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BIQUATERNION ALGEBRAS AND QUARTIC EXTENSIONS

by TSIT YUEN LAM ([1]), DAVID B. LEEP ([2]), JEAN-PIERRE TIGNOL ([3])

CONTENTS

§ 1. Introduction

In this paper, we study a class of central simple algebras of dimension 16 (over a field F) called biquaternion algebras. These are, by definition, tensor product algebras of the form $B \otimes_F C$, where B and C are F-quaternion algebras. For convenience of exposition, we shall assume throughout that the characteristic of the ground field F is not equal to 2.

There are many examples of biquaternion algebras $B \otimes_F C$ which are cyclic algebras over F. On the other hand, according to Albert [A_3], there are also, over certain fields, biquaternion division algebras which are *not* cyclic algebras. (These were, in fact, the first examples of central division algebras which fail to be cyclic algebras.) It is, therefore, natural to ask: *When is a biquaternion algebra* $B \otimes_F C$ *cyclic over a field* F? Although biquaternion algebras have been considered since the 1930's and are known to be an interesting source of examples of central simple algebras, a complete answer to the question above seemed to be unknown. In this work, we shall fill this gap by giving two explicit criteria for a biquaternion algebra $B \otimes_F C$ to be cyclic. The first criterion is quadratic-form-theoretic: the condition for $B \otimes_F C$ to be cyclic is expressed in terms of the splitting

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properties of its Albert form, which is a 6-dimensional F-quadratic form associated with $B \otimes_F C$ (see § 2). The second criterion is, on the other hand, purely algebra-theoretic, and is expressed in terms of the corestriction of central simple algebras. Of course, these two criteria are mathematically equivalent; however, the proofs that they characterize the cyclicity of $B \otimes_F C$ involve different notions and techniques, respectively from the theory of quadratic forms and the theory of algebras. Therefore, it will be convenient to present them separately, independently of each other. (For the detailed statements of these criteria, see (4.13) and (5.11).) Both criteria are simple enough to permit explicit computations: in a sequel to this work they will be used to give various nontrivial examples of cyclic as well as noncyclic biquaternion algebras. However, these results are peculiar to the case of algebras of degree 4, as it can be shown that the would-be analogues of these results for algebras of higher 2-power degree are all false.

If A is a biquaternion division algebra over F, the smallest splitting fields for A are of degree 4 over F. Thus the study of biquaternion algebras is closely linked to that of quartic extensions of F. In fact, a substantial portion of this paper is devoted to the study of quartic 2-extensions, i.e. those quartic extensions $L \supset F$ which contain an intermediate quadratic extension of F. In the quadratic-form-theoretic § 3, we shall study the functorial map of the Witt rings, $W(F) \to W(L)$, for a quartic 2-extension $L \supset F$ and compute explicitly the Witt ring kernel $W(L/F)$. This kernel turns out to be a $\{1, 2\}$-Pfister ideal in $W(F)$, i.e. it is generated by the 1-fold and 2-fold Pfister forms over F which split over L, and these Pfister forms can be explicitly determined. In general, the Witt ring kernels for finite field extensions seem very hard to determine. For a general ground field F, the only finite extensions $E \supset F$ of even degree for which $W(E/F)$ was known were quadratic extensions [L_1: p. 200] and biquadratic extensions [1] [ELW_1: (2.12)]. Our result on $W(L/F)$ mentioned above subsumes, of course, that in the case of biquadratic extensions, and represents a generalization thereof. Using these results, we can establish necessary and sufficient conditions for a biquaternion algebra $B \otimes_F C$ to have a splitting field L which is a quartic 2-extension of F with a prescribed nonsquare discriminant (cf. (4.7), (4.16), (5.10) and (5.12)). The cyclicity criteria mentioned in the last paragraph are simply obtained as special cases of these results when we restrict ourselves to Galois splitting fields. Some further applications of the results of this paper to discriminants of involutions on biquaternion algebras can be found in [KLST].

In general, for even degree extensions $E \supset F$, the computation of $W(E/F)$ remains difficult. For instance, the structure of $W(E/F)$ when $E \supset F$ is a dihedral or a quaternion extension of degree 8 seems to be unknown. Hopefully, the computation of $W(L/F)$ for quartic 2-extensions $L \supset F$ presented in this paper will have some bearing on the

(1) By a biquadratic extension of F we mean an extension of the form $F(\sqrt{a}, \sqrt{b})$, where a, b represent different square classes in $\dot{F}$. This deviates from the usage in the older literature where a biquadratic extension usually meant any extension of degree 4.

ultimate solution of these cases. Indeed, in the case when F is a global field, a complete determination of W(E/F) for several kinds of Galois extensions of degree 8 (including dihedral extensions) has been obtained in [LLT].

Throughout this paper, we shall use freely the standard terminology and notation from the theory of quadratic forms and the theory of finite-dimensional algebras. For these, as well as for other relevant background information, we refer the reader to the books [L_1] and [Pi].

§ 2. Biquaternion Algebras and SAP Fields

In this mainly expository section, we shall set the stage for the present work by recalling some basic facts in the literature about central simple algebras of dimension 16. We shall also construct various examples of noncyclic biquaternion division algebras from the viewpoint of modern quadratic form theory. Most of the results reviewed in the first half of this section go back to the work of A. A. Albert in the 1930's. More historical notes on this subject can be found in § 3 of [L_3].

As stated in the Introduction, all fields considered in this paper are assumed to have characteristic different from 2. Recall that, for any central simple algebra A over a field F, $\dim_F A$ is always a perfect square; the positive square root of $\dim_F A$ is called the *degree* of A. Central simple algebras of degree 2 are precisely the (generalized) quaternion algebras $(a, b)_F$ $(a, b \in \dot{F})$. For central simple algebras of degree 4, we have the following classical result of Albert ([A_5: Ch. 11, Th. 2], [A_1], [A_4]).

Theorem **2.1.** — *Let* A *be a central simple* F*-algebra of degree* 4. *Then the following are equivalent*:

(1) A *is a biquaternion algebra (i.e. isomorphic to a tensor product of two quaternion algebras over* F*)*;
(2) A *has an involution which is the identity on* F;
(3) exp[A] *(the exponent of the class of* A*) is* $\leqslant 2$ *in the Brauer group* B(F).

Independently of these conditions, if A *is a division algebra, then* A *is always a crossed product with respect to the Klein* 4*-group.*

Here, (2) and (3) are in fact equivalent without any assumption on the degree of A. In the case when A has degree 4, the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious, so the substance of the first part of Theorem (2.1) lies in the implications (3) $\Rightarrow$ (1) and (2) $\Rightarrow$ (1). We should note, however, that these implications *do not* generalize to algebras of degree 8. In fact, Amitsur, Rowen and the third author [ART] have found examples, over the field $F = \mathbf{Q}(w, x, y, z)$ (for instance), of central division algebras of degree 8 and of exponent 2 in B(F) which are *not* isomorphic to a tensor product of three F-quaternion algebras. On the other hand, the celebrated result of A. Merkurjev

[Me] implies that any central simple algebra A of exponent $\leqslant 2$ in B(F) (for any field F) is always *similar* to a tensor product of a number of F-quaternion algebras.

For a quaternion algebra $B = (b_1, b_2)_F$ over F, let $q_B = \langle 1, -b_1, -b_2, b_1 b_2 \rangle$ denote its norm form. This is a 2-fold Pfister form over F, usually written as $\langle\langle -b_1, -b_2 \rangle\rangle$. It is well-known that the isomorphism class of B determines, and is determined by, the isometry class of its norm form q_B; moreover, B is a division algebra if and only if q_B is anisotropic, if and only if q'_B is anisotropic, where $q'_B := \langle -b_1, -b_2, b_1 b_2 \rangle$ denotes the pure subform of q_B. Now consider a biquaternion algebra $B \otimes_F C$ where B is as above, and $C = (c_1, c_2)_F$. Following Albert, we shall associate to $B \otimes_F C$ the 6-dimensional form

(2.2) $$q'_B \perp \langle -1 \rangle q'_C \cong \langle -b_1, -b_2, b_1 b_2, c_1, c_2, -c_1 c_2 \rangle$$

of determinant -1. Note that, in the Witt ring W(F), this form is equal to the difference $q_B - q_C$. It turns out that the facts about quaternion algebras recalled above have the following analogues for biquaternion algebras.

Theorem **2.3** (Albert [A_2: Th. 3]; see also [Pf: p. 123], [Ta]). — *Let* $A = B \otimes_F C$ *and let q be the form defined in* (2.2). *Then*:

(1) $B \otimes_F C$ *is split if and only if q has Witt index* 3 *(i.e. q is hyperbolic)*;
(2) $B \otimes_F C \cong M_2(H)$ *where* H *is a quaternion division algebra if and only if q has Witt index* 1;
(3) $B \otimes_F C$ *is a division algebra if and only if q has Witt index* 0 *(i.e. q is anisotropic)*.

Note that, by Wedderburn's Theorem or by the theory of quadratic forms, for any pair of quaternion algebras B, C over F, exactly one of the above conditions holds.

Theorem **2.4.** — (Jacobson [Ja: Theorem 3.12]). *Let* B, B^*, C, C^* *be quaternion algebras over* F. *Then* $B \otimes_F C \cong B^* \otimes_F C^*$ *(as* F*-algebras) if and only if* $q'_B \perp \langle -1 \rangle q'_C$ *is homothetic to* $q'_{B^*} \perp \langle -1 \rangle q'_{C^*}$ *(i.e.* $q'_B \perp \langle -1 \rangle q'_C \cong \alpha . (q'_{B^*} \perp \langle -1 \rangle q'_{C^*})$ *for some constant* $\alpha \in \dot{F}$*)*.

Jacobson's original proof of (2.4) used the theory of Jordan norms on central simple algebras with involution. Later, a purely quadratic-form-theoretic proof was found by Mammone and Shapiro [MSh]. More recently, a third proof using the viewpoint of pfaffians was given by Knus, Parimala and Sridharan [KPS], who also extended this theorem to the case of biquaternion algebras over commutative rings.

According to Theorem (2.4), if A is a biquaternion algebra, say $A \cong B \otimes_F C$, then the six-dimensional quadratic form $q'_B \perp \langle -1 \rangle q'_C$ is determined up to homothety by the isomorphism class of the algebra A. By a slight abuse of notation, we shall write q_A for $q'_B \perp \langle -1 \rangle q'_C$, and call this the *Albert form* of A. This is not liable to cause confusion as long as we keep in mind that the quadratic form q_A is defined only up to homothety. (For instance, if we compute the Albert form using the isomorphism $A \cong C \otimes_F B$ instead, we get $q'_C \perp \langle -1 \rangle q'_B \cong \langle -1 \rangle (q'_B \perp \langle -1 \rangle q'_C)$.) Note that, in the Witt ring W(F), q_A

lies in $I^2 F$, the square of the fundamental ideal IF of W(F); moreover, the image of q_A in $I^2 F/I^3 F$ is uniquely determined by A, and its Clifford invariant is exactly the class of A in the Brauer group of F.

If A is a biquaternion division algebra, A clearly has a splitting field which is a biquadratic extension of F. It is of interest to ask whether A also has a splitting field which is a cyclic extension of degree 4 over F; this is equivalent to asking if A is a cyclic F-algebra. In 1932, Albert [A_3] constructed the first example of an A for which this is not the case. For the purposes of the present work, it will be useful to present a modern rendition of Albert's construction. Recall that a field F is said to be *pythagorean* if $F^2 + F^2 = F^2$. We begin with the following elementary quadratic form-theoretic lemma.

Lemma **2.5.** — *Let q be a quadratic from over a nonpythagorean field* F. *Then the following are equivalent*:

(1) *q becomes isotropic over some quadratic extension* $K \supset F$ *of the form* $K = F(\sqrt{r^2 + s^2})$ *where* $r, s \in F$;
(2) *$2q$ is isotropic over* F.

Proof. — (1) $\Rightarrow$ (2). Clearly we may assume that q is anisotropic over F. By [L_1: p. 200], (1) implies that q has a binary subform $\langle b \rangle \langle 1, -(r^2 + s^2) \rangle$ where $b \in \dot{F}$. Therefore, $2q$ contains the 4-dimensional form $\langle b \rangle . 2 \langle 1, -(r^2 + s^2) \rangle$, which is easily seen to be hyperbolic [L_1: p. 25, Exer. 6]. Therefore, $2q$ has in fact Witt index $\geqslant 2$ over F. (2) $\Rightarrow$ (1). We may assume again that q is anisotropic over F, for, if otherwise, we can choose any $r^2 + s^2 \notin F^2$ and take K to be $F(\sqrt{r^2 + s^2})$. If $2q$ is isotropic over F, Proposition 2.2 in [EL_2] implies that q contains a subform $\langle a, b \rangle$ such that $2 \langle a, b \rangle$ is hyperbolic. Upon a scaling, this leads to $\langle -ab, -ab \rangle \cong \langle 1, 1 \rangle$, so $-ab = r^2 + s^2$ for some $r, s \in F$. Since q is anisotropic over F, we have $-ab \notin \dot{F}^2$, and q becomes isotropic over the quadratic extension $K = F(\sqrt{r^2 + s^2}) \supset F$, as desired. ■

To come up with examples of noncyclic biquaternion algebras, we shall make use of the notion of a SAP field. There are many equivalent definitions for a SAP field (see [L_2: §§ 16-17]). The most convenient one for us here is the following: A field F is SAP if and only if, for any $x, y \in \dot{F}$, the four dimensional form $\langle 1, x, y, -xy \rangle$ is weakly isotropic i.e. there exists a natural number n depending on x, y, such that the n-fold sum $n . \langle 1, x, y, -xy \rangle$ is isotropic over F. (For instance, any nonreal field is always SAP.) The following Proposition shows that there exist noncyclic biquaternion division algebras over *any* non-SAP field.

Proposition **2.6.** — *Let* F *be a field which is not SAP, say $x, y \in \dot{F}$ are such that $n . \langle 1, x, y, -xy \rangle$ is anisotropic for any $n \geqslant 1$. (We say that $\langle 1, x, y, -xy \rangle$ is "strongly anisotropic".) Then, for* $B = (-1, -1)_F$ *and* $C = (x, y)_F$, *the biquaternion algebra* $A := B \otimes_F C$ *is a noncyclic division* F-*algebra.*

Proof. — The Albert form of A is

$$q_A \cong \langle 1, 1, 1 \rangle \perp \langle -1 \rangle \langle -x, -y, xy \rangle$$
$$\cong \langle 1, 1, 1, x, y, -xy \rangle.$$

Since this form is anisotropic, (2.3) (3) implies that A is a division algebra. Assume, for the moment, that A is a cyclic algebra. Then A contains a maximal subfield L which is a cyclic extension of degree 4 over F. As is well-known [L_1: p. 217, Exer. 8], the unique quadratic extension $K \supset F$ inside L has the form $K = F(\sqrt{r^2+s^2})$ where $r, s \in \dot{F}$. The K-algebra $A^K (= K \otimes_F A)$ contains $K \otimes_F K \cong K \times K$, so $A^K \cong B^K \otimes_K C^K$ is not a division algebra. By (2.3) (3) applied to K, we see that q_A becomes isotropic over K, so by (2.5), $2q_A$ is isotropic over F. But then $6.\langle 1, x, y, -xy \rangle$ is also isotropic over F, a contradiction. Therefore, A cannot be a cyclic algebra over F. ∎

In Albert's original construction in [A_3], he used the base field $F = \mathbf{R}(x, y)$. This is a standard non-SAP field: in fact, by going up to the bigger field $F' = \mathbf{R}((x))((y))$ and applying Springer's Theorem [L_1: p. 145], it can be seen that $\langle 1, x, y, -xy \rangle$ is strongly anisotropic over F′, and hence over F. Thus, by (2.6), $(-1, -1)_F \otimes (x, y)_F$ is a noncyclic division algebra over F. Albert's original example was close to ours, but his proof of its noncyclicity was much more complicated since, not having Springer's Theorem at his disposal, Albert had to use *ad hoc* arguments to handle isotropic forms over $\mathbf{R}((x))((y))$ (cf. also the proof in [Pi: pp. 290-292]). It may be said, however, that Albert's ideas in [A_3] have, to some extent, anticipated the modern notion of SAP fields.

Our new rendition of Albert's construction has a second advantage, since it can be used to give examples of noncyclic biquaternion division algebras over some SAP fields as well.

Proposition **2.7.** — *Let* $F = k((y))$ *where k is a field with an element x which is neither a sum of two squares nor the negative of a sum of six squares. Then, for* $B = (-1, -1)_F$ *and* $C = (x, y)_F$, *the biquaternion F-algebra* $A = B \otimes_F C$ *is a noncyclic division algebra.*

Proof. — The proof here follows the same outline as that of (2.6), so we shall use the same notations as in the earlier proof. With respect to the natural discrete valuation on F, the Albert form of A, $q_A \cong \langle 1, 1, 1, x, y, -xy \rangle$, has first residue form $\langle 1, 1, 1, x \rangle$ and second residue form $\langle 1, -x \rangle$. Since both are anisotropic over k, q_A is anisotropic over F by Springer's Theorem, so A is a division algebra. If A is cyclic, then, as in the proof of (2.6), we can show that $2q_A$ is isotropic over F. However, $2q_A$ has first residue form $6 \langle 1 \rangle \perp \langle x, x \rangle$ and second residue form $\langle 1, 1, -x, -x \rangle$, both of which are anisotropic over k in view of the assumptions on x (and the 2-square identity). This contradicts Springer's Theorem. Therefore, A cannot be a cyclic algebra. ∎

Now it is easy to construct examples of noncyclic biquaternion division algebras over some SAP fields. In fact, take k to be a field with a unique ordering "$>$", and with an element $x > 0$ which is not a sum of two squares. (For instance, take $k = \mathbf{Q}$

and x to be any prime $\equiv 3 \pmod 4$.) By [ELP: Prop. 1], $F = k((y))$ is SAP, and (2.7) above guarantees that $A = (-1, -1)_F \otimes (x, y)_F$ is a noncyclic biquaternion division algebra over F. Note that here, $2 \langle 1, x, y, -xy \rangle$ and $2q_A$ are anisotropic, but, as we would expect from a SAP field, $n \langle 1, x, y, -xy \rangle$ is isotropic for sufficiently large n. This follows, for instance, from the fact that, since " $>$ " is the only ordering on k, $x > 0 \Rightarrow x \in \Sigma k^2$ [L_1: p. 227].

Notice that, in the examples given so far, the noncyclicity proof for the division algebra $A = B \otimes_F C$ depended only on working with quadratic extensions of F of the type $K = F(\sqrt{r^2 + s^2})$ within A. Of course, a more effective analysis of the cyclicity of A (or the lack of it) should involve the cyclic maximal subfields L of A and not just their quadratic subextensions. In later sections, we shall try to explain how these cyclic maximal subfields of A (if they exist) can be exploited more fully.

§ 3. Witt Kernels for Quartic 2-Extensions

In order to study quartic splitting fields for biquaternion algebras, it will be useful to study first in this section the behavior of quadratic forms under a quartic 2-extension. By a quartic 2-extension of a field F, we mean a field L which is a quadratic extension of some quadratic extension K of F. Since we are assuming that F has characteristic not 2, we can write $K = F(\sqrt{a})$ where $a \in \dot{F} - \dot{F}^2$ and $L = K\left(\sqrt{b + 2c\sqrt{a}}\right)$ where $b, c \in F$, $b + 2c\sqrt{a} \notin \dot{K}^2$. Notice that we can always arrange the notation so that $c \neq 0$. Indeed, if L/F is not a biquadratic extension, this is automatic, and if L/F is a biquadratic extension, say $L = F(\sqrt{a}, \sqrt{c})$, then, since $(1 + \sqrt{a})^2 c = (1 + a)c + 2c\sqrt{a}$, we can write

$$L = F(\sqrt{a})(\sqrt{c}) = F(\sqrt{a})\left(\sqrt{(1 + a)c + 2c\sqrt{a}}\right) = F\left(\sqrt{b + 2c\sqrt{a}}\right) \tag{3.1}$$

with $b := (1 + a)c$ and $c \neq 0$. Thus, whenever we deal with a quartic 2-extension $L \supset F$, we shall fix the notations

$$K = F(\sqrt{a}), \qquad L = F\left(\sqrt{b + 2c\sqrt{a}}\right), \quad \text{with } c \neq 0. \tag{3.2}$$

In the biquadratic case, we shall always use the representation (3.1). Note, however, that in (3.1) and (3.2), the element $b \in F$ may be zero. For instance, in (3.1), the case $b = 0$ corresponds to $L = F(\sqrt{-1}, \sqrt{c})$. And in general, the case $b = 0$ in (3.2) corresponds to $L = F\left(\sqrt{2c\sqrt{a}}\right)$; we shall refer to the latter as a " pure " quartic extension of F, since $\sqrt{2c\sqrt{a}}$ satisfies the pure equation $t^4 - 4ac^2 = 0$. An important special case of this is $L = F(\sqrt[4]{a})$ when we take $c = 1/2$.

The primitive element $\theta := \sqrt{b + 2c\sqrt{a}}$ for the extension L/F in (3.2) has the minimal polynomial

$$m(t) = t^4 - 2bt^2 + b^2 - 4ac^2 \in F[t].$$

Using this, the field discriminant $\mathscr{D}(\mathrm{L/F})$ can be computed via the well-known formula $\mathscr{D}(\mathrm{L/F}) = \mathrm{N}_{\mathrm{L/F}}(m'(\theta))$. We shall leave it to the reader to verify that, up to a square in F, we have $\mathscr{D}(\mathrm{L/F}) = b^2 - 4ac^2$. Alternatively, we can also compute $\mathscr{D}(\mathrm{L/F})$ by using a " transitivity formula " for the discriminant which can be read off, for instance, from [Sch: Th. (5.12), p. 51]. Since L/K is of even degree here, this transitivity formula gives directly

$$\mathscr{D}(\mathrm{L/F}) = \mathrm{N}_{\mathrm{K/F}}(\mathscr{D}(\mathrm{L/K})) = \mathrm{N}_{\mathrm{K/F}}(b + 2c\sqrt{a}) = b^2 - 4ac^2.$$

Depending on the square class of the discriminant $\mathscr{D}(\mathrm{L/F})$ in the field F, we have the three possibilities described in the Proposition below for the extension L/F.

Proposition **3.3.**

(1) *If* $b^2 - 4ac^2 \in \dot{\mathrm{F}}^2$, *then* L/F *is a biquadratic extension*;

(2) *If* $b^2 - 4ac^2 \in a\dot{\mathrm{F}}^2$, *then* L/F *is a cyclic extension of degree* 4 *(and* a *is a sum of two squares in* F*)*;

(3) *If* $b^2 - 4ac^2 \notin \dot{\mathrm{F}}^2 \cup a\dot{\mathrm{F}}^2$, *then* L/F *is not a Galois extension. In this case, the Galois hull of* L/F *is given by* $\mathrm{E} = \mathrm{L}(\sqrt{b^2 - 4ac^2})$, *with* Gal(E/F) *isomorphic to the dihedral group of order* 8.

Proof. — This Proposition is part of the folklore in Galois Theory; for the sake of completeness, we shall include a proof. The four conjugates of $\theta = \sqrt{b + 2c\sqrt{a}}$ are $\pm\,\theta$, $\pm\,\theta'$, where $\theta' = \sqrt{b - 2c\sqrt{a}}$, with $(\theta\theta')^2 = b^2 - 4ac^2 \in \dot{\mathrm{F}}$. Thus, if

$$b^2 - 4ac^2 \in \dot{\mathrm{F}}^2 \cup a\dot{\mathrm{F}}^2,$$

we have $\theta' \in \mathrm{L}$ and so L/F is Galois. Conversely, if L/F is Galois, then $\theta' \in \dot{\mathrm{L}}$ and so $b - 2c\sqrt{a} \in \dot{\mathrm{L}}^2$. Since $\mathrm{L} = \mathrm{K}\left(\sqrt{b + 2c\sqrt{a}}\right)$, it follows (cf. [$\mathrm{L}_1$: p. 202]) that

$$b - 2c\sqrt{a} \in \dot{\mathrm{K}}^2 \cup (b + 2c\sqrt{a})\,\dot{\mathrm{K}}^2.$$

If $b - 2c\sqrt{a} \in \dot{\mathrm{K}}^2$, taking the norm from K to F gives $b^2 - 4ac^2 \in \dot{\mathrm{F}}^2$; if

$$b - 2c\sqrt{a} \in (b + 2c\sqrt{a})\,\dot{\mathrm{K}}^2,$$

multiplication by $b + 2c\sqrt{a}$ gives $b^2 - 4ac^2 \in \dot{\mathrm{K}}^2$, and so (again from [$\mathrm{L}_1$: p. 202]) $b^2 - 4ac^2 \in \dot{\mathrm{F}}^2 \cup a\dot{\mathrm{F}}^2$. Next, we shall compute Gal(L/F). If $b^2 - 4ac^2 \in \dot{\mathrm{F}}^2$, then $\theta\theta' \in \dot{\mathrm{F}}$ and so the F-automorphism of L sending θ to θ' has order 2. Since the automorphism sending θ to $-\,\theta$ also has order 2, we have clearly $\mathrm{Gal(L/F)} \cong \mathbf{Z}_2 \times \mathbf{Z}_2$, i.e. L/F is a biquadratic extension. On the other hand, if $b^2 - 4ac^2 \in a\dot{\mathrm{F}}^2$, then $\theta\theta' \in \sqrt{a}.\dot{\mathrm{F}}$, and the F-automorphism of L sending θ to θ' has order 4, and so $\mathrm{Gal(L/F)} \cong \mathbf{Z}_4$. For the last part of the proof, we shall now work in the case when $d := b^2 - 4ac^2 \notin \dot{\mathrm{F}}^2 \cup a\dot{\mathrm{F}}^2$; by the above, this is exactly the case when L/F fails to be Galois. Since $\theta\theta' = \pm\sqrt{d}$,

$E := L(\sqrt{d})$ is clearly the splitting field for the minimal polynomial of θ over F, so it is the Galois hull of L/F. The Galois group Gal(E/F) has order 8 and has a non-normal subgroup Gal(E/L) of order 2, so it must be (isomorphic to) the dihedral group of order 8. ∎

Remarks **3.4.** — (a) In Case (3) above, there are five involutions in Gal(E/F). An easy calculation shows that their respective fixed fields are:

$$F\left(\sqrt{b \pm 2c\sqrt{a}}\right), \qquad F(\sqrt{a}, \sqrt{d}), \quad \text{and} \quad F\left(\sqrt{2(b \pm \sqrt{d})}\right),$$

where d is as defined above. These are therefore the five subfields of codimension 2 in E. The fact that $2(b \pm \sqrt{d})$ are perfect squares in E can be seen, for instance, from the equation

$$[\sqrt{d} \pm (b \pm 2c\sqrt{a})]^2 = 2(b \pm \sqrt{d})(b \pm 2c\sqrt{a}),$$

which can be verified by direct expansion of the left side.

(b) By elementary Galois theory, it is easy to see that any dihedral extension of degree 8 over a field F arises exactly in the above fashion (as the field E in (3.3) (3)). Fröhlich's criterion in [Fr: (7.7)] for the embeddability of a biquadratic extension into a dihedral extension of degree 8 can also be deduced easily from Theorem 3.3 (3).

For a given element $d \in \dot{F}$, *when does there exist a quartic 2-extension* L/F *of discriminant d*? (In the following, we shall often confuse a nonzero element with its square class.) If $d \in \dot{F}^2$, such an L must be a biquadratic extension by (3.3), so it exists if and only if F has at least four square classes. In the following, we shall tackle the more interesting case when $d \notin \dot{F}^2$. In this case, the answer to the above question is again: " almost always ".

Theorem **3.5.** — *Let $d \in \dot{F} \backslash \dot{F}^2$. Then* F *has no quartic* 2*-extension* L/F *of discriminant d if and only if* F *is formally real pythagorean and $d \in -\dot{F}^2$.*

Proof. — For the " if " part, suppose F is formally real pythagorean and $d \in -\dot{F}^2$. If there exists a quartic 2-extension L/F as in (3.2) with discriminant d, then

$$b^2 - 4ac^2 \in d\dot{F}^2 = -\dot{F}^2.$$

But then $4ac^2 \in b^2 + \dot{F}^2 \subseteq \dot{F}^2$, contradicting the fact that a is a nonsquare.

For the " only if " part, suppose we are not in the special case when F is formally real pythagorean and $d \in -\dot{F}^2$. *We shall construct a quartic* 2*-extension* L/F *of discriminant d.* First suppose $d \notin -\dot{F}^2$. Writing $a := -d \notin \dot{F}^2$, we note that $\sqrt{a}$ is not a square in $F(\sqrt{a})$ since it has norm $-a$ which is not a square. Therefore, the quartic 2-extension $L := F(\sqrt[4]{a})$ has discriminant $-a = d$, as desired. Finally, assume that $d \in -\dot{F}^2$. (In particular, $\dot{F} \neq -\dot{F}^2$.) Then, by assumption, F cannot be pythagorean (for, if it is, it must be formally real [L_1: p. 234], since $\dot{F} \neq \dot{F}^2$). Fix a nonsquare element $a = r^2 + s^2$ in F. Then $r + \sqrt{a}$ is a nonsquare in $F(\sqrt{a})$, since its norm is $r^2 - a = -s^2 \notin \dot{F}^2$. Now $L := F\left(\sqrt{r + \sqrt{a}}\right)$ is a quartic 2-extension of discriminant $-s^2 \in d\dot{F}^2$, as desired. ■

Before we move on, let us record a consequence of (3.5). A familiar result in the literature of field theory (usually attributed to Diller and Dress [DD: Satz 1], but known much earlier to Garver and Albert) states the following: *A field* F *has no cyclic quartic extension if and only if* F *is pythagorean.* This fact is easy to prove using the general notations for quartic 2-extensions set up above. The following is some kind of a twin to the Diller-Dress Theorem, which does not seem to have appeared in the literature before.

Corollary **3.6.** — *A field* F *has no quartic* 2*-extension of discriminant* -1 *if and only if either* F *is formally real pythagorean, or* $-1 \in \dot{F}^2$ *and* $|\dot{F}/\dot{F}^2| \leqslant 2$.

Proof. — For the " if " part, first assume F is formally real pythagorean. Then (3.5) applies to $d = -1$ to show that F has no quartic 2-extension of discriminant -1. If $-1 \in \dot{F}^2$ and $|\dot{F}/\dot{F}^2| \leqslant 2$, the same conclusion also holds, for F will not have any biquadratic extensions. For the " only if " part, assume that $(*)$ F has no quartic 2-extension of discriminant -1. If $-1 \notin \dot{F}^2$, then (3.5) applies to $d = -1$, and we conclude that F is formally real pythagorean. Finally, assume that $-1 \in \dot{F}^2$. Then, by $(*)$, F has no quartic 2-extension of discriminant 1, so we must have $|\dot{F}/\dot{F}^2| \leqslant 2$. ■

To any quartic 2-extension $L \supset F$ expressed as in (3.2), we shall associate the following quadratic polynomial

$$f(t) = at^2 + bt + c^2 \in F[t].$$

Note that this polynomial has discriminant $b^2 - 4ac^2$, which is exactly the field discriminant of L/F. From (3.3), we see that

$$f(t) \text{ is reducible over } F \Leftrightarrow L/F \text{ is biquadratic.}$$

In this case, it will be particularly convenient to use the expression (3.1) for L, for then

$$b^2 - 4ac^2 = (1 + a)^2 c^2 - 4ac^2 = [(1 - a)\, c]^2,$$

and $f(t)$ has the simple factorization:

$$at^2 + bt + c^2 = at^2 + (1 + a)\, ct + c^2 = (t + c)\,(at + c), \tag{3.7}$$

with the two distinct roots $-c$ and $-c/a$ in F. The role played by $f(t)$ in the investigation of the behavior of quadratic forms under the extension L/F will become clear shortly.

We shall now begin our computation of W(L/F), the kernel of the natural Witt ring map $W(F) \to W(L)$, where L/F is a quartic 2-extension expressed as in (3.2). The first step in this computation is to determine the 2-fold Pfister forms over F which lie in this kernel. In the following, we shall use the standard notation

$$\langle\langle a_1, \ldots, a_n \rangle\rangle \tag{3.8}$$

for the n-fold Pfister forms over F. When all the a_i's are nonzero, (3.8) means the n-fold tensor product $\langle 1, a_1 \rangle \otimes \ldots \otimes \langle 1, a_n \rangle$; when some of the a_i's are zero, (3.8) is taken to mean the hyperbolic n-fold Pfister form.

Theorem **3.9.** — *Any 2-fold* F*-Pfister form in the family* $\mathscr{C} = \{\langle\langle e, -f(e)\rangle\rangle : e \in \dot{\mathrm{F}}\}$ *becomes hyperbolic over* L. *Conversely, let* σ *be any 2-fold* F*-Pfister form in* W(L/F).

(1) *If* L/F *is not biquadratic, then* $\sigma \in \mathscr{C}$. *(In particular,* $\langle\langle -a, z\rangle\rangle \in \mathscr{C}\ \forall\, z \in \dot{\mathrm{F}}$*).*

(2) *If* L/F *is biquadratic, say* $\mathrm{L} = \mathrm{F}(\sqrt{a}, \sqrt{c})$ *(as in* (3.1)*), then either* $\sigma \in \mathscr{C}$, *or* $\sigma \cong \langle\langle -c, *\rangle\rangle$, *or* $\sigma \cong \langle\langle -ac, *\rangle\rangle$.

Proof. — Write $f(e) = ae^2 + be + c^2 = (e\sqrt{a} - c)^2 + (b + 2c\sqrt{a})\,e$. Since $b + 2c\sqrt{a}$ is a square in L, we see that $f(e)$ is represented by $\langle 1, e\rangle$ over L. Therefore, $\langle\langle e, -f(e)\rangle\rangle \in \mathrm{W(L/F)}$ for any $e \in \dot{\mathrm{F}}$. Conversely, let $\sigma = \langle\langle x, y\rangle\rangle$ be any 2-fold F-Pfister form in W(L/F). We may assume that σ is anisotropic over F (for otherwise σ is hyperbolic and we have $\sigma \cong \langle\langle -1, -f(-1)\rangle\rangle$). Since the K-form σ_{K} splits over $\mathrm{L} = \mathrm{K}\left(\sqrt{b + 2c\sqrt{a}}\right)$, its pure subform $\sigma'_{\mathrm{K}} \cong \langle x, y, xy\rangle_{\mathrm{K}}$ represents $-(b + 2c\sqrt{a})$ over K (by [L_1: p. 200]). Thus we have an equation

$$x(u_1 + v_1\sqrt{a})^2 + y(u_2 + v_2\sqrt{a})^2 + xy(u_3 + v_3\sqrt{a})^2 = -(b + 2c\sqrt{a}), \tag{3.10}$$

where $u_i, v_i \in \mathrm{F}$. Letting $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ in the F-quadratic space (F^3, σ'), we can express the left-hand side above as $\sigma'(u) + a\sigma'(v) + 2\sqrt{a}\mathrm{B}_{\sigma'}(u, v)$ where $\mathrm{B}_{\sigma'}$ is the associated symmetric bilinear form of σ'. Comparing rational and irrational parts in (3.10), we have therefore

$$\sigma'(u) + a\sigma'(v) = -b \quad \text{and} \quad \mathrm{B}_{\sigma'}(u, v) = -c.$$

Since $c \neq 0$, we see that $u \neq 0 \neq v$ in F^3; thus, $e := \sigma'(v) \neq 0$ (since σ' is anisotropic). The inner product matrix of the two vectors u, v has determinant

$$\begin{vmatrix} \sigma'(u) & \mathrm{B}_{\sigma'}(u, v) \\ \mathrm{B}_{\sigma'}(u, v) & \sigma'(v) \end{vmatrix} = \begin{vmatrix} -b - ae & -c \\ -c & e \end{vmatrix} = -(b + ae)\,e - c^2 = -f(e). \tag{3.11}$$

We now go into the following two cases.

Case 1. — L/F *is not biquadratic.* In this case, $f(t)$ has no root over F, so $f(e) \neq 0$. This guarantees, in particular, that the two vectors u, v are linearly independent in F^3. Since the binary quadratic subspace $\mathrm{F}.u \oplus \mathrm{F}.v$ of (F^3, σ') represents e and has determinant $-f(e)$ by (3.11), it has a diagonalization $\langle e, -ef(e)\rangle$, and therefore

$$\sigma \cong \langle\langle x, y\rangle\rangle \cong \langle 1, e, -ef(e), -f(e)\rangle \cong \langle\langle e, -f(e)\rangle\rangle \in \mathscr{C}.$$

Case 2. — L/F *is biquadratic, as in* (3.1). If $f(e) \neq 0$, we proceed as above and get $\sigma \in \mathscr{C}$. If, instead, $f(e) = 0$, we have $e \in \{-c, -c/a\}$ in view of (3.7). Since $e = \sigma'(v)$, this implies that $\sigma \cong \langle\langle -c, *\rangle\rangle$ or $\sigma \cong \langle\langle -c/a, *\rangle\rangle \cong \langle\langle -ac, *\rangle\rangle$ (i.e. σ splits already over $\mathrm{F}(\sqrt{c})$ or over $\mathrm{F}(\sqrt{ac})$). ■

In Case (2) above, the 2-fold Pfister forms in the family $\mathscr{C}$ are of the shape $\langle\langle e, -(e+c)(ae+c)\rangle\rangle$ (in view of (3.7)) where $L = F(\sqrt{a}, \sqrt{c})$. This expression for the forms in $\mathscr{C}$ can be further simplified if we use the more natural representation for L as $K(\sqrt{c})$ $\left(\text{rather than as } K\left(\sqrt{b+2c\sqrt{a}}\right)\right)$. Indeed, repeating the same argument in the proof of (3.9) (assuming, as before, that σ is anisotropic over F), we get an equation

$$x(u_1 + v_1\sqrt{a})^2 + y(u_2 + v_2\sqrt{a})^2 + xy(u_3 + v_3\sqrt{a})^2 = -c,$$

and hence $B_{\sigma'}(u, v) = 0$, $\sigma'(u) + a\sigma'(v) = -c$. If $v = 0$, then $\sigma'(u) = -c$ and so $\sigma \cong \langle\langle -c, * \rangle\rangle$. If $u = 0$, then $\sigma'(v) = -c/a$ and so $\sigma \cong \langle\langle -ac, * \rangle\rangle$. If $u \neq 0 \neq v$, then, since σ' is anisotropic, $B_{\sigma'}(u, v) = 0$ implies that $\{u, v\}$ are linearly independent. Now (F^3, σ') contains the binary subspace $F.u \oplus F.v$ which has a diagonalization $\langle \sigma'(u), \sigma'(v)\rangle \cong \langle -c-ae, e\rangle$ where $e := \sigma'(v) \neq 0$, and hence $\sigma \cong \langle\langle e, -(ae+c)\rangle\rangle$.

Corollary **3.12.** — *If* $L = F(\sqrt{a}, \sqrt{c})$, *the 2-fold Pfister forms in* $W(L/F)$ *are precisely the following:*

$$\langle\langle e, -(ae+c)\rangle\rangle \quad (e \neq -c/a), \ \langle\langle -c, * \rangle\rangle, \ \langle\langle -ac, * \rangle\rangle.$$

All of these Pfister forms belong to the ideal $\langle\langle -a \rangle\rangle W(F) + \langle\langle -c \rangle\rangle W(F)$.

Proof. — For $e \neq -c/a$, $\langle\langle e, -(ae+c)\rangle\rangle$ does split in L since, over L, $ae + c = (\sqrt{c})^2 + (\sqrt{a})^2 e$ is represented by $\langle 1, e\rangle_L$. The last statement of the Corollary is seen from the following straightforward Witt ring calculations:

$$\begin{aligned}
\langle\langle -ac \rangle\rangle &= \langle 1, -a, a, -ac\rangle = \langle\langle -a \rangle\rangle \perp \langle a \rangle \langle\langle -c \rangle\rangle \in W(F),\\
\langle\langle e, -(ae+c)\rangle\rangle &= \langle 1, -(ae+c)\rangle \perp \langle e \rangle \langle 1, -c\rangle \perp \langle ec, -e(ae+c)\rangle\\
&= \langle 1, -(ae+c)\rangle \perp \langle e \rangle \langle 1, -c\rangle \perp \langle -a, ac(ae+c)\rangle\\
&= -\langle ae+c\rangle \langle\langle -ac \rangle\rangle \perp \langle\langle -a \rangle\rangle \perp \langle e \rangle \langle\langle -c \rangle\rangle \in W(F).
\end{aligned}$$

∎

Next, we shall refine the method of proof of Theorem 3.9 to give a complete determination of the Witt ring kernel $W(L/F)$ for an arbitrary quartic 2-extension L/F. Recall that, for any set N of natural numbers, an N-Pfister ideal in $W(F)$ means an ideal of the shape $\Sigma \varphi_i W(F)$ where each φ_i is an n_i-fold Pfister form with $n_i \in N$. In the special case when N is a singleton, say $\{n\}$, we shall speak of such an ideal as an n-Pfister ideal (instead of an $\{n\}$-Pfister ideal). The theory of Pfister ideals in Witt rings was developed in [ELW$_2$]; however, the results in [ELW$_2$] will not be needed here.

Theorem **3.13.** — *Let* L/F *be a quartic 2-extension (represented as in* (3.2)*). Then* $W(L/F)$ *is a* $\{1, 2\}$*-Pfister ideal in* $W(F)$. *To be more precise, we have the following:*

(1) *If* L/F *is not biquadratic, then* $W(L/F) = \langle\langle -a \rangle\rangle W(F) + \Sigma_{\sigma\in\mathscr{C}}\, \sigma W(F)$, *where* $\mathscr{C}$ *is as defined in* (3.9);

(2) *If* $L = F(\sqrt{a}, \sqrt{c})$, *then* $W(L/F) = \langle\langle -a \rangle\rangle W(F) + \langle\langle -c \rangle\rangle W(F)$. *(In this case,* $W(L/F)$ *is in fact a 1-Pfister ideal in* $W(F)$.*)*

Proof. — First note that, if L/F is not biquadratic, then $F(\sqrt{a})$ is the only quadratic extension of F in L, while if $L = F(\sqrt{a}, \sqrt{c})$, then $F(\sqrt{a})$, $F(\sqrt{c})$ and $F(\sqrt{ac})$ are the only quadratic extensions of F in L. From this, it follows that the 1-fold F-Pfister forms splitting in L are $\{\langle\langle -1 \rangle\rangle, \langle\langle -a \rangle\rangle\}$ in the former case, and $\{\langle\langle -1 \rangle\rangle, \langle\langle -a \rangle\rangle, \langle\langle -c \rangle\rangle, \langle\langle -ac \rangle\rangle\}$ in the latter case. Therefore, once we know that W(L/F) is a $\{1, 2\}$-Pfister ideal, (1) and (2) in the Theorem will follow from (3.9) and (3.12). Note that (2) here recovers the earlier result of Elman-Lam-Wadsworth in [ELW$_1$: (2.12)], with a substantially different proof.

For any F-form $\sigma \in W(L/F)$, let us now show, by induction on dim σ, that σ belongs to the ideal of W(F) generated by the 1-fold and 2-fold Pfister forms splitting in L. We may assume that σ is anisotropic over any quadratic extension of F in L (for otherwise σ contains a binary subform splitting in some quadratic extension of F in L [L$_1$: p. 200], and we are done by induction). We first work over $K = F(\sqrt{a}) \subset L$. Since σ_K is anisotropic and splits over L, we have $\sigma_K \cong \gamma \langle\langle -(b + 2c\sqrt{a})\rangle\rangle$ for some K-form $\gamma \cong \langle x_1, \ldots, x_n \rangle$. Let φ be the K-form $\langle\langle -(b + 2c\sqrt{a})\rangle\rangle$, so

$$\sigma_K \cong \langle x_1 \rangle \varphi \perp \ldots \perp \langle x_n \rangle \varphi.$$

After a scaling, we may assume that σ represents 1 over F, so there exists an equation $1 = x_1 y_1 + \ldots + x_n y_n$, where, say, $y_1, \ldots, y_m (m \leqslant n)$ are elements of $\dot{K}$ represented by φ, and the remaining y_i are zero. From this, we have an isometry

$$\langle x_1 y_1, \ldots, x_m y_m \rangle \cong \langle 1, x_2', \ldots, x_m' \rangle$$

for suitable $x_2', \ldots, x_m' \in \dot{K}$. Using the fact that $\varphi \cong \langle y_i \rangle \varphi$ for $i \leqslant m$, we have

$$\begin{aligned}\sigma_K &\cong \langle x_1 y_1 \rangle \varphi \perp \ldots \perp \langle x_m y_m \rangle \varphi \perp \langle x_{m+1} \rangle \varphi \perp \ldots \\ &\cong \langle x_1 y_1, \ldots, x_m y_m \rangle \varphi \perp \langle x_{m+1} \rangle \varphi \perp \ldots \\ &\cong \langle 1, x_2', \ldots, x_m' \rangle \varphi \perp \ldots \\ &\cong \langle 1, -(b + 2c\sqrt{a}) \rangle \perp \ldots .\end{aligned}$$

Writing $\sigma \cong \langle 1 \rangle \perp \sigma'$ over F, and cancelling $\langle 1 \rangle$, we see that σ_K' represents $-(b + 2c\sqrt{a})$ over K. Arguing as in the proof of (3.9), we can find F-vectors u, v such that

$$\sigma'(u) + a\sigma'(v) = -b, \quad \text{and} \quad B_{\sigma'}(u, v) = -c.$$

Writing $e = \sigma'(v)$ as before, we have

$$\begin{vmatrix} \sigma'(u) & B_{\sigma'}(u, v) \\ B_{\sigma'}(u, v) & \sigma'(v) \end{vmatrix} = -f(e)$$

as in (3.11). If $f(e) = 0$, we must be in the situation $L = F(\sqrt{a}, \sqrt{c})$ (as in (3.1)) with $e \in \{-c, -c/a\}$. But then $\sigma \cong \langle 1, e \rangle \perp \ldots$ becomes isotropic over $F(\sqrt{c})$ or $F(\sqrt{ac})$, contrary to our assumption. Thus, $f(e) \neq 0$, and our earlier argument in the proof of (3.9) gives a decomposition $\sigma' \cong \langle e, -ef(e)\rangle \perp \sigma''$ over F. Calculating in W(F), we then have

(3.14) $$\sigma \cong \langle 1 \rangle \perp \langle e, -ef(e)\rangle \perp \sigma'' = \langle\langle e, -f(e)\rangle\rangle \perp \langle f(e)\rangle \perp \sigma''$$

where $\dim(\langle f(e)\rangle \perp \sigma'') = \dim \sigma - 2$. Since $\langle f(e)\rangle \perp \sigma'' = \sigma - \langle\langle e, -f(e)\rangle\rangle \in W(L/F)$, we are done by induction.

Remark **3.15.** — Let $L' = F\left(\sqrt{b - 2c\sqrt{a}}\right)$. Then clearly L and L′ have the same associated quadratic polynomial $f(t)$, and therefore the Theorem above implies that $W(L/F) = W(L'/F)$. This fact is to be expected since L and L′ are quartic extensions which are isomorphic over F. (In the case when L is biquadratic, we have, of course, $L = L'$.)

Corollary **3.16.** — *Let* L/F *be as above. Then* $W(L/F) \cap I^2 F$ *is a 2-Pfister ideal in* W(F). *In fact, if* L/F *is not biquadratic, then* $W(L/F) \cap I^2 F = \Sigma_{\sigma \in \mathscr{C}} \sigma W(F)$; *if* $L = F(\sqrt{a}, \sqrt{c})$, *then* $W(L/F) \cap I^2 F = \langle\langle -a \rangle\rangle IF + \langle\langle -c \rangle\rangle IF$.

Proof. — This follows by a standard determinant argument from the explicit computation of W(L/F) as a $\{1, 2\}$-Pfister ideal in W(F). ∎

It seems plausible that, for any $r \geqslant 2$, $W(L/F) \cap I^r F$ is an r-Pfister ideal. Unfortunately, we do not have a proof of this. We can, however, prove the following nice fact about the r-fold F-Pfister forms splitting in L.

Corollary **3.17.** — *Let* L/F *be as above, and let* $r \geqslant 2$. *Then every* r*-fold* F*-Pfister form* $\sigma \in W(L/F)$ *can be written as* $\langle\langle z_1, z_2, \ldots, z_r \rangle\rangle$, *where* $\langle\langle z_1, z_2 \rangle\rangle \in W(L/F)$.

Proof. — If σ becomes isotropic over a quadratic extension of F in L, we can choose the z_i's such that $\langle\langle z_1 \rangle\rangle$ splits in this quadratic extension. If otherwise, the argument in the proof of the Theorem shows that σ' (the pure part of σ) has a subform $\langle e, -ef(e)\rangle$. By [EL_1: p. 192, Remark (1)], we can write σ as

$$\langle\langle e, -ef(e), *, \ldots, * \rangle\rangle \cong \langle\langle e, -f(e), *, \ldots, * \rangle\rangle. \quad \blacksquare$$

In view of the last two Corollaries, it can be easily seen that the statement that $W(L/F) \cap I^r F$ is an r-Pfister ideal ($r \geqslant 2$) is equivalent to the equation

(3.18) $$W(L/F) \cap I^r F = (W(L/F) \cap I^2 F) . I^{r-2} F.$$

To conclude this section, let us record a special case of (3.13) (1) for the class of fields known as " excellent fields ". Recall that a field F is called *excellent* [ELW_3: § 4] if every anisotropic 4-dimensional form over F has determinant 1 (in $\dot{F}/\dot{F}^2$). For instance, local fields are excellent, by [L_1: p. 149].

Proposition **3.19.** — *Let* L/F *be as in* (3.2), *but assume this is not a biquadratic extension. If* F *is an excellent field, then we have* $W(L/F) = \langle\langle -a \rangle\rangle W(F)$.

Proof. — For any two elements x, y in $\dot{F}$, the 4-dimensional form $\langle x, y, xy, a \rangle$ has determinant a and therefore must be isotropic over F. Thus, $\langle x, y, xy \rangle$ represents $-a$, so we can write $\langle\langle x, y \rangle\rangle$ as $\langle\langle -a, z \rangle\rangle$ for some $z \in \dot{F}$. This fact and (3.13) (1) clearly imply that $W(L/F) = \langle\langle -a \rangle\rangle W(F)$. ∎

In the special case when F is a global field, the results in this section can be used to determine the Witt ring kernel W(E/F) for a Galois extension E/F of degree 8 with Gal(E/F) isomorphic to the dihedral group, or $\mathbf{Z}_4 \oplus \mathbf{Z}_2$, or $\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$. For global fields, it also turns out that one can prove equations such as (3.18), and their analogues for the degree 8 extensions mentioned above [LLT].

§ 4. Quartic Splitting Fields for Pfister Forms and Biquaternion Algebras

In this section, we shall study quartic splitting fields of the kind (3.2) for Pfister forms, quaternion and biquaternion algebras over a given field F. Here, we think of the quaternion or biquaternion algebra(s) as given, and study the problem of finding quartic 2-extensions L/F with a given discriminant which are splitting fields for the given algebra(s). Recall that, for any quartic extension L/F, L is a splitting field for a biquaternion algebra $B \otimes_F C$ if and only if L embeds (as a maximal subfield) in $B \otimes_F C$, and L is a splitting field for a quaternion algebra S if and only if L embeds (as a maximal subfield) in the matrix algebra $\mathbf{M}_2(S)$. Thus, the problem we proposed to study above may also be described as that of finding quadratic field extension towers $F \subset K \subset L$ with a given discriminant $\mathscr{D}(L/F)$, inside the central simple algebras $B \otimes_F C$ and $\mathbf{M}_2(S)$. In this section, however, we shall carry out our investigations using the terminology and the techniques of quadratic form theory. Thus, we shall replace the quaternion algebra $S = (-x, -y)_F$ by its associated 2-fold Pfister form $\sigma = \langle\langle x, y \rangle\rangle$, and replace the biquaternion algebra $A = B \otimes_F C$ by its associated six-dimensional Albert form q_A. The beginning point of our investigations in this section is the following important observation on Witt ring kernels.

Proposition **4.1.** — *Let* L/F *be a quartic* 2*-extension represented as in* (3.2), *with discriminant* $d = b^2 - 4ac^2$. *Then* $W(L/F) . \langle\langle -d \rangle\rangle = 0$.

Proof. — Since $\theta = \sqrt{b + 2c\sqrt{a}}$ is a primitive element for the extension L/F, a known result on transfers of quadratic forms ([L_1: p. 197]) implies that W(L/F) is annihilated by the 1-fold Pfister form $\langle\langle -N_{L/F}(\theta) \rangle\rangle$. By the transitivity of norm maps,

$$N_{L/F}(\theta) = N_{K/F}(N_{L/K}(\theta)) = N_{K/F}(-(b + 2c\sqrt{a})) = b^2 - 4ac^2 = d.$$

Hence, $\langle\langle -d \rangle\rangle$ annihilates W(L/F). Alternatively, we can also give a proof of this fact strictly within the context of this paper, thus avoiding the transfer result quoted above. First, we may assume that $d \notin \dot{F}^2$ (for otherwise $\langle\langle -d \rangle\rangle = 0 \in W(F)$ already). By (3.13) (1), W(L/F) is then generated (as an ideal) by $\langle\langle -a \rangle\rangle$ and $\langle\langle e, -f(e) \rangle\rangle$ for all $e \in \dot{F}$. Since $a = (b^2 - d)/4c^2$ and

$$f(e) = ae^2 + be + c^2 = \frac{b^2 - d}{4c^2}e^2 + be + c^2 = \left(\frac{be}{2c} + c\right)^2 - d\left(\frac{e}{2c}\right)^2$$

are both represented by $\langle\langle -d \rangle\rangle$, it follows that

$$\langle\langle -a, -d \rangle\rangle = 0 \quad \text{and} \quad \langle\langle -f(e), -d \rangle\rangle = 0.$$

Thus, $W(L/F) . \langle\langle -d \rangle\rangle = 0$. ∎

Using the last Proposition, we shall now try to study quartic splitting fields (as in (3.2)) for n-fold Pfister forms. In the case $n = 2$, this corresponds to the study of quartic splitting fields for quaternion algebras. In fact, our results were first obtained for the splitting of quaternion algebras, but a careful look at the proof showed that these results also hold for the splitting of n-fold Pfister forms when $n \geqslant 2$. Therefore, we shall present our results in the more general setting of Pfister forms. Later in this section, we shall also study quartic splitting fields (as in (3.2)) for a " difference " of two n-fold Pfister forms ($n \geqslant 2$). In the case $n = 2$, such a " difference " is simply the Albert form of a biquaternion algebra, so our work will give results on quartic splitting fields of biquaternion algebras.

Theorem **4.2.** — *Let a, d be nonsquares in a field* F, *and let* $K = F(\sqrt{a})$. *For any given n-fold Pfister form σ over* F $(n \geqslant 2)$, *the following are equivalent*:

(1) *σ splits over some quadratic extension* $L \supset K$ *with* $\mathscr{D}(L/F) = d$;

(2) $\langle\langle -a, -d \rangle\rangle = \sigma \langle\langle -d \rangle\rangle = 0 \in W(F)$ (1).

Proof. — (1) $\Rightarrow$ (2) Let L be as represented in (3.2). Then $d = b^2 - 4ac^2$ implies that $\langle a, d \rangle$ represents 1 over F, and by (4.1), $\sigma \in W(L/F)$ implies that

$$\sigma \langle\langle -d \rangle\rangle = 0 \in W(F).$$

(2) $\Rightarrow$ (1) Since $\langle a, d \rangle$ represents 1, there exists an equation $a = r^2 - ds^2$ where $r, s \in F$. Here, $s \neq 0$ since $a \notin F^2$. Let β be the binary F-quadratic form associated with the symmetric matrix $\begin{pmatrix} 1 & -r \\ -r & a \end{pmatrix}$, and let $\sigma' \cong \langle z_2, z_3, \ldots, z_{2^n} \rangle$ be the pure subform of σ. Then β has the diagonalization $\langle 1, a - r^2 \rangle \cong \langle 1, -d \rangle$, and we have $\sigma' \otimes \beta \cong \sigma' \otimes \langle\langle -d \rangle\rangle$. Since $n \geqslant 2$, the dimension of $\sigma' \otimes \langle\langle -d \rangle\rangle$ is more than half the dimension of its ambient form $\sigma \otimes \langle\langle -d \rangle\rangle \cong 2^n \mathbf{H}$. Hence $\sigma' \otimes \beta$ is isotropic. This implies that there

(1) Recall that, for a Pfister form σ, $\sigma\langle\langle -d \rangle\rangle = 0$ means that σ represents d; also $\langle\langle -a, -d \rangle\rangle = 0$ simply means that $\langle a, d \rangle$ represents 1.

is an equation $\Sigma_{i \geq 2} z_i(u_i^2 - 2ru_i v_i + av_i^2) = 0$, where the u_i, v_i's are not all zero. Regrouping the terms in this equation, we get

$$\sigma'(u) + a\sigma'(v) - 2r\,\mathrm{B}_{\sigma'}(u, v) = 0, \tag{4.3}$$

where $\mathrm{B}_{\sigma'}$ is the symmetric bilinear form associated to σ', and $u = (u_2, \ldots, u_{2n})$, $v = (v_2, \ldots, v_{2n})$. Writing $b := -(\sigma'(u) + a\sigma'(v))$ and $c := -\mathrm{B}_{\sigma'}(u, v)$, we have the following two cases.

Case 1. — $c \neq 0$. From (4.3), we have $r = b/2c$ and so $a = (b/2c)^2 - ds^2$. Thus,

$$\mathrm{N}_{\mathrm{K/F}}(b + 2c\sqrt{a}) = b^2 - 4ac^2 = d(2cs)^2 \in d\dot{\mathrm{F}}^2.$$

Since $d \notin \dot{\mathrm{F}}^2$, this implies that $b + 2c\sqrt{a} \notin \dot{\mathrm{K}}^2$ and so $\mathrm{L} = \mathrm{K}\left(\sqrt{b + 2c\sqrt{a}}\right)$ is a quadratic extension of K, with $\mathscr{D}(\mathrm{L/F}) \in d\dot{\mathrm{F}}^2$. Over K, σ' represents

$$\sigma'(u + \sqrt{a}v) = \sigma'(u) + a\sigma'(v) + 2\sqrt{a}\mathrm{B}_{\sigma'}(u, v) = -(b + 2c\sqrt{a}).$$

Thus, over L, σ' represents -1, and this implies that $\sigma \in \mathrm{W(L/F)}$, as desired.

Case 2. — $c = 0$. In this case $\sigma'(u) = -a\sigma'(v)$. *We claim that σ splits over* K. We may assume that $\sigma'(u) \neq 0 \neq \sigma'(v)$ (for otherwise σ already splits over F). Then $\{u, v\}$ must be linearly independent, and hence σ' has the binary subform

$$\langle \sigma'(u), \sigma'(v) \rangle \cong \langle -a\sigma'(v), \sigma'(v) \rangle \cong \langle \sigma'(v) \rangle \langle 1, -a \rangle.$$

This implies that σ' is isotropic over K, and hence σ splits over K. Thus, all we need to show is that K has a quadratic extension L with $\mathscr{D}(\mathrm{L/F}) \in d\dot{\mathrm{F}}^2$. But

$$\mathrm{N}_{\mathrm{K/F}}(r + \sqrt{a}) = r^2 - a \in d\dot{\mathrm{F}}^2,$$

so, just as in Case (1), we see that $\mathrm{L} = \mathrm{K}\left(\sqrt{r + \sqrt{a}}\right)$ is quadratic over K with $\mathscr{D}(\mathrm{L/F}) \in d\dot{\mathrm{F}}^2$. ∎

Remark. — The implication (1) ⇒ (2) above remains valid for $n = 1$, since the proof did not make use of the hypothesis that $n \geq 2$. However, (2) ⇒ (1) is false in general for $n = 1$, as the following example shows. Let $\mathrm{F} = \mathbf{Q}$, $a = d = 2$, and $\sigma = \langle\langle -7 \rangle\rangle$. Since $\langle\langle -d \rangle\rangle = \langle 1, -2 \rangle$ represents both 2 and $7 (= 3^2 - 2)$, we have $\langle\langle -a, -d \rangle\rangle = \sigma \langle\langle -d \rangle\rangle = 0$. But σ cannot split in a quadratic extension $\mathrm{L} \supset \mathbf{Q}(\sqrt{2})$ with $\mathscr{D}(\mathrm{L/F}) = d$, for otherwise $\mathbf{Q}(\sqrt{7})$ would be a subfield of L, but L/F is a cyclic quartic extension with a *unique* quadratic subfield $\mathbf{Q}(\sqrt{2})$.

Using the Theorem, we can deduce in the following a new characterization for the values represented by a (non-hyperbolic) Pfister form. Fortunately, in spite of the above remark, we do not need to exclude the case of 1-fold Pfister forms.

Corollary **4.4.** — *Let σ be a non-hyperbolic n-fold Pfister form over* F $(n \geqslant 1)$, *and let* $d \in \dot{F} \backslash \dot{F}^2$. *Then σ represents d over* F *if and only if σ splits over some quartic* 2*-extension* L/F *of discriminant d.*

Proof. — The "if" part follows from (1) $\Rightarrow$ (2) in (4.2) (recalling that $n \geqslant 2$ was not needed in that proof). For the "only if" part, we first work in the case $n \geqslant 2$. Assume that σ represents d. We are done if we can show that $\langle 1, -d \rangle$ represents *some* nonsquare element $a \in \dot{F}$, for then $\langle a, d \rangle$ represents 1 and we can apply (2) $\Rightarrow$ (1) in (4.2). If $d \notin -\dot{F}^2$, we can choose $a = -d$, so we may assume that $d \in -\dot{F}^2$. In this case, $\langle 1, -d \rangle \cong \langle 1, 1 \rangle$. If this represents only squares, F would be a pythagorean field, and it must be formally real since $\dot{F} \neq \dot{F}^2$. But then W(F) is torsion-free ([L_1: p. 236]), and $0 = \sigma \langle\langle -d \rangle\rangle = 2\sigma \in W(F)$ would imply that $\sigma = 0 \in W(F)$, which is not the case. This completes the proof that $\langle 1, -d \rangle$ always represents some nonsquare. Finally, if $n = 1$, write $\sigma = \langle\langle -x \rangle\rangle$ $(x \notin \dot{F}^2)$. The assumption that σ represents d means that $\langle x, d \rangle$ represents 1. In this case, we can embed $F(\sqrt{x})$ in a quartic extension L/F with discriminant d, and, since $F(\sqrt{x})$ splits σ, L also does. ∎

Using the well-known fact that the values of σ form a group under multiplication [L_1: p. 279], it follows that, *if the form σ in* (4.4) *splits in two quartic* 2*-extensions with discriminants* $d\dot{F}^2 \neq d'\dot{F}^2$, *then σ also splits in some quartic* 2*-extension of discriminant* $dd'\dot{F}^2$. This fact does not seem easy to prove without the Corollary above. An analogous result for nonsplit Albert forms will be given later in (4.18).

What happens in the case when σ is the hyperbolic Pfister form? If we exclude the case when F is a formally real pythagorean field and $d \in -\dot{F}^2$, then by (3.5) F *does* have quartic 2-extensions of discriminant d; in this case both conditions in the main statement of the Corollary hold, and the Corollary survives. But if F is formally real pythagorean and $d \in -\dot{F}^2$, then the Corollary fails (more or less as a "freak accident"), for σ would represent all elements of $\dot{F}$, but by (3.5) F has *no* quartic 2-extension of discriminant d.

In the proof of Theorem 4.2, we have used only rather lightly the assumption that σ there was a Pfister form. In fact, we have the following "analogue" of (4.2) which holds for *arbitrary* forms σ of dimension $\geqslant 3$.

Theorem **4.2'.** — *Let a, d be nonsquares in* F *such that $\langle a, d \rangle$ represents* 1, *and let* $K = F(\sqrt{a})$. *If a form σ of dimension $\geqslant 3$ represents both* 1 *and d over* F, *then σ becomes isotropic over some quadratic extension* $L \supset K$ *with* $\mathscr{D}(L/F) = d$.

Proof. — Fix a diagonalization $\langle 1, x, y, \dots \rangle$ for σ such that $\langle 1, x \rangle$ represents d. Then $\langle\langle x, y \rangle\rangle$ represents d, so by (4.2) it splits over some quadratic extension $L \supset K$ with $\mathscr{D}(L/F) = d$. It follows that $\langle 1, x, y \rangle$, and hence σ, are isotropic over L. ∎

Theorem 4.2′ is essentially the special case of (2) $\Rightarrow$ (1) in Theorem 4.2 for 2-fold Pfister forms. On the other hand, if we known (2) $\Rightarrow$ (1) in (4.2) for 2-fold Pfister forms, the general case also follows in view of the well-known factorization theory of Pfister forms. Thus, for all intents and purposes, (4.2′) is equivalent to (2) $\Rightarrow$ (1) in (4.2). For this reason, it is of interest to give below a direct proof of (4.2′).

Write $\sigma \cong \langle 1, x, y, \dots \rangle$ as before such that we have an equation

$$d = s^2 + xt^2 \ (s, t \in \mathrm{F}).$$

Here, $t \neq 0$ since $d \notin \mathrm{F}^2$. Using the assumption that $\langle a, d \rangle$ represents 1 over F, we have also an equation $d = b^2 - 4ac^2$ for some $b, c \in \mathrm{F}$. In K, let $\alpha = b + 2c\sqrt{a}$, $\alpha' = b - 2c\sqrt{a}$. Then $\alpha\alpha' = d$ and

$$\textbf{(4.5)} \qquad (\alpha + s)^2 + xt^2 = \alpha^2 + 2\alpha s + d = \alpha(\alpha + 2s + \alpha') = 2(b + s)\,\alpha.$$

If $b + s \neq 0$, let $\mathrm{L} := \mathrm{K}\big(\sqrt{-2(b+s)y\alpha}\big)$. Then, by (4.5), $\langle 1, x \rangle$ represents $-y$ over L, and hence σ becomes isotropic over L. This completes the proof since

$$\mathscr{D}(\mathrm{L/F}) = \mathrm{N}_{\mathrm{K/F}}(2(b+s)\,y\alpha) \in \mathrm{N}_{\mathrm{K/F}}(\alpha)\,.\,\dot{\mathrm{F}}^2 = d\dot{\mathrm{F}}^2.$$

If $b + s = 0$, (4.5) implies that the subform $\langle 1, x \rangle$ of σ is already isotropic over K. In this case, we can simply choose L to be $\mathrm{K}(\sqrt{\alpha})$.

(The key idea in the proof above is that, if $\langle 1, x \rangle$ represents $\mathrm{N}_{\mathrm{K/F}}(\alpha)$ over F, then $\langle 1, x \rangle$ represents some element in $\alpha.\dot{\mathrm{F}}$ over K. This is a well-known Norm Principle for quadratic extensions (cf. [EL_3: (2.13)]). We have, however, managed to avoid a reference to this Norm Principle by using a direct computation.)

We can now get results on the existence of cyclic quartic splitting fields of n-fold Pfister forms by simply specializing Theorem 4.2 to the case when $a\dot{\mathrm{F}}^2 = d\dot{\mathrm{F}}^2$. In this case, the condition that $\langle a, d \rangle$ represents 1 simply means that a is a sum of two squares in F. Thus, we obtain the following special case of (4.2):

Corollary **4.6.** — *Let $a \in \dot{\mathrm{F}} \backslash \dot{\mathrm{F}}^2$ be a sum of two squares in* F, *and let* $\mathrm{K} = \mathrm{F}(\sqrt{a})$. *Then, for $n \geqslant 2$, an n-fold Pfister form σ over* F *has a cyclic quartic splitting field containing* K *if and only if σ represents a over* F. *In particular, σ has a cyclic quartic splitting field if and only if it represents a sum of two squares which is not a square in* F. *(For instance, if* F *is not a pythagorean field, then any $\sigma \cong \langle\langle 1, a_2, \dots, a_n \rangle\rangle$ $(n \geqslant 2)$ has a cyclic quartic splitting field.)*

Note that, in the special case when σ is the hyperbolic n-fold Pfister form, the last statement in the Corollary recaptures the Diller-Dress Theorem (mentioned in the paragraph preceding (3.6)).

Now let us try to get a partial extension of the above results to a pair of n-fold F-Pfister forms β, γ, with $n \geqslant 2$. We shall write $q = \beta' \perp \langle -1 \rangle \gamma'$ where β', γ' denote the pure subforms of β and γ. Recall that β and γ are said to be *linked* over F if $\beta \cong \delta \langle\langle y \rangle\rangle$ and $\gamma \cong \delta \langle\langle z \rangle\rangle$ for some $(n-1)$-fold Pfister form δ over F, and suitable elements $y, z \in \dot{\mathrm{F}}$ (see [EL_1: p. 197]).

Theorem **4.7.** — *Let a, d be nonsquares in a field* F, *and let* $K = F(\sqrt{a})$. *For* β, γ *and* q *as above, consider the following statements*:

(1) q *splits over some quadratic extension* $L \supset K$ *with* $\mathscr{D}(L/F) \in d\dot{F}^2$;

(2) $\langle\langle -a, -d \rangle\rangle = q \langle\langle -d \rangle\rangle = 0 \in W(F)$, *and* β, γ *are linked over* K.

In general, (2) $\Rightarrow$ (1), *and* (1) $\Rightarrow$ *the first part of* (2). *If* $n = 2$, *then we have* (1) $\Leftrightarrow$ (2).

Remark **4.8.** — If $\beta \cong \langle\langle -b_1, \ldots, -b_n \rangle\rangle$ and $\gamma \cong \langle\langle -c_1, \ldots, -c_n \rangle\rangle$, the condition $q \langle\langle -d \rangle\rangle = 0$ in (2) above is, of course, equivalent to

$$\langle\langle -b_1, \ldots, -b_n, -d \rangle\rangle \cong \langle\langle -c_1, \ldots, -c_n, -d \rangle\rangle$$

over F.

Proof. — Assume (1) and let $L = K\left(\sqrt{b + 2c\sqrt{a}}\right)$. Then the first part of (2) follows as in the proof of (4.2). The last part of (2) can be seen as follows. If q is anisotropic over K, then, since $\dim q = 2(2^n - 1)$ and q is hyperbolic over L, we have a K-isometry $q \cong \langle 1, -(b + 2c\sqrt{a}) \rangle . \varphi$ for some $(2^n - 1)$-dimensional form φ over K [L_1: p. 200]. Taking determinants, we get $-(b + 2c\sqrt{a}) \in -\dot{K}^2$, a contradiction to $[L : K] = 2$. Thus, q is isotropic over K, which means that β_K and γ_K can be written with a common 1-fold Pfister factor [L_1: p. 278]. In the case when $n = 2$, this simply says that β and γ are linked over K.

(2) $\Rightarrow$ (1) (for any $n \geqslant 2$). Since β, γ are linked over K, a straightforward calculation shows that $\beta \perp \langle -1 \rangle \gamma \cong 2^{n-1}\mathbf{H} \perp y.\tau$ for some $y \in \dot{K}$ and some n-fold Pfister form τ over K. This isometry implies that $y.\tau \in \operatorname{im}(W(F) \to W(K))$, so by [$EL_3$: (2.2)], $y.\tau \cong (x.\sigma)_K$ for some $x \in \dot{F}$ and some n-fold Pfister form σ over F. Thus,

$$q \perp \langle -x \rangle \sigma \in W(K/F) = \langle\langle -a \rangle\rangle W(F).$$

By the assumptions in (2), we see from this that $\sigma \langle\langle -d \rangle\rangle = 0$. Thus, by Theorem 4.2, σ splits over some quadratic extension $L \supset K$ with $\mathscr{D}(L/F) \in d\dot{F}^2$. Since

$$q \perp \langle -x \rangle \sigma \in W(K/F) \subseteq W(L/F),$$

it follows that q also splits over L. ■

Specializing Theorem 4.7 to the case when $a\dot{F}^2 = d\dot{F}^2$, and using some results from the theory of transfers of quadratic forms, we shall now prove the following theorem about cyclic splitting fields of higher dimension for the form $q = \beta' \perp \langle -1 \rangle \gamma'$.

Theorem **4.9.** — *Let q be as above, $a \in \dot{F} \backslash \dot{F}^2$ be a sum of two squares in* F, *and let* $K = F(\sqrt{a})$. *Assuming that* β *and* γ *are linked over* K, *the following statements are equivalent*:

(1) $q \langle\langle -a \rangle\rangle = 0 \in W(F)$;

(2) q *has a cyclic quartic splitting field* L *containing* K;

(3) q *has a cyclic splitting field* K_m *containing* K, *of degree* $2^m \geqslant 4$ *over* F.

Proof. — (1) $\Rightarrow$ (2) follows from Theorem 4.7. (2) $\Rightarrow$ (3) is easy as we can take $m = 2$ and $K_2 = L$ in (3). Now assume (3), and let $F = K_0 \subset K = K_1 \subset K_2 \subset \ldots \subset K_m$ be the chain of subfields between F and K_m, and let $K_{i+1} = K_i(\sqrt{a_i})$, where $a_i \in \dot{K}_i - \dot{K}_i^2$ (and $a_0 = a$). Then a_{i+1} has the form $b_i + 2c_i\sqrt{a_i}$ for some $b_i, c_i \in K_i$. Since K_{i+2}/K_i is a cyclic quartic extension, we know by (3.3) that

$$N_{K_{i+1}/K_i}(a_{i+1}) = b_i^2 - 4a_i c_i^2 = \mathscr{D}(K_{i+2}/K_i) \in a_i \dot{F}^2.$$

Applying (4.7) to the cyclic quartic extension K_m/K_{m-2}, we have an equation

(4.10) $$q \langle\langle - a_{m-2} \rangle\rangle = 0 \in W(K_{m-2}).$$

If $m \geqslant 3$, let s be the K_{m-3}-linear functional on K_{m-2} defined by $s(1) = 0$, $s(\sqrt{a_{m-3}}) = 1$, and let $s_* : W(K_{m-2}) \to W(K_{m-3})$ be the transfer map on Witt rings induced by s. Then, by [EL_3: (2.4)], we know that $s_*(\langle\langle - a_{m-2} \rangle\rangle)$ is a scalar multiple of

$$\langle\langle - N_{K_{m-2}/K_{m-3}}(a_{m-2})\rangle\rangle \cong \langle\langle - a_{m-3} \rangle\rangle.$$

Thus, applying s_* to (4.10), we get $q \langle\langle - a_{m-3} \rangle\rangle = 0 \in W(K_{m-3})$ (using " Frobenius Reciprocity " [L_1: p. 192] and the fact that q is defined over F). Repeating this transfer argument, we'll get at the end: $q \langle\langle - a_0 \rangle\rangle = 0 \in W(F)$ (where $a_0 = a$). Thus (3) $\Rightarrow$ (1). ■

Remarks **4.11.** — (a) In the proof of (3) $\Rightarrow$ (1) above, we have not made full use of the fact that K_m/F is a cyclic extension. All we needed was the existence of a chain of fields $F = K_0 \subset K = K_1 \subset K_2 \subset \ldots \subset K_m$ such that each subextension K_{i+2}/K_i is a cyclic quartic extension. (b) If γ in (4.9) is chosen to be the hyperbolic Pfister form, then of course β and γ are linked over any field containing F. In this case, (4.9) implies that *the three conditions* (1), (2) *and* (3) *above are equivalent if q there is replaced by the n-fold Pfister form* β ($n \geqslant 2$). However, starting with a cyclic splitting field K_m (of degree $2^m > 4$) for β, with the chain of subfields $K_m \supset \ldots \supset K_2 \supset K \supset F$, the Theorem *does not* imply that K_2 is a splitting field for β. For instance, over the rational field $\mathbf{Q}$, a classical construction of Brauer and Noether showed that the quaternion division algebra $(-1, -1)_{\mathbf{Q}}$ has cyclic splitting fields K_m with arbitrarily large degree 2^m over $\mathbf{Q}$, such that no proper subfield $K_i \neq K_m$ splits the quaternion algebra (see [Pi: p. 242]).

Applying the (b) part of the Remark above to the norm form of a quaternion algebra, we obtain

Corollary **4.12.** — *Let* S *be a quaternion algebra over* F, *and let* $r \geqslant 1$. *If* $\mathbf{M}_{2^r}(S)$ *is a cyclic algebra, then* $\mathbf{M}_2(S)$ *is a cyclic algebra.*

For convenience of reference, we also restate here the $n = 2$ case of the results (4.7) and (4.9) in the form of a cyclicity criterion for a biquaternion algebra. Recall that a biquaternion algebra is cyclic if and only if it has a cyclic quartic splitting field.

Corollary **4.13.** — *Let* $K = F(\sqrt{a})$ *be a quadratic extension of* F. *Then an* F*-biquaternion algebra* A *has a cyclic quartic splitting field containing* K *if and only if* a *is a sum of two squares in* F, $q_A \langle\langle -a \rangle\rangle = 0 \in W(F)$, *and* q_A *is isotropic over* K. *In particular,* A *is a cyclic algebra if and only if* q_A *has a nonsquare similarity factor* (1) $x^2 + y^2$ *such that* q_A *is isotropic over* $F((x^2+y^2)^{1/2})$.

To prepare ourselves for the last main result in this section, we now prove the following lemma on the transfer of 2-fold Pfister forms under a quadratic extension.

Lemma **4.14.** — *Let* $M = F(\sqrt{d})$ *be a quadratic extension of* F, *and let* $s_* : W(M) \to W(F)$ *be the transfer map associated with the* F*-linear functional on* M *defined by* $s(1) = 0$, $s(\sqrt{d}) = 1$.

a) For any $x, y \in \dot{M}$, *there exist* $f, g \in \dot{F}$ *such that*

$$(4.15a) \qquad s_*\langle\langle -x, -y \rangle\rangle = s_*\langle\langle -f, -x \rangle\rangle + s_*\langle\langle -g, -y \rangle\rangle \in W(F).$$

b) For any $f, g \in \dot{F}$, *and* $u, v \in \dot{M}$, *there exist* $h, k \in \dot{F}$ *such that:*

$$(4.15b) \qquad gh\, s_*\langle\langle -f, -u \rangle\rangle \perp fk\, s_*\langle\langle -g, -v \rangle\rangle = s_*\langle\langle -fkv, -ghu \rangle\rangle \in W(F).$$

In particular, $s_*\langle\langle -f, -u \rangle\rangle + s_*\langle\langle -g, -v \rangle\rangle \equiv s_*\langle\langle -x, -y \rangle\rangle \pmod{I^3 F}$ *for suitable* $x, y \in \dot{M}$.

Proof. — *a)* If $x \in \dot{F}$, we are done by choosing $f = 1$, $g = x$. We may, therefore, assume that $x \notin \dot{F}$, and similarly $y \notin \dot{F}$. The elements $1, x, y$ are linearly dependent over F so there exist $f, g \in F$, not both zero, such that $gx + fy \in \{0, 1\}$. Clearly, neither g nor f can be zero, and $\langle\langle -gx, -fy \rangle\rangle = 0$. Multiplying $g\langle\langle -gx \rangle\rangle = \langle\langle -x \rangle\rangle - \langle\langle -g \rangle\rangle$ with $f\langle\langle -fy \rangle\rangle = \langle\langle -y \rangle\rangle - \langle\langle -f \rangle\rangle$, we get the equation

$$\langle\langle -x, -y \rangle\rangle - \langle\langle -f, -x \rangle\rangle - \langle\langle -g, -y \rangle\rangle + \langle\langle -f, -g \rangle\rangle = 0 \in W(M).$$

Since $\langle\langle -f, -g \rangle\rangle$ is defined over F, $s_*\langle\langle -f, -g \rangle\rangle = 0$ [L_1: p. 201]. (4.15a) now follows by applying s_* to the above equation.

Recall that $s_* : W(M) \to W(F)$ is a homomorphism of W(F)-modules. This implies that $s_*\langle\langle -f, -x \rangle\rangle = \langle\langle -f \rangle\rangle s_*\langle\langle -x \rangle\rangle = \langle\langle -f \rangle\rangle s_*\langle -x \rangle \in I^2 F$, and similarly, $s_*\langle\langle -g, -y \rangle\rangle \in I^2 F$. Therefore, (4.15$a$) implies that $s_*(I^2 M) \subseteq I^2 F$.

b) If $u \in \dot{F}$, then $\langle\langle -f, -u \rangle\rangle$ is defined over F, so $s_*\langle\langle -f, -u \rangle\rangle = 0$. In this case, (4.15$b$) holds trivially by choosing $h = u^{-1}$ and $k = f^{-1}$ in $\dot{F}$. We may, therefore, assume that $u \notin \dot{F}$, and similarly $v \notin \dot{F}$. As above, we can find $h, k \in \dot{F}$ such that $hu + kv \in \{1, 0\}$, and so $\langle\langle -hu, -kv \rangle\rangle = 0 \in W(M)$. Using this and the identity $\langle\langle -xy \rangle\rangle = \langle\langle -x \rangle\rangle + x\langle\langle -y \rangle\rangle$ in W(M), we get easily:

$$\begin{aligned}\langle\langle -fkv, -ghu \rangle\rangle &= \langle\langle -f, -g \rangle\rangle + g\langle\langle -f, -h \rangle\rangle \\ &\quad + f\langle\langle -k, -g \rangle\rangle + gh\langle\langle -f, -u \rangle\rangle + fk\langle\langle -g, -v \rangle\rangle.\end{aligned}$$

(1) A similarity factor of a quadratic form q is a nonzero scalar e such that $q \cong e.q$.

Noting that the first three forms on the right-hand side are defined over F, (4.15*b*) follows by applying the W(F)-module homomorphism s_*. The congruence in the last statement of *b)* now follows by using the fact that $s_*(\mathrm{I}^2\,\mathrm{M}) \subseteq \mathrm{I}^2\,\mathrm{F}$. ∎

We are now ready to prove the following result describing the discriminants of the quartic 2-extensions of F which split a given biquaternion algebra A. (Notice that, unlike the situation in (4.2), no intermediate quadratic extension of F is given beforehand.) This result can also be interpreted as a characterization theorem for the similarity factors of the Albert form of such a biquaternion algebra A.

Theorem **4.16.** — *Let* $\mathrm{M} = \mathrm{F}(\sqrt{d})$ *and* s_* *be as above, and let* A *be a non-split* F*-biquaternion algebra. Then the following statements are equivalent*:

(1) *A splits over some quartic* 2*-extension* L/F *of discriminant* d;
(2) $q_{\mathrm{A}}\langle\langle -d\rangle\rangle = 0 \in \mathrm{W}(\mathrm{F})$;
(3) $q_{\mathrm{A}} \in s_*(\mathrm{W}(\mathrm{M}))$;
(4) $q_{\mathrm{A}} \in s_*(\mathrm{W}(\mathrm{M})) + \mathrm{I}^3\,\mathrm{F}$;
(5) $q_{\mathrm{A}} \equiv s_*\langle\langle -x, -y\rangle\rangle \pmod{\mathrm{I}^3\,\mathrm{F}}$ *for some* $x, y \in \dot{\mathrm{M}}$;
(6) $\mathrm{A} \cong (f_1, a_1)_{\mathrm{F}} \otimes (f_2, a_2)_{\mathrm{F}}$, *where* $\langle\langle -a_i, -d\rangle\rangle = 0 \in \mathrm{W}(\mathrm{F})$ *for* $i = 1, 2$.

(There is, of course, no loss in restricting our attention to the non-split case. If A is split, then (2) through (6) are trivially true, and (1) holds if and only if F *has* some quartic 2-extension of discriminant d, which is true except in the special case when F is formally real pythagorean and $d \in -\dot{\mathrm{F}}^2$: see (3.5).)

Proof. — First, (2) ⇔ (3) holds (even when q_{A} is replaced by any quadratic form over F) by [EL_3: (2.5)]. Next, we shall show (5) ⇒ (4) ⇒ (2) ⇒ (6) ⇒ (5). The first implication here is trivial. For (4) ⇒ (2), multiply the relation in (4) by $\langle\langle -d\rangle\rangle$ to get $q_{\mathrm{A}}\langle\langle -d\rangle\rangle \in \mathrm{I}^4\,\mathrm{F}$. Since $q_{\mathrm{A}}\langle\langle -d\rangle\rangle$ has dimension 12, the Hauptsatz of Arason-Pfister [L_1: p. 289] yields (2). Next, assume (2). By the β-decomposition Theorem [EL_2: (2.3)], we can write

$$q_{\mathrm{A}} \cong e_1\langle 1, -a_1\rangle \perp e_2\langle 1, -a_2\rangle \perp e_3\langle 1, -a_3\rangle, \tag{4.17}$$

where $\langle\langle -a_i, -d\rangle\rangle = 0$ for all i. Here, we have $a_1 a_2 a_3 \in \dot{\mathrm{F}}^2$ since $\det q_{\mathrm{A}} = -1$. Scaling (4.17) by $-a_1 e_3$, we get

$$\begin{aligned} -a_1 e_3 \cdot q_{\mathrm{A}} &\cong \langle -a_1 e_1 e_3, e_1 e_3, -a_1 e_2 e_3, a_1 a_2 e_2 e_3, -a_1, a_2\rangle \\ &\cong \langle\langle e_1 e_3, -a_1\rangle\rangle' \perp \langle -1\rangle\langle\langle a_1 e_2 e_3, -a_2\rangle\rangle'. \end{aligned}$$

Thus, the Albert forms of the biquaternion algebras

$$\mathrm{A} \text{ and } (-e_1 e_3, a_1)_{\mathrm{F}} \otimes (-a_1 e_2 e_3, a_2)_{\mathrm{F}}$$

are homothetic. By (2.4), we have then $\mathrm{A} \cong (-e_1 e_3, a_1)_{\mathrm{F}} \otimes (-a_1 e_2 e_3, a_2)_{\mathrm{F}}$, proving (6). Finally, we have to show (6) ⇒ (5). Given (6), we can write $a_i = \mathrm{N}_{\mathrm{M/F}}(u_i)$

for suitable elements $u_i \in \dot{M}$. Since $s_* \langle\langle -u_i \rangle\rangle \equiv \langle\langle -a_i \rangle\rangle \pmod{I^2 F}$ (cf. [EL$_3$: (2.4)]), we have

$$\begin{aligned} q_A &= \langle\langle -f_1, -a_1 \rangle\rangle - \langle\langle -f_2, -a_2 \rangle\rangle \\ &\equiv \langle\langle -f_1 \rangle\rangle s_* \langle\langle -u_1 \rangle\rangle - \langle\langle -f_2 \rangle\rangle s_* \langle\langle -u_2 \rangle\rangle \\ &\equiv s_* \langle\langle -f_1, -u_1 \rangle\rangle + s_* \langle\langle -f_2, -u_2 \rangle\rangle \pmod{I^3 F}. \end{aligned}$$

By the last part of (4.14) *b)*, this implies (5).

Having now proved the equivalence of (2) through (6), we finish by proving (1) $\Rightarrow$ (2) $\Rightarrow$ (1). The first implication follows from (1) $\Rightarrow$ (2) of (4.7) (for $n = 2$). For the second implication, assume (2). By the β-decomposition Theorem, we can write q_A as in (4.17), where $\langle\langle -a_i, -d \rangle\rangle = 0$ for all i. Since q_A is not hyperbolic, some a_i (say a_1) is not a square in F. Then q_A is isotropic over the quadratic extension $F(\sqrt{a_1})$. It follows by (4.7) (for $n = 2$ and $K := F(\sqrt{a_1})$) that q_A splits over some quadratic extension $L \supset K$ with $\mathscr{D}(L/F) \in d\dot{F}^2$. ■

Corollary **4.18.** — *Let* A *be as above. If* A *splits in two quartic* 2*-extensions of discriminants* d *and* d', *then* A *splits in some quartic* 2*-extension of discriminant* dd'.

Proof. — If $d\dot{F}^2 \neq d'\dot{F}^2$, the Corollary follows from the Theorem since the similarity factors of q_A form a group under multiplication. Now assume $d\dot{F}^2 = d'\dot{F}^2$. In this case, our job is to find a biquadratic extension L/F which splits A. If A is a division algebra, this is easy, so the remaining case is when $A \cong \mathbf{M}_2(B)$ where $B = (b, b')_F$ is a division algebra. It suffices to show in this case that $|\dot{F}/\dot{F}^2| \geqslant 4$, for then we can choose $L = F(\sqrt{b}, \sqrt{c})$, where c is any element outside of $\dot{F}^2 \cup b\dot{F}^2$. If F is formally real, the desired inequality is certainly true, for otherwise F would have to be an Euclidean field [1], but such a field cannot have a quartic 2-extension to begin with (see [L$_1$: p. 254, Exer. 18]). Finally, if F is nonreal, the desired inequality follows (for instance) from Kneser's Lemma [L$_1$: p. 318], since the (anisotropic) norm form of B already represents at least four square classes. ■

Using some of the ideas in the proof of (4.16), we can also get information on abelian Galois splitting fields of biquaternion algebras in some cases.

Corollary **4.19.** — *Let* $a \notin \dot{F}^2$ *be a sum of two squares in* F, *and* A *be an* F*-biquaternion algebra. If* $q_A \langle\langle -a \rangle\rangle = 0 \in W(F)$, *then* A *has a Galois splitting field* T *containing* $K = F(\sqrt{a})$ *such that* $\mathrm{Gal}(T/F) \cong \mathbf{Z}_4$ *or* $\mathbf{Z}_4 \oplus \mathbf{Z}_2$.

Proof. — By (2) $\Rightarrow$ (6) in (4.16), we can write $A \cong (f_1, a_1)_F \otimes (f_2, a_2)_F$, where $\langle\langle -a_i, -a \rangle\rangle = 0$. By (4.6), $(f_i, a_i)_F$ has a cyclic quartic splitting field

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[1] An Euclidean field is a formally real field wih exactly two square classes.

$L_i \supset K$ ($i = 1, 2$). Then A splits over the compositum $T := L_1 . L_2$, which is an abelian extension over F of degree 4 or 8. An easy Galois-theoretic argument shows that $\mathrm{Gal}(T/F) \cong \mathbf{Z}_4$ or $\mathbf{Z}_4 \oplus \mathbf{Z}_2$ accordingly. ∎

§ 5. Algebra-theoretic Approach

In this section, we shall present new proofs for some of the results in § 4 by using techniques from the theory of algebras. This section is, therefore, aimed at readers who are more familiar with the theory of algebras than with the theory of quadratic forms. One principal tool we need from the theory of (finite dimensional) central simple algebras is the corestriction of such algebras from a field M to a subfield F of finite codimension in M. We shall write $\mathrm{Cor}_{M/F}$ for this corestriction, and shall suppress the subscripts M/F whenever the fields involved are clear from the context. (The case of main interest to us here is when $[M : F] = 2$.) For details concerning the corestriction of central simple algebras, we refer the reader to [Dr: § 8] and [T_2].

We shall write B(F) for the Brauer group of a field F, and identify B(F) with the cohomology group $H^2(G_F, \dot{F}_s)$, where F_s denotes the separable closure of F and G_F denotes the profinite Galois group $\mathrm{Gal}(F_s/F)$. We shall write $B \sim C$ to refer to the fact that the central simple algebras B, C are similar. If B is an F-quaternion algebra, we shall write B′ to denote the subspace of pure quaternions in B. As is well-known, $B' = \{0\} \cup \{ b \in B \backslash F : b^2 \in F \}$ [L_1: p. 53]. We shall also need the following Proposition concerning the descent of quaternion algebras; a self-contained proof of it can be found in [T_1: (2.6)].

Proposition **5.0.** — *Let* K *be a quadratic extension of* F, *and let* $w \in \dot{K}$, $b \in \dot{F}$. *Then* $(N_{K/F}(w), b)_F \cong \mathbf{M}_2(F)$ *if and only if* $(w, b)_K \cong (g, b)_K$ *for some* $g \in \dot{F}$.

If A is any (finite dimensional) central simple F-algebra with an F-involution, and $E \subseteq A$ is any simple F-subalgebra, a theorem of M. Kneser says that any F-involution of E can be extended to an F-involution of A. The proof of this is not easy, but can be found in [Sch: p. 311]. The following is a variation of Kneser's result, which will suffice for our purposes, and for which we can offer a very simple proof, following an idea of Racine [Ra].

Proposition **5.1.** — *Let* A *be a central simple* F*-algebra with an* F*-involution* σ, *and let* $E \subseteq A$ *be any simple* F*-subalgebra such that its centralizer* $C_A(E)$ *is a division algebra* (1). *Then any* F*-involution* τ *of* E *extends to an* F*-involution* φ *of* A.

(1) This assumption is automatic, for instance, when A itself is a division algebra. On the other hand, this assumption is also satisfied when E is a *strictly* maximal subfield of A in the sense that $(\dim_F E)^2 = \dim_F A$, for i n this case $C_A(E) = E$.

Proof. — By the Skolem-Noether Theorem there exists a unit $a \in \mathrm{A}$ such that

$$ae = \sigma\tau(e)\, a \quad \forall\, e \in \mathrm{E}. \tag{1}$$

Replacing e by $\tau(e)$, we get $a\tau(e) = \sigma(e)\, a$. Applying σ to this, we get

$$\sigma(a)\, e = \sigma\tau(e)\, \sigma(a) \quad \forall\, e \in \mathrm{E}. \tag{2}$$

We may assume that $\sigma(a) \neq -a$ for otherwise $\varphi(x) := a^{-1}\, \sigma(x)\, a$ $(\forall\, x \in \mathrm{A})$ gives the desired involution. Adding (1), (2) and writing $a' = a + \sigma(a) \neq 0$, we get

$$a'\, e = \sigma\tau(e)\, a' \quad \forall\, e \in \mathrm{E}. \tag{3}$$

Writing further $a' = ab (b \in \mathrm{A}\backslash\{0\})$, we have $abe = \sigma\tau(e)\, ab = aeb$ by (1). This shows that $b \in \mathrm{C_A(E)}$. Since $\mathrm{C_A(E)}$ is a division ring, b and hence a' are *units* in A. Using $\sigma(a') = a'$, we see that $\varphi(x) = a'^{-1}\, \sigma(x)\, a'$ defines an F-involution on A, and by (3), $\varphi(\tau(e)) = e(\forall\, e \in \mathrm{E})$, so $\varphi|_{\mathrm{E}} = \tau$, as desired. ■

The next Proposition deals with the problem of decomposing a biquaternion algebra as a tensor product of two quaternion algebras. Part (2) of this Proposition is a folklore result, while part (1) is a special case of the " $\mathrm{P}_2(2)$ property " of a field established in [T_1: Cor. 2.8]. For the convenience of the reader, we shall include uniform proofs for both parts, avoiding the $\mathrm{P}_i(n)$ terminology of [T_1]. Our arguments below are again modelled upon those of Racine in [Ra].

Proposition **5.2.** — *Let* $\mathrm{E} = \mathrm{F}(\sqrt{a}, \sqrt{b})$ *be a biquadratic extension of* F, *and let* K *be the quadratic extension* $\mathrm{F}(\sqrt{a})$. *Let* A *be any* F-*biquaternion algebra. Then:*

(1) E *can be embedded (as an* F-*algebra) into* A *if and only if* A *can be written in the form* $(f, a)_{\mathrm{F}} \otimes_{\mathrm{F}} (g, b)_{\mathrm{F}}$ *where* $f, g \in \dot{\mathrm{F}}$;

(2) K *can be embedded into* A *if and only if* A *can be written in the form* $(f, a)_{\mathrm{F}} \otimes_{\mathrm{F}} (g, c)_{\mathrm{F}}$ *where* $f, g, c \in \dot{\mathrm{F}}$.

Proof. — We need only prove the " only if " parts.

(1) Assume that $\mathrm{E} \subseteq \mathrm{A}$, and let $\alpha, \beta \in \mathrm{E}$ be such that $\alpha^2 = a$ and $\beta^2 = b$. Since E is a strictly maximal subfield of A, (5.1) applies, so there is an F-involution φ on A such that $\varphi(\alpha) = -\alpha$, and (say) $\varphi(\beta) = \beta$. Let $\mathrm{B} = \mathrm{C_A(K)}$. Since $\dim_{\mathrm{K}} \mathrm{B} = 4$, B is a K-quaternion algebra. The restriction of φ to B is then an involution of the second kind on B. Since $\beta \in \mathrm{B}$ and $\beta^2 = b$, we can write $\mathrm{B} \cong (w, b)_{\mathrm{K}}$ for some $w \in \dot{\mathrm{K}}$. Using the well-known fact that $\mathrm{B} \sim \mathrm{A^K}$, we have

$$\mathrm{Cor_{K/F}\, B} \sim \mathrm{Cor_{K/F}(A^K)} \sim \mathrm{A}^2 \sim 1.$$

On the other hand, by the projection formula for the corestriction (see [T_2: (3.2)]), $\mathrm{Cor_{K/F}\, B} = \mathrm{Cor_{K/F}}(w, b)_{\mathrm{K}} = (\mathrm{N_{K/F}}(w), b)_{\mathrm{F}}$. Thus, $(\mathrm{N_{K/F}}(w), b)_{\mathrm{F}} \sim 1$, so by (5.0), $\mathrm{B} \cong (w, b)_{\mathrm{K}} \cong (g, b)_{\mathrm{K}}$ for some $g \in \dot{\mathrm{F}}$. From $\mathrm{A^K} \sim \mathrm{B} \sim (g, b)_{\mathrm{K}}$, we see that $\mathrm{A} \otimes_{\mathrm{F}} (g, b)_{\mathrm{F}}$

is split by K. Hence $A \otimes_F (g, b)_F \sim (f, a)_F$ for some $f \in \dot{F}$, and by dimension considerations we conclude that $A \cong (f, a)_F \otimes_F (g, b)_F$, as desired.

(2) Here we assume $K = F(\alpha) \subseteq A$ (but there is no given E). We may assume that A is a division algebra for, if otherwise, A has a tensor factor $\mathbf{M}_2(F) \cong (1, a)_F$ and we are done. By (5.1) again, we can find an F-involution φ on A such that $\varphi(\alpha) = -\alpha$. The fixed points of φ in $B := C_A(K)$ form a 4-dimensional F-space B_0 (see [Sch: (7.5) (ii), p. 303]), while, in the K-quaternion algebra B, the pure quaternions form a 6-dimensional F-space B'. Since $\dim_F B = 8$, there exists a nonzero $\gamma \in B_0 \cap B'$. Let $c = \gamma^2 \in \dot{K}$. Then $B \cong (w, c)_K$ for some $w \in \dot{K}$. But $\varphi(c) = c$ implies that $c \in \dot{F}$, so we can finish the argument as before. ∎

Remark. — The fact that the existence of an involution of the second kind on B implies that B is defined over F is Theorem 21 in Chapter 10 of [A_5]. In the argument above, we have avoided a reference to this result by using the more modern tool of corestriction.

As a consequence of (5.2) (2), we have the following explicit description of the exponent two part of the relative Brauer group $B(L/F) := \ker(B(F) \to B(L))$ for any quartic 2-extension L/F.

Corollary **5.3.** — *Let* L/F *be a quartic* 2*-extension containing* $K = F(\sqrt{a})$. *Then a central simple* F*-algebra* A *of exponent* 2 *lies in* B(L/F) *if and only if* $A \sim (f, a)_F \otimes (g, c)_F$ *where* $f, g, c \in \dot{F}$ *and* $(g, c)_F$ *splits over* L. *(Recall that the family of quaternion algebras* $(g, c)_F$ *splitting over* L *has been completely determined in* (3.9).*)*

Proof. — It suffices to prove the "only if" part, so suppose A splits over L. Without loss of generality, we may assume that A is a division algebra. By [Pi: Lemma, p. 242], we know that deg A divides $[L : F] = 4$, so it is either 2 or 4. If $\deg A = 2$, then $A \cong (g, c)_F$ and we are done. If, instead, $\deg A = 4$, then (by (2.1)) A is a biquaternion algebra. By [Pi: Lemma, p. 242] again, A^K cannot remain a division algebra. This implies that K can be embedded into A (see [Pi: Cor., p. 243]), so by (5.2) (2), $A \cong (f, a)_F \otimes (g, c)_F$ for some $f, g, c \in \dot{F}$. Since A and $(f, a)_F$ both split over L, $(g, c)_F$ also does. ∎

In the rest of this section, we shall *not* be working with a fixed quartic 2-extension L/F. Instead, we shall fix a biquaternion (or quaternion) algebra A, and investigate the possible quartic 2-extensions L/F splitting A. We first handle the case of quaternion algebras.

Proposition **5.4.** — *Let* a, d *be nonsquares in* F, *and let* $K = F(\sqrt{a})$, $M = F(\sqrt{d})$. *For any* F*-quaternion algebra* C, *the following are equivalent*:

(1) C *splits over some quadratic extension* L *of* K *with* $\mathscr{D}(L/F) \in d\dot{F}^2$;

(2) d *is a norm from* K *and a reduced norm from* C;

(3) d *is a norm from* K, *and* $C \cong (g, N_{M/F}(v))_F$ *for some* $g \in \dot{F}$ *and* $v \in \dot{M}$.

Proof. — (3) is just a slight reformulation of (2).

(3) $\Rightarrow$ (1) Write $d = N_{K/F}(w)$, where $w \in \dot{K}$. Then

$$(N_{K/F}(w), N_{M/F}(v))_F \cong (d, N_{M/F}(v))_F \cong \mathbf{M}_2(F).$$

By (5.0), we have $(w, N_{M/F}(v))_K \cong (f, N_{M/F}(v))_K$ for some $f \in \dot{F}$. Thus,

$$C^K \cong (g, N_{M/F}(v))_K \cong (gfw, N_{M/F}(v))_K$$

and they split over $L := K(\sqrt{gfw})$. Since

$$N_{K/F}(gfw) = g^2 f^2 N_{K/F}(w) \in d\dot{F}^2,$$

we have $[L : K] = 2$, and $\mathscr{D}(L/F) \in d\dot{F}^2$.

(1) $\Rightarrow$ (3) Let $L = K(\sqrt{z})$, where z is a nonsquare in K. Then $\mathscr{D}(L/F)$ is given by the square class of $N_{K/F}(z)$, so we have $N_{K/F}(z) \in d\dot{F}^2$, proving the first part of (3). Since C^K splits over $L = K(\sqrt{z})$, there exists an element $y \in C^K$ such that $y^2 = z$. Write $y = y_1 \otimes 1 + y_2 \otimes \sqrt{a} \in C^K$, where $y_i \in C$, and let $y_i = e_i + y_i'$, where $e_i \in F$, and $y_i' \in C'$. With respect to the decomposition $C^K = (1 \otimes_F K) \oplus (C' \otimes_F K)$, the element y^2 has component $2(e_1 y_1' + a e_2 y_2') \otimes 1 + 2(e_1 y_2' + e_2 y_1') \otimes \sqrt{a}$ in $C' \otimes_F K$. (This follows from a direct computation, using the fact that $y_1' y_2' + y_2' y_1' \in F$.) Since $y^2 \in K = 1 \otimes_F K$, it follows that

$$\textbf{(5.5)} \qquad e_1 y_1' + a e_2 y_2' = 0, \quad \text{and} \quad e_2 y_1' + e_1 y_2' = 0.$$

Here, y_1' and y_2' are not both zero. For, if $y_1' = y_2' = 0$, then

$$y = e_1 \otimes 1 + e_2 \otimes \sqrt{a} = 1 \otimes (e_1 + e_2 \sqrt{a}) \in 1 \otimes K$$

and hence $z = y^2 \in K^2$, in contradiction to $[L : K] = 2$. Thus, (5.5) implies that

$$0 = \begin{vmatrix} e_1 & a e_2 \\ e_2 & e_1 \end{vmatrix} = e_1^2 - a e_2^2 = N_{K/F}(e_1 + e_2 \sqrt{a}).$$

Therefore, $e_1 = e_2 = 0$, so $y_i \in C'$ and $y_i^2 \in F$. Using the latter, we have

$$(y_1 y_2 + y_2 y_1)^2 - (y_1 y_2 - y_2 y_1)^2 = 2 y_1 y_2 y_2 y_1 + 2 y_2 y_1 y_1 y_2 = 4 y_1^2 y_2^2.$$

Applying $N_{K/F}$ to the element

$$\begin{aligned} z = y^2 &= y_1^2 \otimes 1 + y_2^2 \otimes a + (y_1 y_2 + y_2 y_1) \otimes \sqrt{a} \\ &= 1 \otimes [(y_1^2 + a y_2^2) + (y_1 y_2 + y_2 y_1) \sqrt{a}], \end{aligned}$$

we get

$$\begin{aligned} N_{K/F}(z) &= (y_1^2 + a y_2^2)^2 - a(y_1 y_2 + y_2 y_1)^2 \\ &= (y_1^2 + a y_2^2)^2 - a[4 y_1^2 y_2^2 + (y_1 y_2 - y_2 y_1)^2] \\ &= (y_1^2 - a y_2^2)^2 - a(y_1 y_2 - y_2 y_1)^2. \end{aligned}$$

Let $w = y_1 y_2 - y_2 y_1$, which is easily seen to be in C'. Since $N_{K/F}(z) \in d\dot{F}^2$, the above equations imply that $a w^2 \in N_{M/F}(\dot{M})$, and since $a \in N_{M/F}(\dot{M})$ also, we have $w^2 = N_{M/F}(v)$ for some $v \in \dot{M}$. It follows that $C = (g, N_{M/F}(v))_F$ for some $g \in \dot{F}$. ∎

Remark. — Of course, the above result is just the algebra-theoretic analogue of Theorem 4.2 (in the case of 2-fold Pfister forms). In fact, using basic facts about 2-fold Pfister forms, it is also easy to see directly that the condition (2) above is equivalent to the condition (2) in (4.2) (for $n = 2$).

Lemma **5.6.** — *Let* $\mathrm{M} = \mathrm{F}(\sqrt{d})$ *be a quadratic extension of* F. *Then a Brauer class in* B(F) *can be written in the form* $\mathrm{Cor}(x, y)_{\mathrm{M}}$ *for some* $x, y \in \dot{\mathrm{M}}$ *if and only if it can be written in the form* $\mathrm{Cor}(f, u)_{\mathrm{M}} . \mathrm{Cor}(g, v)_{\mathrm{M}}$ *for some* $f, g \in \dot{\mathrm{F}}$ *and* $u, v \in \dot{\mathrm{M}}$.

Proof. — This follows by exactly the same " linearization " procedure as used in the proof of (4.14), noting that the corestriction of an M-quaternion algebra defined over F is a split algebra. To avoid repetition, we shall suppress the details here. (In fact, the lemma may also be deduced directly from (4.14) by applying the Clifford invariant map: see the commutative diagram in the proof of (A6) in the Appendix.) ■

Before we come to the main theorems of this section, we need to develop one more result on biquaternion algebras. Let B and C be F-quaternion algebras, and let $\mathrm{A} = \mathrm{B} \otimes_{\mathrm{F}} \mathrm{C}$. The direct sum decompositions $\mathrm{B} = \mathrm{F} \oplus \mathrm{B}'$ and $\mathrm{C} = \mathrm{F} \oplus \mathrm{C}'$ yield the equation $\mathrm{A} = \mathrm{F} \oplus (\mathrm{B}' \otimes 1) \oplus (1 \otimes \mathrm{C}') \oplus (\mathrm{B}' \otimes \mathrm{C}')$. With respect to this decomposition of A, the following Proposition gives a useful description of the elements $z \in \mathrm{A}$ with reduced trace zero and with square in F. Since this Proposition is of independent interest, and does not seem to have appeared in the literature before, we shall include a detailed proof below.

Proposition **5.7.** — *Let* $z \in \mathrm{A}$ *be an element of reduced trace zero. Then* z *has the form* $b_0 \otimes 1 + 1 \otimes c_0 + z'$ *where* $b_0 \in \mathrm{B}'$, $c_0 \in \mathrm{C}'$ *and* $z' \in \mathrm{B}' \otimes \mathrm{C}'$. *Assume moreover that* $z^2 \in \mathrm{F}$. *Then*:

(1) z' *(and hence* z*) commutes with* $b_0 \otimes c_0$;

(2) *if* $b_0 = c_0 = 0$, *then* $z = z' = b \otimes c$ *for some* $b \in \mathrm{B}'$ *and* $c \in \mathrm{C}'$.

Proof. — For the proof of the first assertion, we consider the reduced trace maps $\mathrm{Trd}_{\mathrm{A}}$, $\mathrm{Trd}_{\mathrm{B}}$ and $\mathrm{Trd}_{\mathrm{C}}$ on A, B and C respectively. These maps are related by the following property, which can be proved by the same argument as in [Dr: Th. 3, p. 149]:

$$\mathrm{Trd}_{\mathrm{A}}(b \otimes c) = \mathrm{Trd}_{\mathrm{B}}(b)\, \mathrm{Trd}_{\mathrm{C}}(c) \quad (\forall\, b \in \mathrm{B}, c \in \mathrm{C}).$$

From this, it follows that

$$\textbf{(5.8)} \qquad (\mathrm{B}' \otimes 1) \oplus (1 \otimes \mathrm{C}') \oplus (\mathrm{B}' \otimes \mathrm{C}') \subseteq \ker(\mathrm{Trd}_{\mathrm{A}}).$$

Writing $z = \alpha + b_0 \otimes 1 + 1 \otimes c_0 + z'$ where $\alpha \in \mathrm{F}$, $b_0 \in \mathrm{B}'$, $c_0 \in \mathrm{C}'$ and $z' \in \mathrm{B}' \otimes \mathrm{C}'$, we get from (5.8) $0 = \mathrm{Trd}_{\mathrm{A}}(z) = 4\alpha$. Since char $\mathrm{F} \neq 2$, this gives $\alpha = 0$, so the first assertion in the Proposition follows.

For part (1), consider the standard involutions σ_{B} and σ_{C} (which change the signs of the pure quaternions) on B and C respectively, and let $\sigma_{\mathrm{A}} = \sigma_{\mathrm{B}} \otimes \sigma_{\mathrm{C}}$. The map σ_{A}

is then an involution on A. Write $z = y + z'$ where $y = b_0 \otimes 1 + 1 \otimes c_0$. Assuming that $z^2 \in \mathrm{F}$, it follows that

$$(y + z')^2 = z^2 = \sigma_{\mathrm{A}}(z^2) = \sigma_{\mathrm{A}}(z)^2 = (-y + z')^2.$$

Therefore, $yz' + z'y = 0$, and

$$z^2 = y^2 + z'^2 = b_0^2 \otimes 1 + 1 \otimes c_0^2 + 2b_0 \otimes c_0 + z'^2. \tag{5.9}$$

Since z^2, b_0^2 and c_0^2 are all in F, (5.9) yields $b_0 \otimes c_0 \in \mathrm{F}(z'^2)$, from which it follows that z' commutes with $b_0 \otimes c_0$. This proves part (1).

For part (2), let $1, i, j, k$ be a standard basis of B (with $i^2, j^2 \in \mathrm{F}$, $ij + ji = 0$ and $ij = k$). The element z can then be written as $i \otimes c_1 + j \otimes c_2 + k \otimes c_3$ for some $c_1, c_2, c_3 \in \mathrm{C}'$. Since $z^2 \in \mathrm{F}$, a direct computation shows that c_1, c_2 and c_3 pairwise commute. Now, the maximal commutative subalgebras in C have dimension 2 over F and contain only one pure quaternion, up to scalar multiples. Therefore, one can find a pure quaternion $c \in \mathrm{C}'$ such that $c_i = \alpha_i c$ for some $\alpha_i \in \mathrm{F}$, for $i = 1, 2, 3$. Then $z = b \otimes c$ with $b = \alpha_1 i + \alpha_2 j + \alpha_3 k \in \mathrm{B}'$ and the proof is complete. ∎

Now we come to one of the main results of this section, which is the following algebra-theoretic analogue of Theorem 4.7 (in the case of $n = 2$).

Theorem **5.10.** — *Let* $\mathrm{K} = \mathrm{F}(\sqrt{a})$ *and* $\mathrm{M} = \mathrm{F}(\sqrt{d})$ *be quadratic extensions of* F *and let* A *be an* F*-biquaternion algebra. Then the following are equivalent:*

(1) *A splits over some quadratic extension* L *of* K *with* $\mathscr{D}(\mathrm{L}/\mathrm{F}) \in d\dot{\mathrm{F}}^2$;

(2) *d is a norm from* K, K *is embeddable in* A, *and* $\mathrm{A} \cong \mathrm{Cor}_{\mathrm{M}/\mathrm{F}}(\mathrm{Q})$ *for some quaternion algebra* Q *over* M.

Proof. — (1) ⇒ (2) Since L is a splitting field of degree 4 for A, L can be embedded as a (maximal) subfield in A. In particular, K can be embedded in A. By (5.2) (2), this implies that A can be written as $\mathrm{B} \otimes \mathrm{C}$ where $\mathrm{B} = (f, a)_{\mathrm{F}}$ $(f \in \dot{\mathrm{F}})$ and C is an F-quaternion algebra. Going up to L, we have $1 \sim \mathrm{A}^{\mathrm{L}} \sim \mathrm{B}^{\mathrm{L}} \otimes_{\mathrm{L}} \mathrm{C}^{\mathrm{L}} \sim \mathrm{C}^{\mathrm{L}}$. Thus, C splits over L, and (5.4) implies that $\mathrm{C} \cong (g, \mathrm{N}_{\mathrm{M}/\mathrm{F}}(v))_{\mathrm{F}}$ for some $g \in \dot{\mathrm{F}}$ and $v \in \dot{\mathrm{M}}$. Since $\mathscr{D}(\mathrm{L}/\mathrm{F}) \in d\dot{\mathrm{F}}^2$, d is a norm from $\mathrm{K} = \mathrm{F}(\sqrt{a})$, and hence a is a norm from $\mathrm{M} = \mathrm{F}(\sqrt{d})$, say $a = \mathrm{N}_{\mathrm{M}/\mathrm{F}}(u)$ $(u \in \dot{\mathrm{M}})$. Then

$$\mathrm{A} \cong (f, \mathrm{N}_{\mathrm{M}/\mathrm{F}}(u))_{\mathrm{F}} \otimes (g, \mathrm{N}_{\mathrm{M}/\mathrm{F}}(v))_{\mathrm{F}} \sim \mathrm{Cor}_{\mathrm{M}/\mathrm{F}}(f, u)_{\mathrm{M}} \otimes \mathrm{Cor}_{\mathrm{M}/\mathrm{F}}(g, v)_{\mathrm{M}},$$

according to the projection formula for the corestriction. But by Lemma 5.6, the right-hand side above is similar to $\mathrm{Cor}_{\mathrm{M}/\mathrm{F}}\,\mathrm{Q}$ for some M-quaternion algebra Q. Thus, $\mathrm{A} \sim \mathrm{Cor}_{\mathrm{M}/\mathrm{F}}\,\mathrm{Q}$, and, since A and $\mathrm{Cor}_{\mathrm{M}/\mathrm{F}}\,\mathrm{Q}$ are both of degree 4, this implies that $\mathrm{A} \cong \mathrm{Cor}_{\mathrm{M}/\mathrm{F}}\,\mathrm{Q}$.

(2) ⇒ (1) *We may assume that, in* (2), Q *is a division (quaternion) algebra.* For otherwise, Q splits over M and hence A splits over F. Writing $d = b^2 - 4ac^2$ with $b, c \in \mathrm{F}$, we have $b + 2c\sqrt{a} \notin \mathrm{K}^2$, and (1) follows by choosing L to be $\mathrm{K}\left(\sqrt{b + 2c\sqrt{a}}\right)$.

We shall identify A with $\mathrm{Cor}_{M/F}\,Q$, and use the definition of the corestriction of algebras given in [Dr: § 8] and [T_2]. Let s be the nontrivial F-automorphism of M, and let Q^* denote the " conjugate quaternion algebra " of Q, namely: $Q^* = \{x^* : x \in Q\}$, with $x^* + y^* = (x+y)^*$, $x^* y^* = (xy)^*$, and $s(m).x^* = (m.x)^*$ for $m \in M$. Then $A = \mathrm{Cor}_{M/F}\,Q$ consists of the elements in $Q \otimes_M Q^*$ which are invariant under the " exchange involution " mapping $x \otimes y^*$ to $y \otimes x^*$. By assumption, A contains a subfield $F(z)$ with $z^2 = a \in F \backslash F^2$. By the Skolem-Noether Theorem, the nontrivial F-automorphism of the field $F(z)$ extends to an inner automorphism of A, so there is a unit u in A such that $uzu^{-1} = -z$. Reading this equation in $Q \otimes_M Q^*$ and letting Trd denote the reduced trace on this M-algebra, we have $-\mathrm{Trd}(z) = \mathrm{Trd}(-z) = \mathrm{Trd}(uzu^{-1}) = \mathrm{Trd}(z)$, so $\mathrm{Trd}(z) = 0$. Therefore, we can apply Proposition 5.7 to the element $z \in Q \otimes_M Q^*$. Write z in the form $b_0 \otimes 1^* + 1 \otimes c_0^* + z'$ as in (5.7) (with $b_0, c_0 \in Q'$, and $z' \in Q' \otimes_M Q'^*$). Since $z \in A$ is invariant under the exchange involution, we must have $b_0 = c_0$. We have the following two cases.

Case 1. — $b_0 = c_0 \neq 0$. In this case, we let $q := b_0$, and $y := q \otimes q^* \in A$. By (5.7) (1), y commutes with z', and hence also with z. Note that $y \notin F \oplus Fz$, since $b_0 \neq 0$.

Case 2. — $b_0 = c_0 = 0$. By (5.7) (2), $z = z'$ has the form $b \otimes c^*$ for some $b, c \in Q'$. By applying the exchange involution, we see further that $b = c$ up to a scalar factor. In this case, let $y = q \otimes q^* \in A$, where q is any nonzero element in Q' which anticommutes with b. Then $yz = zy$, and again, it is easy to see that $y \notin F \oplus Fz$.

In both cases, $v := q^2 \in \dot{M}$ (since Q is a division algebra), and we have

$$y^2 = q^2 \otimes (q^2)^* = v \otimes v^* = v \otimes s(v) = N_{M/F}(v) \in \dot{F}.$$

If $N_{M/F}(v)$ is not in $\dot{F}^2 \cup a\dot{F}^2$, we see easily that $F[y, z]$ is isomorphic to the biquadratic extension $F(\sqrt{a}, \sqrt{N_{M/F}(v)})$. In this case, (5.2) (1) implies that we can write A in the form $(f, a) \otimes (g, N_{M/F}(v))$ (for suitable $f, g \in \dot{F}$). By (5.4), $(g, N_{M/F}(v))$ splits over a quadratic extension L of K with $\mathscr{D}(L/F) \in d\dot{F}^2$, and hence $A^L \sim 1$, as desired. Finally, we have to deal with the case when $N_{M/F}(v) \in \dot{F}^2 \cup a\dot{F}^2$. After scaling y and/or replacing y by yz, we are down to the case when $y^2 = a$. In this case, $F[y, z] \cong K \times K$, and so A^K contains $(K \times K) \otimes_F K \cong K \times K \times K \times K$. This implies readily that A^K splits (*cf.* [A_5: Ch. 4, Th. 2]). Hence $A \sim (h, a)_F$ for some $h \in \dot{F}$, and we can choose L (as at the beginning of the proof of (2) $\Rightarrow$ (1)) to be any quadratic extension of K with $\mathscr{D}(L/F) \in d\dot{F}^2$. ∎

Corollary **5.11.** — *An* F-*biquaternion algebra* A *is a cyclic algebra if and only if* A *contains a quadratic extension* K *of* F *such that a)* -1 *is a norm from* K, *and b)* $A \cong \mathrm{Cor}_{K/F}\,Q$ *for some quaternion algebra* Q *over* K. *More precisely, given any quadratic extension* K *of* F *in* A, A *has a maximal subfield containing* K *which is cyclic quartic over* F *if and only if a) and b) hold.*

It is worth noting that, if we wish to prove (5.11) directly (without first proving the general result (5.10)), somewhat shorter arguments are possible. For instance, if we know that A is a cyclic algebra $(L/F, \sigma, x)$ for some (quartic) cyclic extension L/F containing K (where σ is a generator of Gal(L/F), and $x \in \dot{F}$), then, since

$$1 \sim A^2 \sim (K/F, \sigma, x),$$

we have $x = N_{K/F}(y)$ for some $y \in \dot{K}$. Then

$$A \cong (L/F, \sigma, N_{K/F}(y)) \cong \mathrm{Cor}_{K/F}(L/K, \sigma, y).$$

This gives a direct construction for the K-quaternion algebra Q in (5.11).

In the balance of this section, we shall combine the quadratic form-theoretic results with the algebra-theoretic results. In the mean time, we shall add a couple of new results of a cohomological nature. For the rest of this paper, we shall write $\mu_2 = \{\pm 1\} \subseteq \dot{F}$, and $G_F = \mathrm{Gal}(F_s/F)$; also, we shall identify A with its class in $H^2(G_F, \mu_2) \cong B_2(F)$ (the subgroup of elements of order $\leqslant 2$ in B(F)), and write (d) to denote the square class of d in $H^1(G_F, \mu_2) \cong \dot{F}/\dot{F}^2$. Finally, we write " $\cup$ " for the cup product in the cohomology ring $H^*(G_F, \mu_2)$.

We now come to the algebra-theoretic analogue of (4.16). In fact the first two conditions in the following theorem are the same as those labeled with the same numbers in (4.16).

Theorem **5.12.** — *Let* $M = F(\sqrt{d})$ *be a quadratic extension of* F *and* A *be a non-split* F-*biquaternion algebra. Then the following conditions are equivalent*:

(1) *A splits over some quartic* 2-*extension* L/F *of discriminant* d;

(6) $A \cong (f_1, N_{M/F}(u_1))_F \otimes (f_2, N_{M/F}(u_2))_F$ *for some* $f_i \in \dot{F}$ *and* $u_i \in \dot{M}$;

(7) $A \cong \mathrm{Cor}_{M/F}\, Q$ *for some quaternion algebra* Q *over* M;

(8) $A \in \mathrm{im}(\mathrm{Cor}_{M/F} : B_2(M) \to B_2(F))$;

(9) $A \cup (d) = 0 \in H^3(G_F, \mu_2)$.

Proof. — We shall first prove (1) $\Rightarrow$ (7) $\Rightarrow$ (6) $\Rightarrow$ (1) within the context of this section. Here, (1) $\Rightarrow$ (7) follows from (5.10), and (7) $\Rightarrow$ (6) follows from (5.6) since $\mathrm{Cor}(f_i, u_i)_M \sim (f_i, N_{M/F}(u_i))_F$. Now assume A has the form in (6). Since A is non-split, at least one of the $N_{M/F}(u_i)$ is not a square in F; say $a := N_{M/F}(u_1) \notin \dot{F}^2$. We see easily that d is a norm from $K := F(\sqrt{a})$, so (5.4) implies that $(f_2, N_{M/F}(u_2))_F$ splits over some quadratic extension L of K with $\mathscr{D}(L/F) \in d\dot{F}^2$. Since K splits $(f_1, a)_F$, it follows that L splits A, proving (1).

The equivalence of (8) and (9) with the other conditions is considerably harder. Of course we have (7) $\Rightarrow$ (8), and (8) $\Rightarrow$ (9) follows from the zero-sequence

$$B_2(M) \xrightarrow{\mathrm{Cor}} B_2(F) \xrightarrow{\cup (d)} H^3(G_F, \mu_2).$$

(This is a part of Arason's long exact sequence for the Galois cohomology of a quadratic extension M/F [Ar_1: Cor. 4.6]. In fact, the exactness of this sequence at $\mathrm{B}_2(\mathrm{F})$ gives directly the equivalence of (8) and (9).) The difficult point is to go from (9) to one of the other conditions in order to close the cycle of implications. In the Appendix, we shall prove that (9) implies $q_{\mathrm{A}} \langle\langle -d \rangle\rangle = 0 \in \mathrm{W}(\mathrm{F})$ (see (A5)). Since the latter is the condition (2) in (4.16), this will show that all the conditions in (4.16) and (5.12) are equivalent. ∎

Corollary **5.13.** — *For any non-split* F-*biquaternion algebra* A, *the following statements are equivalent*:

(0) A *is a square in the Brauer group* B(F);
(1) A *splits over some quartic* 2-*extension* L/F *of discriminant* -1;
(2) $2q_{\mathrm{A}} = 0 \in \mathrm{W}(\mathrm{F})$;
(8) $\mathrm{A} \in \mathrm{im}(\mathrm{Cor}_{\mathrm{F}(\sqrt{-1})/\mathrm{F}} : \mathrm{B}_2(\mathrm{F}(\sqrt{-1})) \to \mathrm{B}_2(\mathrm{F}))$;
(9) $\mathrm{A} \cup (-1) = 0 \in \mathrm{H}^3(\mathrm{G}_{\mathrm{F}}, \mu_2)$.

If $-1 \notin \dot{\mathrm{F}}^2$, *these conditions are further equivalent to*:

(7) $\mathrm{A} \cong \mathrm{Cor}_{\mathrm{F}(\sqrt{-1})/\mathrm{F}}\, \mathrm{Q}$ *for some quaternion algebra* Q *over* $\mathrm{F}(\sqrt{-1})$.

Proof. — The equivalence of (0) and (9) will be proved in the Appendix (see (A4)). The other equivalences follow from (5.12) and (4.16) if $-1 \notin \dot{\mathrm{F}}^2$. If $-1 \in \dot{\mathrm{F}}^2$, (2), (8) and (9) are tautologies, and (1) also holds by repeating the argument (in the nonreal case) in the proof of (4.18). ∎

The Corollary above can be used in conjunction with some results in [J] to give new information on the structure of the Brauer group of fields F with $\mathrm{I}^2\mathrm{F}$ torsion-free. This work will be reported later elsewhere. We conclude this section by recording the following analogue of (5.13) for the case of a single quaternion algebra.

Corollary **5.14.** — *For any quaternion division algebra* B *over* F *with norm form* q, *the following are equivalent*:

(0) B *is a square in the Brauer group* B(F);
(0′) $\mathrm{B} \sim \mathrm{S}^2$ *for some cyclic* F-*algebra* S *of degree* 4;
(1) B *splits over some quartic* 2-*extension* L/F *of discriminant* -1;
(2) $2q = 0 \in \mathrm{W}(\mathrm{F})$;
(2′) *Some element in* B *has reduced norm* -1;
(7) $\mathrm{B} \sim \mathrm{Cor}_{\mathrm{F}(\sqrt{-1})/\mathrm{F}}\, \mathrm{Q}$ *for some quaternion algebra* Q *over* $\mathrm{F}(\sqrt{-1})$;
(9) $\mathrm{B} \cup (-1) = 0 \in \mathrm{H}^3(\mathrm{G}_{\mathrm{F}}, \mu_2)$;
(10) B *splits over some quadratic extension* $\mathrm{K} = \mathrm{F}(\sqrt{r^2+s^2})$, *where* $r, s \in \mathrm{F}$.

Moreover, any of these conditions implies:

(∗) B *splits over some cyclic quartic extension* T *of* F.

Proof. — (2) $\Leftrightarrow$ (2′) is obvious, and (2) $\Leftrightarrow$ (10) is an easy consequence of (2.5). The equivalence of (0), (1), (2), (7) and (9) follows by applying (5.13) to the biquaternion algebra $A \cong B \otimes (1, 1)_F \cong \mathbf{M}_2(B)$.

(0′) $\Rightarrow$ (10): Represent S as $(T/F, \sigma, z)$, where T/F is a cyclic quartic extension with $\mathrm{Gal}(T/F) = \langle \sigma \rangle$ and $z \in \dot{F}$. It is well-known (see the beginning of § 3) that the unique quadratic extension K of F in T has the form $F(\sqrt{r^2 + s^2})$ $(r, s \in F)$. Since $B \sim (T/F, \sigma, z)^2 \sim (K/F, \sigma, z)$, we see that B splits over K.

(10) $\Rightarrow$ (0′) and (10) $\Rightarrow$ (∗): Since $K = F(\sqrt{r^2 + s^2})$, K can be embedded in a cyclic quartic extension T of F (see the beginning of § 3); this implies (∗). Now, write $B \cong (K/F, \sigma_0, z)$, where σ_0 is a generator for Gal(K/F). If σ is a generator for Gal(T/F), then $\sigma \mid K = \sigma_0$ and we have $(T/F, \sigma, z)^2 \sim (K/F, \sigma_0, z) \sim B$. ■

Remark **5.15.** — In general, (∗) does not imply the other conditions in (5.14). For instance, for $F = \mathbf{Q}$, the quaternion algebra $B_0 = (-1, -1)_{\mathbf{Q}}$ splits over some cyclic quartic extension of $\mathbf{Q}$ (see [Pi: p. 242]), but (2′) certainly does not hold since the norm form of B_0 is positive definite. On the other hand, if the field F has level 2 (i.e. -1 is a sum of two squares in F but is not a square), then, in view of (4.2), the conditions in (5.14) (with the exception of (∗)) are all equivalent to:

(11) B *splits over some cyclic quartic extension of* F *containing* $F(\sqrt{-1})$.

Appendix: Some Cohomological Results

For the convenience of the reader, we shall include here the proofs of two cohomological results which were used in the main body of this paper. Both results are known to the experts working in the area of Brauer groups and Galois cohomology; however, there seems to be no place in the literature where these results are stated explicitly and proved in a reasonably self-contained manner. It is our hope, therefore, that this Appendix will help fill this gap. Throughout the Appendix, the main notations used in this paper will remain in force; in particular, we shall continue to assume char $F \neq 2$, and F_s/F, G_F, μ_2, $B_2(F) \subseteq B(F)$, etc., will have the same meaning as in § 5. To simplify the notations further, we shall sometimes write $x_1 \cup \ldots \cup x_n$ for the cup product $(x_1) \cup \ldots \cup (x_n)$ in $H^n(G_F, \mu_2)$.

The first result concerns the cohomology long exact sequence of G_F associated with the short exact sequence of G_F-modules:

$$\text{(A1)} \qquad 1 \to \mu_2 \to \dot{F}_s \xrightarrow{\varepsilon} \dot{F}_s \to 1.$$

Here, ε is the squaring map: $\varepsilon(x) = x^2$ for every $x \in \dot{F}_s$.

Proposition **A2.** — *Let* $\delta : H^2(G_F, \dot{F}_s) \to H^3(G_F, \mu_2)$ *be the connecting homomorphism from the second cohomology to the third cohomology associated with* (A1). *Then, for any* $B = \bigotimes_i (x_i, y_i)_F$, *viewed as an element of* $H^2(G_F, \dot{F}_s)$, *we have* $\delta(B) = \Sigma_i\, x_i \cup y_i \cup (-1) \in H^3(G_F, \mu_2)$.

Remark **A3.** — (1) According to the Theorem of Merkurjev [Me], the classes of quaternion algebras in $B(F) = H^2(G_F, \dot{F}_s)$ generate the subgroup $B_2(F) = H^2(G_F, \mu_2)$. Therefore, the Proposition above implies that the restriction of δ to $H^2(G_F, \mu_2)$ is given by the cup product with (-1). However, we do not need this general result in this paper. (2) There is another connecting homomorphism, say δ', from $H^2(G_F, \mu_2)$ to $H^3(G_F, \mu_2)$, arising from the long exact cohomology sequence associated with the sequence of trivial G_F-modules

$$0 \to (\mu_2 =)\ \mathbf{Z}/2\mathbf{Z} \to \mathbf{Z}/4\mathbf{Z} \to \mathbf{Z}/2\mathbf{Z}\ (= \mu_2) \to 0.$$

However, J.-L. Colliot-Thélène has pointed out to us that δ' is in fact zero; see [MS_1: (16.5.1)] or [Se]. (This computation also assumes Merkurjev's Theorem.)

Proof of (A2). — Clearly, it suffices to prove the Proposition in the case when B is a single quaternion division algebra $(x, y)_F$. Let G_x (resp. G_y) denote the subgroup of G_F corresponding to the subfield $F(\sqrt{x})$ (resp. $F(\sqrt{y})$) of F_s. Then, the class of B in $H^2(G_F, \dot{F}_s)$ is given by the 2-cocycle $f: G_F \times G_F \to \dot{F}_s$ defined by

$$f(\sigma, \tau) = \begin{cases} 1 & \text{if } \sigma \in G_x \text{ or } \tau \in G_x, \\ y & \text{if } \sigma, \tau \notin G_x. \end{cases}$$

In order to calculate $\delta(B)$, we must first "lift" f through the squaring map ε. Thus, we define $f': G_F \times G_F \to \dot{F}_s$ by

$$f'(\sigma, \tau) = \begin{cases} 1 & \text{if } \sigma \in G_x \text{ or } \tau \in G_x, \\ \sqrt{y} & \text{if } \sigma, \tau \notin G_x. \end{cases}$$

Here, $\sqrt{y}$ denotes a *fixed* square root of y in $\dot{F}_s$. We have now $f'(\sigma, \tau)^2 = f(\sigma, \tau)$ for all $\sigma, \tau \in G_F$; hence, $\delta(B)$ is represented by the 3-cocycle $g: G_F \times G_F \times G_F \to \mu_2$ defined by

$$g(\sigma, \tau, \nu) = \sigma(f'(\tau, \nu))\, f'(\sigma\tau, \nu)^{-1} f'(\sigma, \tau\nu)\, f'(\sigma, \tau)^{-1}$$

for $\sigma, \tau, \nu \in G_F$. If τ or ν lies in G_x, we see easily that $g(\sigma, \tau, \nu) = 1$. If $\tau, \nu \notin G_x$, then $\tau\nu \in G_x$ (since $[G_F : G_x] = 2$), and we have

$$g(\sigma, \tau, \nu) = \sigma(\sqrt{y})\, f'(\sigma\tau, \nu)^{-1} f'(\sigma, \tau)^{-1} = \sigma(\sqrt{y})/\sqrt{y},$$

since, for every $\sigma \in G_F$, exactly one of σ, $\sigma\tau$ is outside of G_x. Therefore,

$$g(\sigma, \tau, \nu) = \begin{cases} -1 & \text{if } \sigma \notin G_y, \text{ and } \tau \notin G_x,\ \nu \notin G_x, \\ 1 & \text{if otherwise.} \end{cases}$$

But this 3-cocycle also represents the class of the cup-product $y \cup x \cup x$, so

$$\delta(B) = y \cup x \cup x = y \cup x \cup (-1) = B \cup (-1),$$

as desired. (Another, apparently more complicated, proof of this can be extracted from the arguments in Lemma 16.5 of [MS_1].) ■

We can now deduce the following Corollary which was used in the proof of (5.13).

Corollary **A4.** — *Let* $B \in B(F)$ *be represented by a tensor product of quaternion algebras* $\otimes_i (x_i, y_i)_F$. *Then* B *is a square in* B(F) *if and only if* $\Sigma_i\, x_i \cup y_i \cup (-1) = 0$ *in* $H^3(G_F, \mu_2)$.

Proof. — Since the map $\varepsilon^* : H^2(G_F, \dot{F}_s) \to H^2(G_F, \dot{F}_s)$ induced by ε in cohomology is just the squaring map, the desired conclusion follows from the Proposition and the exactness of the cohomology sequence:

$$\dots \to H^2(G_F, \dot{F}_s) \xrightarrow{\varepsilon^*} H^2(G_F, \dot{F}_s) \xrightarrow{\delta} H^3(G_F, \mu_2) \to \dots.$$

The next main result of this Appendix is the following theorem due to A. Merkurjev relating cup products in $H^3(G_F, \mu_2)$ to 3-fold Pfister forms over F (see, for instance, [Ar_2: p. 129]). Note that, in the special case when $d = d'$, this result shows that, for any F-biquaternion algebra A (viewed as an element in $H^2(G_F, \mu_2)$),

$$A \cup (d) = 0 \in H^3(G_F, \mu_2)$$

implies that $q_A \langle\langle -d \rangle\rangle = 0 \in W(F)$. This is the missing implication in the proof of (5.12) in § 5.

Theorem **A5.** — *If* $b \cup c \cup d = b' \cup c' \cup d' \in H^3(G_F, \mu_2)$, *then*

$$\langle\langle -b, -c, -d \rangle\rangle \cong \langle\langle -b', -c', -d' \rangle\rangle.$$

The converse of this result is true as well (even for higher folds), and was known quite a bit earlier. It is an easy consequence of the fact that isometric Pfister forms are always chain p-equivalent ([EL_1: (3.2)]). Also, we should note that Arason's well-defined homomorphism $\varphi : I^3 F/I^4 F \to H^3(G_F, \mu_2)$ (sending $\langle\langle -b, -c, -d \rangle\rangle + I^4 F$ to $b \cup c \cup d$) is now known to be an *isomorphism* ([Ro], [MS_2]), so (A5) is just a special case of the injectivity of φ (since any pair of 3-fold Pfister forms congruent modulo $I^4 F$ must be isometric). However, we do not want to assume this much deeper result here. Instead, we shall offer a reasonably self-contained proof of (A5), using ideas of Arason, Elman and Jacob. In fact, in [AEJ: Th. 1], these authors have proved the analogue of (A5) for 4-fold Pfister forms, and observed to us that their arguments also work for 3-fold Pfister forms. In the following, we shall give the technical details behind this observation, in order to complete the proof of (5.12), and also to popularize the nice ideas in [AEJ: Th. 1].

To begin with, note that (A5) can be reduced to the following special case:

Theorem **A5'.** — *If* $b \cup c \cup d = 0 \in H^3(G_F, \mu_2)$, *then* $\langle\langle -b, -c, -d \rangle\rangle$ *is a hyperbolic form.*

Indeed, if (A5') is known (for all fields), then (A5) follows by going to the function field K of $\langle\langle -b', -c', -d' \rangle\rangle$, using the fact that $4.\mathbf{H}$ and $\langle\langle -b', -c', -d' \rangle\rangle$ are the only 3-fold Pfister forms over F which become hyperbolic over K (see, for instance, [L_4]).

To prepare ourselves for the proof of (A5′), we first make a notational simplification. Instead of writing our cohomology groups as $H^i(G_F, \mu_2)$, we shall simply write $H^i F$. For any field F, we write e_F for the Clifford invariant map from $I^2 F$ to $H^2 F$, and write $\bar{I}^2 F$ for $I^2 F/I^3 F$. Merkurjev's main theorem in [Me] states that the induced map $\bar{e}_F : \bar{I}^2 F \to H^2 F$ is an *isomorphism*, but, as we have said in Remark (A3), we do not assume this result in this Appendix. Instead, we shall treat the injectivity and surjectivity of $\bar{e}_F$ as *conditions* on a field F, and try to get what we want by working around these conditions. The following lemma makes it clear that if we assumed Merkurjev's result for all fields, then Theorem A5′ would indeed follow immediately.

Lemma **A6.** — *Let* $b, c, d \in \dot{F}$, *and* $M = F(\sqrt{d})$. *If* $\bar{e}_F$ *is injective and* $\bar{e}_M$ *is surjective, then the statement in* (A5′) *holds for* F.

Proof. — We may of course assume that $d \notin \dot{F}^2$, so M/F is a quadratic extension. We have the following commutative diagram:

$$\begin{array}{ccc} \bar{I}^2 M & \xrightarrow{s_*} & \bar{I}^2 F \\ {\scriptstyle \bar{e}_M}\downarrow & & \downarrow{\scriptstyle \bar{e}_F} \\ H^2 M & \xrightarrow{\mathrm{Cor}} & H^2 F \end{array}$$

Here, " Cor " is the corestriction in cohomology, and s_* is the transfer map for quadratic forms induced by the F-linear functional s on M sending 1 to 0, and $\sqrt{d}$ to 1 By Arason's exact sequence in cohomology [Ar$_1$: (4.6)], $b \cup c \cup d = 0$ implies that $b \cup c = \mathrm{Cor}(x)$ for some $x \in H^2 M$. Since $\bar{e}_M$ is assumed surjective, we can write $x = \bar{e}_M(y)$ for some $y \in \bar{I}^2 M$. Now $b \cup c = \mathrm{Cor}(x) = \mathrm{Cor}(\bar{e}_M(y)) = \bar{e}_F(s_*(y))$, so the injectivity of $\bar{e}_F$ implies that $s_*(y) \equiv \langle\langle -b, -c \rangle\rangle \pmod{I^3 F}$. Multiplying this by $\langle\langle -d \rangle\rangle$, and using the fact that the image of the transfer map is killed by $\langle\langle -d \rangle\rangle$, we get $\langle\langle -b, -c, -d \rangle\rangle \equiv s_*(y)\langle\langle -d \rangle\rangle \equiv 0 \pmod{I^4 F}$, so $\langle\langle -b, -c, -d \rangle\rangle$ is hyperbolic. ■

For a fixed quadratic extension $M = F(\sqrt{d})$, let us enlarge the above diagram into the following, where r_* is the functorial map, and f and g are maps induced by $\bar{e}_F$:

$$\begin{array}{ccccccccc} & & \dfrac{I^2 F}{\langle\langle -d \rangle\rangle IF + I^3 F} & \xrightarrow{r_*} & \dfrac{I^2 M}{I^3 M} & \xrightarrow{s_*} & \dfrac{s_*(I^2 M) + I^3 F}{I^3 F} & \longrightarrow & 0 \\ & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar{e}_M} & & \downarrow{\scriptstyle g} & & \\ 0 & \longrightarrow & \dfrac{H^2 F}{(d) \cup \dot{F}} & \longrightarrow & H^2 M & \xrightarrow{\mathrm{Cor}} & H^2 F & & \end{array}$$

Here, the bottom row is well-known to be exact in the context of Brauer groups (special case of [Ar$_1$: (4.6)]), and the top row is exact except possibly at $I^2 M/I^3 M$.

Lemma **A7.** — *Assume that* $I^4 F = 0$. *Then:*

(1) $s_* : I^3 M \to I^3 F$ *is surjective;*

(2) *The diagram above is exact;*

(3) *If* $\bar{e}_F$ *is injective, so are* $\bar{e}_M$ *and the map* $\operatorname{coker} \bar{e}_F \to \operatorname{coker} \bar{e}_M$.

Proof. — For any 3-fold Pfister form σ over F, $\langle\langle -d \rangle\rangle \sigma \in I^4 F$ is hyperbolic. From this, it is known that $\sigma = s_* \tau$ for some 3-fold Pfister form τ over M ([Ar_1: Zusatz, p. 459]). This gives (1). For (2), we only have to prove the exactness of the top row at $I^2 M/I^3 M$. Let $x \in I^2 M$ be such that $s_*(x) \in I^3 F$. By (1), $s_*(x) = s_*(y)$ for some $y \in I^3 M$. Then $x - y \in I^2 M \cap \ker(s_*) = r_*(I^2 F)$ [EL_4: Th. 3.3], so $x \in I^3 M + r_*(I^2 F)$, as desired. From the exact diagram, we have by the Snake Lemma [Pi: p. 202] the following exact sequence:

$$(*) \qquad \ker(f) \to \ker(\bar{e}_M) \to \ker(g) \to \operatorname{coker}(f) \to \operatorname{coker}(\bar{e}_M) \to \operatorname{coker}(g).$$

If we assume that $\bar{e}_F$ is injective, it is clear that f and g are also injective, and $\operatorname{coker}(f)$ is just $\operatorname{coker}(\bar{e}_F)$. The kernel-cokernel exact sequence $(*)$ now yields the conclusion (3). ∎

Corollary **A8.** — *Assume that* $I^4 F = 0$, *and that* $\bar{e}_F$ *is injective. Then, for any finite* 2-*extension* K/F, $\bar{e}_K$ *is injective, and* $\operatorname{coker} \bar{e}_F \to \operatorname{coker} \bar{e}_K$ *is injective.*

Proof. — This follows by repeated application of (A7), in view of the fact that the property $I^4 F = 0$ " goes up " to 2-extensions of F ([Ar_1: (3.6)], [EL_3: (4.5)]). ∎

The following lemma is a special case of [AEJ: Lemma 1]. For the sake of completeness, we shall include its proof.

Lemma **A9.** — *For any field* F_0, *there exists an extension field* F/F_0 *such that: a) any anisotropic* 3-*fold Pfister form over* F_0 *remains anisotropic over* F, *b)* F *has u-invariant* $u(F) \leqslant 8$ *(i.e. any* 9-*dimensional quadratic form over* F *is isotropic), and c)* F *has no proper odd degree extensions.*

Proof. — Let F_0' be the compositum of the function fields of all 9-dimensional forms over F_0, and let F_1 be a maximal odd extension of F_0' (which exists by Zorn's Lemma). Then, any 9-dimensional form over F_0 is isotropic over F_1, and F_1 has no proper odd extensions. Moreover, by [L_4: Cor. 9.11], any anisotropic 3-fold Pfister form over F_0 remains anisotropic over F_0', and hence over F_1 by Springer's Theorem [L_1: p. 198]. Repeating this construction, we get a tower of fields $F_0 \subset F_1 \subset F_2 \ldots$. Their union F is the field we want. ∎

In order to prove (A5′) for a field F_0, it clearly suffices to prove the same thing for the extension field F constructed above (in view of the property *a)*). The following lemma, therefore, gives the final step.

Lemma **A10.** — (A5′) *is true for any field* F *with* $u(F) \leqslant 8$ *having no proper odd degree extensions.*

Proof. — The hypothesis $u(F) \leqslant 8$ implies that $I^4 F = 0$. By Pfister's classical result [Pf: Satz 14], it also implies that $\bar{e}_F$ is *injective*. Therefore, (A8) applies. To get the desired conclusion, it suffices (by (A6)) to show that $\bar{e}_M$ is *surjective* for any quadratic extension M/F. Take any cohomology class $\alpha \in H^2 M$, and split it by a Galois extension K/F containing M. By the hypothesis on F, K/F (and hence K/M) must be a 2-extension. Applying (A8) to M, we see that coker $\bar{e}_M \to$ coker $\bar{e}_K$ is injective, so we have already $\alpha = 0 \in \operatorname{coker} \bar{e}_M$, i.e. $\alpha \in \operatorname{im}(\bar{e}_M)$. ■

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