VLADIMIR TURAEV

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QUANTUM INVARIANTS OF LINKS AND 3-VALENT GRAPHS IN 3-MANIFOLDS

by VLADIMIR TURAEV

CONTENTS

INTRODUCTION

1. The principal aim of this paper is to give a 3-dimensional computation and interpretation of the Jones polynomial of links in the 3-sphere S^3. The original definition of Jones [J] is 2-dimensional in the sense that it is based on the link projections in the plane. A 3-dimensional interpretation of the Jones polynomial was first given by Witten [W1] using Feynman path integrals. This interpretation is simple and beautiful, but its rigorous mathematical treatment calls for considerable efforts. In this paper I use quantum $6j$-symbols to give a rigorous and essentially self-contained computation of the Jones polynomial of a (framed) link in terms of the link exterior. Though the main ideas of the paper are independent of Witten's approach one may view this paper as an attempt to understand his work.

The classical $6j$-symbols naturally appear in the representation theory of the Lie algebra $s\ell_2(\mathbf{C})$. These symbols together with more familiar Clebsch-Gordan coefficients play a fundamental role in the study of tensor products of irreducible finite-dimensional $s\ell_2(\mathbf{C})$-modules. The $6j$-symbols ensure the compatibility of the splittings of the tensor products in the direct sum of irreducible modules on the one hand, and the associativity of the tensor product on the other hand. The $6j$-symbols have been extensively studied by physicists in the frameworks of the quantum theory of angular momentum (for more information and references see [BL]). A one-parameter deformation of the classical $6j$-symbols was introduced by Askey and Wilson [AW] in connection with their study of q-orthogonal polynomials. Here q may be treated either as a variable or as a complex deformation parameter. The resulting $q-6j$-symbols (they are also called quantum $6j$-symbols) were shown by Kirillov and Reshetikhin [KR] to play the same role in the representation theory of the quantum group $U_q(s\ell_2(\mathbf{C}))$ as the classical $6j$-symbols play in the representation theory of $s\ell_2(\mathbf{C})$. (The classical theory corresponds to $q = 1$.)

Kirillov and Reshetikhin [KR] used the quantum $6j$-symbols to compute the Jones polynomial of a link in S^3 via a 2-dimensional state sum model. More exactly, they constructed a face model on any plane link diagram and showed that plugging in this model the quantum $6j$-symbols we get the Jones polynomial (and its generalizations for colored links).

A crucial step in creating a 3-dimensional state sum model for Jones-type invariants of 3-manifolds was made in the paper of Viro and the author [TV]. There we introduced a state sum model based on triangulations of 3-manifolds. As the main algebraic ingredient this model involves $q-6j$-symbols where q is a complex root of unity of a certain degree $r \geqslant 3$. The model is defined on an arbitrary triangulation X of a compact 3-manifold. Assume for simplicity that the manifold has no boundary. We consider "colorings" of X which assign to edges of X elements of the set $\{0, 1/2, 1, \ldots, (r-2)/2\}$. Having a coloring of X we associate with each 3-simplex of X the $q-6j$-symbol

$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} \in \mathbf{C}$$

where (i, l), (j, m), (k, n) are the pairs of colors of the opposite edges of this simplex. We multiply these symbols overs all 3-simplices of X and sum up the resulting products (with certain weights) over the colorings of X. According to [TV] this model yields a topological invariant of compact 3-manifolds. (This invariant does not depend on the orientation of the manifold.) Later on it was shown in [T3], [T4] and independently by Walker [Wa] that for any closed oriented 3-manifold this invariant coincides with the square of the absolute value of the invariant defined in [RT2].

The technique of [TV] is not sufficient to recover the Jones polynomial of links from the link exteriors. One may already see this from the fact that the Jones polynomial of a link in S^3 and of its mirror image may differ whereas the exteriors of these two links are homeomorphic and therefore can not be distinguished by the invariants of [TV].

2. In this paper we compute the Jones polynomial of a link in S^3 in terms of the exterior, say, M of this link. More exactly, we express the values of this polynomial in the roots of unity as certain state sums on an arbitrary triangulation of M. On the algebraic part these state sums are based on the $q-6j$-symbols where q is a root of unity. On the geometric part our model refines the one of [TV]: we take into account the geometric picture on the boundary ∂M of M. On ∂M we have quite a number of objects involved in the model. Among them are the meridians and the longitudes of the link, the orientation of ∂M induced by that of S^3, and the triangulation of ∂M induced by the one in M.

The properties of the classical and quantum $6j$-symbols associated with $s\ell_2(\mathbf{C})$ are especially simple since the representation theory of $s\ell_2(\mathbf{C})$ is multiplicity-free. Therefore the associated 3-dimensional state sum models are simplest models of this kind. In particular, they yield invariants of unoriented links. In more general models corresponding to other simple Lie algebras (or modular Hopf algebras, or modular tensor categories) one has to consider generalized $6j$-symbols depending on 6 irreducible modules which may be geometrically viewed as sitting on the edges of a tetrahedron and on 4 intertwiners which sit on the faces of the tetrahedron. Such models will be treated in [T5]. In this paper we consider only multiplicity-free models.

To make the paper more independent from the theory of quantum groups we construct our models on the ground of axiomatically defined algebraic initial data. The definition of the initial data axiomatizes the properties of the quantum $6j$-symbols associated with $U_q(s\ell_2(\mathbf{C}))$. Following a suggestion of the referee one may call our algebraic initial data a (multiplicity-free) " tetrahedral interaction ".

3. On the geometric part the constructions of the paper are actually more general and extend to links in arbitrary compact 3-manifolds. The constructions apply also to

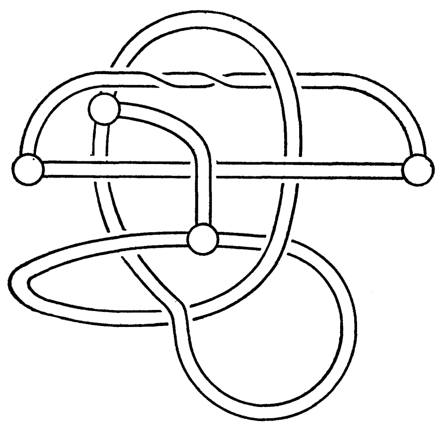

Fig. 1

colored fat graphs in these manifolds. By a fat graph in a 3-manifold N we mean a finite graph in N whose vertices and edges are extended to small 2-disks and narrow bands respectively (see, for example, Fig. 1). We will consider only 3-valent fat graphs which means that every 2-disk is incident either to 2 or 3 bands counted with multiplicities. In particular, fat graphs may consist of annuli, as in Fig. 2. This enables us to treat (framed) links as fat graphs and to include in this way the theory of links in the theory of fat graphs. (The fat graphs should not be confused with ribbon graphs used in [RT2]; the vertices of ribbon graphs have a more complicated structure: they are oriented rectangles with preferred opposite bases.)

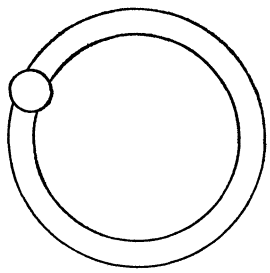

Fig. 2

Fix an integer $r \geqslant 3$. A coloring of a fat graph associates to each of its band a non-negative integer or half-integer lying between 0 and $(r-2)/2$. For such a colored fat graph in a compact 3-manifold we define a state sum model on the graph exterior. (In fact we need an orientability assumption: a regular neighborhood of the graph in the 3-manifold must be oriented.) The model is shown to produce isotopy invariants of the graph depending of the choice of a primitive complex root of unity of degree $4r$. In the case of colored fat graphs in S^3 these invariants coincide with the values of the generalized Jones polynomial in the corresponding roots of unity. For the empty graph in a compact 3-manifold our invariants coincide with the invariants of this manifold introduced in [TV].

4. Besides the state sum models on the graph exteriors in 3-manifolds we develop another, so to say diagrammatical approach to the invariants of the fat graphs. With this view we generalize the classical Reidemeister theory of plane link diagrams and Reidemeister moves to the case of links and fat graphs in arbitrary 3-manifolds. The corresponding graph diagrams live on special spines of the manifolds. We construct a certain " face " model on these diagrams generalizing the model of Kirillov and Reshetikhin [KR] on S^2. Our face model is necessarily more complicated since in general the special spines are not locally flat but rather have local singularities. The resulting partition functions are shown to be essentially the same as the invariants obtained from the 3-dimensional models on the graph exteriors. It is in this way that we establish equivalence of our invariants with the generalized Jones polynomial for links in S^3.

Note, however, that the diagrammatical approach applies only to graphs in oriented 3-manifolds.

In the Appendix to the paper we show how to compute the invariants of fat graphs in 3-manifolds in terms of Heegaard decompositions of these manifolds.

5. This paper should be viewed in the broader frameworks of topological quantum field theory (TQFT; see [At]). Our results extend the non-oriented 3-dimensional TQFT introduced in [TV]. Namely, in generalization of 3-cobordisms one may consider 3-cobordisms with fat graphs sitting inside. (The graphs should be equipped with oriented regular neighborhoods.) We remark that this TQFT differs from the oriented 3-dimensional TQFT which was introduced in [RT2] and which is currently considered as the mathematical version of the Witten's physical TQFT. There are arguments suggesting that the TQFT developed here is related to $2+1$-dimensional quantum gravity.

One may further extend our invariants to include fat graphs with free ends lying on the boundary of the ambient 3-manifold but we do not pursue this line here.

6. The paper is divided into three parts and an Appendix. Part I is purely algebraic. It is concerned with the algebraic initial data used in our state models. In Section 1 we give axiomatic definition of the initial data and formulate some algebraic conditions on the data (Conditions I-IV). In Section 2 the quantum $6j$-symbols associated with the quantum group $U_q(s\ell_2(\mathbf{C}))$ are recalled and shown to yield initial data satisfying Conditions I-IV (the " quantum initial data "). Part II deals with fat graphs and the state sums on their exteriors. Section 3 presents a summary of the results of [TV] used in the present paper. In Section 4 we define certain auxiliary state sum invariants for fat graphs lying in the product of a surface and a segment. In Section 5 we introduce the state sum models on the triangulated exteriors of fat graphs and show independence of the associated partition functions of the choice of triangulation. In Section 6 the invariants corresponding to the quantum initial data are discussed in more detail. Part III is concerned with the diagrammatical approach to the invariants. In Section 7 we define graph diagrams on simple spines of 3-manifolds and discuss the analogues of the Reidemeister moves. In Section 8 we introduce the relevant face model on the graph diagrams and verify invariance of the partition function under the moves. In Section 9 we establish equivalence of this approach to the one introduced in Part II and prove several theorems stated in Sections 5 and 6. The Appendix presents a computation of the graph invariants in terms of the Heegaard splittings.

7. A part of this paper was written while the author was visiting Ohio State University. The support of this university is gratefully acknowledged. The author thanks the referee for suggestions to improve the exposition and valuable remarks.

PART I

INITIAL DATA

1. Abstract initial data

1.a. *Definition of initial data.* — Following the lines of [T2], [TV] we introduce axioms for " algebraic initial data ". The data will be used below to construct our state sum models. The definition of initial data essentially axiomatizes quantum $6j$-symbols (cf. [KR] and Section 2).

Fix a commutative associative ring K with unit. Denote by K^* the group of invertible elements of K. Assume that we are given a finite set I, an element $w \in K^*$ and two functions $I \to K^*$ which associate with each $i \in I$ elements $q_i, w_i \in K^*$. Assume that we have distinguished a set adm of unordered triples of elements of I. The triples belonging to this set will be called *admissible triples*. Note that we put no conditions on the set adm; in particular, elements of an admissible triple may coincide.

An ordered 6-tuple $(i, j, k, l, m, n) \in I^6$ will be called admissible if the triples (i, j, k), (k, l, m), (m, n, i) and (j, l, n) are admissible. Assume that with each admissible 6-tuple (i, j, k, l, m, n) one associates an element of K, called the symbol of this tuple and denoted by

$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}. \tag{1.a.1}$$

Note that if the 6-tuple (i, j, k, l, m, n) is admissible then the 6-tuples (j, i, k, m, l, n), (i, k, j, l, n, m) and (i, m, n, l, j, k) are also admissible. We assume the following symmetry identities of the symbol:

$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} = \begin{vmatrix} j & i & k \\ m & l & n \end{vmatrix} = \begin{vmatrix} i & k & j \\ l & n & m \end{vmatrix} = \begin{vmatrix} i & m & n \\ l & j & k \end{vmatrix}. \tag{1.a.2}$$

(These identities generate a group of permutations of the tuple (i, j, k, l, m, n) consisting of 24 elements.) This completes the definition of the initial data.

We extend the symbol to non-admissible 6-tuples by assuming the symbol (1.a.1) corresponding to any non-admissible tuple (i, j, k, l, m, n) to be equal to zero. The identities (1.a.2) obviously extend to all 6-tuples.

The initial data is called irreducible if for any $j, k \in I$ there exists a sequence $l_1, l_2, \ldots, l_r$ with $l_1 = j$, $l_r = k$ such that the triple (l_t, l_{t+1}, l_{t+2}) is admissible for all $t = 1, 2, \ldots, r-2$.

1.b. *Conditions I-IV.* — Here we formulate some algebraic conditions on the initial data. The geometric meaning of these conditions will be clear in Sections 4, 5 and 8. An example of initial data satisfying these conditions will be given in Section 2.

Condition I. — For any $j_1, j_2, j_3, j_4, j_5 \in \mathrm{I}$ *with* (j_1, j_3, j_4), $(j_2, j_4, j_5) \in \mathrm{adm}$ *we have*

$$w_{j_4}^2 \sum_{j \in \mathrm{I}} w_j^2 \begin{vmatrix} \dot{j_5} & \dot{j_1} & \dot{j} \\ \dot{j_3} & \dot{j_2} & \dot{j_4} \end{vmatrix}^2 = 1, \tag{1.b.1}$$

and for any $j_1, j_2, \ldots, j_6 \in \mathrm{I}$ *with* $j_4 \neq j_6$

$$\sum_{j \in \mathrm{I}} w_j^2 \begin{vmatrix} \dot{j_5} & \dot{j_1} & \dot{j} \\ \dot{j_3} & \dot{j_2} & \dot{j_4} \end{vmatrix} \cdot \begin{vmatrix} \dot{j_5} & \dot{j_1} & \dot{j} \\ \dot{j_3} & \dot{j_2} & \dot{j_6} \end{vmatrix} = 0. \tag{1.b.2}$$

Condition II. — For any $a, b, c, e, f, j_1, j_2, j_3, j_{23} \in \mathrm{I}$

$$\sum_{d \in \mathrm{I}} w_d^2 \begin{vmatrix} \dot{j_2} & \dot{a} & \dot{d} \\ \dot{j_1} & \dot{c} & \dot{b} \end{vmatrix} \cdot \begin{vmatrix} \dot{j_3} & \dot{d} & \dot{e} \\ \dot{j_1} & \dot{f} & \dot{c} \end{vmatrix} \cdot \begin{vmatrix} \dot{j_3} & \dot{j_2} & \dot{j_{23}} \\ \dot{a} & \dot{e} & \dot{d} \end{vmatrix} = \begin{vmatrix} \dot{j_{23}} & \dot{a} & \dot{e} \\ \dot{j_1} & \dot{f} & \dot{b} \end{vmatrix} \cdot \begin{vmatrix} \dot{j_3} & \dot{j_2} & \dot{j_{23}} \\ \dot{b} & \dot{f} & \dot{c} \end{vmatrix}. \tag{1.b.3}$$

Condition III. — For any $j \in \mathrm{I}$

$$w^2 = w_j^{-2} \sum_{\substack{k, l \in \mathrm{I} \\ (j, k, l) \in \mathrm{adm}}} w_k^2 w_l^2. \tag{1.b.4}$$

Conditions I-III were used in [TV] to construct state sum invariants of compact 3-manifolds.

Conditions I and II axiomatize respectively the orthogonality relation and the Biedenharn-Eliott identity for $6j$-symbols (see [BL], [KR, § 6] and references therein).

For irreducible initial data satisfying Condition I the right hand side of (1.b.4) is known to be independent of the choice of j (see [TV, Lemma 1.1.A]). Therefore in this case Condition III together with the inclusion $w \in \mathrm{K}^*$ mean exactly that the right hand side of (1.b.4) presents an invertible element of K with a preferred square root w.

To define state models for invariants of links generalizing the Jones polynomial we need one more condition involving the function $i \mapsto q_i$. This condition is an analog of the Racah identity for $6j$-symbols.

Condition IV. — For any $j, j_1, j_2, j_3, j_{12}, j_{23} \in \mathrm{I}$ *we have*

$$\sum_{j_{13} \in \mathrm{I}} w_{j_{13}}^2 q_{j_{13}} \begin{vmatrix} \dot{j_3} & \dot{j_1} & \dot{j_{13}} \\ \dot{j_2} & \dot{j} & \dot{j_{12}} \end{vmatrix} \begin{vmatrix} \dot{j_2} & \dot{j_3} & \dot{j_{23}} \\ \dot{j_1} & \dot{j} & \dot{j_{13}} \end{vmatrix} = q_j\, q_{j_1}\, q_{j_2}\, q_{j_3}\, q_{j_{12}}^{-1}\, q_{j_{23}}^{-1} \begin{vmatrix} \dot{j_3} & \dot{j_2} & \dot{j_{23}} \\ \dot{j_1} & \dot{j} & \dot{j_{12}} \end{vmatrix}. \tag{1.b.5}$$

Note that only even powers of $\{ w_i \}_i$ appear in these conditions. However, the invariants of graphs in 3-manifolds defined below depend on the choice of $\{ w_i \}_i$. The signs of $\{ w_i \}_i$ are used when the boundary of the manifold is non-empty: in a sense the factor w_i^2 is divided between the manifold under consideration and a hypothetical manifold with the same boundary which one might glue to the other side.

1.c. *Corollaries of Conditions I-IV.*

Lemma **1.1.** — *If the initial data satisfy Conditions I-IV, then for any* $j, j_1, j_2, j_3, j_{12}, j_{23} \in \mathrm{I}$ *we have*

$$\textbf{(1.c.1)} \qquad \sum_{j_{13}\in \mathrm{I}} w_{j_{13}}^2\, q_{j_{13}}^{-1} \begin{vmatrix} j_3 & j_1 & j_{13} \\ j_2 & j & j_{12} \end{vmatrix} \begin{vmatrix} j_2 & j_3 & j_{23} \\ j_1 & j & j_{13} \end{vmatrix} = q_j^{-1}\, q_{j_1}^{-1}\, q_{j_2}^{-1}\, q_{j_3}^{-1}\, q_{j_{12}}\, q_{j_{23}} \begin{vmatrix} j_3 & j_2 & j_{23} \\ j_1 & j & j_{12} \end{vmatrix}.$$

Proof. — Condition I implies that

$$\begin{aligned} &\begin{vmatrix} j_3 & j_2 & j_{23} \\ j_1 & j & j_{12} \end{vmatrix} \\ &= \sum_{j_{13},\, k\in \mathrm{I}} \left(q_{j_{13}}\, q_{j_{12}}^{-1}\, w_{j_{13}}^2\, w_k^2 \begin{vmatrix} j_3 & j_2 & j_{23} \\ j_1 & j & j_{13} \end{vmatrix} \cdot \begin{vmatrix} j_3 & j_1 & k \\ j_2 & j & j_{13} \end{vmatrix} \cdot \begin{vmatrix} j_3 & j_1 & k \\ j_2 & j & j_{12} \end{vmatrix} \right) \\ &= \sum_{k\in \mathrm{I}} \left(q_{j_{12}}^{-1}\, w_k^2 \begin{vmatrix} j_3 & j_1 & k \\ j_2 & j & j_{12} \end{vmatrix} \sum_{j_{13}\in \mathrm{I}} w_{j_{13}}^2\, q_{j_{13}} \begin{vmatrix} j_3 & j & j_{13} \\ j_1 & j_2 & j_{23} \end{vmatrix} \cdot \begin{vmatrix} j_1 & j_3 & k \\ j & j_2 & j_{13} \end{vmatrix} \right). \end{aligned}$$

In view of Condition IV the latter expression is equal to

$$q_j\, q_{j_1}\, q_{j_2}\, q_{j_3}\, q_{j_{12}}^{-1}\, q_{j_{23}}^{-1} \sum_{k\in \mathrm{I}} \left(w_k^2\, q_k^{-1} \begin{vmatrix} j_3 & j_1 & k \\ j_2 & j & j_{12} \end{vmatrix} \cdot \begin{vmatrix} j_2 & j_3 & j_{23} \\ j_1 & j & k \end{vmatrix} \right).$$

This implies the claim of the lemma.

Lemma **1.2.** — *If the initial data satisfy Conditions I, II and IV then for any* $a, b, c, d, e, j_1, j_2, j_3 \in \mathrm{I}$ *and* $\varepsilon = \pm 1$ *we have*

$$\textbf{(1.c.2)} \qquad \sum_{g\in \mathrm{I}} w_g^2 (q_b\, q_d\, q_f\, q_g)^{\varepsilon} \begin{vmatrix} j_2 & a & g \\ j_1 & c & b \end{vmatrix} \cdot \begin{vmatrix} j_3 & g & e \\ j_1 & d & c \end{vmatrix} \cdot \begin{vmatrix} j_3 & a & f \\ j_2 & e & g \end{vmatrix} = \sum_{g\in \mathrm{I}} w_g^2 (q_a\, q_c\, q_e\, q_g)^{\varepsilon} \begin{vmatrix} j_3 & b & g \\ j_2 & d & c \end{vmatrix} \cdot \begin{vmatrix} j_3 & a & f \\ j_1 & g & b \end{vmatrix} \cdot \begin{vmatrix} j_2 & f & e \\ j_1 & d & g \end{vmatrix}.$$

A similar formula is well known in the theory of $6j$-symbols; it is a counterpart of the Yang-Baxter equation (without spectral parameter).

Proof of Lemma **1.2.** — Condition IV and Lemma 1.1 imply that

$$\textbf{(1.c.3)} \qquad \begin{vmatrix} j_3 & g & e \\ j_1 & d & c \end{vmatrix} = \begin{vmatrix} j_1 & c & g \\ j_3 & e & d \end{vmatrix} = \sum_{j_{13}\in \mathrm{I}} w_{j_{13}}^2 (q_e\, q_c\, q_{j_1}\, q_{j_3}\, q_d^{-1}\, q_g^{-1}\, q_{j_{13}}^{-1}) \begin{vmatrix} j_1 & j_3 & j_{13} \\ c & e & d \end{vmatrix} \cdot \begin{vmatrix} c & j_1 & g \\ j_3 & e & j_{13} \end{vmatrix}.$$

Condition II implies that

$$\sum_{g\in I} w_g^2 \begin{vmatrix} j_2 & a & g \\ j_1 & c & b \end{vmatrix} \cdot \begin{vmatrix} j_3 & a & f \\ j_2 & e & g \end{vmatrix} \cdot \begin{vmatrix} c & j_1 & g \\ j_3 & e & j_{13} \end{vmatrix}$$

$$= \sum_{g\in I} w_g^2 \begin{vmatrix} j_2 & a & g \\ j_1 & c & b \end{vmatrix} \cdot \begin{vmatrix} e & g & j_3 \\ j_1 & j_{13} & c \end{vmatrix} \cdot \begin{vmatrix} e & j_2 & f \\ a & j_3 & g \end{vmatrix}$$

$$= \begin{vmatrix} f & a & j_3 \\ j_1 & j_{13} & b \end{vmatrix} \cdot \begin{vmatrix} e & j_2 & f \\ b & j_{13} & c \end{vmatrix}.$$

Therefore substituting the expression (1.c.3) for

$$\begin{vmatrix} j_3 & g & e \\ j_1 & d & c \end{vmatrix}$$

into the left hand side of (1.c.2) one obtains that the left hand side of (1.c.2) equals

$$\text{(1.c.4)} \qquad \sum_{j_{13}\in I} w_{j_{13}}^2 (q_b\, q_c\, q_e\, q_f\, q_{j_1}\, q_{j_3}\, q_{j_{13}}^{-1})^{\varepsilon} \cdot \begin{vmatrix} j_1 & j_3 & j_{13} \\ c & e & d \end{vmatrix} \cdot \begin{vmatrix} f & a & j_3 \\ j_1 & j_{13} & b \end{vmatrix} \cdot \begin{vmatrix} e & j_2 & f \\ b & j_{13} & c \end{vmatrix}.$$

Similarly, substituting in the right hand side of (1.c.2) the following expression

$$\begin{vmatrix} j_3 & a & f \\ j_1 & g & b \end{vmatrix} = \begin{vmatrix} j_1 & b & a \\ j_3 & f & g \end{vmatrix}$$

$$= \sum_{j_{13}\in I} w_{j_{13}}^2 (q_b\, q_f\, q_{j_1}\, q_{j_3}\, q_a^{-1}\, q_g^{-1}\, q_{j_{13}}^{-1})^{\varepsilon} \begin{vmatrix} j_1 & j_3 & j_{13} \\ b & f & g \end{vmatrix} \cdot \begin{vmatrix} b & j_1 & a \\ j_3 & f & j_{13} \end{vmatrix}$$

and applying the formula

$$\sum_{g\in I} w_g^2 \begin{vmatrix} j_3 & b & g \\ j_2 & d & c \end{vmatrix} \cdot \begin{vmatrix} j_1 & g & f \\ j_2 & e & d \end{vmatrix} \cdot \begin{vmatrix} j_1 & j_3 & j_{13} \\ b & f & g \end{vmatrix} = \begin{vmatrix} j_{13} & b & f \\ j_2 & e & c \end{vmatrix} \cdot \begin{vmatrix} j_1 & j_3 & j_{13} \\ c & e & d \end{vmatrix}$$

one easily obtains that the right hand side of (1.c.2) also equals (1.c.4). This completes the proof of the Lemma.

Lemma **1.3.** — *If the initial data satisfy conditions I, II, IV then for any admissible triple* $(i, a, b) \in \operatorname{adm}$ *and for any* $\varepsilon = \pm 1$ *we have*

$$\text{(1.c.5)} \qquad \sum_{j\in I} w_j^2 (q_a\, q_j\, q_b^{-2})^{\varepsilon} \begin{vmatrix} i & b & j \\ i & b & a \end{vmatrix} = q_i^{2\varepsilon}.$$

The equality (1.c.5) is the Markov relation for R-matrices rewritten in terms of $6j$-symbols (cf. [KR], [T1], [T2]).

Proof of Lemma **1.3.** — Let $f \in I$. Applying Condition II with $a, b, c, d, e, f, j_1, j_2, j_3, j_{23}$ replaced respectively by $b, a, f, d, j, b, i, a, b, i$ we get

$$\left(\sum_{j\in I} w_j^2 (q_a q_j q_b^{-2})^\varepsilon \begin{vmatrix} i & b & j \\ i & b & a \end{vmatrix}\right) \begin{vmatrix} b & a & i \\ a & b & f \end{vmatrix}$$

$$= \sum_{j,d\in I} w_j^2 w_d^2 (q_a q_j q_b^{-2})^\varepsilon \begin{vmatrix} a & b & d \\ i & f & a \end{vmatrix} \cdot \begin{vmatrix} b & d & j \\ i & b & f \end{vmatrix} \cdot \begin{vmatrix} b & a & i \\ b & j & d \end{vmatrix}$$

$$= \sum_{d\in I} \left(w_d^2 (q_a q_b^{-2})^\varepsilon \begin{vmatrix} a & b & d \\ i & f & a \end{vmatrix} \sum_{j\in I} w_j^2 q_j^\varepsilon \begin{vmatrix} b & i & j \\ d & b & f \end{vmatrix} \cdot \begin{vmatrix} d & b & a \\ i & b & j \end{vmatrix} \right)$$

$$= \sum_{d\in I} w_d^2 (q_i q_a q_f^{-1})^\varepsilon \begin{vmatrix} a & b & d \\ i & f & a \end{vmatrix} \cdot \begin{vmatrix} b & d & a \\ i & b & f \end{vmatrix},$$

where the latter equality follows from Condition IV and Lemma 1.1. The latter expression equals

$$q_i^\varepsilon q_f^{-\varepsilon} \sum_{d\in I} w_d^2 q_d^\varepsilon \begin{vmatrix} f & i & d \\ a & b & b \end{vmatrix} \cdot \begin{vmatrix} a & f & a \\ i & b & d \end{vmatrix} = q_i^{2\varepsilon} \begin{vmatrix} f & a & a \\ i & b & b \end{vmatrix} = q_i^{2\varepsilon} \begin{vmatrix} b & a & i \\ a & b & f \end{vmatrix}.$$

Here we have again used Condition IV and Lemma 1.1. Combining all these equalities together, multiplying by

$$w_i^2 w_f^2 \begin{vmatrix} b & a & i \\ a & b & f \end{vmatrix},$$

summing up over all $f \in I$, and taking into account Condition I, we get (1.c.5).

1.d. *Remark.* — The proofs of Lemmas 1.1, 1.2 and 1.3 given above have simple visual interpretation (see Section 7.e).

2. Quantum initial data

2.a. *Definition of the quantum initial data.* — Following the lines of [T2], [TV] we introduce here " quantum " initial data over **C**, which satisfy the conditions **I-IV** formulated in Section 1. The main ingredient of the data are the $q-6j$-symbols associated with the quantum group $U_q(s\ell_2(\mathbf{C}))$ and a complex root of unity q (see [KR]). These symbols play an important role in the representation theory of $U_q(s\ell_2(\mathbf{C}))$. Here we treat these symbols in a rather formal way addressing the reader to [KR] for a conceptual definition.

Fix an integer $r \geqslant 3$ and denote by $\mathbf{I}$ the set $\{0, 1/2, 1, \ldots, (r-3)/2, (r-2)/2\}$. Fix a primitive complex root of unity t of degree $4r$ so that $t = \exp(\pi\sqrt{-1}\,h/2r)$ with $h \in \mathbf{Z}$. (The number t is related to q by the formula $q = t^4$). For each integer $n \geqslant 1$ put

$$[n] = \frac{t^{2n} - t^{-2n}}{t^2 - t^{-2}} \in \mathbf{R},$$

and

$$[n]! = [n]\,[n-1] \ldots [2]\,[1].$$

In particular $[1]! = [1] = 1$. Put also $[0]! = [0] = 1$. Note that $[n]! = 0$ for $n \geqslant r$ and $[n]! \neq 0$ for $n < r$.

A triple $(i, j, k) \in \mathbf{I}^3$ is admissible if $i + j + k$ is an integer, $i + j + k \leqslant r - 2$, and $i \leqslant j + k$, $j \leqslant i + k$, $k \leqslant i + j$. For each admissible triple i, j, k set

$$\Delta(ijk) = \left(\frac{[i+j-k]!\,[i-j+k]!\,[-i+j+k]!}{[i+j+k+1]!}\right)^{1/2}.$$

Here by the square root of a real number a we mean $a^{1/2} \geqslant 0$ if $a \geqslant 0$ and $\sqrt{-1}\,|a|^{1/2}$ if $a < 0$.

Recall the notion of admissible 6-tuple (see Sect. 1.a). For any admissible 6-tuple $(i, j, k, l, m, n) \in \mathbf{I}^6$ one defines the Racah-Wigner $6j$-symbol which is computed as follows:

$$\begin{Bmatrix} i & j & k \\ l & m & n \end{Bmatrix}^{\mathrm{RW}} = \Delta(ijk)\,\Delta(imn)\,\Delta(ljn)\,\Delta(lmk) \sum_z (-1)^z\,[z+1]!$$

$$\times \{[z-i-j-k]!\,[z-i-m-n]!\,[z-l-j-n]!\,[z-l-m-k]!$$

$$[i+j+l+m-z]!\,[i+k+l+n-z]!\,[j+k+m+n-z]!\}^{-1}.$$

Here z runs over non-negative integers such that all expressions in the square brackets are non-negative.

Set

$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} = \sqrt{-1}^{\,-2(i+j+k+l+m+n)} \begin{Bmatrix} i & j & k \\ l & m & n \end{Bmatrix}^{\mathrm{RW}}.$$

This complex number is either real or purely imaginary. The identities (1.a.1) are straightforward.

For $j \in \mathbf{I}$ put

$$q_j = \exp(\pi\sqrt{-1}\,(j - j(j+1)\,hr^{-1})),$$

$$w_j = \sqrt{-1}^{\,2j}\,[2j+1]^{1/2},$$

$$w = \sqrt{2r}/|t^2 - t^{-2}| \quad \text{or} \quad w = -\sqrt{2r}/|t^2 - t^{-2}|.$$

This completes the description of the quantum initial data.

Theorem **2.1.** — *The initial data described above are irreducible and satisfy Conditions I-IV.*

Proof. — Irreducibility is obvious: for any $j, k \in \mathrm{I}$ the triple $(j, |j-k|, k)$ is admissible. Conditions I-III have been verified in [TV], using the results of [KR]. The formula (1.b.5) follows from the formula 6.17 of [KR]:

$$\textbf{(2.a.1)} \qquad \sum_{j_{13}} (-1)^{j_{13}} q^{-c_{j_{13}}/2} \begin{Bmatrix} j_3 & j_1 & j_{13} \\ j_2 & j & j_{12} \end{Bmatrix} \begin{Bmatrix} j_2 & j_3 & j_{23} \\ j_1 & j & j_{13} \end{Bmatrix}$$
$$= (-1)^{j+j_1+j_2+j_3-j_{12}-j_{23}} q^{(c_{j_{23}}+c_{j_{12}}-c_j-c_{j_1}-c_{j_2}-c_{j_3})/2} \begin{Bmatrix} j_3 & j_2 & j_{23} \\ j_1 & j & j_{12} \end{Bmatrix}.$$

Here $q = t^4$, $c_i = i(i+1)$ so that

$$(-1)^i q^{-c_i/2} = (-1)^i t^{-2i(i+1)} = q_i,$$

and

$$\begin{Bmatrix} i & j & k \\ l & m & n \end{Bmatrix} = w_k w_n (-1)^{l+m+k-i-j-n} \begin{Bmatrix} i & j & k \\ l & m & n \end{Bmatrix}^{\mathrm{RW}} = w_k w_n \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}.$$

Substituting these formulas in (2.a.1) we get (1.b.5).

2. b. *Remarks.* — It is easy to see that for the quantum initial data the formula (1.c.1) may be obtained from (1.b.5) by the complex conjugation. Similarly, the formulas (1.c.2, 1.c.5) with $\varepsilon = -1$ are conjugated to the same formulas with $\varepsilon = 1$.

Restricting the quantum initial data to $\mathrm{I} \cap \mathbf{Z}$ one gets another example of initial data satisfying Conditions I-IV.

PART II

THREE-DIMENSIONAL STATE SUM MODELS FOR LINK AND GRAPH INVARIANTS

Throughout Part II we will use initial data $(K, I, w, \dots)$ as defined in Section 1.a. We will always assume that the data satisfy Conditions I-IV of Section 1.b.

3. Summary of the results of [TV]

3.a. *Colorings of manifolds.* — Let M be a triangulated manifold of dimension $\geqslant 2$. By a *coloring* (or I-coloring) of M we mean a function which assigns to each edge of M an element of I (the color) such that for every 2-simplex of M the colors of its three edges form an admissible triple. Denote the set of colorings of M by col(M). This set, of course, strongly depends on the triangulation of M. For compact M this set is finite.

3.b. *An invariant of colored 3-manifolds.* — Let M be a compact triangulated 3-manifold and let $\mu \in \operatorname{col}(M)$. With each 3-simplex T of (the triangulation of) M we associate an element $|T|_\mu$ of the ground ring K as follows. If i, j, k are μ-colors of the edges of a 2-face of T and l, m, n are μ-colors of the opposite edges of T then

$$|T|_\mu = \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} \in K.$$

The validity of this definition follows from the symmetry relations (1.a.2). One defines an invariant $|M|_\mu \in K$ of the colored 3-manifold (M, μ) by the formula

$$|M|_\mu = w^{-2\alpha+\beta} \prod_e w^2_{\mu(e)} \prod_{e'} w_{\mu(e')} \prod_T |T|_\mu,$$

where: α is the number of vertices of M, β is the number of vertices of ∂M, e runs over edges of M which do not lie on ∂M, e' runs over edges of ∂M, and T runs over all 3-simplices of M. (Note that the triangulation of ∂M in question is the one induced by the triangulation of M.)

In § 5 we will need a more general invariant $|M, F|_\mu \in K$ where F is a certain union of components of ∂M. We put

$$|M, F|_\mu = w^{-2\alpha+\beta'} \prod_e w^2_{\mu(e)} \prod_{e''} w_{\mu(e'')} \prod_T |T|_\mu,$$

where: α is the number of vertices of M, β' is the number of vertices of $\partial M \backslash F$, e runs over edges of M not lying on ∂M, e'' runs over edges of $\partial M \backslash F$, and T runs over all 3-simplicies of M.

3.c. *A combinatorial invariant of 3-manifolds.* — For simplicity I first state the result of [TV] pertaining to closed 3-manifolds.

Theorem **3.1** *(theorem* 1.3.A *of* [TV]*). — Let* M *be a closed 3-manifold. Provide* M *with a triangulation and put*

$$| \mathrm{M} | = \sum_{\mu \in \mathrm{col(M)}} | \mathrm{M} |_\mu \in \mathrm{K}.$$

Then $|\mathrm{M}|$ *does not depend on the choice of the triangulation of* M.

Thus $|\mathrm{M}|$ is a topological invariant of M. (Note that every 3-manifold may be triangulated and any two triangulations of a 3-manifold are combinatorially equivalent up to ambient isotopy.)

The following theorem generalizes Theorem 3.1 to the case of compact 3-manifolds.

Theorem **3.2** *(theorem* 1.4.A *of* [TV]*). — Let* M *be a compact 3-manifold with triangulated boundary. Let* $\lambda \in \mathrm{col}(\partial \mathrm{M})$. *Extend the triangulation of* ∂M *to a triangulation of* M *(this is always possible) and put*

$$\langle \mathrm{M} \mid \lambda \rangle = \sum_{\substack{\mu \in \mathrm{col(M)} \\ \mu |_{\partial \mathrm{M}} = \lambda}} | \mathrm{M} |_\mu \in \mathrm{K}.$$

Then $\langle \mathrm{M} \mid \lambda \rangle$ *does not depend on the extension of the triangulation of* ∂M *to* M.

Thus $\langle \mathrm{M} \mid \lambda \rangle$ is a topological invariant of the triple (M, the triangulation of ∂M, λ). Note that $\langle \mathrm{M} \mid \lambda \rangle$ is defined for non-oriented (and even non-orientable) M.

4. Fat graphs

4.a. *Graphs and fat graphs.* — By a graph we mean a finite graph with unoriented edges and, possibly, with loops. A graph will be called *3-valent* if each vertex of the graph is incident either to 3 or to 2 edges of the graph counted with multiplicities. A *coloring* (or I-coloring) of a 3-valent graph φ is a function which associates with each edge of φ an element of the set I (the color) such that for any vertex of φ incident to 3 (resp. 2) edges of φ the colors of these edges form an admissible triple (resp. are equal to each other).

By an (abstract) *fat graph* we mean a finite collection of disjoint 2-disks and disjoint bands which meet the disks along their bases lying on the boundaries of the disks (and otherwise do not meet the disks). Examples of fat graphs are given in Fig. 1, 2.

Each fat graph Γ has a *core* $c(\Gamma)$ which is an ordinary graph. The vertices of $c(\Gamma)$ are the centers of the 2-disks of Γ, and the edges of $c(\Gamma)$ are the cores of the bands of Γ extended slightly to meet the vertices. The fat graph Γ is called 3-valent if the graph $c(\Gamma)$ is 3-valent, i.e. if each 2-disk of Γ is incident to either 2 or 3 bands counted with multiplicities. By abuse of language the vertices, edges, and colorings of $c(\Gamma)$ will be called resp. vertices, edges, and colorings of Γ. The set of all colorings of Γ will be denoted by $\mathrm{col}(\Gamma)$. A colored 3-valent fat graph is a 3-valent fat graph provided with a coloring.

The union of disks and bands of a fat graph Γ is, obviously, a compact surface with boundary, called the surface of Γ. A fat graph with oriented surface will be called an oriented fat graph. (These orientations of fat graphs should not be confused with orientations of the edges of the graphs.)

4.b. *Fat graphs and links in 3-manifolds.* — By a fat graph in a 3-manifold N we mean a fat graph smoothly embedded in N. Two fat graphs in N are isotopic if they may be smoothly deformed into each other in the class of fat graphs in N. (Its is understood that the splitting into 2-discs and bands is preserved in the course of deformation.) By an isotopy of colored fat graphs we mean a color-preserving isotopy.

In the context of state model topology the colored 3-valent fat graphs in 3-manifolds present a convenient and natural generalization of colored framed links in 3-manifolds. Recall that a link in a 3-manifold N is a finite collection of disjoint (unoriented) circles smoothly embedded in N. A coloring of a link L is a function associating with each component of L an element of the set I. A framed link in N is a link $L \subset N$ provided with a non-singular normal vector field (a framing) on L. The notion of isotopy readily extends to colored framed links in N.

Every m-component framed link L in a 3-manifold N gives rise to a fat graph $\Gamma_L \subset N$ consisting of m annuli and Möbius bands which contain the components of L as their cores and which are orthogonal to the framing. (If N is orientable, then Γ_L consists of annuli.) Each of the annuli and Möbius bands of Γ_L is treated as a union of a 2-disk and a band. Thus, Γ_L has m vertices and m edges, which all are loops.

Each coloring of L induces a coloring of Γ_L: the color of an edge of Γ_L is defined to be equal to the color of the corresponding component of L. It is easy to see that the formula $L \mapsto \Gamma_L$ yields an injective mapping of the set of isotopy types of colored framed links in N into the set of isotopy types of colored fat graphs in N. Using this mapping one may view colored framed links in N as (rather special) colored fat graphs in N.

4.c. *Graphs in cylinders and their diagrams.* — Let F be a surface. Fat graphs in the cylinder $F \times [-1, 1]$ may be presented by graph diagrams on F in the way similar to the usual presentation of framed links in $\mathbf{R}^3$ by plane link diagrams. By a *graph diagram on* F we mean a graph immersed in F with only double transversal crossings (of edges), each crossing being provided with the following additional structure: one of the two edges traversing the crossing is cut out at the crossing and considered to be the locally lower edge (undercrossing), the second edge being considered as the locally upper one (overcrossing). (Of course, two branches of a graph diagram traversing a crossing may actually lie on the same edge.) Each graph diagram D on F determines a fat graph $\Gamma(D)$ in $F \times [-1, 1]$: the disks of $\Gamma(D)$ are small disk neighborhoods of the vertices of D in $F = F \times 0$, the bands of $\Gamma(D)$ are narrow neighborhoods of the edges of D where at each crossing of the diagram the band corresponding to the over-crossing is slightly pushed into $F \times (0, 1]$.

It is easy to show that two graph diagrams on F present isotopic fat graphs in $F \times [0, 1]$ if and only if they may be obtained from each other by the local Reidemeister type moves

(4.c.1) $\Omega_0, \Omega_2, \Omega_3, \omega_1, \omega_2, \omega_{-1}, \omega_{-2}$

and their inverses. The first five moves are pictured in Fig. 3. The symbols $j_1, j_2, \ldots, a, b, c, \ldots$ which appear in Fig. 3 should be ignored at the moment, they

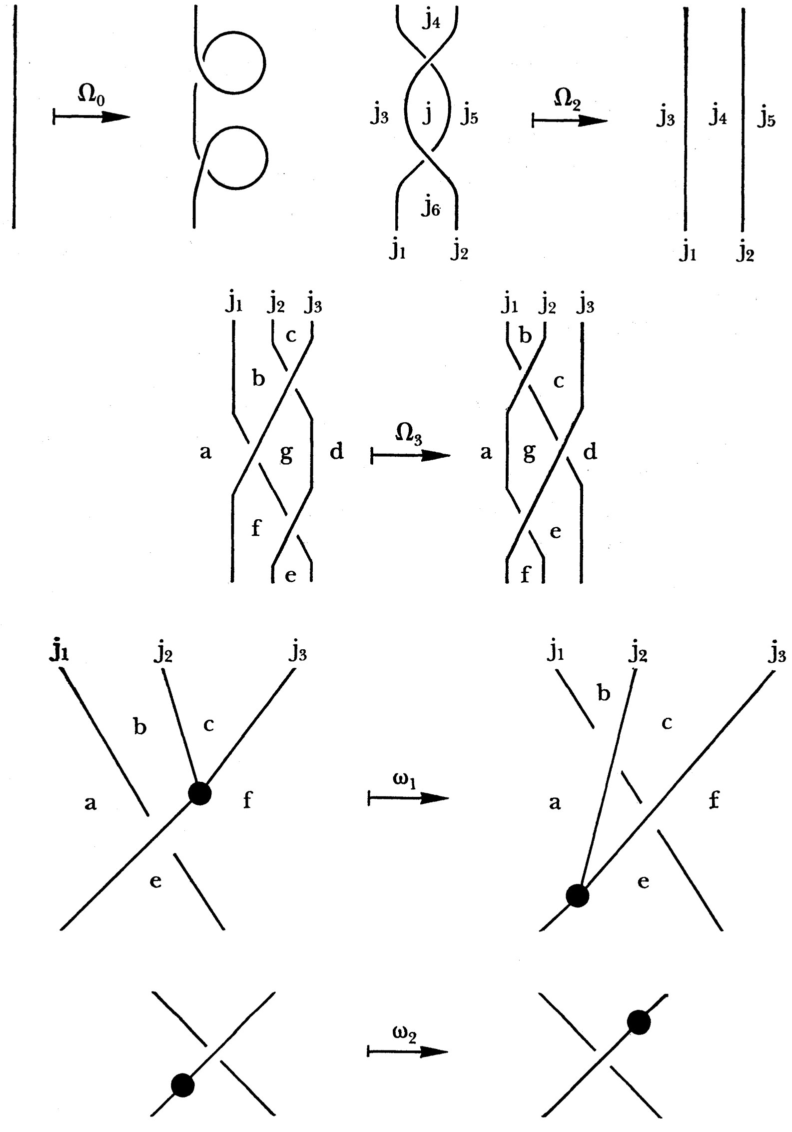

Fig. 3

will be used in Section 4.e. The pictures of ω_{-1}, ω_{-2} are obtained from those of ω_1, ω_2 by the mirror reflection with respect to the plane of the page.

If the surface F is oriented then the orientation of F induces an orientation in the surface of the fat graph presented by any graph diagram on F. It is easy to see that each oriented fat graph in $F \times [-1, 1]$ may be presented in this way by a graph diagram on F. In particular, oriented 3-valent fat graphs in $F \times [-1, 1]$ are presented by 3-valent graph diagrams on F.

One easily transfers the results of this subsection into the setting of colored fat graphs and their colored diagrams. Note that the color of a branch of a colored graph diagram is preserved when this branch traverses a crossing point of the diagram or a vertex of valence 2.

4.d. *A state model associated with graphs in a cylinder.* — Let F be an oriented compact surface. Let φ and ψ be two colored 3-valent graphs embedded in F. (Possibly, φ and ψ intersect each other.) Let Γ be an oriented colored 3-valent fat graph in $F \times [-1, 1]$. We present here a state sum model which produces an element $\langle \varphi \mid \Gamma \mid \psi \rangle$ of the ground ring K. The main properties of this construction are given below in Theorems 4.1 and 4.2.

Let D be a colored graph diagram of Γ on F. Deforming φ by an ambient isotopy of F and deforming ψ by another ambient isotopy of F we may assume that φ, ψ, D lie in general position so that all crossings of $\varphi \cup \psi \cup D$ are double transversal crossings of edges. We form a graph diagram from $\varphi \cup \psi \cup D$ assuming that φ lies everywhere over $\psi \cup D$, and ψ lies under $\varphi \cup D$. Denote the resulting 3-valent graph diagram on F by σ. Equip σ with the coloring induced by the (given) colorings of φ, ψ, and D.

Denote by Σ the graph in F obtained from σ by forgetting the over/under-crossing information. In other words, Σ is the union of $\varphi \cup \psi$ with the vertices and underlying edges of D. The set of vertices of Σ may be split into five disjoint subsets: (i) the 2-valent vertices of φ, ψ, D; (ii) the 3-valent vertices of φ, ψ, D; (iii) crossings of φ with ψ; (iv) crossings of D with φ or ψ; (v) self-crossings of D. The vertices of Σ of types (iii-v) are 4-valent. Note that each edge e of Σ is contained in an edge of either φ, or ψ, or D. By the color of e we will mean the (given) color of this latter edge.

By a *region* of D (with respect to φ and ψ) we will mean a connected component of $F \backslash \Sigma$. By an *area-coloring* of D (with respect to φ and ψ) we mean an arbitrary mapping from the set of regions of D into the set I. An area-coloring η of D is called *admissible* if for each edge e of the graph Σ the color of e together with the η-colors of the 2 regions of D adjacent to e form an admissible triple. Denote the set of admissible area-colorings of D by $\mathrm{adm}(D, \varphi, \psi)$ or, briefly, by $\mathrm{adm}(D)$.

With each $\eta \in \mathrm{adm}(D)$ we will associate an element $| D |_\eta$ of the ring K. First, we associate certain elements of K with vertices of Σ and regions of D. For a region y of D put

$$(4.d.1) \qquad | y |_\eta = w_{\eta(y)}^{2\chi(y)}$$

where $\eta(y)$ and $\chi(y)$ are respectively the η-color and the Euler characteristic of y.

Let a be a vertex of Σ. We distinguish five cases in accordance with the five possible types (i-v) of the vertex a. In case (i) put $|\,a\,|_\eta = 1$. In case (ii), denote by i, j, k the colors of the three edges of Σ incident to a. Let l, m, n be the η-colors of the opposite regions (see Fig. 4). Put

$$(4.d.2) \qquad |\,a\,|_\eta = \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}.$$

In cases (iii-v), a is a crossing of two branches of either φ, or ψ, or D. Let l be the color of the upper branch and i be the color of the lower branch. Let j, k, m, n be the η-colors of the 4 regions of D incident to a: see Fig. 5 where the given orientation of F is the counterclockwise one. In the case (iii) we define $|\,a\,|_\eta$ by the formula (4.d.2). In the case (iv) put

$$|\,a\,|_\eta = q_k^{1/2}\, q_n^{1/2}\, q_j^{-1/2}\, q_m^{-1/2} \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}.$$

Here the symbols $q_k^{1/2}$, $q_k^{-1/2}$ should be understood as formal expressions whose product is equal to 1 and whose squares are equal respectively to q_k and q_k^{-1}. In the final formula for the invariant, only integer powers of q_k actually appear.

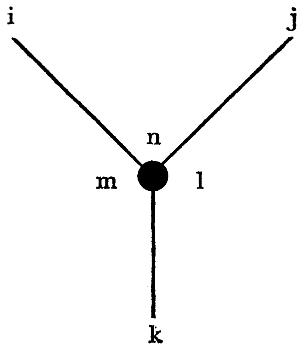

Fig. 4

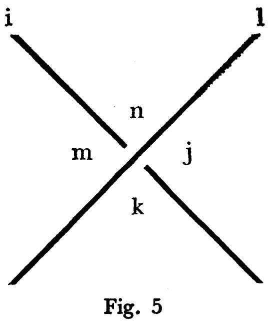

Fig. 5

In the case (v) put

$$| a |_\eta = q_k\, q_n\, q_j^{-1}\, q_m^{-1} \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}.$$

Finally, put

$$\langle \varphi \mid D \mid \psi \rangle_\eta = \prod_y | y |_\eta \cdot \prod_a | a |_\eta$$

where y runs over all regions of D with respect to φ, ψ, and a runs over all vertices of Σ. Put

$$\langle \varphi \mid \Gamma \mid \psi \rangle = \sum_{\eta \in \mathrm{adm(D)}} \langle \varphi \mid D \mid \psi \rangle_\eta \in K.$$

Theorem **4.1.** — *For any* φ, ψ, Γ *as above* $\langle \varphi \mid \Gamma \mid \psi \rangle$ *is invariant under ambient isotopies of* φ *and* ψ *in* F *and under isotopies of* Γ *in* $F \times [-1, 1]$.

Theorem 4.1 provides a large stock of isotopy invariants of oriented colored 3-valent fat graphs in $F \times [-1, 1]$. These invariants are nontrivial even when φ and ψ are empty graphs: $\varphi = \psi = \varnothing$. For example, when F is the 2-disc, the invariant $\langle \varnothing \mid \Gamma \mid \varnothing \rangle$ generalizes the Jones polynomial of links in $\mathbf{R}^3$ to the graphs in $\mathbf{R}^3$.

Remark that if F is a disjoint union of n surfaces $F_1, \ldots, F_n$ then

$$\langle \varphi \mid \Gamma \mid \psi \rangle = \prod_{k=1}^{n} \langle \varphi_k \mid \Gamma_k \mid \psi_k \rangle$$

where φ_k, ψ_k are the parts of φ, ψ lying on F_k, and Γ_k is the part of Γ lying in $F_k \times [-1, 1]$.

It is important to trace the behavior of the invariant $\langle \varphi \mid \Gamma \mid \psi \rangle$ under the local operations $\tau_1 - \tau_4$ on Γ shown in Fig. 6. It is understood that the product orientation in $F \times [-1, 1]$ corresponds to the right-hand orientation of the ambient space $\mathbf{R}^3$ in Fig. 6. The symbols j, j_2, j_3, j_{23} denote the colors of the corresponding bands. Note that τ_1 is just the positive twist of the band around its core. The operations $\tau_1, \ldots, \tau_4$ are invertible up to isotopy.

Theorem **4.2.** — *If the oriented colored fat graphs* $\Gamma_1, \ldots, \Gamma_4$ *are obtained from* Γ *respectively by* $\tau_1, \ldots, \tau_4$ *as in Fig. 6, then*

(4.d.3) $\quad \langle \varphi \mid \Gamma_1 \mid \psi \rangle = q_j^{-2} \langle \varphi \mid \Gamma \mid \psi \rangle,$

(4.d.4) $\quad \langle \varphi \mid \Gamma_2 \mid \psi \rangle = q_j^{-2} \langle \varphi \mid \Gamma \mid \psi \rangle,$

(4.d.5) $\quad \langle \varphi \mid \Gamma_3 \mid \psi \rangle = (q_{j_2}\, q_{j_3}\, q_{j_{23}})^{-1} \langle \varphi \mid \Gamma \mid \psi \rangle,$

(4.d.6) $\quad \langle \varphi \mid \Gamma_4 \mid \psi \rangle = q_{j_2}\, q_{j_3}\, q_{j_{23}}^{-1} \langle \varphi \mid \Gamma \mid \psi \rangle.$

Note that the formulas (4.d.4), (4.d.5) follow from the formulas (4.d.3), (4.d.6) and the isotopy invariance of $\langle \varphi \mid \Gamma \mid \psi \rangle$. Indeed, τ_2 is a composition of τ_1 and an isotopy; τ_3 is a composition of τ_4, two τ_1 (applied to the upper tails) and an isotopy.

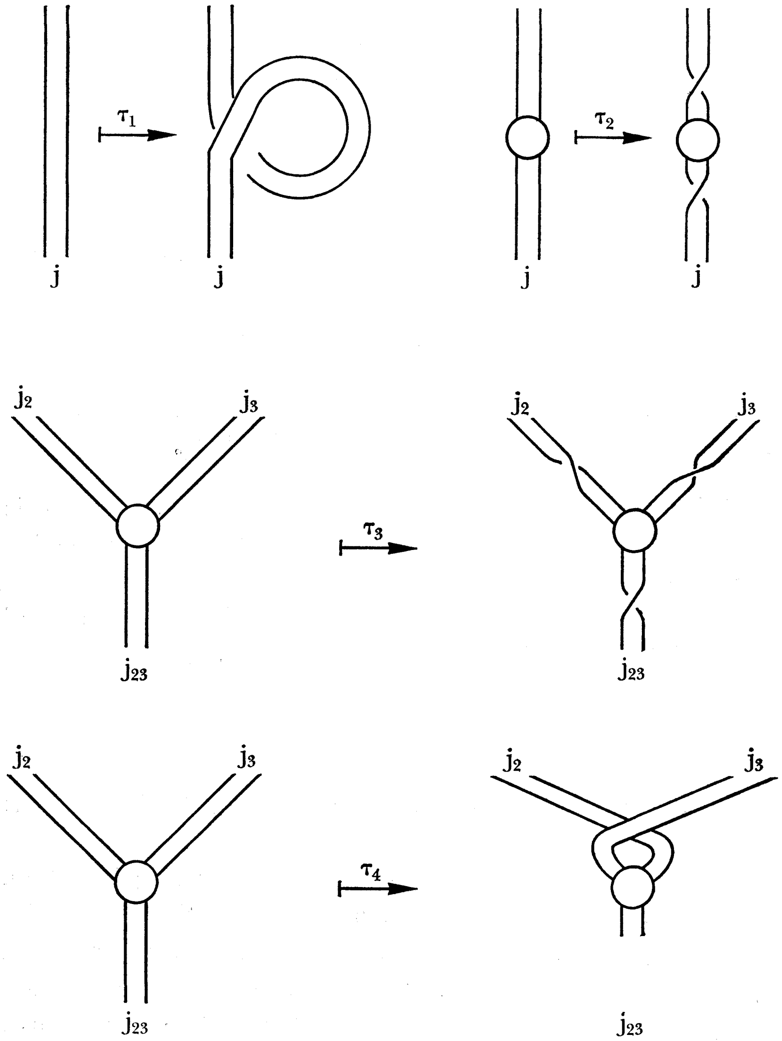

Fig. 6

Corollary **4.3.** — *The element* $\langle \varphi \mid \Gamma \mid \psi \rangle$ *does not depend on the choice of orientation in the surface of* Γ.

Proof. — Let Γ' be a connected component of Γ. Applying τ_2, τ_3 to all 2-disks of Γ' and τ_1^{-1} to all bands of Γ' we get from Γ just the same colored fat graph but with the opposite orientation in the surface of Γ' (and the same orientation in other components). Since each band of Γ' is incident to two (possibly coinciding) 2-disks of Γ', the formulas (4.d.3 - 4.d.5) imply that these operations do not change $\langle \varphi \mid \Gamma \mid \psi \rangle$. This yields our claim.

4.e. *Proof of Theorems* 4.1 *and* 4.2. — Let us first prove (4.d.3). One positive (resp. negative) twist of a band of Γ amounts to applying the Reidemeister move Ω_1

(resp. Ω_{-1}) to the corresponding edge of a graph diagram of Γ on F (see Fig. 7). It is easy to see that the equality (1.c.5) with $\varepsilon = -1$ (resp. $\varepsilon = 1$) implies that this twist of a band of color i leads to multiplication of $\langle \varphi \mid \Gamma \mid \psi \rangle$ by q_i^{-2} (resp. q_i^2). A convenient notation for the colors of regions establishing a correspondence with (1.c.5) is specified in Fig. 7.

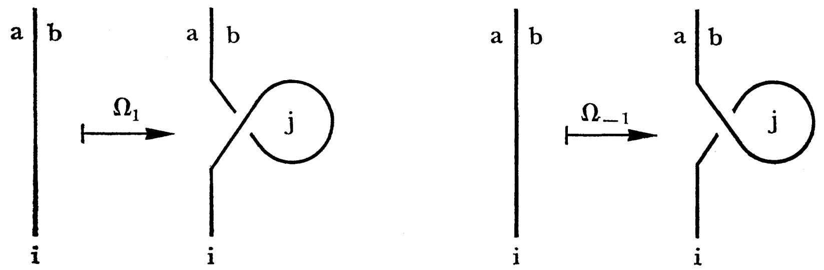

Fig. 7

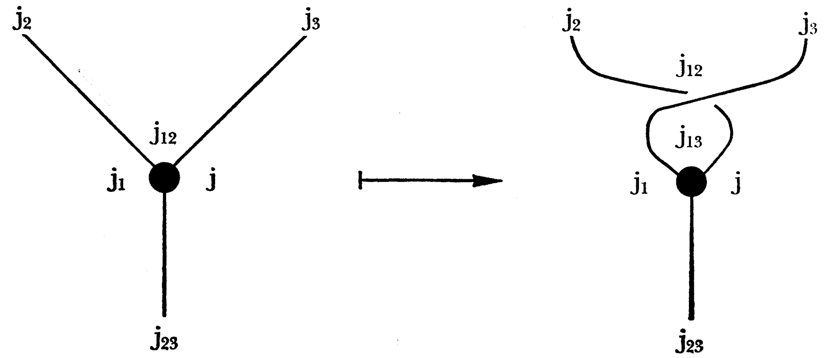

Fig. 8

Similarly, the formula (4.d.6) follows from (1.b.5). For notation establishing the correspondence between (1.b.5) and (4.d.6) see Fig. 8. As it is explained after the statement of Theorem 4.2, the formulas (4.d.3, 4.d.6) imply (4.d.4, 4.d.5) modulo Theorem 4.1.

To prove Theorem 4.1, present Γ by a colored graph diagram D on F, and form a colored graph diagram σ on F as in Sect. 4.d. It is easy to trace the effect of isotopies of φ, ψ, Γ on the diagram σ. Namely, σ is changed by the Reidemeister moves (4.c.1) and their inverses. Here Ω_0 may be applied only to those edges of σ which correspond to bands of Γ. Since Ω_0 is a composition of Ω_1 and Ω_{-1} the results of the first paragraph of this subsection imply invariance of $\langle \varphi \mid \Gamma \mid \psi \rangle$ under Ω_0. Invariance under Ω_2 and ω_1, ω_{-1} directly follows from Conditions I and II respectively. For notation establishing these implications see Fig. 3. Invariance of $\langle \varphi \mid \Gamma \mid \psi \rangle$ under ω_2, ω_{-2} is straightforward.

Let us prove invariance of $\langle \varphi \mid \Gamma \mid \psi \rangle$ under Ω_3. Among the three branches of σ changed by Ω_3 one branch is the higher one, another branch is the lower one and the third branch lies on the medium level. In our setting the higher branch may lie on φ or D, the medium branch always lies on D, and the lower branch may lie on ψ or D. This gives us four cases which should be considered separately. It is straightforward to see that in all these cases the equality (1.c.2) with $\varepsilon = 1$ ensures the invariance. (For notation establishing the correspondence between (1.c.2) and invariance under Ω_3 see Fig. 3). This completes the proof of Theorems 4.1 and 4.2.

4.f. *Triangulated surfaces and graphs.* — Let F be a closed triangulated surface. The triangulation of F canonically gives rise to a graph γ_F in F dual to the 1-skeleton of the triangulation. More precisely, the vertices of γ_F are the barycenters of the 2-simplices of F. Each edge e of F gives rise to a " dual " edge e^* of γ_F which crosses e once and connects the barycenters of the two 2-simplices of F adjacent to e. Clearly, γ_F is a 3-valent graph embedded in F (see Fig. 9 where γ_F is drawn boldface).

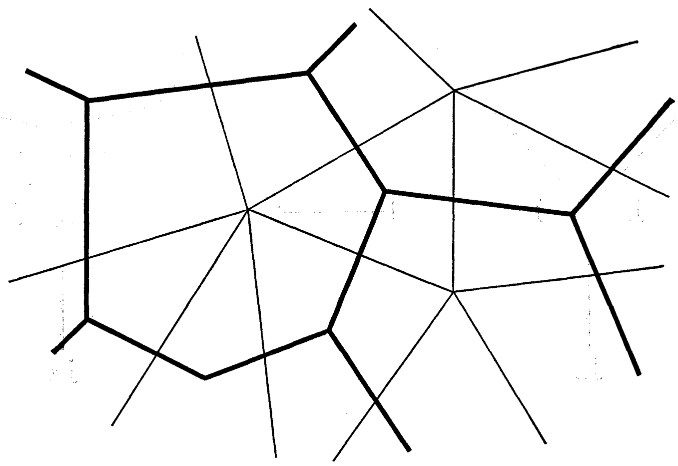

Fig. 9

Each coloring λ of F induces a coloring of γ_F which associates with the edge e^* the color $\lambda(e) \in I$. We will denote the resulting colored graph in F by γ_F^λ.

5. State sum invariants of graphs in 3-manifolds

Throughout Section 5 the symbol N denotes a compact 3-manifold with triangulated boundary. The symbol Γ denotes an I-colored 3-valent fat graph lying in Int N. We assume that a neighborhood of Γ in N is oriented. (The manifold N may be unoriented and even non-orientable.)

5.a. *The case of orientable* Γ. — In this subsection we assume that the surface of the fat graph Γ is orientable. Orient this surface in an arbitrary way.

Put $F = \partial U$ where U is an oriented closed regular neighborhood of Γ in N. Provide F with the orientation induced by that of U (so that the normal vector field on F directed outwards U together with the orientation of F determine the given orientation of U). Consider a non-singular normal vector field on the surface of Γ which together with the fixed orientation of this surface determines the orientation of U. Shifting Γ along this vector field we get a parallel copy Γ' of Γ lying on F.

Remark that U is a handlebody consisting of 3-balls which are regular neighborhoods of the 2-discs of Γ and of solid cylinders which are regular neighborhoods of the bands of Γ. Choose in each of these cylinders a meridianal disk transversal to the corresponding band of Γ. Let $\psi_1, \ldots, \psi_m$ be the boundaries of these discs (see Fig. 10). Here m is the number of edges of Γ, and $\psi_1, \ldots, \psi_m$ are simple disjoint loops on the surface F. We will treat each ψ_i as a graph with one (arbitrarily chosen) vertex and one edge. For a sequence $J = (j_1, \ldots, j_m) \in I^m$ we denote by ψ_J the colored graph $\psi = \psi_1 \cup \ldots \cup \psi_m$ in F whose edges lying on $\psi_1, \ldots, \psi_m$ are colored respectively with $j_1, \ldots, j_m$. Put $w_J = \prod_{k=1}^{m} w_{j_k}^2$.

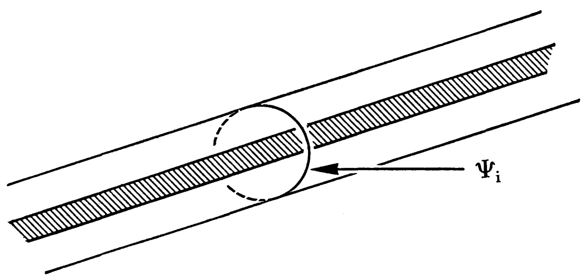

Fig. 10

Let M be the compact 3-manifold $N \backslash \operatorname{Int} U$ bounded by $\partial M = F \cup \partial N$. Provide M with an arbitrary triangulation which extends the given triangulation of ∂N. Provide F with the induced triangulation. Denote by s the number of vertices of Γ.

For each $\lambda \in \operatorname{col}(\partial N)$ we define a " relative " invariant of the pair N, Γ with respect to λ:

$$\langle N, \Gamma \mid \lambda \rangle = w^{2-2s} \sum_{\substack{\mu \in \operatorname{col}(M),\, \mu|_{\partial N} = \lambda \\ J \in I^m}} w_J . \mid M, F \mid_\mu . \langle \gamma_F^{\mu_F} \mid \Gamma' \mid \psi_J \rangle.$$

Here γ_F is the dual graph of the 1-skeleton of the triangulation of F, and $\mu_F = \mu|_F$ (cf. Sect. 4.f). The colored fat graph $\Gamma' \subset F$ is embedded into $F \times [-1, 1]$ via the canonical identification $F = F \times 0$.

Theorem **5.1.** — $\langle N, \Gamma \mid \lambda \rangle$ *does not depend on the choice of triangulation in* M *extending the given triangulation of* ∂N.

This Theorem is proven in Sect. 5.f using the results of Sect. 5.e.

The factor w^{2-2s} in the definition of $\langle N, \Gamma \mid \lambda \rangle$ plays the role of a normalizing scalar: it ensures exact correspondence with the Jones polynomial (see Theorem 6.1 below) and also guarantees invariance of our state sum under the operations of exclusion of a redundant 2-valent vertex of a colored fat graph (see Fig. 11; this invariance easily follows from Theorem 9.2 of Section 9).

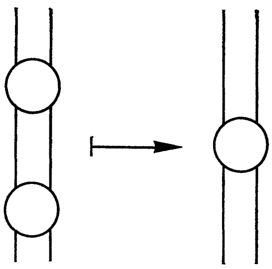

Fig. 11

Theorem 5.1 implies that $\langle N, \Gamma \mid \lambda \rangle$ is a homeomorphism invariant of the triple (N, Γ, λ) and, in particular, an ambient isotopy invariant of Γ. It is understood that the homeomorphisms and isotopies preserve the 2-disks, the bands, and the coloring of Γ and also preserve the orientation of a regular neighborhood of Γ in N and of the surface of Γ (the latter condition is in fact redundant; see Corollary 5.3).

In the case $\partial N = \varnothing$ we get an invariant of the pair (N, Γ) corresponding to the only coloring of the empty surface. (We accept the convention that there is exactly one mapping of the empty set into I.) This invariant is defined by the same formula as above where μ runs over all colorings of M. This invariant of the pair (N, Γ) with $\partial N = \varnothing$ will be denoted by $\langle N, \Gamma \rangle$.

The invariant $\langle N, \Gamma \mid \lambda \rangle$ behaves nicely under the operations $\tau_1, \ldots, \tau_4$ on Γ shown in Fig. 6. It is understood that the fixed orientation in a regular neighborhood of Γ corresponds to the right-hand orientation of the ambient space $\mathbf{R}^3$ in Fig. 6. Note that the orientation of (the surface of) Γ induces in the obvious way orientations of $\Gamma_1, \ldots, \Gamma_4$.

Theorem **5.2.** — *Let* $\lambda \in \operatorname{col}(\partial N)$. *If the oriented colored fat graphs* $\Gamma_1, \ldots. \Gamma_4$ *are obtained from* Γ *respectively by* $\tau_1, \ldots, \tau_4$ *as in Fig. 6, then*

(5.a.1) $\langle N, \Gamma_1 \mid \lambda \rangle = q_j^{-2} \langle N, \Gamma \mid \lambda \rangle,$

(5.a.2) $\langle N, \Gamma_2 \mid \lambda \rangle = q_j^{-2} \langle N, \Gamma \mid \lambda \rangle,$

(5.a.3) $\langle N, \Gamma_3 \mid \lambda \rangle = (q_{j_2} q_{j_3} q_{j_{23}})^{-1} \langle N, \Gamma \mid \lambda \rangle,$

(5.a.4) $\langle N, \Gamma_4 \mid \lambda \rangle = q_{j_2} q_{j_3} q_{j_{23}}^{-1} \langle N, \Gamma \mid \lambda \rangle.$

This theorem will be proven in Section 9. The same argument as in Section 4 shows that the formulas (5.a.1), (5.a.4) imply the formulas (5.a.2), (5.a.3).

The next assertion eliminates dependence of our invariants on the choice of orientations in the surfaces of fat graphs.

Corollary **5.3.** — *For each* $\lambda \in \mathrm{col}(\partial \mathrm{N})$ *the element* $\langle \mathrm{N}, \Gamma \mid \lambda \rangle$ *of* K *does not depend on the choice of orientation in the surface of* Γ.

The proof of this corollary follows the proof of Corollary 4.3 with the obvious changes.

5.b. *The case of non-orientable* Γ. — Assume that the surface of Γ is non-orientable. Let $e_1, \ldots, e_m$ be the bands of Γ. For each sequence of integers $k_1, \ldots, k_m$ denote by $\Gamma(k_1, \ldots, k_m)$ the colored fat graph in N obtained from Γ by applying k_i positive half-twists to e_i for all $i = 1, \ldots, m$. A picture of a positive half-twist is given in Fig. 6 where τ_3 is the composition of three such half-twists.

Let $n_1, \ldots, n_m$ be the colors of $e_1, \ldots, e_m$. Choose integers $k_1, \ldots, k_m$ so that the fat graph $\Gamma(k_1, \ldots, k_m)$ is orientable. Thus, we may consider the invariant $\langle \mathrm{N}, \Gamma(k_1, \ldots, k_m) \mid \lambda \rangle$ with $\lambda \in \mathrm{col}(\partial \mathrm{N})$. It is easy to deduce from the formula (5.a.1) that the product

$$\prod_{i=1}^{m} q_{n_i}^{k_i} \langle \mathrm{N}, \Gamma(k_1, \ldots, k_m) \mid \lambda \rangle$$

does not depend on the choice of $k_1, \ldots, k_m$. We define $\langle \mathrm{N}, \Gamma \mid \lambda \rangle$ to be the common value of such products. It follows from the results of Sect. 5.a that $\langle \mathrm{N}, \Gamma \mid \lambda \rangle$ is a topological invariant of the triple $(\mathrm{N}, \Gamma, \lambda)$ and, in particular, an ambient isotopy invariant of Γ. It is understood that the homeomorphisms and isotopies preserve the 2-disks, the bands, and the coloring of Γ as well as the orientation in a regular neighborhood of Γ in N.

The invariant $\langle \mathrm{N}, \Gamma \rangle$ defined when $\partial \mathrm{N} = \varnothing$ and Theorem 5.2 immediately extend to non-orientable fat graphs.

5.c. *Multiplicativity of the invariant.* — Let G be a closed triangulated surface which lies in $\mathrm{Int}\, \mathrm{N} \backslash \Gamma$ and splits N into two compact 3-manifolds N_1 and N_2 so that $\mathrm{N}_1 \cap \mathrm{N}_2 = \mathrm{G}$, $\mathrm{N}_1 \cup \mathrm{N}_2 = \mathrm{N}$. Let Γ_i be the part of Γ lying in N_i, $i = 1, 2$. Let λ be a coloring of $\partial \mathrm{N}$ and let λ_i be the reduction of λ to $\partial \mathrm{N}_i \backslash \mathrm{G}$. It follows directly from definitions that

$$\langle \mathrm{N}, \Gamma \mid \lambda \rangle = w^{-2} \sum_{\nu \in \mathrm{col(G)}} \langle \mathrm{N}, \Gamma_1 \mid \lambda_1 \amalg \nu \rangle \cdot \langle \mathrm{N}_2, \Gamma_2 \mid \lambda_2 \amalg \nu \rangle.$$

5.d. *Invariants of non-fat graphs.* The formulas (5.a.1)-(5.a.4) show that considered up to multiplication by $\{ q_j \mid j \in \mathrm{I} \}$ the invariants $\langle \mathrm{N}, \Gamma \mid \lambda \rangle$ and $\langle \mathrm{N}, \Gamma \rangle$ are determined by the colored core $c(\Gamma) \subset \mathrm{N}$ of Γ and the orientation of its regular neighborhood. This observation produces non-trivial invariants of colored 3-valent (non-fat) graphs in 3-manifolds. A similar observation applies to the invariants $\langle \varphi \mid \Gamma \mid \psi \rangle$ considered in Section 4 and to more general invariants $\langle \mathrm{N}, \Gamma \mid \varphi \rangle$ introduced in Section 5.g.

5.e. *Alexander transformations and their dualization.* — To prove Theorem 5.1 we will use the Alexander transformations (moves) of polyhedra (stellar subdivisions) and their description in the language of dual cell subdivisions. For simplicity we restrict ourselves to combinatorial manifolds.

Let X be a triangulated combinatorial manifold of dimension $n \geqslant 1$ (possibly with boundary). The Alexander transformation along a simplex e of X changes the triangulation of X inside the star $\mathrm{St}(e)$ of e. (Recall that $\mathrm{St}(e)$ is the union of simplices of X containing e; by simplices we mean closed simplices.) The transformation goes by subdividing the ball $\mathrm{St}(e)$ as a cone over its boundary with the cone point (which is the only new vertex) inside e. For example, if $\dim(e) = 1$, then this transformation splits each simplex containing e into two halves and does not change the simplices which do not contain e. Denote the resulting triangulation by X_e.

Alexander [Al] proved that if X is compact, then any two (combinatorially equivalent) triangulations of X may be related by a finite sequence of Alexander transformations and their inverses. It was remarked in [TV] that if the two triangulations of X coincide on a union $\partial_0 X$ of several components of ∂X, then one may use only those Alexander transformations (and their inverses) which correspond to simplices not lying on $\partial_0(X)$.

Note that if $n = \dim X = 2$, then there are two combinatorial types of Alexander moves along edges (in accordance with the options $e \subset \partial X$ and $e \not\subset \partial X$). If $n \geqslant 3$ then there is an infinite number of combinatorial types of these moves due to the infinite number of triangulations of spheres and balls of dimension $\geqslant 1$. The picture becomes simpler when one passes to the dual cell subdivision X^* of X.

Recall the construction of X^*. With each strictly increasing sequence $A_0 \subset A_1 \subset \ldots \subset A_m$ of simplices of X one associates a m-dimensional linear simplex $[A_0, A_1, \ldots, A_m]$ whose vertices are the barycenters of $A_0, A_1, \ldots, A_m$. Such simplices corresponding to all sequences $A_0 \subset \ldots \subset A_m$ form a subdivision of X, called the first barycentric subdivision and denoted by X^1. For a simpex A of X one denotes by A^* the union of all simplices $[A_0, A_1, \ldots, A_m]$ as above with $A_0 = A$. It is well known that A^* is a closed combinatorial cell of dimension $n - \dim A$. The sphere ∂A^* is the union of cells B^* where B runs over all simplices of X containing A and distinct from A. If $A \subset \partial X$ then one may consider also the dual cell A^*_∂ of A in ∂X. It is obvious that $A^*_\partial = A^* \cap \partial X$. The two sets of cells $\{ A^* \mid A \text{ runs over simplices of } X \}$ and $\{ A^*_\partial \mid A \text{ runs over simplices of } \partial X \}$ form the dual cell subdivision X^* of X. Clearly, the cells $\{ A^*_\partial \}$ form the dual cell subdivision $\partial(X^*) = (\partial X)^*$ of the (triangulated) boundary of X.

It is easy to trace the changes in X^* which accompany the Alexander transformations. For example, if e is an edge of X, then $(X_e)^*$ is obtained from X^* by thickening the $(n-1)$-cell e^*. In other words, e^* is replaced by $e^* \times [-1, 1]$ with the product cell structure.

5.f. *Proof of Theorem* **5.1.** — Theorem 5.1 directly follows from the next lemma.

Lemma **5.3.** — *Let* M *be a compact 3-manifold whose boundary splits into disjoint union of closed surfaces* F *and* G. *Let the surface* G *be triangulated and equipped with a coloring* λ. *Let* ψ *be a colored 3-valent graph in* F *and let* Γ' *be an oriented colored 3-valent fat graph in* $\mathrm{F} \times [-1, 1]$. *Extend the triangulation of* G *to a triangulation of* M. *Then the state sum*

$$\textbf{(5.f.1)} \qquad \sum_{\substack{\mu \in \mathrm{col(M)} \\ \mu|_{\mathrm{G}} = \lambda}} |\, \mathrm{M}, \mathrm{F}\,|_{\mu} \cdot \langle\, \gamma_{\mathrm{F}}^{\mu_{\mathrm{F}}} \mid \Gamma' \mid \psi \,\rangle$$

(where $\mu_{\mathrm{F}} = \mu\,|_{\mathrm{F}}$*) does not depend on the choice of the extension.*

Proof of the Lemma. — It suffices to verify the invariance of (5.f.1) under the Alexander transformation along a simplex e (of the triangulation of M) not lying in G.

Let us rewrite the sum (5.f.1) in the following form

$$\textbf{(5.f.2)} \qquad \sum_{\nu \in \mathrm{col(F)}} \Big(\sum_{\substack{\mu \in \mathrm{col(M)} \\ \mu|_{\mathrm{G}} = \lambda,\ \mu|_{\mathrm{F}} = \nu}} |\, \mathrm{M}, \mathrm{F}\,|_{\mu} \cdot \langle\, \gamma_{\mathrm{F}}^{\nu} \mid \Gamma' \mid \psi \,\rangle \Big).$$

If e does not lie on F, then Theorem 3.2 implies that for any $\nu \in \mathrm{col(F)}$ the sum staying in (5.f.2) in the parentheses is invariant under the Alexander move along e. (Note that $|\, \mathrm{M}, \mathrm{F}\,|_{\mu} = d \,|\, \mathrm{M}\,|_{\mu}$ where d depends only on the triangulation of F and $\mu\,|_{\mathrm{F}}$.) This implies our claim.

In the case $e \subset \mathrm{F}$ the proof goes essentially along the same lines as the proof of Theorem 1.4.A in [TV]. The idea is to rewrite the state sum (5.f.1) as a state sum on the 2-skeleton of M^* (rel. G). Consider first the case of empty Γ' and ψ. The regions of the graph γ_{F} on F are exactly the 2-cells of M^* lying in F. Each pair ($\mu \in \mathrm{col(M)}$, $\eta \in \mathrm{adm}(\gamma_{\mathrm{F}})$) determines a coloring $\mu\eta$ of the 2-cells of M^* not lying in G. Namely, if y is a 2-cell of M^* lying in F, then $(\mu\eta)\,(y) = \eta(y) \in \mathrm{I}$. If y is a 2-cell of M^* not lying in $\partial\mathrm{M}$ and dual to an edge e' of M, then $(\mu\eta)\,(y) = \mu(e') \in \mathrm{I}$. The coloring $\mu\eta$ has the property that the colors of any three 2-cells having a common edge form an admissible triple. The formula $(\mu, \eta) \mapsto \mu\eta$ establishes a bijection of the set of pairs (μ, η) as above and the colorings of the relative 2-skeleton of $(\mathrm{M}^*, \mathrm{G})$ with this property. Now it is easy to rewrite (5.f.1) as a state sum on the relative 2-skeleton of $(\mathrm{M}^*, \mathrm{G})$. Here each vertex of M^* (not lying on G) will contribute a $6j$-symbol determined by the colors of the six adjacent 2-cells. Each 2-cell of $(\mathrm{M}^*, \mathrm{G})$ of color i will also contribute the factor w_i^2.

Now we may compare the state sums on the 2-skeletons of $(\mathrm{M}^*, \mathrm{G})$ and $((\mathrm{M}_e)^*, \mathrm{G})$ where M_e is the triangulation obtained by the Alexander transformation of M along $e \subset \partial\mathrm{M}$. Let $\dim(e) = 1$. The 2-skeleton of $(\mathrm{M}_e)^*$ is obtained from the 2-skeleton of M_e by gluing one 2-disk $e^* \times 1$ along its boundary. The gluing map is homotopic to zero in $e^* \times (-1) \cup \partial e^* \times [-1, 1]$. The homotopy may be presented as a composition of standard local deformations of the circle $\partial e^* \times 1$ in $e^* \times (-1) \cup \partial e^* \times [-1, 1]$. There are two types of such local deformations reflecting the polygonal structure of e^*. A deformation of the first type pushes a small piece of the circle through an edge of

$e^* \times (-1)$ inside $e^* \times (-1)$. A deformation of the second type pushes the circle through a vertex of $e^* \times (-1)$ inside $e^* \times (-1)$. Applying such deformations we may deform the boundary circle of $e^* \times 1$ to a small embedded circle in $e^* \times (-1)$. It is easy to check that Conditions I, II ensure invariance of our state sum under the two local deformations described above. (For a detailed argument see [TV].) Condition III ensures invariance of the state sum under elimination of a 2-disc whose boundary lies inside another 2-disc of M_e^*. Since these operations transform the 2-skeleton of M_e^* into the 2-skeleton of M^* we get the invariance of our state sum under the Alexander transformation along e. The case $\dim e = 2$ is treated similarly.

In the case of non-empty Γ' and ψ we may isotop these graphs out of the star of e in F without changing the sum (5.f.1). (This follows from Theorem 4.1.) Now, Γ' and ψ do not interfere with the Alexander transformation along e and its dualization so that we may apply the same arguments as above.

5.g. *Remark.* — The colorings of a triangulated closed surface G generate the vector space Q(G) introduced in [TV]. Denote the generator corresponding to $\lambda \in \mathrm{col}(G)$ by $|\,\lambda\,\rangle$. Each colored 3-valent graph φ embedded in G gives rise to an element $|\,\varphi\,\rangle$ of this vector space:

$$|\,\varphi\,\rangle = \sum_{\lambda \in \mathrm{col}(G)} w_\lambda \langle\, \gamma_G^\lambda \,|\, \varnothing \,|\, \varphi \,\rangle \cdot |\,\lambda\,\rangle.$$

Here $\varnothing$ is the empty graph in $G \times [-1, 1]$, and

$$\textbf{(5.g.1)} \qquad w_\lambda = w^{-\beta} \prod_{e'} w_{\lambda(e')}$$

where β is the number of vertices of G and e' runs over all edges of G.

Note that the state sum invariant $\langle\, \gamma \,|\, \varnothing \,|\, \varphi \,\rangle$ may be defined for any colored 3-valent graphs γ, φ on G without any assumptions on the orientability of G (cf. Lemma 9.3 below). One may show that the element $|\,\varphi\,\rangle \in Q(G)$ does not depend on the choice of triangulation of G up to canonical isomorphisms relating the vector spaces Q(G) defined via different triangulations of G (see [TV]). In particular, one has an important canonical element $|\,\varnothing\,\rangle$ of Q(G). Note also that $|\,\gamma_G^\lambda\,\rangle = w_\lambda^{-1}\,|\,\lambda\,\rangle$.

If G is the boundary of a compact 3-manifold N and Γ is a colored 3-valent fat graph in Int N with oriented neighborhood, then, for a colored 3-valent graph φ on G, we define

$$\langle\, N, \Gamma \,|\, \varphi \,\rangle = \sum_{\lambda \in \mathrm{col}(G)} w_\lambda \langle\, \gamma_G^\lambda \,|\, \varnothing \,|\, \varphi \,\rangle \cdot \langle\, N, \Gamma \,|\, \lambda \,\rangle.$$

Here one has to fix a triangulation of G, but the result does not depend on the choice of this triangulation. In particular, one has an "absolute" invariant $\langle\, N, \Gamma \,\rangle$ of the pair (N, Γ) defined to be $\langle\, N, \Gamma \,|\, \varnothing \,\rangle$. This latter invariant generalizes the invariant $\langle\, N, \Gamma \,\rangle$ introduced in Section 5.a in the case of closed N. The invariant $\langle\, N, \Gamma \,|\, \varphi \,\rangle$ also generalizes the invariant $\langle\, \varphi \,|\, \Gamma \,|\, \psi \,\rangle$ considered in Section 4.

6. Quantum invariants of graphs in 3-manifolds

In this section we consider the invariants of fat graphs introduced in Section 5 for the quantum initial data corresponding to a primitive complex root of unity $\exp\left(\pi\sqrt{-1}\,h/2r\right)$ of degree $4r$ (see Section 2). Note that these invariants of fat graphs are complex numbers (depending on the choice of the root).

6.a. *Comparison with the Jones polynomial.* — V. Jones [J] introduced for each oriented link $L \subset S^3$ a Laurent polynomial $V_L(t) \in \mathbf{Z}\left[\sqrt{t}, \sqrt{t}^{-1}\right]$ which is an isotopy invariant of L. Kauffmann [K] constructed a bracket version $\langle L \rangle (t) \in \mathbf{Z}[t, t^{-1}]$ of the Jones polynomial which is defined for every framed (unoriented) link $L \subset S^3$ and which is invariant under framing preserving isotopies. If the link L is both oriented and framed then

$$\langle L \rangle (t) = -(t^2 + t^{-2})\, t^m\, V_L(t^4)$$

where the integer m is determined by the framing and the linking numbers of the components of L.

The bracket polynomial is characterized by the following three properties:

(i) if L is a trivial knot lying in $\mathbf{R}^2$ with the framing orthogonal to $\mathbf{R}^2$, then $\langle L \rangle = -(t^2 + t^{-2})$;

(ii) if a framed link L' is obtained from a framed link $L \subset S^3$ by one positive twist of the framing, then $\langle L' \rangle = -t^3 \langle L \rangle$;

(iii) if three framed links L, L_+, L_- in S^3 are presented by link diagrams in $\mathbf{R}^2$ (with the framings orthogonal to $\mathbf{R}^2$) such that the diagrams coincide outside some disc and look as in Fig. 12 inside the disc, then

(6.a.1) $$\langle L \rangle = t \langle L_+ \rangle + t^{-1} \langle L_- \rangle.$$

Note that by the positive twist of the framing we mean the twist which is positive with respect to the right-hand orientation of S^3.

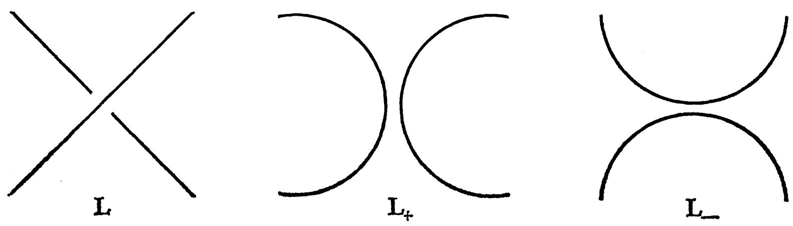

Fig. 12

Theorem **6.1.** — *Let* L *be a framed link in* S^3 *and let* Γ_L *be the corresponding fat graph in* S^3 *consisting of annuli. Provide all edges of* Γ_L *with the color* $1/2 \in I$ *and provide a regular neighborhood of* Γ_L *with the right-hand orientation. Then*

$$\langle S^3, \Gamma_L \rangle = \langle L \rangle \left(\exp\left(\pi\sqrt{-1}\,h/2r\right)\right)$$

where the left hand side is the state sum invariant of the pair (S^3, Γ_L) *defined in Section 5 and based on the quantum initial data corresponding to the root of unity* $\exp(\pi\sqrt{-1}h/2r)$.

Theorem 6.1 shows that the values of the Jones polynomial of a link in the roots of unity may be computed as the partition functions of the corresponding state sum models on the link exterior. This theorem will be proven in Section 9.

6.b. *Generalization of the formula* (6.a.1). — The formula (6.a.1) may be generalized to quantum invariants of fat graphs in an arbitrary compact 3-manifold N. Let $\Gamma, \Gamma_+, \Gamma_-$ be three colored fat graphs in N with oriented regular neighborhoods. We say that these three graphs form a splitting triple if there exists a ball $B \subset N$ such that:

(i) $\Gamma, \Gamma_+, \Gamma_-$ coincide outside B (together with their colorings and with orientations in their neighborhoods);

(ii) $\Gamma, \Gamma_+, \Gamma_-$ look as in Fig. 13 inside B;

(iii) the orientations in the neighborhoods of $\Gamma, \Gamma_+, \Gamma_-$ are compatible with the orientation in B corresponding to the right-hand orientation in Fig. 13;

(iv) the edges of $\Gamma, \Gamma_+, \Gamma_-$ meeting B are colored with $1/2 \in I$.

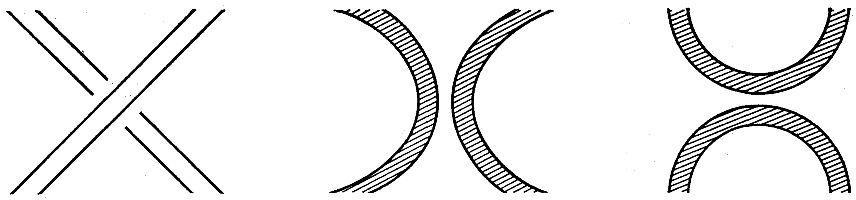

Fig. 13

Theorem **6.2.** — *For any splitting triple* $\Gamma, \Gamma_+, \Gamma_-$ *as above, for any triangulation of* ∂N *and any* $\lambda \in \mathrm{col}(\partial N)$ *we have*

$$\langle N, \Gamma \mid \lambda \rangle = t \langle N, \Gamma_+ \mid \lambda \rangle + t^{-1} \langle N, \Gamma_- \mid \lambda \rangle$$

where $t = \exp(\pi\sqrt{-1}h/2r)$.

This theorem will be proven in Section 9.

Corollary **6.3.** — *For any splitting triple* $\Gamma, \Gamma_+, \Gamma_-$ *in a closed 3-manifold* N *we have*

$$\langle N, \Gamma \rangle = t \langle N, \Gamma_+ \rangle + t^{-1} \langle N, \Gamma_- \rangle$$

where $t = \exp(\pi\sqrt{-1}h/2r)$.

6.c. *Remarks.* — 1. For links in homology 3-spheres one may combine the invariants considered in this section with linking numbers to get numerical invariants of oriented (non-framed) links which satisfy the original Jones relation (cf. [J], [K]).

2. One may generalize Theorem 6.1 to colored 3-valent fat graphs in S^3. With such a graph Γ one may associate an isotopy invariant $\langle \Gamma \rangle$ which is a rational function on one variable t (see [KR], [RT1], [W2]). Then the value of this function at $t = \exp\left(\pi\sqrt{-1}\,h/2r\right)$ coincides with the partition function $\langle S^3, \Gamma \rangle$ corresponding to the quantum initial data and the right-hand orientation in a neighborhood of Γ. Note that the denominator of $\langle \Gamma \rangle$ is a product of expressions $t^{4i+2} - t^{-4i-2}$ where i runs over the colors of certain edges of Γ. Therefore if the colors of all edges of Γ belong to the set $\{0, 1/2, 1, \ldots, (r-3)/2, (r-2)/2\}$, then $t = \exp\left(\pi\sqrt{-1}\,h/2r\right)$ is not a root of this denominator.

3. One may refine the invariant $\langle N, \Gamma \rangle$ to an invariant $\langle N, \Gamma; x \rangle$ where $x \in H^1(N\backslash\Gamma; \mathbf{Z}/2)$. It is defined along the same lines as $\langle N, \Gamma \rangle$ but with μ running over the colorings of M such that the 1-cocycle $e \mapsto \mu(e) \pmod 2$ presents x (cf. [TV]). Clearly,

$$\langle N, \Gamma \rangle = \sum_x \langle N, \Gamma; x \rangle.$$

The results of Part III also may be refined to this setting.

PART III

FACE MODELS FOR LINK AND GRAPH INVARIANTS

7. Simple skeletons of 3-manifolds and enriched graph diagrams

7. a. *Simple 2-polyhedra.* — A 2-dimensional polyhedron X is called a simple polyhedron if each point of X has a neighborhood homeomorphic either to (i) the plane $\mathbf{R}^2$, to (ii) the union of three half-planes in $\mathbf{R}^3$ meeting along their common boundary line, or to (iii) the cone over the 1-skeleton of the 3-dimensional simplex.

The points of a simple 2-polyhedron X which have neighborhoods homeomorphic to $\mathbf{R}^2$ form a two-dimensional manifold denoted by Int X. It is easy to see that the complement of Int X in X is a graph whose vertices are those points of X which have neighborhoods of type (iii) and correspond to the cone points of the cones. In particular, each vertex of the graph $X \backslash \mathrm{Int}\, X$ is incident to four edges (counted with multiplicites).

Note that some components of $X \backslash \mathrm{Int}\, X$ may be topological circles without vertices. By an edge of X we will mean either a circle component of $X \backslash \mathrm{Int}\, X$ without vertices or a honest edge of this graph which connects two (possibly coinciding) vertices. By a vertex of X we will mean a vertex of the graph $X \backslash \mathrm{Int}\, X$.

By an orientation of a simple 2-polyhedron X we mean an orientation of Int X.

7. b. *Simple 2-skeletons of 3-manifolds.* — Let X be a compact simple 2-polyhedron embedded in a compact 3-manifold N. We say that X is a simple 2-skeleton of N if $X \subset \mathrm{Int}\, N$ and the complement in N of an open regular neighborhood of X is a disjoint union of a closed regular neighborhood of ∂N and several closed 3-balls. Note that the closed regular neighborhood of ∂N in N is a collar $\partial N \times [0, 1] \subset N$ of $\partial N = \partial N \times 0$. Thus, for any simple 2-skeleton X of N a regular neighborhood of X in N may be obtained from N by cutting out a collar of ∂N and several disjoint closed 3-balls. Denote the number of these balls by $\sharp(N \backslash X)$.

For example, if N is a closed triangulated 3-manifold, then the 2-skeleton of the dual cell subdivision is a simple 2-skeleton of N. The complement of its regular neighborhood consists of disjoint 3-balls centered in the vertices of the triangulation. To perform a similar construction for a compact 3-manifold N with non-void boundary we fix a collar $\partial N \times [0, 1] \subset N$ of ∂N, triangulate $N \backslash (\partial N \times [0, 1))$ and take the union of $\partial N \times 1$ with the 2-skeleton of the dual cell subdivision. This union is a simple 2-skeleton of N. The complement of its regular neighborhood consists of a collar of ∂N (which is strictly smaller than the first one) and disjoint 3-balls centered in the vertices of the triangulation of $N \backslash (\partial N \times [0, 1))$ not lying on the boundary $\partial N \times 1$ of this manifold.

Another example of a simple 2-skeleton of a 3-manifold presents a closed surface $F = F \times 0$ lying in $F \times [-1, 1]$.

7.c. *Graph diagrams on simple 2-polyhedra.* — In this subsection the notion of graph diagram on a surface is generalized to the notion of graph diagram on a simple 2-polyhedron. We also introduce enriched graph diagrams and show that they present fat graphs in 3-manifolds.

Let X be a simple 2-polyhedron. By a graph diagram on X we mean a graph d immersed in X such that: d does not meet vertices of X; d meets the edges of X transversally; all self-crossings and vertices of d lie in Int X; the self-crossings of d are double transversal crossings of edges of d; each self-crossing is provided with the following additional structure: one of the two edges of d traversing the crossing is cut out at the crossing and considered to be the " locally lower edge ", the second edge being considered as the " locally upper edge ". The abstract graph d will be called the underlying graph of the diagram. The vertices and edges of d will be called respectively the vertices and the edges of the diagram. By abuse of language we will speak of vertices and edges of the diagram as if they lay on X meaning their images under the immersion $d \to X$ specified by the diagram.

By an *enriched graph diagram* on X we mean a pair (a graph diagram on X; a function which associates with each edge of the diagram an integer or half-integer, called the pretwist of this edge).

Let X be an oriented simple 2-polyhedron embedded in an oriented 3-manifold N. Equip Int X with the normal direction which together with the orientation of X determines the orientation of N.

Each graph diagram d on X gives rise to a fat graph $\Gamma(d)$ in X. The 2-disks of $\Gamma(d)$ are small disk neighborhoods of the vertices of d in Int X. The bands of $\Gamma(d)$ contain the edges of d as the cores and lie in X except in small neighborhoods of the self-crossing points of d where the bands corresponding to the locally upper edges are slightly pushed into $N \backslash X$ along the specified normal direction.

Every enriched graph diagram $D = (d$, the pretwist function on the set of edges of $d)$ gives rise to a fat graph $\Gamma(D)$ in N. It is obtained from $\Gamma(d)$ via the twisting of all bands of $\Gamma(d)$ around their cores as many times as are the pretwists of the corresponding edges. Here the positive direction of twisting is determined by the orientation of N. Note that the pretwists $1/2$ and $-1/2$ correspond to one positive and one negative half-twist of a band.

It is straightforward to see that if X is a 2-skeleton of N then each fat graph in N is isotopic to a fat graph presented by some enriched graph diagram on X.

7.d. *Moves on enriched graph diagrams.* — Let X be an oriented simple 2-polyhedron embedded in an oriented 3-manifold N. We consider local moves

(7.d.1) $$\Omega_1 - \Omega_8, \omega_1 - \omega_4, \omega_{-1}, \omega_{-2}$$

on the enriched graph diagrams in X. The first twelve moves are drawn in Figures 3, 14, 15*a*-15*c*, 16*a*, and 16*b*. Here $\Omega_1 - \Omega_3$, ω_1, ω_2 proceed in Int X, whereas $\Omega_4 - \Omega_8$ and ω_3, ω_4 proceed in a neighborhood of sing(X). The moves ω_{-1}, ω_{-2} also proceed in Int X, they are presented by pictures obtained from the pictures of ω_1, ω_2 by the mirror reflection with respect to the plane of the page. The moves Ω_1 and Ω_4 decrease

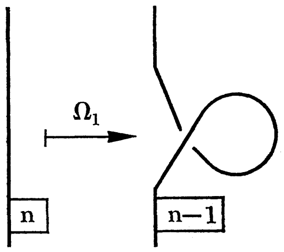

Fig. 14

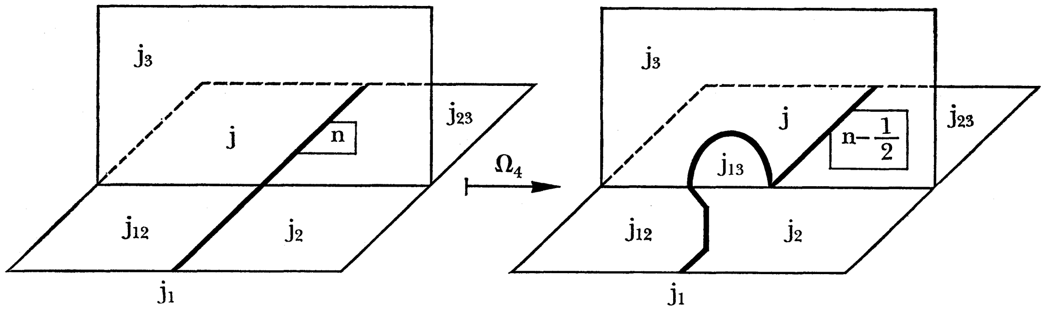

Fig. 15*a*

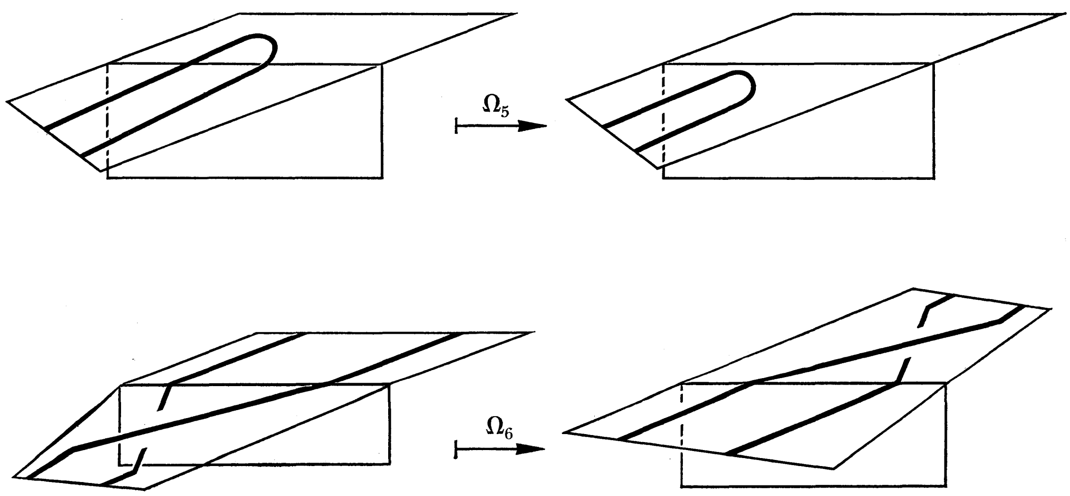

Fig. 15*b*

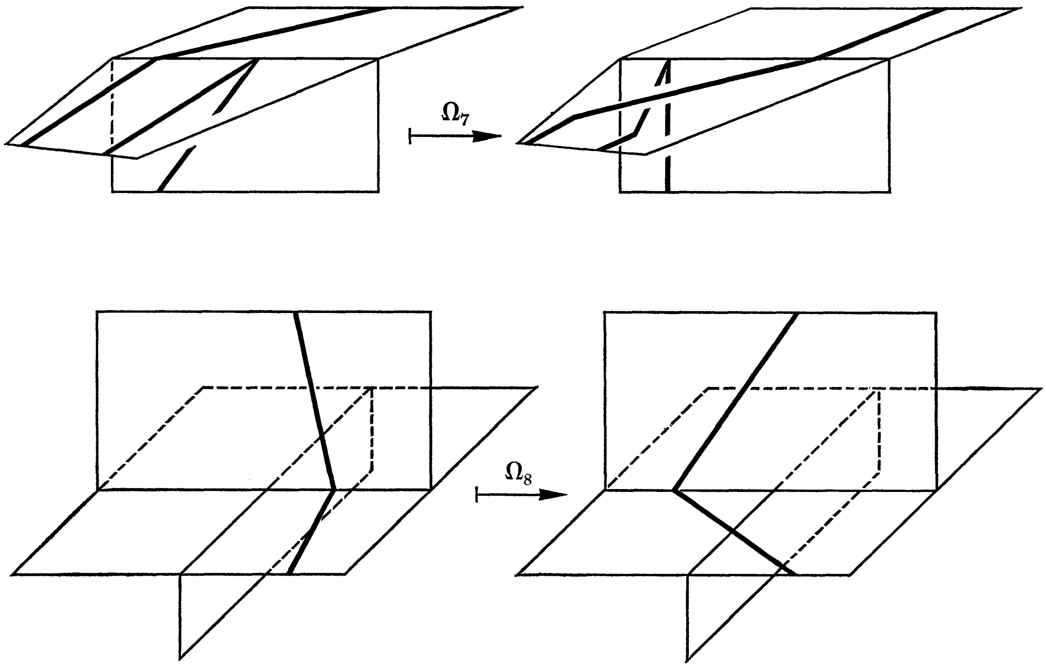

Fig. 15*c*

the pretwist by 1 and 1/2 respectively; in our figures the pretwists are represented by numbers located in small boxes. (We assume that in the pictures for Ω_1, Ω_4 the orientation of N corresponds to the right-hand orientation in $\mathbf{R}^3$.) Other moves do not change the pretwists. The symbols $j, j_1, j_2, \ldots$ entering into Fig. 15*a* should be disregarded at the moment.

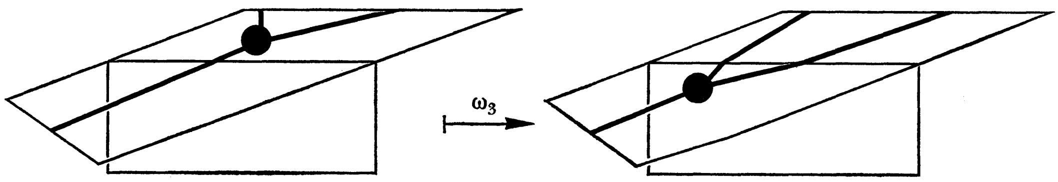

Fig. 16*a*

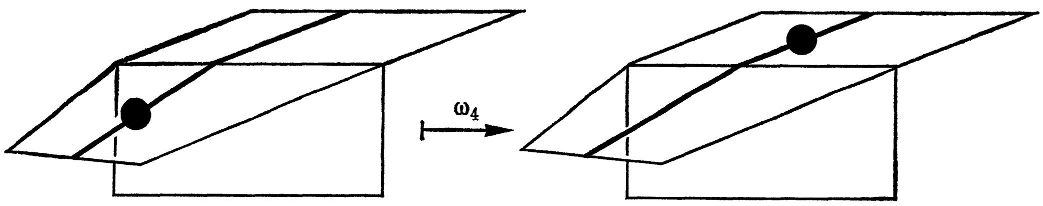

Fig. 16*b*

Note that the move Ω_1 is a lifting to the class of enriched diagrams of the classical Reidemeister move drawn in Fig. 7. One may similarly lift the second transformation drawn in Fig. 7 to a move on enriched diagrams increasing the pretwist by 1. It is obvious that Ω_0 may be presented as a composition of the latter move with Ω_1.

It is straightforward to see that the moves $\Omega_1 - \Omega_8$, $\omega_1 - \omega_4$, ω_{-1}, ω_{-2} preserve the isotopy types of the fat graphs in N presented by the enriched graph diagrams in X.

Theorem **7.1.** — *Let* X *be an oriented simple 2-skeleton of an oriented compact 3-manifold* N. *Two enriched graph diagrams on* X *give rise to isotopic fat graphs in* N *if and only if these diagrams may be obtained from each other by the moves* (7.d.1) *and their inverses.*

Proof. — The proof of the theorem is based on standard arguments similar to the classical arguments of Reidemeister [R] and their well-known extensions to graphs in $\mathbf{R}^3$. First one notes that any isotopy in N between fat graphs presented by enriched graph diagrams on X may be deformed into a regular neighborhood of X. Then one splits the isotopy into a composition of " elementary " local isotopies. It is easy to describe all types of elementary local isotopies and to present them as compositions of the moves (7.d.1) and their inverses.

7.e. *Remarks.* — The isotopy shown in Fig. 17 may be easily presented as a composition of Ω_4, Ω_8, Ω_8^{-1} and Ω_4^{-1}. This presentation motivates the proof of Lemma 1.2 given in Section 1. Similarly, the move Ω_1 applied near an edge of X to a branch of a graph diagram traversing this edge may be presented as a composition of two Ω_4, Ω_7 and Ω_5. This decomposition motivates the proof of Lemma 1.3 given in Section 1. As an exercise the reader may find an analogous visual interpretation of the proof of Lemma 1.1. Note finally that, if each connected component of X has at least one edge, then using our moves one may transform any enriched graph diagram on X into a diagram with zero pretwists of all edges.

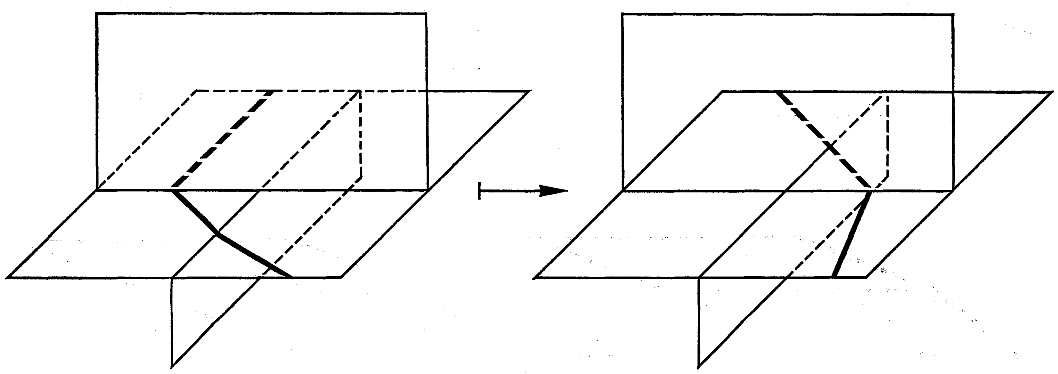

Fig. 17

8. State sum invariants via graph diagrams

In Section 8 and in the next Section 9 we use certain fixed initial data (K, I, w, ...) satisfying Conditions I-IV of Section 1.

8.a. *Colored graph diagrams.* — Let D be an enriched graph diagram on a simple 2-polyhedron X. We say that D is colored if the underlying (abstract) graph of D is 3-valent and I-colored (cf. Sect. 4.a and 7.c).

If X is oriented and embedded into an oriented 3-manifold N, then each coloring of D in the obvious way determines a coloring of the fat graph $\Gamma(D) \subset N$. Here and in the rest of Section 8 we consider only 3-valent graph diagrams.

8.b. *The state model.* — Let X be an oriented simple 2-polyhedron embedded in an oriented 3-manifold N. We present here a state sum model which associates with each colored enriched graph diagram D on X an element $\langle\langle D \rangle\rangle$ of the ground ring K. This element is invariant under the moves (7.d.1).

Let D be a colored enriched graph diagram on X. Let $\overline{D}$ be the graph in X obtained from D by forgetting the over/under-crossing information (see Fig. 18). Put

$$\Sigma = \overline{D} \cup (X \backslash \mathrm{Int}\, X) \subset X.$$

The transversality conditions put on graph diagrams in Section 7.c imply that Σ is a finite graph whose set of vertices may be split into five disjoint subsets: (i) 2-valent vertices of D, (ii) 3-valent vertices of D, (iii) self-crossings of D, (iv) crossings of D with $X\backslash \mathrm{Int}\, X$, (v) vertices of $X\backslash \mathrm{Int}\, X$. The vertices of Σ of types (iii-v) have valency 4.

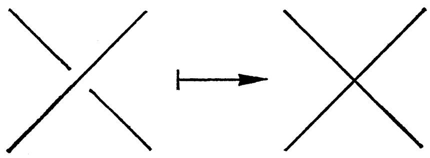

Fig. 18

By a *region* of D we mean a connected component of the 2-manifold $X\backslash\Sigma$. By an *area-coloring* of D we mean an arbitrary mapping of the set of regions of D into I. An area-coloring η of D is called *admissible* if for any edge e of Σ the following holds true: either e lies on an edge of X and the η-colors of the three adjacent regions of D form an admissible triple, or e lies on an edge of D and then the (fixed) color of this edge together with the η-colors of two regions of D adjacent to e form an admissible triple. Denote the set of admissible area-colorings of D by adm(D).

With each $\eta \in \mathrm{adm}(D)$ we associate an element $\langle\langle D \rangle\rangle_\eta$ of K as follows. First we associate certain elements of K with regions and edges of D and with vertices of Σ.

For a region y of D we define $|y|_\eta \in K$ by the formula (4.d.1). For an edge e of D with color $i \in I$ and with the pretwist n we put

$$|e| = q_i^{-2n}.$$

(Clearly, $|e|$ does not depend on the choice of η.)

For a vertex a of Σ we define $|a|_\eta$ in accordance with the type (i-v) of a. In case (i) $|a|_\eta = 1$. In case (ii)

$$|a|_\eta = \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} \tag{8.b.1}$$

where i, j, k are the colors of the edges of D incident to a and l, m, n are the η-colors of the opposite regions of D (see Fig. 4). In case (iii)

$$|a|_\eta = q_k\, q_n\, q_j^{-1}\, q_m^{-1} \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}$$

where l and i are the colors of the locally upper and lower edges of D and j, k, m, n are the η-colors of the regions of D as in Fig. 5. In case (iv)

$$|a|_\eta = q_k^{1/2}\, q_n^{1/2}\, q_j^{-1/2}\, q_m^{-1/2} \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}$$

where i is the color of the edge of D traversing a and j, k, l, m are the η-colors of the regions of D shown in Fig. 19 where the orientation of N is assumed to be the right-hand one. The symbols $q_k^{1/2}$, $q_k^{-1/2}$ are understood here as formal expressions as in Section 4.d. In the final formula for the partition function each q_k appears in an integer power. In case (v) $|a|_\eta$ is defined by the formula (8.b.1) where i, j, k, l, m, n are the η-colors of the regions of D incident to a as in Fig. 20. Note that the inclusion $\eta \in \mathrm{adm}(D)$ and the definition of colorings of graphs ensure admissibility of the 6-tuple (i, j, k, l, m, n) in all cases considered here.

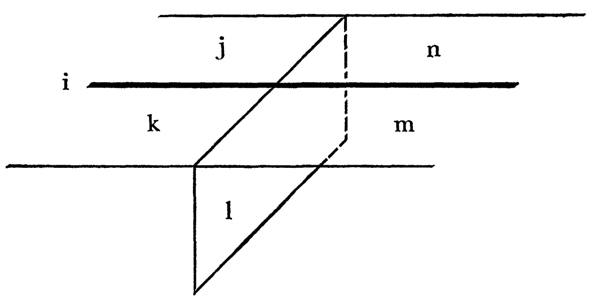

Fig. 19

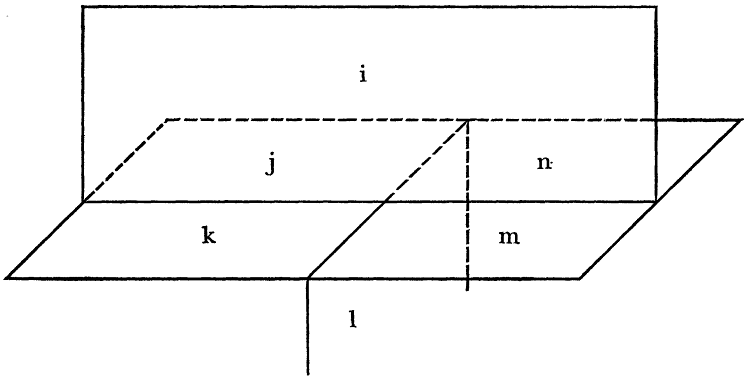

Fig. 20

We put

$$\text{(8.b.2)} \qquad \langle\langle \mathrm{D} \rangle\rangle_\eta = \prod_e | e | . \prod_y | y |_\eta . \prod_a | a |_\eta$$

where e runs over all edges of D, a runs over all vertices of D, y runs over all regions of D.

Here is the state sum invariant of D:

$$\langle\langle \mathrm{D} \rangle\rangle = \sum_{\eta \in \mathrm{adm(D)}} \langle\langle \mathrm{D} \rangle\rangle_\eta .$$

Theorem **8.1.** — *For any colored 3-valent enriched graph diagram* D *the element* $\langle\langle \mathrm{D} \rangle\rangle$ *of* K *is invariant under the moves* (7.d.1) *on* D.

Proof. — Invariance under the moves (4.c.1) is proven along the same lines as Theorem 4.1. Invariance under the remaining moves is proven similarly. Specifically, invariance under Ω_5 follows from Condition I. Invariance under the moves Ω_7, Ω_8, ω_8 follows from Condition II. Invariance under the moves Ω_1, Ω_4, Ω_6 follows from Lemmas 1.3, 1.1, and 1.2 respectively. Invariance under ω_4 is straightforward.

To give a sample argument I will verify invariance of $\langle\langle \mathrm{D} \rangle\rangle$ under Ω_4. Let a diagram D′ be obtained from D by a single application of Ω_4. Fix the colors of all regions of D′ except the one marked by j_{13} in Fig. 15*a* and vary the color j_{13} of this region. In this way we get a finite set, say, ∇ of admissible area-colorings of D′. (We automatically exclude non-admissible colorings so it may happen that $\nabla = \varnothing$.)

If $\nabla \neq \varnothing$, then all colorings $\eta \in \nabla$ induce in the obvious way an admissible coloring η_0 of D (see Fig. 15*a*). We claim that

$$\text{(8.b.3)} \qquad \sum_{\eta \in \nabla} \langle\langle \mathrm{D}' \rangle\rangle_\eta = \langle\langle \mathrm{D} \rangle\rangle_{\eta_0} .$$

The numbers $\langle\langle \mathrm{D}' \rangle\rangle_\eta$, $\langle\langle \mathrm{D} \rangle\rangle_{\eta_0}$ are products of certain factors, most of which are the same. We compare the factors which differ. The edges of D, D′ shown in Fig. 15*a* contribute $q_{j_1}^{1-2n}$ to $\langle\langle \mathrm{D}' \rangle\rangle_\eta$ and $q_{j_1}^{-2n}$ to $\langle\langle \mathrm{D} \rangle\rangle_{\eta_0}$, where j_1 is the color of these edges. The regions of D, D′ contribute the same with only one exception: D′ has one additional disk region marked by j_{13} which contributes the factor $w_{j_{13}}^2$ to $\langle\langle \mathrm{D}' \rangle\rangle_\eta$. The two vertices of D′ shown in Fig. 15*a* contribute the following factor

$$q_{j_{12}}^{-1/2}\, q_{j_{23}}^{-1/2}\, q_{j_{13}}^{-1}\, q_{j}^{1/2}\, q_{j_2}^{1/2}\, q_{j_3} \begin{vmatrix} j_3 & j_1 & j_{13} \\ \dot{j_2} & \dot{j} & \dot{j_{12}} \end{vmatrix} \begin{vmatrix} j_2 & j_3 & j_{23} \\ \dot{j_1} & \dot{j} & \dot{j_{13}} \end{vmatrix}.$$

The vertex of D shown in Fig. 15*a* contributes the following factor to $\langle\langle \mathrm{D} \rangle\rangle_{\eta_0}$

$$q_{j}^{-1/2}\, q_{j_2}^{-1/2}\, q_{j_{12}}^{1/2}\, q_{j_{23}}^{1/2} \begin{vmatrix} j_3 & j_2 & j_{23} \\ \dot{j_1} & \dot{j} & \dot{j_{12}} \end{vmatrix}.$$

All other vertices contribute the same to $\langle\langle \mathrm{D}' \rangle\rangle_\eta$ and $\langle\langle \mathrm{D} \rangle\rangle_{\eta_0}$. Substituting all these contributions into (8.b.3) we immediately get that (8.b.3) is a direct corollary of Lemma 1.1.

We need one more remark. Let η_0 be an admissible area-colouring of D shown in Fig. 15*a*. In principle, it may happen that η_0 does not extend to an admissible area colouring of D′: there may be no j_{13} such that both triples (j_3, j_1, j_{13}) and (j_2, j, j_{13}) are admissible. Condition IV shows that if it is the case then

$$\begin{vmatrix} j_3 & j_2 & j_{23} \\ \dot{j_1} & \dot{j} & \dot{j_{12}} \end{vmatrix} = 0$$

so that $\langle\langle \mathrm{D} \rangle\rangle_{\eta_0} = 0$ and we again have (8.b.3). Summing up equalities (8.b.3) over all $\eta_0 \in \mathrm{adm(D)}$ we get the desired equality $\langle\langle \mathrm{D} \rangle\rangle = \langle\langle \mathrm{D}' \rangle\rangle$.

8.c. *Invariants of colored fat graphs in 3-manifolds.* — Let N be an oriented compact 3-manifold. Let Γ be a colored 3-valent fat graph in N. We define a state sum invariant

$$\langle\langle \mathrm{N}, \Gamma \rangle\rangle \in \mathrm{K}$$

as follows. Choose an arbitrary oriented simple 2-skeleton X of N and present Γ by a colored enriched graph diagram D on X. Put

$$\langle\langle \mathrm{N}, \Gamma \rangle\rangle = w^{2(1-\#(\mathrm{N}\setminus\mathrm{X}))} \langle\langle \mathrm{D} \rangle\rangle.$$

For the definition of the number $\#(\mathrm{N}\setminus\mathrm{X})$ see Section 7.b. Theorems 7.1 and 8.1 imply that $\langle\langle \mathrm{N}, \Gamma \rangle\rangle$ is an isotopy invariant of Γ.

Theorem **8.2.** — *The element* $\langle\langle \mathrm{N}, \Gamma \rangle\rangle$ *of* K *does not depend on the choice of* X.

Proof. — First we show that $\langle\langle \mathrm{N}, \Gamma \rangle\rangle$ does not depend on the orientation of X. Let A be a connected component of Int X and let X′ be the same oriented polyhedron X with inverted orientation in A. Present Γ by an enriched diagram D on X. Let D′ be the enriched diagram on X′ obtained from D by trading all overcrossings/undercrossings lying in A respectively for undercrossings/overcrossings. It is easy to see that the diagram D′ on X′ also presents Γ. It follows directly from definitions that $\langle\langle \mathrm{D} \rangle\rangle = \langle\langle \mathrm{D}' \rangle\rangle$. Thus $\langle\langle \mathrm{N}, \Gamma \rangle\rangle$ is independent of the choice of orientation in X.

Let us show independence of $\langle\langle$ N, Γ $\rangle\rangle$ of the choice of X. It is known that any two simple 2-skeletons of N may be related by a finite sequence of local moves shown in Fig. 21 and the inverse moves. For a proof of this assertion see [TV, Theorem 6.2.A] where it is deduced from the results of Matveev [M] and Piergallini [P] concerned with cellular simple spines of 3-manifolds. The proof actually shows that any two orientable simple 2-skeletons of N may be related by these local moves in the class of orientable simple 2-skeletons of N. For each of these three moves a certain diagram D of Γ may be assumed to lie outside the 3-ball where the move proceeds. Now the same arguments as in the proof of Theorem 8.1 show that the move does not affect $\langle\langle$ D $\rangle\rangle$. This implies the claim of the theorem.

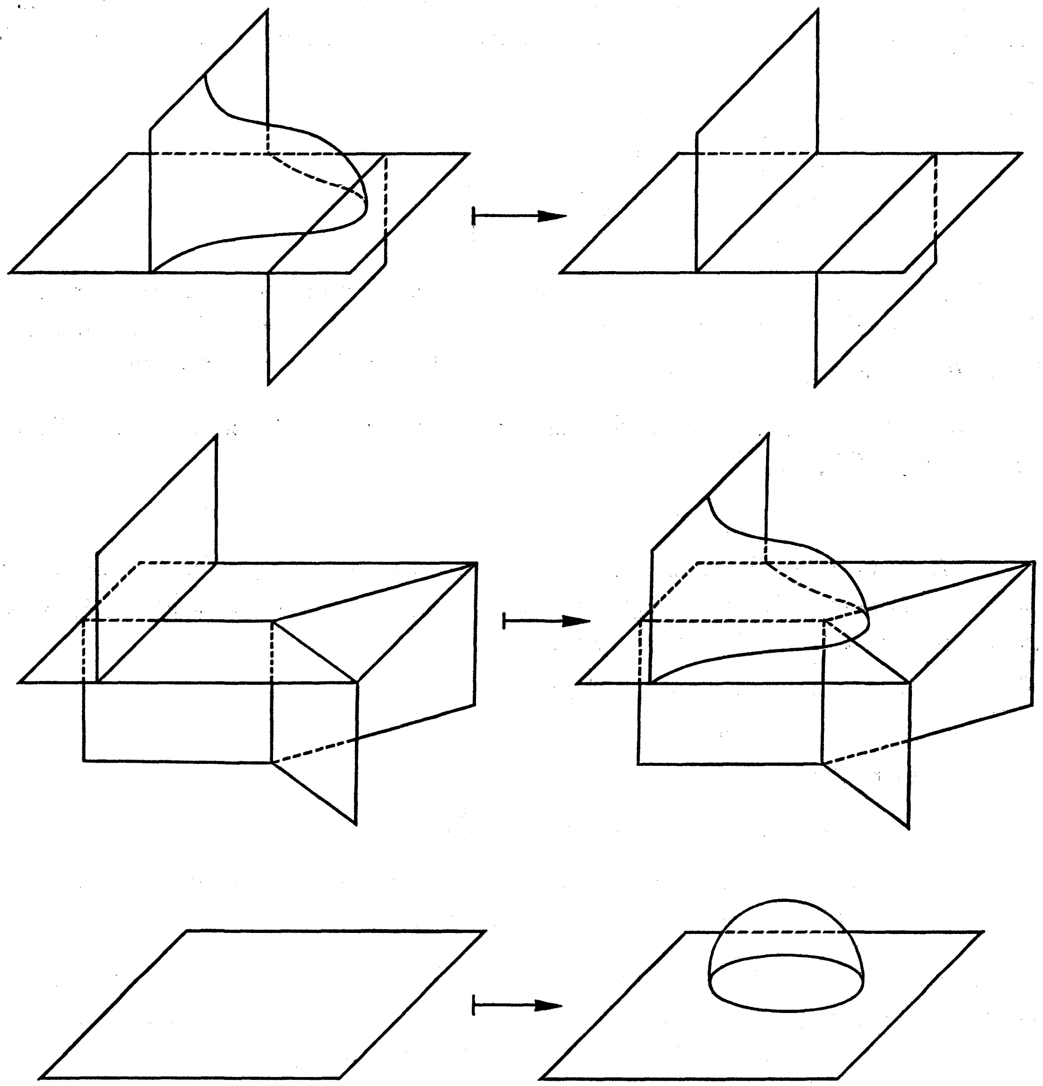

Fig. 21

8.d. *Properties of the invariants.*

Theorem **8.3.** — *Let* Γ *be a colored 3-valent fat graph in an oriented compact 3-manifold* N. *Let* B *be a closed 3-ball lying in* $\mathrm{Int}\,\mathrm{N}\backslash\Gamma$. *Then*

$$\langle\langle\, \mathrm{N}\backslash \mathrm{Int}\, \mathrm{B}, \Gamma \,\rangle\rangle = w^2 \langle\langle\, \mathrm{N}, \Gamma \,\rangle\rangle.$$

Proof. — If X is an oriented simple 2-skeleton of $\mathrm{N}\backslash\mathrm{Int}\,\mathrm{B}$, then X also is a simple 2-skeleton of N. The state sum $\langle\langle\, \mathrm{D} \,\rangle\rangle$ associated with a diagram D of Γ on X depends only on X, D and a regular neighborhood of X in the ambient 3-manifold. Thus $\langle\langle\, \mathrm{D} \,\rangle\rangle$ is preserved under the passage from $\mathrm{N}\backslash\mathrm{Int}\,\mathrm{B}$ to N. This implies the claim.

Theorem **8.4.** — *Let* Γ *be a colored 3-valent fat graph in an oriented compact 3-manifold* N. *If colored fat graphs* $\Gamma_1, \ldots, \Gamma_4$ *in* N *are obtained from* Γ *respectively by* $\tau_1, \ldots, \tau_4$ *(see Section* 4.d*) then*

$$(\mathbf{8.d.1})\qquad \begin{aligned} \langle\langle\, \mathrm{N}, \Gamma_1 \,\rangle\rangle &= q_j^{-2} \langle\langle\, \mathrm{N}, \Gamma \,\rangle\rangle,\\ \langle\langle\, \mathrm{N}, \Gamma_2 \,\rangle\rangle &= q_j^{-2} \langle\langle\, \mathrm{N}, \Gamma \,\rangle\rangle,\\ \langle\langle\, \mathrm{N}, \Gamma_3 \,\rangle\rangle &= (q_{j_2}\, q_{j_3}\, q_{j_{23}})^{-1} \langle\langle\, \mathrm{N}, \Gamma \,\rangle\rangle,\\ \langle\langle\, \mathrm{N}, \Gamma_4 \,\rangle\rangle &= q_{j_2}\, q_{j_3}\, q_{j_{23}}^{-1} \langle\langle\, \mathrm{N}, \Gamma \,\rangle\rangle. \end{aligned}$$

Proof. — Present the graphs by graph diagrams in an oriented simple 2-skeleton X of N which coincide outside a disk in Int X and look as in Fig. 7 inside this disk. Now the arguments given in the beginning of Section 4.e apply word for word.

Theorem **8.5.** — *Let* Γ_i *be the fat graph in* S^3 *shown in Fig. 2 with* $i \in \mathrm{I}$ *being the color of the only edge of* Γ_i. *Then*

$$\langle\langle\, \mathrm{S}^3, \Gamma_i \,\rangle\rangle = \omega_i^2.$$

Proof. — Clearly $\mathrm{S}^2 \subset \mathrm{S}^3$ is a simple 2-skeleton of S^3. The graph Γ_i may be presented by the diagram D on S^2 consisting of one simple loop with zero pretwist and the color i. It follows from the definitions that

$$\langle\langle\, \mathrm{D} \,\rangle\rangle = \sum_{\substack{k,\, l \in \mathrm{I} \\ (i,\, k,\, l) \in \mathrm{adm}}} w_k^2\, w_l^2.$$

In view of Condition III,

$$\langle\langle\, \mathrm{S}^3, \Gamma_i \,\rangle\rangle = w^{-2} \langle\langle\, \mathrm{D} \,\rangle\rangle = w_i^2.$$

8.e. *Properties of quantum invariants.* — In this subsection we assume our initial data to be the quantum ones described in Section 2.

Theorem **8.6.** — *Let* N *be an oriented compact 3-manifold. Let* $\Gamma, \Gamma_+, \Gamma_-$ *be a splitting triple of colored 3-valent fat graphs in* N *(see Sect.* 6.b, *the orientations in the regular neighborhoods of* $\Gamma, \Gamma_+, \Gamma_-$ *are assumed to be the ones induced by the orientation of* N*). Then*

$$\langle\langle\, \mathrm{N}, \Gamma \,\rangle\rangle = t \langle\langle\, \mathrm{N}, \Gamma_+ \,\rangle\rangle + t^{-1} \langle\langle\, \mathrm{N}, \Gamma_- \,\rangle\rangle.$$

Proof. — Present the graphs by graph diagrams in an oriented simple 2-skeleton X of N which coincide outside a disk in Int X and look as in Fig. 12 inside this disk. Now the arguments given in [T2, proof of Theorem 6.2] apply word for word.

Theorem **8.7.** — *Under the conditions of Theorem* 6.1

$$\langle\langle S^3, \Gamma_L \rangle\rangle = \langle L \rangle \left(\exp\left(\pi\sqrt{-1}\,h/2r\right)\right).$$

Proof. — We will show that the function $L \mapsto \langle\langle S^3, \Gamma_L \rangle\rangle$ satisfies the conditions (i-iii) of Section 6.a with $t = \exp\left(\pi\sqrt{-1}\,h/2r\right)$. Since these conditions uniquely characterize the function $L \mapsto \langle L \rangle (t)$ this will imply the claim of the theorem.

The condition (i) follows from Theorem 8.5 since $w_{1/2}^2 = -(t^2 + t^{-2})$. The condition (ii) follows from the formula (8.d.1) since

$$q_{1/2}^{-2} = -t^3.$$

The condition (iii) follows from Theorem 8.6.

9. Comparison of two approaches

9.a. *The case of graphs in closed manifolds.*

Theorem **9.1.** — *Let* N *be a closed connected oriented 3-manifold. Let* Γ *be a colored 3-valent fat graph in* N. *Provide a regular neighborhood of* Γ *in* N *with the orientation induced by that of* N. *Then*

$$\langle N, \Gamma \rangle = \langle\langle N, \Gamma \rangle\rangle. \tag{9.a.1}$$

Proof. — Consider first the case where the surface of Γ is orientable. Orient this surface in an arbitrary way. Let U, $F = \partial U$, M, Γ', $\psi = \psi_1 \cup \ldots \cup \psi_m$ be the same objects as in Section 5.a. Provide $M = N \backslash \mathrm{Int}\, U$ with an arbitrary triangulation and provide $F = \partial M$ with the induced triangulation.

It follows from the definitions that for any edge e of M not lying in F the dual 2-cell e^* does not meet F. For an edge $e \subset F$ the dual 2-cell e^* intersects F in the segment

$$e^*_\partial = e^* \cap F \subset \partial e^*$$

dual to e in F. The union of such segments is the graph $\gamma = \gamma_F$ defined in Section 4.f. We will assume that the loops $\psi_1, \ldots, \psi_m$ are transversal to the edges of γ and do not cross vertices of γ.

Let Z be the 2-skeleton of the dual cell subdivision M^* of M. Note that $F = \partial M \subset Z$. Let $B_1, \ldots, B_m$ be the (embedded disjoint) meridianal disks of the handlebody U bounded respectively by the loops $\psi_1, \ldots, \psi_m$. Put

$$X = Z \cup \bigcup_{i=1}^{m} B_i \subset N.$$

Clearly X is a simple 2-skeleton of N whose complement is the disjoint union of $s + \alpha$ balls where s is the number of vertices of Γ and α is the number of vertices of M.

Let $D = c(\Gamma')$ be the core of the fat graph Γ' lying in F. Clearly, D is a colored 3-valent graph embedded in F. Deforming if necessary D in F we may assume that D lies in general position with respect to γ and ψ. Let $\bar{D}$ be the enriched colored graph diagram on X obtained from $D \subset X$ by providing all edges of D with zero pretwists. (This diagram has no self-crossings.) Clearly, $\bar{D}$ is a graph diagram of Γ. Thus

(9.a.2) $$\langle\langle N, \Gamma \rangle\rangle = w^{2-2s-2\alpha} \langle\langle \bar{D} \rangle\rangle.$$

Let us compute $\langle\langle \bar{D} \rangle\rangle$. It is obvious that the set of regions of $\bar{D}$ splits into three subsets:

(i) the disks $\mathrm{Int}\, B_1, \ldots, \mathrm{Int}\, B_m$;
(ii) the interiors of the 2-cells e^* where e runs over the edges of M;
(iii) the regions of D in F with respect to γ and ψ (see Sect. 4.d).

With each admissible coloring η of $\bar{D}$ we associate three objects:

(i) the sequence $J(\eta) = (j_1, \ldots, j_m)$ with $j_1 = \eta(\mathrm{Int}\, B_1), \ldots, j_m = \eta(\mathrm{Int}\, B_m)$;
(ii) the coloring μ_η of M defined by the rule $\mu_\eta(e) = \eta(e^*)$;
(iii) the area-coloring $\bar{\eta}$ of the diagram D on F (with respect to γ^{μ_F} and $\psi_{J(\eta)}$ where $\mu_F = \mu|_F$) which is just the restriction of η to the set of regions of D with respect to γ and ψ.

It is easy to compute that

$$\langle\langle \bar{D} \rangle\rangle_\eta = w^{2\alpha}\, w_{J(\eta)} \mid M, F \mid_{\mu_\eta} \langle \gamma^{\mu_F} \mid D \mid \psi_{J(\eta)} \rangle_{\bar{\eta}}$$

(for definitions of the factors entering the right hand side, see Sect. 3 and 4).

A comparison of definitions shows that the formula

$$\eta \mapsto (J(\eta), \mu_\eta, \bar{\eta})$$

establishes a bijection of the set $\mathrm{adm}(\bar{D})$ onto the set of triples $J \in I^m$, $\mu \in \mathrm{col}(M)$, $\nu \in \mathrm{adm}(D, \gamma^{\mu_F}, \psi_J)$. Therefore

$$\langle\langle \bar{D} \rangle\rangle = \sum_{\eta \in \mathrm{adm}(\bar{D})} \langle\langle \bar{D} \rangle\rangle = w^{2s+2\alpha-2} \langle N, \Gamma \rangle.$$

These formulas together with (9.a.2) imply (9.a.1) in the case of oriented Γ.

Since $\langle\langle N, \Gamma \rangle\rangle$ does not depend on the choice of orientation in (the surface of) Γ, the formula (9.a.1) implies that for orientable Γ the invariant $\langle N, \Gamma \rangle$ also does not depend on this choice. The equalities (9.a.1) and (8.d.1) imply (5.a.1) (in the case of orientable Γ and $\partial N = \varnothing$). This makes the definitions of Section 5.b consistent (in this case). These definitions and (8.d.1) imply the equality (9.a.1) for non-orientable Γ.

9.b. *Proof of Theorem* 6.1. — Theorem 6.1 directly follows from Theorems 8.7 and 9.1.

9.c. *Graphs in manifolds with boundary.*

Theorem **9.2.** — *Let* N *be a compact connected oriented 3-manifold with triangulated boundary. Let* $\lambda \in \mathrm{col}(\partial \mathrm{N})$. *Let* Γ *be a colored 3-valent fat graph in* Int N *and* $\gamma^{\lambda}_{\partial \mathrm{N}}$ *be the graph in* ∂N *dual to the 1-skeleton of the triangulation of* ∂N *and colored via* λ *(cf. Sect.* 4.f*). Let* γ^{λ} *be the colored fat graph in* N *obtained from a narrow regular neighborhood of* $\gamma^{\lambda}_{\partial \mathrm{N}}$ *in* ∂N *by a slight shift in* Int N$\backslash\Gamma$. *Then*

$$\langle \mathrm{N}, \Gamma \mid \lambda \rangle = w_{\lambda} \langle\langle \mathrm{N}, \Gamma \cup \gamma^{\lambda} \rangle\rangle$$

where w_{λ} *is defined by the formula* (5.g.1) *applied to* G $= \partial$N.

Here it is understood that the invariant in the left hand side corresponds to the orientation of a regular neighborhood of Γ induced by the orientation of N.

Theorem 9.2 will be proven in Section 9.f. It is easy to generalize Theorem 9.2 to include invariants $\langle \mathrm{N}, \Gamma \mid \varphi \rangle$ introduced in Section 5.g. Namely for each colored 3-valent graph φ in ∂N one has

$$\langle \mathrm{N}, \Gamma \mid \varphi \rangle = \langle\langle \mathrm{N}, \Gamma \cup \varphi \rangle\rangle.$$

9.d. *Deduction of Theorems* 5.2, 6.2 *from Theorem* 9.2.

The results of Section 5.c show that it suffices to consider the case when N is the oriented (closed) regular neighborhood of Γ. In this case all assertions of Theorem 5.2 directly follow from Theorems 9.2 and 8.4. By a similar reasoning we deduce Theorem 6.2 from Theorems 9.2 and 8.6.

9.e. *Preparations to the proof of Theorem* 9.2.

Lemma **9.3.** — *Let* X *be a simple oriented compact 2-polyhedron embedded in an oriented 3-manifold* N. *Let* ξ *be the 4-valent graph* X$\backslash$Int X. *Let* G *be an orientable closed surface contained in* X *such that in a neighborhood of* G *in* N *the polyhedron* X *lies on one side of* G. *Let* D *be a colored enriched graph diagram in* G *with all pretwists equal to zero and without self-intersections. Let* η *be an admissible area-coloring of* D *in* X. *For a vertex* a *of the graph* $\xi \cup \mathrm{D} \subset \mathrm{X}$ *put* $\| a \|_{\eta} = | a |_{\eta}$ *if* a *is a vertex of* ξ *or a vertex of* D *(see Sect.* 7*), and put*

$$\| a \|_{\eta} = \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}$$

if a *is a crossing point of* ξ *with* D *where* i *is the color of the edge of* D *passing through* a *and* j, k, l, m, n *are the* η*-colors of the regions of* D *shown in Fig. 19. Then*

(9.e.1) $$\langle\langle \mathrm{D} \rangle\rangle_{\eta} = \prod_{y} | y |_{\eta} \cdot \prod_{a} \| a \|_{\eta},$$

where y *runs over the regions of* D *in* X *and* a *runs over the vertices of* $\xi \cup$ D.

Proof. — Since the pretwists of D are zero, the formula (9.e.1) differs from the definition (8.b.2) of $\langle\langle D \rangle\rangle_\eta$ only in one item: for crossing points $a \in \xi \cap D$ the element $\| a \|_\eta$ differs from $| a |_\eta$. Namely, in the notation of Fig. 19

$$| a |_\eta = q_k^{1/2} q_n^{1/2} q_j^{-1/2} q_m^{-1/2} \| a \|_\eta.$$

We prove that the product of the expressions

$$(\mathbf{9.e.2}) \qquad q_k^{1/2} q_n^{1/2} q_j^{1/2} q_m^{1/2}$$

corresponding to all $a \in \xi \cap D$ is equal to 1. Each of the factors $q_k^{1/2}$, $q_n^{1/2}$, $q_j^{-1/2}$, $q_m^{-1/2}$ is associated with a connected component of $G \backslash (\xi \cup D)$ adjacent to a. We show that each component of $G \backslash (\xi \cup D)$, say, y contributes the same number of $q_{\eta(y)}^{1/2}$ and $q_{\eta(y)}^{-1/2}$. Indeed, moving along $\partial \bar{y}$ one goes successfully along D and ξ. Clearly, the number of switches from D to ξ is equal to the number of switches from ξ to D. Therefore the points of $\xi \cap D$ lying on a component of $\partial \bar{y}$ contribute the same number of factors $q_{\eta(y)}^{1/2}$ and $q_{\eta(y)}^{-1/2}$ to the product of the expressions (9.e.2). Hence this product is equal to 1. This implies (9.e.1).

9.f. *Proof of Theorem* 9.2. — Consider first the case $\Gamma = \varnothing$. Fix a triangulation of N extending the given triangulation of ∂N. Subdividing N, if necessary, we may assume that ∂N is a full sub-complex of N, i.e. that any simplex of N with vertices in ∂N lies in ∂N.

Let X be the union of closed 2-cells dual to the edges of N not lying in ∂N. It is well known that X is a simple 2-skeleton of N. The complement $N \backslash X$ consists of open 3-balls and a collar of ∂N. More exactly, let G be the union of closed 2-cells e^* where e runs over the edges of N with one end in ∂N and the second in Int N. It is easy to see that G is a surface in N. The part of N bounded by $G \cup \partial N$ is a collar $\partial N \times [0, 1]$ of ∂N (see [RS, Corollary 3.9]). Clearly $G \subset X$ and X lies on the side of G opposite to the side of this collar.

Denote the 4-valent graph $X \backslash \operatorname{Int} X$ by ξ. The vertices of ξ are the barycenters of the 3-simplices of N which either do not meet ∂N or meet ∂N in one vertex. The vertices of ξ of the second type are exactly those which lie in G. The edges of ξ are either segments B^* where B runs over the 2-faces of N disjoint from ∂N, or unions of segments B^* where B runs over the 2-faces of N meeting ∂N exactly in one vertex. The edges of ξ of the second type are exactly those which lie in G. Note that $\xi \cap G$ is a 3-valent subgraph of ξ, whose vertices and edges are the vertices and edges of ξ lying in G and described above.

Let us construct a certain colored 3-valent graph D on G. I will describe the part of D lying in each 3-simplex T of N.

If T does not meet ∂N than $D \cap T = G \cap T = \varnothing$. If T meets ∂N in one vertex then $D \cap T = \varnothing$. (In the last case $G \cap T$ is a 2-disk.)

Let T be a 3-simplex of N which meets ∂N along an edge e. Let B_1, B_2 be the 2-faces of T containing e, and let B_3, B_4 be the other 2-faces of T. Then $D \cap T$ is defined to be the union of the two segments $B_1^* \cap T$ and $B_2^* \cap T$ which have the barycenter of T as a common end-point. Note that $D \cap T \subset G \cap T$. We provide the edge of D containing $B_1^* \cap T$ and $B_2^* \cap T$ with the color $\lambda(e)$. Remark that

$$\xi \cap T = (B_3^* \cap T) \cup (B_4^* \cap T).$$

Therefore, the segment $D \cap T \subset G$ intersects the segment $\xi \cap T \subset G$ transversally in the barycenter of T.

Let T be a 3-simplex of N meeting ∂N along a 2-face. Let B_1, B_2, B_3 be the other 2-faces of T. Then $D \cap T$ is defined to be the union of three segments $B_i^* \cap T$, $i = 1, 2, 3$ which have the barycenter of T as a common end-point. We color the edge of D containing $B_i^* \cap T$ in the color $\lambda(e_i)$ where e_i is the edge $B_i \cap \partial N$ of B_i. Remark that $\xi \cap T = \varnothing$.

It is straightforward to see that D is a colored 3-valent graph embedded in G. The vertices of D are the barycenters of the 3-simplices of N meeting ∂N in 2-faces. The graph D lies in X in general position and crosses ξ transversally in the barycenters of the 3-simplices of N meeting ∂N along one edge.

We extend D to an enriched colored graph diagram on X (without self-crossings) providing all edges of D with zero pretwists. It is easy to observe that D is a graph diagram of γ^λ. Looking at intersections with 3-simplices one easily observes that the regions of D in X are exactly the open 2-cells Int e^* where e runs over the edges of N not lying on ∂N.

Let E be the set of colorings $\mu \in \mathrm{col}(N)$ such that $\mu|_{\partial N} = \lambda$. Each coloring $\mu \in E$ gives rise to an area-coloring η_μ of D defined by the formula

$$\eta_\mu(\mathrm{Int}\, e^*) = \mu(e)$$

where e is an edge of N not contained in ∂N. A comparison of definitions shows that the formula $\mu \mapsto \eta_\mu$ yields a bijection of E onto adm(D).

It follows from what was said above that the set of vertices of the graph $\xi \cup D$ coincides with the set of barycenters of the 3-simplices of N. For each vertex a of $\xi \cup D$ which is the barycenter of a 3-simplex T and any $\mu \in E$ we have

$$\| a \|_{\eta_\mu} = | T |_\mu .$$

(For the definitions of the right and left hand sides, see respectively Sect. 3.b and 9.e.) Lemma 9.3 implies that

$$\textbf{(9.f.1)} \qquad \langle\langle D \rangle\rangle_{\eta_\mu} = \prod_e w^2_{\mu(e)} \cdot \prod_T | T |_\mu$$

where e runs over the edges of N not lying on ∂N and T runs over the 3-simplices of N. According to the definitions of Sect. 3.b, the right hand side of (9.f.1) equals

$$w^{2\alpha - 2\beta}\, w_\lambda^{-1}\, | N |_\mu ,$$

where α (resp. β) is the number of vertices of N (resp. of ∂N). Then

$$\langle\langle \mathrm{N}, \gamma^{\lambda} \rangle\rangle = w^{2-2(\alpha-\beta)} \sum_{\eta \in \mathrm{adm(D)}} \langle\langle \mathrm{D} \rangle\rangle_{\eta} = w^{2}\, w_{\lambda}^{-1} \sum_{\mu \in \mathrm{E}} | \mathrm{N} |_{\mu}$$
$$= w_{\lambda}^{-1} \langle \mathrm{N}, \varnothing \,|\, \lambda \rangle.$$

This implies the claim of the Theorem in the case $\Gamma = \varnothing$.

The case $\Gamma \neq \varnothing$ is considered along the same lines combining the arguments above with the arguments used in the proof of Theorem 9.1. As in the latter proof one first considers the case of oriented Γ, and then applies the same argument as at the end of the proof of Theorem 9.1.

APPENDIX

COMPUTATION OF THE INVARIANTS ON HEEGAARD DIAGRAMS

Let us fix initial data satisfying Conditions I-IV. Let N be a compact connected oriented 3-manifold. We show how to compute the invariant $\langle\langle N, \Gamma \rangle\rangle$ of a colored fat graph $\Gamma \subset N$ using a diagram of Γ on a Heegaard surface of N.

a. *Heegaard surfaces and Heegaard diagrams.* — The notion of Heegaard surface in N is more common in the case of closed N, and therefore we start with this case. By a *Heegaard surface* in the closed (connected oriented) 3-manifold N we mean a closed connected oriented surface $F \subset N$ which splits N into the union of two handlebodies U and V bounded by F. We distinguish these handlebodies assuming that the orientation of F together with the normal vector field on F directed outwards U determines the orientation of N. Let $\varphi_1, \ldots, \varphi_g$ (resp. $\psi_1, \ldots, \psi_g$) be the boundaries of a system of meridianal disks of V (resp. of U), where g is the genus of F. Clearly, $\{\varphi_1, \ldots, \varphi_g\}$ and $\{\psi_1, \ldots, \psi_g\}$ are two collections of disjoint simple loops on F. The surface F together with these collections is called a *Heegaard diagram* of N.

Consider the case $\partial N \neq \emptyset$. In this case by a *Heegaard surface* in N we mean a closed connected surface $F \subset \mathrm{Int}\, N$ which splits N into a handlebody V bounded by F and a 3-manifold U obtained from the cylinder $F \times [0, 1]$ (with $F \times 0 = F$) by attaching several 2-handles to $F \times 1$ along certain disjoint annuli. Clearly $\partial U = F \cup \partial N$. Such a surface F is always orientable (since $F = \partial V$ and V is orientable) and we orient F so that the orientation of F together with the normal vector field directed outwards U determines the orientation of N. Let $\psi_1, \ldots, \psi_r$ be the cores of the annuli of U mentioned above transferred to F via the canonical homeomorphism $F \times 1 \to F$. (Here r is the number of 2-handles of U.) Let $\varphi_1, \ldots, \varphi_g$ be the boundaries of a system of meridianal disks of V, where g is the genus of F. As above $\{\varphi_1, \ldots, \varphi_g\}$ and $\{\psi_1, \ldots, \psi_r\}$ are two collections of disjoint simple loops on F. The surface F together with these collections is called a *Heegaard diagram* of N.

It is well known that each compact connected oriented 3-manifold N has a Heegaard surface. Here is one of the simplest constructions. Consider a Morse function $f: N \to \mathbf{R}$ such that all critical points of f of index $i = 0, 1, 2, 3$ lie on $f^{-1}(i)$. If $\partial N \neq \emptyset$ assume additionally that $f(\partial N) = 3$ and f has no critical points of index 3. Then $f^{-1}(3/2)$ is a Heegaard surface of N.

b. *Computation of the invariant.* — Let F, $\{\varphi_1, \ldots, \varphi_g\}$, $\{\psi_1, \ldots, \psi_r\}$ be a Heegaard diagram of N. We will treat the loops $\varphi_1, \ldots, \varphi_g, \psi_1, \ldots, \psi_r$ as graphs on F each having one vertex and one edge. For a sequence $J = (j_1, \ldots, j_r) \in I^r$ denote by ψ_J the colored graph on F obtained from $\psi_1 \cup \ldots \cup \psi_r$ by assigning $j_1, \ldots, j_r$ to the edges of $\psi_1, \ldots, \psi_r$ respectively. For a sequence $H = (h_1, \ldots, h_g) \in I^g$ denote by φ_H the colored graph on F obtained from $\varphi_1 \cup \ldots \cup \varphi_g$ by assigning $h_1, \ldots, h_g$ to the edges of $\varphi_1, \ldots, \varphi_g$ respectively.

Any colored 3-valent fat graph $\Gamma \subset N$ may be deformed into a colored fat graph Γ' lying in a cylindrical neighborhood $F \times [-1, 1] \subset N$ of F. (It is understood that the orientation of N induces in this cylinder the product of the orientation of F and the canonical orientation in $[-1, 1]$.) According to the results of Section 8

$$\langle\langle N, \Gamma \rangle\rangle = \langle\langle N, \Gamma' \rangle\rangle.$$

Theorem **A.1.** — *Let* Γ' *be a colored 3-valent fat graph lying in* $F \times [-1, 1] \subset N$. *Then*

$$\langle\langle N, \Gamma' \rangle\rangle = w^{\nu} \sum_{\substack{J=(j_1, \ldots, j_r) \in I^r \\ H=(h_1, \ldots, h_g) \in I^g}} \prod_{i=1}^{r} w_{j_i}^2 \prod_{k=1}^{g} w_{h_k}^2 \langle \varphi_H \mid \Gamma' \mid \psi_J \rangle$$

where $\nu = -2$ *if* $\partial N = \varnothing$, *and* $\nu = 0$ *if* $\partial N \neq \varnothing$.

Proof. — We will use the notation of Section a. Deforming $\varphi_1, \ldots, \varphi_g$ if necessary we may assume that these loops are transversal to $\psi_1, \ldots, \psi_p$. If $\partial N = \varnothing$ then the union of F and the $2g$ meridianal disks of U, V bounded by $\varphi_1, \ldots, \varphi_g, \psi_1, \ldots, \psi_g$ is obviously a simple 2-skeleton of N. Its complement consists of two 3-balls. Any diagram of Γ' on F in the sense of Section 4 gives rise in the obvious way to a diagram of Γ' on this simple 2-skeleton of N. A direct comparison of definitions implies the claim of the theorem. In the case $\partial N \neq \varnothing$ the argument is the same though instead of the meridianal disks of U one has to use the cores of the 2-handles of U.

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URA-CNRS
Département de Mathématiques
Université Louis-Pasteur
7, rue René-Descartes
67084 Strasbourg Cedex

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