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REDUCTIVE GROUP ACTIONS WITH ONE-DIMENSIONAL QUOTIENT

HANSPETER KRAFT, GERALD W. SCHWARZ

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Chapter I. INTRODUCTION

1. The Linearization Problem

Let G be a reductive complex algebraic group acting algebraically on complex affine n-space $\mathbf{A}^n =: X$. The *Linearization Problem* asks whether every such action is algebraically equivalent to a linear action, i.e., whether there is a G-equivariant isomorphism $\varphi : X \xrightarrow{\sim} V$ where V is a G-module. Another way to state this problem is the following: Is every reductive subgroup of the affine Cremona group $\mathrm{Aut}(\mathbf{A}^n)$ conjugate to a subgroup of GL_n? A detailed report on this question can be found in [Kr1], where we also describe connections with other classical problems such as the *Fixed Point Problem*, the *Cancellation Problem* and the *Equivariant Serre Problem*.

There are a number of positive results, all of which require some kind of "smallness". Linearization always holds in dimension $n=2$ (see [Ka]), and it holds for $n \leqslant 4$ in case G is semisimple ([KP], [Pa]). However, the question remains open in dimension $n=3$ for G finite and for G one-dimensional (cf. [Kr3], [KoR1], [KoR2]).

Another approach is to require that the quotient space $X/\!\!/G$ (see §4) be of small dimension. This constraint works well in the analogous situation of a smooth action of a compact Lie group K on $\mathbf{R}^n$. If the orbit space $\mathbf{R}^n/K$ has dimension $\leqslant 2$, then the action is linearizable ([Br, IV.8.5]). In the algebraic case, it is a corollary of Luna's slice theorem that linearization holds whenever $\dim X/\!\!/G = 0$, i.e., whenever the only G-invariant functions on X are constant.

Our present work arose out of the attempt to prove linearizability in the case that $\dim X/\!\!/G = 1$. In 1981, Luna outlined an attack on this problem which has been our guide. Surprisingly (to us), our work has led to the discovery of counterexamples ([Sch5]). We also have many criteria for linearizability to hold. See the next section for a more precise accounting of our results.

We wish to thank D. Luna for generously sharing his ideas and notes on the linearization problem. We thank J.-P. Serre for crucial help with Galois and group cohomology. Finally, we thank F. Knop for helpful conversations.

2. Main results

(2.1) We consider the following situation: We have an action of the reductive complex algebraic group G on a smooth affine variety X, where X is *acyclic* (i.e., has the **Z**-homology of a point). Denote by $\pi_X : X \to X/\!\!/G$ the quotient map (see §4). We assume that the quotient $X/\!\!/G$ has dimension 1. Note that these hypotheses generalize slightly those in section 1, where we assumed that $X = \mathbf{A}^n$. In the following, parenthetical references (e.g. (II.0.1)) indicate the location where a result (e.g. Theorem 1) is proved.

Theorem **1** (II.0.1). — *We have* $X/\!/G \simeq \mathbf{A}$, *the affine line, and exactly one of the following occurs*:

(1) *The action is fix-pointed, i.e., every closed orbit is a fixed point.*

(2) *There is a unique fixed point* $x_0 \in X$.

In the fix-pointed case it is known that we have linearizability (see Corollary II.0.3), so we concentrate on the second possibility. If one allows *holomorphic* equivalence, then even these actions are always linearizable:

Theorem **2** (VI.2.11(4)). — *There is a holomorphic* G-*equivariant isomorphism of* X *onto a* G-*module* V.

M. Jiang (Thesis, Brandeis, 1992) has extended Theorem 2 to the more natural case where the G-action on X is holomorphic.

(2.2) We assume for now that $X^G = \{x_0\}$. Let V denote the G-module given by the canonical action of G on the tangent space $T_{x_0}X$. It is easy to see that dim $V/\!/G = 1$ and that $V^G = \{0\}$. Let $\pi_V : V \to \mathbf{A}$ be the quotient mapping, where we arrange that $\pi_V(0) = 0$. Of course, π_V is a homogeneous polynomial function on V, and we denote its degree by d.

Our idea now is to classify all X which give rise to the same G-module V. Let $\mathscr{M}_{V,\mathbf{A}}$ denote the set of isomorphism classes of smooth acyclic affine G-varieties X with fixed quotient mapping $\pi_X : X \to \mathbf{A} \simeq X/\!/G$ such that

(1) $X^G = \{x_0\}$ is a single fixed point,
(2) $T_{x_0}X$ is G-isomorphic with V,
(3) $\pi_X(x_0) = 0 \in \mathbf{A}$.

(If φ is an allowable isomorphism of G-varieties X, X′ satisfying our conditions, then φ induces the identity on **A**.) The isomorphism class containing X is denoted $\{X\}$. Let $\mathscr{M}_V$ be defined in the same way as $\mathscr{M}_{V,\mathbf{A}}$, except that we do not fix an isomorphism of $X/\!/G$ with **A**. Clearly, $\mathscr{M}_V$ is the orbit space of $\mathscr{M}_{V,\mathbf{A}}$ by an action of $\mathbf{C}^*$, and $\mathscr{M}_V$ is trivial (i.e. a point) if and only if $\mathscr{M}_{V,\mathbf{A}}$ is trivial.

(2.3) Let F be a G-variety and Y a variety with trivial G-action. A G-*fiber bundle* (over Y) with fiber F is a G-equivariant morphism $\beta : \mathfrak{F} \to Y$ such that every fiber is G-isomorphic to F and β is locally trivial in the étale topology. This means that there is an étale surjective map $\eta : \tilde{Y} \to Y$ and a G-equivariant isomorphism $\mathfrak{F}_{\tilde{Y}} := \tilde{Y} \times_Y \mathfrak{F} \xrightarrow{\sim} \tilde{Y} \times F$ over $\tilde{Y}$ (see IV.1.3-1.5 for a more detailed discussion of this notion).

If U is an open subset of **A**, let V_U and X_U denote $\pi_V^{-1}(U)$ and $\pi_X^{-1}(U)$, respectively. Let $\dot{\mathbf{A}}$ denote $\mathbf{A} \setminus \{0\}$, and set $\dot{V} := V_{\dot{\mathbf{A}}}$, $\dot{X} := X_{\dot{\mathbf{A}}}$.

Theorem **3**. — *Let* $\{X\} \in \mathscr{M}_{V,\mathbf{A}}$.

(1) (IV.0.2) *The morphisms* $\dot{X} \to \dot{A}$ *and* $\dot{V} \to \dot{A}$ *are* G-*fiber bundles with fiber* $F := \pi_V^{-1}(1)$. *The bundles are isomorphic, hence there is a* G-*isomorphism* $\dot{\varphi} : \dot{X} \xrightarrow{\sim} \dot{V}$ *which induces the identity on* $\dot{A}$.

(2) (VI.2.11(3)) *There is an open set* $U \subseteq A$, $0 \in U$ *such that* X_U *and* V_U *are* G-*isomorphic over* U.

Thus X *is obtained from* $\dot{V}$ *and* V_U *identified by a* G-*isomorphism* $\dot{\varphi}_U$ *over* $\dot{U} := U \setminus \{0\}$.

(3) (VI.2.13) *There is a bijection* $\mathcal{M}_{V,A} \xrightarrow{\sim} D/\Gamma$ *where* Γ *denotes the dth roots of unity* ($d = \deg \pi_V$) *and* D *is a* Γ-*module.*

We give $\mathcal{M}_{V,A}$ *the structure of affine variety coming from the bijection above.*

(4) (VI.2.12) *There is an affine* G-*variety* $\mathcal{E}$ *and an equivariant smooth surjective morphism* $\mu : \mathcal{E} \to A \times \mathcal{M}_{V,A}$ *where* G *acts trivially on* $A \times \mathcal{M}_{V,A}$, *with the following property: Let* $\{Y\} \in \mathcal{M}_{V,A}$ *and set* $X := \mu^{-1}(A \times \{Y\})$. *Then* $\{Y\} = \{X\}$ *and* $\pi_X = pr_1 \circ \mu : X \to A$.

An intriguing question is the following: Is every element in $\mathcal{M}_{V,A}$ represented by a variety which is isomorphic to A^n? All the examples we give are of this type.

(2.4) To describe the moduli space $\mathcal{M}_{V,A}$ we need to determine the Γ-module D. This is done in Chapter VI. For the present, we restrict ourselves to describing criteria for D to be trivial.

Let $\Delta(V)$ denote set of all the polynomial vector fields on V, and let $\Delta_t(V)$ denote the elements of $\Delta(V)$ annihilating the generator $t (= \pi_V)$ of the G-invariant polynomials $\mathcal{O}(V)^G$ (see VI.1 for this and the following). The *degree* of an element of $\Delta(V)$ is its degree as a derivation of the graded algebra $\mathcal{O}(V)$. Now $\Delta(V)^G$ is a Lie algebra over $\mathcal{O}(A) = \mathcal{O}(V)^G$, and $\Delta_t(V)^G$ is a subalgebra. Moreover, $\Delta(V)^G$ and $\Delta_t(V)^G$ are free graded $\mathcal{O}(A)$-modules. Set $F := \pi_V^{-1}(1)$ and let L denote the (linear algebraic) group $\mathrm{Aut}(F)^G$ of G-equivariant automorphisms of F. Then $\mathfrak{l} := \mathrm{Lie}(L)$ is the restriction of $\Delta_t(V)^G$ to F. Let $\mathfrak{l}'$ denote the inverse image in $\mathfrak{l}$ of the semisimple part of the reductive Lie algebra $\mathrm{Lie}(L/\mathrm{Rad}_u(L))$.

Theorem **4** (VI.2.4(2)). — *Let* $\mathfrak{k}$ *denote the set of restrictions to* F *of the homogeneous elements of* $\Delta_t(V)^G$ *of degree at most* $d (= \deg t)$. *Then* D (*hence* $\mathcal{M}_{V,A}$) *is trivial if and only if* $\mathfrak{k} + \mathfrak{l}' = \mathfrak{l}$.

We are able to apply Theorem 4 to several classes of representations.

Theorem **5** (VI.3.2). — *The moduli space* $\mathcal{M}_{V,A}$ *is trivial if*

(1) V *is a semifree* G-*module,*

(2) G *is a torus*,

(3) $\dim V^{G^0}=1$,

(4) $\dim V \leqslant 3$,

(5) G^0 *is a simple group, or*

(6) V *is self dual as* G^0*-module.*

(A G-module V is *semifree* if the only closed G-orbits in V are fixed points or have trivial stabilizer).

(2.5) Our examples of non-trivial moduli spaces arise from G-*vector bundles* (see VII.1.1 for definitions). Consider G-vector bundles whose base is a G-module P with one-dimensional quotient. The fiber at $0 \in P$ is a G-module, and we let $\mathrm{Vec}_G(P, Q)$ denote the collection of G-vector bundles over P whose fiber at $0 \in P$ is isomorphic to the G-module Q. Let $\mathrm{VEC}_G(P, Q)$ denote the set of G-isomorphism classes in $\mathrm{Vec}_G(P, Q)$; the class of $E \in \mathrm{Vec}_G(P, Q)$ is denoted by [E]. The trivial class is represented by the product $P \times Q$, which we denote by Θ_Q. If $G=\{e\}$ is trivial, then the solution of the Serre Problem by Quillen and Suslin shows that every element of $\mathrm{Vec}(\mathbf{A}^p, \mathbf{C}^q)$ is trivial, so that every element $E \in \mathrm{Vec}_G(P, Q)$ can be considered as a G-action on some $X=\mathbf{A}^n$.

Let $E \in \mathrm{Vec}_G(P, Q)$. Let $\mathbf{C}^*$ act via scalar multiplication on the fibers of E. Then we obtain an action of $\tilde{G}:=G \times \mathbf{C}^*$ on $E \simeq \mathbf{A}^n$. It is easy to see that $E/\!/\tilde{G} \simeq P/\!/G \simeq \mathbf{A}$. The following result allows us to relate the linearization problem to moduli of G-vector bundles.

Proposition **6** (cf. VII.1.2, 1.3). – *Let* E, $E' \in \mathrm{Vec}_G(P, Q)$ *and let* $\tilde{G}$ *be as above. Then*

(1) ([Kr2]) *The vector bundle* E *is non-trivial if and only if the* $\tilde{G}$*-action on* $E \simeq \mathbf{A}^n$ *is not linearizable.*

(2) ([BH2]) *If* $E \oplus \Theta_P \in \mathrm{Vec}_G(P, Q \oplus P)$ *is non-trivial, then the* G*-action on* E *is not linearizable.*

(3) ([MP]) *Suppose that* H *is a subgroup of* G *such that* $(P \oplus Q)^H = P$. *Then* E *and* E′ *are isomorphic as* G*-varieties if and only if* E *is isomorphic to a pull-back* $\varphi^* E'$ *for some* G*-automorphism* φ *of* P.

(2.6) It turns out that $\mathrm{VEC}_G(P, Q)$ has a pleasant structure.

Theorem **7** (VII.3.4). – *The moduli space* $\mathrm{VEC}_G(P, Q)$ *has a natural structure of vector group. Moreover, there is a* G*-vector bundle* $\mu : \mathscr{B} \to P \times \mathrm{VEC}_G(P, Q)$ *such that* $\mu^{-1}(P \times [E]) \simeq E$ *for all* $E \in \mathrm{Vec}_G(P, Q)$.

Theorem **8** (VII.4.8). — *Let* Q, Q_1 *and* Q_2 *be* G-*modules, and let* H *be a principal isotropy group of* P (*see* II.1.2).

(1) *The map* $\mathrm{VEC}_G(P,Q) \to \mathrm{VEC}_G(P, Q \oplus Q)$ *sending the class* $[E] \in \mathrm{VEC}_G(P,Q)$ *into* $[E \oplus \Theta_Q]$ *is bijective.*

(2) *Let* $[E_1], [E_2] \in \mathrm{VEC}_G(P,Q)$. *Then their sum* $[E_1]+[E_2]=:[E_3]$ *in* $\mathrm{VEC}_G(P,Q)$ *is uniquely determined by the condition:* $E_1 \oplus E_2 \simeq E_3 \oplus \Theta_Q \in \mathrm{Vec}_G(P, Q \oplus Q)$.

(3) *Whitney sum induces an epimorphism of vector groups*

$$\mathrm{WS}: \mathrm{VEC}_G(P, Q_1) \times \mathrm{VEC}_G(P, Q_2) \to \mathrm{VEC}_G(P, Q_1 \oplus Q_2).$$

(4) *If* $\mathrm{Hom}(Q_1, Q_2)^H = \{0\}$, *then* WS *is bijective.*

(2.7) Let $V := P \oplus Q \simeq \Theta_Q$ with the $\tilde{G}$-action of 2.5. We want to compare $\mathrm{VEC}_G(P,Q)$ with $\mathscr{M}_{V,\mathbf{A}}$. As before, let Γ denote the group of *d*th roots of unity, where *d* is the degree of π_P. Let $[E] \in \mathrm{VEC}_G(P,Q)$ and let $\{E\}$ denote E considered as an element of $\mathscr{M}_{V,\mathbf{A}}$. Note that $\{E\} = \{\gamma^* E\}$, $\gamma \in \Gamma$, where $\gamma^* E$ denotes the pull-back of E by $\gamma: P \xrightarrow{\sim} P$. Thus we have a natural map $\lambda: \mathrm{VEC}_G(P,Q)/\Gamma \to \mathscr{M}_{V,\mathbf{A}}$.

Theorem **9** (VII.3.7). — *Suppose that* $\mathscr{M}_{P,\mathbf{A}}$ *is trivial* (*i.e., there are no non-linearizable actions modelled on P*). *Then*

$$\lambda: \mathrm{VEC}_G(P, Q)/\Gamma \to \mathscr{M}_{V,\mathbf{A}}$$

is a bijection.

(2.8) Let F_P denote $\pi_P^{-1}(1)$ and let M denote the (linear algebraic) group $\mathrm{Mor}(F_P, \mathrm{GL}(Q))^G$ (see VII.2.3). The vector group $\mathrm{VEC}_G(P,Q)$ can be computed from the Lie and Artin algebra $\mathfrak{m} := \mathrm{Lie}(M) = \mathrm{Mor}(F_P, \mathrm{End}\,Q)^G$. Let $\mathfrak{m}'$ be defined as in the case of $\mathfrak{l}$ in 2.4, and let $\mathfrak{k}$ denote the restriction to F_P of the elements of $\mathrm{Mor}(P, \mathrm{End}\,Q)^G$ which are homogeneous of degree at most $d(=\deg \pi_P)$.

Theorem **10** (VII.3.4(1)). — *The moduli space* $\mathrm{VEC}_G(P,Q)$ *is trivial if and only if* $\mathfrak{k} + \mathfrak{m}' = \mathfrak{m}$.

It is quite easy to come up with examples where $\mathrm{VEC}_G(P,Q)$ is non-trivial. Using Proposition 6 we then obtain examples of non-linearizable actions of G on $\mathbf{A}^n$ and of non-linearizable actions of $G \times \mathbf{C}^*$ on $\mathbf{A}^n$ with one-dimensional quotient.

Theorem **11** (VII.5.9, 5.4, 5.7). — (1) *Let* G *be a simple classical group, a spin group,* G_2, E_6 *or* E_7. *Then* G *has a non-linearizable faithful action on* $\mathbf{A}^n$ *for some n.*

(2) *There are non-linearizable actions of* O_2 *on* $\mathbf{A}^4$, *of* SL_2 *on* $\mathbf{A}^7$ *and of* SO_3 *on* $\mathbf{A}^{10}$.

(2.9) *Remarks.* — (1) With our methods we obtain explicit families of non-trivial G-vector bundles which give rise to non-linearizable actions on affine space (VII.5). But only with the later work of Masuda and Petrie (see Proposition 6(3)) was it realized that these families of G-vector bundles contain families of non-equivalent G-actions on affine space (VII.5.4, 5.7(2)).

(2) Knop [Kn] has shown that for every semisimple G there exist non-trivial G-vector bundles E with base Lie(G) such that the G-actions on E are not linearizable. Again, one can show that these vector bundles lead to families of non-equivalent G-actions, provided that G has a non-trivial center (VII.5.8).

(3) Masuda, Moser-Jauslin and Petrie ([MP], [MMP]) have constructed families of non-trivial G-vector bundles and families of non-linearizable G-actions. They have shown that some of our examples of $O_2 \times \mathbf{C}^*$-actions on $\mathbf{A}^4$ remain non-linearizable when restricted to certain finite subgroups of $O_2 \times \mathbf{C}^*$, providing the first examples of non-linearizable actions of finite groups.

(4) There are no known examples of non-trivial G-vector bundles or non-linearizable G-actions on affine space for commutative groups G (cf. [Kr3], [KoR1], [KoR2]).

3. Methods

We discuss some of the ideas and methods we use.

(3.1) In Chapter II we establish the topological part of our results, following an outline of Luna. The fact that $X/\!/G \simeq \mathbf{A}$ is quite easy. A careful study of the Leray spectral sequence of $\pi_X : X \to X/\!/G$, using properties of Luna's stratification of $X/\!/G$ (see II.1), gives Theorem 1. If $X^G \simeq \mathbf{A}$, we have linearizability, so we assume that $X^G = \{ x_0 \}$ where $\pi_X(x_0) = 0 \in \mathbf{A}$. Then the Luna strata of $X/\!/G \simeq \mathbf{A}$ are $\dot{\mathbf{A}}$ and $\{0\}$. As before, let V denote the G-module $T_{x_0}X$. Then $\pi_X : \dot{X} \to \dot{\mathbf{A}}$ and $\pi_V : \dot{V} \to \dot{\mathbf{A}}$ are G-fiber bundles (see 2.3) with fiber $F = \pi_V^{-1}(1)$.

The next task is to establish that the two G-fiber bundles $\dot{X}$ and $\dot{V}$ are isomorphic. We show that the G-automorphisms of F form a linear algebraic group, denoted L, so that $\dot{X}$ and $\dot{V}$ correspond as usual to principal L-bundles P_X and P_V over $\dot{\mathbf{A}}$, respectively (see IV.1.1-1.4). Let us denote by $H^1_{et}(Y, L)$ the set of isomorphism classes of principal L-bundles over the variety Y.

Theorem **12** (IV.5.4). — *For any linear algebraic group* M *the canonical map*

$$H^1_{et}(\dot{\mathbf{A}}, M) \xrightarrow{\sim} H^1_{et}(\dot{\mathbf{A}}, M/M^0)$$

is a bijection.

It is not difficult to see that this result implies Theorem 3(1). Note that the principal L/L^0-bundles over $\dot{\mathbf{A}}$ are the same in the topological and algebraic categories. In fact, they are just the finite covers with Galois group L/L^0. It follows from Luna's slice theorem that there is a neighborhood U of $0 \in \mathbf{A}$ (classical topology) such that X_U and V_U are analytically G-isomorphic over U. Then P_X and P_V are isomorphic over $\dot{U}$, hence they are topologically isomorphic over $\dot{\mathbf{A}}$. Thus the principal L/L^0-bundles P_X/L^0 and P_V/L^0 are isomorphic (algebraically!), and so $P_X \simeq P_V$ by Theorem 12.

(3.2) To continue our discussion, we need to consider a small menagerie of spaces. Identify $\mathcal{O}(\mathbf{A})$ with $\mathbf{C}[t]$. Then $\mathcal{O}(\dot{\mathbf{A}}) = \mathbf{C}[t, t^{-1}]$ and set $\hat{\mathbf{A}} := \operatorname{Spec} \mathbf{C}[\![t]\!]$. The schematic intersection $\dot{\mathbf{A}} \cap \hat{\mathbf{A}}$ is $\dot{\hat{\mathbf{A}}} := \operatorname{Spec} \mathbf{C}((t))$. Set

$$\hat{V} := V \times_{\mathbf{A}} \hat{\mathbf{A}} = \operatorname{Spec}(\mathcal{O}(V) \otimes_{\mathcal{O}(\mathbf{A})} \mathcal{O}(\hat{\mathbf{A}})),$$

and define $\hat{X}$, $\dot{\hat{V}}$ and $\dot{\hat{X}}$ similarly.

If Y is an $\mathbf{A}$-scheme, let $\mathfrak{A}(Y)$ denote the group of G-automorphisms of $Y \times_{\mathbf{A}} V$ which induce the identity on the quotient Y. Then $\mathfrak{A}$ is a group valued functor, and we show that it is represented by a group scheme (also denoted $\mathfrak{A}$) over $\mathbf{A}$. For now it is most important to note that $\mathfrak{A}(\dot{\mathbf{A}}) = \{$G-automorphisms of $\dot{V}$ inducing the identity on $\dot{\mathbf{A}}\}$, and similarly for $\mathfrak{A}(\hat{\mathbf{A}})$ and $\mathfrak{A}(\dot{\hat{\mathbf{A}}})$. Equivalently, the opposite group to $\mathfrak{A}(\dot{\mathbf{A}})$ is the group of G- and $\mathcal{O}(\dot{\mathbf{A}})$-automorphisms of $\mathcal{O}(\dot{V})$, etc.

Let $\{X\} \in \mathcal{M}_{V, \mathbf{A}}$ (see 2.2). The slice theorem gives a G-isomorphism $\hat{\varphi}: \hat{X} \xrightarrow{\sim} \hat{V}$ which induces the identity on $\hat{\mathbf{A}}$ (see II.0.4). By Theorem 3(1) we have a G-isomorphism $\dot{\varphi}: \dot{X} \xrightarrow{\sim} \dot{V}$ which induces the identity on $\dot{\mathbf{A}}$. The composition $\dot{\hat{\varphi}} := \dot{\varphi}\hat{\varphi}^{-1}$ lies in $\mathfrak{A}(\dot{\hat{\mathbf{A}}})$. Now $\dot{\varphi}$ is only determined up to composition with an element $\dot{\alpha} \in \mathfrak{A}(\dot{\mathbf{A}})$, and similarly for $\hat{\varphi}$. Thus the double coset of $\dot{\hat{\varphi}}$ in

$$D\mathfrak{A} := \mathfrak{A}(\dot{\mathbf{A}}) \backslash \mathfrak{A}(\dot{\hat{\mathbf{A}}}) / \mathfrak{A}(\hat{\mathbf{A}})$$

is well-defined, and we denote it by $[\dot{\hat{\varphi}}(X)]$. In this way, we obtain a map

$$[\dot{\hat{\varphi}}]: \mathcal{M}_{V, \mathbf{A}} \to D\mathfrak{A}, \quad \{X\} \mapsto [\dot{\hat{\varphi}}(X)].$$

(3.3) We eventually are able to show that $[\dot{\hat{\varphi}}]$ is an isomorphism (VI.2.13), and to identify $D\mathfrak{A}$ with a quotient D/Γ (VI.2.7). Simultaneously, we obtain that there is an isomorphism $\varphi_U: X_U \xrightarrow{\sim} V_U$ over a neighborhood $U \ni 0$ (VI.2.11), and we can then construct the moduli space (VI.2.12). Here are some of the main steps.

Let $\mathbf{B}$ denote $\operatorname{Spec} \mathbf{C}[s]$, where $t = s^d$. Then we have a canonical morphism $\mathbf{B} \to \mathbf{A}$, $z \mapsto z^d$, which identifies $\mathbf{A}$ with $\mathbf{B}/\Gamma$, where $\Gamma = \{d\text{th roots of unity}\}$ acts by scalar multiplication on $\mathbf{B}$. The group Γ also acts on F by scalar multiplication, commuting with the action of G. Thus Γ is a subgroup of the (linear algebraic) group

$L\,(=\mathrm{Aut}\,(F)^G)$. Let Γ act on $\mathbf{B}\times F$ by $\gamma(z,v)=(z\gamma^{-1},\gamma v)$, $\gamma\in\Gamma$, $z\in\mathbf{B}$, $v\in F$. Denote the corresponding quotient by $\mathbf{B}*^{\Gamma}F$. Then we have a G-equivariant morphism

$$\rho:\mathbf{B}*^{\Gamma}F\to V,\qquad [z,v]\mapsto zv,$$

over $\mathbf{A}=\mathbf{B}/\Gamma$, and clearly ρ induces an isomorphism over $\dot{\mathbf{A}}$.

Note that L is a Γ-group, i.e., it is a linear algebraic group together with a homomorphism $\tau:\Gamma\to\mathrm{Aut}(L)$. Here $\tau(\gamma)l$ is just $\gamma l\gamma^{-1}$, $\gamma\in\Gamma$, $l\in L$. Let $L(\mathbf{B})$ denote the group of morphisms from $\mathbf{B}$ to L. Then Γ acts on $L(\mathbf{B})$ by ${}^{\gamma}\varphi(b)=\tau(\gamma)\,\varphi(b\gamma)=\gamma\varphi(b\gamma)\gamma^{-1}$, $\gamma\in\Gamma$, $\varphi\in L(\mathbf{B})$, $b\in\mathbf{B}$. We let $L(\mathbf{B})^{\Gamma}$ denote the set of fixed points of the Γ-action. One defines groups $L(\dot{\mathbf{B}})^{\Gamma}$, etc. similarly. The morphism ρ above induces a bijection ρ_* of $L(\hat{\mathbf{B}})^{\Gamma}$ onto $\mathfrak{A}(\hat{\mathbf{A}})$, and its inverse, denoted σ_*, induces bijections

$$\sigma_*:\mathfrak{A}(\hat{\mathbf{A}})\xrightarrow{\sim}L(\hat{\mathbf{B}})^{\Gamma},\qquad \sigma_*:\mathfrak{A}(\dot{\mathbf{A}})\xrightarrow{\sim}L(\dot{\mathbf{B}})^{\Gamma},$$

and an inclusion (III.4.6)

$$\sigma_*:\mathfrak{A}(\mathscr{A})\hookrightarrow\mathfrak{L}(\mathscr{B})^{\Gamma}.$$

Thus we obtain an isomorphism

$$(*)\qquad D\mathfrak{A}\simeq L(\dot{\mathbf{B}})^{\Gamma}\backslash L(\hat{\mathbf{B}})^{\Gamma}/\sigma_*\mathfrak{A}(\hat{\mathbf{A}}).$$

(3.4) We say that the Γ-group L has the *decomposition property* if

$$L(\hat{\mathbf{B}})^{\Gamma}=L(\dot{\mathbf{B}})^{\Gamma}\,L(\hat{\mathbf{B}})^{\Gamma},$$

and we similarly say that $\mathfrak{A}$ has the decomposition property if

$$\mathfrak{A}(\hat{\mathbf{A}})=\mathfrak{A}(\dot{\mathbf{A}})\,\mathfrak{A}(\hat{\mathbf{A}}).$$

Note that $\mathfrak{A}$ has the decomposition property if and only if the double coset space $D\mathfrak{A}$ is trivial.

Theorem 13 (V.2.6). – *Let* M *be a* Γ*-group. Then* M *has the decomposition property.*

Using $(*)$ and the fact that $L(\dot{\mathbf{B}})^{\Gamma}\cap L(\hat{\mathbf{B}})^{\Gamma}=L(\mathbf{B})^{\Gamma}$ we obtain

$$D\mathfrak{A}\simeq L(\mathbf{B})^{\Gamma}\backslash L(\hat{\mathbf{B}})^{\Gamma}/\sigma_*\mathfrak{A}(\hat{\mathbf{A}}).$$

Clearly, the key to determining $D\mathfrak{A}$ is to understand the image of $\mathfrak{A}(\hat{\mathbf{A}})$ in $L(\hat{\mathbf{B}})^{\Gamma}$.

(3.5) We now study the double coset spaces above using the "Lie algebras" of the corresponding groups. Clearly $\mathrm{Lie}(L(\hat{\mathbf{B}}))=\mathfrak{l}(\hat{\mathbf{B}})$, where $\mathfrak{l}=\mathrm{Lie}(L)$, $\mathfrak{l}(\hat{\mathbf{B}})=\mathrm{Mor}(\hat{\mathbf{B}},\mathfrak{l})$, and the Γ-action is induced from that on L and $\hat{\mathbf{B}}$. There is an *exponential map*, induced from the exponential map of $\mathfrak{l}$. Let $\mathfrak{l}(\hat{\mathbf{B}})_r$, $r \geqslant 0$, denote the algebra of morphisms which vanish to order r at $0 \in \hat{\mathbf{B}}$ (so elements of $\mathfrak{l}(\hat{\mathbf{B}})_1$ send $0 \in \hat{\mathbf{B}}$ to $0 \in \mathfrak{l}$; see V.0.6 and V.3.1). Define $L(\hat{\mathbf{B}})_r$ similarly. Then exp induces isomorphisms of $\mathfrak{l}(\hat{\mathbf{B}})_r$ with $L(\hat{\mathbf{B}})_r$, $r \geqslant 1$, and of $\mathfrak{l}(\hat{\mathbf{B}})_r^\Gamma$ with $L(\hat{\mathbf{B}})_r^\Gamma$, $r \geqslant 1$ (V.3.2).

In the case of the group $\mathfrak{A}(\hat{A})$, the Lie algebra is $\mathfrak{X}(\hat{A}) := \Delta_t(V) \otimes_{\mathcal{O}(A)} \mathcal{O}(\hat{A})$, the algebra of G-derivations of $\mathcal{O}(\hat{V})$ which annihilate $\mathcal{O}(\hat{V})^G = \mathcal{O}(\hat{A})$ (see 2.4). Let $\mathbf{m}$ denote the ideal in $\mathcal{O}(\hat{V})$ of functions vanishing at the origin. We give $\mathcal{O}(\hat{V})$ the $\mathbf{m}$-adic filtration and $\mathfrak{X}(\hat{A})$ the induced filtration, so that $\mathfrak{X}(\hat{A})_r$ consists of the elements sending $\mathbf{m}^j$ to $\mathbf{m}^{j+r}$ for all j. We filter $\mathfrak{A}(\hat{A})$ similarly. In particular, $\mathfrak{A}(\hat{A})_1$ consists of automorphisms fixing the origin of $\hat{V}$. We show that there is a natural exponential map $\mathfrak{X}(\hat{A})_r \xrightarrow{\sim} \mathfrak{A}(\hat{A})_r$, $r \geqslant 1$ (VI.1.6).

(3.6) We now study the morphism $\sigma_* : \mathfrak{A}(\hat{A})_r \to L(\hat{\mathbf{B}})_r^\Gamma$ via the corresponding morphism of Lie algebras $\sigma_\# : \mathfrak{X}(\hat{A})_r \to \mathfrak{l}(\hat{\mathbf{B}})_r^\Gamma$ (see VI.1.14):

$$\begin{array}{ccc} \mathfrak{A}(\hat{A})_r & \xrightarrow[\sigma_*]{\subset} & L(\hat{\mathbf{B}})_r^\Gamma \\ \text{exp}\uparrow\wr & & \wr\uparrow\text{exp} \\ \mathfrak{X}(\hat{A})_r & \xrightarrow[\sigma_\#]{\subset} & \mathfrak{l}(\mathscr{B})_r^\Gamma \end{array}$$

Theorem **14**. — (1) (VI.1.13(2), VI.1.14(2)) *There is an integer* $r_0 \geqslant 1$ *such that* $\sigma_\# : \mathfrak{X}(\hat{A})_r \xrightarrow{\sim} \mathfrak{l}(\mathscr{B})_r^\Gamma$ *is bijective for* $r \geqslant r_0$. *Thus* $\sigma_* : \mathfrak{A}(\hat{A})_r \xrightarrow{\sim} L(\hat{\mathbf{B}})_r^\Gamma$ *is a bijection for* $r \geqslant r_0$.

(2) (V.3.5) *Let* M *be a* Γ*-group such that* $\mathrm{Rad}(M^0) = \mathrm{Rad}_u(M^0)$. *Then*

$$M(\hat{\mathbf{B}})_1^\Gamma = M(\mathbf{B})_1^\Gamma M(\hat{\mathbf{B}})_r^\Gamma \quad \textit{for all } r \geqslant 1.$$

The property in part (2) above we call the *approximation property* for M. Note that if L has the approximation property, then $\mathfrak{A}$ has the decomposition property, since $L(\hat{\mathbf{B}})_r^\Gamma \subseteq \sigma_* \mathfrak{A}(\hat{A})_r$ for $r \geqslant r_0$ and $L(\hat{\mathbf{B}})^\Gamma \subseteq L(\hat{\mathbf{B}})^\Gamma$. In particular, $D\mathfrak{A}$ and $\mathscr{M}_{V,A}$ are trivial in this case.

(3.7) Write the identity component L^0 of L as $Z \cdot L'$ where Z is the central torus in a Levi factor $\tilde{L}$ of L and L' is generated by the semisimple part of $\tilde{L}$ and the unipotent radical of L. We may arrange that Z, etc. are Γ-stable. Now L' has the approximation property by Theorem 14(2), hence $D\mathfrak{A}$ is the image of the vector group $D' := Z(\hat{\mathbf{B}})_1^\Gamma / Z(\hat{\mathbf{B}})_{r_0}^\Gamma$. It follows that $D\mathfrak{A}$ is isomorphic to D/Γ, where D is a quotient vector group of D' with linear Γ-action (VI.2.7).

Let $\hat{z} \in Z(\hat{\mathbf{B}})_1^\Gamma$ and $r \geqslant 0$. One can show that, modulo $Z(\hat{\mathbf{B}})_r^\Gamma$, $\hat{z}$ can be represented by a rational morphism from $\mathbf{B}$ to Z (VI.2.10(1)). It then follows from the results above that every double coset $[\dot{\alpha}] \in D\mathfrak{A}$ can be represented by a rational section of $\mathfrak{A}$. But such a rational section is a G-automorphism $\dot{\varphi}_U$ of $V_{\dot{U}}$ over $\dot{U} = U \setminus \{0\}$ for some neighborhood U of $0 \in \mathbf{A}$ (which gives Theorem 3 (2)). Using $\dot{\varphi}_U$ we glue $\dot{V}$ and V_U to obtain a G-variety X such that $[\hat{\dot{\varphi}}(X)] = [\hat{\dot{\alpha}}]$. Thus the map $[\hat{\dot{\varphi}}] : \mathscr{M}_{V, \mathbf{A}} \to D\mathfrak{A}$ is surjective. It is easy to show that it is also injective, so we have that

$$\mathscr{M}_{V, \mathbf{A}} \simeq D\mathfrak{A} \simeq D/\Gamma.$$

4. Conventions and notation

Our base field is the field $\mathbf{C}$ of complex numbers. Let G be a reductive algebraic group acting on an affine variety X. We denote by $\mathcal{O}(X)$ the $\mathbf{C}$-algebra of regular functions on X and by $\mathcal{O}(X)^G$ the subalgebra of G-invariants. A celebrated theorem of Hilbert shows that $\mathcal{O}(X)^G$ is finitely generated (see [Kr, II.3.2]). Let $X/\!/G$ denote the corresponding affine variety, and let $\pi_X : X \to X/\!/G$ denote the morphism corresponding to the inclusion $\mathcal{O}(X)^G \subseteq \mathcal{O}(X)$.

Proposition (see [Kr, II.3.2]). — *Let* G, X, *etc. be as above. Then*

(1) π_X *is surjective.*

(2) π_X *separates disjoint closed* G-*stable subsets of* X.

(3) *Every orbit contains a unique closed orbit in its closure, and* π_X *sets up a bijection between the closed orbits of* X *and the points of* $X/\!/G$.

If W is a module for the algebraic group M, we will use the notation (W, M) (in place of just W) when it is necessary to emphasize the group involved. We use μ_d to denote $\{d\text{th roots of unity}\} \subset \mathbf{C}^* = GL_1(\mathbf{C})$.

Chapter II. EXISTENCE OF FIXED POINTS

0. Résumé

(0.1) Let G be a reductive group acting on a smooth affine variety X. Assume that X is acyclic, i.e., that X has the **Z**-homology of a point, and that the quotient X//G is one-dimensional. The aim of this chapter is to prove the following result:

Theorem. — The quotient X//G *is isomorphic to* **A**, *and exactly one of the following occurs*:

(1) $X^G \simeq \mathbf{A}$, *the quotient map* $\pi := \pi_X : X \to \mathbf{A}$ *is a* G-*fiber bundle* (*see* I.2.3), *and every fiber contains a fixed point.*

(2) *There is exactly one fixed point* $x_0 \in X$, *and the induced map* $\dot{\pi} : \dot{X} \to \dot{\mathbf{A}}$ *is a* G-*fiber bundle, where* $\dot{X} := X \setminus \pi^{-1}(\pi(x_0))$ *and* $\dot{\mathbf{A}} := \mathbf{A} \setminus \{\pi(x_0)\}$.

(0.2) *Remark.* — It is shown in [KPR, Theorem 0.1 B] that for every acyclic (resp. contractible) G-variety Z, the quotient Z//G is again acyclic (resp. contractible) (cf. [Sch4, 5.7]). In our case the quotient X//G is normal and one-dimensional, hence smooth, and so $X/\!/G \simeq \mathbf{A}$, since **A** is the only acyclic smooth curve ([KPR, Lemma 5.6]).

If $X \simeq \mathbf{A}^n$, then there is a more direct argument (Luna): In this case X//G is a unirational curve, hence rational (Lüroth), and it admits no non-constant invertible functions. It follows that $X/\!/G \simeq \mathbf{A}$.

(0.3) *Corollary.* — (1) *The fixed point set* X^G *is either a single point or is isomorphic to* **A**.

(2) *If* $X^G \simeq \mathbf{A}$, *then* X *is* G-*isomorphic to* $\mathbf{C} \times W$ *where* W *is a* G-*module and* **C** *the trivial* G-*module.*

Proof. — Part (1) is clear from the theorem. If $X^G \simeq \mathbf{A}$ then the quotient map $\pi_X : X \to \mathbf{A}$ has the structure of a G-vector bundle (see 3.3 below), and this bundle is trivial by [Kr2, 2.1 Corollary]. ■

Another proof of part (2) will be given in IV.3.9.

(0.4) Let us consider the case of an isolated fixed point $x_0 \in X$, and let $\pi_X : X \to \mathbf{A}$ be the quotient map, where $\pi_X(x_0) = 0$. Denote by V the G-module $T_{x_0}X$, the tangent space at x_0. It follows from the slice theorem that V//G is one-dimensional, and so the quotient map $\pi_V : V \to \mathbf{A}$ is given by a homogeneous polynomial t. Define $\hat{\mathbf{A}} := \operatorname{Spec} \mathbf{C}[[t]]$, and denote by $\hat{X}$ and $\hat{V}$ the fiber products $X \times_{\mathbf{A}} \hat{\mathbf{A}}$ and $V \times_{\mathbf{A}} \hat{\mathbf{A}}$, respectively (see I.3.2).

Proposition. — *There is an étale neighborhood* $U \to \mathbf{A}$ *of* $0 \in \mathbf{A}$ *and a G-equivariant isomorphism* $\varphi_U : X \times_{\mathbf{A}} U \xrightarrow{\sim} V \times_{\mathbf{A}} U$ *which induces the identity on* U. *In particular, the general fibers of* π_X *and* π_V *are isomorphic, and we get a* G-*equivariant isomorphism* $\hat{\varphi} : \hat{X} \xrightarrow{\sim} \hat{V}$ *over* $\hat{\mathbf{A}}$.

Proof. — It follows from the slice theorem that there exists a G-equivariant morphism $\eta : X \to V$ which induces an isomorphism $\hat{\eta} : \hat{X} \xrightarrow{\sim} \hat{V}$, hence an automorphism $\bar{\eta} : \hat{\mathbf{A}} \xrightarrow{\sim} \hat{\mathbf{A}}$ of the quotient. Clearly, $\bar{\eta}$ is given by multiplication with a unit $e \in \mathbf{C}[[t]]^* \cap \mathbf{C}[t]$. There is an $f \in \mathbf{C}[[t]]$ such that $f^d = e^{-1}$, where $d = \deg t$. It follows that the isomorphism $\hat{\varphi} := (f \circ \pi_X) \cdot \hat{\eta} : \hat{X} \xrightarrow{\sim} \hat{V}$ has the required property. In addition, since f is algebraic over $\mathbf{C}[t]$, we can find an affine étale neighborhood $U \to \mathbf{A}$ of $0 \in \mathbf{A}$ such that $f \in \mathcal{O}(U)^*$ and such that $\eta_U : X_U \to V_U$ is an isomorphism, where $X_U := X \times_{\mathbf{A}} U$, etc. It follows that the induced map $\varphi_U := (f \circ \pi_X) \cdot \eta_U : X_U \to V_U$ is an isomorphism over U. ■

(0.5) In order to establish assertions (1) and (2) of Theorem 0.1 we use the Leray spectral sequence of the quotient map $\pi := \pi_X : X \to X/\!\!/G$ (§ 3). It turns out that the higher direct images $R^q \pi_* \mathbf{k}$, where $\mathbf{k}$ is the constant sheaf with fiber a field k, are locally constant on the strata of the Luna stratification (1.4). The acyclicity of these sheaves on special open sets (2.1) enables us to calculate their cohomology by using a Leray covering of the quotient (4.1). It follows that the Leray spectral sequence of π collapses, and we can then establish the theorem.

1. Stratification and direct images

(1.1) Let Z be a smooth affine G-variety with quotient map $\pi_Z : Z \to Z/\!\!/G$. If $z \in Z$ is a point whose orbit Gz is closed, then the stabilizer $H = G_z$ is a reductive subgroup of G (Theorem of Matsushima, cf. [Lu, I.2]). It follows that there is an H-stable complement of the tangent space $T_z(Gz)$ in $T_z(Z)$:

$$T_z(Z) = T_z(Gz) \oplus N_z.$$

N_z is the *normal space* at z, and the representation of the stabilizer H on N_z is called the *slice representation* at z.

We denote by $G *^H N_z$ the *associated bundle* of the principal H-bundle $G \to G/H$: It is the quotient of $G \times N_z$ by the free action of H given by $h(g, v) := (gh^{-1}, hv)$, $h \in H$, $g \in G$, $v \in N_z$. The image of an element $(g, v) \in G \times N_z$ in $G *^H N_z$ will be denoted by $[g, v]$. Clearly, $G *^H N_z$ is the *normal bundle* of the closed orbit Gz.

As a consequence of the slice theorem ([Lu, III], see [S1]) there exist neighborhoods U of $\pi_Z(z) \in Z/\!\!/G$ and U′ of $\pi_{N_z}(0) \in N_z/\!\!/H$ (classical topology) and a G-equivariant analytic isomorphism

$$G *^H \pi_{N_z}^{-1}(U') \xrightarrow{\sim} \pi_Z^{-1}(U).$$

Moreover, we have the following:

(1.2) *Proposition* (Luna). — *There is a finite stratification* $Z/\!\!/G = \bigcup Y_i$ *into locally closed subvarieties* Y_i *with the following properties:*

(1) *The isotropy groups of points* z_i *of the closed orbits over* Y_i *are all conjugate to a fixed reductive subgroup* $H_i \subseteq G$, *and the corresponding slice representations* N_{z_i} *are all equivalent to a fixed representation* N_i *of* H_i.

(2) *The quotient map* $\pi_i : \pi_Z^{-1}(Y_i) \to Y_i$ *is a* G-*fiber bundle* (*see* I.2.3) *with fiber* $G *^{H_i} \mathfrak{N}(N_i)$, *where* $\mathfrak{N}(N_i)$ *is the null cone of the representation* N_i *of* H_i, *i.e.*, $\mathfrak{N}(N_i) := \{ v \in N_i \mid \overline{H_i v} \ni 0 \}$.

(3) *If* $\bar{Y}_j \supseteq Y_i$, *then* H_j *is conjugate to a subgroup of* H_i. (*In this case we will always arrange that* $H_j \subseteq H_i$.)

The stratification above is called the *Luna stratification* of $Z/\!\!/G$. If $Z/\!\!/G$ is connected (hence irreducible), then the open stratum is called the *principal stratum*, and the corresponding isotropy groups are called *principal isotropy groups*.

(1.3) Let k be any field and denote by $\mathscr{H}^q$ the qth direct image of the constant sheaf $\mathbf{k} := k \times Z$ under the quotient map $\pi : Z \to Z/\!\!/G$:

$$\mathscr{H}^q := R^q \pi_* \mathbf{k}.$$

This is a sheaf of k-vector spaces on $Z/\!\!/G$: it is the associated sheaf to the presheaf

$$U \mapsto H^q(\pi^{-1}(U), k)$$

where H^q denotes ordinary (singular) cohomology (see [Go, II.4.17]). We call $\mathscr{H}^q$ the *Leray sheaf* of the quotient map $\pi : Z \to Z/\!\!/G$. It figures in the *Leray spectral sequence*

$$H^p(Z/\!\!/G, \mathscr{H}^q) \Rightarrow H^{p+q}(Z, k)$$

of π.

Proposition. — *Let* $y_0 \in Y$, *where* Y *is a stratum of* $Z/\!\!/G$. *Then there is a fundamental system of neighborhoods* $\{U_i\}$ *of* y_0 *in* $Z/\!\!/G$ *such that for every* $y \in Y \cap U_i$ *the inclusion* $j_y : O_y \hookrightarrow \pi^{-1}(U_i)$ *induces an isomorphism* $j_y^* : H^*(\pi^{-1}(U_i), k) \xrightarrow{\sim} H^*(O_y, k)$, *where* O_y *is the closed orbit in the fiber* $\pi^{-1}(y)$. *Moreover, the inverse of* j_y^* *is induced by a* G-*retraction* $\rho_y : \pi^{-1}(U_i) \to O_y$.

(1.4) *Corollary.* — *The Leray sheaves $\mathscr{H}^q$ of π are locally constant on the strata of $Z/\!/G$ with stalks $\mathscr{H}^q_y \simeq H^q(O_y, k)$.*

In fact, for every U_i as above and every stratum Y, the sheaf $\mathscr{H}^q|_{Y \cap U_i}$ is constant.

Proof of Proposition 1.3. — Let $z_0 \in O_{y_0}$, let $H := G_{z_0}$ and let N be the slice representation of H at z_0. Denote by W an H-stable complement of N^H in N and consider the G-variety $\tilde{Z} := (G *^H W) \times Y$. Its quotient map is $\tilde{\pi} = \pi' \times \mathrm{id} : \tilde{Z} \to \tilde{Z}/\!/G = W/\!/H \times Y$, where $\pi' : G *^H W \to W/\!/H$ is given by $\pi'[g, w] = \pi_W(w)$. It follows from the slice theorem (see 1.1) that there are neighborhoods U of y_0 in $Z/\!/G$ and $\tilde{U}$ of $\tilde{y}_0 := (\pi'(0), y_0)$ in $\tilde{Z}/\!/G$ and an analytic G-isomorphism $\pi^{-1}(U) \xrightarrow{\sim} \tilde{\pi}^{-1}(\tilde{U})$. In addition, the stratum in $\tilde{Z}/\!/G$ containing $\tilde{y}_0$ is $\{\pi'(0)\} \times Y$. As a consequence, we may reduce to the case $Y = \{\pi_W(0)\} \subset W/\!/H$.

Using the scalar action of $\mathbf{R}^+$ on W, it is easy to see that there is a fundamental system of neighborhoods $\{B_i\}$ of $\pi_W(0) \in W/\!/H$ such that each $\pi_W^{-1}(B_i) \subset W$ is a starlike H-stable neighborhood of 0. Hence the inclusion $G/H \simeq G *^H \{0\} \hookrightarrow G *^H \pi_W^{-1}(B_i)$ induces an isomorphism in cohomology, whose inverse map is induced by the bundle projection $G *^H \pi_W^{-1}(B_i) \to G/H$. ■

(1.5) *Remarks.* — (1) Assume that U is a simply connected neighborhood of a point $y_0 \in Z/\!/G$ such that $U \setminus \{y_0\}$ is contained in the principal stratum. Then we have a canonical isomorphism

$$\sigma_0 : \mathscr{H}^q(U) \xrightarrow{\sim} \mathscr{H}^q_{y_0}.$$

(2) Let U and $y_0 \in U$ be as above and choose $x \in U \setminus \{y_0\}$. Denote by O_x and O_{y_0} the closed orbits in the fibers of x and $y_0 \in Z/\!/G$, respectively. It follows from Proposition 1.3 that the composition

$$\mu_x : H^q(O_{y_0}, k) = \mathscr{H}^q_{y_0} \underset{\sim}{\xrightarrow{\sigma_0^{-1}}} \mathscr{H}^q(U) \xrightarrow{j_x^*} \mathscr{H}^q_x = H^q(O_x, k)$$

is induced by a G-equivariant map

$$O_x \to O_{y_0}.$$

The kernel of μ_x does not depend on the choice of $x \in U \setminus \{y_0\}$: If $x' \in U \setminus \{y_0\}$, then there is a path C in $U \setminus \{y_0\}$ from x to x'. Now $\pi^{-1}(C) \to C$ is a trivial G-fiber bundle, hence we have a G-equivariant homotopy of the inclusions of $\pi^{-1}(x)$ and $\pi^{-1}(x')$ into $\pi^{-1}(U)$. Thus $\mathrm{Ker}\,\mu_x = \mathrm{Ker}\,\mu_{x'}$.

2. Acyclicity

The following was first proved by C. Berger:

(2.1) *Proposition. — Let $\mathscr{F}$ be a sheaf of k-vector spaces on* **A** *which is locally constant on* $\dot{\mathbf{A}} := \mathbf{A} \setminus \{0\}$. *Then $\mathscr{F}$ is acyclic, i.e.,*

$$H^0(\mathbf{A}, \mathscr{F}) = \mathscr{F}_0 \quad \textit{and} \quad H^q(\mathbf{A}, \mathscr{F}) = 0 \quad \textit{for} \quad q > 0.$$

Proof. — Define $U_r := \{z \in \mathbf{A} \mid |z| < r\}$. It suffices to show that the canonical maps $\rho_r : H^q(\mathbf{A}, \mathscr{F}) \to H^q(U_r, \mathscr{F})$ are isomorphisms for all $r > 0$ and all q. Then

$$H^q(\mathbf{A}, \mathscr{F}) \xrightarrow{\sim} \varinjlim_r H^q(U_r, \mathscr{F}) \xrightarrow{\sim} H^q(\{0\}, \mathscr{F}) = \begin{cases} \mathscr{F}_0 & \text{for } q = 0, \\ 0 & \text{for } q > 0, \end{cases}$$

where the second isomorphism follows from continuity (see [Go, II, Theorem 4.11.1]).

Now define $\dot{U}_r := U_r \setminus \{0\}$ and choose $0 < \varepsilon < \tau$. From Mayer-Vietoris we obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccc}
\ldots H^q(\mathbf{A}, \mathscr{F}) & \to & H^q(\dot{\mathbf{A}}, \mathscr{F}) \oplus H^q(U_\varepsilon, \mathscr{F}) & \to & H^q(\dot{U}_\varepsilon, \mathscr{F}) \ldots \\
\downarrow \rho_r & & \downarrow \dot{\rho}_r \oplus \mathrm{id} & & \| \\
\ldots H^q(U_r, \mathscr{F}) & \to & H^q(\dot{U}_r, \mathscr{F}) \oplus H^q(U_\varepsilon, \mathscr{F}) & \to & H^q(\dot{U}_\varepsilon, \mathscr{F}) \ldots
\end{array}$$

Hence it suffices to prove that $\dot{\rho}_r : H^q(\dot{\mathbf{A}}, \mathscr{F}) \to H^q(\dot{U}_r, \mathscr{F})$ is an isomorphism for all $r > 0$ and all q. Define

$$V_1 := \{z \in \mathbf{A} \mid z \neq 0, \ \pi/4 < \arg z < 7\pi/4\} \quad \text{and} \quad V_2 := -V_1.$$

Since V_1, V_2 and both components of $V_1 \cap V_2$ are contractible, the open covering $\mathscr{V} := \{V_1, V_2\}$ is a Leray cover for $\mathscr{F}|_{\dot{\mathbf{A}}}$. Similarly $\mathscr{V}_r := \{V_1 \cap U_r, V_2 \cap U_r\}$ is a Leray cover for $\dot{U}_r$. Hence we have a commutative diagram

$$\begin{array}{ccc}
H^q(\dot{\mathbf{A}}, \mathscr{F}) & \xrightarrow{\sim} & \check{H}^q(\mathscr{V}, \mathscr{F}) \\
\downarrow \dot{\rho}_r & & \downarrow \check{\rho}_r \\
H^q(\dot{U}_r, \mathscr{F}) & \xrightarrow{\sim} & \check{H}^q(\mathscr{V}_r, \mathscr{F})
\end{array}$$

where $\check{\rho}_r$ is induced from the canonical map

$$\tilde{\rho}_r : \check{C}(\mathscr{V}, \mathscr{F}) \to \check{C}(\mathscr{V}_r, \mathscr{F})$$

between the corresponding Čech complexes. But clearly $\tilde{\rho}_r$ is an isomorphism. ■

(2.2) *Remark.* — It is easy to describe the sheaves of k-vector spaces on $\mathbf{A}$ which are locally constant on $\dot{\mathbf{A}}$. They are given by the following data:

> *The fibers* $\mathscr{F}_0$, $\mathscr{F}_1$, *a linear map* $\alpha:\mathscr{F}_0 \to \mathscr{F}_1$ *and an automorphism* $\beta:\mathscr{F}_1 \xrightarrow{\sim} \mathscr{F}_1$ *which is the identity on* $\alpha(\mathscr{F}_0)$.

The map α comes from the canonical isomorphism $\mathscr{F}(\mathbf{A}) \xrightarrow{\sim} \mathscr{F}_0$, and β from the monodromy.

3. The Leray spectral sequence of quotient maps

(3.1) In the general setting of 1.1 consider the *Leray spectral sequence* of the quotient map $\pi: Z \to Z/\!/G$ (1.3):

$$H^p(Z/\!/G, \mathscr{H}^q) \Rightarrow H^{p+q}(Z, k).$$

We first assume that there is only one stratum. Then the sheaves $\mathscr{H}^q := R^q\pi_* \mathbf{k}$ are locally constant with fibers $H^q(G/H, k)$ (Corollary 1.4), where H is a principal isotropy group. If $Z/\!/G$ is contractible, then $H^p(Z/\!/G, \mathscr{H}^q) = 0$ for $p > 0$ and the spectral sequence degenerates:

$$H^q(Z, k) \simeq H^0(Z/\!/G, \mathscr{H}^q) = \mathscr{H}^q(Z/\!/G) \simeq H^q(G/H, k).$$

If, in addition, Z is acyclic, then

$$H^q(G/H, k) \simeq H^q(Z, k) \simeq \begin{cases} 0 & \text{if } q > 0, \\ k & \text{if } q = 0. \end{cases}$$

By Remark 3.4 below this implies that $H = G$. Hence we have a fixed point in every fiber. This proves the following result, which generalizes the first part of Theorem 0.1.

(3.2) *Proposition.* — *Let* Z *be a smooth acyclic* G*-variety with contractible quotient* $Z/\!/G$. *If there is only one stratum in the Luna stratification, then the action is fix-pointed* (*i.e., the closed orbits are fixed points*).

This proposition applies in particular when Z is contractible (see 0.2).

(3.3) *Corollary* ([BH1, 10.3], [Kr3, 6.3, 6.5]; see [Kr1, 5.5]). — *Under the assumptions of the proposition above the variety* Z *is a* G*-vector bundle over* $Z/\!/G$, *i.e.,* $\pi: Z \to Z/\!/G$ *has the structure of a vector bundle such that* G *acts linearly on each fiber.*

(3.4) The following lemma is well known ([KPR, 2.3]).

Lemma. — *Let* G *be a linear algebraic group and* $H \subset G$ *a closed subgroup. Let* $K \subset G$ *be a maximal compact subgroup such that* $K' := K \cap H$ *is a maximal compact subgroup of* H. *Then the inclusion* $K/K' \hookrightarrow G/H$ *is a homotopy equivalence.*

(3.5) *Remark.* — As a consequence of the lemma we see that for reductive groups $H \subset G$ we always have

$$H^q(G/H, k) = 0 \qquad \text{for} \quad q > d := \dim G/H,$$

since $\dim_{\mathbf{R}} K/K' = d$. Assume that G (hence K) is connected. Then $H^d(K/K', \mathbf{Z}/2) = \mathbf{Z}/2$, and $H^d(K/K', k) = k$ for all fields k if K/K' is orientable (Poincaré duality).

4. Proof of Theorem 0.1

(4.1) Let G, X, etc. be as in 0.1. We first assume that G is connected. The principal stratum is of the form $\mathbf{A} \setminus \{y_1, \ldots, y_r\}$ with $r \geqslant 0$. The case $r = 0$ has been handled in 3.2, so we assume $r > 0$.

We cover **A** with open parallel strips S_i, $i = 1, 2, \ldots, r$, with the following properties:

(1) $\mathbf{A} = \bigcup_{i=1}^{r} S_i$ and $y_i \in S_i$,

(2) $S_i \cap S_{i+1}$ is a non-empty open strip containing no y_j,

(3) $S_i \cap S_j = \varnothing$ for $|j - i| > 1$.

(By a "strip" we mean an open subset of $\mathbf{A} = \mathbf{R} \times \mathbf{R}$ which is isometric to $\mathbf{R} \times I = \{(x, y) \mid y \in I\}$, where I is an open interval in **R**.)

For every strip S_i the restrictions $\mathscr{H}^q|_{S_i}$ are acyclic by 2.1. This allows us to calculate the cohomology of $\mathscr{H}^q$ as the Čech cohomology with respect to the covering $\mathscr{S} = \{S_i\}$:

$$H^p(\mathbf{A}, \mathscr{H}^q) = \check{H}^p(\mathscr{S}, \mathscr{H}^q).$$

The (alternating) Čech complex has the following form:

$$(*) \qquad 0 \to \bigoplus_{i=1}^{r} H^0(S_i, \mathscr{H}^q) \xrightarrow{\varphi_q} \bigoplus_{i=1}^{r-1} H^0(S_i \cap S_{i+1} \mathscr{H}^q) \to 0$$

This shows that $H^p(\mathbf{A}, \mathscr{H}^q) = 0$ for $p \geqslant 2$, hence the Leray spectral sequence for the quotient map $\pi : X \to \mathbf{A}$ collapses. As a consequence, we get

$$H^p(\mathbf{A}, \mathscr{H}^q) = \begin{cases} k & \text{for } p = q = 0, \\ 0 & \text{otherwise,} \end{cases}$$

which implies that the map φ_q of (∗) is an isomorphism for $q>0$.

(4.2) Let $H \subset G$ be the principal isotropy group and denote by O_i the closed orbit in the fiber $\pi^{-1}(y_i)$, $i=1, \ldots, r$. By 1.4 and 1.5(1) we can rewrite the sequence (∗) in the form

$$(**) \qquad 0 \to \bigoplus_{i=1}^{r} H^q(O_i, k) \xrightarrow{\varphi_q} \bigoplus_{i=1}^{r-1} H^q(G/H, k) \to 0$$

where the components of the maps φ_q are induced by G-equivariant maps $G/H \to O_i$ (see 1.5(2)).

(4.3) By assumption we have $\dim G/H > 0$, else $H = G$ and there is only one stratum.

Lemma. – $\dim O_i < \dim G/H$ *for all i.*

Proof. – Assume that $\dim O_i = \dim G/H$ for some i. Then, in a neighborhood of $O_i \simeq G/H_i$, the G-variety X has the form $G *^{H_i} W$, where W is a representation of H_i with the following properties: $W/\!/H_i \simeq \mathbf{A}$ and the principal isotropy group $H \subset H_i$ is of finite index. It follows that the identity component H_i^0 and H^0 coincide and that $W/\!/H^0 \simeq \mathbf{A}$. Now $W/\!/H_i = (W/\!/H^0)/(H_i/H^0)$ and H_i/H^0 acts on $W/\!/H^0 \simeq \mathbf{A}$ with principal isotropy group H/H^0. But the principal isotropy group is the noneffective part for a finite group action, and any finite group acting effectively on $\mathbf{A}$ is cyclic. Thus H is normal in H_i with cyclic quotient, i.e., $G/H \to O_i$ is a cyclic covering.

Let $K \subset G$ be a maximal compact subgroup such that $K' := K \cap H$ and $K_i := K \cap H_i$ are maximal compact subgroups of H, H_i, respectively. Then $K/K' \to K/K_i$ is a non-trivial cyclic covering of compact manifolds with covering group $K_i/K' = H_i/H$, and we obtain an exact sequence of homotopy groups

$$\pi_1(K/K') \to \pi_1(K/K_i) \to \pi_0(K_i/K') = K_i/K' \to 1$$

Now we set $k = \mathbf{Z}/p$ where p divides the order of H_i/H. Then, using the canonical isomorphism $\mathrm{Hom}(\pi_1(-), \mathbf{Z}/p) \simeq H^1(-, Z/p)$, we see that the induced map

$$H^1(K/K_i, \mathbf{Z}/p) \to H^1(K/K', \mathbf{Z}/p)$$

has a non-trivial kernel. This implies that the map φ_1 in (∗∗) has a non-trivial kernel, too (3.4 and 1.5(2)), contracting the fact that φ_q is an isomorphism for $q>0$ (4.1). ■

(4.4) Now we can finish the proof of the theorem in case G is connected. Choosing $q = d (= \dim G/H > 0)$ and $k = \mathbf{Z}/2$ in the sequence (∗∗) we get $H^d(O_i, \mathbf{Z}/2) = 0$ by the lemma above and 3.5. Since φ_d is an isomorphism we must have $r=1$. Consequently, $H^q(O_1, k)=0$ for $q>0$ and all fields k, and

$$H^0(O_1, k) \simeq \check{H}^0(\mathcal{S}, \mathcal{H}^0) = H^0(\mathbf{A}, \mathbf{k}) = k.$$

This implies that O_1 is a fixed point (3.5), and the claim follows.

(4.5) If G is not connected, we apply the above results to G^0. We find that $X/\!/G^0 \simeq \mathbf{A}$ and that either

(1°) $\pi: X \to \mathbf{A}$ is a G^0-fiber bundle and every fiber contains a fixed point,

or

(2°) G^0 has an isolated fixed point $x_0 \in X$ and $\pi: \dot{X} \to \dot{\mathbf{A}}$ is a G^0-fiber bundle.

In both case G/G^0 acts on the quotient $X/\!/G^0 \simeq \mathbf{A}$ via a finite cyclic subgroup $\bar{G}$ of $\operatorname{Aut}(\mathbf{A})$.

In case (2°), G fixes x_0 and $\bar{G}$ acts freely on $\dot{\mathbf{A}}$, and we are in case (2) of 0.1. In case (1°), if $\bar{G}$ is trivial we are clearly in case (1). If not, then $\bar{G}$ has a unique fixed point in $\mathbf{A}$ (which we can assume to be 0), $\bar{G}$ acts freely on $\dot{\mathbf{A}}$, and we are in case (2). ■

Chapter III. AUTOMORPHISM GROUP SCHEMES

0. Résumé

Let V be a representation of G with quotient map $\pi : V \to \mathbf{A}$ given by a homogeneous polynomial t of degree d. In this chapter we study the group functor which associates to every open subset $U \subset \mathbf{A}$ the group of all G-equivariant automorphisms $\varphi : \pi^{-1}(U) \xrightarrow{\sim} \pi^{-1}(U)$ which induce the identity on U. We show that this functor is represented by an affine algebraic group scheme $\mathfrak{Aut}_V^G$ over $\mathbf{A}$ (2.2).

The general fiber of $\mathfrak{Aut}_V^G$ is the group $L := \mathrm{Aut}(F)^G$ of G-equivariant automorphisms of the fiber $F := \pi^{-1}(1)$ which is a linear algebraic group (2.5). In section 3 we analyze the structure of L. In section 4 we show that $\mathfrak{Aut}_V^G$ is *isotrivial* over $\dot{\mathbf{A}}$. This means that $\mathfrak{Aut}_V^G|_{\dot{\mathbf{A}}}$ pulls back to the trivial group scheme $\dot{\mathbf{A}} \times L$ under a covering $\dot{\mathbf{A}} \to \dot{\mathbf{A}}$, $z \mapsto z^d$ (4.2). We introduce a group scheme $\mathfrak{L}_{\mathbf{B}}^{\Gamma}$ over $\mathbf{A}$ (which depends only upon d and L) which is "simpler" than $\mathfrak{Aut}_V^G$ and agrees with it over $\dot{\mathbf{A}}$. The interplay between $\mathfrak{L}_{\mathbf{B}}^{\Gamma}$ and $\mathfrak{Aut}_V^G$ is a fundamental theme in our work.

1. Affine group schemes

(1.1) Let Y be a variety. In this paragraph we recall the definition of a group scheme over Y and of the corresponding group functor. A standard reference is [DG]. We are mainly interested in the case where the base Y is the affine line $\mathbf{A}$.

Definition. — An (*affine algebraic*) *group scheme over* Y is a variety $\mathfrak{G}$ together with an affine morphism $\pi_{\mathfrak{G}} : \mathfrak{G} \to Y$ and the structure of an algebraic group on each fiber $\mathfrak{G}_y := \pi_{\mathfrak{G}}^{-1}(y)$. The group structure depends algebraically on $y \in Y$, i.e., the two maps

$$\mu : \mathfrak{G} \times_Y \mathfrak{G} \to \mathfrak{G},\ (g, h) \mapsto gh^{-1}, \qquad \text{and} \qquad \varepsilon : Y \to \mathfrak{G},\ y \mapsto e_y,$$

are algebraic morphisms, where e_y is the neutral element in the group $\pi^{-1}(y)$.

The definition of a *group homomorphism* $\mathfrak{p} : \mathfrak{H} \to \mathfrak{G}$ between two group schemes over Y is clear: It is a morphism which induces the identity on Y such that for every $y \in Y$ the map $\mathfrak{p}_y : \mathfrak{H}_y \to \mathfrak{G}_y$ is a homomorphism of algebraic groups.

Since every affine algebraic group scheme over Spec k, where k is a field of characteristic zero, is reduced (Theorem of Cartier, cf. [DG, II, §6.1.1]), it follows that all fibers of π are reduced and smooth. So the fibers are linear algebraic groups. In particular, if $\mathfrak{G}$ is connected and Y a smooth curve, then π is a smooth morphism.

Clearly, for every morphism $Z \to Y$ the fiber product $Z \times_Y \mathfrak{G}$ is a group scheme over Z. We denote it by $\mathfrak{G}_Z$.

(1.2) *Examples.* – (*a*) The easiest examples are *trivial group schemes* $\mathfrak{G}=Y\times G$, where G is an (affine) algebraic group and $\pi: Y\times G\to Y$ the projection.

(*b*) A modification of the first example gives the following *twisted group schemes*: Let Γ be a finite group which acts freely on the affine variety Y on the right, and denote by Y/Γ the quotient. Assume in addition that Γ acts on G by group automorphisms. Then Γ acts on $Y\times G$ on the right by $(y,g)\mapsto(y\gamma,\gamma^{-1}g)$, $\gamma\in\Gamma$, $y\in Y$, $g\in G$. The quotient $Y*^{\Gamma}G$ is a group scheme over Y/Γ whose fiver over every point is isomorphic to G. This group scheme is said to be *isotrivial*, since it becomes trivial under the étale cover $Y\to Y/\Gamma$.

(*c*) Let (t,x) be coordinates on $\mathbf{C}^2$, and let $\mathfrak{G}_n$ be the open subvariety of $\mathbf{C}^2$ where $1+t^n x\neq 0$, $n\geqslant 1$. Let $\pi: \mathfrak{G}_n\to\mathbf{A}$ be the projection of (t,x) to t, and define a group scheme structure on $\mathfrak{G}_n$ by setting

$$(t, s).(t, x')=(t, x+x'+t^n xx').$$

Over $\dot{\mathbf{A}}$, the map sending (t,x) to $(t,1+t^n x)$ gives an isomorphism of $\mathfrak{G}_n|_{\dot{\mathbf{A}}}$ onto the trivial group scheme $\dot{\mathbf{A}}\times\mathbf{C}^*$. In particular, the fibers $\pi^{-1}(t)$ for $t\neq 0$ are isomorphic to $\mathbf{C}^*$. On the other hand, the fiber over $0\in\mathbf{A}$ is just the additive group $\mathbf{C}^+$.

(1.3) Let $\mathfrak{G}$ be a group scheme over Y. For any open set $U\subset Y$ the sections

$$\mathfrak{G}(U):=\{\sigma: U\to\mathfrak{G}\mid\pi(\sigma(y))=y \text{ for all } y\in U\}$$

form a group in an obvious way: $(\sigma\cdot\lambda)(y):=\sigma(y)\lambda(y)$. More generally, we obtain a *group functor*, also denoted by $\mathfrak{G}$, which associates to every morphism $\varphi: Z\to Y$ the group

$$\mathfrak{G}(Z):=\{\sigma: Z\to\mathfrak{G}\mid\pi\circ\sigma=\varphi\}.$$

Clearly, $\mathfrak{G}(Z)$ is just the set of sections of the group scheme $\mathfrak{G}_Z$ over Z. It is well known that the functor $Z\mapsto\mathfrak{G}(Z)$ completely determines the group scheme $\mathfrak{G}$. It is even sufficient to restrict to Z affine.

(1.4) *Examples.* – For the examples in 1.2 above we have the following descriptions of the corresponding group functors:

(*a*) If $\mathfrak{G}$ is the trivial group scheme $Y\times G$ and $\varphi: Z\to Y$ any morphism, then

$$\mathfrak{G}(Z)=G(Z):=\operatorname{Mor}(Z,G).$$

(*b*) Let $\mathfrak{G}$ be the twisted group scheme $Y*^{\Gamma}G$ of 1.2(b). For any morphism $\varphi: Z\to Y/\Gamma$ we denote by Z_Γ the fiber product $Z\times_{Y/\Gamma}Y$. Then Γ acts (freely, on the right) on Z_Γ with quotient Z. This induces a left action of Γ on $G(Z_\Gamma)$ by group homomorphisms, defined by $(\gamma\sigma)(z):=\gamma(\sigma(z\gamma))$. Then

$$(Y *^{\Gamma} G)(Z) = G(Z_{\Gamma})^{\Gamma}.$$

(*c*) Consider the group schemes $\mathfrak{G}_n$ over $\mathbf{A}$ defined in 1.2 (c), and let $\operatorname{Spec} R \to \mathbf{A} = \operatorname{Spec} \mathbf{C}[t]$ be a morphism, i.e., let R be a $\mathbf{C}[t]$-algebra. Denote by $r \in R$ the image of $t \in \mathbf{C}[t]$. Then $\mathfrak{G}_n(R) := \mathfrak{G}_n(\operatorname{Spec} R)$ is the group of $\mathbf{C}[t]$-homomorphisms from $\mathcal{O}(\mathfrak{G}_n) = \mathbf{C}[t, x]_{1+t^n x}$ to R. Any such homomorphism sends x to some $s \in R$ such that $1 + r^n s$ lies in the group of units R^* of R (since $1 + t^n x$ is a unit in $\mathcal{O}(\mathfrak{G}_n)$). Thus there is an isomorphism of $\mathfrak{G}_n(R)$ with

$$\{ s \in R \mid 1 + r^n s \in R^* \} \subset R,$$

where the group structure is given by $s . s' = s + s' + r^n ss'$. If $r = 0$, then $\mathfrak{G}_n(R) \simeq (R, +)$. If $r \in R^*$, then there is an isomorphism $\tau : \mathfrak{G}_n(R) \to R^*$ sending s into $1 + r^n s$.

As an example we calculate $\mathfrak{G}_n(\dot{\mathbf{A}})$ and $\mathfrak{G}_n(\hat{\mathbf{A}})$ as subgroups of $\mathfrak{G}_n(\hat{\dot{\mathbf{A}}})$, where $\hat{\dot{\mathbf{A}}} := \operatorname{Spec} \mathbf{C}((t))$. Applying the isomorphism τ (with $R = \mathcal{O}(\hat{\dot{\mathbf{A}}})$) we find that

$$\tau(\mathfrak{G}_n(\dot{\mathbf{A}})) = \mathcal{O}(\dot{\mathbf{A}})^* = \mathbf{C}[t, t^{-1}]^* = \{ ct^i \mid c \in \mathbf{C}^*, i \in \mathbf{Z} \}.$$
$$\tau(\mathfrak{G}_n(\hat{\mathbf{A}})) = \{ 1 + t^n h \mid h \in \mathbf{C}[[t]] \},$$
$$\tau(\mathfrak{G}_n(\hat{\dot{\mathbf{A}}})) = \mathcal{O}(\hat{\dot{\mathbf{A}}})^* = \mathbf{C}((t))^*.$$

(1.5) We now describe another important example of a group scheme, the *automorphism group scheme of a* G-vector bundle $\mathscr{W}$ over Y (with trivial G-action on Y). Here G-vector bundle means that $\mathscr{W}$ is a vector bundle over Y together with a G-action which is linear on each fiber, i.e., every $g \in G$ induces an automorphism of the vector bundle $\mathscr{W}$ (cf. [Kr2, § 1]). Assume that $\mathscr{W} = \mathscr{V} \otimes W$ where $\mathscr{V}$ is a vector bundle (with trivial G-action) and W a G-module. We consider the group functor which associates to a morphism $\varphi : Z \to Y$ the group $GL(\varphi^* \mathscr{W})^G$ of G-equivariant automorphisms of the G-vector bundle $\varphi^* \mathscr{W}$. (As usual, $\varphi^* \mathscr{W}$ denotes the pull back of $\mathscr{W}$; it inherits a natural G-vector bundle structure.) We claim that this functor represents a group scheme over Y.

Let $\mathscr{P}$ be the principal GL_n-bundle associated to $\mathscr{V}$, $n := \operatorname{rk} \mathscr{V}$, and let $L := GL(\mathbf{C}^n \otimes W)^G$ be the group of G-linear automorphisms of the fiber of $\mathscr{W}$. There is a canonical injective homomorphism $\mu : GL_n \to L$. We define a group scheme $\mathfrak{L}_{\mathscr{W}}$ over Y as the associated bundle

$$\mathfrak{L}_{\mathscr{W}} := \mathscr{P} *^{GL_n} L$$

where GL_n acts on L by conjugation via the homomorphism μ.

(1.6) *Lemma. — For any morphism* $\varphi : Z \to Y$ *we have*

$$\mathfrak{L}_{\mathscr{W}}(Z) = GL(\varphi^* \mathscr{W})^G.$$

This is obvious for a trivial vector bundle $\mathscr{V}$, and the general case follows easily. We leave it as an exercise (cf. [DG, II, § 1.2.4]).

2. The automorphism group scheme of a quotient

(2.1) Let G be a reductive group and X an affine G-variety. Assume that the quotient map $\pi: X \to X/\!/G$ is flat. Given a morphism $Z \to X/\!/G$ we denote by X_Z the fiber product $Z \times_{X/\!/G} X$. Then G acts on X_Z, and the projection $\pi_Z: X_Z \to Z$ is the quotient map. Define the group

$$\mathrm{Aut}(X_Z/Z)^G := \{ \varphi : X_Z \xrightarrow{\sim} X_Z \mid \varphi \text{ is G-equivariant}, \pi_Z \circ \varphi = \pi_Z \}.$$

Clearly, this group depends functorially on Z.

(2.2) *Proposition. – The group functor* $Z \mapsto \mathrm{Aut}(X_Z/Z)^G$ *is represented by an affine algebraic group scheme* $\mathfrak{Aut}_X^G$ *over* $X/\!/G$.

Proof. – Let $A := \mathcal{O}(X)^G$. The coordinate ring $\mathcal{O}(X)$ is a direct sum of isotypic components $\mathcal{O}(X)_\omega$ which are all finitely generated A-modules. Hence, by flatness, $\mathcal{O}(X)$ and all the $\mathcal{O}(X)_\omega$ are projective over A. Moreover, there is a finitely generated A-submodule $P \subseteq \mathcal{O}(X)$ which generates $\mathcal{O}(X)$ as an algebra and is a (finite) direct sum of isotypic components $\oplus_i \mathcal{O}(X)_{\omega_i}$. There is an exact sequence

$$0 \to \mathbf{a} \to S_A(P) \to \mathcal{O}(X) \to 0$$

of A-modules, where $S_A(P)$ is the symmetric algebra of P and all maps are G-equivariant. Since $\mathcal{O}(X)$ is projective over A, for every A-algebra R, we obtain an exact sequence

$$0 \to R \otimes_A \mathbf{a} \to S_R(P_R) \to \mathcal{O}(X_{\mathrm{Spec}\,R}) \to 0$$

where

$$P_R := R \otimes_A P \subseteq R \otimes_A \mathcal{O}(X) = \mathcal{O}(X_{\mathrm{Spec}\,R}).$$

Every G-automorphism of $S_{\mathrm{Spec}\,R}$ over $\mathrm{Spec}\,R$ induces an R-linear and G-equivariant automorphism of $P_R = \oplus_i R \otimes \mathcal{O}(X)_{\omega_i}$, and is clearly determined by this automorphism. Hence we get, in a functorial way, an inclusion

$$\iota_R\colon \mathrm{Aut}(X_{\mathrm{Spec}\,R}/\mathrm{Spec}\,R)^G \hookrightarrow \mathrm{GL}_R(P_R)^G = \prod_i \mathrm{GL}_R(R \otimes_A \mathcal{O}(X)_{\omega_i})^G.$$

By [Kr2, 2.1 Proposition 1], every $\mathcal{O}(X)_{\omega_i}$ determines a vector bundle of the form $\mathscr{V} \otimes W_i$, where W_i is an irreductible G-module of type ω_i. Thus Lemma 1.6 implies that $R \mapsto GL_R(P_R)^G$ is represented by an affine group scheme $\mathfrak{L}_P$ over Spec $A = X/\!/G$. The image of the homomorphism ι_R is exactly the set of those R-G-automorphisms of P_R which stabilize the ideal $R \otimes_A \mathbf{a}$ in the symmetric algebra $S_R(P_R)$. This clearly defines a closed subgroup scheme

$$\mathfrak{Aut}_X^G \subseteq \mathfrak{L}_P$$

over $X/\!/G$. ■

(2.3) We describe in more detail the case of a representation V with a one-dimensional quotient $V/\!/G \simeq \mathbf{A}$. Here $A = \mathcal{O}(V)^G = \mathbf{C}[t]$, where $t \in \mathcal{O}(V)$ is homogeneous. Since $\mathcal{O}(V)$ is a free graded A-module (see [Sch2]) we can find a G-stable homogeneous subspace $S \subseteq \mathcal{O}(V)$ such that multiplication induces an isomorphism

$$(*) \qquad A \otimes_{\mathbf{C}} S \xrightarrow{\sim} \mathcal{O}(V).$$

Let $V_1, V_2, \ldots, V_r$ be the non-equivalent irreducible subrepresentations of V. Denote by S_i the sum of all subspaces of S which are isomorphic to V_i^*, $i = 1, 2, \ldots, r$. Then each S_i is finite dimensional and

$$A \otimes_{\mathbf{C}} S_i \xrightarrow{\sim} AS_i \subseteq \mathcal{O}(V)$$

where AS_i is the isotypic component of $\mathcal{O}(V)$ of type V_i^*. Let S_i^0 be the linear part of S_i: $S_i^0 = S_i \cap V^*$. Then $S^0 := \oplus S_i^0 = V^*$. We choose a homogeneous G-stable complement S_i' of S_i^0 in S_i.

Let R be an A-algebra and let $\alpha \in \mathrm{Aut}_{R\text{-alg}}(R \otimes_A \mathcal{O}(V))^G$, the group of G-equivariant R-algebra automorphisms of $R \otimes_A \mathcal{O}(V)$. Note that $\mathrm{Aut}_{R\text{-alg}}(R \otimes_A \mathcal{O}(V))^G$ is the opposite group to $\mathrm{Aut}(V_{\mathrm{Spec}\,R}/\mathrm{Spec}\,R)^G$. Since $R \otimes_A \mathcal{O}(V) \simeq R \otimes_{\mathbf{C}} S$ by $(*)$, the automorphism α preserves every $R \otimes_{\mathbf{C}} S_i$. If we choose an isomorphism $S_i \simeq n_i V_i^*$, then $\alpha|_{R \otimes S_i}$ corresponds to a matrix $\sigma_i(\alpha) \in GL_{n_i}(R)$. Now V^* generates the R-algebra $R \otimes_A \mathcal{O}(V)$ and so the homomorphism $\alpha \mapsto \sigma(\alpha) := \prod_i \sigma_i(\alpha)$ is injective.

(2.4) Clearly, the R-matrices $\sigma_i(\alpha)$ are not arbitrary: First of all, they are invertible. From the decomposition $S_i = S_i^0 \oplus S_i'$ we obtain a block decomposition

$$\sigma_i(\alpha) = \begin{pmatrix} A_i & C_i \\ B_i & D_i \end{pmatrix}.$$

Since the elements of $\oplus_i S_i'$ are polynomials in $V^* = \oplus_i S_i^0$, the matrices C_i and D_i are polynomials in the entries of the matrices A_j, B_j, $j = 1, \ldots, r$, with coefficients in A (and are independent of α).

Finally, α is the identity map on $R = R \otimes 1 \subset R \otimes_A \mathcal{O}(V)$. In particular, it preserves the element $\bar{t} :=$ the image of t in R. Now t is a polynomial function on V with coefficients in **C**, and so $\alpha(\bar{t})$ is a polynomial in the coordinates of V and the entries of the matrices $\sigma_i(\alpha)$, $i = 1, \ldots, r$. This polynomial has to equal $\bar{t}$.

In sum, the matrices $\sigma_i(\alpha) = \begin{pmatrix} A_i & C_i \\ B_i & D_i \end{pmatrix}$ have to satisfy the following three conditions:

(i) The $\sigma_i(\alpha)$ are invertible R-matrices.

(ii) There are polynomials with coefficients in A, independent of α, which express the matrices C_i and D_i in terms of the matrices A_j and B_j, $j = 1, \ldots, r$.

(iii) There is a polynomial, independent of α, in the coordinates of V and the entries of the matrices A_i, B_i, C_i and D_i which is equal to $\bar{t}$.

Conversely, it is not hard to see that every element

$$\prod_i \beta_i \in \prod_i GL_{n_i}(R), \qquad \beta_i = \begin{pmatrix} A_i & C_i \\ B_i & D_i \end{pmatrix}$$

which satisfies the three conditions (i), (ii) and (iii) above, comes from $\mathrm{Aut}_{R\text{-alg}}(R \otimes_A \mathcal{O}(V))^G$. In fact, the matrices $\begin{pmatrix} A_i \\ B_i \end{pmatrix}$ determine a G-equivariant **C**-linear map $V^* \to \oplus_i R \otimes_A S_i \subseteq R \otimes_A \mathcal{O}(V)$, hence a G-equivariant **C**-algebra homomorphism

$$\beta' : \mathcal{O}(V) \to R \otimes_A \mathcal{O}(V).$$

Condition (ii) makes sure that $\beta'|_{S_i} : S_i \to R \otimes_A S_i$, $i = 1, \ldots, r$, is given by the matrix β_i. Condition (iii) implies that β' is A-linear, hence it induces a G-equivariant R-algebra homomorphism

$$\beta : R \otimes_A \mathcal{O}(V) \to R \otimes_A \mathcal{O}(V).$$

This is an automorphism because of condition (i).

Geometrically, we see that

$$\mathfrak{Aut}_V^G \subseteq \mathbf{A} \times \prod_i GL_{n_i}$$

is the closed subvariety defined by the conditions (ii) and (iii).

(2.5) A special case of Proposition 2.2 is the following. We say that an affine G-variety Z is *without invariants* if $\mathcal{O}(Z)^G = \mathbf{C}$.

Corollary. — *Let* Z *be an affine* G*-variety without invariants. Then the group* $\mathrm{Aut}(Z)^G$ *of* G*-equivariant automorphisms of* Z *is a linear algebraic group. In particular, in the situation of* 2.2 *we have* $(\mathfrak{Aut}_X^G)_y = \mathrm{Aut}(F)^G$ *where* $y \in X/\!/G$ *and* $F := \pi_X^{-1}(y)$.

(2.6) *Remark.* — Let V be as in 2.3 and let $y \in \mathbf{A} \simeq V/\!/G$. Then the isomorphism (∗) of 2.3 shows that the restriction $\mathcal{O}(V) \to \mathcal{O}(\pi^{-1}(y))$ induces a G-equivariant isomorphism $S \xrightarrow{\sim} \mathcal{O}(\pi^{-1}(y))$.

If the fiber $\pi^{-1}(y)$ is an orbit $\simeq G/H$, then we obtain, by *Frobenius reciprocity*, that

$$\mathrm{mult}_M S = \dim (M^*)^H$$

for every simple G-module M. In general, the fiber $\pi^{-1}(y)$ for $y \neq 0$ is of the form $G *^H W$ where W is an H-module without invariants (see II.1.1), and we get

$$\mathrm{mult}_M S = \dim (\mathcal{O}(W) \otimes M^*)^H$$

(cf. [Sch1, Proposition 4.6]).

(2.7) *Example.* — Let $V = \mathbf{C}^2$ and G be the subgroup of SL_2 generated by $\left\{\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix} \middle| \lambda \in \mathbf{C}^*\right\}$ and $\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$. Let $\{x, y\}$ be the standard dual basis of $V = \mathbf{C}^2$, and let $s := xy \in \mathcal{O}(V)$. Then $A = \mathcal{O}(V)^G$ is generated by $t = s^2$. Now V is a simple G-module, and there are 2 (= dim V) copies of V* in S (see Remark 2.6). One is $V^* = \mathrm{span}\{x, y\}$ and the other is $\mathrm{span}\{sx, sy\} \subseteq S^3(V^*)$. There is a G-isomorphism between these two, which sends x to sx and y to $-sy$.

Let $\alpha \in \mathrm{Aut}_{R\text{-alg}}(R \otimes_A \mathcal{O}(V))^G$. Then

$$\sigma(\alpha) = \begin{pmatrix} a & c\\ b & d\end{pmatrix} \in GL_2(R)$$

where

(1) $\alpha(x) = ax + bsx$, $\alpha(y) = ay - bsy$;

(2) $\alpha(sx) = cx + dsx$, $\alpha(-sy) = cy - dsy$.

Since $sx = x^2 y$ and $-sy = -xy^2$, we can use (1) to compute the coefficients occuring in (2), and we obtain

(3) $c = (a^2 - b^2 \bar{t})\, b\bar{t}$, $d = (a^2 - b^2 \bar{t})\, a$.

The condition 2.4 (ii) of the general case reduces to (3) in this example, 2.4 (iii) reduces to

(4) $(a^2 - b^2 \bar{t})^2 \bar{t} = \bar{t}$,

and 2.4 (i) to

(5) $\det\begin{pmatrix} a & c \\ b & d \end{pmatrix} = (a^2 - b^2 \bar{t})^2 \in \mathrm{R}^*$.

Hence we get the following description of the group scheme $\mathfrak{Aut}_V^G$.

$$\mathfrak{Aut}_V^G = \{ (t, a, b) \mid (a^2 - b^2 t)^2 t = t,\ a^2 - b^2 t \neq 0 \} \subset \mathbf{A}^3$$

with multiplication

$$(t, a, b) . (t, a', b') = (t, aa' + (a^2 - b^2 t) bb' t,\ ba' + (a^2 - b^2 t) ab')$$

and unit $\varepsilon : t \mapsto (t, 1, 0)$. The fibers over $\lambda \neq 0$ are all isomorphic to G: By (3) and (4),

$$(\mathfrak{Aut}_V^G)_\lambda \simeq G_\lambda := \left\{ \begin{pmatrix} a & \delta b \lambda \\ b & \delta a \end{pmatrix} \mid a^2 - \lambda b^2 =: \delta = \pm 1 \right\}.$$

Choose μ such that $\mu^2 = \lambda$, and set $S := \begin{pmatrix} 1 & \mu \\ 1 & -\mu \end{pmatrix}$. Then $SG_\lambda S^{-1} = G$.

It is not true that $\mathfrak{Aut}_V^G|_{\dot{\mathbf{A}}}$ is a trivial group scheme. It becomes trivial after the base change $\dot{\mathbf{A}} \to \dot{\mathbf{A}}$, $z \mapsto z^2$, hence it is a "twisted form" of $\dot{\mathbf{A}} \times G$ (see Example 1.2(b)).

The fiber over 0 is isomorphic to $\mathbf{C}^* \ltimes \mathbf{C}$:

$$(\mathfrak{Aut}_V^G)_0 = \{ (a, b) \mid a \neq 0 \} \xrightarrow{\sim} \left\{ \begin{pmatrix} a & 0 \\ b & a^3 \end{pmatrix} \in GL_2(\mathbf{C}) \right\},$$

where the isomorphism is given in the obvious way. In particular, this fiber is of dimension 2.

The variety $\mathfrak{Aut}_V^G$ has three irreducible components:

$$\mathfrak{G}^0 := \{ (t, a, b) \mid a^2 - b^2 t = 1 \},$$
$$\mathfrak{G}' := \{ (t, a, b) \mid a^2 - b^2 t = -1 \},$$
$$\mathfrak{G}_0 := (\mathfrak{Aut}_V^G)_0.$$

The component $\mathfrak{G}^0$ is a subgroup scheme whose general fiber is $\mathbf{C}^*$ and whose zero-fiber is $\mathbf{C}^+$. The union $\mathfrak{G}^0 \cup \mathfrak{G}' = \mathfrak{G}^0 \cup \sigma \mathfrak{G}^0$ is also a subgroup scheme, where σ is the section $\sigma(t) = (t, i, 0)$. It has two connected components. The general fiber is G and the zero fiber is a semidirect product $\mathbf{Z}/2 \ltimes \mathbf{C}^+$.

(2.8) *Example.* — We identify O_2 with $\mathbf{C}^* \ltimes \mathbf{Z}/2$. Let V_j denote the irreducible two-dimensional O_2-module with weights j and $-j$ relative to the action of $\mathbf{C}^*$, $j \geqslant 1$. The generator of $\mathbf{Z}/2$ interchanges the two weight spaces. Let $G = O_2 \times \mathbf{C}^*$ and define $W_i := V_i \otimes \mathbf{C}_1$, where $\mathbf{C}_1$ is the representation of $\mathbf{C}^*$ with weight 1. Consider the representation $V = V_1 \oplus W_n$ of G and denote by $\{u, v, x, y\}$ the dual basis of V corresponding to the weights $1, -1, n, -n$ of $\mathbf{C}^* \subset O_2$.

The invariant ring A is generated by $t = uv$. The multiplicity of V_1^* in S is one, that of W_n^* is 2 (see 2.6), and the copies of these representations in S are given by span $\{u, v\}$, span $\{x, y\}$ and span $\{u^{2n}y, v^{2n}x\}$, respectively. If R is an A-algebra and $\alpha \in \mathrm{Aut}_{\text{R-alg}}(\mathrm{R} \otimes_{\mathrm{A}} \mathcal{O}(\mathrm{V}))^{\mathrm{G}}$, then $\sigma(\alpha)$ is the pair

$$\left((r), \begin{pmatrix} a & c \\ b & d \end{pmatrix}\right), \qquad r, a, b, c, d \in \mathrm{R},$$

where

$$\alpha\left(\begin{pmatrix} u \\ v \end{pmatrix}\right) = r\begin{pmatrix} u \\ v \end{pmatrix},$$

$$\alpha\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = a\begin{pmatrix} x \\ y \end{pmatrix} + b\begin{pmatrix} u^{2n}y \\ v^{2n}x \end{pmatrix},$$

$$\alpha\left(\begin{pmatrix} u^{2n}y \\ v^{2n}x \end{pmatrix}\right) = c\begin{pmatrix} x \\ y \end{pmatrix} + d\begin{pmatrix} u^{2n}y \\ v^{2n}x \end{pmatrix}.$$

Now condition (iii) of 2.4 imposes $r^2\bar{t} = \bar{t}$, and condition (ii) forces $c = \bar{t}^{2n}b$ and $d = r^{2n}a$. From condition (i) we obtain $r \in \mathrm{R}^*$ and $r^{2n}a^2 - \bar{t}^{2n}b^2 \in \mathrm{R}^*$. Hence we get the following description of $\mathfrak{Aut}_{\mathrm{V}}^{\mathrm{G}}$:

$$\mathfrak{Aut}_{\mathrm{V}}^{\mathrm{G}} = \{(t, r, a, b) \mid r \neq 0,\ r^2 t = t,\ r^{2n}a^2 - t^{2n}b^2 \neq 0\} \subset \mathbf{A}^4$$

with multiplication

$$(t, r, a, b) . (t, r', a', b') = (t, rr', aa' + t^{2n}bb', a'b + r^{2n}ab')$$

and unit $\varepsilon : t \mapsto (t, 1, 1, 0)$. It follows that $\mathfrak{Aut}_{\mathrm{V}}^{\mathrm{G}}|_{\dot{\mathbf{A}}}$ is a trivial group scheme with fiber $\mathbf{Z}/2 \times (\mathbf{C}^*)^2$ and that the zero fiber is isomorphic to a semidirect product of $\mathbf{C}^*$ with $\mathbf{C}^* \times \mathbf{C}^+$. We leave it as an exercise to work out the details.

3. Structure of Aut(F)$^{\mathrm{G}}$

(3.1) Let V, $t = \pi : \mathrm{V} \to \mathbf{A}$, F and $\mathrm{L} = \mathrm{Aut}(\mathrm{F})^{\mathrm{G}}$ be as in section 0. Note that L is a linear algebraic group, and $\mathrm{L} = (\mathfrak{Aut}_{\mathrm{V}}^{\mathrm{G}})_1$, the fiber of our group scheme at 1 (Corollary 2.5). Let $\Gamma := \mu_d$, the group of d-th roots of unity, acting as scalars on V. Then Γ preserves F, so there is a canonical inclusion $\Gamma \subseteq \mathrm{L}$.

Choose $v_0 \in \mathrm{F}$ such that $\mathrm{G}v_0$ is closed, and set $\mathrm{H} := \mathrm{G}_{v_0}$. Let N be the slice representation of H (II.1.1), i.e., let N be an H-stable complement in V to the tangent space $\mathrm{T}_{v_0}(\mathrm{G}v_0)$. Then N has an H-stable decomposition $\mathrm{N} = \mathbf{C} \oplus \mathrm{W}$, where W is an

H-representation without invariants. We have an isomorphism $G *^H W \xrightarrow{\sim} F$ defined by $[g, w] \mapsto g(w+v_0)$. In other words, the choice of N (hence W) determines a G-vector bundle structure on F.

Let $L_{vb} \subset L$ denote the subgroup of G-*vector bundle automorphisms* of $F = G *^H W$. Any $r \in L$ preserves the closed orbit $G v_0$, which is the zero section $Z \subset G *^H W$, and its differential dr induces a G-automorphism of the normal bundle $\mathcal{N}(Z)$ of Z in F. But $\mathcal{N}(Z)$ is canonically G-isomorphic to F, so there is a homomorphism $\varepsilon : L \to L_{vb}$, which we can assume to be the identity on L_{vb}.

Let $\bar{L}$ denote $\operatorname{Aut}(G v_0)^G \simeq \operatorname{Aut}(G/H)^G \simeq N_G(H)/H$. Since L preserves the closed orbit $G v_0$, there is a natural homomorphism $\beta : L \to \bar{L}$.

(3.2) Now consider the following commutative diagram:

$$
(*) \qquad
\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & U_1 & \to & U_L & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \operatorname{Aut}(W)^H & \to & L & \xrightarrow{\beta} & \bar{L} & \to & 1 \\
 & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\varepsilon} & & \| & & \\
1 & \to & GL(W)^H & \to & L_{vb} & \xrightarrow{\beta'} & \bar{L} & \to & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}
$$

Here δ is defined similarly to ε, using differentials. We put $U_L := \operatorname{Ker} \varepsilon$ and $U_1 := \operatorname{Ker} \delta$. The other homomorphisms in (*) are clear from what we said above, given our identification $F = G *^H W$. The vertical sequences are exact by construction, and clearly $GL(W)^H = \operatorname{Ker} \beta'$.

(3.3) *Proposition.* – (1) *The groups* L_{vb} *and* $GL(W)^H$ *are reductive, and* $GL(W)^H$ *is a product of* GL_n*'s.*

(2) $U_1 = \operatorname{Rad}_u(\operatorname{Aut}(W)^H)$, $U_L = \operatorname{Rad}_u(L)$, *and* $\operatorname{Aut}(W)^H$ *is connected.*

(3) *The lower horizontal sequence in the diagram* (*) *is exact. In particular,* β *and* β' *are both surjective.*

(4) *The homomorphisms* $\Gamma \hookrightarrow L$ *and* $L \xrightarrow{\beta} \bar{L}$ *induce isomorphisms*

$$\Gamma/\Gamma' \xrightarrow{\sim} L/L^0 \xrightarrow{\sim} \bar{L}/\bar{L}^0$$

where $\Gamma' := \mathrm{Ker}(\Gamma \to L/L^0)$.

Proof. — It follows from Schur's lemma that $GL(W)^H$ is a product of GL_n's. We now show that $\beta(L^0_{vb}) = \bar{L}^0$. Since $\bar{L}^0$ and $GL(W)^H$ are reductive, this implies that L^0_{vb} and L_{vb} are reductive.

Let $h \in \mathrm{Cent}_G(H)$. Then h acts on F (from the right) by $[g, w]h = [gh, w]$, where $g \in G$ and $w \in W$. Thus $\mathrm{Cent}_G(H)$ maps to L_{vb}, and its image in $\bar{L}$ is $\mathrm{Cent}_G(H) H/H \supseteq \bar{L}^0$. Thus $\beta'(L^0_{vb}) = \bar{L}^0$, and (1) is proved.

Let M be a Levi factor of $\mathrm{Aut}(W)^H$, i.e., a reductive subgroup such that $\mathrm{Aut}(W)^H = M . \mathrm{Rad}_u(\mathrm{Aut}(W)^H)$, and let K denote $M \cap \mathrm{Ker}\,\delta$. Then K is a reductive group, K fixes $0 \in W$ and acts trivially on the tangent space $T_0(W)$. This implies by Luna's slice theorem that K acts trivially on W. Thus $M \cap \mathrm{Ker}\,\delta = \{e\}$. Since δ is surjective and $GL(W)^H$ reductive, $\mathrm{Ker}\,\delta \supseteq \mathrm{Rad}_u(\mathrm{Aut}(W)^H)$. Hence $\mathrm{Ker}\,\delta = \mathrm{Rad}_u(\mathrm{Aut}(W)^H)$. The proof that $\mathrm{Ker}\,\varepsilon = \mathrm{Rad}_u(L)$ is similar. Finally, $\mathrm{Aut}(W)^H$ is connected, since U_1 and $GL(W)^H$ are, and the proof of (2) is complete.

By the Luna-Richardson theorme [LR], the inclusion $V^H \hookrightarrow V$ induces an isomorphism $\mathcal{O}(V)^G \xrightarrow{\sim} \mathcal{O}(V^H)^{\bar{L}}$. Since $\dim V^H/\!/\bar{L} = 1$, we get $\dim V^H/\!/\bar{L}^0 = 1$ and $\mathcal{O}(V^H)^{\bar{L}^0} = \mathbf{C}[t_0]$ for some homogeneous t_0 of degree d_0. By construction, the generic $\bar{L}$-orbit in V^H is closed and isomorphic to $\bar{L}$. It follows that $\bar{L}/\bar{L}_0$ acts faithfully on $\mathbf{C}[t_0]$ and is therefore cyclic. Thus $d = \deg t = d_0 . |\bar{L}/\bar{L}^0|$. Now Γ acts on V by scalar multiplication and clearly $t_0(\gamma v_0) = \gamma^{d_0} t_0(v_0)$ for $\gamma \in \Gamma$. Thus $\mathbf{C}[t_0]^\Gamma = \mathbf{C}[t_0]^{\bar{L}/\bar{L}^0}$, hence $\beta(\Gamma) . \bar{L}^0 = \bar{L}$. It follows that β and β' are surjective, which proves (3). Since $L/L^0 \simeq L_{vb}/L^0_{vb} \simeq \bar{L}/\bar{L}^0$ we obtain (4). ■

(3.4) *Remarks.* — (1) (Vust) In general, $\mathrm{Rad}_u(\mathrm{Aut}(W)^H)$ is not trivial. For example, consider the representation $(W, H) = (\Lambda^2 \mathbf{C}^5 \oplus (\mathbf{C}^5)^*, SL_5)$. Then $S^2(\Lambda^2 \mathbf{C}^5) \supseteq \Lambda^4 \mathbf{C}^5 = (\mathbf{C}^5)^*$, and the mapping

$$\Lambda^2 \mathbf{C}^5 \oplus (\mathbf{C}^5)^* \ni (\omega, \xi) \mapsto (\omega, \xi + \omega \wedge \omega)$$

is in $\mathrm{Aut}(W)^H$, but not in $GL(W)^H$.

(2) In general, $U_L \neq U_1$, i.e., $\mathrm{Aut}(W)^H \neq \mathrm{Ker}\,\beta$. Consider $(V, G) = (\Lambda^3 \mathbf{C}^6 \oplus 2\mathbf{C}^6, SL_6)$. Then $(W, H) = (2\mathbf{C}^3 \oplus 2\bar{\mathbf{C}}^3, SL_3 \times \overline{SL}_3)$, and

$$\mathrm{Aut}(W)^H = GL(W)^H \simeq GL_2 \times \overline{GL}_2.$$

Moreover, $\bar{L}^0 = \mathbf{C}^*$, and one can compute that $\dim L = 13$ (see VI.3.14 Table I*a*). Thus $U_1 = \{e\}$, while $\dim U_L = 4$.

(3.5) We can say a bit more about the structure of L. Let $G' := G \times \Gamma$ acting on V in the obvious way. Then $\mathcal{O}(V)^G = \mathcal{O}(V)^{G'} = \mathbf{C}[t]$. Thus $F \simeq G' *^{H'} W$, where $H' := G'_{v_0}$ and W is H'-stable and chosen as in 3.1. (From now on we always assume W to be H'-stable.) Note that $[H' : H] = d$, and H' is generated by H and elements (a_γ, γ^{-1}), $\gamma \in \Gamma$, where $a_\gamma \in N_G(H)$ and $a_\gamma v_0 = \gamma v_0$. We may assume that $a_{\gamma^{-1}} = a_\gamma^{-1}$. Now H is obviously normal in H'. It follows that $\Gamma \subseteq L_{vb}$. In fact, the isomorphism $\varphi : G *^H W \xrightarrow{\sim} F$ sends $[g, w]$ to $g(v_0 + w)$, hence

$$\begin{aligned} \gamma[g, w] &= \varphi^{-1}(\gamma\,\varphi[g, w]) = \varphi^{-1}(\gamma\, g(v_0 + w)) \\ &= \varphi^{-1}(g a_\gamma v_0 + g a_\gamma a_\gamma^{-1} \gamma w) \\ &= [g a_\gamma, a_\gamma^{-1} \gamma w] \end{aligned}$$

where $w \mapsto a_\gamma^{-1} \gamma w$ is the action of $(a_\gamma^{-1}, \gamma) \in H'$ on $w \in W$. Thus Γ acts via G-vector bundle automorphisms. This proves part (1) of the following proposition; part (2) is obvious from 3.3.

Proposition. — (1) $\Gamma \subseteq L_{vb}$.

(2) $L = L_{vb} \ltimes U_L$ *is a semidirect product of* Γ*-groups.*

(3) $L_{vb} = \tilde{L} \ltimes GL_H(W)$ *is a semidirect product of* Γ*-groups, where* $\tilde{L} \subset L_{vb}$ *is generated by* $\bar{L}^0$ *and* Γ, *and* β' *induces an isomorphism* $\tilde{L} \xrightarrow{\sim} \bar{L}$.

Proof of (3). — From the proof of 3.3 we know that the elements of $\bar{L}^0$ act on $G *^H W \simeq F$ by $[g, w] a_0 = [g a_0, w]$. Then $\gamma^{-1} a_0 \gamma$ sends $[g, w]$ to $[g a_\gamma a_0 a_\gamma^{-1}, w]$ where $a_\gamma a_0 a_\gamma^{-1} \in \bar{L}^0$. Thus Γ normalizes the image $\tilde{L}^0 \subseteq L_{vb}$ of $\bar{L}^0$, and (3) follows easily. Note that $\gamma \in \Gamma$ acts on $GL(W)^H$ via conjugation by $(a_\gamma^{-1}, \gamma) \in H'$. ■

(3.6) *Remark* (Luna). — The connected component L^0 of L is a *special group* (IV.2.6).

4. The group scheme $\mathfrak{L}_\mathbf{B}^\Gamma$

We now define a group scheme $\mathfrak{L}_\mathbf{B}^\Gamma$ associated to L which coincides with $\mathfrak{Aut}_V^G$ over $\dot{\mathbf{A}}$, but which is much easier to deal with. In chapter V we will establish the *decomposition property* (see I.3.4) for $\mathfrak{L}_\mathbf{B}^\Gamma$. The sticky point will then be to compare the points over $\hat{\mathbf{A}}$, i.e., $\mathfrak{Aut}_V^G(\hat{\mathbf{A}}) = \mathrm{Aut}(\hat{V}/\hat{\mathbf{A}})^G$ and $\mathfrak{L}_\mathbf{B}^\Gamma(\hat{\mathbf{A}})$. We will see in 4.6 that $\mathrm{Aut}(\hat{V}/\hat{\mathbf{A}})^G \subseteq \mathfrak{L}_\mathbf{B}^\Gamma(\hat{\mathbf{A}})$.

(4.1) Let **B** denote Spec $\mathbf{C}[s]$, and represent **A** as Spec $\mathbf{C}[t]$ where $t = s^d$. Then we have a canonical morphism $\mathbf{B} \to \mathbf{A}$, $z \mapsto z^d$, which identifies **A** with $\mathbf{B}/\Gamma$, where

$\Gamma := \mu_d$ (I.4) acts by scalar multiplication on $\mathbf{B}$. As before, we denote by F the fiber $\pi^{-1}(1)$. Let Γ act on $\dot{\mathbf{B}} \times \mathrm{F}$ (on the right) by $(z, v)\gamma = (z\gamma, \gamma^{-1}v)$. There is a canonical Γ-G-equivariant isomorphism

$$(\ast) \qquad \tilde{\rho} : \dot{\mathbf{B}} \times \mathrm{F} \xrightarrow{\sim} \dot{\mathbf{B}} \times_{\dot{\mathbf{A}}} \mathrm{V}, \qquad (z, v) \mapsto [z, zv]$$

over $\dot{\mathbf{B}}$, which induces a G-isomorphism

$$\dot{\rho} : \dot{\mathbf{B}} \ast^{\Gamma} \mathrm{F} \xrightarrow{\sim} \dot{\mathrm{V}}, \qquad [z, v] \mapsto zv$$

over $\dot{\mathbf{A}} = \dot{\mathbf{B}}/\Gamma$ (cf. I.3.3).

(4.2) *Proposition. — The morphism $\tilde{\rho}$ induces a Γ-equivariant isomorphism of group schemes*

$$(1) \qquad \mathfrak{Aut}^{\mathrm{G}}_{\dot{\mathbf{B}} \times_{\dot{\mathbf{A}}} \dot{\mathrm{V}}} \xrightarrow{\sim} \dot{\mathbf{B}} \times \mathrm{L}$$

over $\dot{\mathbf{B}}$, and $\dot{\rho}$ induces an isomorphism

$$(2) \qquad \mathfrak{Aut}^{\mathrm{G}}_{\dot{\mathrm{V}}} = \mathfrak{Aut}^{\mathrm{G}}_{\mathrm{V}}|_{\dot{\mathbf{A}}} \xrightarrow{\sim} \dot{\mathbf{B}} \ast^{\Gamma} \mathrm{L}$$

over $\dot{\mathbf{A}}$.

Proof. — The isomorphism (1) is clear from $(\ast)$. By definition of the group scheme $\mathfrak{Aut}^{\mathrm{G}}_{\dot{\mathrm{V}}}$ we have in a canonical way

$$\dot{\mathbf{B}} \times_{\dot{\mathbf{A}}} \mathfrak{Aut}^{\mathrm{G}}_{\dot{\mathrm{V}}} = \mathfrak{Aut}^{\mathrm{G}}_{\dot{\mathbf{B}} \times_{\dot{\mathbf{A}}} \dot{\mathrm{V}}}.$$

Hence, by the isomorphism (1),

$$\mathfrak{Aut}^{\mathrm{G}}_{\dot{\mathrm{V}}} = \mathfrak{Aut}^{\mathrm{G}}_{\dot{\mathbf{B}} \times_{\dot{\mathbf{A}}} \dot{\mathrm{V}}}/\Gamma \xrightarrow{\sim} \dot{\mathbf{B}} \ast^{\Gamma} \mathrm{L}$$

(cf. Example 1.2(b)). ■

(4.3) *Remark.* — The Γ-action on the sections $\varphi : \dot{\mathbf{B}} \to \mathrm{L}$ has the following description:

$$\varphi \mapsto {}^{\gamma}\varphi := \gamma \cdot (\varphi \circ \gamma) \cdot \gamma^{-1}$$

(see 1.4(b)).

(4.4) Define a group functor $\mathfrak{L}^{\Gamma}_{\mathbf{B}}$ over $\mathbf{A}$ by

$$\mathfrak{L}^{\Gamma}_{\mathbf{B}}(\mathrm{Y}) := \mathrm{L}(\mathbf{B} \times_{\mathbf{A}} \mathrm{Y})^{\Gamma} \quad \text{for a morphism } \mathrm{Y} \to \mathbf{A}.$$

Then $\mathfrak{L}^{\Gamma}_{\dot{\mathbf{B}}} := \mathfrak{L}^{\Gamma}_{\mathbf{B}}|_{\dot{\mathbf{A}}}$ equals $\dot{\mathbf{B}} *^{\Gamma} \mathrm{L}$ (see 1.4(*b*)). One can show that $\mathfrak{L}^{\Gamma}_{\mathbf{B}}$ is represented by an affine group scheme over $\mathbf{A}$. In fact, $\mathfrak{L}^{\Gamma}_{\mathbf{B}}$ is the set of Γ-fixed points of the so-called *Weil restriction*, denoted by $\prod_{\mathbf{B}/\mathbf{A}} (\mathbf{B} \times \mathrm{L})$. (See [DG, I, § 1.6.6].)

Proposition 4.2(2) can now be rephrased in the following way:

(4.5) *Corollary. — There is a canonical isomorphism*

$$\dot{\sigma} : \mathfrak{Aut}^{G}_{\dot{V}} \xrightarrow{\sim} \mathfrak{L}^{\Gamma}_{\dot{\mathbf{B}}}$$

of group schemes over $\dot{\mathbf{A}}$. *In particular, we have an induced isomorphism* $\sigma_* : \mathrm{Aut}(\hat{\mathrm{V}}/\dot{\hat{\mathbf{A}}})^{G} \xrightarrow{\sim} \mathrm{L}(\dot{\hat{\mathbf{B}}})^{\Gamma}$ *where* $\sigma_*(\mathrm{Aut}(\dot{\mathrm{V}}/\dot{\mathbf{A}})^{G}) = \mathrm{L}(\dot{\mathbf{B}})^{\Gamma}$.

Later on it will be important to compare the points over $\hat{\mathbf{A}}$. The next proposition is a first step in this direction. Let $\bar{\mathfrak{A}}$ denote the closure of $\mathfrak{Aut}^{G}_{\dot{V}}$ in $\mathfrak{Aut}^{G}_{V}$. This is a (closed) subgroup scheme of $\mathfrak{Aut}^{G}_{V}$ such that $\bar{\mathfrak{A}}|_{\dot{\mathbf{A}}} = \mathfrak{Aut}^{G}_{\dot{V}}$ and $\bar{\mathfrak{A}}(\hat{\mathbf{A}}) = \mathrm{Aut}(\hat{\mathrm{V}}/\hat{\mathbf{A}})^{G}$.

(4.6) *Proposition. — The isomorphism* $\dot{\sigma} : \mathfrak{Aut}^{G}_{\dot{V}} \xrightarrow{\sim} \mathfrak{L}^{\Gamma}_{\dot{\mathbf{B}}}$ *of* 4.5 *extends to a homomorphism of group schemes*

$$\sigma : \bar{\mathfrak{A}} \to \mathfrak{L}^{\Gamma}_{\mathbf{B}}.$$

In particular, σ_* *induces an injection* $\mathrm{Aut}(\hat{\mathrm{V}}/\hat{\mathbf{A}})^{G} \hookrightarrow \mathrm{L}(\hat{\mathbf{B}})^{\Gamma}$.

Proof. — Clearly, $\sigma : \bar{\mathfrak{A}} \to \mathfrak{L}^{\Gamma}_{\mathbf{B}}$ is given as a rational map defined on $\bar{\mathfrak{A}}|_{\dot{\mathbf{A}}}$. If σ were not regular then it would have poles on a non-empty open subset of the zero fiber $\bar{\mathfrak{A}}_0$. For every point $a \in \bar{\mathfrak{A}}_0$ there is a section $\psi \in \bar{\mathfrak{A}}(\hat{\mathbf{A}}) = \mathrm{Aut}(\hat{\mathrm{V}}/\hat{\mathbf{A}})^{G}$ such that $\psi(0) = a$. Hence, it suffices to prove that the isomorphism $\mathrm{Aut}(\dot{\hat{\mathrm{V}}}/\dot{\hat{\mathbf{A}}})^{G} \xrightarrow{\sim} \mathrm{L}(\dot{\hat{\mathbf{B}}})^{\Gamma}$ induces an inclusion $\mathrm{Aut}(\hat{\mathrm{V}}/\hat{\mathbf{A}})^{G} \hookrightarrow \mathrm{L}(\hat{\mathbf{B}})^{\Gamma}$.

For this consider the diagram

$$\begin{array}{ccccc}
\hat{\mathrm{V}} & \longleftarrow & \hat{\mathbf{B}} *^{\Gamma} \mathrm{F} & \longleftarrow & \hat{\mathbf{B}} \times \mathrm{F} \\
\downarrow & & \downarrow & & \downarrow \mathrm{pr} \\
\hat{\mathbf{A}} & = & \hat{\mathbf{A}} & \longleftarrow & \hat{\mathbf{B}}
\end{array}$$

Every scheme in this diagram has a natural $\mathbf{C}^*$-action coming from the scalar multiplication on V, in such a way that all morphisms are $\mathbf{C}^*$-equivariant (the action on $\hat{\mathbf{B}} \times \mathrm{F}$ is trivial on F and by scalar multiplication on $\hat{\mathbf{B}}$). As a consequence, we obtain $\mathbf{C}^*$-actions on the automorphism groups $\mathrm{Aut}(\hat{\mathrm{V}}/\hat{\mathbf{A}})$, $\mathrm{L}(\hat{\mathbf{B}})$, $\mathrm{L}(\hat{\mathbf{B}})^{\Gamma}$:

$$\psi \to {}^{\lambda}\psi := \lambda^{-1} \circ \psi \circ \lambda, \qquad \lambda \in \mathbf{C}^*, \qquad \psi \in \mathrm{Aut}(\hat{\mathrm{V}}/\hat{\mathbf{A}}), \text{ etc.}$$

Similarly, we have $\mathbf{C}^*$-actions on $\operatorname{Aut}(\hat{\mathrm{V}}/\hat{\mathbf{A}})$, $\mathrm{L}(\hat{\mathbf{B}})$, $\mathrm{L}(\hat{\mathbf{B}})^\Gamma$. For a given $\psi \in \operatorname{Aut}(\hat{\mathrm{V}}/\hat{\mathbf{A}})^{\mathrm{G}} = \mathrm{L}(\hat{\mathbf{B}})^\Gamma$ we say that $\lim_{\lambda \to 0} {}^\lambda\psi$ *exists* if the corresponding morphism $\mathbf{C}^* \times \hat{\mathrm{V}} \to \hat{\mathrm{V}}$, $(\lambda, v) \mapsto {}^\lambda\psi(v)$, extends to a morphism $\mathbf{C} \times \hat{\mathrm{V}} \to \hat{\mathrm{V}}$. It is easy to see that

$$\mathrm{L}(\hat{\mathbf{B}}) = \{ \psi \in \mathrm{L}(\hat{\mathbf{B}}) \mid \lim_{\lambda \to 0} {}^\lambda\psi \text{ exists} \}.$$

(In fact, considering $\psi \in \mathrm{L}(\hat{\mathbf{B}})$ as a morphism $\psi : \hat{\mathbf{B}} \to \mathrm{L}$ we have ${}^\lambda\psi = \psi \circ \lambda$, which implies the claim.) Therefore, we have to show that $\lim_{\lambda \to 0} {}^\lambda\psi$ exists for every $\psi \in \operatorname{Aut}(\hat{\mathrm{V}}/\hat{\mathbf{A}})^{\mathrm{G}}$.

Consider the automorphism $({}^\lambda\psi)^*$ of the coordinate ring $\mathcal{O}(\hat{\mathrm{V}})$. We have

$$\begin{aligned}({}^\lambda\psi)^*(x_i) &= \lambda^* \psi^* (\lambda^{-1} x_i) = \lambda^* (\lambda^{-1} \cdot \psi^*(x_i)) \\ &= \lambda^{-1} \cdot (\psi^* x_i)(\lambda x_1, \lambda x_2, \ldots, \lambda x_n).\end{aligned}$$

Since $\psi^*(x_i)$ belongs to the maximal ideal $\mathbf{m}$ of $0 \in \hat{\mathrm{V}}$ and since all weights of $\mathbf{C}^*$ on $\mathbf{m}$ are $\geqslant 1$, we see that $({}^\lambda\psi)^*(x_i)$ is a polynomial in λ. Hence $\lim_{\lambda \to 0} {}^\lambda\psi$ exists. ■

(4.7) *Remarks.* — (1) A less direct proof of 4.6 can be obtained from our results in Chapter VI (see VI.1.17).

(2) Suppose that $\mathrm{Z} \xrightarrow{\rho} \mathbf{A}$ is a flat $\mathbf{A}$-scheme (for example, Z is irreducible and ρ is dominant). Then $\dot{\mathrm{Z}} := \rho^{-1}(\dot{\mathbf{A}})$ is dense in Z and any element of $\mathfrak{Aut}_{\mathrm{V}}^{\mathrm{G}}(\mathrm{Z})$ is determined by its restriction to $\dot{\mathrm{Z}}$. Hence $\mathfrak{Aut}_{\mathrm{V}}^{\mathrm{G}}(\mathrm{Z}) = \bar{\mathfrak{A}}(\mathrm{Z})$.

Chapter IV. FIBER BUNDLES AND COHOMOLOGY

0. Résumé

(0.1) Let X be a smooth affine acyclic G-variety with fixed point set $X^G = \{x_0\}$ and quotient map $\pi_X : X \to \mathbf{A}$ where $\pi_X(x_0) = 0$. Denote by V the G-module $T_{x_0}X$ and fix a homogeneous quotient map $\pi_V : V \to \mathbf{A}$. Let $\dot{X} := X \setminus \pi_X^{-1}(0)$ and $\dot{V} := V \setminus \pi_V^{-1}(0)$ be the complements of the zero fiber, and let $F := \pi_V^{-1}(1)$ be the general fiber.

(0.2) *Theorem. — There exists a* G*-equivariant isomorphism* $\dot{\varphi} : \dot{X} \xrightarrow{\sim} \dot{V}$ *over* $\dot{\mathbf{A}}$:

$$\begin{array}{ccc} \dot{X} & \xrightarrow[\sim]{\dot{\varphi}} & \dot{V} \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle \pi_V} \\ \dot{\mathbf{A}} & = & \dot{\mathbf{A}} \end{array}$$

Proof (cf. discussion following Theorem 12 of I.3.1). — We will reduce the proof to a statement about cohomology sets. The two quotients $\dot{X} \to \dot{\mathbf{A}}$ and $\dot{V} \to \dot{\mathbf{A}}$ are G-fiber bundles with fiber F (II.0.1 and II.0.4). They correspond to principal L-bundles over $\dot{\mathbf{A}}$ where L is the G-automorphism group $\mathrm{Aut}(F)^G$ of F (see 1.4 and 1.5(d)). We have to show that the two classes $[\dot{X}]$ and $[\dot{V}]$ in $H^1_{et}(\dot{\mathbf{A}}, L)$, the set of isomorphism classes of principal L-bundles over $\dot{\mathbf{A}}$, are equal. Since $\hat{X}$ and $\hat{V}$ are isomorphic over $\hat{\mathbf{A}}$ (II.0.4) the bundles become isomorphic over $\hat{\dot{\mathbf{A}}} := \dot{\mathbf{A}} \cap \hat{\mathbf{A}} = \mathrm{Spec}\, \mathbf{C}((t))$. Consider the following diagram:

$$\begin{array}{ccc} H^1_{et}(\dot{\mathbf{A}}, L) & \longrightarrow & H^1_{et}(\dot{\mathbf{A}}, L/L^0) \\ \downarrow & & \downarrow \\ H^1_{et}(\hat{\dot{\mathbf{A}}}, L) & \longrightarrow & H^1_{et}(\hat{\dot{\mathbf{A}}}, L/L^0) \end{array}$$

The right vertical map is bijective, because L/L^0 is finite and $\dot{\mathbf{A}}$ and $\hat{\dot{\mathbf{A}}}$ have the same (algebraic) fundamental group (i.e., they have corresponding étale coverings; see 4.4). So the proof is reduced to showing that the canonical map $H^1_{et}(\dot{\mathbf{A}}, L) \to H^1_{et}(\dot{\mathbf{A}}, L/L^0)$ is injective. In 5.4 we show that the map is, in fact, bijective. ■

(0.3) There is a more general approach to the results of this chapter using *locally constant group schemes over* $\dot{\mathbf{A}}$, i.e., group schemes over $\dot{\mathbf{A}}$ which become trivial under an étale base change $Y \to \dot{\mathbf{A}}$. For example, the automorphism group scheme $\dot{\mathfrak{A}} := \mathfrak{Aut}^G_V|_{\dot{\mathbf{A}}}$ is locally constant, and even isotrivial (III.4.2). This is a general fact:

Proposition. — Every locally constant group scheme over $\dot{\mathbf{A}}$ *is isotrivial.*

(0.4) For any locally constant group scheme $\mathfrak{G}$ over $\dot{A}$ one has the notion of a *principal* $\mathfrak{G}$*-bundle* generalizing Definition 1.1 below (cf. [DG], III, §4]; we denote by $H^1_{et}(\dot{A}, \mathfrak{G})$ the set of isomorphism classes of these bundles.

In this setting the two quotients $\dot{X} \to \dot{A}$ and $\dot{V} \to \dot{A}$ correspond to principal $\dot{\mathfrak{A}}$-bundles, and the second represents the trivial element of $H^1_{et}(\dot{A}, \dot{\mathfrak{A}})$. Define $\dot{\mathfrak{A}}^0 \subseteq \dot{\mathfrak{A}}$ to be the connected component containing the unit section. It is a locally constant normal subgroup scheme with connected fiber, and the quotient $\dot{\mathfrak{A}}/\dot{\mathfrak{A}}^0$ exists and is a *finite isotrivial* group scheme over $\dot{A}$. One can prove the following result which immediately implies Theorem 0.2.

Theorem. — Let $\mathfrak{G}$ *be a locally constant group scheme over* $\dot{A}$. *Then the canonical map* $H^1_{et}(\dot{A}, \mathfrak{G}) \to H^1_{et}(\dot{A}, \mathfrak{G}/\mathfrak{G}^0)$ *is bijective.*

1. Fiber bundles

Let M be an algebraic group and Y a variety.

(1.1) *Definition.* — A *principal* M*-bundle* over Y is a variety $\mathfrak{P}$ with a (right) M-action together with a morphism $\pi_{\mathfrak{P}}: \mathfrak{P} \to Y$ such that there is a surjective étale map $\eta: \tilde{Y} \to Y$ and an M-equivariant isomorphism $\tilde{Y} \times_Y \mathfrak{P} \xrightarrow{\sim} \tilde{Y} \times M$ over $\tilde{Y}$, where M acts on the second factor by right multiplication. The bundle $\mathfrak{P}$ is called *trivial* if it is isomorphic to $Y \times M$, and *isotrivial* if it can be trivialized by a finite covering $\eta: \tilde{Y} \to Y$ (i.e. by a finite étale surjective morphism).

We denote by $H^1_{et}(Y, M)$ the set of isomorphism classes of principal M-bundles, and by $H^1_{iso}(Y, M)$ the subset represented by the isotrivial bundles. They are *pointed sets* where the distinguished element $*$ is the isomorphism class of the trivial M-bundle.

(1.2) The sets $H^1_{et}(Y, M)$ have a number of well-known functorial properties (cf. [DG, III, §4]). For any morphism $\alpha: Z \to Y$ there is a canonical map $\alpha^*: H^1_{et}(Y, M) \to H^1_{et}(Z, M)$ given by the pull-back $\mathfrak{P} \mapsto \alpha^*(\mathfrak{P}) := Z \times_Y \mathfrak{P}$. If $N \subseteq M$ is a closed subgroup we get an exact sequence (of pointed sets)

$$1 \to N(Y) \to M(Y) \to (M/N)(Y) \to H^1_{et}(Y, N) \xrightarrow{i_*} H^1_{et}(Y, M)$$

where the image of i_* is given by those principal M-bundles $\mathfrak{P}$ for which the quotient $\mathfrak{P}/N$ has a section. If N is a normal subgroup then the exact sequence can be extended by one term:

$$1 \to N(Y) \to M(Y) \to (M/N)(Y) \to H^1_{et}(Y, N) \to H^1_{et}(Y, M) \to H^1_{et}(Y, M/N)$$

(1.3) Let E be a variety. A *fiber bundle* over Y with fiber E is a morphism $\beta: \mathfrak{F} \to Y$ such that there is an étale surjective map $\eta: \tilde{Y} \to Y$ and an isomorphism $\mathfrak{F}_{\tilde{Y}} := \tilde{Y} \times_Y \mathfrak{F} \xrightarrow{\sim} \tilde{Y} \times E$ over $\tilde{Y}$.

If the automorphism group of E is an algebraic group M then there is a well-known equivalence between the fiber bundles over Y with fiber E and the principal M-bundles over Y: If $\mathfrak{P}$ is a principal M-bundle then $\mathfrak{P} *^M E := (\mathfrak{P} \times E)/M$ is a fiber bundle over Y with fiber E. Conversely, if $\mathfrak{F}$ is a fiber bundle over Y with fiber E, then the functor $\mathfrak{Iso}(Y \times E, \mathfrak{F})$ which associates to each morphism $U \to Y$ the set of isomorphisms $U \times E \xrightarrow{\sim} \mathfrak{F}_U$ over U, is represented by a principal M-bundle over Y. In particular, $H^1_{et}(Y, M)$ describes the isomorphism classes of fiber bundles over Y with fiber E.

(1.4) In most applications, the variety E has some additional structure, e.g., E is a group or a vector space, or there is a group G acting on E. One then wants to study the fiber bundles $\mathfrak{F}$ with this additional structure, e.g., *locally trivial group schemes*, *vector bundles*, or *G-fiber bundles* (cf. I.2.3). The same approach as above gives an equivalence with the principal M-bundles over Y, where now M is the group of those automorphisms of E which preserve the additional structure.

(1.5) *Examples.* – (*a*) Usually, vector bundles are defined to be locally trivial in the Zariski-topology. But every principal GL_n-bundle is automatically locally trivial in the Zariski-topology (see 2.3). Hence the set of isomorphism classes of vector bundles of rank *n* over Y is given by $H^1_{et}(Y, GL_n)$.

(*b*) If T is a torus, $T \simeq (\mathbf{C}^*)^r$, then $H^1_{et}(Y, T) \simeq \operatorname{Pic}(Y)^r$, where $\operatorname{Pic}(Y)$ denotes the group of isomorphism classes of line bundles on Y.

(*d*) Let X, V, F, etc. be as in 0.1. Then $\dot{X} \to \dot{A}$ and $\dot{V} \to \dot{A}$ are G-fiber bundles with fiber F (II.0.1 and II.0.4), and the automorphism group $L := \operatorname{Aut}(F)^G$ is a linear algebraic group (III.2.5). Hence, the two G-fiber bundles correspond to principal L-bundles over $\dot{A}$ and determine elements $[\dot{X}], [\dot{V}] \in H^1_{et}(\dot{A}, L)$. We have $[\dot{X}] = [\dot{V}]$ if and only if $\dot{X}$ and $\dot{V}$ are G-isomorphic over $\dot{A}$.

2. Special groups

In this section we collect results about special groups. The references are [Se3] and some unpublished notes of Luna. Let us first recall the definition from which the importance of this notion for our work is clear.

(2.1) *Definition.* – A linear algebraic group M is called *special* if every principal M-bundle is locally trivial in the Zariski-topology.

Clearly, special groups are connected. The main result of this section is Corollary 2.6, stating that L^0 is special where L is as in 1.5(d). The only part of this section needed later on is Proposition 2.3.

(2.2) *Remark.* – Consider an exact sequence

$$1 \to M' \to M \to M'' \to 1$$

of algebraic groups. If M′ and M″ are special, then so is M. (This is an easy consequence of the second exact sequence in 1.2.)

The following result can be found in ([Se3]). It can also be deduced from Lemma 2.4 below.

(2.3) *Proposition.* – (1) *The groups* GL_n, SL_n *and* Sp_n *are special.*

(2) *Every connected solvable group is special.*

Remark. – Grothendieck [Gr] has shown that SL_n and Sp_n are the only simple groups which are special.

The following result is due to Luna (unpublished).

(2.4) *Lemma.* – *Let* M *be a reductive group and* W *an* M-*module. Assume that* W *contains a closed orbit* $M v_0$ *with trivial stabilizer, and that there is an* M-*equivariant retraction* $\rho : U \to M v_0$ *where* U *is an open* M-*stable neighborhood of* $M v_0$. *Then* M *is special.*

Proof. – Let $\pi : \mathfrak{P} \to Y$ be a principal M-bundle and consider a fiber $P := \pi^{-1}(y_0)$. We may assume that Y and $\mathfrak{P}$ are affine. Choose an M-equivariant isomorphism $\alpha : P \xrightarrow{\sim} M v_0$. Then α extends to an M-equivariant morphism $\bar{\alpha} : \mathfrak{P} \to W$ because W is a vector space and M is reductive. It follows that $\tilde{U} := \tilde{\alpha}^{-1}(U)$ is an M-stable open neighborhood of P and that $\tilde{\rho} := \alpha^{-1} \circ \rho \circ \tilde{\alpha} : \tilde{U} \to P$ is an M-equivariant retraction. If S is a fiber of $\tilde{\rho}$, then the map $M \times S \to \tilde{U}$, $(m, s) \mapsto ms$, is an M-equivariant isomorphism. Hence the bundle $\mathfrak{P}$ is trivial over the neighborhood $\pi(\tilde{U})$ of y_0. ■

In the rest of this section V denotes any G-module with one-dimensional quotient. Let F and L be defined as in 0.1 and 0.2.

(2.5) *Proposition* (Luna). – *Assume that* G *is connected and acts semi-freely on* V. *Then* G *is special.*
(Recall that "semifree" means that every closed orbit different from $\{0\}$ has a trivial stabilizer.)

Proof. — The quotient map $\pi: V \to \mathbf{A}$ is homogeneous of degree d. Choose $v_0 \in F$. Clearly, $\{\lambda \in \mathbf{C}^* \mid \lambda F = F\} = \mu_d$, the group of d-th roots of unity. Since $g \mapsto g v_0$ is an isomorphism $G \xrightarrow{\sim} F$, there is a unique homomorphism $\rho: \mu_d \to G$ such that $\gamma \cdot v_0 = \rho(\gamma) v_0$ for all $\gamma \in \mu_d$. Since G is connected, ρ extends to a homomorphism $\tilde{\rho}: \mathbf{C}^* \to G$. It follows that the map $(g, t) \mapsto t \cdot (g \rho(t)^{-1} v_0)$ induces a G-equivariant isomorphism $G \times (\mathbf{C}^*/\mu_d) \xrightarrow{\sim} \dot{V}$. Hence there is a G-equivariant retraction $\dot{V} \to F \simeq G$, and the claim follows from 2.4. ■

(2.6) *Corollary* (Luna). — L^0 *is special.*

Proof. — Let H be the principal isotropy group of V. Using 2.2, 2.3 and III.3.3 we reduce to proving that $(N_G(H)/H)^0$ is special. But this follows from 2.5, since $(N_G(H)/H)^0$ acts semifreely on V^H with a one-dimensional quotient (cf. [LR]). ■

3. Bundles over curves

Let M denote a linear algebraic group. First we recall some well-known facts about fiber bundles over curves, and then we establish that $H^1_{et}(\dot{\mathbf{A}}, M) = H^1_{iso}(\dot{\mathbf{A}}, M)$, the main result of this section. In addition, we show that $H^1_{et}(\mathbf{A}, M)$ is trivial for every M.

(3.0) Let Y be a variety. We denote by $H^1_{Zar}(Y, M)$ the subset of $H^1_{et}(Y, M)$ consisting of those principal M-bundles which are locally trivial in the Zariski-topology.

(3.1) *Theorem.* — *Let* C *be a smooth curve. Assume that* M *is connected, and let* B *be a Borel subgroup of* M. *Then*

(1) $H^1_{et}(C, B) \to H^1_{et}(C, M)$ *is surjective.*

(2) *If* C *is affine and rational then* $H^1_{et}(C, M)$ *is trivial.*

The proof will be given in 3.7. We first draw some consequences.

(3.2) *Corollary.* — *Let* C *and* M *be as above. Then*

(1) $H^1_{et}(C, M) = H^1_{Zar}(C, M)$.

(2) *If* C *is affine and rational and* M *is a closed subgroup of the linear algebraic group* N, *then* $N(C) \to (N/M)(C)$ *is surjective.*

Proof. — Since B is special (2.3), $H^1_{et}(C, B) = H^1_{Zar}(C, B)$ and so 3.1 (1) implies the first claim. Part (2) follows from 3.1 (2) and the exact sequence of $H^{\bullet}_{et}$ (see 1.2). ■

(3.3) *Theorem.* — *Every element of* $H^1_{et}(\dot{A}, M)$ *becomes trivial on an* $|M/M^0|$*-fold cover of* $\dot{A}$. *In particular,* $H^1_{et}(\dot{A}, M) = H^1_{iso}(\dot{A}, M)$.

Proof. — Every element of $H^1_{et}(\dot{A}, M/M^0)$ is represented by a (not necessarily connected) Galois covering of $\dot{A}$ with group M/M^0. All such covers become trivial when lifted to the n-fold cover of $\dot{A}$, where $n := |M/M^0|$. Given $[P] \in H^1_{et}(\dot{A}, M)$, consider its n-fold lift $\tilde{P}$. Then the class $[\tilde{P}]$ has trivial image in $H^1_{et}(\dot{A}, M/M^0)$. Hence $[\tilde{P}]$ comes from an element of $H^1_{et}(\dot{A}, M^0)$, and $H^1_{et}(\dot{A}, M^0)$ is trivial by 3.1 (2). ■

Our proof of Theorem 3.1 will use the following results:

(3.4) *Proposition.* — *Let* Z *be an affine variety,* U *a unipotent algebraic group, and* T *a torus of rank r. Then*

(1) $H^1_{et}(Z, U) = \{*\}$.

(2) $H^1_{et}(Z, T) \simeq \mathrm{Pic}(Z)^r$.

Proof. — Since U is unipotent there is a surjective homomorphism to the additive group $\mathbf{C}^+$. Using the exact sequence of $H^{\bullet}_{et}$ and induction on dim U we reduce to proving that $H^1_{et}(Z, \mathbf{C}^+) = \{*\}$. But $\mathbf{C}^+$ is special (2.3), hence $H^1_{et}(Z, \mathbf{C}^+) = H^1_{Zar}(Z, \mathbf{C}^+) = H^1(Z, \mathcal{O}_Z) = 0$. This establishes (1), and (2) follows from the isomorphisms

$$H^1_{et}(Z, T) \simeq H^1_{et}(Z, (\mathbf{C}^*)^r) \simeq H^1_{Zar}(Z, (\mathbf{C}^*)^r) \simeq H^1_{Zar}(Z, \mathbf{C}^*)^r = (\mathrm{Pic}\, Z)^r$$

(see 1.5 (b)). ■

The exact sequence of $H^{\bullet}_{et}(Z, -)$ now gives:

(3.5) *Corollary.* — *Suppose that* Pic Z = 0 *and that* B *is a connected solvable group. Then* $H^1_{et}(Z, B)$ *is trivial.*

(3.6) *Corollary.* — *Suppose that* U *is a normal unipotent subgroup of* M. *Then* $M \to M/U$ *is a trivial* U*-bundle, hence it has a section.*

(3.7) *Proof of* (3.1). — We may assume that C is irreductible. Let $[P] \in H^1_{et}(C, M)$. The field $\mathscr{K}$ of rational functions on C is a (C_1)-field and so $H^1_{et}(\mathrm{Spec}\, \mathscr{K}, M)$ is trivial ([Se1, II.3.3 and III.2.2]). In other words, P has a rational section, i.e., there is a section $\sigma : C' \to P$ where C′ is the complement of a finite set in C.

Now σ induces a section $\tau : C' \to P/B$ which extends to a global section on C, because the projection $p : P/B \to C$ is proper and C is smooth. (In fact, consider the closure D of the image of τ in P/B. Then p induces an isomorphism $D \xrightarrow{\sim} C$.) Thus

[P] lies in the image of the map $H^1_{et}(C, B) \to H^1_{et}(C, M)$. Hence we obtain (1), and (2) follows from (1) and 3.5. ■

(3.8) *Remark.* — Since $H^1_{et}(\mathbf{A}, M/M^0)$ is trivial we have $H^1_{et}(\mathbf{A}, M) = \{*\}$ for any M. As a consequence, we obtain the following version of Corollary II.0.3(2).

(3.9) *Proposition.* — *Consider an action of* G *on a smooth affine variety* X. *Assume that* $X/\!/G \simeq \mathbf{A}$ *and that the fixed point set is one-dimensional. Then* X *is* G-*isomorphic to* $\mathbf{C} \times W$ *where* W *is a* G-*module and* $\mathbf{C}$ *denotes the trivial* G-*module.*

Proof. — The slice theorem implies that the quotient $\pi : X \to \mathbf{A}$ is a G-fiber bundle whose fiber F has the structure of a G-module W (see II.1.2). These bundles are classified by the set $H^1_{et}(\mathbf{A}, \operatorname{Aut}(F)^G)$ which is trivial by the remark above. ■

The following decomposition result will be useful in sections 5-6.

(3.10) *Proposition.* — *Let* $X = \operatorname{Spec} \mathbf{C}(t)$ *or let* $X \subset \mathbf{A}$ *be a non-empty open subset. If* $B \subset M$ *is a Borel subgroup, then* $M(X) = M(\mathbf{A}) \cdot B(X)$.

Proof. — Let $\sigma \in M(X)$. Since M/B is complete, the composition $\rho \circ \sigma : X \to M \to M/B$ extends to a morphism $\tau : \mathbf{A} \to M/B$. By 3.2(2), τ lifts to a morphism $\tilde{\sigma} : \mathbf{A} \to M$. Clearly, $\tilde{\sigma}^{-1} \cdot \sigma \in B(X)$, hence $\sigma \in M(\mathbf{A}) \cdot B(X)$. ■

4. Non-abelian cohomology

We first identify $H^1_{et}(\dot{\mathbf{A}}, M)$ with certain (non-abelian) cohomology sets arising from actions of cyclic groups. Then we use Galois cohomology and group cohomology to establish that $H^1_{et}(\dot{\mathbf{A}}, M) \xrightarrow{\sim} H^1_{et}(\dot{\mathbf{A}}, M/M^0)$. As we have already seen, this will finish our proof of Theorem 0.2.

(4.1) Let Y be an irreducible affine variety, and let Γ be a finite group acting freely on Y on the right. We denote by $H^1(\Gamma, Y, M)$ the subset of those elements of $H^1_{et}(Y/\Gamma, M)$ which become trivial when lifted to Y.

Let $[P] \in H^1(\Gamma, Y, M)$. Then P, being trivial on Y, is isomorphic to the quotient $(Y \times M)/\Gamma$, where $\gamma \in \Gamma$ acts by

$$(y, m)\gamma = (y\gamma, h_\gamma(y)^{-1} m)$$

for a suitable $h_\gamma \in M(Y)$. One easily verifies that

$$(*) \qquad h_{\gamma\gamma'} = h_\gamma(\gamma h_{\gamma'}), \qquad \gamma, \gamma' \in \Gamma,$$

where Γ acts on $M(Y)$ by $(\gamma f)(y) := f(y\gamma)$, $\gamma \in \Gamma$, $f \in M(Y)$, $y \in Y$. The condition $(*)$ defines the set $Z^1(\Gamma, M(Y))$ of 1-*cocycles of* Γ *with values in* $M(Y)$.

(4.2) Let $\{h'_\gamma\}$ be a 1-cocycle arising from $[P'] \in H^1(\Gamma, Y, M)$. Then one easily shows:

$(**)$ $[P] = [P']$ if and only if there is an $f \in M(Y)$ such that $h'_\gamma = f^{-1} h_\gamma (\gamma f)$ for all $\gamma \in \Gamma$,

i.e., if and only if $\{h_\gamma\}$ and $\{h'_\gamma\}$ give the same element of the cohomology set $H^1(\Gamma, M(Y))$ (cf. [Se1, I.5)]. Moreover, given $\{h_\gamma\} \in Z^1(\Gamma, M(Y))$ one easily constructs a principal M-bundle P over Y/Γ whose associated cohomology class [P] is the one defined by $\{h_\gamma\}$. Thus we have the following result:

(4.3) *Proposition. – The association* $P \mapsto \{h_\gamma\}$ *of* 4.1 *gives a bijection*

$$H^1(\Gamma, Y, M) \xrightarrow{\sim} H^1(\Gamma, M(Y)).$$

(4.4) *Remark.* – Suppose that M is finite. Then $M(Y) = M$ and $H^1(\Gamma, M(Y)) = H^1(\Gamma, M)$. Clearly $Z^1(\Gamma, M)$ consists of group homomorphisms $\sigma: \Gamma \to M$, and $[\sigma] = [\sigma']$ in $H^1(\Gamma, M)$ if and only if $\sigma' = f\sigma f^{-1}$ for some $f \in M$. Now let $\Gamma = \mu_n$ act on $Y = \dot{\mathbf{B}}$ by multiplication with quotient $\dot{\mathbf{A}}$ as in III.4.1. If $|M|$ divides n, then $H^1(\Gamma, M)$ is isomorphic to M modulo conjugation. By 3.3 and 4.3 we get

$$\begin{aligned} H^1_{et}(\dot{\mathbf{A}}, M) = H^1_{iso}(\dot{\mathbf{A}}, M) = H^1(\Gamma, \dot{\mathbf{B}}, M) \simeq H^1(\Gamma, M(\dot{\mathbf{B}})) \\ = H^1(\Gamma, M) \simeq \{\text{conjugacy classes in } M\}. \end{aligned}$$

By Galois theory, we also have $H^1_{et}(\hat{\dot{\mathbf{A}}}, M) \simeq \{\text{conjugacy classes in } M\}$, and so $H^1_{et}(\dot{\mathbf{A}}, M) \simeq H^1_{et}(\hat{\dot{\mathbf{A}}}, M)$.

5. The twist construction

(5.1) We use the notation of 4.1. Suppose that $\{h_\gamma\}$ is in $Z(\Gamma, M(Y))$. Then one can define a new action of Γ on $M(Y)$ as follows:

$$({}^\gamma f)(y) := h_\gamma(y)(\gamma f)(y) h_\gamma(y)^{-1}, \qquad f \in M(Y), y \in Y.$$

One easily verifies that ${}^{\gamma\gamma'}f = {}^\gamma({}^{\gamma'}f)$. Let ${}_hM(Y)$ denote $M(Y)$ with the new action of Γ, so we have a new cohomology set $H^1(\Gamma, {}_hM(Y))$. A straightforward verification gives the following:

(5.2) *Proposition. – Let* $h = \{h_\gamma\}$, ${}_hM(Y)$, *etc. be as above. Let* $\{h'_\gamma\} \in Z(\Gamma, M(Y))$ *and define* $\bar{h}_\gamma := h'_\gamma h_\gamma^{-1}$. *Then*

(1) $\{\bar{h}_\gamma\} \in H^1(\Gamma, {}_hM(Y))$.

(2) $\{\bar{h}_\gamma\}$ *is the trivial element in* $H^1(\Gamma, {}_h M(Y))$ *if and only if* $\{h'_\gamma\} = \{h_\gamma\}$ *in* $H^1(\Gamma, M(Y))$.

(5.3) Consider the exact sequence

$$H^1_{et}(\dot{A}, M^0) \to H^1_{et}(\dot{A}, M) \to H^1_{et}(\dot{A}, M/M^0)$$

of pointed sets. As advertised, we will prove the following result:

(5.4) *Theorem.* – $H^1_{et}(\dot{A}, M) \xrightarrow{\sim} H^1_{et}(\dot{A}, M/M^0)$.

Note that we have already shown in 3.1 (2) that $H^1_{et}(\dot{A}, M^0)$ is trivial, but this is not sufficient to establish 5.4. We need to use the twist construction applied to the following situation (see 4.4): $\Gamma = \mu_n$ acts on $Y = \dot{\mathbf{B}}$ by multiplication with quotient $\dot{\mathbf{A}}$ (cf. III.4.1).

(5.5) *Proof of* (5.4). – Let $z \in H^1_{et}(\dot{A}, M/M^0)$. Choose n such that $|M/M^0|$ divides n. Then $z = [\bar{\sigma}]$, where $\bar{\sigma}: \Gamma \to M/M^0$ is a homomorphism (see 4.4). Choosing n big enough one can always find a homomorphism $\sigma: \Gamma \to M$ lifting $\bar{\sigma}$. (In fact, one easily sees that every connected component of an algebraic group contains elements of finite order). Clearly, σ defines an element $\{\sigma(\gamma)\} \in H^1(\Gamma, M(\dot{\mathbf{B}}))$ whose image in $H^1(\Gamma, M/M^0)$ is $[\bar{\sigma}]$. In particular, $H^1_{et}(\dot{A}, M) \to H^1_{et}(\dot{A}, M/M^0)$ is surjective.

Let $[P] \in H^1_{et}(\dot{A}, M)$ have image $[\bar{\sigma}]$ in $H^1_{et}(\dot{A}, M/M^0)$, and let $\{h_\gamma\}$ be the cocycle in $Z(\Gamma, M(\dot{\mathbf{B}}))$ corresponding to [P] (see 3.3). We need to show that $\{h_\gamma\} = \{\sigma(\gamma)\}$ in $H^1(\Gamma, M(\dot{\mathbf{B}}))$. Clearly, the cocycle $\bar{h}_\gamma(y) := h_\gamma(y)\sigma(\gamma)^{-1}$ has values in M^0. Therefore, by 5.2, it suffices to prove the following result:

(5.6) *Proposition.* – $H^1(\Gamma, {}_\sigma M^0(\dot{\mathbf{B}})) = \{*\}$.

Proof. – By [Se1, III.2.2] we obtain this result if we replace ${}_\sigma M^0(\dot{\mathbf{B}})$ by ${}_\sigma M^0(\mathrm{Spec}\,\mathbf{C}(s))$. Thus there is a rational function $f: \dot{\mathbf{B}} \to M^0$ with $\bar{h}_\gamma = f({}^\gamma f)^{-1}$, $\gamma \in \Gamma$. From [St2, Theorem 7.5] we obtain a Γ-stable Borel subgroup B of M^0 and a Γ-stable torus $T \subset B$ (see V.2.5). By 3.10 we can find $\tilde{f} \in M^0(\mathbf{B}) \subset M^0(\dot{\mathbf{B}})$ such that $\tilde{f}^{-1} f$ takes values in B. Hence

$$\tilde{f}^{-1}\bar{h}_\gamma({}^\gamma\tilde{f}) = (\tilde{f}^{-1}f)({}^\gamma(\tilde{f}^{-1}f))^{-1}$$

takes values in B, and we can reduce to proving 5.6 with M^0 replaced by B.

Let U denote the unipotent radical of B. Then U is Γ-stable, hence so is its center Z(U). We claim that $H^1(\Gamma, {}_\sigma U(\dot{\mathbf{B}})) = \{*\}$. By 3.6 we have an exact sequence

$$1 \to Z(U)(\dot{\mathbf{B}}) \to U(\dot{\mathbf{B}}) \to (U/Z(U))(\dot{\mathbf{B}}) \to 1$$

which gives rise to a corresponding exact sequence for $H^{\cdot}(\Gamma, {}_{\sigma}-)$ (see [Se1, I.5.5]). By induction on dimension, we reduce to the case that U is abelian. Then $U(\dot{\mathbf{B}})$ is a direct sum of copies of $\mathcal{O}(\dot{\mathbf{B}})$, and multiplication with $|\Gamma|$ is invertible in $\mathcal{O}(\dot{\mathbf{B}})$. Hence $H^1(\Gamma, {}_{\sigma}U(\dot{\mathbf{B}})) = \{*\}$ ([Se2, p. 138]). Applying $H^{\cdot}(\Gamma, {}_{\sigma}-)$ to

$$1 \to U(\dot{\mathbf{B}}) \to B(\dot{\mathbf{B}}) \to T(\dot{\mathbf{B}}) \to 1$$

we see that 5.6 follows from the next proposition. ■

(5.7) *Proposition.* – $H^1(\Gamma, {}_{\sigma}T(\dot{\mathbf{B}})) = \{*\}$.

Proof (Serre). – Let $r = \operatorname{rank} T$, so that $T \simeq \mathbf{C}^{*r}$ and $\operatorname{Aut}(T) \simeq GL_r(\mathbf{Z})$. Let $\tau : \Gamma \to \operatorname{Aut}(T)$ be the mapping sending γ into conjugation by $\sigma(\gamma)$. Let E denote $\mathbf{C}^*(\dot{\mathbf{B}})$ with the usual Γ-action (coming from the action on $\dot{\mathbf{B}}$), and let E_τ denote $E \otimes_{\mathbf{Z}} \mathbf{Z}^r$ where Γ acts on $\mathbf{Z}^r$ via τ. Then $E_\tau \simeq {}_{\sigma}T(\dot{\mathbf{B}})$ as Γ-module.

For the Γ-modules E and E_τ we have the ordinary (reduced) cohomology groups $\hat{H}^i(\Gamma, E)$ and $\hat{H}^i(\Gamma, E_\tau)$, $i \in \mathbf{Z}$, where $\hat{H}^1(\Gamma, E_\tau) = H^1(\Gamma, E_\tau)$ ([Se2, p. 131]). A Γ-module M is called *cohomology trivial* if $\hat{H}^i(\Delta, M) = 0$ for all i and all subgroups Δ of Γ. Now any subgroup Δ of Γ is cyclic, and one can compute by hand ([Se2, p. 141]) that $\hat{H}^0(\Delta, E) = \hat{H}^1(\Delta, E) = 0$. The $\hat{H}^i(\Delta, E)$ only depend upon the parity of i, hence E is cohomologically trivial. Since $\mathbf{Z}^r$ is torsion free this implies that $E_\tau = E \otimes_{\mathbf{Z}} \mathbf{Z}^r$ is cohomologically trivial, too ([Se2, p. 152]). Thus $H^1(\Gamma, {}_{\sigma}T(\dot{\mathbf{B}})) = \{*\}$. ■

(5.8) *Remark.* – In 5.6 the action of Γ on $M^0(\dot{\mathbf{B}})$ is induced by a homomorphism $\sigma : \Gamma \to M$ (and the natural action of $\Gamma = \mu_n$ on $\dot{\mathbf{B}}$). It is easy to see that 5.6 holds for a general Γ-group structure on M^0, rather than only for those induced (via conjugation) by homomorphisms $\sigma : \Gamma \to M$.

6. Group cohomology over the line

We carry out some cohomology calculations over **B** similar to those in section 5. They will be used in the proof of the decomposition property for linear algebraic groups (see Theorem V.2.6).

(6.1) Consider an action of $\Gamma = \mu_n$ on M given by a group homomorphism $\sigma : \Gamma \to \operatorname{Aut} M$. Then Γ acts on $M(\mathbf{B}) = \operatorname{Mor}(\mathbf{B}, M)$ by group automorphisms:

$$m \mapsto {}^{\gamma}m := \sigma(\gamma) \circ m \circ \gamma, \qquad \gamma \in \Gamma, \quad m \in M(\mathbf{B}).$$

As before the corresponding Γ-groups will be denoted by ${}_{\sigma}M$ and ${}_{\sigma}M(\mathbf{B})$, respectively.

We want to calculate the group cohomology $H^1(\Gamma, {}_\sigma M(\mathbf{B}))$ of ${}_\sigma M(\mathbf{B})$. The evaluation map $e: m \mapsto m(0)$ induces a split exact sequence of Γ-groups

$$1 \to {}_\sigma M(\mathbf{B})_1 \to {}_\sigma M(\mathbf{B}) \overset{e}{\to} {}_\sigma M \to 1$$

where ${}_\sigma M(\mathbf{B})_1 = \operatorname{Ker} e$, and the section $s: {}_\sigma M \to {}_\sigma M(\mathbf{B})$ of e sends h to the constant map $\mathbf{B} \to M$ with image h.

(6.2) *Lemma. — The following are equivalent:*

(1) $H^1(\Gamma, {}_\sigma M(\mathbf{B})_1)$ *is trivial for every* σ.

(2) *The canonical map* $H^1(\Gamma, {}_\sigma M(\mathbf{B})) \to H^1(\Gamma, {}_\sigma M)$ *is bijective for all* σ.

Proof. — We have the usual exact sequence of pointed sets (see [Se1, I.5.5], cf. 12):

$$\ldots \to {}_\sigma M(\mathbf{B})^\Gamma \overset{\alpha}{\to} {}_\sigma M^\Gamma \to H^1(\Gamma, {}_\sigma M(\mathbf{B})_1) \to$$

$$\to H^1(\Gamma, {}_\sigma M(\mathbf{B})) \overset{\beta}{\to} H^1(\Gamma, {}_\sigma M).$$

The existence of a Γ-equivariant section $s: {}_\sigma M \to {}_\sigma M(\mathbf{B})$ (6.1) implies that α is surjective. If β is bijective, then $H^1(\Gamma, {}_\sigma M(\mathbf{B})_1)$ is trivial and so (2) implies (1).

For the other implication we use the twist construction. Let $\{h_\gamma\}$, $\{h'_\gamma\}$ be two cocycles in $Z^1(\Gamma, {}_\sigma M(\mathbf{B}))$. Then $\{h_\gamma(0)\} \in Z^1(\Gamma, {}_\sigma M)$. Define $\tau(\gamma) \in \operatorname{Aut} M$ by $\tau(\gamma)(m) := h_\gamma(0)\,\sigma(\gamma)(m)\,h_\gamma(0)^{-1}$. One easily checks that $\tau: \Gamma \to \operatorname{Aut} M$ is a group homomorphism. Put $\bar{h}_\gamma(z) := h_\gamma(z)\,h_\gamma(0)^{-1}$ and $\bar{h}'_\gamma(z) := h'_\gamma(z)\,h_\gamma(0)^{-1}$. Then we have $\{\bar{h}_\gamma\}, \{\bar{h}'_\gamma\} \in Z(\Gamma, {}_\tau M(\mathbf{B}))$.

Assume now that the images of $\{h_\gamma\}$ and $\{h'_\gamma\}$ in $Z^1(\Gamma, {}_\sigma M)$ are equivalent. Then the images of $\{\bar{h}_\gamma\}$ and $\{\bar{h}'_\gamma\}$ in $Z^1(\Gamma, {}_\tau M)$ are equivalent, where, by construction, $\{\bar{h}_\gamma\}$ defines the trivial element in $H^1(\Gamma, {}_\tau M)$. Since $H^1(\Gamma, {}_\tau M(\mathbf{B})_1)$ is trivial the cocycles $\{\bar{h}_\gamma\}$ and $\{\bar{h}'_\gamma\}$ are equivalent in $Z^1(\Gamma, {}_\tau M(\mathbf{B}))$. By 5.2 this implies that $\{h_\gamma\} = \{h'_\gamma\}$ in $H^1(\Gamma, {}_\sigma M(\mathbf{B}))$. ■

(6.3) *Proposition. — Let* M *be a* Γ*-group. Then*

(1) $H^1(\Gamma, {}_\sigma M(\mathbf{B})_1)$ *is trivial.*

(2) *The canonical map* $H^1(\Gamma, {}_\sigma M(\mathbf{B})) \to H^1(\Gamma, {}_\sigma M)$ *is bijective.*

Proof. — (*a*) We first consider the case where M is connected and solvable. Let $T \subset M$ be a maximal torus. Then the evaluation map $T(\mathbf{B}) \to T$ is an isomorphism. Hence $M(\mathbf{B})_1 = U(\mathbf{B})_1$ where U is the unipotent radical of M. If U is commutative then $U(\mathbf{B})$ and $U(\mathbf{B})_1$ both have the structure of a vector space over $\mathbf{C}$ and so

$\mathrm{H}^1(\Gamma, {}_\sigma\mathrm{U}(\mathbf{B})_1)=0$. The long exact sequence for $\mathrm{H}^\bullet(\Gamma, {}_\sigma-)$ then gives (1), and (2) follows by 6.2.

(*b*) Using 6.2 we can assume that M is connected (since $\mathrm{M}(\mathbf{B})_1=\mathrm{M}^0(\mathbf{B})_1$). Let $\mathrm{B}\subset\mathrm{M}$ be a Γ-stable Borel subgroup ([St2, Theorem 7.5]). We have the following commutative diagram

$$\begin{array}{ccc} \mathrm{H}^1(\Gamma, {}_\sigma\mathrm{M}(\mathbf{B})) & \xleftarrow{s_\mathrm{M}} & \mathrm{H}^1(\Gamma, {}_\sigma\mathrm{M}) \\ {\scriptstyle j}\uparrow & & \uparrow \\ \mathrm{H}^1(\Gamma, {}_\sigma\mathrm{B}(\mathbf{B})) & \xleftarrow{s_\mathrm{B}} & \mathrm{H}^1(\Gamma, {}_\sigma\mathrm{B}) \end{array}$$

where s_B is bijective by (*a*) and s_M is injective (since it has a left inverse). It suffices to show that j is surjective, since this clearly implies that s_M is surjective also.

Let $\{h_\gamma\}\in\mathrm{Z}^1(\Gamma, {}_\sigma\mathrm{M}(\mathbf{B}))$. We know that $\{h_\gamma\}$ is a coboundary in $\mathrm{Z}^1(\Gamma, {}_\sigma\mathrm{M}(\dot{\mathbf{B}}))$ since $\mathrm{H}^1(\Gamma, {}_\sigma\mathrm{M}(\dot{\mathbf{B}}))$ is trivial (5.6). It follows that there is a morphism $f:\dot{\mathbf{B}}\to\mathrm{M}$ such that $h_\gamma(z)=f(z)^{-1}\cdot{}^\gamma f(z)$ for $z\in\mathbf{B}$. As in the proof of 5.6, Proposition 3.10 implies that $h_\gamma(z)$ is equivalent to a cocycle with values in B. Hence j is surjective. ■

(6.4) *Remark.* – Recall that $\hat{\mathbf{B}}=\mathrm{Spec}\,\mathbf{C}[[s]]$ and $\dot{\hat{\mathbf{B}}}=\mathrm{Spec}\,\mathbf{C}((s))$. Using the techniques above we can show the following:

(1) *Let* M *be a linear algebraic group and* B *a Borel subgroup of* M. *Then* $\mathrm{M}(\dot{\hat{\mathbf{B}}})=\mathrm{M}(\hat{\mathbf{B}})\cdot\mathrm{B}(\dot{\hat{\mathbf{B}}})$.

(2) *For every* $\sigma:\Gamma\to\mathrm{Aut}\,\mathrm{M}$, *the cohomology* $\mathrm{H}^1(\Gamma, {}_\sigma\mathrm{M}(\hat{\mathbf{B}})_1)$ *is trivial, and the map* $\mathrm{H}^1(\Gamma, {}_\sigma\mathrm{M}(\hat{\mathbf{B}}))\to\mathrm{H}^1(\Gamma, {}_\sigma\mathrm{M})$ *is bijective.*

Part (1) is a variant of 3.10. For (2) one needs the following two facts: (*a*) $\mathbf{C}((s))$ is a (C_1)-field, hence $\mathrm{H}^1(\Gamma, {}_\sigma\mathrm{M}^0(\dot{\hat{\mathbf{B}}}))$ is trivial and $\mathrm{H}^1(\Gamma, {}_\sigma\mathrm{B}(\hat{\mathbf{B}}))\to\mathrm{H}^1(\Gamma, {}_\sigma\mathrm{M}^0(\hat{\mathbf{B}}))$ is surjective. (*b*) For a torus $\mathrm{T}=(\mathbf{C}^*)^r$ the group $\mathrm{T}(\hat{\mathbf{B}})_1\simeq(1+s\,\mathbf{C}[[s]])^r$ is divisible and torsion free, and so $\mathrm{H}^1(\Gamma, {}_\sigma\mathrm{T}(\hat{\mathbf{B}})_1)$ is trivial.

Chapter V. DECOMPOSITION AND APPROXIMATION

0. Résumé

(0.1) We begin with some general remarks on the contents of Chapters V and VI: Let V be a G-module such that $V/\!/G \simeq \mathbf{A}$ and $V^G = (0)$. Let $\mathfrak{A} := \mathfrak{Aut}_V^G$ be the automorphism group scheme of V (III.2.2-4), and let $\mathcal{M}_{V,\mathbf{A}}$ denote the set of isomorphism classes of smooth acyclic affine G-varieties X "modelled" on V (see I.2.2). The isomorphism class containing X is denoted $\{X\}$. Our main goal is to compute $\mathcal{M}_{V,\mathbf{A}}$.

(0.2) Let $\{X\} \in \mathcal{M}_{V,\mathbf{A}}$. Then there are G-isomorphisms $\dot\varphi : \dot X \to \dot V$ and $\hat\varphi : \hat X \to \hat V$ over $\dot{\mathbf{A}}$ and $\hat{\mathbf{A}}$, respectively (IV.0.2 and II.0.4). As we saw in I.3.2, the double coset of $\hat{\dot\varphi} = \dot\varphi\hat\varphi^{-1}$ in $D\mathfrak{A} := \mathfrak{A}(\dot{\mathbf{A}})\backslash\mathfrak{A}(\hat{\dot{\mathbf{A}}})/\mathfrak{A}(\hat{\mathbf{A}})$ is well-defined, and we denote it by $[\hat{\dot\varphi}(X)]$. Thus we obtain a map

$$[\hat{\dot\varphi}] : \mathcal{M}_{V,\mathbf{A}} \to D\mathfrak{A}, \qquad \{X\} \mapsto [\hat{\dot\varphi}(X)].$$

(0.3) In section 1 we show that $[\hat{\dot\varphi}]$ is injective. Define $\tilde{\mathbf{A}} := \operatorname{Spec} \mathbf{C}[t]_0$ (localization at 0), $\tilde{\dot{\mathbf{A}}} := \operatorname{Spec} \mathbf{C}(t)$ and $\tilde{D}\mathfrak{A} := \mathfrak{A}(\dot{\mathbf{A}})\backslash\mathfrak{A}(\tilde{\dot{\mathbf{A}}})/\mathfrak{A}(\tilde{\mathbf{A}})$. Since $\mathfrak{A}(\tilde{\dot{\mathbf{A}}}) \cap \mathfrak{A}(\hat{\mathbf{A}}) = \mathfrak{A}(\tilde{\mathbf{A}})$, there is a natural inclusion $\tilde{D}\mathfrak{A} \subseteq D\mathfrak{A}$. Moreover, it is easy to establish that every class in $\tilde{D}\mathfrak{A}$ arises from an element of $\mathcal{M}_{V,\mathbf{A}}$ (see Theorem VI.2.12 and its proof). In VI.2.11 we show that $\tilde{D}\mathfrak{A} = D\mathfrak{A}$, hence $[\hat{\dot\varphi}]$ is a bijection:

$$\mathcal{M}_{V,\mathbf{A}} \xrightarrow{\sim} \tilde{D}\mathfrak{A} = D\mathfrak{A}.$$

(0.4) Since $[\hat{\dot\varphi}]$ is injective, $\mathcal{M}_{V,\mathbf{A}}$ is trivial if every element $\hat{\dot\varphi} \in \mathfrak{A}(\hat{\dot{\mathbf{A}}})$ can be written as a product $\hat{\dot\varphi} = \dot\varphi\hat\varphi$ where $\dot\varphi \in \mathfrak{A}(\dot{\mathbf{A}})$, $\hat\varphi \in \mathfrak{A}(\hat{\mathbf{A}})$. We formalize this property in the following:

Definition. — Let $\mathfrak{C}$ be a group scheme over $\mathbf{A}$. We say that $\mathfrak{C}$ has the *decomposition property* if the double coset space $D\mathfrak{C} := \mathfrak{C}(\dot{\mathbf{A}})\backslash\mathfrak{C}(\hat{\dot{\mathbf{A}}})/\mathfrak{C}(\hat{\mathbf{A}})$ is trivial, or equivalently, if $\mathfrak{C}(\hat{\dot{\mathbf{A}}}) = \mathfrak{C}(\dot{\mathbf{A}}) \cdot \mathfrak{C}(\hat{\mathbf{A}})$.

(0.5) Let Γ, $\mathbf{B}$, $\dot{\mathbf{B}}$, etc. be as in III.4.1, and consider the group scheme $\mathfrak{L}_{\mathbf{B}}^{\Gamma}$ constructed from the Γ-group $L = \operatorname{Aut}(F)^G$ (III.4.4). In section 2 we show that $\mathfrak{L}_{\mathbf{B}}^{\Gamma}$ always has the decomposition property. In fact, in 2.6 we show that for any Γ-group M we have:

$$(*) \qquad M(\hat{\dot{\mathbf{B}}})^{\Gamma} = M(\dot{\mathbf{B}})^{\Gamma} \cdot M(\hat{\mathbf{B}})^{\Gamma}.$$

We call $(*)$ the *decomposition property for the Γ-group* M. The decomposition property for $\mathfrak{L}_{\mathbf{B}}^{\Gamma}$ is, of course, just the case $M = L$. We already have an isomorphism σ_* of $\mathfrak{A}(\hat{\dot{\mathbf{A}}})$ with $L(\hat{\dot{\mathbf{B}}})^{\Gamma}$ (III.4.5) which induces an isomorphism $\mathfrak{A}(\dot{\mathbf{A}}) \simeq L(\dot{\mathbf{B}})^{\Gamma}$ and an inclusion $\sigma_*: \mathfrak{A}(\hat{\mathbf{A}}) \hookrightarrow \mathrm{L}(\hat{\mathbf{B}})^\Gamma$ (III.4.6). We need to know about the cokernel of the inclusion.

(0.6) *Notation.* – Let M be a Γ-group, and by abuse of notation, let I denote the constant map to the identity I of M. Let $r \geqslant 0$ and define $\mathrm{M}(\hat{\mathbf{B}})_r := \{ \mu \in \mathrm{M}(\hat{\mathbf{B}}) \mid \mu = \mathrm{I} + \mathrm{O}(s^r) \}$. Equivalently,

$$\mathrm{M}(\hat{\mathbf{B}})_r = \mathrm{Ker}\{ \mathrm{M}(\hat{\mathbf{B}}) = \mathrm{M}(\mathbf{C}[\![s]\!]) \to \mathrm{M}(\mathbf{C}[\![s]\!]/\mathbf{n}^r) \},$$

where $\mathbf{n}$ denotes the maximal ideal of $\mathbf{C}[\![s]\!]$. Note that $\mathrm{M}(\hat{\mathbf{B}})_0 = \mathrm{M}(\hat{\mathbf{B}})$. We set $\mathrm{M}(\hat{\mathbf{B}})_r^\Gamma = \mathrm{M}(\hat{\mathbf{B}})^\Gamma \cap \mathrm{M}(\hat{\mathbf{B}})_r$, and similarly for $\mathrm{M}(\mathbf{B})_r$, $\mathrm{M}(\mathbf{B})_r^\Gamma$, $\mathrm{M}(\tilde{\mathbf{B}})_r$, etc.

(0.7) Clearly $\mathrm{M}^\Gamma \subseteq \mathrm{M}(\mathbf{B})^\Gamma \subseteq \mathrm{M}(\dot{\mathbf{B}})^\Gamma$, and $\mathrm{M}(\hat{\mathbf{B}})^\Gamma = \mathrm{M}^\Gamma \cdot \mathrm{M}(\hat{\mathbf{B}})_1^\Gamma$. Thus M has the decomposition property if and only if

$$(**) \qquad \mathrm{M}(\hat{\mathbf{B}})^\Gamma = \mathrm{M}(\dot{\mathbf{B}})^\Gamma \cdot \mathrm{M}(\hat{\mathbf{B}})_1^\Gamma.$$

(0.8) Let $\sigma_*: \mathfrak{A}(\hat{\mathbf{A}}) \hookrightarrow \mathrm{L}(\hat{\mathbf{B}})^\Gamma$ be as above. In Chapter VI we show that there is an $r_0 \geqslant 1$ such that $\mathrm{L}(\hat{\mathbf{B}})_{r_0}^\Gamma \subseteq \sigma_*(\mathfrak{A}(\hat{\mathbf{A}}))$. If $r_0 = 1$, then the decomposition property for L and 0.7 give that

$$\mathrm{L}(\hat{\mathbf{B}})^\Gamma = \mathrm{L}(\dot{\mathbf{B}})^\Gamma \cdot \sigma_*(\mathfrak{A}(\hat{\mathbf{A}})).$$

Applying $(\sigma_*)^{-1}$ we see that $\mathfrak{A}$ has the decomposition property, i.e., $\mathrm{D}\mathfrak{A}$ is trivial.

Suppose that $r_0 > 1$. Our argument shows that one still has $\mathrm{D}\mathfrak{A} = \{ * \}$ if one can replace $\mathrm{L}(\hat{\mathbf{B}})_1^\Gamma$ by $\mathrm{L}(\hat{\mathbf{B}})_{r_0}^\Gamma$ in $(**)$ (with L in place of M).

(0.9) *Definition.* – Let M be a Γ-group. We say that M has the *approximation property* if

$$\mathrm{M}(\hat{\mathbf{B}})_1^\Gamma = \mathrm{M}(\mathbf{B})_1^\Gamma \cdot \mathrm{M}(\hat{\mathbf{B}})_r^\Gamma \qquad \text{for all} \quad r \geqslant 1.$$

Since $\mathrm{M}(\mathbf{B})_1^\Gamma \subseteq \mathrm{M}(\dot{\mathbf{B}})^\Gamma$, the approximation property for M implies that we can replace $\mathrm{M}(\hat{\mathbf{B}})_1^\Gamma$ with $\mathrm{M}(\hat{\mathbf{B}})_r^\Gamma$ in $(**)$ for any $r \geqslant 1$. In particular, *the approximation property for* L *implies the decomposition property for* $\mathfrak{A}$.

In section 2 of this chapter we prove the approximation property for unipotent Γ-groups, and in section 3 we establish the approximation property for semisimple Γ-groups. Thus, the possible failure of the approximation property and decomposition property is due to the toral part of the radical of L.

1. Injectivity of $[\hat{\dot{\varphi}}]$

(1.1) We consider $\mathcal{O}(\mathbf{A})$, $\mathcal{O}(\dot{\mathbf{A}})$ and $\mathcal{O}(\hat{\mathbf{A}})$ as subalgebras of $\mathcal{O}(\hat{\dot{\mathbf{A}}})$ in the canonical way, and this gives rise to inclusions $\mathcal{O}(\dot{\mathrm{X}}) \subseteq \mathcal{O}(\hat{\dot{\mathrm{X}}})$, etc. Since $\mathcal{O}(\mathbf{A}) = \mathcal{O}(\dot{\mathbf{A}}) \cap \mathcal{O}(\hat{\mathbf{A}})$, we have an exact sequence of $\mathcal{O}(\mathbf{A})$-modules

$$(***) \qquad 0 \to \mathcal{O}(\mathbf{A}) \to \mathcal{O}(\dot{\mathbf{A}}) \oplus \mathcal{O}(\hat{\mathbf{A}}) \to \mathcal{O}(\hat{\dot{\mathbf{A}}}).$$

(1.2) *Lemma.* — *We have* $\mathcal{O}(\mathrm{X}) = \mathcal{O}(\dot{\mathrm{X}}) \cap \mathcal{O}(\hat{\mathrm{X}})$.

Proof. — Since $\pi : \mathrm{X} \to \mathbf{A}$ is flat, we obtain the desired result by tensoring $(***)$ with $\mathcal{O}(\mathrm{X})$ over $\mathcal{O}(\mathbf{A})$. ■

(1.3) *Theorem.* — *The map* $[\hat{\dot{\varphi}}] : \mathcal{M}_{\mathrm{V},\mathbf{A}} \to \mathrm{D}\mathfrak{A} = \mathfrak{A}(\dot{\mathbf{A}}) \backslash \mathfrak{A}(\hat{\dot{\mathbf{A}}}) / \mathfrak{A}(\hat{\mathbf{A}})$ *is injective.*

Proof. — Let $\hat{\dot{\varphi}} = \dot{\varphi}\hat{\varphi}^{-1} \in \mathfrak{A}(\hat{\dot{\mathbf{A}}})$ correspond to X as in 0.2, and let $\hat{\dot{\varphi}}' = \dot{\varphi}'(\hat{\varphi}')^{-1}$ similarly correspond to X′. Suppose that $\hat{\dot{\varphi}} = \dot{\alpha}\hat{\dot{\varphi}}'\hat{\alpha}$, where $\dot{\alpha} \in \mathfrak{A}(\dot{\mathbf{A}})$ and $\hat{\alpha} \in \mathfrak{A}(\hat{\mathbf{A}})$. Let $\dot{\psi} := (\dot{\varphi}')^{-1}\dot{\alpha}^{-1}\dot{\varphi} : \dot{\mathrm{X}} \to \dot{\mathrm{X}}'$ and $\hat{\psi} := (\hat{\varphi}')^{-1}\hat{\alpha}\hat{\varphi} : \hat{\mathrm{X}} \to \hat{\mathrm{X}}'$. Then $\dot{\psi}$ and $\hat{\psi}$ restrict to the same isomorphism of $\hat{\dot{\mathrm{X}}}$ and $\hat{\dot{\mathrm{X}}}'$. It follows from 1.2 (applied to both X and X′) that $\dot{\psi}$ and $\hat{\psi}$ induce a G-isomorphism of X and X′. ■

(1.4) *Example.* — Let V, G, α, $\sigma(\alpha)$, etc. be as in Example III.2.7. We show that the automorphism group scheme $\mathfrak{A}$ has the decomposition property: Let $\hat{\dot{\alpha}} \in \mathfrak{A}(\hat{\dot{\mathbf{A}}})$ with

$$\sigma(\hat{\dot{\alpha}}) = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in \mathrm{GL}_2(\mathbf{C}((t))).$$

Then III.2.7(4) implies that $a^2 - b^2 t = \pm 1$ since $\mathcal{O}(\hat{\dot{\mathbf{A}}})$ is a field. If a contains negative powers of t, then so must $a^2 - b^2 t$, and similarly for b. Hence $a, b \in \mathcal{O}(\hat{\mathbf{A}})$, and $c, d \in \mathcal{O}(\hat{\mathbf{A}})$ by III.2.7(3). Thus $\mathfrak{A}(\hat{\dot{\mathbf{A}}}) = \mathfrak{A}(\hat{\mathbf{A}})$, and the decomposition property holds. Similar reasoning shows that $\mathfrak{A}(\dot{\mathbf{A}}) = \Gamma =$ the group of 4th roots of unity.

(1.5) *Examples.* — Consider the group schemes $\mathfrak{G}_n$ of III.1.2(c) and III.1.4(c). Every element $\hat{\dot{\alpha}}$ of $\mathbf{C}((t))^* = \tau(\mathfrak{G}_n(\hat{\dot{\mathbf{A}}}))$ has a unique decomposition $\hat{\dot{\alpha}} = \dot{\alpha}(1 + t\hat{\beta})$ where $\dot{\alpha} \in \mathbf{C}[t, t^{-1}]^* = \tau(\mathfrak{G}_n(\dot{\mathbf{A}}))$ and $\hat{\beta} \in \mathcal{O}(\hat{\mathbf{A}})$. Since $\tau(\mathfrak{G}_n(\hat{\mathbf{A}})) = 1 + t^n \mathbf{C}[[t]]$, we see that

$$\mathrm{D}\mathfrak{G}_n \simeq (1 + t\mathbf{C}[[t]]) / (1 + t^n \mathbf{C}[[t]]) \simeq \mathbf{C}^{n-1}.$$

Thus $\mathfrak{G}_n$ has the decomposition property if and only if $n = 1$.

We now present some comments on the structure of $\mathrm{D}\mathfrak{A}$.

(1.6) *Notation.* – Let $\mathfrak{C}$ be a group scheme over $\mathbf{A}$ which is isotrivial over $\dot{\mathbf{A}}$ (IV.1.1). We denote by $\mathfrak{C}^0$ the closure in $\mathfrak{C}$ of the connected component of the identity of $\mathfrak{C}|_{\dot{\mathbf{A}}}$. The identity element of the fiber of $\mathfrak{C}$ at 0 lies in $\mathfrak{C}^0$, and since $\mathfrak{C}^0 \to \mathbf{A}$ must be flat, $\mathfrak{C}^0$ is a group scheme all of whose fibers have the same dimension.

(1.7) *Remark.* – In Example III.2.7, the fibers of $\mathfrak{A}$ do not all have the same dimension. The fibers over $\dot{\mathbf{A}}$ are all isomorphic to the original one-dimensional group G, while the fiber at 0 is a semidirect product $\mathbf{C}^* \ltimes \mathbf{C}$.

(1.8) *Lemma.* – *Let $\mathfrak{C}$ be isotrivial over $\dot{\mathbf{A}}$. Consider the exact sequence of group schemes over $\dot{\mathbf{A}}$*

$$1 \to \mathfrak{C}^0|_{\dot{\mathbf{A}}} \xrightarrow{i} \mathfrak{C}|_{\dot{\mathbf{A}}} \to \mathfrak{D} \to 1,$$

where $\mathfrak{D}$ is the cokernel of i. Then $\mathfrak{C}(\dot{\mathbf{A}})$ maps onto $\mathfrak{D}(\dot{\mathbf{A}})$, and $\mathfrak{D}(\dot{\mathbf{A}}) = \mathfrak{D}(\hat{\dot{\mathbf{A}}})$.

Proof. – Since $\mathfrak{C}$ is isotrivial, we have $\mathfrak{C} \simeq \dot{\mathbf{B}} *^{\Gamma} \mathrm{M}$, $\mathfrak{C}^0 \simeq \dot{\mathbf{B}} *^{\Gamma} \mathrm{M}^0$, and $\mathfrak{D} \simeq \dot{\mathbf{B}} *^{\Gamma} \mathrm{M}/\mathrm{M}^0$, where M is an algebraic group and $\Gamma = \mu_d$ acts as usual on $\dot{\mathbf{B}}$ and on M via a homomorphism $\tau: \Gamma \to \operatorname{Aut} \mathrm{M}$. Then $\mathfrak{C}(\dot{\mathbf{A}}) \simeq {}_{\tau}\mathrm{M}(\dot{\mathbf{B}})^{\Gamma}$, where Γ acts by $h \mapsto {}^{\gamma}h := \tau(\gamma) \circ h \circ \gamma$ for $\gamma \in \Gamma$, $h \in {}_{\tau}\mathrm{M}(\dot{\mathbf{B}})$, and similarly for $\mathfrak{C}^0(\dot{\mathbf{A}})$ and $\mathfrak{D}(\dot{\mathbf{A}})$ (see III.1.2(b) and 1.4(b); cf. IV.6.1). By IV.3.1(2), $\mathrm{H}^1_{\text{et}}(\dot{\mathbf{B}}, \mathrm{M}^0) = \{*\}$, so we get an exact sequence of Γ-groups

$$1 \to {}_{\tau}\mathrm{M}^0(\dot{\mathbf{B}}) \to {}_{\tau}\mathrm{M}(\dot{\mathbf{B}}) \to {}_{\tau}(\mathrm{M}/\mathrm{M}^0)(\dot{\mathbf{B}}) \to 1.$$

Since $\mathrm{H}^1(\Gamma, {}_{\tau}\mathrm{M}^0(\dot{\mathbf{B}})) = \{*\}$ (see IV.5.6 and 5.8), the sequence remains exact when we take the fixed points of Γ. Thus $\mathfrak{C}(\dot{\mathbf{A}})$ maps onto $\mathfrak{D}(\dot{\mathbf{A}})$. Clearly $\mathfrak{D}(\dot{\mathbf{A}}) = \mathfrak{D}(\hat{\dot{\mathbf{A}}})$. ■

(1.9) *Theorem.* – (1) *Suppose that $\mathfrak{C}$ is isotrivial over $\dot{\mathbf{A}}$. Then the inclusion $\mathfrak{C}^0 \subseteq \mathfrak{C}$ induces a surjection $\mathrm{D}\mathfrak{C}^0 \twoheadrightarrow \mathrm{D}\mathfrak{C}$.*

(2) *The action of Γ on V gives a homomorphism $\Gamma \to \mathfrak{A}(\mathbf{A})$ such that $\Gamma\mathfrak{A}^0(\dot{\mathbf{A}}) = \mathfrak{A}(\dot{\mathbf{A}})$. Hence $\Gamma\mathfrak{A}^0(\dot{\mathbf{A}}) = \mathfrak{A}(\dot{\mathbf{A}})$ and $\Gamma\mathfrak{A}^0(\hat{\dot{\mathbf{A}}}) = \mathfrak{A}(\hat{\dot{\mathbf{A}}})$. Consequently,*

$$\mathrm{D}\mathfrak{A} = \mathrm{D}\mathfrak{A}^0/\Gamma,$$

where Γ acts on $\mathrm{D}\mathfrak{A}$ via conjugation on $\mathfrak{A}(\hat{\dot{\mathbf{A}}})$. In particular, $\mathfrak{A}$ has the decomposition property if $\mathfrak{A}^0$ does.

Proof. – Let $\mathfrak{C}$ be as in (1). Then $\mathfrak{C}(\dot{\mathbf{A}})\mathfrak{C}^0(\hat{\dot{\mathbf{A}}}) = \mathfrak{C}(\hat{\dot{\mathbf{A}}})$ by 1.8, so $\mathrm{D}\mathfrak{C}^0 \to \mathrm{D}\mathfrak{C}$ is surjective.

When $\mathfrak{C} = \mathfrak{A}$, the group scheme $\mathfrak{D}$ of 1.8 becomes $\dot{\mathbf{B}} *^{\Gamma} \mathrm{L}/\mathrm{L}^0$ since $\mathfrak{A}|_{\dot{\mathbf{A}}} \simeq \dot{\mathbf{B}} *^{\Gamma} \mathrm{L}$. Let Γ' denote $\operatorname{Ker}(\Gamma \to \mathrm{L}/\mathrm{L}^0)$. By III.3.3, $\dot{\mathbf{B}} *^{\Gamma} \mathrm{L}/\mathrm{L}^0 \simeq \dot{\mathbf{B}} *^{\Gamma} \Gamma/\Gamma' \simeq \dot{\mathbf{A}} \times \Gamma/\Gamma'$ since Γ is abelian. Clearly the composition $\Gamma \to \mathfrak{A}(\mathbf{A}) \to \mathfrak{D}(\hat{\mathbf{A}}) \simeq \mathfrak{D}(\dot{\mathbf{A}}) \simeq \Gamma/\Gamma'$ is surjective. Hence $\Gamma \cdot \mathfrak{A}^0(\hat{\mathbf{A}}) = \mathfrak{A}(\hat{\mathbf{A}})$, and (2) follows. ■

(1.10) *Corollary. — Suppose that* L *is finite. Then* $\mathfrak{A}^0 \simeq \mathbf{A} \times \{e\}$, *and* $\mathfrak{A}$ *has the decomposition property.*

2. The decomposition property

We will show that any Γ-group M has the decomposition property, and that unipotent Γ-groups have the approximation property.

(2.1) *Remarks.* — (1) $M^0(\hat{\mathbf{B}})_1 \simeq M(\hat{\mathbf{B}})_1$ and $M^0(\hat{\mathbf{B}})_r^\Gamma = M(\hat{\mathbf{B}})_r^\Gamma$, $r \geqslant 1$.

(2) M^0 is a Γ-group, and it is obvious that the approximation property for M is equivalent to that for M^0.

(3) The decomposition property is obvious for finite Γ-groups, since sections over $\dot{\mathbf{A}}$ and $\hat{\mathbf{A}}$ coincide (cf. 1.8).

The following lemma shows that the decomposition property for M^0 implies the decomposition property for M.

(2.2) *Lemma. — Let*

$$1 \to P \xrightarrow{i} Q \xrightarrow{j} R \to 1$$

be an exact sequence of Γ*-groups. Assume that there is a* Γ*-equivariant section* $\sigma : R \to Q$ *of* j, *considered only as a morphism of varieties, or that* $P = Q^0$. *If* P *and* R *have the decomposition property* (*resp. approximation property*), *then so does* Q.

Proof. — Assume that the section σ exists. Let $\hat{\alpha} \in Q(\hat{\mathbf{B}})^\Gamma$. Then $\hat{\alpha}$ gives rise to $j(\hat{\alpha}) \in R(\hat{\mathbf{B}})^\Gamma$, and we may write $j(\hat{\alpha}) = \dot{\beta}\hat{\beta}$ where $\dot{\beta} \in R(\dot{\mathbf{B}})^\Gamma$ and $\hat{\beta} \in R(\hat{\mathbf{B}})^\Gamma$. Replacing $\hat{\alpha}$ by $\sigma(\dot{\beta})^{-1}\hat{\alpha}\sigma(\hat{\beta})^{-1}$ we may reduce to the case that $j(\hat{\alpha}) = \mathrm{I}$. But then $\hat{\alpha} \in \mathrm{Im}\, i$, and P has the decomposition property, hence so does Q. The proof in the case of the approximation property is analogous. If $P = Q^0$, then $j(\hat{\alpha}) = j(\dot{\alpha})$ for some $\dot{\alpha}$ by Lemma 1.8, and we can continue as above. ■

(2.3) *Proposition. — Let* M *be a unipotent* Γ*-group. Then* M *has the approximation property and decomposition property.*

Proof. — Identify $\Gamma \ltimes M$ with an algebraic subgroup of some GL_p such that M is a subgroup of the upper triangular matrices. Let $\mathfrak{m}$ denote the Lie algebra of M, a subalgebra of $\mathfrak{gl}_p$. The exponential map gives an isomorphism $\exp : \mathfrak{m} \simeq M$ of varieties with Γ-action, where Γ acts linearly on $\mathfrak{m}$. The exact sequence

$$0 \to [\mathfrak{m}, \mathfrak{m}] \to \mathfrak{m} \to \mathfrak{m}/[\mathfrak{m}, \mathfrak{m}] \to 0$$

has a Γ-equivariant splitting which, via the exponential map, gives rise to a Γ-splitting for the exact sequence

$$1 \to [M, M] \to M \to M/[M, M] \to 1.$$

Using Lemma 2.2 and induction on dim M, we can reduce to the case where M is abelian. Then $M \simeq (\mathbf{C}^q, +)$ for some q, and Γ acts linearly on $\mathbf{C}^q$. Now

$$\mathbf{C}((s)) = \mathbf{C}[s, s^{-1}] + s\,\mathbf{C}[\![s]\!],$$

hence there is a surjection

$$(\mathbf{C}[s, s^{-1}] \otimes_{\mathbf{C}} \mathbf{C}^q) + (s\,\mathbf{C}[\![s]\!] \otimes_{\mathbf{C}} \mathbf{C}^q) \to \mathbf{C}((s)) \otimes_{\mathbf{C}} \mathbf{C}^q,$$

where $\mathbf{C}[s, s^{-1}] \otimes_{\mathbf{C}} \mathbf{C}^q \simeq M(\dot{\mathbf{B}})$, etc. Since $M(\dot{\mathbf{B}})$, etc. are direct sums of finite dimensional Γ-representations, $M(\dot{\mathbf{B}}) + M(\hat{\mathbf{B}})_1 = M(\hat{\dot{\mathbf{B}}})$ implies that $M(\dot{\mathbf{B}})^\Gamma + M(\hat{\mathbf{B}})_1^\Gamma = M(\hat{\dot{\mathbf{B}}})^\Gamma$. Hence M has the decomposition property. The approximation property similarly follows from the fact that

$$\mathbf{C}[\![s]\!] = \mathbf{C}[s] + s^r\,\mathbf{C}[\![s]\!]$$

for all $r \geq 1$. ■

(2.4) *Proposition. — Let* M *be a* Γ*-group with* M^0 *a torus. Then* M *has the decomposition property; in fact, the multiplication mapping*

$$M(\dot{\mathbf{B}})^\Gamma \times M(\hat{\mathbf{B}})_1^\Gamma \to M(\hat{\dot{\mathbf{B}}})^\Gamma$$

is a bijection.

Proof. — Since $M(\dot{\mathbf{B}})/M^0(\dot{\mathbf{B}}) = M(\hat{\dot{\mathbf{B}}})/M^0(\hat{\dot{\mathbf{B}}})$, we may reduce to the case that M is connected. Now every element of $\mathbf{C}((s))^*$ can be written uniquely in the form $\dot{\alpha}\hat{\alpha}$ where $\dot{\alpha} \in \mathbf{C}[s, s^{-1}]^*$ and $\hat{\alpha} \in 1 + s\,\mathbf{C}[\![s]\!]$. Thus the multiplication map $M(\dot{\mathbf{B}}) \times M(\hat{\mathbf{B}})_1 \to M(\hat{\dot{\mathbf{B}}})$ is a bijection, and the proposition follows. ■

We now need the following structure result.

(2.5) *Proposition* [St2]. — *Let* M *be a connected* Γ*-group, and let* R *denote the unipotent radical of* M. *Then* $M = M_{rd} \ltimes R$ *where* M_{rd} *is a connected reductive* Γ*-subgroup of* M. *There is a* Γ*-invariant Borel subgroup* $B = T \ltimes U$ *of* M_{rd}, *where* U *is the unipotent radical of* B *and* T *is a* Γ*-stable maximal torus.*

(2.6) *Theorem.* – *Any* Γ-*group* M *has the decomposition property, i.e.,*

$$M(\hat{\dot{\mathbf{B}}})^{\Gamma} = M(\dot{\mathbf{B}})^{\Gamma} \cdot M(\hat{\mathbf{B}})_1^{\Gamma}.$$

Proof. – By 2.2, we may assume that M is connected. First we assume that $\Gamma = \{1\}$. Let $B_0 = B \ltimes R$ (see 2.5). Then B_0 is a Borel subgroup of M, and $M(\hat{\mathbf{B}}) \cdot B_0(\hat{\dot{\mathbf{B}}}) = M(\hat{\dot{\mathbf{B}}})$ by IV.6.4(1). By 2.2, 2.3 and 2.4, B_0 has the decomposition property, hence so does M.

We now consider the case where Γ is non-trivial. Let $\hat{\dot{\alpha}} \in M(\hat{\dot{\mathbf{B}}})^{\Gamma}$. Then $\hat{\dot{\alpha}} = \dot{\alpha}\hat{\alpha}$ where $\dot{\alpha} \in M(\dot{\mathbf{B}})$ and $\hat{\alpha} \in M(\hat{\mathbf{B}})_1$. Note that for any $\gamma \in \Gamma$, $\gamma \mapsto \dot{\alpha}^{-1}({}^{\gamma}\dot{\alpha}) = ({}^{\gamma}\hat{\alpha})^{-1}\hat{\alpha}$ is a 1-cocycle with values in $M(\dot{\mathbf{B}}) \cap M(\hat{\mathbf{B}})_1 = M(\mathbf{B})_1$. By Proposition IV.6.3(1), this cocycle is trivial, hence we can modify $\dot{\alpha}$ and $\hat{\alpha}$ by an element of $M(\mathbf{B})_1$ so that they become Γ-invariant. Thus M has the decomposition property. ■

(2.7) *Example.* – Let $(V, G) = (n\mathbf{C}^n, SL_n)$. Then (V, G) is semifree. Here the group scheme $\mathfrak{A}^0$ (see 1.6) is easy to describe: Consider the representation $(W, M) = (\mathbf{C}^n \otimes \bar{\mathbf{C}}^n, SL_n \times \overline{SL}_n)$ where $(\bar{\mathbf{C}}^n, \overline{SL}_n)$ is another copy of $(\mathbf{C}^n, SL_n)$. Then $W \simeq V$ as a representation of SL_n. The action of $\overline{SL}_n$ commutes with that of SL_n, and we have an inclusion $\mathbf{A} \times \overline{SL}_n \subseteq \mathfrak{A}$, where $\dot{\mathbf{A}} \times \overline{SL}_n \simeq \mathfrak{A}|_{\dot{\mathbf{A}}}$ and $\mathbf{A} \times \overline{SL}_n$ is closed in $\mathfrak{A}$. Thus $\mathfrak{A}^0 \simeq \mathbf{A} \times \overline{SL}_n$. From 2.6, with $\Gamma = \{1\}$, we see that $\mathfrak{A}^0$ has the decomposition property. Hence, by 1.9, so does $\mathfrak{A}$, and $\mathscr{M}_{V, \mathbf{A}} = \{*\}$. In fact, $\mathfrak{A}(\hat{\dot{\mathbf{A}}}) = \mathfrak{A}^0(\hat{\dot{\mathbf{A}}})$ so that $D\mathfrak{A} = D\mathfrak{A}^0 = \{*\}$.

(2.8) *Remark.* – We will need to use the analogues of the decomposition property and approximation property for Γ-groups with the triple $\{\dot{\mathbf{B}}, \hat{\mathbf{B}}, \hat{\dot{\mathbf{B}}}\}$ replaced by $\{\dot{\mathbf{B}}, \tilde{\mathbf{B}}, \tilde{\dot{\mathbf{B}}}\}$, where $\tilde{\mathbf{B}} = \operatorname{Spec} \mathbf{C}[s]_0$ (localization at 0) and $\tilde{\dot{\mathbf{B}}} = \operatorname{Spec} \mathbf{C}(s)$ (see 0.3). For example, *for any* Γ-*group* M *we have* $M(\tilde{\dot{\mathbf{B}}})^{\Gamma} = M(\dot{\mathbf{B}})^{\Gamma} \cdot M(\tilde{\mathbf{B}})_1^{\Gamma}$.

The proofs follow exactly the same lines. We need only observe the following:

(1) $\mathbf{C}(s) = \mathbf{C}[s, s^{-1}] + s\,\mathbf{C}[s]_0$.

(2) $\mathbf{C}[s]_0 = \mathbf{C}[s] + s^r \mathbf{C}[s]_0$ for all $r \geqslant 1$.

(3) $\mathbf{C}(s)^* = \mathbf{C}[s, s^{-1}]\,(1 + s\,\mathbf{C}[s]_0)$, where the decomposition on the right hand side is unique.

3. The approximation property

(3.1) We need some easy results concerning the exponential map: Let M be a linear algebraic group and $\mathfrak{m}$ its Lie algebra. Let $\hat{M} = \operatorname{Spec}(\hat{\mathcal{O}}_{M, e})$, where $\mathcal{O}_{M, e}$ is the local ring of M at the identity e and $\hat{\mathcal{O}}_{M, e}$ its completion, and similarly define $\hat{\mathfrak{m}}$. The exponential map from $\mathfrak{m}$ to M is a local analytic isomorphism, and thus it induces an isomorphism of affine schemes $\exp: \hat{\mathfrak{m}} \xrightarrow{\sim} \hat{M}$.

Define $\mathfrak{m}(\hat{\mathbf{B}})_r$, $\hat{\mathfrak{m}}(\hat{\mathbf{B}})_r$, $\mathrm{M}(\hat{\mathbf{B}})_r$ and $\hat{\mathrm{M}}(\hat{\mathbf{B}})_r$ as in 0.6. It is easy to see that the canonical morphism $\hat{\mathfrak{m}} \to \mathfrak{m}$ induces an isomorphism $\hat{\mathfrak{m}}(\hat{\mathbf{B}})_r \simeq \mathfrak{m}(\hat{\mathbf{B}})_r$ for $r \geqslant 1$, and similarly $\hat{\mathrm{M}}(\hat{\mathbf{B}})_r \simeq \mathrm{M}(\hat{\mathbf{B}})_r$ for $r \geqslant 1$. If $\varphi \in \hat{\mathfrak{m}}(\hat{\mathbf{B}})_r$, then $\exp \circ \varphi : \hat{\mathbf{B}} \to \hat{\mathrm{M}}$ lies in $\hat{\mathrm{M}}(\hat{\mathbf{B}})_r$. It is obvious that we have the following result:

(3.2) *Proposition. — Let* M *be as above,* $r \geqslant 1$.

(1) $\exp : \mathfrak{m}(\hat{\mathbf{B}})_r \to \mathrm{M}(\hat{\mathbf{B}})_r$ *is a bijection.*

(2) *If* M *is a* Γ*-group, then the exponential map is* Γ*-equivariant, and* $\exp : \mathfrak{m}(\hat{\mathbf{B}})_r^\Gamma \xrightarrow{\sim} \mathrm{M}(\hat{\mathbf{B}})_r^\Gamma$.

(3.3) Let M be a Γ-group. We show that M has the approximation property if $\mathrm{Rad}(\mathrm{M}^0) = \mathrm{Rad}_u(\mathrm{M}^0)$, e.g., if M is semisimple.

Recall that L^0 is special (IV.2.6), so we really only need to consider the approximation property for special groups. However, the general case is no harder.

(3.4) *Lemma. — Let* $r \geqslant 1$ *and let* K *be a finite central* Γ*-subgroup of the* Γ*-group* M.

(1) *The canonical map* $\mathrm{M} \to \mathrm{M}/\mathrm{K}$ *induces isomorphisms* $\mathrm{M}(\mathbf{B})_r^\Gamma \simeq (\mathrm{M}/\mathrm{K})(\mathbf{B})_r^\Gamma$, $\mathrm{M}(\tilde{\mathbf{B}})_r^\Gamma \simeq (\mathrm{M}/\mathrm{K})(\tilde{\mathbf{B}})_r^\Gamma$ *and* $\mathrm{M}(\hat{\mathbf{B}})_r^\Gamma \simeq (\mathrm{M}/\mathrm{K})(\hat{\mathbf{B}})_r^\Gamma$.

(2) M *has the approximation property if and only if* M/K *does.*

Proof. — Applying $\mathrm{H}^{\bullet}_{\mathrm{et}}(\mathbf{B}, -)$ to the exact sequence

$$1 \to \mathrm{K} \to \mathrm{M} \to \mathrm{M}/\mathrm{K} \to 1$$

we obtain the exact sequence

$$1 \to \mathrm{K} \simeq \mathrm{K}(\mathbf{B}) \to \mathrm{M}(\mathbf{B}) \to (\mathrm{M}/\mathrm{K})(\mathbf{B}) \to 1$$

since $\mathrm{H}^1_{\mathrm{et}}(\mathbf{B}, \mathrm{K})$ is trivial. It follows that $\mathrm{M}(\mathbf{B})_r \simeq (\mathrm{M}/\mathrm{K})(\mathbf{B})_r$, and similarly for $\mathbf{B}$ replaced by $\tilde{\mathbf{B}}$ and $\hat{\mathbf{B}}$. (The $\hat{\mathbf{B}}$ case also follows from 3.2). We have proved (1), and (2) is immediate from (1). ■

(3.5) *Theorem. — Let* M *be a* Γ*-group such that* $\mathrm{Rad}(\mathrm{M}^0) = \mathrm{Rad}_u(\mathrm{M}^0)$. *Then* M *has the approximation property.*

Proof. — By 2.1(2), 2.2, 2.3 and 2.5 we may reduce to the case that M is connected and semisimple. Let $\mathrm{B} = \mathrm{T} \ltimes \mathrm{U}$ be a Borel subgroup of M, where T is a maximal torus and U a maximal unipotent subgroup of B, all being Γ-subgroups of M (see 2.5). It suffices to show:

$(*)$ *Given* $\hat{g}(s) \in \mathrm{M}(\hat{\mathbf{B}})_r^\Gamma$, $r > 0$, *there is a* $g(s) \in \mathrm{M}(\mathbf{B})^\Gamma$ *such that* $g(s)^{-1}\hat{g}(s) \in \mathrm{M}(\hat{\mathbf{B}})_{r+1}^\Gamma$.

Choose an embedding $M \subseteq GL_n$, for some n. Then we may consider $\mathfrak{m}$ as a subalgebra of $\mathfrak{gl}_n$. Let $\hat{g}(s) \in M(\hat{\mathbf{B}})_r^\Gamma$ with $\hat{g}(s) = I + s^r A + O(s^{r+1})$. Then, for $z \in \mathbf{C}$ small, $z \mapsto I + zA$ is a curve in GL_n which is tangent to M at I, hence $A \in \mathfrak{m}$. Clearly $s^r A$ is Γ-invariant. Hence (∗) follows from

(∗∗) *Let* $A \in \mathfrak{m}$ *such that* s^r A *is* Γ-*invariant. Then there is a* $g(s) \in M(\mathbf{B})^\Gamma$ *with* $g(s) = I + s^r A + O(s^{r+1})$.

Note that, by Lemma 3.4, properties (∗) and (∗∗) are really properties of $\mathfrak{m}$ alone and do not depend upon the various possible M's.

Let γ denote a generator of Γ. We consider the action of Γ on $\mathfrak{m}$: Clearly we may reduce to the case where $\mathfrak{m} = \mathfrak{m}_1 \oplus \ldots \oplus \mathfrak{m}_n$, each $\mathfrak{m}_i$ is simple and isomorphic to $\mathfrak{m}_1$, and $\gamma\mathfrak{m}_1 = \mathfrak{m}_2, \ldots, \gamma\mathfrak{m}_{n-1} = \mathfrak{m}_n$, $\gamma\mathfrak{m}_n = \mathfrak{m}_1$. Thus $\tau := \gamma^n$ preserves each $\mathfrak{m}_i$. Choose τ-stable subalgebras $\mathfrak{b}_1$, $\mathfrak{t}_1$ and $\mathfrak{u}_1$ in $\mathfrak{m}_1$ where $\mathfrak{b}_1 = \mathfrak{t}_1 \oplus \mathfrak{u}_1$ is a Borel subalgebra, $\mathfrak{t}_1$ a maximal toral subalgebra, etc. Set $\mathfrak{b}_i = \gamma^{i-1}\mathfrak{b}_1$, etc. Then the $\mathfrak{b}_i$, $\mathfrak{t}_i$ and $\mathfrak{u}_i$ are τ-stable, $\mathfrak{b} = \oplus\,\mathfrak{b}_i$, $\mathfrak{t} = \oplus\,\mathfrak{t}_i$ and $\mathfrak{u} = \oplus\,\mathfrak{u}_i$ are Γ-stable, and $\mathfrak{b}$, etc. is a Borel subalgebra, etc. of $\mathfrak{m}$.

Given A as in (∗∗), we may write $A = A_0 + A_+ + A_-$ where $A_0 \in \mathfrak{t}$, $A_+ \in \mathfrak{u}$ and $A_- \in \mathfrak{u}_-$ (the opposite nilpotent subalgebra). Then $s^r A_+$ and $s^r A_-$ are Γ-invariant, so $g(s) := \exp(s^r A_+)\exp(s^r A_-) \in M(\mathbf{B})^\Gamma$, and $g(s) = I + s^r(A_+ + A_-) + O(s^{r+1})$. Hence we may reduce to the case where $A \in \mathfrak{t}$.

Let $\alpha_i^1, \ldots, \alpha_i^l$ be the simple roots of $\mathfrak{m}_i$, and let x_i^j, y_i^j, h_i^j be corresponding $\mathfrak{sl}_2$-triples, $j = 1, \ldots, l$. Then we may arrange that $\gamma x_i^j = x_{i+1}^j$, $\gamma y_i^j = y_{i+1}^j$, $\gamma h_i^j = h_{i+1}^j$ for $i < n$.

(3.6) *Lemma.* – *Let* $\alpha = \alpha_1^j$ *be a simple root of* $\mathfrak{m}_1$, *and let* $\mathfrak{m}_1'$ *be the subalgebra of* $\mathfrak{m}_1$ *generated by* $\{\tau^k x, \tau^k y, \tau^k h \mid k \geq 0\}$ $(\tau = \gamma^n)$, *where* $x = x_1^j$, *etc. Then* $\mathfrak{m}_1' \simeq \mathfrak{sl}_2$, $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$, $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ *or* $\mathfrak{sl}_3$.

Proof. – From τ we get an automorphism $\bar{\tau}$ of the Dynkin diagram of $\mathfrak{m}_1$. The possibilities are:

(1) $\mathfrak{m}_1 \leftrightarrow G_2$, F_4, E_7, E_8, B_n or C_n: Then $\bar{\tau}$ is trivial, i.e., $\bar{\tau}\alpha = \alpha$, and we get a copy of $\mathfrak{sl}_2$.

(2) $\mathfrak{m}_1 \leftrightarrow D_4$: Then $\bar{\tau}\alpha = \alpha$, or $\bar{\tau}\alpha = \beta$ and $\bar{\tau}\beta = \alpha$, or $\bar{\tau}\alpha = \beta$, $\bar{\tau}\beta = \gamma$, $\bar{\tau}\gamma = \alpha$ where α, β and γ are distinct. So we get one, two or three copies of $\mathfrak{sl}_2$. (The simple roots involved are all non-adjacent).

(3) $\mathfrak{m}_1 \leftrightarrow D_n$, $n \geq 5$ or A_{2k-1} or E_6: Then $\bar{\tau}\alpha = \alpha$, or $\bar{\tau}^2\alpha = \alpha$ where $\bar{\tau}\alpha \neq \alpha$ and α, $\bar{\tau}\alpha$ are not adjacent. So we get one or two copies of $\mathfrak{sl}_2$.

(4) $\mathfrak{m}_1 = A_{2k}$: If $\bar{\tau}\alpha \neq \alpha$, then in the case of the middle two simple roots we have adjacency. Thus we can have $\mathfrak{m}_1' = \mathfrak{sl}_3$, $\mathfrak{sl}_2$ or $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$. ■

Let $\mathfrak{m}'_i = \gamma^{i-1}\mathfrak{m}'_1$. Then $\mathfrak{m}' := \oplus\, \mathfrak{m}'_i$ is Γ-stable. If we can prove (**) for $\mathfrak{m}'$, then we can prove it for $\mathfrak{m}$, since the toral subalgebras of the various $\mathfrak{m}'$ span a toral subalgebra of $\mathfrak{m}$. Thus we may assume that $\mathfrak{m}_1 = \mathfrak{sl}_2$ or $\mathfrak{m}_1 = \mathfrak{sl}_3$, i.e., that M is a product of SL_2's or SL_3's. We may also easily reduce to the case that Γ acts faithfully on M, i.e., d is the order of $\gamma \in \mathrm{Aut}(M)$. Let ξ denote the standard primitive d-th root of unity, and set $m = d/n$.

Case A. – Suppose that $\mathfrak{m}_1 = \mathfrak{sl}_2$. Let $\{x_i, y_i, h_i\}$ denote the $\mathfrak{sl}_2$-triple of $\mathfrak{m}_i$, where $\gamma x_i = x_{i+1}$, $\gamma y_i = y_{i+1}$ and $\gamma h_i = h_{i+1}$ for $i < n$. Since $\tau \in \mathrm{Aut}(\mathfrak{m}_1)$ preserves $\mathfrak{t}_1$ and $\mathfrak{b}_1$, $\tau h_1 = h_1$ and $\gamma h_n = h_1$, while γx_n is a multiple of x_1. Let $h'_1, \ldots, h'_n$; $x'_1, \ldots, x'_n$; $y'_1, \ldots, y'_n$ be a basis of eigenvectors for the action of γ on span $\{h_i\}$, etc. One easily shows:

(A.1) Each h'_i is a sum $\sum c_j h_j$ where no c_j is zero, and similarly for the x'_i and y'_i. The eigenvalues of the h'_i are the n-th roots of unity.

(A.2) No two x'_i have the same eigenvalue (else a linear combination violates A.1), and similarly for the y'_i.

(A.3) For all i and j, $1 \leqslant i, j \leqslant n$, the brackets $[h'_i, x'_j]$, $[h'_i, y'_j]$ and $[x'_i, y'_j]$ are non-zero, and span $\{[x'_i, y'_j]\} = \text{span}\{h'_k\}$.

(A.4) We may choose the h'_i, etc. so that

(*a*) The eigenvalue of h'_i is $\xi^{m(i-1)}$.

(*b*) $2x'_i = [h'_i, x'_1]$, $i = 1, \ldots, n$.

(*c*) $-2y'_i = [h'_i, y'_1]$, $i = 1, \ldots, n$.

Let $\mathfrak{m}_{(k)}$ denote the γ-eigenspace of $\mathfrak{m}$ with eigenvalue ξ^{-k}. Since γ acts on $\mathcal{O}(\mathbf{B})$ sending the generator s to $s \circ \gamma$, the elements of $s^k \mathfrak{m}_{(k)}$ are Γ-invariant.

(3.7) *Lemma.* – *Let* $1 \leqslant k \leqslant n$. *Then* $h'_k \in \mathfrak{m}_{(r)}$, *where* $0 < r \leqslant d$. *There are elements* $x'_i \in \mathfrak{m}_{(a)}$ *and* $y'_j \in \mathfrak{m}_{(b)}$ *such that* $h'_k = [x'_i, y'_j]$, *where* $0 < a \leqslant r$, $0 \leqslant b < d$ *and* $a + b = r$.

Proof. – Note that r is a multiple of m. It follows from A.1, A.2 and A.3 that $h'_k = [x'_i, y'_j]$ for some i and j. By A.4, keeping $i + j$ constant mod n, we can arrange that $x'_i \in \mathfrak{m}_{(a)}$, $0 < a \leqslant r$. Then $y'_j \in \mathfrak{m}_{(b)}$ where b is as required. ■

Proof of (**) *in Case A.* – Let h, x and y (without primes) be as in 3.7, i.e., $[s^a x, s^b y] = s^r h$ where $s^r h$, $s^a x$ and $s^b y$ are Γ-invariant, $r > 0$. By construction, $a > 0$. If $b > 0$, then

$$\exp(s^a x)\exp(s^b y)\exp(-s^a x)\exp(-s^b y) = \mathrm{I} + s^r h + \mathrm{O}(s^{r+1}),$$

and $(**)$ is established. (Recall that we are inside a product of SL_2's, so that $\exp(s^a x) = I + s^a x$!) If $b=0$, then we can replace a by $a-m$ and b by $b+m$, as long as $a>m=d/n$. Hence the only problem is the case $a=r=m$, $b=0$. But then we have:

$$\exp(s^r x)\exp(y)\exp(-s^r x)\exp(-y) = I + s^r h + s^r(yxy) + O(s^{r+1}),$$

where yxy is strictly lower triangular and in $\mathfrak{m}_{(r)}$. Multiplying by $\exp(-s^r yxy)$ we obtain the required element of $M(\mathbf{B})^\Gamma$. ■

Case B. – Suppose that $\mathfrak{m}_1 = \mathfrak{sl}_3$. First consider the case $n=1$, so that $\mathfrak{m} = \mathfrak{sl}_3$ and $\tau = \gamma$ has even order $d = 2m$. Let α, β and $\alpha+\beta$ be the positive roots. Then $\bar{\tau}\alpha = \beta$, $\bar{\tau}\beta = \alpha$ and $\bar{\tau}(\alpha+\beta) = \alpha+\beta$. If x_α, y_α, h_α and x_β, y_β, h_β are the usual $\mathfrak{sl}_2$-triples in $\mathfrak{sl}_3$, then $\tau h_\alpha = h_\beta$, τx_α is a multiple of x_β, etc. There are linear combinations $h^\pm$, x^1, x^2, y^1, y^2 of the elements above which are eigenvectors of τ such that we have:

(B.1) $h^+ \in (\mathfrak{m} = \mathfrak{sl}_3)_{(0)}$, $h^- \in \mathfrak{m}_{(m)}$. The $h^\pm$, x^i and y^j are linear combinations of the h_α, h_β, etc. with non-zero coefficients (e.g. $x^1 = a x_\alpha + b x_\beta$ where $ab \neq 0$).

(B.2) x^1 and x^2 correspond to distinct eigenvalues of τ, and similarly for y^1 and y^2.

(B.3) $[h^\pm, x^i]$, $[h^\pm, y^j]$, $[x^i, y^j]$ are non-zero for all i, j, and we have $\operatorname{span}\{[x^i, y^j]\} = \operatorname{span}\{h^\pm\}$.

(B.4) $2x^1 = [h^+, x^1]$, $2x^2 = [h^-, x^1]$, etc.

We now prove $(**)$ in this case ($\mathfrak{m} = \mathfrak{sl}_3$). Note that h^+ presents no problem, as $h^+ = [x_{\alpha+\beta}, y_{\alpha+\beta}]$ (up to a constant), and we can use our argument for SL_2. As in 3.7, we can find $x^i \in \mathfrak{m}_{(k)}$, $y^j \in \mathfrak{m}_{(l)}$ such that $h^- = [x^i, y^j]$ and $k+l=m$.

Let x denote x^i, y denote y^j. Consider

$$g(s) := \exp(s^k x)\exp(s^l y)\exp(-s^k x)\exp(-s^l y).$$

If k, $l>0$, then $g(s) = I + s^m h^- + O(s^{m+1})$. Suppose that $k=0$ (the case $l=0$ is similar). Then we find that $g(s) = I + s^m h^- + s^m A + O(s^{m+1})$, where A is a sum of products $x^p y x^q$ and $p+q \geqslant 2$. All such matrices are strictly upper triangular, so we may "correct" $g(s)$ as in the SL_2 case. This completes our argument in case $n=1$.

If $n>1$, we proceed as in the case of a product of SL_2's: Let x_1^1, x_1^2, y_1^1, y_1^2, $h_1^\pm$ be the eigenvectors for τ in $\mathfrak{m}_1$. Let $x_i^1 = \gamma^{i-1} x_1^1$, etc. be the corresponding elements of $\mathfrak{m}_i$. There are linear combinations $x_j^{(1)}$, $x_j^{(2)}$, $y_j^{(1)}$, $y_j^{(2)}$, $h_j^{(\pm)}$ of the x_i^1, etc. which are eigenvectors for γ and satisfy analogues of the properties **B**.1-**B**.4 above. One proceeds exactly as before. This completes our proof of Theorem 3.5. ■

Chapter VI. THE MODULI SPACE

0. Résumé

In section 1 we use a type of exponential map to determine the cokernel of $\sigma_* : \mathfrak{A}(\hat{\mathbf{A}}) \hookrightarrow \mathrm{L}(\hat{\mathbf{B}})^\Gamma$ (see III.4.6). In section 2 we are then able to show that $[\hat{\varphi}] : \mathscr{M}_{\mathrm{V},\mathbf{A}} \to \mathrm{D}\mathfrak{A} = \mathfrak{A}(\mathring{\mathbf{A}}) \backslash \mathfrak{A}(\hat{\mathbf{A}})/\mathfrak{A}(\hat{\mathbf{A}})$ is a bijection (2.13) and that, holomorphically, $\mathfrak{A}$ has the decomposition property (2.11). In other words, for every $\{\mathrm{X}\} \in \mathscr{M}_{\mathrm{V},\mathbf{A}}$, there is a G-equivariant *complex analytic* isomorphism of X with V. Moreover, we show that $\mathrm{D}\mathfrak{A} \simeq \mathbf{C}^p/\Gamma$, where $\Gamma \to \mathrm{GL}_p$ is a representation (2.7), and we construct a universal family over $\mathscr{M}_{\mathrm{V},\mathbf{A}} \simeq \mathrm{D}\mathfrak{A}$ (2.12). In section 3 we find several sufficient conditions for $\mathrm{D}\mathfrak{A}$ to be trivial, and we give examples. Except in section 3, we assume that $\mathrm{V}^\mathrm{G} = (0)$.

1. Exponential maps

(1.1) Let t denote a homogeneous generator of $\mathcal{O}(\mathrm{V})^\mathrm{G} = \mathcal{O}(\mathbf{A})$, $d := \deg t$. Let $\Delta(\mathrm{V})$ denote the Lie algebra of *polynomial vector fields on* V. Then $\Delta(\mathrm{V}) \simeq \mathrm{Mor}(\mathrm{V},\mathrm{V}) \simeq \mathrm{Der}_\mathbf{C}(\mathcal{O}(\mathrm{V}))$, the algebra of $\mathbf{C}$-linear derivations of $\mathcal{O}(\mathrm{V})$. Let $\Delta_t(\mathrm{V})$ denote the set of all elements of $\Delta(\mathrm{V})$ annihilating t. Then $\Delta_t(\mathrm{V})$ consists of the elements of $\Delta(\mathrm{V})$ which are tangent to the fibers of t. We will always identify $\Delta(\mathrm{V})$ with $\mathrm{Der}_\mathbf{C}(\mathcal{O}(\mathrm{V}))$ and $\Delta_t(\mathrm{V})$ with $\mathrm{Der}_{\mathcal{O}(\mathbf{A})}(\mathcal{O}(\mathrm{V}))$ without explicit mention. Define $\Delta(\hat{\mathrm{V}}) := \mathrm{Der}_\mathbf{C}(\mathcal{O}(\hat{\mathrm{V}}))$, and define $\Delta_t(\hat{\mathrm{V}})$, $\Delta(\mathring{\mathrm{V}})$, etc. in the obvious way. Note that $\Delta(\hat{\mathrm{V}}) \simeq \Delta(\mathrm{V}) \otimes_{\mathcal{O}(\mathrm{V})} \mathcal{O}(\hat{\mathrm{V}})$, etc. and that the G-invariant vector fields $\Delta(\mathrm{V})^\mathrm{G} \simeq \mathrm{Mor}(\mathrm{V},\mathrm{V})^\mathrm{G}$ form a free graded $\mathcal{O}(\mathbf{A})$-module.

(1.2) *Lemma.* – *As* $\mathcal{O}(\mathbf{A})$*-module,* $\Delta(\mathrm{V})^\mathrm{G} = \Delta_t(\mathrm{V})^\mathrm{G} \oplus \mathcal{O}(\mathbf{A})\mathrm{A}_0$, *where* $\mathrm{A}_0 = \sum x_i\, \partial/\partial x_i$ *is the Euler vector field on* V. *Hence* $\Delta_t(\mathrm{V})^\mathrm{G}$ *is a free graded* $\mathcal{O}(\mathbf{A})$*-module.*

Proof. – Let $\mathrm{A} \in \Delta(\mathrm{V})^\mathrm{G}$. Since $\mathrm{V}^\mathrm{G} = (0)$, A vanishes at 0 and $\mathrm{A}(t) = h(t)\,t$ for some polynomial h in one variable. Then $\mathrm{A} = \mathrm{A}' + (1/d)\,h(t)\,\mathrm{A}_0$ where $\mathrm{A}' := \mathrm{A} - (1/d)\,h(t)\,\mathrm{A}_0 \in \Delta_t(\mathrm{V})^\mathrm{G}$. ■

(1.3) From $\Delta_t(\mathrm{V})^\mathrm{G}$ we construct the Lie algebra scheme of $\mathfrak{A}^0 \subseteq \tilde{\mathfrak{A}} \subseteq \mathfrak{A}$ (see III.4.5-4.6 and V.1.6). Let $\mathrm{A}_1, \ldots, \mathrm{A}_m$ be a basis of $\Delta_t(\mathrm{V})^\mathrm{G}$. Then $[\mathrm{A}_i, \mathrm{A}_j] = \sum c_{ijk}\mathrm{A}_k$ for some $c_{ijk} \in \mathcal{O}(\mathbf{A})$. Define $\mathfrak{X}$ to be $\mathbf{A} \times \mathbf{A}^m$ with projection $\mathrm{pr}_\mathfrak{X} := \mathrm{pr}_1$ to $\mathbf{A}$. Then $\mathfrak{X}$ is a scheme over $\mathbf{A}$, and we give the fibers of $\mathrm{pr}_\mathfrak{X}$ the Lie algebra structure:

$$[(t, y_1, \ldots, y_m), (t, z_1, \ldots, z_m)] = (t, w_1, \ldots, w_m),$$

where $w_k = \sum_{ij} c_{ijk}(t) y_i z_j$. If $Z \xrightarrow{\rho} A$ is flat, then we have (see III.4.7(2))

$$\mathrm{Der}_{\mathcal{O}(Z)}(\mathcal{O}(Z) \otimes_{\mathcal{O}(A)} \mathcal{O}(V))^G \simeq \mathcal{O}(Z) \otimes_{\mathcal{O}(A)} \Delta_t(V)^G = \mathfrak{X}(Z),$$

i.e., $\mathfrak{X}(Z)$ is the set of G-invariant vector fields on $V_Z := Z \times_A V$ which preserve (i.e., are tangent to) the fibers of the quotient map $\pi : V_Z \to Z$. In particular, we have

$$\Delta_t(\hat{V})^G = \mathrm{Der}_{\mathcal{O}(\hat{A})}(\mathcal{O}(\hat{V}))^G = \mathfrak{X}(\hat{A}) = \mathcal{O}(\hat{A}) \otimes_{\mathcal{O}(A)} \mathfrak{X}(A),$$
$$\Delta_t(\hat{V})^G = \mathrm{Der}_{\mathcal{O}(\hat{A})}(\mathcal{O}(\hat{V}))^G = \mathfrak{X}(\hat{A}) = \mathcal{O}(\hat{A}) \otimes_{\mathcal{O}(A)} \mathfrak{X}(A).$$

(1.4) We give $\mathcal{O}(\hat{V})$ the $\mathbf{m}$-adic filtration $\{\mathcal{O}(\hat{V})_j := \mathbf{m}^j\}_{j \geq 0}$, where $\mathbf{m}$ denotes the ideal of $0 \in \hat{V}$. We give $\mathcal{O}(\hat{A})$ the induced filtration $\{\mathcal{O}(\hat{A})_j = (\mathbf{m}^j)^G\}_{j \geq 0}$.

Let $\mathfrak{X}(\hat{A})_r := \{A \in \mathfrak{X}(\hat{A}) \mid A(\mathcal{O}(\hat{V})_s) \subseteq \mathcal{O}(\hat{V})_{s+r}$ for all $s\}$, $r \geq 0$, and similarly define $\mathfrak{X}(A)_r$. Let $x_1, \ldots, x_n$ be coordinates on V and let $A \in \mathfrak{X}(\hat{A})$. Then $A \in \mathfrak{X}(\hat{A})_r$ if and only if $A(x_i) \in \mathcal{O}(\hat{V})_{r+1}$ for all i. Define $\mathfrak{X}(A)_{(r)}$ (the elements of $\mathfrak{X}(A)$ *homogeneous of degree* r) to be $\{A \in \mathfrak{X}(A) \mid A(x_i)$ is homogeneous of degree $r+1$ for all $i\}$. Then $\mathfrak{X}(A)_r = \mathfrak{X}(A)_{(r)} + \mathfrak{X}(A)_{(r+1)} + \ldots$

We will confuse elements of $\mathfrak{A}(\hat{A})$ with the automorphisms of $\mathcal{O}(\hat{V})$ that they induce. Let $x_1, \ldots, x_n$ be as above, and define $\mathfrak{A}(\hat{A})_r := \{\hat{\alpha} \in \mathfrak{A}(\hat{A}) \mid \hat{\alpha}(x_i) \in x_i + \mathcal{O}(\hat{V})_{r+1}$ for all $i\}$, $r \geq 0$. Let $GL_t(V)^G$ denote the group of all elements of $GL(V)^G$ which preserve t. Then $GL_t(V)^G \subseteq \mathfrak{A}(A)$. Since $V^G = (0)$, all elements of $\mathfrak{A}(\hat{A})$ preserve $\mathbf{m} \subseteq \mathcal{O}(\hat{V})$, and there is a canonical (surjective) morphism $\mathfrak{A}(\hat{A}) \to GL_t(T_0(\hat{V}) = V)^G$. We have the following:

(1.5) *Remarks.* — (1) $\mathfrak{A}(\hat{A}) = \mathfrak{A}(\hat{A})_0$ and $\mathfrak{X}(\hat{A}) = \mathfrak{X}(\hat{A})_0$.

(2) $\mathfrak{A}(\hat{A})_1 = \{\hat{\alpha} \in \mathfrak{A}(\hat{A}) \mid \partial/\partial x_i(\hat{\alpha}(x_j))(0) = \delta_{ij}\}$.

(3) There is a split exact sequence

$$1 \to \mathfrak{A}(\hat{A})_1 \to \mathfrak{A}(\hat{A}) \to GL_t(V)^G \to 1.$$

If $A \in \mathfrak{X}(\hat{A})$, then we define $\exp(A) : \mathcal{O}(\hat{V}) \to \mathcal{O}(\hat{V})$ to be the usual series $I + A + (A \circ A)/2 + \ldots$, where $\circ$ denotes composition of derivations. Similarly, we define the *logarithm* of an endomorphism of $\mathcal{O}(\hat{V})$. Recall that F denotes the fiber $\pi^{-1}(1)$ and $L = \mathrm{Aut}(F)^G$ the group of G-automorphisms of F (III.0).

(1.6) *Proposition.* — (1) *The exponential* $A \mapsto \exp(A)$ *induces a bijection* (*with inverse* log) *from* $\mathfrak{X}(\hat{A})_r$ *onto* $\mathfrak{A}(\hat{A})_r$ *for* $r \geq 1$.

(2) *If* $A \in \mathfrak{X}(A)$, *then* $\exp(A)$ *is an entire complex analytic section of* $\mathfrak{A}$, *and* $\exp(A)|_F = \exp(A|_F) \in L^0$.

Proof. — Let $A \in \mathfrak{X}(\hat{A})_r$, $r \geq 1$. Consider an isotypic component $\mathcal{O}(\hat{V})_{(\omega)} = \mathcal{O}(\hat{A}) \otimes_{\mathbf{C}} S_{(\omega)}$ of $\mathcal{O}(\hat{V})$ (see III.2.3). Here $S_{(\omega)} \simeq k\, W_{(\omega)}$ where $W_{(\omega)}$ is irreducible. Let q denote the maximal degree of the elements of $S_{(\omega)} \subseteq \mathcal{O}(V)$. Now A preserves $\mathcal{O}(\hat{V})_{(\omega)} \simeq \mathcal{O}(\hat{A}) \otimes_{\mathbf{C}} k\, W_{(\omega)}$, hence the action of A on $\mathcal{O}(\hat{V})_{(\omega)}$ is given by a matrix $(a_{ij}) \in M_k(\mathcal{O}(\hat{A}))$. The action of A^p corresponds to the p-th power $(a_{ij}^{(p)})$ of (a_{ij}), where the $a_{ij}^{(p)}$ lie in $\mathcal{O}(\hat{A})_{\max\{0,\, pr-q+1\}}$. Since $r \geq 1$, the series $\exp((a_{ij}))$ converges in the $\mathbf{m}$-adic topology, i.e., $\exp(A)$ gives an automorphism of $\mathcal{O}(\hat{V})_{(\omega)}$. It follows easily that $\exp(A)(h)$ exists for all $h \in \mathcal{O}(\hat{V})$, and that the corresponding automorphism of $\mathcal{O}(\hat{V})$ is an element of $\mathfrak{A}(\hat{A})_r$.

Conversely, let $\hat{\alpha} \in \mathfrak{A}(\hat{A})_r$, $r \geq 1$. On $\mathcal{O}(\hat{V})_{(\omega)}$, $\hat{\alpha}$ corresponds to a matrix $(a_{ij}) \in GL_k(\mathcal{O}(\hat{A}))$, and arguments as above show that the series $\log((a_{ij}))$ converges to an element of $M_k(\mathcal{O}(\hat{A}))$ which sends $\mathcal{O}(\hat{V})_{(\omega)} \cap \mathcal{O}(\hat{V})_s$ to $\mathcal{O}(\hat{V})_{(\omega)} \cap \mathcal{O}(\hat{V})_{s+r}$. It follows that $A := \log(\hat{\alpha})$ is a derivation of $\mathcal{O}(\hat{V})$, and clearly $A \in \mathfrak{X}(\hat{A})_r$. We have (1).

Let A be as in (2). Then $A \in \mathfrak{X}(A)_0$ (see 1.5), and by arguments as above, $\exp(A)$ gives an entire analytic G-automorphism of V preserving t. Moreover, $\exp(\lambda A)|_F \in L$ for all $\lambda \in \mathbf{C}$, hence $\exp(A)|_F = \exp(A|_F) \in L^0$. ■

(1.7) There is a natural Γ-action on $\mathfrak{X}(A)$ given by $(\gamma A)(v) := \gamma A(\gamma^{-1} v)$ for $\gamma \in \Gamma$, $A \in \mathfrak{X}(A)$, $v \in V$. Thus γ acts via multiplication by γ^{-r} on $\mathfrak{X}(A)_{(r)}$. The action extends in the obvious way to $\mathfrak{X}(\hat{A}) = \mathfrak{X}(A) \otimes_{\mathcal{O}(A)} \mathcal{O}(\hat{A})$.

Let $\mathfrak{l}$ denote Lie(L). We give $\mathfrak{l}$ the Γ-action induced from the one on L.

(1.8) *Proposition.* — (1) $\mathfrak{l} = \mathfrak{X}(A)|_F$, *and the* Γ-*actions coincide.*

(2) *If* $\mathfrak{k}$ *is a* Γ-*stable subspace of* $\mathfrak{l}$, *then* $\mathfrak{k}$ *has a basis* $\{A'_1, \ldots, A'_p\}$ *where* $A'_i = A_i|_F$ *and* $A_i \in \mathfrak{X}(A)$ *is homogeneous,* $i = 1, \ldots, p$. *Similarly, there are homogeneous* A_i *whose restrictions to* F *project to a basis of* $\mathfrak{l}/\mathfrak{k}$.

Proof. — It is clear that $A \mapsto A|_F$ is a Γ-equivariant Lie algebra homomorphism $\mathfrak{X}(A) \to \mathfrak{l}$. Let $A' \in \mathfrak{l} \subseteq \mathrm{Der}_{\mathbf{C}}(\mathcal{O}(F))$. Then A' is the restriction to F of some element $B \in \Delta(V)$. By averaging over G we may arrange that $B \in \Delta(V)^G$. Then $B = C \oplus h A_0 \in \mathfrak{X}(A) \oplus \mathcal{O}(A) A_0$ (see 1.2). By 1.6, $\mathfrak{X}(A)|_F \subseteq \mathfrak{l}$, hence we may assume that $A' = h A_0|_F$. Since $A_0(t) = \deg t \cdot t$ and $A'(t)$ is zero, we have $h|_F = 0$. Hence $A' = 0$, and (1) holds.

If $\mathfrak{k}$ is as in (2), then $\mathfrak{k}$ has a basis $A'_1, \ldots, A'_p$ where the A'_i are eigenvectors for the Γ-action. Thus $A'_i = B_i|_F$, where the Γ-eigenvectors $B_i \in \mathfrak{X}(A)$ have the form $B_i = B_i^{(r)} + B_i^{(r+d)} + \ldots + B_i^{(r+k_i d)}$ and $B_i^{(j)} \in \mathfrak{X}(A)_{(j)}$. Set

$$A_i := t^{k_i} B_i^{(r)} + t^{k_i - 1} B_i^{(r+d)} + \ldots + B_i^{(r+k_i d)}.$$

The argument for $\mathfrak{l}/\mathfrak{k}$ is similar. ■

(1.9) Recall the branched cover $\mathbf{B}$ of $\mathbf{A}$, where $\mathcal{O}(\mathbf{B})=\mathbf{C}[s]$, $\mathcal{O}(\mathbf{A})=\mathbf{C}[t]$ and $s^d=t$. The canonical map $\rho:\mathbf{B}*^{\Gamma}\mathrm{F}\to\mathrm{V}$, $[z,v]\mapsto zv$ (see I.3.3) gives rise to an isomorphism $\rho_*:\mathrm{L}(\hat{\mathbf{B}})^{\Gamma}\xrightarrow{\sim}\mathfrak{A}(\hat{\mathbf{A}})$ with inverse σ_* (III.4.5). From III.4.6 we know that $\sigma_*(\mathfrak{A}(\hat{\mathbf{A}}))\subseteq\mathrm{L}(\hat{\mathbf{B}})^{\Gamma}$.

Now ρ also gives an isomorphism $\rho_{\#}$ of $\mathrm{Der}_{\mathcal{O}(\hat{\mathbf{A}})}(\mathcal{O}(\hat{\mathbf{B}}*^{\Gamma}\mathrm{F}))^{\mathrm{G}}(\simeq\mathfrak{l}(\hat{\mathbf{B}})^{\Gamma})$ with $\mathfrak{X}(\hat{\mathbf{A}})$, where

$$(*)\qquad (\rho_{\#}\mathrm{B})(h)\circ\rho=\mathrm{B}(h\circ\rho),\qquad h\in\mathcal{O}(\hat{\mathrm{V}}),\qquad \mathrm{B}\in\mathrm{Der}_{\mathcal{O}(\hat{\mathbf{A}})}(\mathcal{O}(\hat{\mathbf{B}}*^{\Gamma}\mathrm{F}))^{\mathrm{G}}.$$

We denote $(\rho_{\#})^{-1}$ by $\sigma_{\#}$. Note that $\rho_{\#}$ and $\sigma_{\#}$ are $\mathcal{O}(\hat{\mathbf{A}})$-module homomorphisms.

(1.10) Let $\mathrm{A}_1,\dots,\mathrm{A}_m$ be minimal homogeneous generators of $\mathfrak{X}(\mathbf{A})$, where $\mathrm{A}_i\in\mathfrak{X}(\mathbf{A})_{(d_i)}$. Write $d_i=k_i\,d+a_i$ where $0\leqslant a_i<d$.

(1.11) *Lemma.* – (1) *The* A_i *are an* $\mathcal{O}(\mathbf{A})$*-module basis of* $\mathfrak{X}(\mathbf{A})$, *the* $\mathrm{A}_i':=\mathrm{A}_i|_{\mathrm{F}}$ *are a basis of* $\mathfrak{l}$, *and the* $s^{a_i}\mathrm{A}_i'$ *are an* $(\mathcal{O}(\mathbf{A})=\mathcal{O}(\mathbf{B})^{\Gamma})$*-module basis of* $\mathfrak{l}(\mathbf{B})^{\Gamma}$.

(2) $\sigma_{\#}\mathrm{A}_i=s^{k_i d+a_i}\mathrm{A}_i'=t^{k_i}(s^{a_i}\mathrm{A}_i')$, *hence* $\sigma_{\#}:\mathfrak{X}(\mathbf{A})\to\mathfrak{l}(\mathbf{B})^{\Gamma}$ *is homogeneous of degree* 0 *and is an injection of free* $\mathcal{O}(\mathbf{A})$*-modules.*

(3) $\sigma_{\#}:\mathfrak{X}(\hat{\mathbf{A}})\to\mathfrak{l}(\hat{\mathbf{B}})^{\Gamma}$ *is an injection of free* $\mathcal{O}(\hat{\mathbf{A}})$*-modules.*

Proof. – The only non-obvious part is the formula in (2). Let $v\in\mathrm{F}$, $z\in\mathbf{B}$ and let $h\in\mathcal{O}(\mathrm{V})$ be homogeneous of degree r. Then $\mathrm{A}_i(h)(\rho[z,v])=z^{r+d_i}\mathrm{A}_i(h)(v)$, i.e., $\mathrm{A}_i(h)\circ\rho=s^{r+d_i}\mathrm{A}_i'(h|_{\mathrm{F}})$. On the other hand, since $(h\circ\rho)[z,v]=z^r h(v)$, we obtain that $(s^{a_i}\mathrm{A}_i')(h\circ\rho)[z,v]=z^{r+a_i}\mathrm{A}_i(h)(v)$, i.e., $(s^{a_i}\mathrm{A}_i')(h\circ\rho)=s^{r+a_i}\mathrm{A}_i'(h|_{\mathrm{F}})$. Thus (see $(*)$ above), $\rho_{\#}s^{a_i}\mathrm{A}_i'=s^{-k_i d}\mathrm{A}_i=t^{-k_i}\mathrm{A}_i$, proving (2). ■

(1.12) *Definition.* – Let $\mathfrak{k}$ be a Γ-invariant subspace of $\mathfrak{l}$. Choose a basis $\mathrm{B}_i'=\mathrm{B}_i|_{\mathrm{F}}$ of $\mathfrak{l}/\mathfrak{k}$ where $\mathrm{B}_i\in\mathfrak{X}(\mathbf{A})_{(e_i)}$, $i=1,\dots,q$ and $e:=\max\{e_i\}$ is minimal. Define $r_0(\mathfrak{l}/\mathfrak{k}):=\max\{0,e-d\}+1$. Define $r_0(\mathfrak{k})$ similarly, where the B_i' are now required to be a basis of $\mathfrak{k}$. If M is a Γ-subgroup of L with Lie algebra $\mathfrak{m}\subseteq\mathfrak{l}$, then we define $r_0(\mathrm{L}/\mathrm{M}):=r_0(\mathfrak{l}/\mathfrak{m})$ and $r_0(\mathrm{M}):=r_0(\mathfrak{m})$.

(1.13) *Proposition.* – *Let* $\mathfrak{k}$ *be a* Γ*-stable subspace of* $\mathfrak{l}$, *and let* $r\geqslant 1$.

(1) $r_0(\mathfrak{l}/\mathfrak{k})=1$ *if and only if* $e_i\leqslant d$, $i=1,\dots,q$.

(2) $\mathfrak{k}(\hat{\mathbf{B}})_r^{\Gamma}\subseteq\sigma_{\#}\mathfrak{X}(\hat{\mathbf{A}})_r$ *if and only if* $r\geqslant r_0(\mathfrak{k})$.

(3) $\mathfrak{l}(\hat{\mathbf{B}})_r^{\Gamma}\subseteq\mathfrak{k}(\hat{\mathbf{B}})_r^{\Gamma}+\sigma_{\#}\mathfrak{X}(\hat{\mathbf{A}})_r$ *if and only if* $r\geqslant r_0(\mathfrak{l}/\mathfrak{k})$.

(4) $r_0(\mathfrak{l})=\max\{r_0(\mathfrak{k}),r_0(\mathfrak{l}/\mathfrak{k})\}$.

Proof. – Let $\mathrm{B}_1,\dots,\mathrm{B}_q$ be as in 1.12, so that the B_i' project to a basis of $\mathfrak{l}/\mathfrak{k}$. Let $\mathfrak{l}(\mathbf{B})_{(r)}^{\Gamma}\subseteq\mathfrak{l}(\mathbf{B})^{\Gamma}$ denote the elements homogeneous of degree r, and similarly for $\mathfrak{l}$

replaced by $\mathfrak{l}/\mathfrak{k}$. Let $C \in \mathfrak{l}(\mathbf{B})^{\Gamma}_{(r)}$. Modulo $\mathfrak{k}(\mathbf{B})^{\Gamma}_{(r)}$, $C = \sum_i c_i s^r B'_i$ where the coefficients $c_i \in \mathbf{C}$ are zero unless $r \equiv e_i \bmod d$. If $r \geqslant r_0(\mathfrak{l}/\mathfrak{k}) > e_i - d$, then $r \equiv e_i \pmod d$ forces $r \geqslant e_i$. Thus $C = \sum_i c_i s^{r-e_i} \sigma_{\#} B_i$, hence $\mathfrak{l}(\hat{\mathbf{B}})^{\Gamma}_r \subseteq \mathfrak{k}(\hat{\mathbf{B}})^{\Gamma}_r + \sigma_{\#} \mathfrak{X}(\hat{\mathbf{A}})_r$. Conversely, if $r < r_0(\mathfrak{l}/\mathfrak{k})$, then for some i, $s^{-d} \sigma_{\#} B_i$ projects to an element of $(\mathfrak{l}/\mathfrak{k})(\mathbf{B})^{\Gamma}_{(r)}$ which does not come from an element of $\sigma_{\#} \mathfrak{X}(\hat{\mathbf{A}})_r$. We have proved (3), (2) is proved similarly, and (1) is obvious.

It is clear that $r_0(\mathfrak{l}) \leqslant \max\{r_0(\mathfrak{k}), r_0(\mathfrak{l}/\mathfrak{k})\}$. But if $e = \max\{d_i\}$, where the d_i are as in 1.10, then $\max\{0, e-d\} + 1 = r_0(\mathfrak{l})$ is an upper bound for $r_0(\mathfrak{k})$ and $r_0(\mathfrak{l}/\mathfrak{k})$. ■

(1.14) *Theorem.* — *Let* M *be a* Γ*-subgroup of* L *with Lie algebra* $\mathfrak{m} \subseteq \mathfrak{l}$.

(1) *For all* $r \geqslant 1$ *the following diagram commutes*:

$$\begin{array}{ccccc}
M(\hat{\mathbf{B}})^{\Gamma}_r & \xrightarrow{\subset} & L(\hat{\mathbf{B}})^{\Gamma}_r & \xleftarrow[\sigma_*]{\supset} & \mathfrak{A}(\hat{\mathbf{A}})_r \\
\uparrow{\scriptstyle \exp} & & \uparrow{\scriptstyle \exp} & & \uparrow{\scriptstyle \exp} \\
\mathfrak{m}(\hat{\mathbf{B}})^{\Gamma}_r & \xrightarrow{\subset} & \mathfrak{l}(\hat{\mathbf{B}})^{\Gamma}_r & \xleftarrow[\sigma_{\#}]{\supset} & \mathfrak{X}(\hat{\mathbf{A}})_r
\end{array}$$

(2) $M(\hat{\mathbf{B}})^{\Gamma}_r \subseteq \sigma_*(\mathfrak{A}(\hat{\mathbf{A}})_r)$ *if and only if* $r \geqslant r_0(M)$.

(3) $L(\hat{\mathbf{B}})^{\Gamma}_r \subseteq M(\hat{\mathbf{B}})^{\Gamma}_r \cdot \sigma_* \mathfrak{A}(\hat{\mathbf{A}})_r$ *if and only if* $r \geqslant r_0(L/M)$.

(4) *If* M *has the approximation property* (*see* V.0.9) *and in addition* $r \geqslant r_0(L/M)$, *then*

$$L(\hat{\mathbf{B}})^{\Gamma}_r \subseteq M(\mathbf{B})^{\Gamma}_1 \cdot \sigma_* \mathfrak{A}(\hat{\mathbf{A}})_r \subseteq \sigma_*(\mathfrak{A}(\dot{\mathbf{A}}) \cdot \mathfrak{A}(\hat{\mathbf{A}})_r).$$

Proof. — Parts (1) and (2) follow from 1.6 and 1.13, and (4) follows from (2) and (3). Suppose that $r \geqslant r_0(L/M)$. Let $\hat{\alpha} \in L(\hat{\mathbf{B}})^{\Gamma}_r$. Then $\hat{\alpha} = \exp(A)$, where, by 1.13(3), $A = B + C$, $B \in \mathfrak{m}(\hat{\mathbf{B}})^{\Gamma}_r$, $C \in \sigma_{\#} \mathfrak{X}(\hat{\mathbf{A}})_r$. Then

$$\exp(-B)\,\hat{\alpha}\exp(-C) = \exp(-B)\exp(B+C)\exp(-C) \in L(\hat{\mathbf{B}})^{\Gamma}_{r+1}.$$

By induction,

$$L(\hat{\mathbf{B}})^{\Gamma}_r \subseteq M(\hat{\mathbf{B}})^{\Gamma}_r \cdot L(\hat{\mathbf{B}})^{\Gamma}_q \cdot \sigma_* \mathfrak{A}(\mathbf{A})_r$$

for all $q \geqslant r$. As soon as $q \geqslant \max\{r_0(L), r\}$ we have that $L(\hat{\mathbf{B}})^{\Gamma}_q \subseteq \sigma_* \mathfrak{A}(\mathbf{A})_r$. Thus the "if" direction of (3) holds.

Conversely, suppose that $L(\hat{\mathbf{B}})^{\Gamma}_r \subseteq M(\hat{\mathbf{B}})^{\Gamma}_r \cdot \sigma_* \mathfrak{A}(\hat{\mathbf{A}})_r$. Let μ denote the projection $L(\hat{\mathbf{B}})^{\Gamma}_r \to L(\hat{\mathbf{B}})^{\Gamma}_r / L(\hat{\mathbf{B}})^{\Gamma}_{r_0(L)}$. Then

$$(G_1 := \mu(M(\hat{\mathbf{B}})^{\Gamma}_r)) \cdot (G_2 := \mu(\sigma_* \mathfrak{A}(\hat{\mathbf{A}})_r)) = (G_3 := L(\hat{\mathbf{B}})^{\Gamma}_r / L(\hat{\mathbf{B}})^{\Gamma}_{r_0(L)}).$$

The G_i have natural structures as (unipotent) complex algebraic groups. Since $G_1 G_2 = G_3$, we have that $\mathrm{Lie}(G_1) + \mathrm{Lie}(G_2) = \mathrm{Lie}(G_3)$. But $\mathrm{Lie}(G_1) = (\mathfrak{m}(\hat{\mathbf{B}})_r^\Gamma + \mathfrak{l}(\hat{\mathbf{B}})_{r_0(L)}^\Gamma)/\mathfrak{l}(\hat{\mathbf{B}})_{r_0(L)}^\Gamma$, etc., and (3) follows. ■

(1.15) Let $\mathfrak{l}_u := \mathrm{Lie}(\mathrm{Rad}_u(L))$, and let $\mathfrak{l}'$ denote the inverse image in $\mathfrak{l}$ of the semi-simple part of $\mathfrak{l}/\mathfrak{l}_u$. By V.3.5 the corresponding subgroup $L' \subseteq L$ has the approximation property. Let $\mathfrak{k}$ denote span $\{A_i' : \deg A_i \leqslant d\}$.

(1.16) *Corollary.* (See Theorem 2.4). – (1) *If* $r_0(\mathfrak{l}/\mathfrak{l}') = 1$, *then* $D\mathfrak{A}$ *is trivial. Equivalently,* $\mathfrak{A}$ *has the decomposition property* (*see* V.0.4) *if* $\mathfrak{k} + \mathfrak{l}' = \mathfrak{l}$. *In particular, if* M *is a* Γ*-subgroup of* L *such that* $r_0(M) = 1$ *and* $\mathfrak{m} + \mathfrak{l}' = \mathfrak{l}$, *then* $\mathfrak{A}$ *has the decomposition property.*

(2) *Let* L_{vb} *be a* Γ*-invariant Levi factor of* L (*see* III.3.1 *and* 3.5), *and let* Z *denote the center of* L_{vb}^0. *If* $r_0(Z) = 1$, *then* $\mathfrak{A}$ *has the decomposition property. In particular, if* $r_0(L_{vb}) = 1$, *then* $\mathfrak{A}$ *has the decomposition property.*

(1.17) *Remarks.* – (1) One can use 1.5(3), 1.11(3) and the exponential map (see 1.14(1)) to establish that $\sigma_* \mathfrak{A}(\hat{\mathbf{A}}) \subseteq L(\hat{\mathbf{B}})^\Gamma$ (cf. III.4.6).

(2) There is a quick proof that there is a number $r_0(L)$ as in 1.14(2): One uses the fact that $\mathcal{O}(\hat{\mathbf{A}})$ is a topological field, where the $\mathcal{O}(\hat{\mathbf{A}})_r$, $r \geqslant 0$, are a fundamental system of neighborhoods of 0. The groups $L(\hat{\mathbf{B}})^\Gamma$ and $\mathfrak{A}(\hat{\mathbf{A}})$ inherit topologies with the obvious fundamental systems of neighborhoods of the identity, and $\rho_* : L(\hat{\mathbf{B}})^\Gamma \xrightarrow{\sim} \mathfrak{A}(\hat{\mathbf{A}})$ is continuous (see 1.9). It follows that $\rho_*(L(\hat{\mathbf{B}})_r^\Gamma) \subseteq \mathfrak{A}(\hat{\mathbf{A}})$ for some $r \geqslant 0$.

(1.18) *Examples.* – (1) In Example III.2.7 we have $d=4$, $L^0 = \mathbf{C}^*$ and $\Delta(V)^G$ is generated by elements of degrees 0 and 2. Thus $\mathfrak{A}$ has the decomposition property.

(2) In Example III.2.8 one has $d=2$, $L^0 = (\mathbf{C}^*)^2$ and $\Delta(V)^G$ is generated by elements of degrees 0 and $2n$. It follows from 2.4 below that $\mathfrak{A}$ has the decomposition property if and only if $n \leqslant 1$.

2. Moduli

We combine the tools at hand to compute $\mathcal{M}_{V, \mathbf{A}}$.

(2.1) *Proposition.* – *Let* $\mathfrak{A}^0 \subseteq \mathfrak{A}$ *be as in* V.1.6, *so that* $D\mathfrak{A} = D\mathfrak{A}^0/\Gamma$ (V.1.9). *Then*

(1) $$D\mathfrak{A} \simeq L(\mathbf{B})^\Gamma \backslash L(\hat{\mathbf{B}})^\Gamma / \sigma_* \mathfrak{A}(\mathbf{A}),$$

(2) $$D\mathfrak{A}^0 \simeq L^0(\mathbf{B})^\Gamma \backslash L^0(\hat{\mathbf{B}})^\Gamma / \sigma_* \mathfrak{A}^0(\mathbf{A}).$$

Proof. — Via σ_*, we have that

$$\mathrm{D}\mathfrak{A} \simeq \sigma_* \mathfrak{A}(\dot{\mathbf{A}}) \backslash \sigma_* \mathfrak{A}(\hat{\mathbf{A}}) / \sigma_* \mathfrak{A}(\hat{\mathbf{A}}) = \mathrm{L}(\dot{\mathbf{B}})^\Gamma \backslash \mathrm{L}(\hat{\mathbf{B}})^\Gamma / \sigma_* \mathfrak{A}(\hat{\mathbf{A}}).$$

Now $\mathrm{L}(\hat{\mathbf{B}})^\Gamma = \mathrm{L}(\dot{\mathbf{B}})^\Gamma \, \mathrm{L}(\hat{\mathbf{B}})^\Gamma$, hence

$$\mathrm{L}(\dot{\mathbf{B}})^\Gamma \backslash \mathrm{L}(\hat{\mathbf{B}})^\Gamma \simeq (\mathrm{L}(\dot{\mathbf{B}})^\Gamma \cap \mathrm{L}(\hat{\mathbf{B}})^\Gamma) \backslash \mathrm{L}(\hat{\mathbf{B}})^\Gamma \simeq \mathrm{L}(\mathbf{B})^\Gamma \backslash \mathrm{L}(\hat{\mathbf{B}})^\Gamma,$$

proving (1). The proof of (2) is similar. ■

(2.2) Let L′ denote the connected subgroup of L corresponding to $\mathfrak{l}' \subseteq \mathfrak{l}$. Then L^0/L' is a torus, and we may choose a Γ-stable torus Z in L such that the canonical map $\tau: \mathrm{L} \to \mathrm{L}/\mathrm{L}'$ induces an isomorphism of $\mathfrak{z} := \mathrm{Lie}(\mathrm{Z})$ with $\mathfrak{l}/\mathfrak{l}'$. Then $\mathrm{K} := \mathrm{Ker}\, \tau|_{\mathrm{Z}}$ is finite. From V.3.4 we see that $\mathrm{Z}(\hat{\mathbf{B}})_r^\Gamma \to (\mathrm{L}/\mathrm{L}')(\hat{\mathbf{B}})_r^\Gamma$ and $\mathrm{Z}(\mathbf{B})_r^\Gamma \to (\mathrm{L}/\mathrm{L}')(\mathbf{B})_r^\Gamma$ are bijective, $r \geqslant 1$. Of course, $\mathrm{Z}(\mathbf{B})_1^\Gamma = \{e\}$. We have:

(2.3) *Lemma.* — *Let τ, etc. be as above and $r \geqslant 1$. There are split exact sequences*

$$1 \to \mathrm{L}'(\hat{\mathbf{B}})_r^\Gamma \to \mathrm{L}(\hat{\mathbf{B}})_r^\Gamma \xrightarrow{\tau_*} (\mathrm{L}/\mathrm{L}')(\hat{\mathbf{B}})_r^\Gamma \to 1,$$

$$0 \to \mathfrak{l}'(\hat{\mathbf{B}})_r^\Gamma \to \mathfrak{l}(\hat{\mathbf{B}})_r^\Gamma \xrightarrow{\tau_\#} (\mathfrak{l}/\mathfrak{l}')(\hat{\mathbf{B}})_r^\Gamma \to 0,$$

where τ induces τ_* and $\tau_\#$. Moreover, $\mathrm{L}(\hat{\mathbf{B}})_r^\Gamma = \mathrm{Z}(\hat{\mathbf{B}})_r^\Gamma \, \mathrm{L}'(\hat{\mathbf{B}})_r^\Gamma$ *and* $\mathrm{L}(\mathbf{B})_r^\Gamma = \mathrm{L}'(\mathbf{B})_r^\Gamma$.

(2.4) *Theorem.* — (1) *There is a canonical bijection*

$$\mathrm{D}\mathfrak{A}^0 \simeq (\mathrm{L}/\mathrm{L}')(\hat{\mathbf{B}})_1^\Gamma / \tau_* \sigma_* \mathfrak{A}(\hat{\mathbf{A}})_1.$$

(2) $\mathrm{D}\mathfrak{A}$ *is trivial if and only if* $r_0(\mathfrak{l}/\mathfrak{l}') = 1$. *Equivalently,* $\mathfrak{A}$ *has the decomposition property if and only if* $\mathfrak{k} + \mathfrak{l}' = \mathfrak{l}$ *where* $\mathfrak{k}$ *is defined as in* 1.15.

Proof. — From 1.14(3), 2.1 and 2.3 we see that (1) gives (2). To establish (1), we simultaneously show that the canonical map

$$\psi: \mathrm{D} := \mathrm{L}(\mathbf{B})_1^\Gamma \backslash \mathrm{L}(\hat{\mathbf{B}})_1^\Gamma / \sigma_* \mathfrak{A}(\hat{\mathbf{A}})_1 \to \mathrm{L}^0(\mathbf{B})^\Gamma \backslash \mathrm{L}^0(\hat{\mathbf{B}})^\Gamma / \sigma_* \mathfrak{A}^0(\hat{\mathbf{A}}) \simeq \mathrm{D}\mathfrak{A}^0$$

is a bijection. It is surjective, since $\mathrm{L}^0(\hat{\mathbf{B}})^\Gamma = (\mathrm{L}^0)^\Gamma \, \mathrm{L}^0(\hat{\mathbf{B}})_1^\Gamma$ and $(\mathrm{L}^0)^\Gamma \subseteq \mathrm{L}^0(\mathbf{B})^\Gamma$. Suppose that $\hat{\beta}, \hat{\beta}' \in \mathrm{L}(\hat{\mathbf{B}})_1^\Gamma$ and that $\hat{\beta}' = \alpha \hat{\beta} \sigma_* \hat{\alpha}$ where $\alpha \in \mathrm{L}^0(\mathbf{B})^\Gamma$ and $\hat{\alpha} \in \mathfrak{A}^0(\hat{\mathbf{A}})$. Then $\alpha(0) = (\sigma_* \hat{\alpha})^{-1}(0) \in \mathrm{J} := \mathrm{GL}_t(\mathrm{V})^{\mathrm{G}} \cap (\mathrm{L}^0)^\Gamma$. Thus

$$\hat{\beta}' = \alpha\alpha(0)^{-1} \alpha(0) \hat{\beta} \alpha(0)^{-1} \sigma_* (\hat{\alpha}(0)^{-1} \hat{\alpha}),$$

where $\alpha\alpha(0)^{-1} \in \mathrm{L}(\mathbf{B})_1^\Gamma$ and $\hat{\alpha}(0)^{-1} \hat{\alpha} \in \mathfrak{A}(\hat{\mathbf{A}})_1$. Since J normalizes $\mathrm{L}(\mathbf{B})_1^\Gamma$, $\mathrm{L}(\hat{\mathbf{B}})_1^\Gamma$ and $\mathfrak{A}(\hat{\mathbf{A}})_1$, it acts by conjugation on D, and $\mathrm{D}/\mathrm{J} \simeq \mathrm{D}\mathfrak{A}^0$.

Now $D = A\backslash B/C$ where $A = L(\mathbf{B})_1^\Gamma$, etc. Let $E_r := L(\hat{\mathbf{B}})_r^\Gamma$, $r \geqslant r_0(L)$. Then $E_r \subseteq C$, E_r is normal in B and $AE_r \supseteq H := L'(\hat{\mathbf{B}})_1^\Gamma \supseteq A$. By 2.3, $B/H = (L/L')(\hat{\mathbf{B}})_1^\Gamma \simeq Z(\hat{\mathbf{B}})_1^\Gamma$ is abelian, and AE_r is normal in B. Thus

$$\begin{aligned} D &= A\backslash B/E_r C = AE_r\backslash B/C = B/AE_r C = B/AC \\ &\simeq (B/H)/(AC/H) = (L/L')(\hat{\mathbf{B}})_1^\Gamma/\tau_* \sigma_* \mathfrak{A}(\hat{\mathbf{A}})_1. \end{aligned}$$

Since L^0/L' is a torus, the action of L^0 on $(L/L')(\hat{\mathbf{B}})_1^\Gamma$ by conjugation is trivial, hence so is that of $J \subseteq L^0$. Thus $D \simeq D\mathfrak{A}^0$. ■

(2.5) Let

$$\begin{aligned} &\pi_r : (L/L')(\hat{\mathbf{B}})_1^\Gamma \to Q_r := (L/L')(\hat{\mathbf{B}})_1^\Gamma/(L/L')(\hat{\mathbf{B}})_r^\Gamma, \\ &(\pi_r)_\# : (\mathfrak{l}/\mathfrak{l}')(\hat{\mathbf{B}})_1^\Gamma \to \mathfrak{q}_r := (\mathfrak{l}/\mathfrak{l}')(\hat{\mathbf{B}})_1^\Gamma/(\mathfrak{l}/\mathfrak{l}')(\hat{\mathbf{B}})_r^\Gamma \end{aligned}$$

be the canonical maps, $r \geqslant r_0(L)$. The exponential map of L induces the exponential maps of L/L' and Q_r. We consider $\mathfrak{q}_r$ as a vector group under $+$, and since $\mathfrak{l}/\mathfrak{l}'$ is abelian, $\exp : \mathfrak{q}_r \xrightarrow{\sim} Q_r$ is an isomorphism of groups. We give Q_r the induced vector group structure.

Since $(\pi_r)_\# \tau_\# \sigma_\# \mathfrak{X}(\hat{\mathbf{A}})_1$ is a linear subspace of $\mathfrak{q}_r$, $\pi_r \tau_* \sigma_* \mathfrak{A}(\hat{\mathbf{A}})_1$ is a vector subgroup of Q_r, and we have an induced structure of vector group on the quotient

$$Q_r/\pi_r \tau_* \sigma_* \mathfrak{A}(\hat{\mathbf{A}})_1 \simeq (L/L')(\hat{\mathbf{B}})_1^\Gamma/\tau_* \sigma_* \mathfrak{A}(\hat{\mathbf{A}})_1,$$

independent of $r \geqslant r_0(L)$.

(2.6) Let $B_i' = B_i|_F$ project to a basis of $\mathfrak{l}/\mathfrak{l}'$, where $B_i \in \mathfrak{X}(\mathbf{A})_{(e_i)}$, $i = 1, \ldots, q$. For each i, choose $0 < b_i \leqslant d$ such that $b_i \equiv e_i \mod d$. Note that $b_i = d$ and $a_i = 0$ (see 1.10) if $e_i \equiv 0 \mod d$, else $b_i = a_i$. The $s^{b_i} B_i'$ project to an $\mathcal{O}(\mathbf{A})$-basis for $(\mathfrak{l}/\mathfrak{l}')(\mathbf{B})_1^\Gamma$ (note the subscript 1!). We may choose integers $m_i \geqslant 0$ such that

$$\{ C_{ij}' := t^{j-1}(s^{b_i} B_i') \mid i = 1, \ldots, q, j = 1, \ldots, m_i \} \subset \mathfrak{l}(\mathbf{B})_1^\Gamma$$

projects to a $\mathbf{C}$-basis of $(\mathfrak{l}/\mathfrak{l}')(\hat{\mathbf{B}})_1^\Gamma/\tau_\# \sigma_\# \mathfrak{X}(\hat{\mathbf{A}})_1$. (Note that if $e_i \leqslant d$, then $m_i = 0$.)

(2.7) *Corollary. — The double coset space* $D\mathfrak{A}^0$ *has a canonical vector group structure as a quotient of the vector group*

$$Q_r = (L/L')(\hat{\mathbf{B}})_1^\Gamma/(L/L')(\hat{\mathbf{B}})_r^\Gamma \simeq Z(\hat{\mathbf{B}})_1^\Gamma/Z(\hat{\mathbf{B}})_r^\Gamma, \qquad r = r_0(L).$$

The group structure is induced by that of the "middle term" $\mathfrak{A}^0(\hat{\mathbf{A}}) = L^0(\hat{\mathbf{B}})^\Gamma$ *of the double coset space. More specifically, let* C_{ij}'*, etc. be as above.*

(1) *The* C_{ij}' *and the exponential map induce an "explicit" bijection*

$$(\mathbf{C}^p,+)\ \tilde{\rightarrow}\ (\mathfrak{l}/\mathfrak{l}')(\hat{\mathbf{B}})_1^\Gamma/\tau_\#\,\sigma_\#\,\mathfrak{X}(\hat{\mathbf{A}})_1 \xrightarrow[\sim]{\exp} \mathrm{D}\mathfrak{A}^0,$$

where $p=\sum_i m_i$.

(2) *Denote the coordinates on* $\mathbf{C}^p$ *by* x_{ij}, $i=1,\ldots,q$, $j=1,\ldots,m_i$. *Define an action of* Γ *on* $\mathbf{C}^p$ *by*

$$\gamma(\ldots,x_{ij},\ldots)=(\ldots,\gamma^{-e_i}x_{ij},\ldots)$$

(*see* 1.7). *Then* $\mathrm{D}\mathfrak{A}\simeq\mathrm{D}\mathfrak{A}^0/\Gamma\simeq\mathbf{C}^p/\Gamma$.

(2.8) *Remark.* — Suppose that $e_i\geqslant d$. Then $m_i\leqslant(e_i-b_i)/d$ since $\sigma_\#(\mathrm{B}_i)=t^{(e_i-b_i)/d}(s^{b_i}\mathrm{B}_i')$. Suppose further that L^0 is a torus, so that $\mathfrak{l}'=0$. Choose the B_i in 2.6 to be an $\mathcal{O}(\mathbf{A})$-basis of $\mathfrak{X}(\mathbf{A})$. Then we have $m_i=(e_i-b_i)/d$.

(2.9) If M is a Γ-group, let $\mathrm{M}(\mathbf{B}^{(h)})_r^\Gamma$ denote the subgroup of $\mathrm{M}(\hat{\mathbf{B}})_r^\Gamma$ consisting of *entire analytic elements* (i.e. power series with infinite radii of convergence). Recall that $\mathrm{M}(\tilde{\mathbf{B}})_r^\Gamma\subset\mathrm{M}(\hat{\mathbf{B}})_r^\Gamma$ denotes the subgroup of *rational* sections.

(2.10) *Lemma.* — (1) *For all* $r\geqslant 1$, $\mathrm{L}(\hat{\mathbf{B}})_1^\Gamma=\mathrm{L}(\tilde{\mathbf{B}})_1^\Gamma\cdot\mathrm{L}(\hat{\mathbf{B}})_r^\Gamma$.

(2) *Let* $\hat{z}\in\mathrm{Z}(\hat{\mathbf{B}})_1^\Gamma$ *and* $r\geqslant 0$. *Then there is a* $z^{(h)}\in\mathrm{Z}(\mathbf{B}^{(h)})_1^\Gamma$ *such that* $\hat{z}z^{(h)}\in\mathrm{Z}(\hat{\mathbf{B}})_r^\Gamma$.

Proof. — Since L′ has the approximation property, we may use 2.3 to reduce (1) to the case L = Z. Let $\hat{z}\in\mathrm{Z}(\hat{\mathbf{B}})_1^\Gamma$. There is an isomorphism $\mathrm{Z}\simeq(\mathbf{C}^*)^e$ for some e, and Γ acts on $(\mathbf{C}^*)^e$ through a homomorphism to $\mathrm{GL}_e(\mathbf{Z})$. Our section $\hat{z}$ is a Γ-invariant e-tuple of series $(z_1,\ldots,z_e)$ where

$$z_i=1+\sum_{j=1}^{r-1}a_{ij}s^j+\mathrm{O}(s^r),\ a_{ij}\in\mathbf{C},\ i=1,\ldots,e.$$

Clearly,

$$\tilde{z}:=1+\sum_{j=1}^{r-1}a_{ij}s^j$$

lies in $\mathrm{Z}(\tilde{\mathbf{B}})_1^\Gamma$ and $\hat{z}\tilde{z}^{-1}\in\mathrm{Z}(\hat{\mathbf{B}})_r^\Gamma$, proving (1).

Now

$$\log\left(1+\sum_{j=1}^{r-1}a_{ij}s^j\right)=\sum_{j=1}^{r-1}b_{ij}s^j+\mathrm{O}(s^r)$$

for some b_{ij}. Set

$$z^{(h)} := \left(\exp\left(-\sum_{j=1}^{r-1} b_{ij} s^j\right)\right).$$

Then $z^{(h)} \in \mathrm{Z}(\mathbf{B}^{(h)})_1^{\Gamma}$ and $\hat{z}z^{(h)} \in \mathrm{Z}(\hat{\mathbf{B}})_r^{\Gamma}$. ■

Recall that $\tilde{\mathrm{D}}\mathfrak{A} := \mathfrak{A}(\dot{\mathbf{A}})\ \mathfrak{A}(\tilde{\mathbf{A}})/\mathfrak{A}(\tilde{\mathbf{A}})$ (see V.0.3), and similarly define $\tilde{\mathrm{D}}\mathfrak{A}^0$.

(2.11) *Corollary.* — (1) *The canonical inclusions* $\tilde{\mathrm{D}}\mathfrak{A}^0 \subsetneq \mathrm{D}\mathfrak{A}^0$ *and* $\tilde{\mathrm{D}}\mathfrak{A} \subsetneq \mathrm{D}\mathfrak{A}$ *are bijections.*

(2) *Holomorphically,* $\mathfrak{A}$ *has the decomposition property.*

Let X *be a* G*-variety with* $\{\mathrm{X}\} \in \mathscr{M}_{\mathrm{V},\mathbf{A}}$.

(3) *There is a neighborhood* U *of* $0 \in \mathbf{A}$ *such that* $\mathrm{X}_{\mathrm{U}} := \pi_{\mathrm{X}}^{-1}(\mathrm{U})$ *and* $\mathrm{V}_{\mathrm{U}} := \pi_{\mathrm{V}}^{-1}(\mathrm{U})$ *are* G*-isomorphic over* U.

(4) *There is a holomorphic G-isomorphism* $\mathrm{X} \simeq \mathrm{V}$.

Proof. — Since L has the decomposition property and L′ has the approximation property, we may reduce (1) and (2) to the case that L = Z, i.e., to 2.10. Parts (3) and (4) are reinterpretation of (1) and (2). ■

(2.12) *Theorem.* — (1) *There is an affine* $(\mathrm{G}\times\Gamma)$*-variety* $\mathfrak{F}$ *and an equivariant surjective morphism* $\eta : \mathfrak{F} \to \mathbf{A} \times \mathrm{D}\mathfrak{A}^0$ *where* G *acts trivially on* $\mathbf{A} \times \mathrm{D}\mathfrak{A}^0$ *and* Γ *acts on* $\mathrm{D}\mathfrak{A}^0$ *as in* V.1.9 (cf. 2.7(2)), *with the following property: Let* $\alpha \in \mathfrak{A}^0(\hat{\mathbf{A}})$ *and set* $\mathrm{X} = \eta^{-1}(\mathbf{A} \times [\alpha])$. *Then* $\{\mathrm{X}\} \in \mathscr{M}_{\mathrm{V},\mathbf{A}}$ *with* $[\hat{\varphi}(\mathrm{X})] = [\alpha]$, *and* $\pi_{\mathrm{X}} = \mathrm{pr}_1 \circ \eta : \mathrm{X} \to \mathbf{A}$.

(2) *The composition*

$$\mathscr{E} := \mathfrak{F}/\Gamma \to \mathbf{A} \times \mathrm{D}\mathfrak{A}^0/\Gamma \simeq \mathbf{A} \times \mathscr{M}_{\mathrm{V},\mathbf{A}} \to \mathscr{M}_{\mathrm{V},\mathbf{A}}$$

is a universal family (cf. I.2.3 *Theorem* 3(4)).

Proof. — Let $\mathrm{C}_1, \ldots, \mathrm{C}_p \in \mathfrak{z}(\mathbf{B})_1^{\Gamma}$ project to a basis of $\mathrm{I}(\hat{\mathbf{B}})_1^{\Gamma}/\mathrm{I}'(\hat{\mathbf{B}})_1^{\Gamma} + \sigma_{\#}\,\mathfrak{X}(\hat{\mathbf{A}})_1$. Let $r \geqslant r_0(\mathrm{L})$, and let $\exp_r$ denote the truncated exponential series sending x into $1 + x + x^2/2 + \ldots + x^{r-1}/(r-1)!$. Identify Z with $(\mathbf{C}^*)^e$ as in the proof of 2.10, so that $\mathfrak{z} \simeq \mathbf{C}^e$. For each $c = (c_1, \ldots, c_p) \in \mathbf{C}^p$ define $\mathrm{C}_c := \sum c_i \mathrm{C}_i$. Clearly, $z = (z_i, \ldots, z_e) := \exp_r(\mathrm{C}_c) \in \mathbf{C}^e(\mathbf{B})_1^{\Gamma}$, and $z \in \mathrm{Z}(\mathbf{B}_c)_1^{\Gamma}$ for the Γ-invariant subset $\mathbf{B}_c$ of points of $\mathbf{B}$ where $\det(z) = \prod z_i \neq 0$. Let $\mathbf{A}_c = \mathbf{B}_c/\Gamma \subseteq \mathbf{A}$. Then $0 \in \mathbf{A}_c$ for all $c \in \mathbf{C}^p$.

Let $\mathscr{A} := \mathbf{A} \times \mathbf{C}^p$, let $\dot{\mathscr{A}} := \dot{\mathbf{A}} \times \mathbf{C}^p$, let $\mathscr{A}' := \{(t, c) \in \mathscr{A} : t \in \mathbf{A}_c\}$ and set $\dot{\mathscr{A}}' := \dot{\mathscr{A}} \cap \mathscr{A}'$. Let $\mathfrak{B}$ denote $\mathrm{V} \times \mathbf{C}^p$, and set $\dot{\mathfrak{B}} := (\pi_{\mathrm{V}} \times \mathrm{id})^{-1}(\dot{\mathscr{A}})$ and $\mathfrak{B}' := (\pi_{\mathrm{V}} \times \mathrm{id})^{-1}(\mathscr{A}')$. Then $(t, c) \mapsto (\rho_* \exp_r(\mathrm{C}_c)(t), c)$ is a Γ-invariant section of $\mathfrak{A} \times \mathbf{C}^p$ defined over $\dot{\mathscr{A}}'$, i.e., it defines a $(\mathrm{G}\times\Gamma)$-invariant automorphism $\dot{\alpha}'$ of

$\dot{\mathfrak{B}}' := \dot{\mathfrak{B}} \cap \mathfrak{B}'$ over $\dot{\mathscr{A}}'$. We may glue $\dot{\mathfrak{B}}$ and $\mathfrak{B}'$ together over $\dot{\mathfrak{B}}'$ using the identification $\dot{\alpha}'$. The resulting scheme $\mathfrak{F}$ is an affine $(\mathrm{G} \times \Gamma)$-variety over $\mathscr{A}$ with $\mathcal{O}(\mathfrak{F})^{\mathrm{G}} \simeq \mathcal{O}(\mathscr{A})$ (see [Ha, p. 81 Ex. 2.17 and p. 91 Ex. 3.3]). Clearly, the canonical map $\eta : \mathfrak{F} \to \mathscr{A} \simeq \mathbf{A} \times \mathrm{D}\mathfrak{A}^0$ has the properties in (1), and (2) follows easily. ■

(2.13) *Corollary.* – *We have* $\mathscr{M}_{\mathrm{V},\mathbf{A}} \simeq \tilde{\mathrm{D}}\mathfrak{A} = \mathrm{D}\mathfrak{A} \simeq \mathbf{C}^p/\Gamma$. *Moreover,* $\mathscr{M}_{\mathrm{V}} \simeq \mathbf{C}^p/\mathbf{C}^*$ *where* $\mathbf{C}^*$ *acts on* $\mathbf{C}^p$ *with the weights given in* 2.7 (2). *In particular,* $\mathscr{M}_{\mathrm{V}} \setminus \{*\} \simeq (\mathscr{M}_{\mathrm{V},\mathbf{A}} \setminus \{*\})/\mathbf{C}^*$ *is a weighted projective space, and* $(\mathfrak{F} \setminus \mathrm{pr}_2 \circ \eta^{-1}(*))/\mathbf{C}^* \to (\mathrm{D}\mathfrak{A}^0 \setminus \{*\})/\mathbf{C}^* \simeq \mathscr{M}_{\mathrm{V}} \setminus \{*\}$ *is a universal family.*

3. Rigid Representations

We find conditions on V which guarantee that $\mathscr{M}_{\mathrm{V},\mathbf{A}} = \mathrm{D}\mathfrak{A} = \{*\}$. It will be convenient to drop our longstanding assumption that $\dim \mathrm{V}^{\mathrm{G}} = 0$. If $\dim \mathrm{V}^{\mathrm{G}} = 1$, then $\mathrm{V} \simeq \mathbf{C} \oplus \mathrm{W}$ as G-module where G acts trivially on $\mathbf{C}$ (cf. II.0.3 or IV.3.9).

We will denote a principal isotropy group of V by H.

(3.1) *Definitions.* – (1) We say that V is G-*rigid* (or just *rigid*) if $r_0(\mathfrak{l}/\mathfrak{l}') = 1$, and *strongly* G-*rigid* (or just *strongly rigid*) if $r_0(\mathfrak{l}/\mathfrak{l}_u) = 1$.

(2) V is *stable* if the generic G-orbit in V is closed.

(3) V is *semifree* if it is stable and the principal isotropy groups are trivial.

(4) We say that V is *unreduced* if $\mathrm{G} = \mathrm{G}' \times \mathrm{G}''$ and $\mathrm{V} = \mathrm{V}' \oplus \mathrm{V}''$ where V′ (resp. V″) is a G′-module (resp. G″-module) and (V′, G′) has a one-dimensional quotient, else we say that V is *reduced.*

Of course, strongly rigid G-modules are rigid. Note that V is rigid if and only if $\mathfrak{A}$ has the decomposition property. The case $\dim \mathrm{V}^{\mathrm{G}} = 1$ is a special type of unreduced representation, where $(\mathrm{V}', \mathrm{G}') = (\mathbf{C}, \{e\})$.

(3.2) *Theorem.* – *Let* V *be a* G-*module with one-dimensional quotient. Then* V *is rigid if*

(1) V *is semifree, or*

(2) *The principal isotropy group of* V *is central in* G (*e.g.*, G *is a torus*),

and strongly rigid if

(3) V *is a stable torus action, or*

(4) $\dim \mathrm{V}^{\mathrm{G}^0} = 1$, *or*

(5) $\dim \mathrm{V} \leqslant 3$, or

(6) G^0 *is a simple group, or*

(7) $(\mathrm{V}, \mathrm{G}^0)$ *is self dual.*

The rest of this section is devoted to the proof of 3.2. Along the way we develop more criteria for (strong) rigidity which are not so neat to state as 3.2.

(3.3) *Remark.* — Let X be a G-variety as in Ch. II, with $T_{x_0}X = V$. By the Luna-Richardson theorem [LR], $V/\!\!/G \simeq V^H/\!\!/N$, where $N = N_G(H)/H$ acts semifreely on V^H, and similarly for X. It follows from 3.2(1) that X^H is N-isomorphic to V^H. It remains to extend this isomorphism to a G-isomorphism of X and V.

(3.4) *Corollary. — Suppose that the restriction map* $\mathrm{res}: \Delta(V)^G \to \Delta(V^H)^N$ *is an isomorphism. Then* V *is rigid.*

Proof. — The group scheme $\mathfrak{A}$ of V is determined by the associative $\mathcal{O}(\mathbf{A})$-algebra $\mathrm{Mor}(V, V)^G$ (see III.2.3-4). Since res is an isomorphism, the group schemes associated to V and (V^H, N) are the same. By 3.2(1), V^H has the decomposition property, hence so does V. ■

(3.5) *Proposition. — Suppose that* $(V, G) = (V' \oplus V'', G' \times G'')$ *is unreduced. Then* (V, G) *is (strongly) rigid if and only if* (V', G') *is (strongly) rigid. In particular,* (V, G) *is strongly rigid if* $\dim V^G = 1$.

Proof. — We have $F = F' \times V''$ and

$$L = \mathrm{Aut}(F)^G = \mathrm{Aut}(F')^{G'} \times \mathrm{Aut}(V'')^{G''} = M \times \mathrm{Aut}(V'')^{G''},$$

where F' and M are defined analogously to F and L, respectively. Then $\mathfrak{l}/\mathfrak{l}_u = \mathfrak{m}/\mathfrak{m}_u \oplus \mathrm{End}(V'')^{G''}$. Now $\mathrm{End}(V'')^{G''}$ consists of degree 0 elements of $\Delta(V'')^{G''}$, and the proposition follows. ■

(3.6) *Proposition. — Let* V *be semifree, where* G *is connected. Then* V *is rigid. If, in addition,* G *is a torus, then* V *is strongly rigid.*

Proof. — Let $v_0 \in F$. Then $F = G v_0$ and $L = G$ acts on F by: $l(gv_0) = gl^{-1} v_0$, $g \in G$, $l \in L$. Let Z_G denote the center of G. Then $Z_G \subseteq GL_t(V)^G \subseteq \mathfrak{A}(\mathbf{A})$. There is a natural homomorphism $\mu: Z_G \to L$, where $\mu(z) g v_0 = z^{-1} g v_0 = g z^{-1} v_0$. Clearly, μ gives an isomorphism from Z_G to $Z := \mathrm{Cent}(L)$ and from $\mathfrak{z}_G$ to $\mathfrak{z}$. Thus $\mathfrak{z}$ is generated by the restriction of elements of $\mathfrak{X}(\mathbf{A})_{(0)}$ and $r_0(\mathfrak{z}) = 1$. Now apply 1.16(2). ■

(3.7) *Theorem. — Let* V *be rigid, and let* $\tilde{G}$ *be a finite normal extension of* G *which acts linearly on* V *such that* $\tilde{G}/G$ *acts faithfully on the quotient* $\mathbf{A} = V/\!\!/G$. Then $(V, \tilde{G})$ *is rigid. If* (V, G) *is strongly rigid, then so is* $(V, \tilde{G})$.

Let L, F, Γ, t, etc. have their usual meaning relative to (V, G), and let $\tilde{\mathrm{L}}$, $\tilde{\mathrm{F}}$, $\tilde{\Gamma}$, $\tilde{t}$, etc. correspond to $(\mathrm{V}, \tilde{\mathrm{G}})$, where $\tilde{t}=t^k$ (so $k=[\tilde{\mathrm{G}}:\mathrm{G}]$).

(3.8) *Lemma.* — *Let* $\tilde{\mathrm{L}}'':=\{\tilde{l}\in\tilde{\mathrm{L}}\,|\,\tilde{l}$ *preserves* F$\}$. *Then the canonical restriction map* res : $\tilde{\mathrm{L}}''\to\mathrm{L}$ *is an isomorphism. In particular,* res : $\tilde{\mathrm{L}}^0\to\mathrm{L}^0$ *is an isomorphism.*

Proof. — Since $\tilde{\mathrm{G}}\mathrm{F}=\tilde{\mathrm{F}}$, any element of $\tilde{\mathrm{L}}''$ is determined by its restriction to F, so res is injective. Now let $l\in\mathrm{L}$. For every k-th root of unity ω_i, choose $g_i\in\tilde{\mathrm{G}}$ such that $g_i\mathrm{F}=t^{-1}(\omega_i)$. Then $\tilde{\mathrm{F}}=\coprod_i g_i\mathrm{F}$. Define $l_i:g_i\mathrm{F}\to g_i\mathrm{F}$ by $l_i(g_i v):=g_i l(v)$, and define $\tilde{l}\in\mathrm{Aut}(\tilde{\mathrm{F}})$ to be l_i on $g_i\mathrm{F}$. Then $\tilde{l}\in\tilde{\mathrm{L}}$: Let $g\in\tilde{\mathrm{G}}$ and given i, choose j such that $g_j^{-1}gg_i\in\mathrm{G}$. Then $\tilde{l}(gg_iv)=\tilde{l}(g_jg_j^{-1}gg_iv)=g_jl(g_j^{-1}gg_iv)$ (by definition) $=g_jg_j^{-1}gg_il(v)$ (by G-invariance of l) $=gg_il(v)=g\tilde{l}(g_iv)$ (by definition). ■

(3.9) *Corollary.* — (1) *The restriction map from* $\mathrm{Der}(\mathcal{O}(\tilde{\mathrm{F}}))^{\tilde{\mathrm{G}}}$ *to* $\mathrm{Der}(\mathcal{O}(\mathrm{F}))^{\mathrm{G}}$ *gives an isomorphism* res: $\tilde{\mathfrak{l}}\xrightarrow{\sim}\mathfrak{l}$.

(2) Let $\mathrm{A}\in\Delta_t(\mathrm{V})^{\mathrm{G}}=\mathfrak{X}(\mathrm{A})$ *be homogeneous, and let* $\mathrm{A}'=\mathrm{A}|_{\mathrm{F}}$. *Then there is an* m, $0\leqslant m<k$, such that $t^m\mathrm{A}\in\Delta_{\tilde{t}}(\mathrm{V})^{\tilde{\mathrm{G}}}$ *and such that* $\mathrm{res}(\mathrm{A}'')=\mathrm{A}'$, *where* $\mathrm{A}''=t^m\mathrm{A}|_{\tilde{\mathrm{F}}}$.

Proof. — Part (1) is immediate from 3.8. Let $\mathrm{A}\in\Delta(\mathrm{V})^{\mathrm{G}}$ be homogeneous. Then, under the action of $\tilde{\mathrm{G}}$, A transforms by a character of $\tilde{\mathrm{G}}/\mathrm{G}\simeq\mathbf{Z}/k\mathbf{Z}$, so that $t^m\mathrm{A}$ is $\tilde{\mathrm{G}}$-invariant for some m, $0\leqslant m<k$. If $\mathrm{A}\in\Delta_t(\mathrm{V})^{\mathrm{G}}$, then $\mathrm{A}(t)=0$ implies that $\mathrm{A}(\tilde{t})=0$, hence $t^m\mathrm{A}\in\Delta_{\tilde{t}}(\mathrm{V})^{\tilde{\mathrm{G}}}$, and (2) follows. ■

Proof of 3.7. — Since (V, G) is rigid, there are homogeneous elements $\mathrm{A}_i\in\Delta_t(\mathrm{V})^{\mathrm{G}}$ of degree $\leqslant d$ whose restrictions to F project to a basis of $\mathfrak{l}/\mathfrak{l}'$. By 3.9, there are integers m_i, $0\leqslant m_i<k$, such that $\mathrm{B}_i:=t^{m_i}\mathrm{A}_i\in\Delta_{\tilde{t}}(\mathrm{V})^{\tilde{\mathrm{G}}}$ and such that the $\mathrm{B}_i|_{\tilde{\mathrm{F}}}$ generate $\tilde{\mathfrak{l}}/\tilde{\mathfrak{l}}'$. But $\deg\mathrm{B}_i=m_id+\deg\mathrm{A}_i\leqslant kd$, so $r_0(\tilde{\mathfrak{l}}/\tilde{\mathfrak{l}}')=1$ and $(\mathrm{V},\tilde{\mathrm{G}})$ is rigid. Similarly, if (V, G) is strongly rigid, then so is $(\mathrm{V},\tilde{\mathrm{G}})$. ■

(3.10) *Example.* — In III.2.7, consider the action of $\mathrm{G}^0=\mathbf{C}^*$. Then $\Delta_{t^0}(\mathrm{V})^{\mathrm{G}^0}$ is generated by $x\dfrac{\partial}{\partial x}-y\dfrac{\partial}{\partial y}$ while $\Delta_t(\mathrm{V})^{\mathrm{G}}$ is generated by $xy\left(x\dfrac{\partial}{\partial x}-y\dfrac{\partial}{\partial y}\right)$.

(3.11) *Corollary.* — *Let* V *be strongly rigid, and let* $\tilde{\mathrm{G}}$ *be an extension of* G *which acts linearly on* V *with one-dimensional quotient. Then* $(\mathrm{V},\tilde{\mathrm{G}})$ *is strongly rigid.*

Proof. — By 3.7 we may reduce to the case where $\mathcal{O}(\mathrm{V})^{\tilde{\mathrm{G}}}=\mathcal{O}(\mathrm{V})^{\mathrm{G}}$, $\tilde{\mathrm{F}}=\mathrm{F}$, etc. Let $\tilde{\mathrm{L}}$, $\tilde{\mathfrak{X}}$, etc. correspond to $\tilde{\mathrm{G}}$. Averaging over $\tilde{\mathrm{G}}$ gives a linear projection $\Delta_t(\mathrm{V})^{\mathrm{G}}\to\Delta_t(\mathrm{V})^{\tilde{\mathrm{G}}}=\tilde{\mathfrak{X}}(\mathrm{A})$ which induces a homogeneous projection $\mathrm{Av}:\mathfrak{X}(\mathrm{A})\to\tilde{\mathfrak{X}}(\mathrm{A})$ and a projection $\mathrm{Av}_*:\mathfrak{l}\to\tilde{\mathfrak{l}}$.

Let $O \subseteq \tilde{O}$ denote the closed orbits of G and $\tilde{G}$ in F, respectively, and let S denote either O or $\tilde{O}$. Then F is homotopic to S, hence $H^q(F, \mathbf{Z}/2)=0$ if $q>\dim S$ and $H^q(F, \mathbf{Z}/2)\neq 0$ for $q=\dim S$ (see II.3.5). Thus $\dim O=\dim \tilde{O}$. It follows that $O=\tilde{O}$ and that there is a $\tilde{G}$-vector bundle structure on F which restricts to a G-vector bundle structure. Consequently, $Av_*(\mathfrak{l}_{vb})=\tilde{\mathfrak{l}}_{vb}$, and we have a commutative diagram (see III.3.2-3)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{l}_u & \longrightarrow & \mathfrak{l} & \longrightarrow & \mathfrak{l}_{vb} & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle Av_*} & & \downarrow{\scriptstyle Av_*} & & \\ 0 & \longrightarrow & \tilde{\mathfrak{l}}_u & \longrightarrow & \tilde{\mathfrak{l}} & \longrightarrow & \tilde{\mathfrak{l}}_{vb} & \longrightarrow & 0 \end{array}$$

Thus $Av_*(\mathfrak{l}_u)=\tilde{\mathfrak{l}}_u$.

By hypothesis,

$$(*) \qquad \mathfrak{l}=\operatorname{span}\{A_i' : A_i \in \mathfrak{X}(A),\ \deg A_i \leqslant d\}+\mathfrak{l}_u.$$

Since Av is degree preserving and $Av_*(\mathfrak{l}_u)=\tilde{\mathfrak{l}}_u$, we obtain the analogue of $(*)$ for $\tilde{\mathfrak{l}}$ and $\tilde{\mathfrak{X}}$. ■

(3.12) *Proposition.* — *Suppose that every* H-*isotypic component in* V *is* G-*stable* (*e.g.*, H *is central in* G). *Then* H *is normal in* G *and* V *is rigid.*

(Recall that H denotes the principal isotropy group of V.)

Proof. — Write $V=V^H \oplus V'$ as H-module. By hypothesis, G stabilizes V^H and V'. The action of G on V^H must be stable with principal isotropy group H, where $H \subseteq G_v$ for all $v \in V^H$. Hence G normalizes H, and G/H acts semifreely on V^H.

Now $F \simeq G/H \times V'$, and there is a split exact sequence of Γ-groups

$$(**) \qquad 1 \to GL(V')^H \to L_{vb} \to G/H \to 1.$$

Write $V'=\bigoplus_{i=1}^{p} n_i V_i$ where the V_i are irreducible and pairwise non-isomorphic H-modules. Then $GL(V')^H \simeq \prod_i GL_{n_i}$. As G-module, each $n_i V_i$ becomes a sum $\oplus\, m_j W_j$, so that the center of GL_{n_i} lies in the center of $\prod_j GL_{m_j}$. In other words, the center of $GL(V')^H$ lies in the center of $GL(V')^G$, hence the center $\mathfrak{z}'$ of $\operatorname{End}(V')^H$ is spanned by degree 0 elements of $\Delta(V')^G=\Delta_t(V')^G$.

The proofs of 3.6 and 3.9 show that $r_0(\mathfrak{z}'')=1$ where $\mathfrak{z}''$ is the center of the Lie algebra of the group "L" of $(V^H, G/H)$. Using the split exact sequence $(**)$ above, we consider $\mathfrak{z}''$ as a subalgebra of $\mathfrak{z} = \text{Cent}(\mathfrak{l}_{vb})$. Then $\mathfrak{z}' + \mathfrak{z}''$ contains $\mathfrak{z}$, so we have $r_0(\mathfrak{z}) = 1$. ∎

(3.13) *Proof of* 3.2 (1)-(2). – These follow from 3.6, 3.7 and 3.12. ∎

Proof of 3.2 (3). – Apply 3.6 to the G/H-module V, where H is the ineffective part of the G-action. ∎

Proof of 3.2(4). – By 3.5, (V, G^0) is strongly rigid. Hence, by 3.11, so is (V, G). ∎

Proof of 3.2(5). – Suppose that $\dim V = 3$ and that G is connected. Assume that the semisimple part G_{ss} of G is non-trivial. Then the only possible (effective) examples of (V, G_{ss}) are easily seen to be $(\mathbf{C}^2 \oplus \mathbf{C}, SL_2)$ and $(\mathbf{C}^3, SO_3)$. In the first case $\dim V^{G_{ss}} = 1$, and in the second the group "L" of (V, G_{ss}) is finite. Thus V is strongly G_{ss}-rigid, hence strongly G-rigid by 3.11.

We may now suppose that G is a torus. By 3.2(3) we need only consider the case that V is not stable. Let H denote the principal isotropy group of V. If $\dim V^H = 1$, then $\dim V^H /\!/ (G/H) = 1$ which implies that G/H is finite. Thus $G = H$, $\dim V^G = 1$ and we have strong rigidity by 3.5.

If $\dim V^H = 2$, then $V = V^H \oplus W$ and $L = \text{Aut}(G *^H W)^G$ where $\dim W = 1$. Thus $GL(W)^H = \mathbf{C}^*$. Since $\dim V^H /\!/ (G/H) = 1$, we have that $G/H \simeq \mathbf{C}^*$. Thus $L_{vb} \simeq (\mathbf{C}^*)^2$ and so $\dim \mathfrak{l}/\mathfrak{l}_u = 2$. Now there are linear actions of $\mathbf{C}^*$ on V^H and on W commuting with the G-action and preserving t. Thus $\mathfrak{l}/\mathfrak{l}_u$ is spanned by degree 0 elements of $\mathfrak{X}(\mathbf{A})$, and V is strongly rigid. We leave the cases when $\dim V = 1$ and $\dim V = 2$ to the reader. ∎

(3.14) We now consider the case where G is simple. In Tables I a and I b below we list the relevant modules V for the simple groups G (modulo outer automorphisms). (See [Schl] for the notation we use. For example, (φ_j, A_{n-1}) denotes the standard SL_n-module structure on $\Lambda^j(\mathbf{C}^n)$.) In each case we list $L = \text{Aut}(F)^G$, the group of G-automorphisms of the fiber $F = \pi^{-1}(1)$ (III.0), the degree $d = \deg t$ of the generator t of the invariant ring $\mathcal{O}(\mathbf{A}) = \mathcal{O}(V)^G$, and the degrees of the minimal homogeneous generators of $\Delta(V)^G$ (as graded $\mathcal{O}(\mathbf{A})$-module). By 1.2, minimal homogeneous generators of $\mathfrak{X}(\mathbf{A})$ together with the Euler vector field A_0 (which has degree 0) form a set of minimal homogeneous generators of $\Delta(V)^G$.

In the column listing L, the symbol N denotes the normalizer of the maximal torus of SL_2. A symbol U_j denotes a unipotent group of dimension j, and $\widetilde{SL}_2$ denotes the elements of GL_2 of determinant ± 1.

One can use the tables and results of [Schl] to obtain the list of relevant V and to compute d. The versions of Frobenius reciprocity discussed in Remark III.2.6

allow one to compute the generators of $\Delta(V)^G$. To compute L, one uses results of III.3. In most cases, $L = N_G(H)/H$.

TABLE I a

	G		φ	d	Degrees	L
1·····	A_{n-1},	$n \geqslant 2$	$n\varphi_1$	n	0	A_{n-1}
2·····		$n \geqslant 2$	$\varphi_1 + \varphi_1^*$	2	0	$\mathbf{C}^*$
3·····		$n \geqslant 2$	φ_1^2	n	0	$\mathbf{Z}/n$
4·····	A_{2m-1},	$m \geqslant 2$	φ_2	m	0	$\mathbf{Z}/m$
5·····		$m \geqslant 2$	$\varphi_2 + \varphi_1$	m	0	$\mathbf{Z}/m \times \mathbf{C}^*$
6·····		$m \geqslant 2$	$\varphi_2 + \varphi_1^*$	m	0	$\mathbf{Z}/m \times \mathbf{C}^*$
7·····	A_{2m},	$m \geqslant 2$	$\varphi_2 + \varphi_1$	$m+1$	0	$\mathbf{C}^*$
8·····		$m \geqslant 2$	$\varphi_2 + 2\varphi_1^*$	3	0,3, $m-1$	$(GL_2 \times \mathbf{C}^*) \ltimes U_2$
9·····	A_1		φ_1^3	4	0, 2	N
10·····	A_5		φ_3	4	0, 2	N
11·····			$\varphi_3 + \varphi_1$	4	0, 2	$N \ltimes (\mathbf{C}^*)^2$
12·····			$\varphi_3 + 2\varphi_1$	4	0, 2, 4, 6	$(N \ltimes (GL_2)^2) \ltimes U_4$
13·····	A_6		φ_3	7	0	$\mathbf{Z}/7$
14·····	A_7		φ_3	16	0	$\mathbf{Z}/16$

TABLE I b

	G	φ	d	Degrees	L
1·······	D_n, $n \geqslant 4$	φ_1	2	0	$\mathbf{Z}/2$
2·······	B_n, $n \geqslant 2$	φ_1	2	0	$\mathbf{Z}/2$
3·······	C_n, $n \geqslant 2$	$2\varphi_1$	2	0	A_1
4·······	B_2	$\varphi_1 + \varphi_2$	2	0, 1, 2	$(\mathbf{Z}/2 \ltimes (\mathbf{C}^*)^2) \ltimes U_1$
5·······	B_3	φ_3	2	0	$\mathbf{Z}/2$
6·······	B_4	φ_4	2	0	$\mathbf{Z}/2$
7·······	B_5	φ_5	4	0, 2	N
8·······	C_3	φ_3	4	0, 2	N
9·······	D_5	$2\varphi_5$	4	0	$\widetilde{SL}_2$
10·······	D_6	φ_6	4	0, 2	N
11·······	D_7	φ_7	8	0	$\mathbf{Z}/8$
12·······	E_6	φ_1	3	0	$\mathbf{Z}/3$
13·······	E_7	φ_1	4	0, 2	N
14·······	G_2	φ_1	2	0	$\mathbf{Z}/2$

Proof of 3.2(6). — It follows from 1.16 and Tables Ia and Ib that there are only two cases to worry about, namely $(\varphi_2+2\varphi_1^*, A_{2m})$, $m>4$, and $(\varphi_3+2\varphi_1, A_5)$. In the latter case, $d=4$, and it is easy to see that the elements of $\mathfrak{X}(\mathbf{A})$ of degrees 0 and 2 restrict to generators of a Levi factor of $\mathfrak{l}$. In the former case, $d=3$, and the elements of degrees 0 and 3 generate a Levi factor. ■

TABLE II

	G		φ	d	Degrees	L
1 ······	$C_n\times C_1'$,	$n\geq 1$	$\varphi_1\otimes\varphi_1'$	2	0	$\mathbf{Z}/2$
2 ······		$n\geq 2$	$\varphi_1\otimes\varphi_1'+\varphi_1$	2	0, 2, 4	$(\mathbf{Z}/2\times(\mathbf{C}^*)^2)\ltimes U_2$
3 ······			$\varphi_1\otimes\varphi_1'+\varphi_1'$	2	0, 2	$(\mathbf{Z}/2\times\mathbf{C}^*)\ltimes U_1$
4 ······	$C_n\times SO_3$,	$n\geq 2$	$\varphi_1\otimes\mathbf{C}^3$	4	0, 2, 4	$N\times\mathbf{C}^*$
5 ······	$SO_n\times C_1'$,	$n\geq 3$	$\mathbf{C}^n\otimes\varphi_1'$	4	0, 2	N
6 ······	$B_3\times C_1'$		$\varphi_3\otimes\varphi_1'$	4	0, 2	N
7 ······	$G_2\times C_1'$		$\varphi_1\otimes\varphi_1'$	4	0, 2	N
8 ······	GL_n,	$n\geq 1$	$\mathbf{C}^n\oplus\mathbf{C}^{n*}$	2	0	$\mathbf{C}^*$
9 ······	$C_n\times SO_2$,	$n\geq 1$	$\varphi_1\otimes\mathbf{C}^2$	2	0	$\mathbf{C}^*$

We now consider the case of self dual representations. By 3.5, we need only consider reduced representations.

(3.15) *Proposition.* — *Let* V *be a reduced self dual* G*-module with one-dimensional quotient, where* G *is connected. Then either* G *is simple (and* V *appears in Table Ia or Ib), or* V *appears in Table II.*

(3.16) *Remarks.* — Table II uses the same notation and conventions as Tables Ia and Ib. The modules in Tables Ia, Ib and II are strongly rigid. (For entry 2 of Table II we have $GL(V)^G\simeq(\mathbf{C}^*)^2$ which is the Levi part of L^0.) Thus 3.11 and 3.15 finish the proof of Theorem 3.2.

(3.17) *Lemma.* — *Let* $(W,H)=(W_1\otimes W_2, H_1\times H_2)$ *be irreducible where the* H_i *are connected semisimple and* $n_1=\dim W_1\geq n_2=\dim W_2\geq 2$. *Then*

(1) *If* W *is orthogonal and* $\dim W/\!/H=1$, *then* $W\simeq(\varphi_1\otimes\varphi_1', C_n\times C_1')$.

(2) *If* W *is symplectic, then* $\dim W/\!/H\geq 1$, *and* $\dim W/\!/H=1$ *implies that* W *is isomorphic to entry* 4, 5, 6 *or* 7 *of Table II.*

(3) *The dimension of* $(W\oplus W^*)/\!/H$ *is at least* 2.

Proof. — Suppose that W is orthogonal. If each (W_i, H_i) is orthogonal, then $\dim W/\!/H \geqslant \dim(\mathbf{C}^{n_1} \otimes \mathbf{C}^{n_2})/\!/(O_{n_1} \times O_{n_2})$. By taking O_{n_1}-invariants and then O_{n_2}-invariants, we see that $\dim W/\!/H \geqslant n_2 \geqslant 2$. Suppose that (W_1, H_1) and (W_2, H_2) are both symplectic. If H_1 or H_2 is not simple, then we may rearrange the factorization of H so that the (W_i, H_i), $i=1, 2$ are again orthogonal. Thus we may assume that H_1 and H_2 are simple. A computation as in the orthogonal case forces $n_2=2$ and $H_2=Sp_1$. A case by case check of the possibilities for (W_1, H_1) gives (1). The proof of (2) is similar. In (3), one computes invariants for the case $H_i=SL_{n_i}$ to derive the inequality. ■

(3.18) *Proof of* 3.15. — Let (V', G') be an irreducible subrepresentation of (V, G) where G' denotes the image of G in $GL(V')$. Then (V', G') is orthogonal (and we have $\dim V'/\!/G'=1$), or (V', G') is symplectic (and $\dim V'/\!/G'=1$ if G' is not simple) or (V, G) contains $(V' \oplus V'^*, G')$ (and $\dim (V' \oplus V'^*)/\!/G'=1$).

Suppose that (V', G') is orthogonal. Then 3.17 shows that G is simple or that $(V', G')=(\varphi_1 \otimes \varphi_1', C_n \times C_1')$. To reconstruct V one adds irreducible symplectic representations of simple groups with zero-dimensional quotient. Since V is reduced, we can only add copies of (φ_1, C_m) where C_m is a factor of G'. Thus G is simple (and V is in Table I a or I b), or $G \simeq C_n \times C_1'$ and V is entry 1, 2 or 3 of Table II.

Suppose that (V', G') is symplectic. If $\dim V'/\!/G'=1$, then using techniques similar to the orthogonal case, one can prove that G is simple or that V is isomorphic to entry 4, 5, 6 or 7 or Table II. If every possible (V', G') is symplectic with zero-dimensional quotient, then there must be a subrepresentation of the form $(2\varphi_1, C_n)$, and we are in the last case:

Suppose that V contains a subrepresentation of the form $(V' \oplus V'^*, G')$. Then by 3.17, $G'=H$ or $H \times \mathbf{C}^*$, where H is simple (V' is irreducible). Case by case checking shows that $(V', H) \simeq (\varphi_1, C_n)$ or (φ_1, A_n), $n \geqslant 1$. One easily sees that nothing can be added, i.e., $(V, G)=(V' \oplus V'^*, G')$. The relevant representations are entry 2 of Table I a, entry 3 of Table I b, and entries 8 and 9 of Table II. ■

Chapter VII. G-VECTOR BUNDLES

0. Résumé

We consider the problem of classifying G-vector bundles whose base is a G-module P with one-dimensional quotient. There are several connections between this problem and the linearization problem considered so far (see I.2.5 Proposition 6, cf. [Kr2]). The G-vector bundles over a fixed P are classified by cohomology sets $H^1_{Zar}(\mathbf{A}, \mathfrak{P})$, where $\mathfrak{P}$ is a group scheme over **A**. We use the techniques of Chapter VI to compute these cohomology sets and to construct universal families. Non-trivial G-vector bundles give rise to examples of non-linearizable actions.

1. Preliminaries

(1.1) Let G be a reductive complex algebraic group and Y a G-variety. A G-*vector bundle* on Y is a vector bundle E over Y such that E is a G-variety, the projection $p: E \to Y$ is equivariant and the action of G is linear on the fibers of p. If y_0 is a fixed point of the G-action on Y, then we obtain a representation of G on the fiber E_{y_0}.

Let P and Q be G-modules. Let $\mathrm{Vec}_G(P, Q)$ denote the class of all G-vector bundles over P whose fiber at $0 \in P$ is Q, and let $\mathrm{VEC}_G(P, Q)$ denote the set of G-isomorphism classes in $\mathrm{Vec}_G(P, Q)$. The trivial class is represented by the product $P \times Q$, which we denote by Θ_Q. If $G = \{e\}$ is trivial, then the solution of the Serre Problem by Quillen and Suslin shows that every element of $\mathrm{Vec}(\mathbf{A}^n, \mathbf{C}^m)$ is trivial, so that every $E \in \mathrm{Vec}_G(P, Q)$ can be considered as a G-action on some $X = \mathbf{A}^r$. Bass and Haboush [BH2] have shown that every element E of $\mathrm{Vec}_G(P, Q$ is *stably trivial*, i.e., $E \oplus \Theta_{Q'}$ is trivial for some G-module Q′, where $\oplus$ denotes Whitney sum. However, our examples show that $\mathrm{VEC}_G(P, Q)$ can be non-trivial.

(1.2) *Proposition.* – *Let* $E, E' \in \mathrm{Vec}_G(P, Q)$.

(1) ([MP]) *Suppose that* H *is a subgroup of* G *such that* $(P \oplus Q)^H = P$. *Then* E *and* E′ *are isomorphic as* G-*varieties if and only if* E *is isomorphic to a pull-back* $\varphi^* E'$ *for some* G-*automorphism* φ *of* P.

(2) ([BH2]) *If* $E \oplus \Theta_P \in \mathrm{Vec}_G(P, Q \oplus P)$ *is non-trivial, then the* G-*action on* E *is non-linearizable.*

(3) ([Kr2]) *There is an open cover* $\{U_i\}$ *of* P//G *and* G-*isomorphisms* $\varphi_i : E|_{\pi_P^{-1}(U_i)} \simeq \pi_P^{-1}(U_i) \times Q$ *for each i.*

Proof. – We show how to obtain (1), since no proof is in the literature: Let $\psi: E \xrightarrow{\sim} E'$ be an isomorphism of G-varieties. Then ψ induces an isomorphism

$\varphi: P \simeq E^H \xrightarrow{\sim} E'^H \simeq P$ of the zero sections, and the derivative of ψ induces an isomorphism of normal bundles: $\nu(E^H) \simeq \varphi^* \nu(E'^H)$. But $\nu(E^H) \simeq E$ and $\nu(E'^H) \simeq E'$. ■

(1.3) *Remarks.* – (1) Given $E \in \mathrm{Vec}_G(P, Q)$, consider the $\tilde{G} := (G \times \mathbf{C}^*)$-action on E, where $\mathbf{C}^*$ acts by scalars on the fiber. Then 1.2(1) gives Proposition 6(1) of Chapter I, i.e., E is trivial as G-vector bundle if and only if the $\tilde{G}$-action on E is not linearizable. Now $E/\!/\tilde{G} \simeq P/\!/G$. If $\dim P/\!/G = 1$, then as soon as we show that $\mathrm{VEC}_G(P, Q) \neq \{*\}$, we have that $\mathcal{M}_{V, \mathbf{A}} \neq \{*\}$, where $V = P \oplus Q$ is considered as a $\tilde{G}$-module with $\mathbf{C}^*$ acting by scalars on Q.

(2) Using 1.2(1) and 1.2(2) we can construct (families of) non-linearizable G-actions on $E \simeq \mathbf{A}^n$, but we lose the fact that the quotient has dimension 1 (see section 5). We are unable to give a classification in these cases.

(3) Let $E \in \mathrm{Vec}_G(P, Q)$. Then 1.2(3) shows that E is obtained by glueing together trivial bundles $\pi^{-1}(U_i) \times Q$ via G-isomorphisms $\alpha_{ij} = \varphi_i \circ \varphi_j^{-1} \in \mathfrak{P}(U_i \cap U_j)$, where $\mathfrak{P}(U) := \mathrm{Mor}(\pi^{-1}(U), \mathrm{GL}(Q))^G$ is the group of G-automorphisms of the trivial bundle $\pi^{-1}(U) \times Q$. When $\dim P/\!/G = 1$ we show that $\mathfrak{P}(U)$ is the group of sections of a group scheme $\mathfrak{P}$ over $\mathbf{A}$, so that $\mathrm{VEC}_G(P, Q) \simeq H^1_{\mathrm{Zar}}(\mathbf{A}, \mathfrak{P}) :=$ Čech cohomology of the sheaf of groups $U \mapsto \mathfrak{P}(U)$.

2. The group scheme $\mathfrak{P}$

(2.1) Let P, Q be as in (1.1). From now on we assume that $\dim P/\!/G = 1$. Let $t: P \to P/\!/G = \mathbf{A}$ be a homogeneous generator of $\mathcal{O}(P)^G$, where $\deg t = d$. We will also denote t by π or π_P. Let $A_1, \ldots, A_m$ generate the (free) $\mathcal{O}(\mathbf{A})$-module $\mathrm{Mor}(P, \mathrm{End}\, Q)^G$. Then, using composition in End Q, we have that $A_i \circ A_j = \sum d_{ijk} A_k$ where the $d_{ijk} \in \mathcal{O}(\mathbf{A})$. As in VI.1.3, these formulas enable us to construct an *algebra scheme* $\mathfrak{E}$ over $\mathbf{A}$. Then $\mathfrak{E}(X) = \mathrm{Mor}(X \times_{\mathbf{A}} P, \mathrm{End}\, Q)^G$ for any $\mathbf{A}$-scheme X. We use the same symbol $\mathfrak{E}$ to denote the Lie algebra scheme associated to the (associative) algebra scheme $\mathfrak{E}$.

Note that $\det: \mathrm{End}\, Q \to \mathbf{C}$ is G-invariant. Choose coordinates $t, x_1 \ldots, x_m$ on $\mathfrak{E} \simeq \mathbf{A} \times \mathbf{A}^m$, and consider a section $\sum x_i A_i$ of $\mathfrak{E}$. Then $\det(\sum x_i A_i)$ is a polynomial $p(x_1, \ldots, x_m)$ in the x_i with coefficients in $\mathcal{O}(\mathbf{A})$. Consider p as a function from $\mathfrak{E}$ to $\mathbf{C}$. Then $\mathfrak{P} := p^{-1}(\mathbf{C} \setminus \{0\})$ is a group scheme representing the functor sending an $\mathbf{A}$-scheme X into $\mathrm{Mor}(X \times_{\mathbf{A}} P, \mathrm{GL}(Q))^G$. We call $\mathfrak{P}$ the *automorphism group scheme of* Θ_Q. One may realize $\mathfrak{P}$ as

$$\{(t, x_1, \ldots, x_m) \in \mathbf{A} \times \mathbf{A}^m \mid p(x_1, \ldots, x_m)(t) \neq 0\}$$

with canonical projection $\mathrm{pr} = \mathrm{pr}_1 : \mathfrak{P} \to \mathbf{A}$. The fiberwise multiplication is given by

$$(t, x_1, \ldots, x_m) . (t, x'_1, \ldots, x'_m) = (t, x''_1, \ldots, x''_m),$$

where $x_k'' = \sum_{i,j} x_i x_j' d_{ijk}(t)$.

(2.2) *Example* (compare III.2.8 and [Sch5]). – Let $G = O_2 \simeq \mathbf{C}^* \ltimes \mathbf{Z}/2$, $P = V_1$ and $Q = V_n$ in the notation of III.2.8. Let $\{u, v\}$, $\{x, y\}$ be coordinates on P and Q corresponding to the $\mathbf{C}^*$-weights 1, -1 and n, $-n$, respectively. Then $t = uv$, and $\operatorname{End} Q \simeq V_{2n} \oplus \mathbf{C} \oplus \varepsilon_1$ where $\mathbf{C}$ is the one-dimensional trivial representation of G and ε_1 its sign representation. It is easy to see that $\operatorname{Mor}(V_1, \operatorname{End} V_n)^G$ is generated by A (corresponding to $\mathbf{C}$) and B (corresponding to $V_{2n} \subseteq \operatorname{End} V_n$), where $A\begin{pmatrix} u \\ v \end{pmatrix}$ multiplies $\begin{pmatrix} x \\ y \end{pmatrix}$ by the identity matrix and $B\begin{pmatrix} u \\ v \end{pmatrix}$ multiplies it by $\begin{pmatrix} 0 & u^{2n} \\ v^{2n} & 0 \end{pmatrix}$. Now $A \circ A = A$, $A \circ B = B = B \circ A$ and $B \circ B = t^{2n} A$. Moreover, $\det(aA + bB) = a^2 - t^{2n} b^2$. Thus $\mathfrak{E} = \{ (t, a, b) \in \mathbf{A}^3 \}$ and

$$\mathfrak{P} = \{ (t, a, b) \mid a^2 - t^{2n} b^2 \neq 0 \},$$

with both schemes having fiberwise multiplication

$$(t, a, b) \,.\, (t, a', b') = (t, aa' + t^{2n} bb', ab' + a'b).$$

(2.3) Let $F_P := \pi_P^{-1}(1) \subseteq P$. Then $M := \mathfrak{P}_1 = \operatorname{Mor}(F_P, GL(Q))^G$ has Lie algebra $\mathfrak{m} := \mathfrak{E}_1 = \operatorname{Mor}(F_P, \operatorname{End} Q)^G$. In fact, M is the group of units in $\mathfrak{m}$ (considered as an Artin algebra), hence M is a connected linear algebraic group. The group $\Gamma = \mu_d$ acts as usual on $\mathbf{B}$ and via composition on M, i.e., via $m \mapsto m \circ \gamma^{-1}$. As in III.4.4, we define a group scheme $\mathfrak{M}_{\mathbf{B}}^{\Gamma}$, where $\mathfrak{M}_{\mathbf{B}}^{\Gamma}(X) = \operatorname{Mor}(X \times_{\mathbf{A}} \mathbf{B}, M)^{\Gamma}$ for any $\mathbf{A}$-scheme X.

(2.4) *Remark.* – Set $\tilde{G} := G \times \mathbf{C}^*$, and let V denote the G-module $P \oplus Q$, where $\mathbf{C}^*$ acts by scalars on the second factor. Let $\tilde{L}$ and $\tilde{\Gamma}$ correspond to $(V, \tilde{G})$ as usual (see III.3.1). Then $\tilde{\Gamma} = \Gamma$, M is a Γ-subgroup of $\tilde{L}$ and the action of Γ on M is the one given above.

(2.5) *Proposition* (cf. III.4.6). – *The canonical morphism* $\rho : \mathbf{B} *^{\Gamma} F_P \to P$ *gives rise to a morphism of group schemes* $\varphi : \mathfrak{P} \to \mathfrak{M}_{\mathbf{B}}^{\Gamma}$, *and* φ *is an isomorphism over* $\dot{\mathbf{A}}$.

Proof. – For any $\mathbf{A}$-scheme X, we have $\mathfrak{P}(X) = \operatorname{Mor}(X \times_{\mathbf{A}} P, GL(Q))^G$. Composition with ρ gives a morphism from $\mathfrak{P}(X)$ to

$$\operatorname{Mor}(X \times_{\mathbf{A}} (\mathbf{B} *^{\Gamma} F_P), GL(Q))^G \simeq \operatorname{Mor}(X \times_{\mathbf{A}} \mathbf{B}, \operatorname{Mor}(F_P, GL(Q))^G)^{\Gamma}$$
$$= \operatorname{Mor}(X \times_{\mathbf{A}} \mathbf{B}, M)^{\Gamma} = \mathfrak{M}_{\mathbf{B}}^{\Gamma}(X). \quad \blacksquare$$

(2.6) *Example* (continuation of 2.2). – We have $M \simeq \{ (a, b) \mid a^2 - b^2 \neq 0 \} \simeq (\mathbf{C}^*)^2$. Note that $\Gamma = \{ \pm 1 \}$ acts trivially on M (since A and B only involve even powers of u

and v), so that $\mathfrak{M}_{\mathbf{B}}^{\Gamma} \simeq \mathbf{A} \times \mathrm{M}$. It is easy to calculate that φ sends (t, a, b) into $(t, a^2 - (t^n b)^2) \in \{t\} \times \mathrm{M}$.

(2.7) *Theorem.* – *Let* $\mathfrak{P}$, *etc. be as above, and let* $\mathrm{E} \in \mathrm{Vec}_{\mathrm{G}}(\mathrm{P}, \mathrm{Q})$.

(1) $\mathrm{H}^1_{\mathrm{Zar}}(\dot{\mathbf{A}}, \mathfrak{P}|_{\dot{\mathbf{A}}}) = \{ * \}$, *hence* $\mathrm{E}|_{\dot{\mathrm{P}}}$ *is trivial, where* $\dot{\mathrm{P}} := \pi_{\mathrm{P}}^{-1}(\dot{\mathbf{A}})$.

(2) $\mathrm{VEC}_{\mathrm{G}}(\mathrm{P}, \mathrm{Q}) \simeq \tilde{\mathfrak{D}}\mathfrak{P} := \mathfrak{P}(\dot{\mathbf{A}}) \backslash \mathfrak{P}(\tilde{\mathbf{A}}) / \mathfrak{P}(\tilde{\mathbf{A}})$.

Proof. – Consider the pull-back $\tilde{\mathrm{E}}$ of $\mathrm{E}|_{\dot{\mathrm{P}}}$ to the d-fold cover $\dot{\mathbf{B}} \times \mathrm{F}_{\mathrm{P}} \to \dot{\mathbf{B}} *^{\Gamma} \mathrm{F}_{\mathrm{P}} \simeq \dot{\mathrm{P}}$. Then $\tilde{\mathrm{E}}$ gives rise to a principal M-bundle over $\dot{\mathbf{B}}$, which is trivial by IV.3.1(2). Thus we may assume that $\tilde{\mathrm{E}} = (\dot{\mathbf{B}} \times \mathrm{F}_{\mathrm{P}}) \times \mathrm{Q}$ is a trivial G-bundle. Now E is the quotient of $\tilde{\mathrm{E}}$ by a free G-equivariant action of Γ. We must have

$$(b, f, q)\gamma = (b\gamma, \gamma^{-1} f, \tilde{h}_\gamma^{-1}(b, f, q)), \qquad b \in \dot{\mathbf{B}}, f \in \mathrm{F}_{\mathrm{P}}, q \in \mathrm{Q}, \gamma \in \Gamma,$$

where $h_\gamma^{-1}(b) := \tilde{h}_\gamma^{-1}(b, \cdot, \cdot)$ lies in M. One easily verifies that

$$h_{\gamma\gamma'} = h_\gamma \cdot (\sigma(\gamma)(h_{\gamma'} \circ \gamma))$$

where $\sigma(\gamma)(m) = m \circ \gamma^{-1}$, $\gamma \in \Gamma$, $m \in \mathrm{M}$. Thus the h_γ give rise to an element of $\mathrm{H}^1(\Gamma, {}_\sigma\mathrm{M}(\dot{\mathbf{B}}))$, where $\sigma : \Gamma \to \mathrm{Aut}\,\mathrm{M}$ is as above. By IV.5.6, we have $\mathrm{H}^1(\Gamma, {}_\sigma\mathrm{M}(\dot{\mathbf{B}})) = \{ * \}$, and it follows that we can change the action of Γ on $\tilde{\mathrm{E}}$ by a G-automorphism so that the h_γ become the identity element of M. Hence $\mathrm{E}|_{\dot{\mathrm{P}}}$ is trivial, proving (1). Part (2) is an easy exercise. ■

Evaluation at $0 \in \hat{\mathrm{P}} := \mathrm{P} \times_{\mathbf{A}} \hat{\mathbf{A}}$ gives a homomorphism from $\mathfrak{P}(\hat{\mathbf{A}})$ to $\mathrm{GL}(\mathrm{Q})^{\mathrm{G}}$. Clearly, $\mathrm{GL}(\mathrm{Q})^{\mathrm{G}} \subseteq \mathfrak{P}(\mathbf{A})$. Thus we have

(2.8) *Proposition* (cf. VI.1.5(3)). – *There are canonical split exact sequences*

$$1 \to \mathfrak{P}(\hat{\mathbf{A}})_1 \to \mathfrak{P}(\hat{\mathbf{A}}) \to \mathrm{GL}(\mathrm{Q})^{\mathrm{G}} \to 1,$$
$$1 \to \mathfrak{P}(\tilde{\mathbf{A}})_1 \to \mathfrak{P}(\tilde{\mathbf{A}}) \to \mathrm{GL}(\mathrm{Q})^{\mathrm{G}} \to 1.$$

(2.9) *Remark.* – The fiber $\mathfrak{P}_t$ of $\mathfrak{P}$ at $t \in \mathbf{A}$ is the group of units in the algebra $\mathfrak{E}_t = \mathrm{span}\{\mathrm{A}_i(t)\}$ (see 2.1). The dimension of this algebra is independent of t, hence $\mathfrak{P} \to \mathbf{A}$ is equidimensional with connected fibers. It follows that the subgroup scheme $\mathfrak{P}^0$ of $\mathfrak{P}$ (V.1.6) actually equals $\mathfrak{P}$.

3. Moduli of vector bundles

We apply the techniques of Chapters V and VI to compute moduli and construct universal families of vector bundles. We review, more than strictly necessary, some of the results of Chapter VI since they become more transparent in the vector bundle setting.

(3.1) Let P, Q, $A_1, \ldots, A_m$, etc. be as in 2.1. Let $d_i := \deg A_i = k_i d + a_i$ as in VI.1.10. Set $A_i' := A_i|_{F_P}$. We know that $\mathrm{VEC}_G(P, Q) \simeq \tilde{D}\mathfrak{P}$ (2.7(2)), and we have the morphism $\varphi : \mathfrak{P} \to \mathfrak{M}_{\mathbf{B}}^{\Gamma}$ which is an isomorphism over $\dot{\mathbf{A}}$ (2.5). Let $\varphi_* : \mathfrak{P}(\hat{\mathbf{A}}) \xrightarrow{\sim} M(\hat{\mathbf{B}})^{\Gamma}$ denote the induced isomorphism. Replacing $\mathfrak{P}$ by $\mathfrak{E}$ and M by $\mathfrak{m}$, we similarly have an isomorphism $\varphi_{\#} : \mathfrak{E}(\hat{\mathbf{A}}) \xrightarrow{\sim} \mathfrak{m}(\hat{\mathbf{B}})^{\Gamma}$.

We obtain an exponential map $\exp : \mathfrak{E}(\hat{\mathbf{A}})_r \to \mathfrak{P}(\hat{\mathbf{A}})_r$, $r \geqslant 1$ in the obvious way from $\exp : \mathrm{End}\, Q \to GL(Q)$. It is easy to establish the following analogues of previous results:

(3.2) *Proposition* (cf. VI.1.11 and VI.1.14(1)). – (1) *The* A_i *are an* $\mathcal{O}(\mathbf{A})$*-module basis of* $\mathfrak{E}(\mathbf{A})$, *the* A_i' *are a basis of* $\mathfrak{m}$, *and the* $s^{a_i} A_i'$ *are an* $\mathcal{O}(\mathbf{A})$*-module basis of* $\mathfrak{m}(\mathbf{B})^{\Gamma}$.

(2) $\varphi_{\#} A_i = t^{k_i}(s^{a_i} A_i')$, *and* $\varphi_{\#} : \mathfrak{E}(\mathbf{A}) \to \mathfrak{m}(\mathbf{B})^{\Gamma}$ *is homogeneous of degree* 0 *and is an injection of free* $\mathcal{O}(\mathbf{A})$*-modules.*

(3) $\varphi_{\#} : \mathfrak{E}(\hat{\mathbf{A}}) \to \mathfrak{m}(\hat{\mathbf{B}})^{\Gamma}$ *is an injection of free* $\mathcal{O}(\hat{\mathbf{A}})$*-modules.*

(4) *For every* $r \geqslant 1$ *there is a commutative diagram*

$$\begin{array}{ccc}
\mathfrak{P}(\hat{\mathbf{A}})_r & \xrightarrow[\varphi^*]{\subset} & M(\hat{\mathbf{B}})_r^{\Gamma} \\
\exp \uparrow \wr & & \wr \uparrow \exp \\
\mathfrak{E}(\hat{\mathbf{A}})_r & \xrightarrow[\varphi\#]{\subset} & \mathfrak{m}(\hat{\mathbf{B}})_r^{\Gamma}
\end{array}$$

(3.3) *Proposition* (cf. VI.2.1). – *We have* $\tilde{D}\mathfrak{P} \simeq M(\mathbf{B})^{\Gamma} \backslash M(\tilde{\mathbf{B}})^{\Gamma} / \varphi_* \mathfrak{P}(\tilde{\mathbf{A}})$.

Proof. – We have $M(\tilde{\dot{\mathbf{B}}})^{\Gamma} = M(\dot{\mathbf{B}})^{\Gamma} M(\tilde{\mathbf{B}})^{\Gamma}$ (see V.2.8). As in VI.2.1, one obtains the desired result. ■

Let M', $\tau : M \to M/M'$ and $\tau_* : M(\tilde{\mathbf{B}})_1^{\Gamma} \to (M/M')(\tilde{\mathbf{B}})_1^{\Gamma}$, etc. be defined as in VI.2.2 and VI.2.3.

(3.4) *Theorem.* – (1) (cf. VI.2.4) *There is a canonical isomorphism*

$$\tilde{D}\mathfrak{P} \simeq (M/M')(\tilde{\mathbf{B}})_1^{\Gamma} / \tau_* \varphi_* \mathfrak{P}(\tilde{\mathbf{A}})_1.$$

Moreover, $\tilde{D}\mathfrak{P} \simeq \mathrm{VEC}_G(P, Q)$ *is trivial if and only if* $\mathfrak{k} + \mathfrak{m}' = \mathfrak{m}$, *where* $\mathfrak{k} = \mathrm{span}\{A_i' : d_i \leqslant d\}$.

(2) (cf. VI.2.7) *The logarithm gives an isomorphism*

$$\tilde{D}\mathfrak{P} \simeq (\mathfrak{m}/\mathfrak{m}')(\tilde{\mathbf{B}})_1^{\Gamma} / \tau_{\#} \varphi_{\#} \mathfrak{E}(\tilde{\mathbf{A}})_1,$$

and, as in VI.2.5, *this induces a structure of vector group on* $\tilde{D}\mathfrak{P} \simeq \mathrm{VEC}_G(P, Q)$.

(3) (cf. VI.2.11(2)) *Holomorphically,* $\mathfrak{P}$ *has the decomposition property, i.e., every element of* $\mathrm{Vec}_G(P, Q)$ *is holomorphically trivial.*

(4) (cf. VI.2.12) *There is a* G-*vector bundle* $\mu: \mathscr{B} \to P \times \mathrm{VEC}_G(P, Q)$ *such that, for every* $E \in \mathrm{Vec}_G(P, Q)$, the vector bundle $\mu^{-1}(P \times [E])$ *is an element of* $\mathrm{Vec}_G(P, Q)$ *isomorphic to* E.

(3.5) *Remark.* — The results in VI.2.6-VI.2.8 on computing dimensions of moduli spaces hold for $\tilde{D}\mathfrak{P}$. Just replace $\mathfrak{A}$ by $\mathfrak{P}$, L by M, etc.

(3.6) Let $(V, \tilde{G}) = (P \oplus Q, G \times \mathbf{C}^*)$ be as in 2.4. We compare $\mathrm{VEC}_G(P, Q)$ with $\mathscr{M}_{V, \mathbf{A}}$. Let $[E] \in \mathrm{VEC}_G(P, Q)$. We may consider E as a $\tilde{G}$-variety, so [E] gives rise to $\{E\} \in \mathscr{M}_{V, \mathbf{A}}$. Note that $\{E\} = \{\gamma^* E\}$, $\gamma \in \Gamma$, where $\gamma^* E$ denotes the pull-back of E by $\gamma: P \to P$. Thus we have a natural morphism $\lambda: \mathrm{VEC}_G(P, Q)/\Gamma \to \mathscr{M}_{V, \mathbf{A}}$.

Let $\mathfrak{A}$ denote the automorphism group scheme of $(V, \tilde{G})$ and $\mathfrak{A}_P$ that of (P, G). We consider $\mathfrak{A}_P$ as the subgroup scheme of $\mathfrak{A}$ of elements acting by the identity on Q. Then $\mathfrak{A}_P$ normalizes $\mathfrak{P} \subseteq \mathfrak{A}$, so we have a semidirect product $\mathfrak{A}_P \ltimes_{\mathbf{A}} \mathfrak{P} \subseteq \mathfrak{A}$.

(3.7) *Theorem.* — (1) $\mathfrak{A} = \mathfrak{A}_P \ltimes_{\mathbf{A}} \mathfrak{P}$.

(2) $\lambda: \mathrm{VEC}_G(P, Q)/\Gamma \hookrightarrow \mathscr{M}_{V, \mathbf{A}}$ *is injective.*

(3) *If* (P, G) *is rigid* (*see VI*.3.1), *then* λ *is an isomorphism.*

Proof. — Let X be an **A**-scheme. Then $\mathfrak{A}(X)$ is the group of $\tilde{G}$-automorphisms of the X-scheme $(X \times_{\mathbf{A}} P) \times Q$. Since $\mathbf{C}^*$ only acts on Q, one easily sees that $\mathfrak{A}(X) = \mathfrak{A}_P(X) \ltimes \mathrm{Mor}(X \times_{\mathbf{A}} P, GL(Q))^G = \mathfrak{A}_P(X) \ltimes \mathfrak{P}(X)$, proving (1). Clearly, we also have $\mathfrak{A}^0 \simeq \mathfrak{A}_P^0 \ltimes \mathfrak{P}$, and using VI.2.7 and 3.4(2) one easily sees that $\tilde{D}\mathfrak{A}^0 \simeq \tilde{D}\mathfrak{A}_P^0 \times \tilde{D}\mathfrak{P}$. By V.1.9, $\tilde{D}\mathfrak{A} \simeq (\tilde{D}\mathfrak{A}_P^0 \times \tilde{D}\mathfrak{P})/\Gamma$, and (2) and (3) follow. ■

4. Additive structure

By Theorem 3.4(2) we have a vector group structure on $\mathrm{VEC}_G(P, Q)$. We relate this structure to the Whitney sum of vector bundles.

(4.1) Let P, Q, M, F_P, etc. be as in section 3. Let H be a principal isotropy group of (P, G). Then $F_P \simeq G *^H W$ where $\mathcal{O}(W)^H = \mathbf{C}$, and $M = \mathrm{Mor}(F_P, GL(Q))^G \simeq \mathrm{Mor}(W, GL(Q))^H$ only depends upon H, W and Q. Let $\psi: M \to GL(Q)^H$ denote evaluation at $0 \in W$. Note that ψ depends upon some choices, but that its kernel does not.

(4.2) *Lemma.* — *The homomorphism* ψ *is surjective, and* $\mathrm{Ker}\,\psi = M_u$.

Proof. — One can deduce the lemma from III.3.3, but here is a direct proof: Clearly ψ is surjective. Now $\mathfrak{m} \simeq (\mathcal{O}(W) \otimes_{\mathbf{C}} \operatorname{End} Q)^H$ is a graded Artin algebra, via the multiplication in End Q. The elements of strictly positive degree form a nilpotent subalgebra $\mathfrak{m}_+$, and clearly $I + \mathfrak{m}_+ = \operatorname{Ker} \psi$ is unipotent and normal in M. ■

(4.3) As H-module, $Q \simeq \bigoplus_{i=1}^{p} n_i W_i$ where the W_i are irreducible and pairwise non-isomorphic. Thus $GL(Q)^H \simeq \prod_{i=1}^{p} GL_{n_i} \simeq M/M_u$ and $M/M' \simeq (\mathbf{C}^*)^p$.

(4.4) *Proposition* (cf. VI.3.12). — *Suppose that every* H-*isotypic component in* Q *is* G-*stable* (*e.g.*, H *is central in* G). *Then* $VEC_G(P, Q) = \{*\}$. *In particular, if* $H = \{e\}$ (*i.e.*, (P, G) *is semifree*) *or* G *is a torus, then* $VEC_G(P, Q) = \{*\}$.

Proof. — For each i, the isotypic component $n_i W_i$ is G-stable, hence $\mathbf{C}^* \simeq \operatorname{Cent}(GL_{n_i}) \subseteq \operatorname{Cent}(GL(n_i W_i)^G)$. Thus the composition

$$GL(Q)^G \hookrightarrow GL(Q)^H \to M \twoheadrightarrow M/M'$$

is surjective, so we have $\operatorname{End}(Q)^G + \mathfrak{m}' = \mathfrak{m}$. The proposition now follows from Theorem 3.4(1) since $\operatorname{End}(Q)^G$ consists of degree 0 elements of $\mathfrak{m}$. ■

(4.5) Let Q_1 and Q_2 be G-modules. We compare $VEC_G(P, Q_1 \oplus Q_2)$ with $VEC_G(P, Q_1)$ and $VEC_G(P, Q_2)$: Set $M_i = \operatorname{Mor}(W, GL(Q_i))^H$, $i = 1, 2$, and let $\tilde{M}$ denote $\operatorname{Mor}(W, GL(Q_1 \oplus Q_2))^H$. The inclusion $GL(Q_1) \times GL(Q_2) \subseteq GL(Q_1 \oplus Q_2)$ induces a canonical morphism $\eta : M_1 \times M_2 \to \tilde{M}$. Let $A' \in \tilde{\mathfrak{m}}$. Then we can write A' uniquely in the form

$$\begin{pmatrix} A'_{11} & A'_{12} \\ A'_{21} & A'_{22} \end{pmatrix}, \qquad A'_{ij} \in \operatorname{Mor}(W, \operatorname{Hom}(Q_j, Q_i))^H.$$

Write $Q_1 = \bigoplus_{i=1}^{p} n_i W_i$, $Q_2 = \bigoplus_{i=1}^{p} m_i W_i$ as H-modules, where $n_i + m_i > 0$, $i = 1, \ldots, p$.

(4.6) *Lemma.* — *Let* Q_1, Q_2, A', A'_{ij}, *etc. be as above.*

(1) *The morphism* $\eta : M_1 \times M_2 \to \tilde{M}$ *induces a surjection of* $M_1 \times M_2$ *onto* $\tilde{M}/\tilde{M}'$.

(2) *If* $n_i \neq 0$, $i = 1, \ldots, p$, *then the canonical map* $M_1/M'_1 \to \tilde{M}/\tilde{M}'$ *is an isomorphism.*

(3) $\tilde{\mathfrak{m}}'$ *contains all elements* B' *of the form*

$$\begin{pmatrix} 0 & B'_{12} \\ B'_{21} & 0 \end{pmatrix}.$$

In particular,

$$A' \in \begin{pmatrix} A'_{11} & 0 \\ 0 & A'_{22} \end{pmatrix} + \tilde{\mathfrak{m}}'.$$

(4) *Suppose that* $Q_1 = Q_2 = Q$. *Then* $\tilde{\mathfrak{m}}'$ *contains all elements* B' *of the form*

$$\begin{pmatrix} B'_{11} & B'_{12} \\ B'_{21} & -B'_{11} \end{pmatrix}.$$

In particular,

$$A' \in \begin{pmatrix} A'_{11} + A'_{22} & 0 \\ 0 & 0 \end{pmatrix} + \tilde{\mathfrak{m}}'.$$

Proof. — Let $M_{1,u} \subseteq M'_1$, etc. denote the unipotent radical of M_1, etc. We have a commutative diagram

$$\begin{array}{ccc} M_1/M_{1,u} \times M_2/M_{2,u} & \xrightarrow{\tilde{\eta}} & \tilde{M}/\tilde{M}_u \\ \downarrow \qquad \downarrow & & \downarrow \\ \prod GL_{n_i} \times \prod GL_{m_i} & \rightarrow & \prod GL_{n_i+m_i} \end{array}$$

where η induces $\tilde{\eta}$. Parts (1) and (2) are now clear.

The subspaces $\mathrm{Mor}(W, \mathrm{Hom}(Q_i, Q_j))^H \subseteq \tilde{\mathfrak{m}}$ are nilpotent for $i \neq j$, hence they lie in $\tilde{\mathfrak{m}}'$, and we have (3). If $Q_1 = Q_2$, then the image of the subalgebra $\{(C, -C) \in \mathfrak{m}_1 \oplus \mathfrak{m}_2\}$ in $\tilde{\mathfrak{m}}/\tilde{\mathfrak{m}}_u$ lies in $\oplus\, \mathfrak{sl}_{2n_i}$, and we have (4). ■

(4.7) From 3.4 we have isomorphisms

$$\mathrm{VEC}_G(P, Q) \simeq (\mathfrak{m}/\mathfrak{m}')(\tilde{\mathbf{B}})_1^\Gamma / \tau_{\#}\, \varphi_{\#}\, \mathfrak{E}(\tilde{\mathbf{A}})_1 \simeq (\mathbf{C}^p, +).$$

We give $\mathrm{VEC}_G(P, Q)$ the additive structure carried over from that of $(\mathbf{C}^p, +)$, and we denote addition of isomorphism classes by +. The next result shows that + comes from Whitney sum, denoted ⊕.

(4.8) *Theorem.* — *Let* Q, Q_1 *and* Q_2 *be* G-*modules.*

(1) *Whitney sum induces an epimorphism of vector groups*

$$\mathrm{WS}: \mathrm{VEC}_G(P, Q_1) \times \mathrm{VEC}_G(P, Q_2) \rightarrow \mathrm{VEC}_G(P, Q_1 \oplus Q_2).$$

(2) *If* $\mathrm{Hom}(Q_1, Q_2)^H = \{0\}$, *then* WS *is an isomorphism.*

(3) *If every simple* H-*submodule of* Q_2 *also occurs in* Q_1, *then the map* $\mathrm{VEC}_G(P, Q_1) \to \mathrm{VEC}_G(P, Q_1 \oplus Q_2)$ *sending* [E] *into* $[E \oplus \Theta_{Q_2}]$ *is surjective.*

(4) *Let* $E_1, E_2 \in \mathrm{Vec}_G(P, Q)$. *Then* $E_1 \oplus E_2 \simeq E_3 \oplus \Theta_Q$ *where* $[E_3] := [E_1] + [E_2]$.

(5) *The map* $\mathrm{VEC}_G(P, Q) \to \mathrm{VEC}_G(P, Q \oplus Q)$ *sending* [E] *into* $[E \oplus \Theta_Q]$ *is bijective.*

Proof. – Let M_1, M_2, η, etc. be as in 4.5-4.6. The inclusion $GL(Q_1) \times GL(Q_2) \subseteq GL(Q_1 \oplus Q_2)$ induces a homomorphism of group schemes $\mathfrak{P}_1 \times_A \mathfrak{P}_2 \to \tilde{\mathfrak{P}}$, where $\mathfrak{P}_1$, etc. is the automorphism group scheme of Θ_{Q_1}, etc. The resulting homomorphism

$$\mathrm{VEC}_G(P_1, Q_1) \times \mathrm{VEC}_G(P_2, Q_2) \to \mathrm{VEC}_G(P, Q)$$

is just Whitney sum. If $\mathrm{Hom}(Q_1, Q_2)^H = \{0\}$, then $\tilde{\mathfrak{P}} \simeq \mathfrak{P}_1 \times_A \mathfrak{P}_2$, proving (2). The canonical inclusion.

$$\mathrm{Mor}(P, \mathrm{End}\, Q_1)^G \oplus \mathrm{Mor}(P, \mathrm{End}\, Q_2)^G \to \mathrm{Mor}(P, \mathrm{End}(Q_1 \oplus Q_2))^G$$

induces a morphism of Lie algebra schemes $\mathfrak{E}_1 \times_A \mathfrak{E}_2 \to \tilde{\mathfrak{E}}$, where $\mathfrak{E}_1$, etc. is the endomorphism scheme of Θ_{Q_1}, etc. One now obtains (1), (3), (4) and (5) from 4.6 using the isomorphism of 3.4(2). ■

5. Examples

We present several examples where $\mathrm{VEC}_G(P, Q) \neq \{*\}$ and examples of non-linearizable actions of simple groups. In particular, we give proofs of the results announced in [Sch5].

Let O_2, V_j, etc. be as in Example III.2.8.

(5.1) *Theorem* (see VII.2.2 and VII.2.6). – *Let* $G = O_2$ *and* $E \in \mathrm{Vec}_G(V_1, V_n)$, $n \geq 1$. *Then*

(1) $\mathrm{VEC}_G(V_1, V_n) \simeq \mathbf{C}^{n-1}$.

(2) $E \oplus \Theta_{V_1}$ *and* $E \oplus \Theta_{\mathbf{C}}$ *are trivial, where* G *acts trivially on* $\mathbf{C}$.

(3) E, *considered as a* $\mathbf{C}^*$-*vector bundle, is trivial.*

(5.2) *Theorem.* – *Let* $G = O_2$ *and* $E \in \mathrm{Vec}_G(V_2, V_n)$, *n odd*, $n \geq 1$. *Then*

(1) $\mathrm{VEC}_G(V_2, V_n) \simeq \mathbf{C}^{(n-1)/2}$.

(2) $E \oplus \Theta_{V_1}$ *is trivial.*

(3) E, *considered as a* $\mathbf{C}^*$-*vector bundle, is trivial.*

(4) *Whitney sum with* Θ_{V_2} *induces an isomorphism of* $\mathrm{VEC}_G(V_2, V_n)$ *and* $\mathrm{VEC}_G(V_2, V_2 \oplus V_n)$.

Proof of (5.1)-(5.2). — Let $P = V_2$, $Q = V_n$ and $n = 2m+1$ be as in 5.2. The principal isotropy group of (P, G) is the Klein 4-group $H = (\mathbf{Z}/2)^2 \subseteq G$, and $(Q, H) \simeq W_1 \oplus W_2$ where W_1 and W_2 are non-isomorphic one-dimensional representations of H. Thus $M = GL(Q)^H \simeq (\mathbf{C}^*)^2$. It is easy to compute that $\mathfrak{E}(A) = \mathrm{Mor}(P, \mathrm{End}\, Q)^G$ has homogeneous generators of degrees 0 and $2m+1$, so that $\mathrm{VEC}_G(P, Q) \simeq (\mathbf{C}^m, +)$ (see VI.2.6-VI.2.8), proving 5.2(1). Now the representation $(Q = V_{2m+1}, H)$ is independent of m, and $\mathrm{VEC}_G(P, V_1) = \{*\}$. It follows from 4.8(3) that direct sum with Θ_Q gives a surjection from $\mathrm{VEC}_G(P, V_1)$ onto $\mathrm{VEC}_G(P, Q \oplus V_1)$, hence we have 5.2(2). Part 5.2(3) follows from 4.4. Now $\{\pm 1\} \subset \mathbf{C}^* \subset O_2$ acts non-trivially on Q and trivially on P, and 5.2(4) follows from 4.8(2).

Let $P = V_1$, $Q = V_n$, etc. be as in 5.1. In the notation used above we have $H = \mathbf{Z}/2$ and $(Q, H) = \mathbf{C} \oplus \varepsilon_1$ where ε_1 is the sign representation of H. Thus $M = (\mathbf{C}^*)^2$. One computes that $\mathfrak{E}(A)$ has homogeneous generators of degrees 0 and $2n$. Now 5.1(1) etc. follow as above. ■

(5.3) *Proposition.* — (1) *Let* K *be a finite group and* U *a reflection representation of* K *with* $U^K = (0)$. *Then* $\mathrm{Aut}(U)^K = GL(U)^K$.

(2) *Let* V *be a stable* G-*module, i.e.,* V *contains an open dense subset of closed orbits. Let* H *be a principal isotropy subgroup of* (V, G) (*see* II.1.2), *and let* $(U, K) = (V^H, N_G(H)/H)$. *If* K *is finite and* (U, K) *is a non-trivial irreducible reflection representation, then* $\mathrm{Aut}(V)^G = GL(V)^G = \mathbf{C}^*$.

(3) Let $(V, G) = (\mathbf{C}^n, O_n)$, $n \geq 1$, *or* (Lie(G), G) *where* G *is simple. Then* $\mathrm{Aut}(V)^G = \mathbf{C}^*$.

Proof. — Let $\varphi \in \mathrm{Aut}(U)^K$ where (U, K) is as in (1). For each reflection σ_i, let l_i denote a linear function which vanishes on the hyperplane fixed by σ_i. Since φ commutes with σ_i for all i, $\varphi^* l_i = \lambda_i l_i$ for some $\lambda_i \in \mathbf{C}^*$. The l_i generate U^* (since $U^K = (0)$), hence φ is linear.

Let (V, G) be as in (2). Then $G \cdot V^H$ is dense in V, hence each $\varphi \in \mathrm{Aut}(V)^G$ is determined by its restriction $\varphi|_{U = V^H} \in \mathrm{Aut}(U)^K$. By (1), $\mathrm{Aut}(U)^K = GL(U)^K$ which equals $\mathbf{C}^*$ by Schur's lemma. It follows that φ is just scalar multiplication by an element of $\mathbf{C}^*$.

The representations (V, G) in (3) satisfy the hypotheses of (2). ■

(5.4) *Corollary.* — (1) *There are families of non-linearizable actions of* $O_2 \times \mathbf{C}^*$ *on* $\mathbf{A}^4$ *with one-dimensional quotient.*

(2) *There are families of non-linearizable actions of* O_2 *on* $\mathbf{A}^4$.

Proof. – Use 5.2, 5.3(3) and 1.2(1). ■

(5.5) *Example.* – Let $G=O_2$, $P=V_2$ and $Q=V_3$. Let u, v, x, y be coordinate functions on $P\oplus Q$ corresponding to the weights 2, -2, 3, -3, respectively, so that $t=uv$. Consider the following section of $\mathfrak{P}(\tilde{A})$:

$$\left[\begin{pmatrix}u\\v\end{pmatrix},\begin{pmatrix}x\\y\end{pmatrix}\right]\mapsto\left[\begin{pmatrix}u\\v\end{pmatrix},\begin{pmatrix}1 & t^{-1}u^3\\ t^{-1}v^3 & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}\right].$$

The corresponding vector bundle E can be trivialized as a $\mathbf{C}^*$-vector bundle. Thus we can find an isomorphism $E\simeq\Theta_Q$ as $\mathbf{C}^*$-vector bundle, where the action of $\mathbf{Z}/2$ transforms to one which is linear on the fibers of Θ_Q, but not constant. One can compute that an explicit such action is:

$$\left[\begin{pmatrix}u\\v\end{pmatrix},\begin{pmatrix}x\\y\end{pmatrix}\right]\mapsto\left[\begin{pmatrix}v\\u\end{pmatrix},\begin{pmatrix}1+t+t^2 & -v^3\\ u^3 & 1-t\end{pmatrix}\begin{pmatrix}y\\x\end{pmatrix}\right].$$

By 1.2(1) (or 1.2(2) and 5.2(4)), this action of O_2 is not linearizable as an action on $\mathbf{A}^4$. Recall that by VI.3.2(5) there are no examples of non-linearizable actions in dimension 3 arising from vector bundles.

Let R_n denote the SL_2-module of binary forms of degree n. For n even, the SL_2-action descends to an action of $SO_3=SL_2/\{\pm 1\}$. Recall that $R_2\simeq\mathfrak{sl}_2$ and that $t=\det:\mathfrak{sl}_2\to\mathbf{A}$ is of degree $d=2$.

(5.6) *Theorem.* – *Let* $G=SL_2$, $n\geqslant 1$.

(1) $VEC_G(R_2,R_n)\simeq\mathbf{C}^p$, *where* $p=[(n-1)^2/4]$.

(2) *Whitney sum with* Θ_{R_2} *gives a surjection*

$$S_n: VEC_G(R_2,R_n)\to VEC_G(R_2,R_2\oplus R_n).$$

(3) *If n is odd, then* S_n *is an isomorphism.*

(4) *If* $n\geqslant 6$, *then* $\operatorname{Im} S_n=VEC_G(R_2,R_2\oplus R_n)$ *is non-trivial.*

(5.7) *Corollary.* – (1) *There are non-linearizable actions of* SL_2 *on* $\mathbf{A}^7$ *and of* SO_3 *on* $\mathbf{A}^{10}$.

(2) *There are families of non-linearizable actions of* SL_2 *on* $\mathbf{A}^{2m+1}$, $m\geqslant 4$.

Proof of (5.6)-(5.7). – Parts (1), (2) and (3) of 5.6 use no new techniques, and we leave them to the reader. We need to establish (4) (with n even), so consider the case $Q=Q_1\oplus Q_2=R_2\oplus R_{2m}$, $m\geqslant 1$. Note that $-I\in SL_2$ acts trivially, so we are actually considering SO_3-vector bundles. Now a principal isotropy group of R_2 is a maximal torus $\mathbf{C}^*$, and the weights of R_{2m} relative to $\mathbf{C}^*\subseteq SO_3$

are m, $m-1, \ldots, -m$. Thus, in the notation of 4.5, $M_1 \simeq (C^*)^3$, $M_2 \simeq (C^*)^3 \times (C^*)^{2m-2}$ and $\tilde{M} \simeq (GL_2)^3 \times (C^*)^{2m-2}$, where M_1 and $(C^*)^3 \subseteq M_2$ map into $(GL_2)^3 \subseteq \tilde{M}$. The copy of $(C^*)^{2m-2}$ in M_2 maps isomorphically onto $\tilde{M}/(GL_2)^3$. Now minimal homogeneous generators A_i of $\mathrm{Mor}(R_2, \mathrm{End}\, R_{2m})^G$ have degree $d_i := i$, $0 \leqslant i \leqslant 2m$. The only way that all the elements of $\mathrm{VEC}_G(P, P \oplus Q)$ could be trivial is if the restrictions $A_i' = A_i|_{F_P}$, $i \leqslant d = 2$, map onto the Lie algebra of $\tilde{M}/(GL_2)^3$ (see 3.4 (1)). This can only happen if $2m-3 \leqslant 2$, hence $\mathrm{VEC}_G(P, Q_1 \oplus Q_2) \neq \{*\}$ as soon as $2m-3 > d = 2$, i.e., as soon as $m \geqslant 3$.

Part (1) of the corollary follows from 5.6 using 1.2(2). Part(2) results from 5.6(1), 5.3 and 1.2(1) with $H = \{\pm 1\} \subseteq SL_2$. ■

(5.8) *Remark.* — Let G be simple with non-trivial center C. Then Knop's construction ([Kn], see I.2.9(2)) shows that one may choose G-modules Q with $Q^C = (0)$ such that $\mathrm{VEC}_G(\mathrm{Lie}(G), Q)$ contains families of arbitrarily large dimension. Applying 1.2(1) and 5.3(3) one obtains families of non-trivial actions of G on affine space. One easily extends these results to the case of semisimple groups G with non-trivial center.

In Tables III a and III b below we list triples (G, P, Q) for which (P, G) is rigid, Q is irreducible, $\mathrm{VEC}_G(P, P) = \{*\}$ and $\mathrm{VEC}_G(P, P \oplus Q) \neq \{*\}$ (hence $\mathrm{VEC}_G(P, Q) \neq \{*\}$). Using 1.2(1) and 1.2(2) we then obtain examples of non-linearizable actions of $G \times C^*$ and G on affine space. In each case one has to argue as in 5.6(4) to establish that $\mathrm{VEC}_G(P, P \oplus Q) \neq \{*\}$. The tables, together with [Sch1], contain all the information needed for verification. Some sample verifications are given below. In each case, the fiber $F_P = \pi_P^{-1}(1)$ is isomorphic to G/H where H is a principal isotropy group of (P, G). In Table III b we list the decomposition of P and Q (as H-modules). From this decomposition one can determine the groups $M_2 := \mathrm{Mor}(F_P, GL(Q))^G \simeq GL(Q)^H$ and $\tilde{M} := \mathrm{Mor}(F_P, GL(P \oplus Q))^G \simeq GL(P \oplus Q)^H$ (so $P = Q_1$ and $Q = Q_2$ in the notation of 4.5). In Table III a, an entry $(A \subseteq B)^2 \times C$, etc. is shorthand for $A \times A \times C \subseteq B \times B \times C$, etc. To compute the generators of $\mathrm{Mor}(P, \mathrm{End}\, Q)^G$ one needs to know End Q as a G-representation, or at least that portion of End Q which occurs in $\mathcal{O}(P)$. In Table III b we list either the decomposition of End Q, or only those components which occur in $\mathcal{O}(P)$ (entries 5, 6, 6′, 7 and 10). Knowledge of the decomposition of $\mathcal{O}(P)$ as an $\mathcal{O}(P)^G$-module then allows one to compute the generators of $\mathrm{Mor}(P, \mathrm{End}\, Q)^G$. We list their degrees in Table III a and also the degree d of the homogeneous generator of $\mathcal{O}(P)^G$. An entry $k\langle l \rangle$ denotes k invariants homogeneous of degree l, and θ_j denotes a trivial representation of dimension j. We have:

(5.9) *Theorem.* — *Let* G *be a simple classical group, a spin group,* G_2, E_6 *or* E_7. *Then* G *has a non-linearizable faithful action on* $\mathbf{A}^n$ *for some n.*

(5.10) *Example.* — Consider entry 4, where $G=\mathbf{A}_m$, $m\geqslant 3$, $P=\varphi_1+\varphi_m$ and $Q=\varphi_1^k$. From the decompositions $(Q,H)=\theta_1+\varphi_1+\bigoplus_{i=2}^{k}\varphi_1^i$ and $(P,H)=\theta_2+\varphi_1+\varphi_{m-1}$, one sees that $M_2=(\mathbf{C}^*)^2\times(\mathbf{C}^*)^{k-1}\subseteq\tilde{M}=GL_3\times GL_2\times\mathbf{C}^*\times(\mathbf{C}^*)^{k-1}$, where the image of $M_1\simeq(\mathbf{C}^*)^3$ lands in the first three factors of $\tilde{M}$. As in the argument of 5.6(4), to show that $VEC_G(P,P\oplus Q)\neq\{*\}$, it suffices to show that the number of generators of $\mathrm{Mor}(P,\mathrm{End}\,Q)^G$ of degree $\leqslant 2$ is less than $k-1$. Now the covariants of type $\varphi_1^i\varphi_m^i$ in $\mathcal{O}(P)$ have multiplicity one (see III.2.6), and the generators clearly occur in the copies of $\varphi_1^i\varphi_m^i\subseteq S^i\varphi_1\otimes S^i\varphi_m\subseteq S^{2i}(\varphi_1+\varphi_m)\subseteq\mathcal{O}(P)$. Thus the generators have degrees $0,2,\ldots,2k$ and $VEC_G(P,P\oplus Q)\neq\{*\}$ as soon as $k-1\geqslant 3$.

(5.11) *Example.* — Consider entry 11, where $G=E_7$, $P=\varphi_1$ and $Q=\varphi_1^4$. From the decompositions $(Q,H)=\theta_5+4(\varphi_1+\varphi_5)+\ldots+(\varphi_1^4+\varphi_1^3\varphi_5+\ldots+\varphi_5^4)$ and $(P,H)=\theta_2+\varphi_1+\varphi_5$ one sees that $M_2=GL_5\times(GL_4)^2\times\ldots\times(GL_1=\mathbf{C}^*)^5\subseteq\tilde{M}=GL_7\times(GL_5)^2\times(GL_3)^3\times\ldots\times(\mathbf{C}^*)^5$. The image of $M_1\simeq GL_2\times(\mathbf{C}^*)^2$ lands in the factors $GL_7\times(GL_5)^2$ of $\tilde{M}$. Dividing $\tilde{M}$ by these factors and also by $\tilde{M}'$ we have a surjection of M_2 onto the quotient $(\mathbf{C}^*)^{12}$. From [Sch1, Table 5 b] there are 8 generators in degree $\leqslant 4$ of the covariants corresponding to elements of $\mathrm{End}\,Q$. Thus $VEC_G(P,P\oplus Q)\neq\{*\}$.

(Added in Proof:) One can replace $Q=\varphi_1^4$ by φ_1^k for any $k\geqslant 4$. One needs to verify the analogues of the formulas in Table III b for End Q and (Q, H). E. Elashvili (private communication) has verified the formula for End Q. Both required formulas follow from Littelmann [Li].

TABLE III a

	G	P	Q	$M_2 \subseteq \tilde{M}$	d	Degrees
1	A_1	φ_1^2	φ_1^{2k+1} $k \geqslant 1$	$(C^*)^{2k+2} \subseteq (C^*)^{2k+5}$	2	$0, 1, \ldots, 2k+1$
2	SO_3	φ_1^2	φ_1^{2k} $k \geqslant 3$	$(C^* \subseteq GL_2)^3 \times (C^*)^{2k-2}$	2	$0, 1, \ldots, 2k$
3	A_2	$\varphi_1 + \varphi_2$	φ_1^k $k \geqslant 4$	$(C^* \subseteq GL_3)^2 \times (C^*)^{k-1}$	2	$0, 2, \ldots, 2k$
4	A_m $m \geqslant 3$	$\varphi_1 + \varphi_m$	φ_1^k $k \geqslant 4$	$(\{e\} \subseteq C^*) \times (C^* \subseteq GL_2)$ $\times (C^* \subseteq GL_3) \times (C^*)^{k-1}$	2	$0, 2, \ldots, 2k$
5	SO_m $m \geqslant 5$	φ_1	φ_1^k $k \geqslant 4$	$(C^* \subseteq GL_2)^2 \times (C^*)^{k-1}$	2	$0, 2, \ldots, 2k$
6	B_m $m \geqslant 2$	φ_1	$\varphi_1^k \varphi_m$ $k \geqslant 1$	$(\{e\} \subseteq C^*)^2 \times (C^*)^{2k+2}$	2	$0, 1, \ldots, 2k+1$
7	D_m $m \geqslant 3$	φ_1	$\varphi_1^k \varphi_m$ $k \geqslant 2$	$(\{e\} \subseteq C^*)^2 \times (C^*)^{k+1}$	2	$0, 2, \ldots, 2k$
8	C_m $m \geqslant 2$	$2\varphi_1$	φ_1^k $k \geqslant 7$	$(GL_{k+1} \subseteq GL_{k+5}) \times$ $(GL_k \subseteq GL_{k+2}) \times$ $\prod_{j=1}^{k-1} GL_j$	2	$(j+1)^2 \langle 2j \rangle$, $0 \leqslant j \leqslant k$
9	G_2	φ_1	φ_1^k $k \geqslant 3$	$(C^* \subseteq GL_2)^3 \times$ $(C^*)^{\binom{k+2}{2}-3}$	2	$\left[\frac{k+2-i}{2}\right] \langle k \pm i \rangle$, $0 \leqslant i \leqslant k$
10	E_6	φ_1	φ_1^k $k \geqslant 2$	$(C^* \subseteq GL_2)^2 \times (C^*)^{k-1}$	3	$(k+1) \langle 2k \rangle$
11	E_7	φ_1	φ_1^4	$(GL_5 \subseteq GL_7) \times$ $(GL_4 \subseteq GL_5)^2 \times$ $(GL_3)^3 \times (GL_2)^4 \times (C^*)^5$	4	$0, 2 \langle 2 \rangle, 5 \langle 4 \rangle$ and higher degree

TABLE III b

	G	End Q	H	(Q, H)	(P, H)
1	A_1	$\bigoplus_{i=0}^{2k+1} \varphi_i^{2i}$	C^*	$\bigoplus_{i=0}^{k} \nu_{\pm(2i+1)}$	$\nu_2+\nu_0+\nu_{-2}$
2	SO_3	$\bigoplus_{i=0}^{2k} \varphi_1^{2i}$	C^*	$\bigoplus_{i=-k}^{k} \nu_{2i}$	$\nu_2+\nu_0+\nu_{-2}$
3	A_2	$\bigoplus_{i=0}^{k} \varphi_1^i\varphi_2^i$	A_1	$\bigoplus_{i=0}^{k} \varphi_1^i$	$2\varphi_1+\theta_2$
4	A_m $m\geqq 3$	$\bigoplus_{i=0}^{k} \varphi_1^i\varphi_m^i$	A_{m-1}	$\bigoplus_{i=0}^{k} \varphi_1^i$	$\varphi_1+\varphi_{m-1}+\theta_2$
5	SO_m $m\geqq 5$	$\supset \bigoplus_{i=0}^{k} \varphi_1^{2i}$	SO_{m-1}	$\bigoplus_{i=0}^{k} \varphi_1^i$	$\varphi_1+\theta_1$
6	B_2	$\supset \bigoplus_{i=0}^{2k+1} \varphi_1^i$	$A_1\times\bar{A}_1$	$\bigoplus_{i=0}^{k} (\varphi_1^{i+1}\bar{\varphi}_1^i+\varphi_1^i\bar{\varphi}_1^{i+1})$	$\varphi_1\otimes\bar{\varphi}_1+\theta_1$
6′	B_m $m\geqq 3$	$\supset \bigoplus_{i=0}^{2k+1} \varphi_1^i$	D_m	$\bigoplus_{i=0}^{k} (\varphi_1^i\varphi_{m-1}+\varphi_1^i\varphi_m)$	$\varphi_1+\theta_1$
7	D_m $m\geqq 3$	$\supset \bigoplus_{i=0}^{k} \varphi_1^i$	B_{m-1}	$\bigoplus_{i=0}^{k} \varphi_1^i\varphi_{m-1}$	$\varphi_1+\theta_1$
8	C_m $m\geqq 2$	$\bigoplus_{i+j\leqslant k} \varphi_1^{2i}\varphi_2^j$	C_{m-1}	$\bigoplus_{i=0}^{k} (k+1-i)\,\varphi_1^i$	$2\varphi_1+\theta_4$
9	G_2	$\bigoplus_{i=1}^{k} \left[\frac{k+2-i}{2}\right]\varphi_1^{k\pm i} + \left[\frac{k+2}{2}\right]\varphi_1^k$	SL_3	$\bigoplus_{i+j\leqslant k} \varphi_1^i\varphi_2^j$	$\varphi_1+\varphi_2+\theta_1$
10	E_6	$\supset \bigoplus_{i+j\leqslant k} \varphi_1^{2i}\varphi_5^j$	F_4	$\bigoplus_{i=0}^{k} \varphi_1^i$	$\varphi_1+\theta_1$
11	E_7	$\bigoplus_{l+m+n\leqslant 4}^{8} \varphi_1^{2l}\varphi_2^m\varphi_6^n$	E_6	$\bigoplus_{i=0}^{4}\bigoplus_{s+t=i} (5-i)\,\varphi_1^s\varphi_5^t$	$\varphi_1+\varphi_5+\theta_2$

INDEX OF NOTATION

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