) \subseteq [1/2, T + 1/2]$, $0 \leqslant g(\xi) \leqslant 1$, and $g(\xi) = 1$ when $\xi \in [1, T]$. Since it is known that $t_1 > 1$ and $|\{t_j : T \leqslant t_j \leqslant T + 1\}| \ll T$, we have

$$\sum_{t_j \leqslant T} X^{it_j} = \sum_{t_j} g(t_j)\, X^{it_j} + O(T).$$

Let $\hat{g}(\xi)$ be the Fourier transform of $g(\xi) \exp(\xi/T)$:

$$\hat{g}(x) = \int_{-\infty}^{+\infty} g(\xi) \exp(\xi/T)\, e(\xi x)\, d\xi.$$

The easy estimate

$$\hat{g}(x) \ll \min\left(T, \frac{1}{|x|}\right)$$

gives

$$\int_{-1}^{1} |\hat{g}(x)|\, dx \ll \log T.$$

For $|x| \geqslant 1$, from partial integration

$$\hat{g}(x) = -\frac{1}{2\pi i x} \int_{-\infty}^{+\infty} \frac{d}{d\xi} (g(\xi) \exp(\xi/T))\, e(\xi x)\, d\xi,$$

we infer that

$$\begin{aligned}&\int_{|x| \geqslant 1} e(-yx)\, \hat{g}(x)\, dx\\ &\qquad = -\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{d}{d\xi} (g(\xi) \exp(\xi/T)) \int_{1}^{+\infty} \frac{\sin(2\pi x(\xi - y))}{x}\, dx\, d\xi\\ &\qquad \ll \frac{1}{|y| + 1} + \frac{1}{|T - y| + 1} + \frac{\log(T + |y|)}{T}.\end{aligned}$$

Thus, from the Fourier inversion formula

$$g(x) \exp(x/T) = \int_{-\infty}^{+\infty} \hat{g}(\xi)\, e(-\xi x)\, d\xi,$$

we deduce that

$$g(x)\exp(x/T) = \int_{-1}^{1} \hat{g}(\xi)\, e(-\xi x)\, d\xi + O\left(\frac{1}{|x|+1} + \frac{1}{|T-x|+1} + \frac{\log(T+|x|)}{T}\right).$$

Therefore

$$\sum_{t_j} g(t_t)\, X^{it_j} = \int_{-1}^{1} \hat{g}(\xi)\, \left(\sum_{t_j} (Xe^{-2\pi\xi})^{it_j} \exp(-t_j/T)\right) d\xi + O\left(\sum_{t_j} \frac{e^{-t_j/T}}{t_j} + \frac{e^{-t_j/T}}{|T-t_j|+1} + \frac{\log(T+t_j)\, e^{-t_j/T}}{T}\right) \ll T^{5/4} X^{1/8} (\log T)^2.$$

Thus,

$$\sum_{t_j \leqslant T} X^{it_j} \ll T^{5/4} X^{1/8} (\log T)^2. \tag{58}$$

Now [I1, lemma 1]

$$\Psi_\Gamma(X) = X + \sum_{t_j \leqslant T} \frac{X^{(1/2)+it_j}}{(1/2)+it_j} + O\left(\frac{X}{T}\log^2 X\right), \tag{59}$$

where $\Psi_\Gamma(X) = \Sigma_{N\{P\} \leqslant X} \Lambda P$ and $\Lambda P = \log NP_0$ if $\{P\}$ is a power of the primitive hyperbolic class $\{P_0\}$. Hence from (58), (59) and partial summation, we deduce that

$$\Psi_\Gamma(X) = X + O(X^{7/10}\log^2 X)$$

on taking $T = X^{3/10}$. Finally Theorem 1.4 follows by partial summation.

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