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SOME GROUPS WHOSE REDUCED C*-ALGEBRA IS SIMPLE

by M. BEKKA, M. COWLING *and* P. DE LA HARPE (*)

1. Introduction

Let Γ be a discrete group. Denote by $\ell^2(\Gamma)$ the Hilbert space of all square-summable complex-valued functions on Γ, and let $\mathscr{L}(\ell^2(\Gamma))$ be the C*-algebra of all bounded linear operators on $\ell^2(\Gamma)$. The group Γ acts on $\ell^2(\Gamma)$ by the left regular representation λ_Γ, defined by the formula

$$(\lambda_\Gamma(\gamma) f)\ (x) = f(\gamma^{-1} x) \quad \forall\, \gamma \in \Gamma,\ \forall f \in \ell^2(\Gamma),\ \forall\, x \in \Gamma.$$

The reduced C*-algebra $C_r^*(\Gamma)$ of Γ is the norm closure in $\mathscr{L}(\ell^2(\Gamma))$ of the linear span of $\lambda_\Gamma(\Gamma)$. It is a C*-algebra with unit. Recall that a normalized trace on a C*-algebra A with unit is a linear map $\sigma : A \to \mathbf{C}$ such that $\sigma(1) = 1$ and $\sigma(a^* a) \geqslant 0$ and $\sigma(ab) = \sigma(ba)$ for all a, b in A. Such a map is automatically continuous (see [Dix], 2.1.8 and 2.1.9). The algebra $C_r^*(\Gamma)$ has a canonical trace $\tau : C_r^*(\Gamma) \to \mathbf{C}$, defined by $\tau(1) = 1$ and $\tau(\lambda_\Gamma(\gamma)) = 0$ for all γ in $\Gamma \backslash \{1\}$.

Suppose Γ is a nonabelian free group. A remarkable result of R. Powers [Pow] is that $C_r^*(\Gamma)$ is simple (i.e., it has no nontrivial two-sided ideals) and τ is the unique normalized trace. This has been generalized by many authors (see, e.g., [Ake], [AkO], [Ha1], [PaS]).

Let G be a connected semisimple Lie group without compact factors and with trivial centre, and let Γ be a lattice in G. A well-known conjecture asserts that $C_r^*(\Gamma)$ is simple. The main result of this paper is that this conjecture is true. In fact, we prove a more general result, from which the conjecture follows immediately, using the Borel density theorem (cf. [Zim], 3.1.5), which shows that lattices are Zariski-dense. A little notation is necessary before we enounce our main result.

In this paper, we let $\underset{\sim}{G}$ denote the adjoint group of the Lie algebra $\mathfrak{g}$ of a semisimple Lie group G; by this, we mean the algebraic group of automorphisms of $\mathfrak{g}$ whose Hausdorff connected component is isomorphic to the quotient of G by its centre. Also, for a topological group H, the symbol H_d indicates the group H with the discrete topology.

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Theorem 1. — Let G *be a connected real semisimple Lie group without compact factors. Let* H *be a subgroup of* G *with trivial centre, whose image in* $\underset{\sim}{\mathrm{G}}$ *under the canonical projection is Zariski-dense. Then* $C_r^*(H_d)$ *is simple and has a unique normalized trace.*

The following property of a discrete group Γ implies that $C_r^*(\Gamma)$ is simple.

Definition 1. — A discrete group Γ *is said to have property* P_{ana} *if, for any finite subset* F *of* $\Gamma\backslash\{1\}$, *there exist* y_0 *in* Γ *and a constant* C *such that*

$$\left\| \sum_{j=1}^{\infty} a_j \lambda_\Gamma(y_0^{-j} x y_0^j) \right\| \leqslant C \| a \|_2 \quad \forall a \in \ell^2(\mathbf{Z}^+), \ \forall x \in F.$$

Here a_j is the jth-term of the sequence a, and $\mathbf{Z}$ and $\mathbf{Z}^+$ denote the sets of integers and positive integers respectively.

It is immediate that, if Γ has property P_{ana}, then $C_r^*(\Gamma)$ has a unique normalized trace (for this, it suffices to consider singleton sets F only). Indeed, for any x in $\Gamma\backslash\{1\}$ and any trace σ, there exist y_0 in Γ and a constant C such that

$$\| \sigma(\lambda_\Gamma(x)) \| = \left\| \sigma\left(\frac{1}{J} \sum_{j=1}^{J} \lambda_\Gamma(y_0^{-j} x y_0^j)\right) \right\| \leqslant \frac{C}{\sqrt{J}} \quad \forall J \in \mathbf{Z}^+,$$

and hence $\sigma(\lambda_\Gamma(x)) = 0$.

We shall show (Lemma 2.1) that, if Γ has property P_{ana}, then $C_r^*(\Gamma)$ is simple. In turn, property P_{ana} is a consequence (see Lemma 2.3) of the following combinatorial property.

Definition 2. — A discrete group Γ *is said to have property* P_{com} *if, for any finite subset* F *of* $\Gamma\backslash\{1\}$, *there exist* y_0 *in* Γ *and subsets* U *and* A_s *(indexed by a finite set* S*) of* Γ *such that*

(i) $\Gamma\backslash U \subseteq \bigcup_{s\in S} A_s$;
(ii) $xU \cap U = \emptyset$ *for all* x *in* F;
(iii) $y_0^{-j} A_s \cap A_s = \emptyset$ *for all* j *in* $\mathbf{Z}^+$ *and all* s *in* S.

This definition should be compared with the " table-tennis criterion " in Lemma 4.1 below, and with the definition of Powers' group in [HaS]. Note that condition (iii) implies that the sets $y_0^{-j} A_s$ and $y_0^{-j'} A_s$ are disjoint if j and j' are two different integers.

In a number of cases, property P_{com} follows readily from geometric data about Γ. To formalise this, we introduce another condition for a group Γ acting on a compact space B.

Definition 3. — Let Γ *be a discrete group* Γ *acting on a compact set* B. *Then* (Γ, B) *is said to have property* P_{geo} *if, for any finite subset* F *of* $\Gamma\backslash\{1\}$, *there exist* y_0 *in* Γ, *a finite subset* $\{ b_s : s \in S \}$ *of* B, *and open neighbourhoods* V_s *of* b_s *in* B *for each* s *in* S, *such that*

(i) $\{ b_s : s \in S \}$ *is the set of fixed points of the action of* y_0 *on* B, *and, for each* b *in* B, *there exists* s *in* S *such that* $\lim_{j\to\infty} y_0^j b = b_s$;
(ii) $xV_s \cap V_{s'} = \emptyset$, *for all* s, s' *in* S *and all* x *in* F;
(iii) *for all* s *in* S *and* j *in* $\mathbf{Z}^+$, *if* $b \in V_s$ *and* $y_0^j b \notin V_s$, *then* $y_0^{j+1} b \notin V_s$.

An easy compactness argument (see Lemma 2.4) shows that if Γ acts on a compact space B, and (Γ, B) has property P_{geo}, then Γ has property P_{com}.

So the real problem is to establish the following result.

Theorem 2. — Let G *and* H *be as in Theorem 1, and let* B *denote the Furstenberg boundary of* G. *Then* (H_d, B) *has property* P_{geo}.

In the real rank one case where the action of G on B is simpler, we can offer a different proof of independent interest of Theorem 1, at least for subgroups which are both Zariski-dense and discrete. Before formulating this result, we introduce one final property of a discrete group.

Definition 4. — A discrete group Γ *is said to have property* P_{nai} *if, for any finite subset* F *of* $\Gamma \backslash \{ 1 \}$, *there exists* y_0 *in* Γ *of infinite order such that, for each* x *in* F, *the canonical epimorphism from the free product* $\langle x \rangle * \langle y_0 \rangle$ *onto the subgroup* $\langle x, y_0 \rangle$ *of* Γ *generated by* x *and* y_0 *is an isomorphism.*

It is easy to show (Lemma 2.2) that property P_{nai} for a discrete group Γ implies property P_{ana}, and hence the simplicity of $C^*_r(\Gamma)$, and uniqueness of the trace thereon (Lemma 2.1). We also prove the following result.

Theorem 3. — Let G *be a connected simple Lie group of* **R**-*rank* 1 *and trivial centre, and let* Γ *be a discrete subgroup of* G, *Zariski-dense in* $\underset{\sim}{G}$. *Then* Γ *has property* P_{nai}.

Essentially the same proof establishes the simplicity of the reduced C*-algebras of all nonelementary, torsion-free groups which are hyperbolic in the sense of Gromov; cf. [Ha3].

Remark 1. — The subscripts ana, com, geo, and nai are abbreviations for analytic, combinatorial, geometric, and naive respectively. We like to think of P as the first letter of " permissive ". For example, a group Γ has property P_{nai}, or is permissive in the naive sense, if it is so free that, for any finite subset F of $\Gamma \backslash \{ 1 \}$, there exists a partner y_0 of infinite order in Γ such that each pair $\{ x, y_0 \}$ (where $x \in F$) generates a subgroup which is as free as possible.

Remark 2. — Subsets $\{ x_j : j \in \mathbf{Z}^+ \}$ of a group Γ such that, for some constant C,

$$\Big|\Big| \sum_{j=1}^{\infty} a_j \lambda_\Gamma(x_j) \Big|\Big| \leqslant C \, || a ||_2 \quad \forall \, a \in \ell^2(\mathbf{Z}^+),$$

have already appeared in the literature (see [Lei], [AkO]).

Remark 3. — Let H be a group as in Theorems 1 and 2, so that H has property P_{com}. We do not know whether H has property P_{nai} in general.

In [HoR] and [Ros], it is proved that $C^*_r(PGL(n, k))$ is simple with a unique normalized trace (concerning the uniqueness of the trace, see also [Kir]), where $n \geqslant 2$ and k is any discrete field which is not an algebraic extension of a finite field. As a consequence of Theorem 1, we are able to generalize this to algebraic groups over arbitrary fields of characteristic 0.

Corollary 1. — Let k be a field of characteristic 0, *and let* $\mathbf{G}$ *be a connected, semisimple algebraic group, defined over k, with trivial centre. Let* Γ *be* $\mathbf{G}(k)$, *the group of the k-rational points of* $\mathbf{G}$, *equipped with the discrete topology. Then* $C^*_r(\Gamma)$ *is simple, and has a unique trace.*

Theorem 1 has two natural generalizations, with similar proofs. The first of these, Theorem 4, describes the structure of $C^*_r(H_d)$, in the case where H has finite centre, as a direct sum of finitely many simple subalgebras. In order to give the precise statement, we introduce some notation. Let Γ be a discrete group with finite centre Z. For χ in $\hat{Z}$, the dual group of Z, let λ_χ be the representation of Γ induced by χ. Denote by $C^*(\Gamma, \chi)$ the C^*-algebra generated by $\{ \lambda_\chi(x) : x \in \Gamma \}$. It has a canonical trace τ_χ defined by $\tau_\chi(\lambda_\chi(x)) = \chi(x)$ for x in Z and $\tau_\chi(\lambda_\chi(x)) = 0$ for x in $\Gamma \setminus Z$. The reduced C^*-algebra $C^*_r(\Gamma)$ decomposes as the direct sum of the algebras $C^*(\Gamma, \chi)$.

Theorem 4. — Let G *be a connected real semisimple Lie group, without compact factors, with finite centre. Let* H *be a subgroup of* G *with finite centre* Z, *whose image in* $\underset{\sim}{G}$ *under the natural projection is Zariski-dense. Then, for every* χ *in* $\hat{Z}$, $C^*(H_d, \chi)$ *is simple and has a unique trace.*

The second generalization of Theorem 1 deals with reduced crossed-product algebras.

Theorem 5. — Let Γ *be a discrete group with property* P_{com}. *Let* A *be a* C^**-algebra with unit, and let* α *be an action of* Γ *on* A. *Denote by* B *the corresponding reduced crossed-product algebra* $A \rtimes_{\alpha, r} \Gamma$. *If the only* Γ*-invariant ideals in* A *are trivial, then* B *is simple. If A has a unique* Γ*-invariant trace, then* B *has a unique trace.*

This paper is organized as follows. In Section 2, we show that when Γ has property P_{ana}, then $C^*_r(\Gamma)$ is simple. We also establish the relationships between the various properties introduced in Definitions 1 to 4. Sections 3 and 4 are devoted to the results about semisimple Lie groups, and Section 5 to the generalizations and corollaries of Theorem 1.

Some of the results in this paper were announced in [BCH].

It is a pleasure to thank Donald Cartwright, who read this paper very carefully, for a number of useful suggestions.

2. Properties P_{ana}, P_{com}, P_{geo}, and P_{nai}

In this section, we show that property P_{ana} implies the simplicity of the C^*-algebra, that P_{nai} and P_{com} both imply P_{ana}, and that P_{geo} implies P_{com}.

Lemma **2.1.** — *Let* Γ *be a discrete group. If* Γ *has property* P_{ana}, *then* $C^*_r(\Gamma)$ *is simple.*

Proof. — Observe that, since $C^*_r(\Gamma)$ is a Banach algebra with unit, the closure of any proper ideal of $C^*_r(\Gamma)$ is still a proper ideal. Hence, it is sufficient to prove that

$C_r^*(\Gamma)$ has no nontrivial closed two-sided ideals. This amounts to showing that any unitary representation of Γ which is weakly contained in λ_Γ is actually weakly equivalent to λ_Γ (for the notion of weak containment, see [Dix], Chapter 18).

Let π be a unitary representation of Γ which is weakly contained in λ_Γ. The Dirac function δ_1 at the group unit is a positive definite function associated with λ_Γ. Since λ_Γ is cyclic, we need only show that δ_1 is the limit, uniformly on finite subsets of Γ, of sums of positive definite functions associated with π (see [Dix], 18.1.4).

Let F be a finite subset of $\Gamma\backslash\{1\}$. By assumption, there exist y_0 in Γ and a constant C such that, for all x in F,

$$\left\| \sum_{j=1}^{\infty} a_j \lambda_\Gamma(y_0^{-j} x y_0^j) \right\| \leqslant C \| a \|_2 \quad \forall\, a \in \ell^2(\mathbf{Z}^+).$$

In particular,

$$\lim_{J\to\infty} \left\| \frac{1}{J} \sum_{j=1}^{J} \lambda_\Gamma(y_0^{-j} x y_0^j) \right\| = 0 \quad \forall\, x \in F.$$

Since π is weakly contained in λ_Γ, this implies that

$$\lim_{J\to\infty} \left\| \frac{1}{J} \sum_{j=1}^{J} \pi(y_0^{-j} x y_0^j) \right\| = 0 \quad \forall\, x \in F.$$

Take a unit vector ξ in the Hilbert space of π, and define the normalized positive definite function φ_j to be $\langle \pi(\cdot)\, \pi(y_0^j)\, \xi, \pi(y_0^j)\, \xi \rangle$. Then φ_j is a matrix coefficient of π, and

$$\lim_{J\to\infty} \frac{1}{J} \sum_{j=1}^{J} \varphi_j(x) = \delta_1(x) \quad \forall\, x \in F \cup \{1\}. \quad \square$$

Lemma **2.2.** — *Let Γ be a discrete group. If Γ has property P_{nai}, then Γ has property P_{ana}.*

Proof. — Let F be a finite subset of $\Gamma\backslash\{1\}$, and let y_0 in Γ of infinite order be such that $\langle x, y_0 \rangle$ is the free product of $\langle x \rangle$ and $\langle y_0 \rangle$ for all x in F.

Fix x in F and write Γ' for the group $\langle x, y_0 \rangle$. Denote by W_0 the subset of Γ' consisting of the words which do not begin with a nontrivial power of y_0, and by W_j the set $y_0^j W_0$, for all j in $\mathbf{Z}$. Observe that the sets W_j are pairwise disjoint. Then, denoting by λ the regular representation of Γ' and by χ_A the characteristic function of a subset A of Γ', we have, for f and g in $\ell^2(\Gamma')$ and j in $\mathbf{Z}$,

$$\begin{aligned} |\langle \lambda(y_0^{-j} x y_0^j) f, g \rangle| &= |\langle \lambda(x y_0^j) f, \lambda(y_0^j)\, g \rangle| \\ &\leqslant |\langle \lambda(x)\, (\chi_{W_0} \lambda(y_0^j) f), \lambda(y_0^j)\, g \rangle| \\ &\quad + |\langle \lambda(x)\, (\chi_{\Gamma'\backslash W_0} \lambda(y_0^j) f), \lambda(y_0^j)\, g \rangle| \\ &= |\langle \lambda(x)\, \lambda(y_0^j)\, (\chi_{W_{-j}} f), \lambda(y_0^j)\, g \rangle| \\ &\quad + |\langle \lambda(x)\, (\chi_{\Gamma'\backslash W_0} \lambda(y_0^j) f), \chi_{W_0} \lambda(y_0^j)\, g \rangle| \\ &\leqslant \| \chi_{W_{-j}} f \| \, \| g \| + \| f \| \, \| \chi_{W_0} \lambda(y_0^j)\, g \| \\ &= \| \chi_{W_{-j}} f \| \, \| g \| + \| f \| \, \| \chi_{W_{-j}}\, g \|, \end{aligned}$$

where we used the fact that $x(\Gamma'\backslash W_0) \subseteq W_0$.

Now take a in $\ell^2(\mathbf{Z}^+)$ and define the operator T_a on $\ell^2(\Gamma')$ by the formula

$$T_a = \sum_{j=1}^{\infty} a_j \lambda(y_0^{-j} x y_0^j).$$

Then

$$|\langle T_a f, g \rangle| \leqslant \sum_{j=1}^{\infty} |a_j| [\| \chi_{W_{-j}} f \| \| g \| + \| f \| \| \chi_{W_{-j}} g \|]$$
$$\leqslant 2 \| a \|_2 \| f \| \| g \| \quad \forall f, g \in \ell^2(\Gamma').$$

Thus $\| T_a \| \leqslant 2 \| a \|_2$. Since

$$\| \sum_{j=1}^{\infty} a_j \lambda(y_0^{-j} x y_0^j) \| = \| \sum_{j=1}^{\infty} a_j \lambda_\Gamma(y_0^{-j} x y_0^j) \|,$$

the required inequality is proved.

Note that T_a is not a priori a bounded operator on $\ell^2(\Gamma')$. One may get around this by considering a with finite support, and then applying a limiting argument. $\square$

Remark 4. — It should be mentioned that Lemma 2.1 is implicit in [Pow] and [AkO], and that Lemma 2.2 can easily be deduced from [Lei] or [AkO]. For a better understanding of later arguments, we preferred to give independent, quick proofs. Since free groups are readily seen to have property P_{nai}, it should also be observed that a combination of both lemmas provides a short proof of Powers' theorem.

Our next lemma is a generalization of Lemma 2.2. In particular, it implies that, if Γ has property P_{com}, then Γ has property P_{ana}.

Lemma **2.3.** — *Let Γ be a discrete group with property P_{com}, and let $\mathscr{H}$ be a Hilbert space. Let $\mathscr{D}$ denote the space of all bounded operators T on the Hilbert space $\ell^2(\Gamma; \mathscr{H})$ of square-integrable $\mathscr{H}$-valued functions on Γ for which there exists a bounded $\mathscr{L}(\mathscr{H})$-valued function B on Γ such that $Tf(x) = B(x) f(x)$ for all x in Γ and all f in $\ell^2(\Gamma; \mathscr{H})$. Suppose that $(T_j)_{j \geqslant 1}$ is a sequence of operators in $\mathscr{D}$. Let F be a finite subset of $\Gamma \backslash \{1\}$, and let y_0, U, and $\{A_s : s \in S\}$ be as in Definition 2. Then*

$$\| \sum_{j=1}^{\infty} T_j \lambda(y_0^{-j} x y_0^j) \| \leqslant 2 |S| (\sum_{j=1}^{\infty} \| T_j \|^2)^{1/2}$$

for all x in F, where λ denotes the regular representation of Γ in $\ell^2(\Gamma; \mathscr{H})$.

Proof. — We need to observe that operators in $\mathscr{D}$ commute with multiplications by characteristic functions of subsets of Γ. Now, for all f and g in $\ell^2(\Gamma; \mathscr{H})$, and T in $\mathscr{D}$,

$$\begin{aligned}
|\langle T\lambda(x) f, g \rangle| &\leqslant |\langle T\lambda(x) \chi_U f, g \rangle| + |\langle T\lambda(x) \chi_{\Gamma \backslash U} f, g \rangle| \\
&= |\langle T\chi_{xU} \lambda(x) f, g \rangle| + |\langle \lambda(x) \chi_{\Gamma \backslash U} f, T^* g \rangle| \\
&= |\langle \chi_{xU} T\lambda(x) f, \chi_{\Gamma \backslash U} g \rangle| + |\langle \lambda(x) \chi_{\Gamma \backslash U} f, T^* g \rangle| \\
&\leqslant \| T\lambda(x) f \| \| \chi_{\Gamma \backslash U} g \| + \| \chi_{\Gamma \backslash U} f \| \| T^* g \| \\
&\leqslant \sum_{s \in S} [\| T\lambda(x) f \| \| \chi_{A_s} g \| + \| \chi_{A_s} f \| \| T^* g \|]
\end{aligned}$$

for all x in F. For each positive integer j, write R_j for $\lambda(y_0^j)\, T_j\, \lambda(y_0^{-j})$. It is clear that $R_j \in \mathcal{D}$. Hence when $j \geqslant 1$,

$$\begin{aligned} |\langle R_j \lambda(x)\, \lambda(y_0^j) f, \lambda(y_0^j)\, g\rangle| &\leqslant \sum_{s\in S} [\| R_j \lambda(xy_0^j) f\| \, \| \chi_{A_s} \lambda(y_0^j)\, g\| \\ &\qquad + \| \chi_{A_s} \lambda(y_0^j) f\| \, \| R_j^* \lambda(y_0^j)\, g\|] \\ &= \sum_{s\in S} [\| R_j \lambda(xy_0^j) f\| \, \| \chi_{y_0^{-j} A_s}\, g\| \\ &\qquad + \| \chi_{y_0^{-j} A_s} f\| \, \| R_j^* \lambda(y_0^j)\, g\|]. \end{aligned}$$

Now

$$\begin{aligned} \Big|\sum_{j=1}^{\infty} \langle T_j \lambda(y_0^{-j} x y_0^j) f, g\rangle\Big| &= \Big|\sum_{j=1}^{\infty} \langle \lambda(y_0^{-j})\, R_j \lambda(xy_0^j) f, g\rangle\Big| \\ &\leqslant \sum_{j=1}^{\infty} |\langle R_j \lambda(x)\, \lambda(y_0^j) f, \lambda(y_0^j)\, g\rangle| \\ &\leqslant \sum_{s\in S} \Big(\Big[\sum_{j=1}^{\infty} \| R_j \lambda(xy_0^j) f\|^2\Big]^{1/2} \Big[\sum_{j=1}^{\infty} \| \chi_{y_0^{-j} A_s} g\|^2\Big]^{1/2} \\ &\qquad + \Big[\sum_{j=1}^{\infty} \| \chi_{y_0^{-j} A_s} f\|^2\Big]^{1/2} \Big[\sum_{j=1}^{\infty} \| R_j^* \lambda(y_0^j)\, g\|^2\Big]^{1/2}\Big) \\ &\leqslant \sum_{s\in S} \Big(\Big[\sum_{j=1}^{\infty} \langle R_j \lambda(xy_0^j) f, R_j \lambda(xy_0^j) f\rangle\Big]^{1/2} \|g\| \\ &\qquad + \|f\| \Big[\sum_{j=1}^{\infty} \langle R_j^* \lambda(y_0^j)\, g, R_j^* \lambda(y_0^j)\, g\rangle\Big]^{1/2}\Big) \\ &\leqslant |S| \Big(\Big\| \sum_{j=1}^{\infty} \lambda(xy_0^j)^* R_j^* R_j \lambda(xy_0^j)\Big\|^{1/2} \\ &\qquad + \Big\| \sum_{j=1}^{\infty} \lambda(y_0^j)^* R_j R_j^* \lambda(y_0^j)\Big\|^{1/2}\Big) \|f\| \, \|g\| \\ &\leqslant 2|S| \Big(\sum_{j=1}^{\infty} \|R_j\|^2\Big)^{1/2} \|f\| \, \|g\| \\ &= 2|S| \Big(\sum_{j=1}^{\infty} \|T_j\|^2\Big)^{1/2} \|f\| \, \|g\|, \end{aligned}$$

since the sets $y_0^{-j} A_s$ and $y_0^{-j'} A_s$ are disjoint for different integers j and j'. □

Lemma **2.4.** — *Let* Γ *be a discrete group, which acts on a compact space* B. *If* (Γ, B) *has property* P_{geo}, *then* Γ *has property* P_{com}.

Proof. — Let F be a finite subset of $\Gamma\backslash\{1\}$, and let y_0, b_s, and $\{V_s : s \in S\}$, be as in Definition 3. Let S_0 be the set of all s in S such that for some b in $B\backslash\bigcup_{s\in S} V_s$, $y_0^j b \to b_s$ as $j \to \infty$. For i in $\mathbf{Z}^+$ and s in S_0, we define $B_{s,i}$ as follows:

$$B_{s,i} = y_0^{-i} V_s \backslash \bigcup_{0\leqslant j<i} y_0^{-j} V_s.$$

Using condition (iii) in Definition 3, it is easy to show that $y_0^{-1} B_{s,i} = B_{s,i+1}$. Consequently, the sets $y_0^{-j} B_{s,i}$ and $B_{s,i}$ are disjoint for all positive integers j.

Define V thus:

$$V = \bigcup_{s \in S} V_s.$$

Then $xV \subseteq B \backslash V$ for all x in F. Since $\lim_{i \to \infty} y_0^i b \in \{ b_s : s \in S_0 \}$ for all b in $B \backslash V$,

$$B \backslash V \subseteq \bigcup_{i \in \mathbf{Z}^+} \bigcup_{s \in S_0} y_0^{-i} V_s,$$

and since $B \backslash V$ is compact, there exists I in $\mathbf{Z}^+$ such that

$$B \backslash V \subseteq \bigcup_{i=1}^{I} \bigcup_{s \in S_0} B_{s,i}.$$

Now write y_1 for y_0^I, and define B_s (for any s in S_0) by the rule

$$B_s = \bigcup_{i=1}^{I} B_{s,i}.$$

It is clear that the sets $y_1^{-j} B_s$ and B_s are disjoint for any positive integer j.

We fix an arbitrary base point b_0 in B, and for any subset A of B, we define the subset $\check{A}$ of Γ by the rule

$$\check{A} = \{ \gamma \in \Gamma : \gamma b_0 \in A \}.$$

Then $y_1^{-j} \check{B}_s$ and $\check{B}_s$ are disjoint for any positive integer j. Further, $x\check{V} \subseteq \Gamma \backslash \check{V}$ for all x in F, and

$$\Gamma \backslash \check{V} \subseteq \bigcup_{s \in S_0} \check{B}_s.$$

Taking A_s to be $\check{B}_s$, for s in S_0, and U to be $\check{V}$, we are done. □

3. Proof of Theorems 1 and 2

In view of the results of the previous section, it suffices to prove Theorem 2.

Before we give the general proof, it may be helpful to consider a particular situation, namely, where Γ is $PSL(n, \mathbf{Z})$.

Example 1. — Let n be an integer greater than 1, and G be the group $PSL(n, \mathbf{R})$. Let H be a subgroup of G containing $PSL(n, \mathbf{Z})$, hereafter written Γ. The group G acts in the usual way on the real projective space $\mathbf{RP}^{n-1}$, henceforth denoted by B. We shall check that (H_d, B) has property P_{geo}. For this, fix a finite subset F of $H_d \backslash \{ 1 \}$.

Choose y' in Γ with eigendirections $b_1, \ldots, b_n$ in B and corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$ satisfying $\lambda_1 > \lambda_2 > \ldots > \lambda_n > 0$; examples of such matrices y' are direct sums of two-by-two blocks of the form $\begin{pmatrix} j+1 & j \\ 1 & 1 \end{pmatrix}$ for pairwise distinct positive integers j, and of a trivial block (1) in case n is odd. Using the fact that Γ is Zariski-dense in G, it is easy to see that Γ contains a conjugate y_0 of y' with eigendirections $b_1, \ldots, b_n$ in B such that

$$x\{ b_1, \ldots, b_n \} \cap \{ b_1, \ldots, b_n \} = \emptyset \quad \forall\, x \in \mathrm{F}.$$

The details of this are in the proof of Theorem 2 below.

Denote by $[x_1 : \ldots : x_n]$ homogeneous coordinates on B with respect to the eigendirections $b_1, \ldots, b_n$ of y_0. So $b_s = [0 : \ldots : 0 : 1 : 0 : \ldots : 0]$, with 1 in the s-th place. For small positive ε, define V_s by the rule

$$\mathrm{V}_s = \left\{ [x_1 : \ldots : x_n] \in \mathrm{B} : x_s \neq 0 \text{ and } \left| \frac{x_t}{x_s} \right| < \varepsilon \text{ when } t \neq s \right\}.$$

One may choose ε so small that condition (ii) in Definition 3 is satisfied. If $b = [x_1 : \ldots : x_n]$ in B, let s_+ (respectively s_-) denote the smallest (respectively the largest) integer for which $x_s \neq 0$. It is clear that

$$\lim_{j \to \infty} y_0^j\, b = b_{s_+} \quad \text{and} \quad \lim_{j \to \infty} y_0^{-j}\, b = b_{s_-}$$

so that condition (i) in Definition 3 holds. Finally, condition (iii) is fulfilled by definition of the sets V_s.

Throughout the remainder of this section, G, Z, and $\underset{\sim}{\mathrm{G}}$ denote respectively a connected real semisimple Lie group without compact factors, its centre, and the associated adjoint group.

Let KAN be an Iwasawa decomposition of G. We denote by M, M′, and W the centralizer and normalizer of A in K, and the Weyl group M′/M. The Lie algebra of a group is denoted by the corresponding lower case gothic letter. Fix a choice of positive roots of $(\mathfrak{g}, \mathfrak{a})$ such that

$$\mathfrak{n} = \sum_{\alpha > 0} \mathfrak{g}_\alpha.$$

Let $\mathrm{A}^+ = \exp \mathfrak{a}^+$, where $\mathfrak{a}^+$ is the positive Weyl chamber in $\mathfrak{a}$. We denote the minimal parabolic subgroup MAN of G by P, and the Furstenberg boundary G/P by B.

The following crucial lemma is proved in [BeL], appendice, as a consequence of results from [GoM] and [GuR]. A proof also appears in [Mos], p. 63, in the case where H is a lattice.

Lemma **3.1.** — *Let* G *be a noncompact semisimple real algebraic group, and let* H *be a Zariski-dense subgroup of* G. *Then there exists an element* y_0 *in* H *which is " maximally hyperbolic ",* i.e., *which is conjugate in* G *to an element of* MA^+.

Note that if H is a subgroup of the real semisimple not necessarily algebraic Lie group G, then any y_0 in H whose image in $\underset{\sim}{G}$ is maximally hyperbolic is itself maximally periodic. Lemma 3.1 therefore implies the following result.

Lemma **3.2.** — *Let* G *be a noncompact connected real semisimple group, and let* H *be a subgroup of* G *whose image in* $\underset{\sim}{G}$ *under the canonical projection is Zariski-dense. Then there exists an element* y_0 *in* H *which is maximally hyperbolic.*

Next, we need some information about the action of an element y_0 in MA^+ on the Furstenberg boundary B. For each w in W, choose a representative s_w of w in M′ and write $b_w = s_w P$. (Note that $s_w P$ is independent of the coset representative chosen for w.) Then, by the Bruhat decomposition (cf. [War], 1.2), B is a disjoint union:

$$B = \bigcup_{w \in W} N b_w.$$

Let $\bar{w}$ be the longest element in W, and $\bar{N}$ be the subgroup opposite to N. Then $\bar{N} = s_{\bar{w}} N s_{\bar{w}}^{-1}$. Since $B = s_{\bar{w}} B$, we have also

$$B = \bigcup_{w \in W} \bar{N} b_w.$$

Let β and θ denote the Killing form and the Cartan involution on $\mathfrak{g}$. Then

$$(X, Y) \mapsto -\beta(X, \theta Y) \quad \forall\, X, Y \in \mathfrak{g}$$

is an inner product on $\mathfrak{g}$ with respect to which M acts by isometries (by Ad, the adjoint representation of G), and A^+ acts by centralizing $\mathfrak{m} \oplus \mathfrak{a}$, by "shrinking" $\bar{\mathfrak{n}}$, and by "expanding" $\mathfrak{n}$. More precisely, if $\|\cdot\|$ denotes the norm corresponding to this inner product, and $y_0 \in A^+$, then $\|\mathrm{Ad}(y_0) X\| < \|X\|$ if $X \in \bar{\mathfrak{n}} \setminus \{0\}$, while $\|\mathrm{Ad}(y_0) X\| > \|X\|$ if $X \in \mathfrak{n} \setminus \{0\}$.

Observe that, for any w in W and X in $\mathfrak{g}$,

$$(1) \qquad y_0 \exp(X)\, b_w = \exp(\mathrm{Ad}(y_0) X)\, b_w.$$

Together with the fact that $s_{\bar{w}}^{-1} y_0^{-1} s_{\bar{w}} \in MA^+$, this shows that

$$\lim_{j \to +\infty} y_0^j b = b_w \quad \forall\, b \in \bar{N} b_w, \qquad \text{and} \qquad \lim_{j \to -\infty} y_0^j b = b_w \quad \forall\, b \in N b_w.$$

It is clear that the fixed point set of the map $b \mapsto y_0 b$ is $\{ b_w : w \in W \}$. We shall need to understand this map; in particular, we need to study the action near all the fixed points.

For any w in W, and any Ad(M)-invariant subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, $\mathrm{Ad}(s_w)\, \mathfrak{h}$ is independent of the coset representative s_w for w in M′/M, so we may denote it by ${}^w\mathfrak{h}$. Clearly

$$s_w \bar{N} b_e = s_w \bar{N} s_w^{-1} b_w = \exp({}^w\bar{\mathfrak{n}})\, b_w.$$

Since $\overline{\mathrm{N}}b_e$ is a neighbourhood of b_e, Zariski-dense in B, the set $s_w\,\overline{\mathrm{N}}b_e$ is a neighbourhood of b_w, Zariski-dense in B, and $\mathrm{X} \mapsto \exp(\mathrm{X})\, b_w$ is a bijection of ${}^w\overline{\mathfrak{n}}$ onto this set, which is biregular (in the algebraic-geometric sense) and diffeomorphic (in the differential-geometric sense). In view of formula (1), this shows that the action of y_0 on B near b_w is equivalent to that of $\mathrm{Ad}(y_0)$ on ${}^w\overline{\mathfrak{n}}$ near 0.

For w in the Weyl group W, the coset representative s_w acts on the Lie algebra (by Ad), stabilizing the subalgebras $\mathfrak{m}$ and $\mathfrak{a}$, and permuting the root spaces $\mathfrak{g}_\alpha$ amongst themselves, so ${}^w\overline{\mathfrak{n}}$ is a sum of root spaces, and we may write

$$ {}^w\overline{\mathfrak{n}} = (\mathfrak{n} \cap {}^w\overline{\mathfrak{n}}) \oplus (\overline{\mathfrak{n}} \cap {}^w\overline{\mathfrak{n}}). $$

If we take neighbourhoods U_ε of 0 in ${}^w\overline{\mathfrak{n}}$ of the form

$$ \mathrm{U}_\varepsilon = \{ \mathrm{X} \in \mathfrak{n} \cap {}^w\overline{\mathfrak{n}} : \|\mathrm{X}\| < \varepsilon \} \times \{ \mathrm{X} \in \overline{\mathfrak{n}} \cap {}^w\overline{\mathfrak{n}} : \|\mathrm{X}\| < \varepsilon \}, $$

where $\varepsilon \in \mathbf{R}^+$, we see immediately that, if $\mathrm{X} \in \mathrm{U}_\varepsilon$ and $\mathrm{Ad}(y_0^j)\,\mathrm{X} \notin \mathrm{U}_\varepsilon$ for some positive integer j, then this is because the projection of $\mathrm{Ad}(y_0^j)\,\mathrm{X}$ into its $\mathfrak{n} \cap {}^w\overline{\mathfrak{n}}$-component has length at least ε, and we deduce that $\mathrm{Ad}(y_0^{j+1})\,\mathrm{X} \notin \mathrm{U}_\varepsilon$.

This is essentially all the information we need about the map $b \mapsto y_0\, b$, but it may be worth pointing out that this line of reasoning can be pushed a little further to show that $\exp({}^w\overline{\mathfrak{n}})\, b_w$ is a MA-invariant neighbourhood of the singular Bruhat cell $\overline{\mathrm{N}}b_w$. Indeed, for a fixed w in W, $\overline{\mathfrak{n}} = (\overline{\mathfrak{n}} \cap {}^w\mathfrak{n}) \oplus (\overline{\mathfrak{n}} \cap {}^w\overline{\mathfrak{n}})$. A standard argument (see, e.g., [Wal], 8.10.2) implies that

$$ \overline{\mathrm{N}} = \exp(\overline{\mathfrak{n}} \cap {}^w\overline{\mathfrak{n}})\, \exp(\overline{\mathfrak{n}} \cap {}^w\mathfrak{n}). $$

Now

$$ \begin{aligned} \overline{\mathrm{N}}b_w &= \exp(\overline{\mathfrak{n}} \cap {}^w\overline{\mathfrak{n}})\; w \exp(\mathrm{Ad}(w^{-1})\, (\overline{\mathfrak{n}} \cap {}^w\mathfrak{n}))\; b_e \\ &\subseteq \exp(\overline{\mathfrak{n}} \cap {}^w\overline{\mathfrak{n}})\; w \exp(\mathfrak{n})\; b_e \\ &= \exp(\overline{\mathfrak{n}} \cap {}^w\overline{\mathfrak{n}})\; b_w \\ &\subseteq \exp({}^w\overline{\mathfrak{n}})\; b_w. \end{aligned} $$

These neighbourhoods therefore provide a finite open cover of the Furstenberg boundary B, and understanding the action of MA on each of them is tantamount to understanding the action of MA on B.

By replacing y_0 by yy_0y^{-1} if necessary, we deduce that any maximal hyperbolic element has a similar action on B.

We summarize the relevant parts of this discussion as a lemma.

Lemma **3.3.** — *Let y_0 in* G *be conjugate to an element of* MA^+. *Then there exists a subset* $\{ b_w : w \in \mathrm{W} \}$ *of* B *such that the following holds. For any b in* B, *there exist w_+ and w_- in* W, *such that*

$$ \lim_{j \to +\infty} y_0^j\, b = b_{w_+} \quad \textit{and} \quad \lim_{j \to -\infty} y_0^j\, b = b_{w_-}. $$

Further, the fixed points b_w all have arbitrarily small neighbourhoods U_w *with the property that if $b \in \mathrm{U}_w$, $j \in \mathbf{Z}^+$ and $y_0^j\, b \notin \mathrm{U}_w$, then $y_0^{j+1}\, b \notin \mathrm{U}_w$.*

Proof of Theorem 2. — Let H be a subgroup of G, with trivial centre, whose image in $\underset{\sim}{G}$ under the canonical projection π is Zariski-dense. Fix a finite subset F of $H\backslash\{1\}$. Take a maximally hyperbolic element y_0 in H, whose existence is assured by Lemma 3.2, and a subset $\{b_w : w \in W\}$ of B, as in Lemma 3.3.

First, we claim that there exists an element y in H such that

$$yxy^{-1}\{b_w : w \in W\} \cap \{b_w : w \in W\} = \emptyset \quad \forall\, x \in F.$$

Indeed, for w', w'' in W and x in F, the set $\{y \in G : yxy^{-1}\, b_{w'} \neq b_{w''}\}$ is clearly the inverse image under π of a Zariski-open subset of $\underset{\sim}{G}$. We prove below that it is nonvoid by contradiction. Our claim then follows because the intersection of all these sets is still the inverse image under π of a nonvoid and Zariski-open subset of $\underset{\sim}{G}$, and the image of H in $\underset{\sim}{G}$ is Zariski-dense.

Suppose that there exist w', w'' in W and x in F, such that $yxy^{-1}\, b_{w'} = b_{w''}$ for all y in G. Then $b_{w''} = xb_{w'}$, so $xy^{-1}\, b_{w'} = y^{-1}\, xb_{w'}$ for all y in G. Then the stabilizers of $b_{w'}$ and $xb_{w'}$ coincide. As P is its own normalizer in G, this implies that $b_{w'} = xb_{w'}$, i.e. $w' = w''$. Now we have $xy^{-1}\, b_{w'} = y^{-1}\, b_{w'}$ for all y in G, and since G acts transitively on B, x fixes every point of B. Hence x is central, since G has no compact factors. This contradiction proves our claim.

By replacing y_0 with $y^{-1} y_0 y$, if necessary, we may therefore assume that $xb_{w'} \neq b_{w''}$ whenever $x \in F$ and $w', w'' \in W$.

Take open neighbourhoods $U_{w'}$ of the points $b_{w'}$ such that $xU_{w'} \cap U_{w''} = \emptyset$ whenever $x \in F$ and $w', w'' \in W$, and such that if $b \in U_{w'}$, $j \geqslant 1$ and $y_0^j\, b \notin U_{w'}$, then $y_0^{j+1}\, b \notin U_{w'}$ (this is possible by Lemma 3.3). This establishes that (H_d, B) has property P_{geo}. $\square$

Remark 5. — It may be of interest to observe that, when one combines the arguments of this section with those of Lemma 2.3, the final conclusion is that

$$\Big\| \sum_{j=1}^{\infty} a_j\, \lambda(y_0^{-j}\, x y_0^j) \Big\| \leqslant 2(|W| - 1)\, \|a\|_2.$$

The point is that one of the fixed points, namely $b_{\overline{w}}$, is dropped out in the passage from P_{geo} to P_{com}, because no point b of $B\backslash\{b_{\overline{w}}\}$ has the property that $\lim_{j\to\infty} y_0^j\, b = b_{\overline{w}}$. In particular, in the rank one case, where $|W| = 2$, we obtain the same constant as for the free group (see the proof of Lemma 2.2).

Moreover, by looking at other boundaries, one may reduce this constant further. The example given earlier of subgroups of $PSL(n, \mathbf{R})$ containing $PSL(n, \mathbf{Z})$ shows that, for these groups, the constant $2(|W| - 1)$, equal to $2(n! - 1)$, can be replaced by $2(n - 1)$.

Remark 6. — In the recent theory of operator Hilbert spaces developed by U. Haagerup and G. Pisier [HaP], there are estimates similar to some which appear in the proof of our Theorem 2. More precisely, by considering in Lemma 2.3 operators T_j

which commute with the translation operators $\lambda(x)$ for all x in Γ, and unravelling the last group of inequalities in the proof, without the last two lines, we obtain the inequality

$$\|\sum_{j=1}^{\infty} T_j \lambda(y_0^{-j} x y_0^j)\| \leqslant (|W| - 1)\,[\|\sum_{j=1}^{\infty} T_j^* T_j\|^{1/2} + \|\sum_{j=1}^{\infty} T_j T_j^*\|^{1/2}].$$

4. Proof of Theorem 3

First, recall the following well-known lemma (see [Ha2], p. 130, or [Tit], Proposition 1.1).

Lemma 4.1 ("Table-tennis criterion"). — *Let* G *be a group acting on a set* X, *and let* H *and* K *be subgroups of* G. *Assume that* K *has at least 3 elements. Suppose that there exist disjoint subsets* A *and* B *of* X *such that* $h(B) \subseteq A$ *for all* h *in* $H\backslash\{1\}$ *and* $k(A) \subseteq B$ *for all* k *in* $K\backslash\{1\}$. *Then the subgroup of* G *generated by* H *and* K *is the free product* $H * K$.

Proof of Theorem 3. — We assume now that G has **R**-rank 1. The Riemannian symmetric space G/K, denoted by X, has strictly negative curvature, and B is the boundary of the compactification $\overline{X}$ of X, as in [BGS], 3.2. The elements of G may be classified by means of their fixed points in $\overline{X}$ (see [BGS], 6.8, or [EbO], Section 6): any x in G is *elliptic*, when x has a fixed point in X, or *parabolic*, when x has no fixed point in X and exactly one fixed point in B, or *hyperbolic*, when x has no fixed point in X and exactly two fixed points in B.

Further, if x in G is parabolic or hyperbolic and if a_1 and a_2 are the fixed points of x in B (of course, $a_1 = a_2$ if x is parabolic), then (permuting a_1 and a_2 if necessary)

$$\lim_{j\to+\infty} x^j b = a_1 \quad \text{and} \quad \lim_{j\to-\infty} x^j b = a_2$$

for all b in $B\backslash\{a_1, a_2\}$.

Let Γ be a discrete subgroup of G, Zariski-dense in $\underset{\sim}{G}$. Observe that any elliptic element x in Γ has finite order, since it is contained in a compact subgroup of G.

We shall now prove that Γ has property P_{nai}. Let F be a finite subset of $\Gamma\backslash\{1\}$ and set

$$F' = \{(x,j) \in F \times \mathbf{Z} : x^j \neq 1\}.$$

Let

$$B_0 = \{b \in B : x^j b \neq b \quad \forall\,(x,j) \in F'\}.$$

Recall that, for any x in G, x^j is of the same type (elliptic, hyperbolic or parabolic) as x for all j in $\mathbf{Z}\backslash\{0\}$ (see [BGS], Lemma 6.5). This shows that B_0 is a finite intersection of Zariski-open nonvoid subsets of B. Hence B_0 is a Zariski-open nonvoid subset of B.

Let y_0 be a hyperbolic element of Γ, with attracting fixed point $b_1 \in B$ and repulsing fixed point $b_2 \in B$. Since Γ is Zariski-dense, the Γ-orbit of b_1 intersects B_0. Hence, by conjugating y_0 if necessary, we may assume that $b_1 \in B_0$. It is clear that we can find a neighbourhood V (with respect to the Hausdorff-topology on B) of b_1 such that $x^j V \cap V = \emptyset$ for all (x, j) in $F^!$.

Since Γ is Zariski-dense, we may choose y_1 in Γ so that

$$\{y_1 b_1, y_1 b_2\} \cap \{b_1, b_2\} = \emptyset.$$

Then $y_1 y_0 y_1^{-1}$ and y_0 have no common fixed points in B. Hence, for a sufficiently large positive integer j, the element y_2, defined by the rule $y_2 = y_0^j (y_1 y_0 y_1^{-1}) y_0^{-j}$, has its fixed points in V.

Replacing y_2 with y_2^i for a sufficiently large i, we may assume that

$$y_2^j(B \backslash V) \subseteq V \quad \forall j \in \mathbf{Z} \backslash \{0\}.$$

Now define U by the formula

$$U = \bigcup_{(x,j) \in F^!} x^j V.$$

Then $V \cap U = \emptyset$, $y_2^j U \subseteq V$ for all nonzero integers j and $x^j V \subseteq U$ for all (x, j) in $F^!$. Hence, by Lemma 4.1, $\langle x, y_2 \rangle$ is isomorphic to $\langle x \rangle * \langle y_2 \rangle$ for all x in F. □

5. Extensions and corollaries of Theorem 1

In this section, we prove Corollary 1 and Theorems 4 and 5. The following simple observation will be useful for the proof of Corollary 1.

Lemma **5.1.** — *Let Γ be a discrete group, and let $\{\Gamma_i : i \in I\}$ be a family of subgroups of Γ with the property that every finite subset of Γ is contained in some Γ_i. Assume that $C_r^*(\Gamma_i)$ is simple and has a unique trace for any i in I. Then $C_r^*(\Gamma)$ is simple and has a unique trace.*

Proof. — Let π be a unitary representation of Γ which is weakly contained in λ_Γ. Let F be a finite subset of Γ, and let i in I be such that F is contained in Γ_i. By assumption, λ_{Γ_i} is weakly contained in the restriction of π to Γ_i. Hence δ_1, the Dirac function at the group unit, is the limit on F of sums of positive definite functions associated to π. This shows that λ_Γ is weakly contained in π.

The assertion concerning the trace is trivial. □

Proof of Corollary 1. — Every finite subset of $\mathbf{G}(k)$ is contained in $\mathbf{G}(k')$ for some finitely generated subfield k' of k. By the lemma above, we may therefore assume that k is a finitely generated field of characteristic 0. It is well-known (and easy to prove) that such a field may be embedded in **C**. So we may further assume that k is a subfield of **C**. There are two cases to distinguish: if k is totally real, then k is dense in **R**, so $\mathbf{G}(k)$ is dense in $\mathbf{G}(\mathbf{R})$, and if not, then k is dense in **C**, so $\mathbf{G}(k)$ is dense in $\mathbf{G}(\mathbf{C})$. A fortiori, in the first case, $\mathbf{G}(k)$ is Zariski-dense in $\mathbf{G}(\mathbf{R})$, and in the second, $\mathbf{G}(k)$ is Zariski-dense in $\mathbf{G}(\mathbf{C})$, considered as a real group, by restriction of scalars. Hence, the claim follows from Theorem 1. □

Proof of Theorem 4. — Write Γ for H_a. For χ in $\hat{Z}$, let $\tilde{\chi}$ denote the trivial extension of χ to Γ (i.e. $\tilde{\chi}(\gamma) = 0$ for all γ in $\Gamma\backslash Z$). Let π be a unitary representation of Γ which is weakly contained in λ_χ, the representation of Γ induced by χ. Then π is weakly contained in λ_Γ and the restriction of π to Z is a multiple of χ.

Let F be a finite subset of Γ. The proof of Theorem 2 combined with that of Lemma 2.3 shows that there exists y_0 in Γ such that

$$(2) \qquad \Big\| \sum_{j=1}^{\infty} a_j \lambda_\Gamma(y_0^{-j} x y_0^j) \Big\| \leqslant 2(|W| - 1) \|a\|_2 \quad \forall\, a \in \ell^2(\mathbf{Z}^+) \quad \forall\, x \in F \cap (\Gamma\backslash Z).$$

Because λ_χ is a subrepresentation of λ_Γ, and π is weakly contained in λ_χ, the same inequality holds with λ_χ or π in place of λ_Γ.

Now, proceeding as in the proof of Lemma 2.1, let ξ be a unit vector in the Hilbert space of π and let φ_j be $\langle \pi(\cdot)\, \pi(y_0^j)\, \xi, \pi(y_0^j)\, \xi \rangle$. Then

$$\lim_{J\to\infty} \frac{1}{J} \sum_{j=1}^{J} \varphi_j(x) = \tilde{\chi}(x) \quad \forall\, x \in F.$$

This shows that λ_χ is weakly contained in π. Hence $C^*(\Gamma, \chi)$ is simple.

Let τ be a trace on $C^*(\Gamma, \chi)$. The version of inequality (2) for λ_χ shows that

$$\tau(\lambda_\chi(x)) = 0 \quad \forall\, x \in \Gamma\backslash Z.$$

Since $\tau(\lambda_\chi(x)) = \chi(x)$ for all x in Z, τ is unique. □

Proof of Theorem 5. — The proof of the simplicity is similar to that of Proposition 10 in [HaS].

We may assume that A acts faithfully on some Hilbert space $\mathscr{H}$. Then the reduced crossed-product $A \rtimes_{\alpha, r} \Gamma$, also known as B, may be defined to be the C*-algebra on $\ell^2(\Gamma; \mathscr{H})$ generated by the operators given by the formulae

$$(a\xi)(x) = \alpha_{x^{-1}}(a)\, \xi(x) \quad \forall\, \xi \in \ell^2(\Gamma; \mathscr{H}) \ \forall\, x \in \Gamma$$

$$(\lambda(\gamma)\, \xi)(x) = \xi(\gamma^{-1} x) \quad \forall\, \xi \in \ell^2(\Gamma; \mathscr{H}) \ \forall\, x \in \Gamma,$$

as a runs over A, and γ runs over Γ. Any element of B may be considered to be an infinite sum $\sum_{\gamma\in\Gamma} a_\gamma \lambda(\gamma)$, where $a_\gamma \in A$. There is a conditional expectation $e : B \to A$, defined by the rule

$$e\Big(\sum_{\gamma\in\Gamma} a_\gamma \lambda(\gamma)\Big) = a_1.$$

Let I be a nonzero ideal in B, and let $b = \sum_{\gamma\in\Gamma} a_\gamma \lambda(\gamma)$ be a nonzero element of I. Replacing b by bb^* if necessary, we may assume that $a_1 \geqslant 0$, and $a_1 \neq 0$.

According to [HaS], Lemma 9, we may even assume that $a_1 \geqslant 1$, by replacing b by the element

$$\sum_{j=1}^{J} a_j \lambda(\gamma_j)\, b\lambda(\gamma_j^{-1})\, a_j^*,$$

for appropriately chosen a_j in A and γ_j in Γ. Hence, upon replacing b by $a_1^{-1} b$, we may assume that $a_1 = 1$.

Now there exist a finite subset F of $\Gamma \backslash \{1\}$ and an element b' in B of the form $1 + \Sigma_{\gamma \in F}\, a'_\gamma \lambda(\gamma)$ such that

$$\| b - b' \| \leqslant \frac{1}{3}.$$

Let y_0 in Γ and S be as in Definition 2. Then, according to Lemma 2.3,

$$\left\| \frac{1}{J} \sum_{j=1}^{J} \lambda(y_0^{-j}) \left(\sum_{\gamma \in F} a'_\gamma \lambda(\gamma)\right) \lambda(y_0^j) \right\| = \left\| \frac{1}{J} \sum_{\gamma \in F} \sum_{j=1}^{J} \alpha_{y_0^{-j}}(a'_\gamma)\, \lambda(y_0^{-j} \gamma y_0^j) \right\| \tag{3}$$
$$\leqslant \frac{2\,|\,S\,|}{\sqrt{J}} \sum_{\gamma \in F} \| a'_\gamma \|.$$

Hence

$$\left\| \frac{1}{J} \sum_{j=1}^{J} \lambda(y_0^{-j})\, (b' - 1)\, \lambda(y_0^j) \right\| \leqslant \frac{1}{3}$$

for J large enough. It follows that

$$\left\| \frac{1}{J} \sum_{j=1}^{J} \lambda(y_0^{-j})\, b\lambda(y_0^j) - 1 \right\| \leqslant \left\| \frac{1}{J} \sum_{j=1}^{J} \lambda(y_0^{-j})\, (b - b')\, \lambda(y_0^j) \right\|$$
$$+ \left\| \frac{1}{J} \sum_{j=1}^{J} \lambda(y_0^{-j})\, (b' - 1)\, \lambda(y_0^j) \right\|$$
$$\leqslant \frac{2}{3},$$

so that $(1/J)\, \Sigma_{j=1}^{J}\, \lambda(y_0^{-j})\, b\lambda(y_0^j)$ is invertible in B. As this is obviously an element of I, the equality I $=$ B is proved.

The uniqueness of the trace is also a consequence of inequality (3) above. Indeed, let τ be a trace on B. Then (3) shows that $\tau(a\lambda(\gamma)) = 0$ for all a in A and γ in $\Gamma \backslash \{1\}$. Hence $\tau = \sigma \circ e$, where σ is the unique Γ-invariant trace on A. □

Remark 7. — In fact, the above proof shows the following somewhat more general result: any trace on B is of the form $\sigma \circ e$ for some Γ-invariant trace σ on A.

Example 2. — Let G be a semisimple Lie group without compact factors and with trivial centre. Let Γ be a lattice in G. Then Γ acts minimally on the compact space G/P for any parabolic subgroup P of G (see [Mos], Lemma 8.5). So the C*-algebra C(G/P) of all continuous functions on G/P has no nontrivial Γ-invariant ideals. Hence, the reduced crossed-product C*-algebra $C(G/P) \rtimes_{\alpha, r} \Gamma$ is simple. Moreover, there is no Γ-invariant probability measure on G/P (see [Zim], 3.2.23). Therefore $C(G/P) \rtimes_{\alpha, r} \Gamma$ has no trace.

Example 3. — Let Γ be a group acting simply transitively on the set of vertices of a building of type $\widetilde{A}_2$, as in [CMSZ1] and [CMSZ2] (some of these are lattices in semisimple algebraic groups, and some are not). Mantero, Steger and Zappa (private communication) have shown that Γ has property P_{geo}, and so the reduced C*-algebras of the group and of the group acting on the boundary of the building are simple. Guyan Robertson (private communication) has obtained related results on this crossed product algebra, including the nuclearity thereof.

REFERENCES

[Ake] C. A. Akemann, Operator algebras associated with Fuchsian groups, *Houston J. Math.*, **7** (1981), 295-301.

[AkO] C. A. Akemann and P. A. Ostrand, Computing norms in group C*-algebras, *Amer. J. Math.*, **98** (1976), 1015-1047.

[BCH] M. Bekka, M. Cowling and P. de la Harpe, Simplicity of the reduced C*-algebra of PSL(n, **Z**), *Inter. Math. Res. Not.*, **7** (1994), 285-291.

[BeL] Y. Benoist and F. Labourie, Sur les difféomorphismes d'Anosov affines à feuilletages stable et instable différentiables, *Invent. Math.*, **111** (1993), 285-308.

[BGS] W. Ballmann, M. Gromov and V. Schroeder, *Manifolds of Nonpositive Curvature*, Birkhäuser, 1985.

[CMSZ1] D. I. Cartwright, A. M. Mantero, T. Steger and A. Zappa, Groups acting simply transitively on the vertices of a building of type $\widetilde{A}_2$, I, *Geom. Ded.*, **47** (1993), 143-166.

[CMSZ2] D. I. Cartwright, A. M. Mantero, T. Steger and A. Zappa, Groups acting simply transitively on the vertices of a building of type $\widetilde{A}_2$, II, *Geom. Ded.*, **47** (1993), 167-223.

[Dix] J. Dixmier, *Les C*-algèbres et leurs représentations*, Gauthiers-Villars, 1969.

[EbO] P. Eberlein and B. O'Neill, Visibility manifolds, *Pacific J. Math.*, **46** (1973), 45-110.

[GoM] I. A. Gol'dsheid and G. A. Margulis, A condition for simplicity of the spectrum of Lyapunov exponents, *Soviet Math. Dokl.*, **35** (1987), 309-313.

[GuR] Y. Guivarc'h and A. Raugi, Propriétés de contraction d'un semi-groupe de matrices inversibles, *Israel J. Math.*, **65** (1989), 165-196.

[Ha1] P. de la Harpe, Reduced C*-algebras of discrete groups which are simple with unique trace, *Springer Lecture Notes in Math.*, **1132** (1985), 230-253.

[Ha2] P. de la Harpe, Free subgroups in linear groups, *L'Enseignement Math.*, **29** (1983), 129-144.

[Ha3] P. de la Harpe, Groupes hyperboliques, algèbres d'opérateurs, et un théorème de Jolissaint, *C. R. Acad. Sci. Paris*, **307**, Série I (1988), 771-774.

[HaP] U. Haagerup and G. Pisier, Bounded linear operators between C*-algebras, *Duke Math. J.*, **71** (1993), 889-925.

[HaS] P. de la Harpe and G. Skandalis, Powers' property and simple C*-algebras, *Math. Ann.*, **273** (1986), 241-250.

[HoR] R. E. Howe and J. Rosenberg, The unitary representation theory of GL(n) of an infinite discrete field, *Israel J. Math.*, **67** (1989), 67-81.

[Kir] A. A. Kirillov, Positive definite functions on a group of matrices with elements from a discrete field, *Soviet Math. Dokl.*, **6** (1965), 707-709.

[Lei] M. LEINERT, Faltungsoperatoren auf gewissen diskreten Gruppen, *Studia Math.*, **52** (1974), 149-158.

[Mos] G. D. MOSTOW, *Strong Rigidity of Locally Symmetric Spaces*, Princeton University Press, 1973.

[PaS] W. PASCHKE and N. SALINAS, C*-algebras associated with the free products of groups, *Pacific J. Math.*, **82** (1979), 211-221.

[Pow] R. T. POWERS, Simplicity of the C*-algebra associated with the free group on two generators, *Duke Math. J.*, **42** (1975), 151-156.

[Ros] J. ROSENBERG, Un complément à un théorème de Kirillov sur les caractères de GL(n) d'un corps infini discret, *C. R. Acad. Sci. Paris*, **309**, Série I (1989), 581-586.

[Tit] J. TITS, Free subgroups in linear groups, *J. Algebra*, **20** (1972), 250-270.

[Wal] N. WALLACH, *Harmonic Analysis on Homogeneous Spaces*, Marcel Dekker, 1973.

[War] G. WARNER, *Harmonic Analysis on Semisimple Lie Groups I*, Springer-Verlag, 1972.

[Zim] R. J. ZIMMER, *Ergodic Theory and Semisimple Groups*, Birkhäuser, 1984.

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