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HARMONIC MAPS INTO SINGULAR SPACES AND p-ADIC SUPERRIGIDITY FOR LATTICES IN GROUPS OF RANK ONE

By Mikhail Gromov and Richard Schoen

Introduction

In Part I of this paper we develop a theory of harmonic mappings into non-positively curved metric spaces. The main application of the theory, which is presented in Part II, is to provide a new approach to the study of p-adic representations of lattices in noncompact semisimple Lie groups. The celebrated work of G. Margulis [Mar] establishes "superrigidity" for lattices in groups of real rank at least two. The fact that superrigidity fails for lattices in the isometry groups of the real and complex hyperbolic spaces is known. In fact, Margulis deduced as a consequence of superrigidity the conclusion that lattices are necessarily arithmetic in groups of rank at least two. Arithmeticity of lattices was conjectured and proved in some cases by A. Selberg (see [Se] for discussion). Constructions of nonarithmetic lattices in the real hyperbolic case were given by Makarov [Mak], Vinberg [V], and Gromov-Piatetski-Shapiro [GPS]. For the complex hyperbolic case, nonarithmetic lattices have been constructed in low dimensions by G. D. Mostow [Mos] and Deligne-Mostow [DM]. In this paper we establish p-adic superrigidity and the consequent arithmeticity for lattices in the isometry groups of Quaternionic hyperbolic space and the Cayley plane (the groups $\mathrm{Sp}(n, 1)$, $n \geqslant 2$ and F_4^{-20}). Archimedian superrigidity for these cases has been established recently by K. Corlette [C] who used harmonic map theory together with a new Bochner formula and vanishing theorem to prove the result. We show here that representations of lattices in $\mathrm{Sp}(n, 1)$ and F_4 in almost simple p-adic algebraic groups have bounded image. This is accomplished by the construction of an equivariant harmonic map from the symmetric space into the Euclidean building of Bruhat-Tits [BT] associated to the p-adic group. We analyze the structure of such maps in detail, and show that their image is locally contained in an apartment at enough points so that differential geometric methods may be applied. In particular, we apply the Corlette vanishing theorem to show that the harmonic map is constant, and conclude that the representation has bounded image.

We also prove that equivariant harmonic maps of finite energy from a Kähler manifold into a class of Riemannian simplicial complexes (referred to as F-connected) are pluriharmonic. The class of F-connected complexes includes Euclidean buildings.

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This result generalizes work of Y. T. Siu [Siu] which implies the same result for maps to manifolds with nonpositive curvature operator.

We now briefly outline the contents of this paper. We consider maps into locally finite Riemannian simplicial complexes, by which we mean simplicial complexes with a smooth Riemannian metric on each face. In the first four sections of this paper we develop methods for constructing Lipschitz maps of least energy in homotopy classes or with the map specified on the boundary provided the receiving space (complex) has non-positive curvature in a suitable sense. This generalizes the theorems of J. Eells and J. H. Sampson [ES] and R. Hamilton [Ham] who proved these results for maps to manifolds of nonpositive curvature. We also prove and use convexity properties of the energy functional along geodesic homotopies to prove uniqueness theorems generalizing those of P. Hartman [Har]. A key property of harmonic maps which we exploit to prove these results is a statement to the effect that harmonic maps can achieve their value at a point only to a bounded order, and near the point they can be approximated by homogeneous maps from the tangent space of the domain manifold to the tangent cone of the image complex at the image point. These homogeneous maps have degree at least one and, at most points, they must have degree equal to one. The homogeneous maps of degree one are compositions of an isometric totally geodesic embedding of a Euclidean space into the tangent complex with a linear map of Euclidean spaces. In particular, these maps identify flat totally geodesic submanifolds of the tangent complex.

In section 5 we define an intrinsic notion of differentiability for harmonic maps based on how well approximated they are near a point by maps which are homogeneous of degree one in an intrinsic sense. We then prove a result which enables us to establish differentiability of a map based on the differentiability of maps into a totally geodesic subcomplex which approximately contains the local image of the map. This result is the main technical tool of the paper as it can be used to show that the local image of a harmonic map under appropriate conditions is actually in a subcomplex whose geometry is simpler than that of the ambient complex. We then apply this result to assert differentiability of harmonic maps into one-dimensional complexes.

In section 6 we define a class of complexes which we refer to as F-connected. A k-dimensional complex is called F-*connected* if each of its simplices is isometric to a linear image of the standard simplex and any two adjacent simplices are contained in a k-flat, by which we mean a totally geodesic subcomplex isometric to a region in $\mathbf{R}^k$. We then show that harmonic maps into F-connected complexes are differentiable, and we give a detailed discussion of the size of the set of nonsmooth points, by which we mean points for which the local image of the map is not contained in a k-flat.

In section 7 we carry through the Bochner method (in particular the Corlette vanishing theorem) for maps into F-connected complexes. We establish pluri-harmonic properties for maps of Kähler manifolds, and show that finite energy equivariant maps are constant from the Quaternionic hyperbolic space or the Cayley plane. We also extend the existence theory to include the construction of finite energy equivariant maps into buildings associated to an almost simple p-adic algebraic group H. We show that either the harmonic map exists or the image of the representation lies in a parabolic subgroup of H. In particular, if the image of the representation is Zariski dense in H, then the harmonic map exists. The hypothesis on the domain manifold is very general here. One requires only that it be complete. In section 8 we establish our p-adic superrigidity results and discuss the arithmeticity of lattices.

Finally in section 9 we discuss the structure of harmonic maps of Kähler manifolds into trees and buildings. We describe an extension of our work to maps in $\mathbf{Z}$-trees and use it to show that the fundamental group of a Kähler manifold cannot be an amalgamated free product unless the manifold admits a surjective holomorphic map to a Riemann surface. Applications of harmonic maps into trees similar to those done in section 9 were also obtained by C. Simpson [Sim].

A technical device which plays an important role in determining the structure of harmonic maps into nonpositively curved complexes is the monotonicity in σ of the ratio $\operatorname{Ord}(x, \sigma, Q)$ defined in section 2. For harmonic functions on $\mathbf{R}^n$ this is a classical fact which is the L^2 version of the Hadamard three spheres theorem. It says that the logarithm of the mean L^2 norm of a harmonic function on a sphere of radius r is a convex function of $\log r$. We prove in section 2 a global geometric version of this result. Its proof relies on the usual monotonicity formula for harmonic maps (which plays an important role in the regularity theory of Schoen-Uhlenbeck [SU] for energy minimizing maps into manifolds) combined with the strong convexity of the distance function on a nonpositively curved complex. A ratio of this type has been used by a variety of authors on various elliptic PDE problems. A partial list includes Agmon [Ag], Almgren [Al], Garofolo-Lin [GL], Landis [La1, La2], Lin [Lin], Miller [Mi]. The first author to realize the importance of this type of result for proving unique continuation properties of solutions of general classes of elliptic equations seems to have been S. Agmon [Ag] in 1965. (The earlier papers of Landis are also quite closely related.) The optimal unique continuation result was proved by this method only recently in [GL].

The work in this paper was initiated by a suggestion of the first author that it might be possible to develop a harmonic map theory into nonpositively curved metric spaces, and that, in interesting cases, the resulting maps might be regular enough so that the Bochner method could be applied. In particular, he had a conjecture on the singular structure of harmonic maps into trees. He also proposed a version of the heat equation method which might be used to produce such harmonic maps. The work in Part I of this paper comprises the second author's solution to this problem. The approach taken is a variational approach rather than a heat flow method. The conjectured behavior of harmonic maps to trees is shown to be substantially correct (with a slightly worse blow-up of derivatives near singular points than conjectured). Note that the theory developed in this paper is largely independent of discrete group theory, and should be viewed as a part of the geometric calculus of variations.

The authors are grateful to Kevin Corlette for pointing out several errors in the first version of § 9 and to Ralf Spatzier for a useful conversation about buildings. The second author thanks Scot Adams and Alex Freire for several suggestions for improving the exposition. Many of these have been incorporated in the final version of the paper.

History, motivation and examples

We develop in this paper a theory of harmonic maps into certain singular spaces with non-positive curvature. The simplest example of such a space is the *tripod* (see Figure 1) that is the union of three copies of the segment [0, 1] identified at zero, such that the distance between every two points a and b lying in different copies of [0, 1] by definition equals $a+b$,

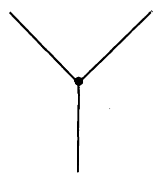

FIG. 1.

Similarly, one may consider n-pods obtained by joining n intervals. Then we see further examples by looking at *graphs* (i.e. connected 1-dimensional simplicial complexes) which are endowed with metrics locally isometric to the above n-pods. Notice that the distance between every two points x and y equals the length of the *shortest path* in X between x and y that is an isometric embedding of the interval $[0, \delta = \text{Dist}(x, y)] \subseteq \mathbf{R}$ into X with $0 \mapsto x$ and $\delta \mapsto y$. If the points x and y lie sufficiently close together, then the minimal path is unique. In fact, if X is a k-pod then this uniqueness holds for all pairs of points and this manifests the (not yet defined) non-positivity of curvature of X.

The next important example is provided by *Bruhat-Tits euclidean buildings* associated to reductive p-adic Lie groups. For us, a *building* is a simplicial complex X which is accompanied by a simplicial action of a (p-adic Lie) group G. One knows that such an X carries a G-invariant metric (with curvature $\leqslant 0$) such that

(a) Every simplex in X is isometric to an affine simplex in some Euclidean space.

(b) Every two points in X can be joined by a unique shortest path (geodesic) lying in the union of some top-dimensional simplices.

Moreover, the buildings have the following remarkable property:

For every two points x and y in X *there exists an isometric embedding of* $\mathbf{R}^k$ *into* X *for* $k=\dim X$, *such that the image of this* $\mathbf{R}^k$, *called a k-flat in* X, *contains x and y.* (*See* [BT] *and* [B] *for information on buildings.*)

One can easily grasp the geometry of 1-dimensional buildings as these are just regular trees i.e., simply connected 1-dimensional simplicial complexes where all edges have the same length and where all vertices have the same number of adjacent edges. The geometry of higher dimensional buildings is somewhat more elaborate but one gains some insight by looking at the Cartesian products of regular trees. (These products are not quite buildings in the above sense as they are built of cubes rather than simplices but they are buildings in the sense of [BT] – which allows polysimplicial complexes –, they do have non-positive curvature and all pairs of points are connected by k-flats.)

We shall not discuss at this stage the general notion of non-positive curvature, expressed by $K(X)\leqslant 0$, for general metric spaces X, but rather indicate the following.

Examples. – (1) If X is a smooth Riemannian manifold, then $K(X)\leqslant 0$ signifies that the sectional curvature of X is everywhere $\leqslant 0$. In particular, the *symmetric spaces of non-compact type* have $K\leqslant 0$.

Recall that these symmetric spaces have the form $X=G/H$ where G is a connected semisimple Lie group with finite center and without non-trivial compact factor groups and H is a maximal compact subgroup. Every such X admits a G-invariant Riemannian metric (since H is compact) and this metric has $K\leqslant 0$ by a theorem of E. Cartan. The basic example here is the space

$$X=SL_n(\mathbf{R})/SO(n)$$

which may be thought of as the space of positive definite quadratic forms on the n-dimensional linear space.

Notice that the buildings discussed earlier are substitutes, for p-adic groups, of the symmetric spaces. In the p-adic case one has several maximal compact subgroups in G corresponding to different G-orbits on the set of vertices of the building. Also observe that the dimension of a building corresponds to the *rank* (rather than dimension) of a symmetric space X.

Recall that rank (X) is the dimension of a maximal *flat* in X, i.e., a totally geodesic submanifold isometric to $\mathbf{R}^k$. For example, rank $(SL_n(\mathbf{R})/SO(n))=n-1$ and a maximal flat consists of the set of quadratic forms which are diagonal with respect to a fixed basis.

(*Super*) *rigidity.* – Consider a symmetric space of non-compact type, called $\tilde{M}$, (it is secretly thought of as the universal covering of a compact manifold M) and let

Γ be a discrete faithful group of isometries acting on $\tilde{M}$ (if Γ acts freely on $\tilde{M}$ then, in fact, $\tilde{M}$ is the universal covering of $M = \tilde{M}/\Gamma$). Then we take another space, say X, and let Γ act isometrically on X, where now the action need not be discrete or faithful. In the cases we are most concerned with, X is either a building or a symmetric space of non-compact type. In this case, X is topologically contractible and therefore there exists a continuous Γ-equivariant map $u_0 : \tilde{M} \to X$ which is unique up to Γ-equivariant homotopy. (If the actions of Γ on $\tilde{M}$ and X are discrete and free then Γ-equivariant maps $\tilde{M} \to X$ correspond to continuous maps $\tilde{M}/\Gamma \to X/\Gamma$.)

We call the above setup *rigid* if the map u_0 is Γ-equivariantly homotopic to a *geodesic* map $u : \tilde{M} \to X$, which means that the graph $\Gamma_u \subset \tilde{M} \times X$ is a totally geodesic subspace of the Cartesian product.

The first instance of rigidity was discovered by Mostow in the case where $\tilde{M}$ and X are manifolds of constant negative curvature of equal dimension $\geqslant 3$ and where the action of Γ is discrete and cocompact on $\tilde{M}$ and on X. ("Co-compact" signifies "the quotient space is compact".) This was extended later by Mostow to other equidimensional symmetric spaces and then a similar result was proven by Prasad and Ragunathan for equidimensional buildings. Finally there came.

Margulis' superrigidity theorem. — *If* $\tilde{M}$ *is an irreducible symmetric space of* rank $\geqslant 2$ *and the action of* Γ *has finite covolume* (*i.e.*, Vol $\tilde{M}/\Gamma < \infty$) *then the above setup is rigid* (*i.e.*, u_0 *is homotopic to a geodesic map whenever* X *is an arbitrary building or a symmetric space*).

The celebrated corollary of this rigidity is *Margulis's arithmeticity theorem* for Γ which says that Γ is obtained from the lattice $SL_N(\mathbf{Z}) \subset SL_N(\mathbf{R})$ by certain elementary algebraic manipulations. (These are: taking the intersection of $SL_N(\mathbf{Z})$ with Lie subgroups in $SL_N(\mathbf{R})$, applying surjective homomorphisms between Lie groups with compact kernels, replacing discrete groups by subgroups of finite index or enlarging groups by finite index extensions.)

Remarks. — (a) Margulis has also proved his theorem for certain *reducible* spaces (which are Cartesian metric products $\tilde{M} = \tilde{M}_1 \times \tilde{M}_2$) but we stick to the irreducible case for the purpose of the exposition. In this case every geodesic map $u : \tilde{M} \to X$ is either constant or is an injective map onto a totally geodesic submanifold $\tilde{M}' \subset X$. In fact, the map $\tilde{M} \to \tilde{M}'$ becomes an isometry if we change the metric in $\tilde{M}$ by a multiplicative constant. Furthermore, if X is a building, then u is necessarily a constant map which sends all of $\tilde{M}$ to a fixed point of Γ acting on X. Thus the p-adic superrigidity amounts to the existence of a fixed point for every action of Γ on a building.

(b) The superrigidity fails to be true for certain symmetric spaces $\tilde{M}$ of rank one. Such examples are easy to construct for $\tilde{M}$ a real hyperbolic space and there are (more complicated examples) for the complex hyperbolic space. What remains of rank one are the quaternionic hyperbolic spaces and the hyperbolic Cayley plane. These long have been suspected to be as rigid as rank $\geqslant 2$ spaces as Kostant has shown that they satisfy Kazhdan's T-property (see [HV]) and also a kind of metric rigidity proven by Pansu (see [GP]). Then, recently, the real superrigidity (i.e., for X a symmetric space) was proven by K. Corlette [C] using *harmonic* maps $\tilde{M} \to X$ and one of the goals of the present paper is to do the same in the p-adic case where X is a Bruhat-Tits building.

(c) Margulis' approach, unlike those by Mostow and Corlette, does not directly involve symmetric spaces and geodesic maps but rather deals with Lie groups and continuous homomorphisms. Notice that a geodesic map between symmetric spaces immediately gives us a homomorphism between the relevant (isometry) groups as central symmetries of a geodesic subspace canonically extend to symmetries of the ambient space. But going from group homomorphisms to geodesic maps (which we do not need for this paper) requires a non-trivial Lie algebraic lemma by Mostow (see [GP]).

Idea of harmonic maps. — The classical Dirichlet energy of a smooth function u, that is $\int \|\operatorname{grad} u\|^2$ or better, $\int \|du\|^2$, can be defined for a smooth map between Riemannian manifolds, $u: M \to Y$, by

$$E(u) = \int_M e(u)\, d\mu$$

where $e(u)$ is the so called energy density of u whose value at $m \in M$ is given by

$$e(u)(m) = \frac{1}{2}\|D_m(u)\|^2,$$

where

$$D_m(u): T_m(M) \to T_{u(m)}(Y)$$

is the differential of u whose norm is defined by $\|D\| = \text{Trace } D^* D$ where D^* denotes the adjoint operator. (If one uses orthonormal bases which diagonalize D the $\|D\|^2$ becomes $\sum_{i=1}^{\dim M} \lambda_i^2$ for the diagonal entries λ_i.)

One should slightly modify the domain of integration for Γ-equivariant maps $\tilde{u}: \tilde{M} \to X$ by first observing that the density function $e(\tilde{u})$ is Γ-equivariant and so descends to a function on $M=\tilde{M}/\Gamma$, also denoted $e(\tilde{u})$. Then we set

$$E(\tilde{u})\overset{\text{def}}{=}\int_M e(\tilde{u})\,d\mu.$$

A smooth map between Riemannian manifolds is *harmonic* if it satisfies the Euler-Lagrange equation for the energy functional. Thus every energy minimizing map $M\to Y$ in a fixed homotopy class (if such a map exists at all) is harmonic. The similar conclusion applies to Γ-equivariant map $\tilde{M}\to X$ minimizing the above equivariant energy.

The theory of harmonic maps into nonpositively curved manifolds starts with the following existence theorems proven by Eells and Sampson [ES] in 1964 which came before the first rigidity result by Mostow.

If X *is simply connected,* $K(X)\leqslant 0$ *and the actions of* Γ *on* $\tilde{M}$ *and* X *are discrete and co-compact then there exists a smooth energy minimizing* Γ*-equivariant map* $\tilde{u}\colon \tilde{M}\to X$.

Moreover, one knows in the above situation that every harmonic map necessarily is energy minimizing, and such a map is unique up to a *parallel translation* in X. This means (apart from some irrelevant pathological examples) that there exists a Γ-invariant totally geodesic submanifold $X'\subset X$ which isometrically splits as $X'=X'_0\times \mathbf{R}'$ such that the image of every Γ-equivariant harmonic map $\tilde{M}\to X$ is contained in X' and any two such maps can be obtained one from another by applying the (obvious) action of $\mathbf{R}'$ on X'. (This uniqueness result is due to Hartman [Har].)

Notice that every geodesic map $\tilde{M}\to X$ is (obviously) harmonic and, moreover, energy minimizing (at least in the case where the total energy is finite). Furthermore, the above parallel translation moves geodesic maps again to geodesic maps and so the existence of a single geodesic Γ-invariant map $\tilde{M}\to X$ implies that every harmonic map is geodesic. This suggests the following approach to the (super) rigidity problem: First construct an energy minimizing Γ-equivariant map $\tilde{M}\to X$ and then show that every harmonic map is geodesic. In fact, such a result appears in the original paper by Eells and Sampson, as they prove that every Γ-invariant harmonic map of $\tilde{M}=\mathbf{R}^n$ into an arbitrary manifold X with $K(X)\leqslant 0$ is geodesic. Then they combine this with their existence theorem and come to the following conclusion.

Let M *and* Y *be closed* (*i.e. compact without boundaries*) *Riemannian manifolds where* M *is flat* (*i.e.,* $K(M)=0$) *and* $K(Y)\leqslant 0$. *Then every continuous map* $M\to Y$ *is homotopic to a geodesic map. In particular, if the fundamental group of* Y *contains a subgroup isomorphic to* $\mathbf{Z}^2$ *then* Y *contains an immersed totally geodesic flat torus.*

The proof of the implication

$$\text{harmonic}\Rightarrow\text{geodesic}$$

is similar to the following classical argument showing that every harmonic function u on a closed manifold M is constant. The basic formula here reads

$$\operatorname{div}(u \operatorname{grad} u) = \| \operatorname{grad} u \|^2 + u \Delta u.$$

If u is harmonic and $\Delta u = 0$, this formula shows that $\| \operatorname{grad} u \|^2$ equals the divergence of a vector field and so integrates to zero

$$\int_M \| \operatorname{grad} u \|^2 = \int_M \operatorname{div}(u \operatorname{grad} u) = 0.$$

Thus grad $u = 0$ which means (as M is assumed connected) u is constant.

The proof of Eells and Sampson uses a more elaborate expression which involves the Hessian of u (rather than the gradient) which measures the totality of the second derivations of u and which vanishes if and only if u is geodesic. Here is the Bochner formula of Eells and Sampson:

If M is flat then every smooth harmonic map $u : M \to X$ satisfies at each point $m \in M$,

$$\| \operatorname{Hess}(u) \|^2 = \Delta e(u) + K^*$$

where e denotes the energy density (function) on M and K^* is a certain (real valued) function on M obtained by pulling back the curvature tensor of Y to M by the differential of u and then by taking an appropriate trace of the resulting tensor on M. The explicit formula for K^* is not important at the moment but we need the following crucial property of K^*:

$$\text{if} \quad K(Y) \leqslant 0 \text{ then also } K^* \leqslant 0$$

(this is true for all C^1-maps u, not only for harmonic ones). Now, since $\Delta = \operatorname{div} \operatorname{grad}$, the integral of Δe over M vanishes and thus

$$\int \| \operatorname{Hess}(u) \|^2 = \int K^* \leqslant 0$$

which for $K(Y) \leqslant 0$ implies that

$$\operatorname{Hess}(u) = 0$$

as well as

$$K^* = 0.$$

The first relation, as we know, tells us that u is a *geodesic* map and the second relation says (once the explicit formula for K^* is written down) that the curvature (2-form) of Y vanishes on the image of the differential of u. (In this particular case the second conclusion can be derived from the first one but for some Bochner-type situations one should keep track of both terms separately.)

The idea of using harmonic maps for rigidity problems was widely discussed since the appearance of the first paper by Mostow. One also was encouraged by the local rigidity results established earlier by Calabi-Vesentini and A. Weil where the Bochner method had been successfully carried through in the infinitesimal (and hence linear) setting. However, it took more than 10 years before the first usable Bochner formula was found by Siu in 1978. Siu's formula applies to harmonic maps between *Kähler* manifolds and shows, in the case where a certain curvature of the target space is nonpositive, that every harmonic map is either holomorphic or antiholomorphic. Siu's formula was modified by Sampson who considered harmonic maps u of a Kähler manifold M into an arbitrary Riemannian manifold. Sampson expressed the *complex Hessian* $\| d\mathrm{J}\, du \|^2$ as a sum of a divergence term and a certain curvature expression (like the above K^*) pulled back from the target manifold. Then, assuming his curvature is $\leqslant 0$, Sampson concludes the vanishing of the Hessian $d\mathrm{J}\, du$ (here J stands for the complex structure operator in M and the two d's are appropriate differentials) which means (more or less by definition) that u is a *pluriharmonic* map, *i.e.*, the restriction of u to every complex submanifold in M is harmonic. Furthermore, Sampson has shown that his curvature is $\leqslant 0$ for symmetric spaces of non-compact type and thus proved the following pluriharmonic (rather than geodesic) rigidity theorem.

Every continuous map of a compact Kähler manifold into a compact locally symmetric space X *with* $\mathrm{K}(\mathrm{X}) \leqslant 0$ (*i.e. of non-compact type*) *is homotopic to a pluriharmonic map.*

(This result together with the circle of surrounding ideas was explained by D. Toledo to the first author some time ago.)

The simplest case where Sampson's theorem applies is that of the flat torus $\mathrm{Y} = \mathrm{T}'$ where the Siu-Sampson formula reduces to the classical Hodge identity $\Delta u = 2\,\bar\partial^* \bar\partial u$ for some function $u : \mathrm{M} \to \mathrm{C}$. (The corresponding pluriharmonicity theorem claiming that every continuous map $\mathrm{M} \to \mathrm{T}^n$ is homotopic to a pluriharmonic map can probably be dated back to Poincaré or maybe to Riemann.)

Maps to singular spaces. — Now we turn to the p-adic (super) rigidity problem where the receiving space X is a Bruhat-Tits building and we want to see what remains of the theory of harmonic maps when the target space is singular. First of all we should define a notion of the energy for maps into non-Riemannian metric spaces. This can be done in a variety of ways in a quite general situation. The most direct definition uses the squared ratio (stretch) between the distances in M and Y, i.e.,

$$\mathrm{S}(m_1, m_2) = \mathrm{Dist}^2_{\mathrm{Y}}(u(m_1), u(m_2)) / \mathrm{Dist}^2_{\mathrm{M}}(m_1, m_2)$$

which makes sense (for $m_1 \neq m_2$) for maps between arbitrary metric spaces. Then one can integrate S over the ε-neighborhood of the diagonal in $\mathrm{M} \times \mathrm{M}$, say $\mathrm{N}_\varepsilon \subset \mathrm{M} \times \mathrm{M}$

(here one needs the Riemannian measure in M and one assumes $\dim M \geqslant 1$) and define the energy by

$$E(u) = \limsup_{\varepsilon \to 0} \varepsilon^{-\dim M} \int_{N_\varepsilon} S(m_1, m_2)\, dm_1\, dm_2.$$

Notice that in the smooth case this agrees with the classical definition up to a normalizing constant. In fact, the above limit formula gives an intuitive explanation for the Dirichlet energy classically defined with infinitesimals. (One may slightly modify the above by using instead of the Riemannian measure restricted to N_ε another approximation to the δ-measure on the diagonal, for example the normalized heat kernel on M for time tending to zero.)

Another more practical (but indirect) definition for maps into buildings X (and similar spaces) can be made by locally isometrically embedding X into some $\mathbf{R}^N$ thus reducing the definition to the classical case (see section 1).

The simplest case to look at is where $M = [0, 1]$ and where one can define the energy density of a map u at m by

$$e(u) = \frac{1}{2} \limsup_{\substack{m' \to m \\ m'' \to m}} S(m', m'')$$

and then set

$$E(u) = \int_0^1 e(u)\, dm.$$

This energy $E(u)$ on maps $u: [0, 1] \to X$ is well behaved under rather general assumptions on X and $E(u)$ assumes its minimum at the geodesic curves in X parametrized by a multiple of the length parameter. (A geometric study of these curves in the situation where the singularity comes from an obstacle inside a smooth manifold Y was conducted in [ABB].)

Notice that one can recapture some properties of the energy $E(u)$ for $\dim M \geqslant 2$ from $\dim M = 1$ by restricting u to the unit geodesic segments in M, taking the energy of the restricted maps and then by integrating over the unit tangent bundle of M. Yet one does not expect a meaningful higher dimensional variational theory for $E(u)$ unless the "curvature" of the receiving (singular!) space X is somehow bounded from above. The most convenient (for us) definition of $K(Y) \leqslant 0$ (suggested by A. D. Alexandrov many years ago) can be best seen in the universal covering X of Y. Here is a list of the properties of X given in the order of increasing strength which can be used as a definition for $K(X) \leqslant 0$ (and thus for $K(Y) \leqslant 0$, compare section 2.1).

1. Every two points in X can be joined by a unique distance minimizing geodesic segment.

2. The distance function $\operatorname{Dist}(x_1, x_2)$ on $X \times X$ is convex on the cartesian products of geodesic segments.

3. The distance function on X is "more convex" than the distance on $X' = \mathbf{R}^2$. Namely, if we take geodesic segments of the same length in X and X′ which are both identified with $[0, \delta] \subset \mathbf{R}$ (for δ the distance between the ends), then the equalities

$$\operatorname*{Dist}_{X'}(x'_0, 0) = \operatorname*{Dist}_{X}(x_0, 0)$$

$$\operatorname*{Dist}_{X'}(x'_0, \delta) = \operatorname*{Dist}_{X}(x_0, \delta)$$

imply that

$$\operatorname*{Dist}_{X'}(x'_0, t) \geqslant \operatorname*{Dist}_{X}(x_0, t)$$

for all $t \in [0, \delta]$.

The convexity of the distance implies that the energy $E(u)$ is convex under geodesic deformations of maps which implies Hartman's uniqueness theorem (see section 4). Furthermore, the convexity of the distance allows one to define a good notion (in fact several non-equivalent notions) of the *center* of mass (see [K]) of a finite measure μ on X. (The standard definition of center (μ) refers to the point $x_0 \in X$ which minimizes $\int_X \operatorname{Dist}^2(x_0, x)\, d\mu_x$. Another possible definition uses the map $c: X \times X \to X$ assigning to (x_1, x_2) the center of the geodesic segment between x_1 and x_2. This c pushes forward the measure $\mu \times \mu$ to a measure on X, say μ_1, and thus, by induction, one has μ_2 coming from $\mu_1 \times \mu_1$, etc. Then one defines the center of μ as the (one point!) support of the weak limit of μ_i for $i \to \infty$.) This gives us a possibility to regularize a map $\tilde{u}: \tilde{M} \to X$ using smoothing kernels $K(m, m')$ on M as follows. Assign to each point $m \in M$ the measure $K_m = K(m, m')\, dm'$ for the Riemannian measure dm' on M, and define the regularization $\bar{u}$ of a given map u by defining $\bar{u}$ (m) to be the center of the u-push-forward of K_m to X, $m \in M$. This operation works particularly nicely if M is a compact flat manifolds, say a flat torus T^n, and K is of the form $K(m, m') = L(m - m')$. In order words, the μ in this case comes from some measure λ (corresponding to L) on the torus acting on itself by parallel translation. In fact, given any family of selfmappings of M (e.g., of isometries or more general selfdiffeomorphisms) and a measure on this family, we obtain by composing with u a family of maps $M \to Y$, say u with a measure $d\lambda$. Now we can "average" this family over the parameter space with the measure λ by applying a center of mass construction at each point $m \in M$.

In the case where the measure λ is supported on the isometries of M, this smoothing decreases the energy density at each point of M (for any conceivable definition of the energy) provided the chosen center of mass operator is contracting in the appropriate sense. (Here it is appropriate to use the center of mass defined above with the map $c: X \times X \to X$ where the needed contraction property directly follows from the convexity of the distance function on $X \times X$.) It follows that every energy minimizing map $u: M \to X$ is invariant under such regularization and consequently, in the case $M = T^n$ (and more generally, for M flat), the minimizing map is geodesic. Thus the "flat $\Rightarrow$ geodesic" theorem of Eells-Sampson extends to maps into singular spaces.

Recall that the basic existence theorem of Eells and Sampson for harmonic maps is based on a certain construction of a *heat flow* (in the space of maps $u: M \to Y$) which can be (at least formally) perforce used for singular spaces Y with $K(Y) \leqslant 0$ using the above smoothing operators. (The smoothing operators, the way we describe them, must be performed for the corresponding maps between the covering manifolds $\tilde{M} \to X = \tilde{Y}$ on the global center of mass defined in X but not Y. Yet this causes no problem as all relevant constructions are invariant under $\Gamma \subset$ Isom. group (X).) Namely, for every $t > 0$ and $i = 1, 2, \ldots$, we consider the ordinary heat kernel $\tilde{K}(\tilde{m}, \tilde{m}', t/i)$ on the covering $\tilde{M}$ on M (corresponding to the universal covering X of Y) and apply the smoothing i times to a given map $\tilde{u}: \tilde{M} \to X$. (Here we speak the language of Γ-equivariant maps $\tilde{M} \to X$ corresponding to maps $M = \tilde{M}/\Gamma \to Y = X/\Gamma$). The limit (or sublimit, whichever existence one is able to prove) of these iterated smoothings for $i \to \infty$ defines the action of the heat flow on u at the time t. Since the (regularity) properties of the Eells-Sampson heat flow are "uniformly good" for $K(Y) \leqslant -\mathscr{H}^2$ for $\mathscr{H} \to \infty$ one may expect that every map $M \to Y$ can be homotoped to a (essentially unique) sufficiently regular (at least Lipschitz) energy minimizing map. Evidence in favor of this conclusion is provided by those singular spaces with $K \leqslant 0$ which can be approximated by Riemannian manifolds with $K \leqslant 0$. For example, take a unit geodesic triangle in the hyperbolic plane with curvature $-\mathscr{H}^2$ and let $\mathscr{H} \to \infty$. Then we obtain in the limit the tripod described at the beginning of the introduction. In fact, an arbitrary finite graph Y admits a similar approximation. Namely, there exists a sequence of compact manifolds $Y_{\mathscr{H}}$ with convex boundaries and with constant curvatures $-\mathscr{H}^2$ for $\mathscr{H} \to \infty$, such that Y admits embeddings $Y \subset Y_{\mathscr{H}}$ for all $\mathscr{H}$ with the following properties:

(i) each edge of Y isometrically goes to a geodesic segment of Y;

(ii) $\sup_{y' \in Y_{\mathscr{H}}} \operatorname{Dist}(y', Y) \to 0$ for $\mathscr{H} \to \infty$;

(iii) for each $\mathscr{H}$ there exists a homotopy retraction $Y_{\mathscr{H}} \to Y$ (which is moreover Lipschitz and the implied Lipschitz constant is independent of $\mathscr{H}$).

Now one can construct Lipschitz harmonic maps $M \in Y$ by using (sub)limits of harmonic maps $M \to Y_{\mathscr{H}}$ for $\mathscr{H} \to \infty$, as the latter maps (in a fixed homotopy class) satisfy uniform Lipschitz bounds by the Eells-Sampson theorem.

Another geometrically significant class of singular surfaces Y which admit a smooth approximation comes from *ramified coverings* of smooth manifolds Y_0 with $K \leqslant 0$. If the ramification locus is totally geodesic (of codimension 2) in Y_0, then the induced singular metric in Y has $K \leqslant 0$ and usually it can be approximated by smooth metrics with $K \leqslant -\varepsilon$ for $\varepsilon \to 0$. (In fact, the same is true for more complicated ramification loci such as unions of totally geodesic submanifolds with 90°-crossings). On the other hand, a smooth approximation of higher dimensional buildings appears more difficult though not inconceivable as certain buildings (and building-like spaces) do appear in the limits of (parts of) symmetric spaces (of rank $\geqslant 2$).

On Bochner formulas in singular spaces. — In order to derive an interesting geometric conclusion from the general theory of harmonic maps $u: M \to Y$ (or $\tilde{M} \to X = \tilde{Y}$) one needs a Bochner formula showing that a map u, *a priori* only harmonic, is, under favorable conditions on the curvature of Y, more special, e.g. geodesic or pluriharmonic. If Y is a piecewise Euclidean polyhedron (e.g. the universal covering X of Y is a Euclidean building) then one may think that the required curvature condition is somehow encoded in the local combinatorial structure of X. For example, one knows a combinatorial formulation for $K(X) \leqslant 0$ in terms of the numbers of different simplices adjacent to every face and similar but stronger conditions might be responsible for Siu-Sampson type curvatures and their generalization. A comprehensive understanding of such conditions appears a rather difficult (and still unresolved) problem but the examples presented below indicate a way out of this difficulty for maps having certain regularity properties.

Consider a map u of the plane $\mathbf{R}^2$ near the origin to the unit tripod Y with the edges numbered 1, 2 and 3, see Figure 2. Here the sectors $\tilde{1}$, $\tilde{2}$ and $\tilde{3}$ represent the

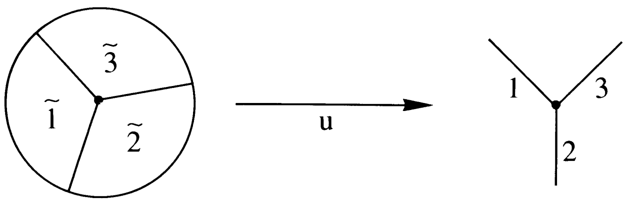

FIG. 2

u-pull-backs of the corresponding edges of Y and the tripod Z in $\mathbf{R}^2$ formed by the boundaries of these sectors equals the pullback of the central point in Y. At first sight the map u seems necessarily singular at Z as Z goes to the singular locus (the center) of Y. Yet if we look at the map u restricted to two out of three sectors, say on $\tilde{1}+\tilde{2}$, we see that this $(\tilde{1}+\tilde{2})$-sector is mapped into the union of the edges 1 and 2 of Y.

This union, from the intrinsic point of view, has no singularity as it is just isometric to the real segment of double length and so our map u to $\tilde{1}+\tilde{2}$ (as well as to $\tilde{1}+\tilde{3}$ and $\tilde{2}+\tilde{3}$) reduces to a real valued function. Thus the "true singularity" of u is concentrated at the origin of $\mathbf{R}^2$, where the three sectors meet.

Now let us see what we need of such a map u in order to prove some integral Bochner formula. We assume we do have some infinitesimal Bochner identity defined on M where the map u in question is smooth and the global identity is obtained by integrating this formula over M. A non-trivial effect is achieved by the presence of some *divergence term* in the identity which integrates to zero if M is a closed manifold and the vector field, call it δ, whose divergence we integrate is smooth. If Y has singularity and δ is not everywhere defined, then, in general, $\int_M \operatorname{div} \delta$ need not be zero. On the other hand if δ decays near the singularity, then one may expect (and prove whenever the decay is sufficiently strong) that div δ does integrate to zero. Here, let us recall that the field δ in all relevant Bochner formulas appears as a bilinear expression in the first and the second derivatives of u, something like $\Sigma\, \partial_i u\, \partial_{kl} u$. (In the Eells-Sampson formula the divergence term is

$$\partial \| \mathscr{D} u \|^2 = \operatorname{div} \operatorname{grad} \textstyle\sum \partial_i u\, \partial_j u).$$

If, for example, the map u in question has a singular set of codimension two or more (as in the above picture of maps from $\mathbf{R}^2$ to Y), and if the first and the second derivatives are bounded, then the divergence of such δ obviously integrates to zero as is seen by integrating δ over the complement of ε-neighborhoods of the singular locus and then letting $\varepsilon \to 0$. In fact what one needs for the vanishing $\int \operatorname{div} \delta$ is the decay of $\varepsilon^{-1} \| \mathscr{D} u \|\, \| \mathscr{D}^2 u \|$ for $\varepsilon \to 0$. (In fact, if the codimension of the singularity is two, one can relax the "bounded derivatives" condition to $\| \mathscr{D} u \|\, \| \mathscr{D}^2 u \| = 0\, (\varepsilon^{-1})$ for the distance $\varepsilon \to 0$ from the singularity of u).

Let us exhibit actual examples of harmonic maps into our tripod Y. To do that we embed Y into a two-dimensional singular space, namely to the unit cone Y′ over the circle of length 3π. The natural ($\mathbf{Z}_3$-symmetric) embedding of Y to Y′ is isometric and Y divides Y′ into three sectors each isometric to the half-plane. The projections of these half-planes to the boundary lines define a projection of Y′ to Y. Then we observe that Y′ is conformally equivalent to the unit disk D^2 and we think of Y′ as D^2 with a singular Kähler metric (with $K \leqslant 0$). It is easy to see that every *holomorphic* map of an arbitrary Kähler manifold M into Y′ is harmonic (in fact pluriharmonic) and by composing with the projection $Y' \to Y$ we obtain harmonic maps $M \to Y$. The simplest of these corresponds to the identity map of $M = D^2$ (with the flat metric) to $Y' = D^2$ (with the singular metric). Notice that the derivative of such a map at zero decays with the rate $\varepsilon^{1/2}$. (If we use the circle of length $k\pi$ we have the derivative $0(\varepsilon^{k/2-1})$).

The above considerations have led the first author to the following:

Conjecture. — If Y is a building, every harmonic map to Y must have a singular set of codimension at least two and $\|\mathscr{D}u\|\,\|\mathscr{D}^2 u\|$ should have a sufficient rate of decay in order to validate Bochner formulas.

It is also natural to expect that harmonic maps into general spaces Y of nonpositive curvature exist and are Lipschitz. These conjectures were proved by the second author and the proof occupies sections 1-6 of the present paper.

Before entering the analytic discussion of singular spaces on the technical level we add a few more motivating examples.

First we observe that the harmonic maps into tripods (and graphs in general) coming from holomorphic maps do satisfy the above regularity properties. Furthermore by looking at the level curves of the simplest such map of the disk $\mathscr{D}^2$ into the tripod Y we recognize (see Fig. 3) the familiar pattern associated to a quadratic differential on a Riemann surface.

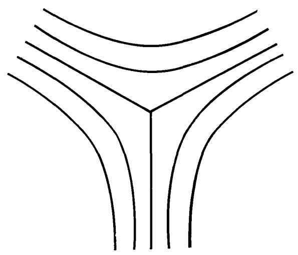

FIG. 3

Then we show that every harmonic map $u\colon D^2 \to Y$ gives rise to a *holomorphic quadratic differential* on D^2 which is equal away from the singular locus of u to the complexification of the form dy^2 for the length parameter y on the nonsingular part of the tripod Y. This is done by approximating Y by (regular) spaces $Y_{\mathscr{H}} \supset Y$ of constant negative curvature $-\mathscr{H}$, for $\mathscr{H} \to \infty$, and then by approximating u by harmonic maps $u^{\mathscr{H}} \to Y_{\mathscr{H}}$. Every such map gives rise to a holomorphic quadratic differential on D^2 coming from the (2, 0)-part of the pullback of the Riemannian metric of $Y_{\mathscr{H}}$ to D^2 and these differentials converge to the desired limit and are associated to the original harmonic map $u\colon D^2 \to Y$. Thus we see that the harmonic maps of surfaces into graphs are non-singular apart from a discrete set (where the corresponding quadratic differential vanishes). Then one can express $\mathscr{D}u$ and $\mathscr{D}^2 u$

(outside the singularity) in terms of the quadratic differential and check the regularity properties.

If dim $M \geqslant 3$ it seems more difficult to obtain local regularity results by the above approximation. Yet the global Bochner type theorems do follow this way. For example,

Every harmonic map of a compact Kähler manifold into a graph is pluriharmonic.

This immediately follows from the corresponding result by Jost-Yau and Carlson-Toledo for harmonic maps into spaces with constant negative curvature.

We conclude with an example of a map into a surface with $K \leqslant 0$ with an isolated singularity where the singular locus of the map in the domain has codimension one. We take the unit cone over the circle S_l^1 of length $l > 2\pi$ for Y and let M be the cylinder $M = M_0 \times [-1, 1]$ for some closed manifold M_0. We map the boundary $\partial M = (M_0 \times -1) \cup (M_0 \times 1)$ to the boundary $S^1 = \partial Y$ in such a way that the angular distance between the images of $M_0 \times -1$ and $M_0 \times 1$ in S_l^1 is at least π. For example we may send $M_0 \times \{-1\}$ to an arc of length $\varepsilon \leqslant \frac{1}{2} l - \pi$ and then send $M_0 \times 1$ symmetrically to the opposite arc in the S_l^1. It is easy to see that the convex hull in Y of two such arcs equals the union of the cones over these arcs and so the harmonic map $M \to Y$ solving the Dirichlet problem will be contained in this union of cones and thus have a singular hypersurface in M (separating $M_0 \times -1$ from $M_0 \times 1$) sent into the common vertex of the cones, where the metric of Y is singular. See Figure 4.

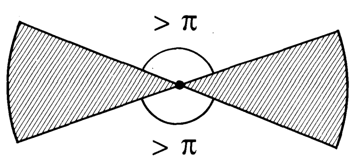

FIG. 4

(Notice that the solvability and Lipschitz regularity of the solutions of the Dirichlet problem follows for this Y by an approximation of the singular metric on Y by regular metrics with $K \leqslant 0$.)

Additional remarks. — (a) The conjectured regularity property of harmonic maps referring to Bochner formulas does not provide such formulas but rather allows us to reduce those to the nonsingular locus of the map. The relevant formula for our p-adic superrigidity is the one discovered by Corlette in the non-singular framework. (One could also use the Kodaira-Siu-Sampson formula applied to an auxiliary Kähler foliation associated to the domain manifold. But the "foliated" approach becomes somewhat cumbersome, though quite interesting from a geometric viewpoint, when the curvature of the receiving space is not strictly negative. Yet there are certain cases where the global conclusion obtained by using foliations cannot be achieved with known Bochner formulas). Notice here that Corlette also arrived at the idea that the harmonic maps into buildings may provide the solution of the p-adic superrigidity problem.

(b) Although there are some instance of harmonic maps into singular spaces which have been considered, the subject has not attracted a great deal of attention. (We should mention that Hodge theory and function theory on singular manifolds has been considered by a number of authors. A cohomology theory on Euclidean buildings was developed by H. Garland [G] to study p-adic group cohomology. The work of J. Cheeger has brought the subject of global analysis on singular spaces into prominence recently.) The uniform Lipschitz bound was exploited by the second author in the late seventies to construct Lipschitz harmonic maps into surfaces with cone metrics of nonpositive curvature. This work was refined and used to characterize the Teichmüller map by M. L. Leite [L]. S. Alexander, I. D. Berg, and R. Bishop [ABB] have studied geodesics for obstacle problems from this point of view and Nikolaev [N] has constructed minimal surfaces in singular spaces with curvature bounded from above. The works of Almgren [Al] and Lin [Lin] both deal with harmonic maps into special singular spaces. The paper by Y. J. Chiang [Ch] discusses harmonic maps into V-manifolds.

Part I: Harmonic maps into singular spaces

In this first part of the paper we develop some basic existence, uniqueness, and regularity results for harmonic maps into a class of nonpositively curved singular spaces.

1. Preliminary results

Let X be a locally compact Riemannian simplicial complex. By this we mean a space which is the geometric realization of a locally finite simplicial complex such that each geometric simplex is endowed with a Riemannian metric which is the restriction to the standard simplex of a smooth Riemannian metric defined in a neighborhood of that simplex. Moreover, assume that the maximal dimension of a simplex in X is k. Assume finally that X is properly isometrically embedded in a Euclidean space $\mathbf{R}^{\mathrm{N}}$ in the sense that the induced Riemannian metric on each simplex coincides with the given metric.

Let M be a smooth Riemannian manifold of dimension n and Riemannian metric g. For a bounded domain $\Omega \subseteq \mathrm{M}$ with smooth boundary, define the space of H^1 maps from Ω to X by

$$H^1(\Omega, X) = \{ u \in H^1(\Omega, \mathbf{R}^N) : u(x) \in X \text{ a.e. } x \in \Omega \}$$

where $H^1(\Omega, \mathbf{R}^N)$ is the Hilbert space of $\mathbf{R}^N$ vector valued L^2 functions on Ω with first distributional derivatives in L^2. The H^1 inner product is given by

$$\langle u, v \rangle_1 = \int_\Omega \left(u.v + \sum_{\alpha, \beta = 1}^n \frac{\partial u}{\partial x^\alpha} \cdot \frac{\partial u}{\partial x^\beta} g^{\alpha\beta}(x) \right) d\mu_g,$$

where we use $u.v$ to denote the Euclidean dot product of vectors in $\mathbf{R}^N$. For $u \in H^1(\Omega, X)$, we denote the *energy* of u by $E(u)$, so that

$$E(u) = \int_\Omega |\nabla u|^2 \, d\mu_g,$$

where $|\nabla u|^2 = \sum_{\alpha, \beta = 1}^n g^{\alpha\beta} \frac{\partial u}{\partial x^\alpha} \cdot \frac{\partial u}{\partial x^\beta}$ denotes the *energy density*. We now observe the following result which allows us to construct energy minimizing maps in the space $H^1(\Omega, X)$ with arbitrarily specified boundary data. Recall that if $u, v \in H^1(\Omega, \mathbf{R}^N)$, we say that $u = v$ on $\partial\Omega$ provided $u - v \in H_0^1(\Omega, \mathbf{R}^N)$ where $H_0^1(\Omega, \mathbf{R}^N)$ is the H^1-norm closure of smooth compactly supported $\mathbf{R}^N$-valued maps on Ω.

Lemma 1.1. – *Let* $\varphi \in H^1(\Omega, X)$. *There exists* $u \in H^1(\Omega, X)$ *such that* $u = \varphi$ *on* $\partial\Omega$, *and* $E(u) \leqslant E(v)$ *for all* $v \in H^1(\Omega, X)$ *with* $v = \varphi$ *on* $\partial\Omega$.

Proof. – Let $\{u_i\}$ be a minimizing sequence of maps in $H^1(\Omega, X)$ with $u_i = \varphi$ on $\partial\Omega$. Since bounded subsets of $H^1(\Omega, \mathbf{R}^N)$ are weakly compact, there is a subsequence again denoted $\{u_i\}$ which converges weakly to a map $u \in H^1(\Omega\ \mathbf{R}^N)$. Since X is a closed subset of $\mathbf{R}^N$ and a subsequence of $\{u_i\}$ converges pointwise almost everywhere, it follows that $u \in H^1(\Omega, X)$. Since the set $\{ v \in H^1(\Omega, \mathbf{R}^N) : v = \varphi \text{ on } \partial\Omega \}$ is a closed affine subspace of $H^1(\Omega, \mathbf{R}^N)$, it is weakly closed. Thus $u = \varphi$ *on* $\partial\Omega$, and we have established Lemma 1.1.

We now discuss the distance function on X and give a treatment of harmonic maps of an interval into X. Assuming that X is connected (which we do without loss of generality), we see that any two points $P_0, P_1 \in X$ can be joined by a path $\gamma : [0, 1] \to X$ which is Lipschitz as a map to $\mathbf{R}^N$. We can then define the Riemannian distance function $d(P_0, P_1)$ by

$$d(P_0, P_1) = \inf \{ L(\gamma) : \gamma \text{ a Lipschitz path from } P_0 \text{ to } P_1 \}.$$

It is immediate that $d(.,.)$ is a metric, and that (X, d) is a complete metric space. We show that the infimum is attained, and describe the associated harmonic map. We consider paths $\gamma : [0, 1] \to X$ with $\gamma(0) = P_0$ and $\gamma(1) = P_1$. Fix such a path φ, and let γ

be an energy minimizing path provided by Lemma 1.1. An easy application of the fundamental theorem of calculus shows that γ is Hölder continuous with exponent 1/2. We let $d\gamma/dt$ denote the L^2-vector valued function on [0, 1] which is the distributional derivative of γ. We will show that $|d\gamma/dt|$ is a constant a.e. on [0, 1]. To see this, let $\zeta(t)$ be a smooth real valued function with compact support in (0, 1), and consider the path of maps $\gamma_s(t)=\gamma(t+s\zeta(t))$. By the minimizing property of $\gamma=\gamma_0$ we have $E(\gamma)\leqslant E(\gamma_s)$ for all sufficiently small s, and hence the function $s\mapsto E(\gamma_s)$ has a minimum at $s=0$. We examine this function more carefully:

$$E(\gamma_s)=\int_0^1\left|\frac{d\gamma_s}{dt}\right|^2 dt=\int_0^1\left|\frac{d\gamma_s}{dt}\right|^2(1+s\zeta'(t))^{-1}\,d\tau$$

where $\tau=t+s\zeta(t)$, s being a *fixed* small number. Now

$$\frac{d\gamma_s}{dt}=(1+s\zeta'(t))\frac{d\gamma}{d\tau}$$

by the chain rule, and hence

$$E(\gamma_s)=\int_0^1\left|\frac{d\gamma}{d\tau}\right|^2(1+s\zeta'(t))\,d\tau.$$

Since the s dependence is explicitly exhibited here, we see that the function $s\mapsto E(\gamma_s)$ is a differentiable function of s. Thus its derivative vanishes at $s=0$, and we have

$$\int_0^1\left|\frac{d\gamma}{dt}\right|^2\zeta'(t)\,dt=0$$

for every smooth function ζ with compact support in (0, 1). This implies that the L^1 function $|d\gamma/dt|^2$ is equal to a constant almost everywhere on [0, 1]. Thus we have established the following result for energy minimizing maps of an interval into X. This will be used later when we develop the theory for $n\geqslant 2$.

Lemma 1.2. — *A map* $\gamma\in H^1([0, 1], X)$ *which minimizes energy among maps which coincide with* γ *at* $t=0, 1$ *is Lipschitz and satisfies* $|d\gamma/dt|=L(\gamma)$ a.e. *on* [0, 1]. *Moreover,* γ *is a length minimizing curve among all Lipschitz curves from* $\gamma(0)$ *to* $\gamma(1)$ *in* X.

We showed above that $|d\gamma/dt|=c$ a.e. for some constant c. Integrating we find $c=L(\gamma)$. To see that γ is length minimizing, let γ_1 be any Lipschitz curve with $\gamma_1(0)=\gamma(0)$, $\gamma_1(1)=\gamma(1)$. Assume that $\gamma_1:[0, 1]\to X$ is parametrized so that $|d\gamma_1/dt|=L(\gamma_1)$. We then have

$$L^2(\gamma)=E(\gamma)\leqslant E(\gamma_1)=L^2(\gamma_1),$$

and hence $L(\gamma)\leqslant L(\gamma_1)$. To see that any Lipschitz curve may be parametrized proportionate to arc length, we can consider all $H^1([0, 1], X)$ which have image in the closed set $\gamma_1([0,1])$ and which agree with γ_1 at $t=0, 1$. We can then minimize energy in this class and repeat our previous argument to show that the minimizer is parametrized with constant speed.

Finally, we make some general remarks about H^1 maps which will be needed in the next section. Let $u\in H^1(\Omega, \mathbf{R}^N)$, and let $x_0\in\Omega$. We will say that u is *approximately differentiable* at x_0 if there is a linear map $l(x)$ of the form $l(x)=A(x-x_0)+B$ with A an $N\times n$ matrix and $B\in\mathbf{R}^N$ such that

$$\lim_{\sigma\downarrow 0}\left\{\sigma^{-2-n}\int_{B_\sigma(x_0)}|u-l|^2\,d\mu_g+\sigma^{-n}\int_{B_\sigma}|\nabla u-\nabla l|^2\,d\mu_g\right\}=0.$$

It is then a general result (see [Z, Theorem 3.4.2]) about H^1 maps that u is approximately differentiable at almost every point $x_0\in\Omega$. Observe also that $l(x_0)=u(x_0)$ for a.e. x_0, and hence for a.e. x_0, $B\in X$ if $u\in H^1(\Omega, X)$. We will need the following result.

Lemma 1.3. — *Let* $u\in H^1(\Omega, X)$ *be a map whose image lies in a compact subset of* X, *and let* $x_0\in\Omega$ *be a point at which* u *is approximately differentiable with linear approximation* $l(x)=A(x-x_0)+B$. *If* $A\neq 0$ *and* $B\in X$, *then we have*

$$\lim_{\sigma\to 0}\left(\int_{\partial B_\sigma(x_0)}d^2(u(x), B)\,d\Sigma\right)^{-1}\sigma\int_{B_\sigma(x_0)}|\nabla u|^2\,d\mu_g=1.$$

Proof. — We first observe that there is no loss of generality in assuming the metric g to be Euclidean near x_0 because we can introduce Riemannian normal coordinates centered at x_0, and compare the integrals in the g metric with the corresponding Euclidean integrals in these coordinates. Note that the balls centered at x_0 are identical in the two metrics, and it is immediate that each of the two integrals appearing in the statement has ratio with the corresponding Euclidean integral which tends to 1. Thus it suffices to consider the Euclidean metric.

By the triangle inequality we have

$$\left|\left(\int_{B_\sigma(x_0)}|\nabla u|^2\,d\mu\right)^{1/2}-\left(\int_{B_\sigma(x_0)}|\nabla l|^2\,d\mu\right)^{1/2}\right|$$
$$\leqslant\left(\int_{B_\sigma(x_0)}|\nabla u-\nabla l|^2\,d\mu\right)^{1/2}\leqslant\varepsilon(\sigma)\,\sigma^{n/2}$$

where the second inequality holds because u is approximately differentiable at x. Since $A\neq 0$, we see that $\int_{B_\sigma(x_0)}|\nabla l|^2\,d\mu$ is a nonzero constant times σ^n. Thus it follows that

$$\lim_{\sigma \downarrow 0} \left(\int_{B_\sigma(x_0)} |\nabla l|^2 \, d\mu \right)^{-1} \int_{B_\sigma(x_0)} |\nabla u|^2 \, d\mu = 1. \tag{1.1}$$

In order to complete the proof we need to compose the distance function of X with the Euclidean distance for points near B. We claim that for any $\varepsilon > 0$, there is $r_0 > 0$ such that

$$\left| \frac{d(\mathrm{B}, \mathrm{P})}{|\mathrm{B} - \mathrm{P}|} - 1 \right| \leqslant \varepsilon \tag{1.2}$$

for $\mathrm{P} \neq \mathrm{B}$, $\mathrm{P} \in \mathrm{X}$ satisfying $|\mathrm{B} - \mathrm{P}| \leqslant r_0$. This follows from the hypothesis that each simplex containing B is smoothly embedded in $\mathbf{R}^{\mathrm{N}}$ so that there is a curve from B to P which is arbitrarily close to a straight line when P is close to B. Moreover, since the image of u lies in a compact subset of X, the function $d^2(u(x), \mathrm{B})$ is bounded and hence

$$\int_{\{x : |\mathrm{B} - u(x)| > r_0\} \cap \partial \mathrm{B}_\sigma(x_0)} d^2(u(x), \mathrm{B}) \, d\Sigma \leqslant c \operatorname{Vol}\{x \in \partial \mathrm{B}_\sigma(x_0) : |u(x) - \mathrm{B}| \geqslant r_0\}.$$

For $\sigma \ll r_0$ we then have

$$\operatorname{Vol}\{x \in \partial \mathrm{B}_\sigma(x_0) : |u(x) - \mathrm{B}| \geqslant r_0\} \leqslant c r_0^{-2} \int_{\partial \mathrm{B}_\sigma(x_0)} |u(x) - l(x)|^2 \, d\Sigma.$$

By the Sobolev trace inequality (see [Z]) we have

$$\int_{\partial \mathrm{B}_\sigma(x_0)} |u(x) - l(x)|^2 \, d\Sigma \leqslant c\sigma \int_{\mathrm{B}_\sigma(x_0)} |\nabla u - \nabla l|^2 \, d\mu + c\sigma^{-1} \int_{\mathrm{B}_\sigma(x_0)} |u - l|^2 \, d\mu.$$

Thus it follows that

$$\begin{aligned} &\lim_{\sigma \downarrow 0} \sigma^{-1-n} \int_{\partial \mathrm{B}_\sigma(x_0)} |u - l|^2 \, d\Sigma = 0 \\ &\lim_{\sigma \downarrow 0} \sigma^{-1-n} \int_{\{x \in \partial \mathrm{B}_\sigma(x_0) : |u(x) - \mathrm{B}| \geqslant r_0\}} d^2(u(x), \mathrm{B}) \, d\Sigma = 0. \end{aligned} \tag{1.3}$$

From (1.2) we have $(1-\varepsilon)|\mathrm{P} - \mathrm{B}| \leqslant d(\mathrm{P}, \mathrm{B}) \leqslant (1+\varepsilon)|\mathrm{P} - \mathrm{B}|$ for $\mathrm{P} \in \mathrm{X}$ with $|\mathrm{P} - \mathrm{B}| \leqslant r_0$, and therefore we will write

$$(1.4)\qquad \int_{\{x\in\partial B_\sigma(x_0):\,|u(x)-B|\leqslant r_0\}} d^2(u,\mathrm{B})\,d\Sigma \approx \int_{\{x\in\partial B_\sigma(x_0):\,|u(x)-B|\leqslant r_0\}} |u-\mathrm{B}|^2\,d\Sigma$$

where $\approx$ means that the ratio of the quantities is arbitrarily close to one. We also have from the triangle inequality and (1.3)

$$\left|\left(\int_{\partial B_\sigma(x_0)} |u-\mathrm{B}|^2\,d\Sigma\right)^{1/2} - \left(\int_{\partial B_\sigma(x_0)} |l-\mathrm{B}|^2\,d\Sigma\right)^{1/2}\right| \leqslant \left(\int_{\partial B_\sigma(x_0)} |u-l|^2\,d\Sigma\right)^{1/2}$$

$$\leqslant o(\sigma^{n+1/2}).$$

Since $\mathrm{A}\neq 0$ we see that $\int_{\partial B_\sigma(x_0)} |l-\mathrm{B}|^2\,d\Sigma$ is a positive constant times σ^{n+1}, and hence it follows that

$$\int_{\partial B_\sigma(x_0)} |u-\mathrm{B}|^2\,d\Sigma \approx \int_{\partial B_\sigma(x_0)} |l-\mathrm{B}|^2\,d\Sigma,$$

and each term is of the order σ^{n+1}. Combining (1.3) and (1.4) we see that

$$\int_{\partial B_\sigma(x_0)} d^2(u,\mathrm{B})\,d\Sigma \approx \int_{\partial B_\sigma(x_0)} |u-\mathrm{B}|^2\,d\Sigma,$$

and hence

$$(1.5)\qquad \int_{\partial B_\sigma(x_0)} d^2(u,\mathrm{B})\,d\Sigma \approx \int_{\partial B_\sigma(x_0)} |l-\mathrm{B}|^2\,d\Sigma.$$

Direct calculation shows that for a linear function $l(x)=\mathrm{A}(x-x_0)+\mathrm{B}$ we have for all $\sigma>0$

$$\left(\int_{\partial B_\sigma(x_0)} |l-\mathrm{B}|^2\,d\Sigma\right)^{-1} \sigma \int_{B_\sigma(x_0)} |\nabla l|^2\,d\mu = 1.$$

Thus combining this with (1.1) and (1.5) we have established the conclusion of Lemma 1.3.

2. Behavior of harmonic maps into nonpositively curved spaces

We showed in section one that minimizing maps of an interval into X are Lipschitz. In order to obtain Lipschitz bounds on minimizing maps $M \to X$ for $\dim M \geqslant 2$ (or even continuity of these maps), it is necessary to require that (X, d) have nonpositive curvature as a metric space. We now discuss this notion. First observe that if $y, x_0, x_1 \in \mathbf{R}^k$, and $x(s)$ is the unit speed geodesic from x_0 to x_1 parametrized on $[0, l]$, $l = |x_0 - x_1|$, then the function $D_0(s) = |x(s) - y|^2$ is a quadratic polynomial in s of the form $D_0(s) = s^2 + as + b$ where the constants a, b are uniquely determined by the boundary conditions $D_0(0) = |x_0 - y|^2$, $D_0(l) = |x_1 - y|^2$. Indeed, this is another way of saying that the function $|x - y|^2$ satisfies $\dfrac{\partial^2 |x-y|^2}{\partial x_i \partial x_j} = 2\,\delta_{ij}$. We say that a simply connected space X has *nonpositive curvature* if for any three points Q, P_0, $P_1 \in X$, the function $D(s) = d^2(P(s), Q)$ satisfies $D(s) \leqslant D_0(s)$ where $P(s)$, $s \in [0, l]$, $l = d(P_0, P_1)$, is a minimizing unit speed geodesic from P_0 to P_1 and $D_0(s)$ is the unique solution of $D_0''(s) = 2$ on $[0, l]$ with $D_0(0) = D(0)$, $D_0(l) = D(l)$. Thus the condition states that points of the side of a geodesic triangle opposite to Q in the space X are at least as close to Q as they would be in a Euclidean triangle with the same side lengths. Note that the statement that the Lipschitz function $D(s)$ lies below the monic quadratic polynomial with the same boundary data for every subinterval of $[0, l]$ is equivalent to the distributional inequality $D''(s) \geqslant 2$ on $[0, l]$. Precisely this means that for any nonnegative function $\zeta(s)$ with compact support in $(0, l)$ the following holds

$$\int_0^l D(s)\,\zeta''(s)\,ds \geqslant 2 \int_0^l \zeta(s)\,ds.$$

Thus X having nonpositive curvature is equivalent to the statement that the distance function is more convex than the Euclidean distance function. For a general space X we say that X has nonpositive curvature if its universal covering space has nonpositive curvature.

The following properties can be derived from the definition of nonpositive curvature (see [B, VI.3 B]). First, any two points in X can be joined by precisely one length minimizing path. Secondly, if P_0, P_1 and Q_0, Q_1 are two pairs of points in X, and we parametrize the geodesic paths from P_0 to P_1 and from Q_0 to Q_1 by $P(t)$, $Q(t)$ for $t \in [0, 1]$ where t is a constant speed parameter along each of the paths, then the function $g(t) = d(P(t), Q(t))$ is a convex function of t. Note that this second property implies that geodesics from a point spread more quickly than Euclidean geodesics since we may take $P_0 = Q_0$ and conclude that

$$d(\mathrm{P}(t), \mathrm{Q}(t)) \geqslant \frac{t}{t'} d(\mathrm{P}(t'), \mathrm{Q}(t'))$$

for $0 \leqslant t' \leqslant t \leqslant 1$. Finally, we observe that for any $\lambda \in [0, 1]$ and $\mathrm{Q} \in \mathrm{X}$ we can define a map $\mathrm{R}_{\lambda, \mathrm{Q}} : \mathrm{X} \to \mathrm{X}$ by

$$\mathrm{R}_{\lambda, \mathrm{Q}}(\mathrm{P}) = \mathrm{P}(\lambda),$$

where $\mathrm{P}(t)$, $t \in [0, 1]$, denotes the constant speed geodesic from Q to P parametrized on $[0, 1]$. By our previous discussion we see that $\mathrm{R}_{\lambda, \mathrm{Q}}$ is a Lipschitz map; in fact

$$d(\mathrm{R}_{\lambda, \mathrm{Q}}(\mathrm{P}_0), \mathrm{R}_{\lambda, \mathrm{Q}}(\mathrm{P}_1)) \leqslant \lambda\, d(\mathrm{P}_0, \mathrm{P}_1),$$

so the Lipschitz constant is at most λ. Moreover, the family of maps $\mathrm{R}_{\lambda, \mathrm{Q}}$, $\lambda \in [0, 1]$, defines a deformation retraction of X to the point Q, so that X is necessarily contractible.

We now digress briefly to present a technical result which will be needed to justify a calculation below. First suppose $\mathrm{F} : \mathbf{R}^{m_1} \to \mathbf{R}^{m_2}$ is a Lipschitz map, for a point $\mathrm{P} \in \mathbf{R}^{m_1}$ and a vector V we define the directional derivative $\mathrm{D}_{\mathrm{V}} \mathrm{F}(\mathrm{P})$ by

$$\mathrm{D}_{\mathrm{V}} \mathrm{F}(\mathrm{P}) = \lim_{h \to 0} \frac{\mathrm{F}(\mathrm{P} + h\mathrm{V}) - \mathrm{F}(\mathrm{P})}{h}$$

assuming the limit exists.

Lemma 2.1. — *Assume* $\gamma : [a, b] \to \mathbf{R}^{m_1}$ *is absolutely continuous and* $\mathrm{F} : \mathbf{R}^{m_1} \to \mathbf{R}^{m_2}$ *is Lipschitz. Then* $\mathrm{F} \circ \gamma$ *is absolutely continuous and at any point* $t_0 \in (a,b)$ *at which both* γ *and* $\mathrm{F} \circ \gamma$ *are differentiable it follows that* $\mathrm{D}_{\gamma'(t_0)} \mathrm{F}$ *exists at* $\gamma(t_0)$ *and*

$$(\mathrm{F} \circ \gamma)'(t_0) = \mathrm{D}_{\gamma'(t_0)} \mathrm{F}(\gamma(t_0)).$$

In particular this holds for almost all $t_0 \in [a, b]$.

Proof. — That $\mathrm{F} \circ \gamma$ is absolutely continuous is clear. Since γ is differentiable at t_0 we have $\gamma(t_0 + h) = \gamma(t_0) + \gamma'(t_0) h + o(h)$. Since $f \circ \gamma$ is differentiable at t_0 we have the existence of the limit

$$\lim_{h \to 0} \frac{\mathrm{F}(\gamma(t_0 + h)) - \mathrm{F}(\gamma(t_0))}{h}.$$

Since F is Lipschitz, we have from above

$$\mathrm{F}(\gamma(t_0 + h)) = \mathrm{F}(\gamma(t_0) + \gamma'(t_0) h + o(h)) = \mathrm{F}(\gamma(t_0) + \gamma'(t_0) h) + o(h).$$

It follows that $D_{\gamma'(t_0)} F(\gamma(t_0))$ exists and is equal to $(F \circ \gamma)'(t_0)$ as required. This proves Lemma 2.1.

Coming back to our situation, let us assume that $u \in H^1(\Omega, X)$ is energy minimizing and that its image lies in a compact subset K of X. Observe that the map $R : [0, 1] \times K \to \mathbf{R}^N$ is Lipschitz with respect to distance measured along X. On the other hand, on compact subsets of X we have $d(P_1, P_2) \leqslant c\,|P_1 - P_2|$, so R is Lipschitz in the Euclidean sense. By Kirzbraun's theorem (see [F1, 2.10.43]) R may be extended as a Lipschitz map from all of $\mathbf{R}^{N+1}$ into $\mathbf{R}^N$. We now consider the family of maps $u_\tau(x) = R_{1-\tau\zeta(x), Q}(u(x))$ for $x \in \Omega$ where $\tau \geqslant 0$ and ζ is a nonnegative smooth function with compact support on Ω. (We assume that τ is so small that $1 - \tau\zeta(x) \in [0, 1]$ for all $x \in \Omega$.) It is easy to see that $u_\tau \in H^1(\Omega, X)$ since u_τ is a composition of a Lipschitz map with an H^1 map. We will justify the following equality of distributional derivatives

$$\frac{\partial u_\tau}{\partial x_i}(x) = D_{\partial u/\partial x_i(x)} R_{1-\tau\zeta(x), Q}(u(x)) - \tau \frac{\partial \zeta}{\partial x_i} \frac{\partial R_{1-\tau\zeta, Q}}{\partial \lambda}(u(x)) \tag{2.1}$$

for $i = 1, \ldots, n$. For example we consider the case $i = 1$, and observe that (2.1) is a local result near a given $x_0 \in \Omega$. Consider a neighborhood of x_0 of the form $I_1 \times \mathcal{O}$ where $I_1 = (x_0^1 - r, x_0^1 + r)$ for some $r > 0$, and $\mathcal{O}$ is an open subset of $\mathbf{R}^{n-1} = \{(0, x^2, \ldots, x^n)\}$. Denote by $\hat{x}$ the point of $\mathcal{O}$ whose final $(n-1)$ coordinates are those of x. It then follows that for $\mathscr{L}^{n-1}$ almost every point $\hat{x}$ of $\mathcal{O}$ the map from $I_1 \to X$ given by $t \mapsto u(t, \hat{x})$ is in $H^1(I_1, X)$. For such an $\hat{x}$, let $\gamma : I_1 \to \mathbf{R}^{N+1}$ be the map

$$\gamma(t) = (1 - \tau\zeta(t, \hat{x}), u(t, \hat{x})).$$

It then follows that $u_\tau(t, \hat{x}) = R \circ \gamma(t)$. Since an H^1 map of an interval is equal a.e. to an absolutely continuous map, we may assume by redefining u on a set of measure zero in $I_1 \times 0$ that the map γ is absolutely continuous for almost every $\hat{x} \in \mathcal{O}$. We then apply Lemma 2.1 to conclude that for almost all $t \in I_1$ we have

$$\frac{\partial u_\tau}{\partial x_1}(t, \hat{x}) = D_{\gamma'(t)} R(\gamma(t)).$$

Now

$$\gamma'(t) = \left(-\tau \frac{\partial \zeta}{\partial x_i}(t, \hat{x}), \frac{\partial u}{\partial x_i}(t, \hat{x})\right) = \frac{\partial u}{\partial x_i}(t, \hat{x}) - \tau \frac{\partial \zeta}{\partial x_i}(t, \hat{x}) \frac{\partial}{\partial \lambda},$$

and hence (2.1) follows.

Next we observe that the function $P \mapsto d^2(P, Q)$ is Lipschitz on X, and therefore its restriction to the compact set K has a global Lipschitz extension to $\mathbf{R}^N$. Thus by Lemma 2.1 the chain rule calculation

$$\frac{\partial}{\partial x_i} d^2(u(x), Q) = D_{\partial u/\partial x_i} d^2(u(x), Q)$$

is justified. We also have for a tangent vector V to X at P

$$D_V R_{\lambda, Q}(P) \cdot \frac{\partial}{\partial \lambda} R_{\lambda, Q}(P) = \frac{1}{2} D_V d^2(R_{\lambda, Q}(P), Q)$$

provided the indicated diectional derivatives exist. (Both sides are equal to $d((R_{\lambda, Q}(P), Q)\, D_V R_{\lambda, Q}(P) \cdot \gamma'(R_{\lambda, Q}(P))$ where γ is the unit speed geodesic from Q to P.) Thus we square (2.1) and use this result on the cross term to get

$$\begin{aligned}\left|\frac{\partial u_\tau}{\partial x_i}(x)\right|^2 &= |D_{\partial u/\partial x(x)} R_{1-\tau\zeta(x), Q}(u(x))|^2 \\ &\quad - \tau \frac{\partial \zeta}{\partial x_i} \frac{\partial d^2(R_{1-\tau\zeta(x), Q}(u(x)), Q)}{\partial x_i} \\ &\quad + \tau^2 \left(\frac{\partial \zeta}{\partial x_i}\right)^2 \left|\frac{\partial R_{1-\tau\zeta, Q}}{\partial \lambda}(u(x))\right|^2.\end{aligned}$$

Using the contracting property of $R_{\lambda, Q}$ we thus have

$$\begin{aligned}E(u_\tau) \leqslant \int_\Omega (1-\tau\zeta)^2 |\nabla u|^2 \, d\mu \\ - \tau \int_\Omega \nabla \zeta \cdot \nabla d^2(R_{1-\tau\zeta(x), Q}(u(x)), Q)\, d\mu + 0(\tau^2).\end{aligned}$$

Since u is minimizing we therefore have

$$0 \leqslant -2\tau \int_\Omega \zeta |\nabla u|^2 \, d\mu + \tau \int_\Omega (\Delta\zeta)\, d^2(u_\tau(x), Q)\, d\mu + 0(\tau^2).$$

It follows that for every smooth nonnegative function ζ with compact support in Ω

$$\int_\Omega [(\Delta\zeta)\, d^2(u(x), Q) - 2\zeta |\nabla u|^2]\, d\mu \geqslant 0.$$

We restate this as a formal result.

Proposition 2.2. — *If* $u \in H^1(\Omega, X)$ *is energy minimizing and has image lying in a compact subset of* X, *then the function* $d^2(u(x), Q)$ *for any* $Q \in X$ *satisfies the differential inequality* $\Delta d^2(u(x), Q) - 2|\nabla u|^2 \geqslant 0$ *in the weak sense.*

As a first application of this result, choose $\zeta(x)$ to approximate the characteristic function of a small geodesic ball $B_\sigma(x_0)$ centered at a point $x_0 \in \Omega$. We then get for almost every σ

$$2\int_{B_\sigma} |\nabla u|^2 \, d\mu \leqslant \int_{\partial B_\sigma} \frac{\partial}{\partial r}(d^2(u(x), Q))\, d\Sigma. \tag{2.2}$$

We now derive the usual monotonicity formula for harmonic maps which can be done for minima of our problem. Note that the nonpositive curvature condition on X is not needed in this derivation. Let $\zeta(x)$ be a smooth function with support in a small neighborhood of a point $x_0 \in \Omega$. For $|\tau|$ small consider the diffeomorphism of Ω given in normal coordinates by $F_\tau(x) = (1 + \tau\zeta(x))\, x$ in a neighborhood of 0 with $F_\tau = \mathrm{id}$ outside this neighborhood. Consider the maps $u_\tau = u \circ F_\tau$. These are clearly in $H^1(\Omega, X)$, so the function $\tau \mapsto E(u_\tau)$ has a minimum at $\tau = 0$. To analyze this condition we perform a change of variable as we did in the geodesic case in section 1. We assume that $\zeta(x)$ has compact support in $B_\sigma(0)$ so that we may work in a single normal coordinate chart. We then set $y = F_\tau(x)$ and use the chain rule to compute

$$|\nabla u_\tau|^2(x) = \sum_{i,j,k,l} g^{ij} \frac{\partial y^k}{\partial x^i} \frac{\partial y^l}{\partial x^j} \left(\frac{\partial u}{\partial y^k} \cdot \frac{\partial u}{\partial y^l} \right).$$

The volume element $\sqrt{g}\, dx$ then becomes

$$\sqrt{g}\, dx = \det\left(\frac{\partial x^p}{\partial y^q} \right) \sqrt{g}\, dy.$$

Thus we may write the energy of u_τ in the form

$$\int_{B_\sigma(0)} |\nabla u_\tau|^2 \, d\mu = \int_{B_\sigma(0)} \Sigma\, a^{ij}(y, \tau)\, \frac{\partial u}{\partial y^i} \cdot \frac{\partial u}{\partial y^j}\, dy,$$

where $a^{ij}(y, \tau)$ is a smooth function of y and τ. Thus it follows that $\tau \mapsto E(u_\tau)$ is a smooth function of τ, and its derivatives may be computed by differentiation under the integral sign. In particular we have $E'(0) = 0$, and this gives us by direct calculation

$$0 = \int_{B_\sigma(0)} \Bigg[|\nabla u|^2 (2-n)\zeta - |\nabla u|^2 \sum_i x^i \frac{\partial \zeta}{\partial x^i} + 2 \sum_{i,j,k} g^{ik} \frac{\partial \zeta}{\partial x^i}\, x^j\, \frac{\partial u}{\partial x^j} \cdot \frac{\partial u}{\partial x^k} \Bigg] d\mu$$

$$+ \text{Remainder},$$

where the remainder term arises from the fact that the metric is not exactly Euclidean. The remainder term is given precisely by

$$\int_{B_\sigma(0)} \left[-\zeta \sum_{i,j,k} \frac{\partial g^{ij}}{\partial x^k} x^k \frac{\partial u}{\partial x^i} \cdot \frac{\partial u}{\partial x^j} \sqrt{g} + |\nabla u|^2 \zeta \sum_i x^i \frac{\partial \sqrt{g}}{\partial x^i} \right] dx.$$

Observe in particular that this term is bounded by a constant $\times \sigma^2 E_{B_\sigma(0)}(u)$. Taking ζ to be an approximation to the characteristic function of the ball $B_\sigma(0)$ we get

$$0 = (2-n+0(\sigma^2)) \int_{B_\sigma(0)} |\nabla u|^2 \, d\mu + \sigma \int_{\partial B_\sigma(0)} |\nabla u|^2 \, d\Sigma - 2\sigma \int_{\partial B_\sigma(0)} \left| \frac{\partial u}{\partial r} \right|^2 d\Sigma. \tag{2.3}$$

We now introduce the notation $E(\sigma)$, $I(\sigma)$ defined by

$$E(\sigma) = \int_{B_\sigma(0)} |\nabla u|^2 \, d\mu, \quad I(\sigma) = \int_{\partial B_\sigma(0)} d^2(u(x), Q) \, d\Sigma(x).$$

Since we are working in normal coordinates observe that if $f(x)$ is a nonnegative function we have

$$\frac{d}{d\sigma} \int_{\partial B_\sigma(0)} f \, d\Sigma = \int_{\partial B_\sigma(0)} \frac{\partial f}{\partial r} d\Sigma + (n-1)\sigma^{-1} \int_{\partial B_\sigma(0)} f \, d\Sigma + 0(\sigma) \int_{\partial B_\sigma(0)} f \, d\Sigma.$$

We now compute logarithmic derivatives

$$\frac{I'(\sigma)}{I(\sigma)} = \frac{n-1}{\sigma} + (I(\sigma))^{-1} \int_{\partial B_\sigma(0)} \frac{\partial}{\partial r}(d^2(u, Q)) \, d\Sigma + 0(\sigma).$$

(We should remark that $I(\sigma)$ is an absolutely continuous function for $\sigma > 0$.) From (2.3) we have

$$\frac{E'(\sigma)}{E(\sigma)} = \frac{n-2}{\sigma} + 2(E(\sigma))^{-1} \int_{\partial B_\sigma(0)} \left| \frac{\partial u}{\partial r} \right|^2 d\Sigma + 0(\sigma).$$

Therefore

$$\frac{I'(\sigma)}{I(\sigma)} - \frac{E'(\sigma)}{E(\sigma)} = \frac{1}{\sigma} + (E(\sigma) I(\sigma))^{-1} \left[E(\sigma) \int_{\partial B_\sigma(0)} \frac{\partial}{\partial r}(d^2(u, Q)) \, d\Sigma \right.$$

$$-2\,\mathrm{I}(\sigma)\int_{\partial B_\sigma}\left|\frac{\partial u}{\partial r}\right|^2 d\Sigma\Bigg]+0(\sigma)$$

which together with (2.2) implies the inequality

$$\frac{d}{d\sigma}\log\left(\frac{\mathrm{I}(\sigma)}{\sigma\,\mathrm{E}(\sigma)}\right)\leqslant 2(\mathrm{E}(\sigma)\,\mathrm{I}(\sigma))^{-1}\Bigg[\left(\int_{\partial B_\sigma(0)} d(u,\mathrm{Q})\frac{\partial}{\partial r}d(u,\mathrm{Q})\,d\Sigma\right)^2 - \left(\int_{\partial B_\sigma(0)} d^2(u,\mathrm{Q})\,d\Sigma\right)\left(\int_{\partial B_\sigma(0)}\left|\frac{\partial u}{\partial r}\right|^2 d\Sigma\right)\Bigg]+0(\sigma). \tag{2.4}$$

Since $\left|\frac{\partial}{\partial r}d(u,\mathrm{Q})\right|\leqslant\left|\frac{\partial u}{\partial r}\right|$, it follows by the Schwarz inequality that

$$\frac{d}{d\sigma}\left[e^{c_1\sigma^2}\frac{\sigma\,\mathrm{E}(\sigma)}{\mathrm{I}(\sigma)}\right]\geqslant 0 \tag{2.5}$$

for a constant c_1 depending on the metric g. Of course (2.5) holds only under the assumption that $\mathrm{I}(\sigma)>0$ for $\sigma>0$. Notice however that from Proposition 2.2 it follows that the function $d^2(u(x),\mathrm{Q})$ is subharmonic, so that if $\mathrm{I}(\sigma)=0$ for some $\sigma>0$, then the map u is equal almost everywhere to Q in a neighborhood of 0.

For any $x\in\Omega$, $\sigma>0$, $\mathrm{Q}\in\mathrm{X}$ we define an order function $\mathrm{Ord}(x,\sigma,\mathrm{Q})$ by

$$\mathrm{Ord}(x,\sigma,\mathrm{Q})=e^{c_1\sigma^2}\frac{\sigma\displaystyle\int_{B_\sigma(x)}|\nabla u|^2\,d\mu}{\displaystyle\int_{\partial B_\sigma(x)} d^2(u(x),\mathrm{Q})\,d\Sigma(x)}.$$

The reason for this notation is that for a harmonic function

$$\lim_{\sigma\downarrow 0}\mathrm{Ord}(x,\sigma,u(x))=\mathrm{Order}(u-u(x)),$$

that is, the order with which u attains its value $u(x)$ at x. Alternatively, it is the degree of the dominant homogeneous harmonic polynomial which approximates $u-u(x)$ near x. In particular, for harmonic functions (or harmonic maps into smooth manifolds of nonpositive curvature) this limit is a positive integer.

Generally, if $x\in\Omega$ and $\sigma>0$, then the function

$$\mathrm{Q}\mapsto\int_{\partial B_\sigma(x)} d^2(u,\mathrm{Q})\,d\Sigma$$

is a convex function on X with compact sublevel sets, and hence has a unique minimum point $Q_{x,\sigma} \in X$. The function $Q \mapsto \operatorname{Ord}(x, \sigma, Q)$ thus has a unique maximum point at $Q_{x,\sigma}$. We now define $\operatorname{Ord}(x)$ by

$$\operatorname{Ord}(x) = \lim_{\sigma \to 0} \operatorname{Ord}(x, \sigma, Q_{x,\sigma}).$$

This limit exists because the function $\sigma \mapsto \operatorname{Ord}(x, \sigma, Q_{x,\sigma})$ is monotone increasing in σ. Moreover, for a fixed $\sigma > 0$, the function $x \mapsto \operatorname{Ord}(x, \sigma, Q_{x,\sigma})$ is a continuous function, and hence it follows that the function $x \mapsto \operatorname{Ord}(x)$ is upper semicontinuous since it is the decreasing limit of a family of continuous functions. We now prove the following result.

Theorem 2.3. — *Suppose* $u \in H^1(\Omega, X)$ *is an energy minimizing map with image in a compact subset of a nonpositively curved complex* X. *Then u is (equal a.e. to) a locally Lipschitz map.*

Proof. — Since $u \in H^1(\Omega, X)$, it is approximately differentiable in the sense of Lemma 1.3 at almost every point of Ω. Consider a point x_0 in the closure of the set of points at which u has *nonzero* approximate derivative. By Lemma 1.3, x_0 is a limit of points x_j at which $\operatorname{Ord}(x_j) \geq 1$. Therefore by the upper semicontinuity of the $\operatorname{Ord}(.)$ function it follows that $\operatorname{Ord}(x_0) \geq 1$. Let $\alpha = \operatorname{Ord}(x_0)$ and fix $\sigma_0 > 0$ so that $B_{\sigma_0}(x_0) \in \Omega$. Let $\sigma_1 \in (0, \sigma_0)$, and note that the monotonicity of the ratio implies

$$\sigma \int_{B_\sigma(x_0)} |\nabla u|^2 \, d\mu \geq \alpha e^{-c_1 \sigma^2} \int_{\partial B_\sigma(x_0)} d^2(u, Q_1) \, d\Sigma$$

for all $\sigma \in [\sigma_1, \sigma_0)$ where $Q_1 = Q_{x_0, \sigma_1}$. Combining this with (2.2) yields

$$\begin{aligned} \alpha e^{-c_1 \sigma^2} I(\sigma) &\leq \frac{1}{2} \sigma \int_{\partial B_\sigma(x_0)} \frac{\partial}{\partial r}(d^2(u(x), Q_1)) \, d\Sigma(x) \\ &\leq \frac{1}{2}(\sigma I'(\sigma) - (n-1) I(\sigma)) + 0(\sigma^2) I(\sigma). \end{aligned}$$

This implies

$$\frac{I'(\sigma)}{I(\sigma)} \geq \frac{n-1+2a}{\sigma} - 0(\sigma),$$

where σ is any radius in $[\sigma_1, \sigma_0)$. Integrating from σ_1 to σ_0 and *fixing* σ_0 we obtain

$$\sigma_1^{-(n-1)} I(\sigma_1) \leq c \, \sigma_1^{2\alpha} I(\sigma_0).$$

Since the function $d^2(u(x), Q_1)$ is a subharmonic function (Proposition 2.2) on Ω, the mean value inequality implies

$$\sup_{x \in B_{\sigma_1/2}(x_0)} d^2(u(x), Q_1) \leqslant c\, \sigma_1^{2\alpha}.$$

(Note that $I(\sigma_0)$ is bounded independent of Q_1 because $u(\Omega) \subseteq K$.) In particular, by the triangle inequality,

$$d(u(x), u(x_0)) \leqslant d(u(x), Q_1) + d(u(x_0), Q_1) \leqslant 2\, c\, \sigma_1^{\alpha}$$

for $x \in B_{\sigma_1/2}(x_0)$. Thus if $x \in B_{\sigma_0/2}(x_0)$, we may choose $\sigma_1 = 2|x - x_0|$ and conclude $d(u(x), u(x_0)) \leqslant c\,|x - x_0|^{\alpha}$ for $x \in B_{\sigma_0/2}(x_0)$. Since $\alpha \geqslant 1$ this certainly implies that for any x_0 at which the approximate derivative exists and is nonzero we have $\left|\dfrac{\partial u}{\partial x_i}\right| \leqslant c$, $i = 1, \ldots, n$. It follows that u has bounded first derivatives locally in Ω, and hence u is (equal a.e. to) a locally Lipschitz function. This completes the proof of Theorem 2.3.

Remark. — The previous result leaves open the possibility that u might be constant on an open subset of Ω. We will show in the next section that this does not occur. Thus it will follow that $\mathrm{Ord}(x)$ is defined for all $x \in \Omega$.

We will need the following local estimate in order to apply compactness arguments to gain more detailed information about harmonic maps.

Theorem 2.4. — *Let* $u \in H^1(B_1(0), X)$ *be a least energy map* (*with image in a compact subset of* X) *for some metric* g *on* $B_1(0) = \{x \in \mathbf{R}^n : |x| \leqslant 1\}$. *There is a constant* c *depending only on* g (*e.g. on the* C^2 *norm of the matrix valued functions* $(g_{ij}(x))$, $(g^{ij}(x))$ *such that*

$$\sup_{B_{1/2}(0)} |\nabla u|^2 \leqslant c \int_{B_1(0)} |\nabla u|^2\, d\mu.$$

Proof. — By Theorem 2.3 the map u is locally Lipschitz. We need to estimate its Lipschitz constant. We first observe that we can replace X by a dilated complex μX where we assume by translation of coordinates in $\mathbf{R}^N$ that $u(0) = 0$. The complex μX still has nonpositive curvature, and we may choose μ so that the map μu has energy equal to 1 on $B_1(0)$. Thus we may assume without loss of generality that

$$\int_{B_1(0)} |\nabla u|^2\, d\mu = 1.$$

We also observe that it suffices to prove $|\nabla u|^2(0) \leqslant c$ where we may assume that $x = 0$ is a point of approximate differentiability of u. This follows just by changing the center of balls. Now we may of course assume $\nabla u(0)\neq 0$, and hence $\mathrm{Ord}(0)\geqslant 1$. It follows that $\mathrm{Ord}(0, \sigma, \mathrm{Q}_{0,\sigma})\geqslant 1$ for $\sigma\in(0,1)$, and therefore

$$\int_{\partial \mathrm{B}_{1/2}(0)} d^2(u, \mathrm{Q}_{0,1/2})\, d\Sigma \leqslant c \int_{\mathrm{B}_{1/2}(0)} |\nabla u|^2\, d\mu \leqslant c$$

for a constant c depending only on g. The fact that $d^2(u(x), \mathrm{Q}_{0,1/2})$ is subharmonic then implies

$$\sup_{x\in \mathrm{B}_{1/4}(0)} d^2(u(x), \mathrm{Q}_{0,1/2})\leqslant c$$

for a new constant c. In particular it follows that the distance from $\mathrm{Q}_{0,\sigma}$ to $0=u(0)$ is bounded for $\sigma\in(0,1/4)$. Taking $\sigma_0=1/4$ we may then apply the argument of Theorem 2.3 to show that for $x\in \mathrm{B}_{1/8}(0)$ we have from above

$$d^2(u(x), 0)\leqslant c|x|^2 \sup_{\sigma\in(0,1/4)} \int_{\partial \mathrm{B}_{1/4}} d^2(u, \mathrm{Q}_{0,\sigma})\, d\Sigma \leqslant c|x|^2.$$

This gives the desired conclusion and completes the proof of Theorem 2.4.

Remark. — For harmonic maps into smooth manifolds of nonpositive curvature the conclusion of Theorem 2.4 is a well known result of Eells and Sampson [ES]. The usual proof of this is based on the Bochner formula for the calculation of $\Delta|\nabla u|^2$. This proof seems to rely heavily on the smoothness of X whereas the proof we have given is a "lower order" proof which works in a setting which allows X to be singular.

3. Approximation by homogeneous maps

To begin this section we consider the case when the image complex X is a geometric cone in $\mathbf{R}^{\mathrm{N}}$; that is, if $\mathrm{Q}\in\mathrm{X}$, $\lambda\in\mathbf{R}_+$, then $\lambda\mathrm{Q}\in\mathrm{X}$. Under this assumption it is natural to consider energy minimizing maps $u\colon \mathbf{R}^n\to \mathrm{X}$ which are homogeneous of some degree $\alpha\geqslant 0$. We make the standing assumption throughout the remainder of this paper that X has nonpositive curvature. Thus we assume that $u(\lambda x)=\lambda^\alpha u(x)$ for $x\in\mathbf{R}^n$, $\lambda\geqslant 0$. From the results of section two we know that the map u is locally Lipschitz on $\mathbf{R}^n$ and hence if the map u is not identically zero we must have $\alpha\geqslant 1$.

There is a special class of homogeneous maps which we call *regular homogeneous maps*. To describe these we consider an embedding $\mathrm{J}:\mathbf{R}^m\to\mathrm{X}$ which is isometric and totally geodesic. This means that $d(\mathrm{J}(x), \mathrm{J}(y))=|x-y|$ and the image of a line under J is a geodesic in X. (Note that the set $\mathrm{J}(\mathbf{R}^m)$ need not be a plane in $\mathbf{R}^{\mathrm{N}}$.) Now suppose $v\colon\mathbf{R}^n\to\mathbf{R}^m$ is a homogeneous harmonic map. This simply means that $v(x)=(v_1(x), \ldots, v_m(x))$ where each v_i is a homogeneous harmonic polynomial of a given degree α. If the map $u = J \circ v$ is homogeneous, then we refer to such a map u as a *regular homogeneous* map. It is regular in the sense that it can be described in terms of a smooth (in fact polynomial) map.

A priori it would seem that regular homogeneous maps would be quite rare. However, the next result implies that they occur in abundance.

Proposition 3.1. — *A homogeneous minimizing map* $u: \mathbf{R}^n \to X$ *is regular if it is of degree* 1.

Before we give the proof of this result we need to introduce a new concept. We will say that a minimizing map $u: B_1(0) \to X$ is *intrinsically homogeneous* if there is $\alpha \geqslant 1$ such that for $x \in B_1(0)$ we have $d(u(x), u(0)) = |x|^\alpha d(u(x/|x|), u(0))$, and for each $x \in \partial B_1(0)$ the curve $t \mapsto u(tx)$ is a geodesic in X. The following result gives us a simple criterion which guarantees that a map is intrinsically homogeneous.

Lemma 3.2. — *If* $u: B_1(0) \to X$ *is a minimizing map from the unit ball in* $\mathbf{R}^n$ *with Euclidean metric such that for each* $\sigma \in (0, 1)$ *we have* $\operatorname{Ord}(0, \sigma, Q_{0,\sigma}) = \alpha$ *for some fixed* $\alpha \geqslant 1$, *then u is intrinsically homogeneous of degree* α.

Proof. — Since the domain metric is Euclidean, inequality (2.4) holds without the $0(\sigma)$ term. Next observe that since u is Lipschitz we have $\lim_{\sigma \to 0} Q_{0,\sigma} = u(0)$ so that if we first fix σ_0 small we have for $\sigma \in [\sigma_0, 1)$

$$\operatorname{Ord}(0, \sigma, Q_{0,\sigma_0}) \leqslant \operatorname{Ord}(0, \sigma, Q_{0,\sigma})$$

by the maximizing property of $Q_{0,\sigma}$. Since the right hand side is equal to α, and the left hand side is equal to α for $\sigma = \sigma_0$ and is monotone increasing, it follows that

$$\operatorname{Ord}(0, \sigma, Q_{0,\sigma_0}) = \alpha \qquad \text{for} \quad \sigma \in [\sigma_0, 1).$$

Letting σ_0 tend to zero we see that $\operatorname{Ord}(0, \sigma, u(0)) = \alpha$ for all $\sigma \in (0, 1)$. We now apply (2.4) with $Q = u(0)$ so that the left hand side vanishes. We have

$$\left(\int_{\partial B_\sigma(0)} d^2(u, u(0))\, d\Sigma\right)\left(\int_{\partial B_\sigma(0)} \left|\frac{\partial u}{\partial r}\right|^2 d\Sigma\right) = \left(\int_{\partial B_\sigma(0)} d(u, u(0))\, \frac{\partial}{\partial r}\, d(u, u(0))\, d\Sigma\right)^2.$$

It follows that $(\partial/\partial r)\, d(u, u(0)) = |\partial u/\partial \tau|$ a.e., and that for $\sigma \in (0, 1)$, there is a constant $h(\sigma)$ such that $(\partial/\partial r)\, d(u, u(0)) = h(\sigma)\, d(u, u(0))$. Integrating the first equality along the ray $\gamma : (\sigma, 1) \to \mathbf{R}^n$ given by $\gamma(r) = r\xi$ for some $\xi \in \partial B_1(0)$, we find

$$L(u(\gamma)) = d(u(\xi), u(0)) - d(u(\sigma\xi), u(0)) \leqslant d(u(\xi), u(\sigma\xi)).$$

In particular it follows that $u(\gamma)$ is a geodesic path in X. We now return to (2.2), and observe that since equality holds in (2.4), we must also have equality in (2.2). This then gives us

$$E(\sigma) = \int_{\partial B_\sigma(0)} d(u, u(0)) \frac{\partial}{\partial r} d(u, u(0))\, d\Sigma = h(\sigma)\, I(\sigma).$$

On the other hand we have $E(\sigma) = \alpha\sigma^{-1} I(\sigma)$, so we conclude that $h(\sigma) = \alpha\sigma^{-1}$. We may then integrate along a ray from x to $x/|x|$ to obtain

$$d(u(x), u(0)) = |x|^\alpha d\left(u\left(\frac{x}{|x|}\right), u(0)\right).$$

This completes the proof of Lemma 3.2.

Proof of Proposition 3.1. — Suppose $u : \mathbb{R}^n \to X$ is homogeneous of degree 1. It follows immediately that $\mathrm{Ord}(0, \sigma, u(0))$ is a constant independent of σ, and in fact that this constant is one. (To see this, observe that equality holds in (2.4) and (2.2) while $(\partial/\partial r)\, d(u, u(0)) = r^{-1} d(u, u(0))$.) Since $Q_{0,\sigma}$ approaches $u(0)$ as σ approaches 0, we have, for $\sigma \geqslant \sigma_0$,

$$1 \leqslant \mathrm{Ord}(0, \sigma_0, Q_{0,\sigma_0}) \leqslant \mathrm{Ord}(0, \sigma, Q_{0,\sigma_0}).$$

Letting $\sigma_0 \to 0$ we then have $\mathrm{Ord}(0) = 1$. The homogeneity of u then implies that $\mathrm{Ord}(0, \sigma, Q_{0,\sigma})$ is a constant independent of σ, and hence this constant is identically one. Therefore we have $Q_{0,\sigma} = u(0)$ for all σ. Because $x \mapsto \mathrm{Ord}(x)$ is uppersemicontinuous, at least one for all x, equal to one for $x = 0$, and homogeneous of degree zero, we have $\mathrm{Ord}(x) = 1$ for all $x \in \mathbf{R}^n$. On the other hand we have, for any $\lambda > 0$,

$$\mathrm{Ord}(x, \sigma, u(x)) = \mathrm{Ord}(\lambda x, \lambda\sigma, \lambda u(x)),$$

so we may take $\lambda = \sigma^{-1}$ and conclude

$$\lim_{\sigma \to \infty} \mathrm{Ord}(x, \sigma, u(x)) = 1.$$

It now follows that for all $x \in \mathbf{R}^n$ and all $\sigma > 0$ we have $\mathrm{Ord}(x, \sigma, u(x)) = 1$. From Lemma 3.2 we conclude that u is intrinsically homogeneous of degree one about every point. It follows that the restriction of u to any line parametrizes a geodesic in X with constant speed. The fact that equality holds in (2.2) for $Q=u(x_0)$ on $B_\sigma(x_0)$ for arbitrary σ implies that for each x_0 the function $d^2(u, u(x_0))$ is a *weak solution* of $\Delta_x d^2(u(x), u(x_0))=2|\nabla u|^2$. Since for all $x_0\in\mathbf{R}^n$ the function $x\mapsto d^2(u(x), u(x_0))$ is homogeneous of degree 2 about x_0, it follows that $|\nabla u|^2$ is homogeneous of degree zero about x_0 for every x_0. Therefore $|\nabla u|^2$ is a constant, say e_0. Thus it follows that $x\mapsto d^2(u(x), u(0))$ has constant Laplacian and quadratic growth. This function is therefore a quadratic polynomial. Since the function is everywhere positive and vanishes quadratically at $x=0$ we must have

$$d^2(u(x), u(0))=\sum_{i, j=1}^{n} g_{ij}x^i x^j$$

for an $n\times n$ symmetric $G=(g_{ij})$. This matrix is positive semi-definite, and we can find an orthonormal basis $e_1, \ldots, e_n$ for $\mathbf{R}^n$ such that ${}^t e_j G e_j=\lambda_i\delta_{ij}$ with $\lambda_i\geqslant 0$. By reordering $e_1, \ldots, e_n$, we may assume that $\lambda_i>0$ for $i=1, \ldots, m$ and $\lambda_i=0$ for $i=m+1, \ldots, n$. By change of coordinates we assume that the $x^1, \ldots, x^n$ are coordinates associated to the basis $e_1, \ldots, e_n$ so that

$$d^2(u(x), u(0))=\sum_{i=1}^{m}\lambda_i(x^i)^2.$$

Let $v:\mathbf{R}^n\to\mathbf{R}^m$ be the linear map given by

$$v(x^1, \ldots, x^n)=(\lambda_1^{1/2}x^2, \ldots, \lambda_m^{1/2}x^m),$$

and let $J:\mathbf{R}^m\to X$ be given by

$$J(y^1, \ldots, y^m)=u(\lambda_1^{-1/2}y_1, \ldots, \lambda_m^{-1/2}y^m, 0, \ldots, 0).$$

We then have $u=J\circ v$, and

$$d^2(J(y), J(0))=\sum_{i=1}^{m}\lambda_i(\lambda_i^{-1/2}y^i)^2=\sum_{i=1}^{m}(y^i)^2,$$

so that J is an isometric totally geodesic embedding. (Note that J is an embedding because $J(x)=J(y)$ implies that the image of the segment $\overline{xy}$ is a geodesic with the same initial and final point, thus $x=y$.) This completes the proof of Proposition 3.1.

Now we return to the general situation of a minimizing map $u:\Omega\to X$. Given a point $x_0\in\Omega$, we will attempt to approximate the map u near x_0 by a homogeneous map. We choose coordinates so that $x_0=0$ and $u(x_0)=0$. We next observe that $\operatorname{Ord}(0)=\lim_{\sigma\to 0}\operatorname{Ord}(0, \sigma, u(0))$. To see this, note that $\operatorname{Ord}(0)\geqslant\lim_{\sigma\to 0}\operatorname{Ord}(0, \sigma, u(0))$ by the choice of $Q_{0,\sigma}$. On the other hand $Q_{0,\sigma}$ approaches $u(0)$ as σ tends to zero, so given $\varepsilon>0$ and $\sigma_0>0$ we have for σ sufficiently small

$$\operatorname{Ord}(0, \sigma, Q_{0,\sigma}) \leqslant \operatorname{Ord}(0, \sigma_0, Q_{0,\sigma}) \leqslant \operatorname{Ord}(0, \sigma_0, u(0)) + \varepsilon.$$

Since σ_0 is arbitrary, it follows that $\operatorname{Ord}(0) \leqslant \lim_{\sigma \to 0} \operatorname{Ord}(0, \sigma, u(0))$ as required. Now let $\alpha = \operatorname{Ord}(0)$ and fix a normal coordinate chart on $B_{\sigma_0}(0)$. For λ, $\mu>0$, define the map $u_{\lambda,\mu}(x) = \mu^{-1} u(\lambda x)$. This is then a minimizing map from $B_{\lambda^{-1}\sigma_0}(0)$ with metric $g_\lambda(x) = g(\lambda x)$ to the complex $\mu^{-1}X = \{\mu^{-1}P : P \in X\}$. Notice that the complex $\mu^{-1}X$ again has non-positive curvature since distances are multiplied by a constant factor. We have, by a change of variable,

$$\int_{B_\sigma(0)} |\nabla u_{\lambda,\mu}|^2_{g_\lambda}\, d\mu_{g_\lambda} = \mu^{-2}\lambda^{2-n} \int_{B_{\lambda\sigma}(0)} |\nabla u|^2_g\, d\mu_g,$$

$$\int_{\partial B_\sigma(0)} d^2_{\mu^{-1}X}(u_{\lambda,\mu}, 0)\, d\Sigma_{g_\lambda} = \mu^{-2}\lambda^{1-n} \int_{\partial B_{\lambda\sigma}(0)} d^2_X(u, 0)\, d\Sigma_g.$$

In particular, $\operatorname{Ord}^{u_{\lambda,\mu}}(0, \sigma, 0) = \operatorname{Ord}^u(0, \lambda\sigma, 0)$ for any $\sigma \in (0, \lambda^{-1}\sigma_0)$. For any small $\lambda>0$, let $\mu = (\lambda^{1-n} I(\lambda))^{1/2}$, so that we then have

$$\int_{\partial B_1(0)} d^2_{\mu^{-1}X}(u_{\lambda,\mu}, 0)\, d\Sigma_{g_\lambda} = 1.$$

Since $\operatorname{Ord}^{u_{\lambda,\mu}}(0, 1, 0)$ tends to $\alpha = \operatorname{Ord}^u(0)$ as $\lambda \to 0$, we also have

$$\int_{B_1(0)} |\nabla u_{\lambda,\mu}|^2_{g_\lambda}\, d\mu_{g_\lambda} \leqslant 2\alpha$$

for λ small. Thus $u_{\lambda,\mu}$ has uniformly bounded energy and then by Theorem 2.4, has uniformly bounded Lipschitz constant on compact subsets of $B_1(0)$. Thus for any sequence $\{\lambda_i\}$ tending to zero, the corresponding sequence of maps $\{u_i\}$ has a uniformly convergent subsequence, again denoted $\{u_i\}$, which has Lipschitz limit which we denote $u_* : B_1(0) \to X_0$ where X_0 denotes the tangent cone of X at 0. The next result shows that u_* is a nonconstant homogeneous minimizing map of degree α. We will refer to such a map u_* as a *homogeneous approximating map* for u at the point 0.

Proposition 3.3. — *The map u_* is a nonconstant homogeneous minimizing map of degree α.*

Proof. — To show that u_* is nonconstant, we will show that $I^{u_i}(\sigma) \geqslant \varepsilon_0 \sigma^{\alpha_1}$ for $\sigma \in (0, 1)$ and constants ε_0, α_1. We are using the obvious notation

$$I^{u_i}(\sigma) = \int_{\partial B_\sigma(0)} d^2_{\mu_i^{-1} X}(u_i, 0)\, d\Sigma_{g_{\lambda_i}}.$$

Since u_i converges uniformly to u_*, it follows that u_* is nonconstant. To derive this lower bound, let $\theta \in (0, 1)$ and $r_0 \in (\theta, 1]$. We then have, by an obvious estimate,

$$\begin{aligned} I^{u_i}(r_0) - I^{u_i}(\theta) &= \int_\theta^{r_0} \frac{d}{d\sigma} I^{u_i}(\sigma)\, d\sigma \\ &\leqslant 2 \int_{B_{r_0}(0) - B_\theta(0)} d_{\mu_i^{-1} X}(u_i, 0)\, |\nabla u_i|_{g_{\lambda_i}} d\mu_{g_i} \\ &\quad + c \int_{B_{r_0}(0) - B_\theta(0)} d^2_{u_i^{-1} X}(u_i, 0)\, d\mu_{g_i}. \end{aligned}$$

Using the bound $2ab \leqslant \varepsilon a^2 + \varepsilon^{-1} b^2$, we then have

$$I^{u_i}(r_0) - I^{u_i}(\theta) \leqslant \varepsilon E^{u_i}(r_0) + c\varepsilon^{-1} \int_\theta^{r_0} I^{u_i}(\sigma)\, d\sigma.$$

Since $\mathrm{Ord}^{u_i}(0, r_0, 0)$ is bounded above, we may fix ε and obtain

$$I^{u_i}(r_0) - I^{u_i}(\theta) \leqslant \frac{1}{2} I^{u_i}(r_0) + c \int_\theta^{r_0} I^{u_i}(\sigma)\, d\sigma.$$

This implies, since $r_0 \in (\theta, 1]$ is arbitrary,

$$\sup_{r \in (\theta, 1]} I^{u_i}(r) \leqslant 2 I^{u_i}(\theta) + c(1-\theta) \sup_{r \in (\theta, 1]} I^{u_i}(r).$$

Therefore, we may fix θ close enough to 1 so that $c(1-\theta) = 1/2$ and we then obtain $I^{u_i}(1) \leqslant 4 I^{u_i}(\theta)$. This gives a lower bound on $I^{u_i}(\theta)$ since $I^{u_i}(1) = 1$ by choice of λ_i, μ_i. This already implies that u_* is nonconstant. The lower bound $I^{u_i}(\sigma) \geqslant \varepsilon_0 \sigma^{\alpha_1}$ for all $\sigma \in (0, 1)$ follows by iterating the previous argument.

In order to show that u_* is minimizing, we use the fact that there is a bi-Lipschitz map, for any $\sigma_0 > 0$, $F_i : (\mu_i^{-1} X) \cap B^{\mathbf{R}^N}_{\sigma_0}(0) \to X_0 \cap B^{\mathbf{R}^N}_{\sigma_0}(0)$ with $F_i(0) = 0$ and with Lipschitz constants of both F_i and F_i^{-1} approaching 1, and F_i converging to the identity as $i \to \infty$. Let v be a minimizing map from $B_\sigma(0)$ into X_0 with $u_* = v$ on $\partial B_\sigma(0)$ for some $\sigma \in (0, 1)$. We must show that $E(u_*) \leqslant E(v)$, so that u_* is also a minimizer. To see this, consider the map $F_i^{-1} \circ v : B_\sigma(0) \to \mu_i^{-1} X$. Let $\sigma_1 \in (\sigma, 1)$, and observe that for i sufficiently large we have $d_{\mu_i^{-1} X}(u_i(x), F_i^{-1} \circ v(x))$ smaller than

$\sigma_1-\sigma$ for all $x\in\partial B_\sigma(0)$. We may then extend the map $F_i^{-1}\circ v$ to the annular region $B_{\sigma_1}(0)-B_\sigma(0)$ so that it agrees with u_i on the outer boundary. We do this by choosing for each $\xi\in S^{n-1}$, the constant speed geodesic $\gamma_\xi:[\sigma,\sigma_1]\to\mu_i^{-1}X$ which satisfies $\gamma_\xi(\sigma)=F_i^{-1}(v(\sigma\xi))$ and $\gamma_\xi(\sigma_1)=u_i(\sigma_1\xi)$. We then define $v_i:B_{\sigma_1}(0)\to\mu_i^{-1}X$ by setting $v_i=F_i^{-1}\circ v$ on $B_\sigma(0)$, and $v_i(r\xi)=\gamma_\xi(r)$ for $r\in[\sigma,\sigma_1]$. The nonpositive curvature condition then implies that the Lipschitz constant of v_i in $B_{\sigma_1}(0)-B_\sigma(0)$ is bounded by a constant depending only on the Lipschitz constants of u_i and u_*. Since u_i is minimizing, we have for any $\varepsilon>0$

$$E^{u_i}(\sigma_1)\leqslant E^{v_i}(\sigma_1)\leqslant E(v)+\varepsilon,$$

for i large. By lower semicontinuity of the energy it then follows that $E(u_*)\leqslant E(v)$ on $B_\sigma(0)$ as required. This shows that u_* is minimizing.

Finally we show that for each $\sigma\in(0,1)$ we have $\lim\limits_{i\to\infty}E^{u_i}(\sigma)=E^{u_*}(\sigma)$. Since $E(v)\leqslant E(u_*)$ on $B_\sigma(0)$ in the previous argument, we have for i large $E^{u_i}(\sigma)\leqslant E^{u_*}(\sigma)+\varepsilon$. This implies $\overline{\lim\limits_{i\to\infty}}\,E^{u_i}(\sigma)\leqslant E^{u_*}(\sigma)$ which, combined with lower semicontinuity, establishes continuity of the energy. In particular, since we have also shown that $I^{u_i}(\sigma)$ has a lower bound for $\sigma\in(0,1)$, we can now conclude that

$$\lim_{i\to\infty}\mathrm{Ord}^{u_i}(0,\sigma,0)=\lim_{i\to\infty}\mathrm{Ord}^{u_*}(0,\sigma,0).$$

In particular it follows that $\mathrm{Ord}^{u_*}(0,\sigma,0)=\alpha$ for all $\sigma\in(0,1)$. Therefore $u_*:B_1(0)\to X_0$ is intrinsically homogeneous of order α. Since X_0 is a geometric cone in $\mathbf{R}^N$, the geodesics from 0 are simply Euclidean rays, and it follows that u_* is homogeneous of order α. This completes the proof of Proposition 3.3.

Recall that Lipschitz functions are differentiable almost everywhere. We now generalize the notion of differentiability for minimizing maps into X to exploit the intrinsic geometry of X. We have seen that an intrinsically homogeneous map of degree 1 is essentially a linear map, so it is natural to consider these as derivatives of maps to X. We make the following definition.

Definition. — We say that u has an *intrinsic derivative* at a point $x_0\in\Omega$ if there is a minimizing map $l:\Omega_{x_0}\to X_{u(x_0)}$ (Ω_{x_0} = tangent space) which is intrinsically homogeneous of degree 1 such that

$$\lim_{\substack{V\to 0\\ V\in\Omega_{x_0}}}|V|^{-1}|u(\exp_{x_0}(V))-u(x_0)-l(V)|=0.$$

Note that if x_0 is the center of a normal coordinate system $x^1,\ldots.x^n$, then this condition reads

$$\lim_{x \to 0} |x|^{-1} |u(x) - u(0) - l(x)| = 0.$$

It is clear that if u has a derivative at x_0, then u has an intrinsic derivative at x_0, and $l = u'_{x_0}$, the usual linear approximation to u. While the best one can hope for is that u is differentiable at almost every $x \in \Omega$ (true since u is Lipschitz), we will give general conditions on X in later sections of this paper which imply that u has an intrinsic derivative at *every* point $x \in \Omega$. It is immediate that l is unique if it exists since two such maps l, l_1 would satisfy

$$\lim_{x \to 0} |x|^{-1} |l(x) - l_1(x)| = 0.$$

If we fix a point $\xi \in S^{n-1}$ and let $\gamma_\xi(r) = l(r\xi)$ and $\gamma_{1,\xi}(r) = l_1(r\xi)$, we then have

$$\lim_{r \to 0} r^{-1} |\gamma_{1,\xi}(r) - \gamma_\xi(r)| = 0,$$

which implies that $\gamma_{1,\xi}$ coincides with $\gamma_\xi(r)$ for r small, since these are constant speed geodesics. It follows that $l = l_1$ in a neighborhood of $x = 0$.

Note that if $\alpha = \operatorname{Ord}(x_0) > 1$, then u has an intrinsic derivative at x_0 because we saw in the proof of Theorem 2.3 that for x near x_0 we have $d(u(x), u(x_0)) \leqslant c\,|x - x_0|^\alpha$, which implies

$$\lim_{x \to x_0} |x - x_0|^{-1} |u(x) - u(x_0)| = 0,$$

so that $u'_{x_0} = 0$. If $\operatorname{Ord}(x_0) = 1$, we are not able to show that u has an intrinsic derivative at x_0 (although this will be shown under additional assumptions on X in later sections). In order to show this it would suffice to show that there is a unique homogeneous approximating map u_* which would then be a nonzero constant times l.

We close this section by establishing the result that a minimizing map which is constant on an open set is identically constant. This then implies that $\operatorname{Ord}(x)$ is defined for all $x \in \Omega$ provided $u: \Omega \to X$ is nonconstant.

Proposition 3.4. — *If* Ω *is connected, and* $u: \Omega \to X$ *is a minimizing map which is constant on an open subset of* Ω, *then* u *is identically constant in* Ω.

Proof. — If u is not constant in Ω but is constant on an open subset of Ω we can find a ball B completely contained in Ω such that u is constant in the interior of B, but for some boundary point $x_0 \in \partial B$, u is not constant in any neighborhood of x_0. Assuming without loss of generality that $u \equiv 0$ inside B, we may then find a homogeneous approximating map u_* at x_0 which satisfies $u_* \equiv 0$ in a half space of $\mathbf{R}^n$. In particular by Proposition 3.1 we know that the degree $\alpha = \operatorname{Ord}(x_0)$ of u_*

is strictly greater than one. Since u_* is not constant, we may find a ball B_1 in $\mathbf{R}^n$ such that $u_* \equiv 0$ in B_1, and there is a point $x_1 \in \partial B_1$ with $|x_1| = 1$ such that u_* is not constant in a neighborhood of x_1. As above we must have $\mathrm{Ord}^{u_*}(x_1) > 1$, and so if we let u_*^1 be a homogeneous approximating map for u_* at x_1, we must have u_*^1 independent of a direction in $\mathbf{R}^n$. (If we take $x_1 = (1, 0, \ldots, 0)$, then $\partial u_*^1 / \partial x_1 = 0$ a.e.). Thus u_*^1 restricts to $\mathbf{R}^{n-1}$ as a nonconstant minimizing homogeneous map which vanishes in a half space. Repeating this argument a finite number of times we produce a nonconstant minimizing homogeneous map of $\mathbf{R}$ to X_0 which vanishes on a half line. This contradicts the fact from section 1 that energy minimizing maps of $\mathbf{R}$ have constant speed parametrization. This contradiction then shows that u must have been constant in Ω. This completes the proof of Proposition 3.4.

4. Existence in a homotopy class and uniqueness

In this section we make the transition from the local problem of minimizers from a domain with given boundary data to the more global problem of existence of minimizers in a homotopy class. This can be done fairly directly with the help of convexity properties of the energy functional along geodesic homotopies. The approach we follow here was developed in [S]. In particular we will generalize the theorems of Eells-Sampson [ES] and P. Hartman [Har]. A non-simply connected complex X will be said to have *nonpositive curvature* if its universal covering space $\tilde{X}$ has nonpositive curvature in the sense we have discussed. Given two Lipschitz maps $u_0, u_1 : M \to X$ which are homotopic, we can construct a unique *geodesic homotopy* $u_t : M \to X$ by replacing each parameter curve of any given homotopy by the unique constant speed geodesic with the same endpoints in the same homotopy class. The convexity result which we need is the following.

Proposition 4.1. — *Each map* $u_t : M \to N$ *is locally Lipschitz, and for any compact domain* $\Omega \subseteq M$ *the function* $t \mapsto E_\Omega(u_t)$ *is a continuous convex function which is a weak solution of the differential inequality*

$$\frac{d^2 E_\Omega(u_t)}{dt^2} \geqslant 2 \int_\Omega |\nabla d(u_0, u_1)|^2 \, d\mu.$$

Proof. — For simplicity of notation we assume that M is compact with boundary and $\Omega = M$. First consider the one-dimensional case in which we have Lipschitz curves $\gamma_0, \gamma_1 : (-\delta, \delta) \to X$ and a geodesic homotopy γ_t for $0 \leqslant t \leqslant 1$. Assume that $s = 0$ is a point at which both γ_0, γ_1 are differentiable and γ_t is differentiable for almost every $t \in [0, 1]$. We are going to do a calculation involving $d\gamma_t / ds$ at $s = 0$ only, so we may replace γ_0, γ_1 by constant speed geodesics with the same tangent vectors. Observe that at any t for which the original γ_t is differentiable, the new curve is differentiable and has the same derivative. Let $l(s)$ be the length of the curve $t \mapsto \gamma_t(s)$, and observe that $l(s) = d(\gamma_0(s), \gamma_1(s))$ is a Lipschitz function of s. Assume that $s \mapsto l(s)$ is differentiable at $s=0$, and reparametrize the homotopy by setting $\bar{\gamma}_\tau(s) = \gamma_{\tau/l(s)}(s)$ for $\tau \in [0, l(s)]$. Thus $\tau \mapsto \bar{\gamma}_\tau(s)$ is now a unit speed geodesic. The fact that X has nonpositive curvature implies that for any h the function $\tau \mapsto d^2(\bar{\gamma}_\tau(h), \bar{\gamma}_\tau(0))$ is convex. At any τ for which $d/ds\, \gamma_\tau(0)$ exists we have

$$\lim_{h \to 0} h^{-2} d^2(\bar{\gamma}_\tau(h), \bar{\gamma}_\tau(0)) = \left| \frac{d\bar{\gamma}_\tau}{ds}(0) \right|^2.$$

Since this derivative exists at $\tau = 0$ and $l(0)$, it follows that there is a sequence h_i tending to zero such that the functions $\tau \mapsto h_i^{-2} d^2(\bar{\gamma}_\tau(h_i), \bar{\gamma}_\tau(0))$ converge uniformly on $[0, l(0)]$ to a convex function which agrees almost everywhere with the function $\tau \mapsto |(d\bar{\gamma}_\tau/ds)(0)|^2$. In particular by redefining it on a set of measure zero we may assume that this function is convex. Now by the chain rule we have

$$\frac{d\bar{\gamma}_\tau}{ds}(0) = -\tau\, l(0)^{-2} \frac{dl}{ds}(0) \frac{\partial \gamma_{\tau/l(0)}}{\partial \tau}(0) + \frac{d}{ds} \gamma_{\tau/l(0)}(0).$$

This may be rewritten in terms of t:

$$\frac{d}{ds} \gamma_t(0) = \frac{d\bar{\gamma}_\tau}{ds}(0) + t l(0)^{-1} \frac{dl(0)}{ds} \frac{\partial \gamma_t}{\partial t}(0).$$

Now for any $\tau_1, \tau_2 \in (0, l(0))$ with $\tau_1 < \tau_2$ we have $d(\bar{\gamma}_{\tau_1}(s), \bar{\gamma}_{\tau_2}(s)) = \tau_2 - \tau_1$. Differentiating with respect to s we then conclude that

$$\left. \frac{d\bar{\gamma}_\tau}{ds} \cdot \frac{\partial \bar{\gamma}_\tau}{\partial \tau} \right|_{\tau=\tau_1} = \left. \frac{d\bar{\gamma}_\tau}{ds} \cdot \frac{\partial \bar{\gamma}_\tau}{\partial \tau} \right|_{\tau=\tau_2}$$

Therefore we have, for almost every $t \in [0, 1]$,

$$\frac{d\gamma_t}{ds}(0) = \mathrm{V}(t) + \left(a + t l(0)^{-1} \frac{dl}{ds}(0) \right) \frac{\partial \gamma_t}{\partial t}(0),$$

for a constant a where $\mathrm{V}(t) \cdot (\partial \gamma_t / \partial t)(0) = 0$ for a.e. t. Since $|\mathrm{V}(t)|^2 = |(d\bar{\gamma}_\tau/ds)(0)|^2 + a^2 l(0)^2$, it follows that $|\mathrm{V}(t)|^2$ is a convex function of t. Therefore we have

$$\left| \frac{d\gamma_t}{ds}(0) \right|^2 = |\mathrm{V}(t)|^2 + \left(a l(0) + t \frac{dl}{ds}(0) \right)^2$$

and hence it follows that, in the weak sense,

$$\frac{d^2}{dt^2}\left(\left|\frac{d\gamma_t}{ds}(0)\right|^2\right) \geqslant 2\left(\frac{d}{ds}d(\gamma_0(s), \gamma_1(s))\right)^2\bigg|_{s=0}.$$

Now to prove the result in higher dimensions, observe that the map $(x, t) \mapsto u_t(x)$ is Lipschitz, and hence for almost every line parallel to the t-axis it is differentiable at almost every point of the line. At such points of differentiability the previous results tell us

$$\frac{d^2}{dt^2}(|\nabla u_t|^2) \geqslant 2|\nabla d(u_0, u_1)|^2$$

in the weak sense. Thus if $\zeta(t)$ is a smooth nonnegative compactly supported function in $(0, 1)$, we have, for almost every $x \in \mathrm{M}$,

$$\int_0^1 |\nabla u_t|^2(x)\zeta''(t)\,dt \geqslant 2\int_0^1 |\nabla d(u_0(x), u_1(x))|^2 \zeta(t)\,dt.$$

Integrating and interchanging the order of integration we have

$$\int_0^1 \mathrm{E}(u_t)\zeta''(t)\,dt \geqslant 2\left(\int_{\mathrm{M}} |\nabla d(u_0, u_1)|^2\,d\mu\right)\int_0^1 \zeta(t)\,dt.$$

This completes the proof of Proposition 4.1.

We now derive two easy corollaries of this result. The first tells us that in case $u_0 = u_1$ on a nontrivial boundary and u_0, u_1 are nearly minimizing, u_0 is close to u_1.

Corollary 4.2. — *Suppose* $u_0, u_1 : \Omega \to \mathrm{X}$ *are Lipschitz maps which agree on* $\partial\Omega\,(\partial\Omega \neq \varnothing)$ *and are homotopic through maps which are fixed on* $\partial\Omega$. *Let* E_0 *be given by*

$$\mathrm{E}_0 = \inf\{\mathrm{E}(v) : v : \Omega \to \mathrm{X} \textit{ Lipschitz, homotopic to } u_0 \textit{ with fixed boundary}\}.$$

Suppose $\varepsilon_0 \geqslant 0$ *such that* $\mathrm{E}(u_0), \mathrm{E}(u_1) \leqslant \mathrm{E}_0 + \varepsilon_0$. *Then it follows that*

$$\int_\Omega d^2(u_0, u_1)\,d\mu \leqslant c\,\varepsilon_0$$

for a constant c depending only on Ω.

Corollary 4.3. — *Suppose* X *is simply connected. There is a unique minimizing map* $u : \Omega \to \mathrm{X}$ *with given Lipschitz boundary data.*

Proof of Corollary 4.2. — Connect u_0 to u_1 with a geodesic homotopy u_t, and let $E(t)=E(u_t)$. Define $\alpha \geqslant 0$ by

$$\alpha=\int_M |\nabla d(u_0, u_1)|^2 \, d\mu.$$

Proposition 4.1 then implies that the function $E(t)$ lies below the appropriate quadratic polynomial with leading term αt^2. This implies

$$E(t) \leqslant \alpha t(t-1)+E(1) t+(1-t) E(0).$$

Since $E(t) \geqslant E_0$ and $E(0)$, $E(1) \leqslant E_0+\varepsilon_0$, it follows that for every t this quadratic polynomial has value between E_0 and $E_0+\varepsilon_0$. Setting $t=1/2$ we find

$$E_0 \leqslant -\frac{1}{4}\alpha+\frac{1}{2}E(1)+\frac{1}{2}E(0) \leqslant -\frac{1}{4}\alpha+E_0+\varepsilon_0$$

which implies $\alpha \leqslant 4\varepsilon_0$. Combining this with the Poincaré inequality we then obtain the conclusion of Corollary 4.2.

Proof of Corollary 4.3. — This is almost immediate modulo a minor technical detail; if u_0, u_1 are both minimizing and equal to a given Lipschitz map φ on $\partial\Omega$, we know that u_0, u_1 are *locally* Lipschitz but not necessarily Lipschitz up to the boundary. First observe that Proposition 4.1 works for such maps since its proof involved integrating a local expression. Thus we have $\int_M |\nabla d(u_0, u_1)|^2 \, d\mu=0$, and therefore $d(u_0, u_1)=\text{const}$. Since $u_0=u_1$ on $\partial\Omega$ and $d(.,.)$ is a Lipschitz function it follows that $u_0 \equiv u_1$, as required. This proves Corollary 4.3.

We are now in a position to solve the homotopy problem for harmonic maps into nonpositively curved complexes.

Theorem 4.4. — *Let* M *be a compact Riemannian manifold without boundary, and let* X *be a compact nonpositively curved complex. Let* $\varphi: M \to X$ *be a Lipschitz map. There exists a Lipschitz map* $u: M \to X$ *which is freely homotopic to* φ *and which minimizes energy in the sense that*

$$E(u)=\inf\{E(v): v: M \to X, \textit{ Lipschitz}, \ v \textit{ homotopic to } \varphi\}.$$

Moreover, on simply connected regions $\Omega \subset M$, *the lift of* u *to the universal cover* $\tilde{X}$ *minimizes in the sense of the previous sections.*

Proof. — Let $\{u_i\}$ be a sequence of Lipschitz maps homotopic to φ with $E(u_i) \to E_0$, where E_0 is given by

$$E_0 = \inf \{ E(v) : v \text{ Lipschitz, homotopic to } \varphi \}.$$

Assume that X is embedded in $\mathbb{R}^N$, and choose a subsequence, again denoted $\{u_i\}$, which converges weakly in $H^1(M, X)$ to an H^1 map u. We claim that u is (equal a.e. to) a Lipschitz map homotopic to φ. First let $x_0 \in M$ and consider a small ball B. We may lift the map u_i to the universal cover $\tilde{X}$. Denote this lift by $\tilde{u}_i : B \to \tilde{X}$. Let v_i be a minimizing map from $B \to \tilde{X}$ which is equal to $\tilde{u}_i$ on ∂B. We then define a replaced map $\hat{u}_i$ by

$$\hat{u}_i(x) = \begin{cases} \pi(v_i(x)) & \text{for } x \in B \\ u_i & \text{for } x \in M - B. \end{cases}$$

Since $E(\hat{u}_i) \leqslant E(u_i)$, the sequence $\{\hat{u}_i\}$ is again a minimizing sequence. In particular, given any $\varepsilon > 0$, for i sufficiently large we have $E_0 \leqslant E(\hat{u}_i) \leqslant E(u_i) \leqslant E_0 + \varepsilon$. Applying Corollary 4.2 in the region B we have

$$\int_B d^2(u_i, \hat{u}_i)\, d\mu \leqslant c\,\varepsilon.$$

On the other hand by Theorem 2.4, the sequence $\hat{u}_i$ is uniformly Lipschitz on compact subsets interior to B. Thus a subsequence of $\{\hat{u}_i\}$ converges uniformly in a neighborhood of x_0 to a Lipschitz map $\hat{u}$. Since ε above was arbitrary we have $\hat{u} = u$ a.e., and hence u is a Lipschitz map. The sequence $\{\hat{u}_i\}$ is then a minimizing sequence which converges uniformly near x_0 to u. If we consider the geodesic homotopy ${}^i v_t$ with ${}^i v_0 = \varphi$, ${}^i v_1 = \hat{u}_i$, we see immediately that this homotopy converges in a neighborhood of x_0 to a geodesic homotopy v_t with $v_0 = \varphi$, $v_1 = u$. If we consider an overlapping ball, since replaced maps are uniformly close on the intersection, the corresponding geodesic homotopies agree. Therefore we have a global geodesic homotopy from φ to u. The fact that the lift of the restriction of u to Ω for a simply connected region $\Omega \subseteq M$ minimizes is a consequence of Corollary 4.3 which shows that the minimizer must agree with the lift of u. This completes the proof of Theorem 4.4.

5. Some smoothness results for harmonic maps

For the main applications of this work it will be important to show that harmonic maps are better than Lipschitz in certain cases. First note that if $u : \Omega \to X$ is minimizing, and for some $x_0 \in \Omega$, $u(x_0)$ is a regular point of X, then the usual regularity theory for harmonic maps (see e.g. [S]) implies that u is C^∞ in a neighborhood of x_0. On the other hand, it happens in important cases that even if $u(x_0)$ is a singular point of X, the map u may be differentiable at x_0 in a strong sense. Of course differentiability should mean that u is well approximated by a linear map near x_0. We have seen previously that for harmonic maps, a map which is homogeneous of degree 1 is essentially linear. Thus it is natural to use such maps to approximate a general harmonic map. In fact, homogeneous maps tend to exist only when the image complex is a cone, so we generalize the notion as follows. Let $x^1, \ldots, x^n$ be a normal coordinate system centered at x_0, and let $r=|x|$, $\xi=x/|x|$ denote polar coordinates in $B_{\tau_0}(x_0)$. We will say that a Lipschitz map $l: B_{\tau_0}(x_0) \to X$ is *essentielly homogeneous of degree* 1 if there is a nonnegative function $\lambda: S^{n-1} \to \mathbb{R}$ and an assignment γ_ξ to each $\xi \in S^{n-1}$ of unit speed geodesic in X with $\gamma_\xi(0)=P$ (where $P=l(0)$) such that $l(r\xi)=\gamma_\xi(\lambda(\xi)r)$ for $x=r\xi \in B_{r_0}(x_0)$. In short, a map is essentially homogeneous of degree 1 if the restriction of u to each ray is a constant speed geodesic. Of course maps of this type exist in great abundance because they are determined by their restriction to $\partial B_{r_0}(x_0)$ and by the assignment of the value at x_0. Thus given any Lipschitz map $l: \partial B_{\tau_0}(x_0) \to X$ and any point $P \in X$, there is a unique essentially homogeneous map of degree 1 which agrees with l on $\partial B_{\tau_0}(x_0)$ and sends x_0 to P.

For a point $x_0 \in \Omega$ and a radius $\sigma>0$ such that $B_\sigma(x_0)$ is compactly contained in Ω we consider the error with which u can be approximated by degree 1 essentially homogeneous maps. Let $l: B_\sigma(x_0) \to X$ be such a map, and consider the quantity

$$d_\sigma(u, l) = \sup_{x \in B_\sigma(x_0)} d(u(x), l(x)).$$

We then define $R(x_0, \sigma)$ by

$$R(x_0, \sigma) = \inf_l d_\sigma(u, l),$$

where the infimum is taken over all essentially homogeneous maps, $l: B_\sigma(x_0) \to X$, of degree 1. Since the constant map $l(x) \equiv u(x_0)$ is a competitor and the map u is Lipschitz we have $R(x_0, \sigma) \leqslant c\,\sigma$.

Definition. — A minimizing map $u: \Omega \to X$ is *intrinsically differentiable* on a compact subset $K \subseteq \Omega$ provided there exists r_0, $c>0$ and $\beta \in (0, 1]$ such that

$$R(x, \sigma) \leqslant c\,\sigma^{1+\beta} R(x, r_0)$$

for all $x \in K$, $\sigma \in (0, r_0]$. The constants c, β, r_0 may depend only on K, Ω, X, and the total energy of the map u.

Definition. — A subset $S \subseteq X$ is *essentially regular* if for any minimizing map $u: \Omega \to X$ with $u(\Omega) \subseteq S$, the restriction of u to any compact subset of Ω is intrinsically differentiable.

It is not difficult to see that a closed subset of the regular set of X is essentially regular; this will be explicitly discussed later in the section. More generally, an isometrically and totally geodesically embedded submanifold of any dimension in X has essentially regular image.

The main result of this section will provide a criterion for a map to be intrinsically differentiable near a point. This criterion will say roughly that the map is well approximated by an essentially homogeneous map of degree 1 whose image is "effectively" contained in a totally geodesic subcomplex X_0 which is essentially regular. In order to define what it means for a map to be effectively contained in X_0 we need to introduce some terminology. Given a Lipschitz map $l:\partial B_{\tau_0}(x_0)\to X$ together with points $x\in B_\tau(x_0)$ and $P\in X$, there is a unique essentially homogeneous map of degree 1 denoted $l_{x,P}$ from $B_{\tau_0}(x_0)$ into X which satisfies $l_{x,P}(x)=P$ and $l_{x,P}=l$ on $\partial B_{\tau_0}(x_0)$. (Note we are assuming that r_0 is small so that $B_{\tau_0}(x_0)$ is convex.)

Definition. — Let X_0 be a totally geodesic subcomplex of X, and let $l:B_{\tau_0}(x_0)\to X$ be a map which is essentially homogeneous of degree 1. We say that l is *effectively contained in* X_0 *near the point* x_0 if for any $\varepsilon>0$ there exists $\delta>0$ such that for all $x\in\Omega$ sufficiently near x_0 and all $P\in X_0$ sufficiently near $P_0=l(x_0)$ and all $\sigma\in(0, r_0/2]$ we have

$$\mathrm{Vol}\{y\in B_\sigma(x):B^X_{\delta\sigma}(l_{x,P}(y))\nsubseteq X_0\}\leqslant\varepsilon\sigma^n.$$

We stress that $B^X_{\cdot}(.)$ is used to denote the full ball in X, so that it follows that for any point $y\in B_\sigma(x)$ such that $B^X_{\delta\sigma}(l_{x,P}(y))\subseteq X_0$, a maximal simplex in X_0 which contains $l_{x,P}(y)$ is also maximal in X. Thus X_0 should be thought of as a top-dimensional (typically of dimension $k=\dim X$) subcomplex of X. We are now ready to prove the main result of this section.

Theorem 5.1. — *Let* $u:\Omega\to X$ *be a minimizing map. Let* $x_0\in\Omega$ *and* $r_0>0$ *be such that* $B_{r_0}(x_0)$ *is compactly contained in* Ω. *Let* $X_0\subseteq X$ *be a totally geodesic subcomplex, and let* $l:B_{r_0}(x_0)\to X_0$ *be an essentially homogeneous degree* 1 *map with* $P_0\equiv l(x_0)\in X_0$. *Assume that a neighborhood of* P_0 *in* X_0 *is essentially regular. There exists* $\delta_0>0$ *depending only on* l, Ω, X, X_0 *such that if* l *is effectively contained in* X_0 *near* x_0 *and*

$$\sup_{x\in B_{r_0}(x_0)} d(u(x), l(x))\leqslant\delta_0,$$

then u *is intrinsically differentiable in a neighborhood of* x_0. *In fact, there exists* $\sigma_0>0$ *such that* $u(B_{\sigma_0}(x_0))\subseteq X_0$.

Proof. — The idea of the proof is to compare u at small scales to a minimizing map having image in X_0. To carry this out, we let $\Pi:X\to X_0$ denote the nearest point projection map. The map Π is then a distance nonincreasing Lipschitz map. We need the following lemma.

Lemma 5.2. — *If* $v \in H^1(\Omega, X)$, *then* $\bar{v} = \Pi \circ v$ *is also in* $H^1(\Omega, X)$, *and* $E(\bar{v}) \leq E(v)$. *Thus if* $\varphi : \partial\Omega \to X_0$ *is Lipschitz, then the minimizer* u_0 *into* X *which agrees with* φ *on* $\partial\Omega$ *has image in* X_0.

We also need a second lemma. We postpone the proofs of both of these until we have completed the proof of the theorem.

Lemma 5.3. — *If* u_1, u_2 *are minimizing maps from a region* Ω_1 *into* X, *then the function* $x \mapsto d(u_1(x), u(x))$ *is a Lipschitz weakly subharmonic function in* Ω_1.

Let x_1 be a point sufficiently close to x_0 such that $u(x_1)$ lies in X_0 and such that $u(x_1)$ is close enough to $u(x_0)$ so that we may find for any $\varepsilon > 0$ a number $\delta > 0$ such that

$$\mathrm{Vol}\{x \in B_\sigma(x_1) : B^X_{\delta\sigma}(l_1(x)) \nsubseteq X_0\} \leq \varepsilon\sigma^n$$

for $\sigma \in (0, r_0/2]$ where we have denoted by l_1 the map $l_{x_1, u(x_1)}$. By translation of coordinates in $\mathbb{R}^{k_1}$ we may assume $u(x_1) = 0$, and by multiplying the metric on Ω by a fixed constant factor we may for convenience take $r_0 = 2$, and hence both u and l_1 are defined on $B_1(x_1)$. Also note that we have

$$(5.1) \qquad \sup_{x \in B_1(x_1)} d(u(x), l_1(x)) \leq 2\delta_0$$

provided x_1 is sufficiently close to x_0.

We choose a normal coordinate system $x^1, \ldots . x^n$ centered at x_1, and for any map $v : B_1(x_1) \to X$ and any $\sigma \in (0, 1]$ we let ${}^\sigma v$ denote the dilated map given by ${}^\sigma v(x) = \sigma^{-1} v(\sigma x)$. Thus we have ${}^\sigma v : B_{\sigma^{-1}}(0) \to \sigma^{-1} X$, and if v is minimizing, then ${}^\sigma v$ is minimizing as a map from $(B_{\sigma^{-1}}(0), \sigma^{-1} g(\sigma x))$ to σ^{-1} X. Note also that the metrics $\sigma^{-1} g(\sigma x)$ become Euclidean as σ tends to zero, and hence are uniformly controlled (in any C^k topology) on $B_1(0)$ while the complex $\sigma^{-1}X$ has nonpositive curvature for each σ and converges to the tangent cone to X at 0 as σ tends to zero.

Now assume that for some $\sigma \in (0, 1]$ there exists an essentially homogeneous degree 1 map $l_2 : B_1(0) \to \sigma^{-1} X_0$ such that D is defined by

$$(5.2) \qquad \sup_{x \in B_1(0)} d({}^\sigma u(x), l_2(x)) = D,$$

and for some $\delta_1 > 0$ assume that

$$(5.3) \qquad \sup_{x \in B_1(0)} d({}^\sigma u(x), {}^\sigma l_1(x)) \leq \delta_1.$$

We assume D, δ_1 are small positive numbers; in fact, for a given $\varepsilon_1 > 0$ assume that δ_1 is so small that

$$\operatorname{Vol}\left\{x \in B_1(0) : B_{\delta_1}^{\sigma^{-1}X}({}^{\sigma}l_1(x)) \nsubseteq \sigma^{-1}X_0\right\} \leqslant \varepsilon_1.$$

In particular from (5.3) we have

$$\operatorname{Vol}\left\{x \in B_1(0) : {}^{\sigma}u(x) \notin \sigma^{-1}X_0\right\} \leqslant \varepsilon_1.$$

We may therefore choose a number $\theta_1 \in [3/4, 1]$ such that

$$\operatorname{Vol}\left\{x \in \partial B_{\theta_1}(0) : {}^{\sigma}u(x) \notin \sigma^{-1}X_0\right\} \leqslant 4\,\varepsilon_1.$$

Now let ${}^{\sigma}\Pi : \sigma^{-1}X \to \sigma^{-1}X_0$ be the nearest point projection map, and let ${}^{\sigma}\varphi = {}^{\sigma}\Pi \circ {}^{\sigma}u$. We then have

$$\operatorname{Vol}\left\{x \in \partial B_{\theta_1}(0) : {}^{\sigma}\varphi(x) \neq {}^{\sigma}u(x)\right\} \leqslant 4\,\varepsilon_1. \tag{5.4}$$

Let $v : B_{\theta_1}(0) \to X_0$ be the least energy map with $v = {}^{\sigma}\varphi$ on $\partial B_{\theta_1}(0)$. Since a neighborhood of P_0 in X_0 is essentially regular, it follows that for any $\theta \in (0, 1/4)$ there is a homogeneous degree 1 map $\hat{l}_2 : B_\theta(0) \to \sigma^{-1}X_0$ such that for some $\beta \in (0, 1)$

$$\sup_{x \in B_\theta(0)} d(v(x), \hat{l}_2(x)) \leqslant c\,\theta^{1+\beta} R_v(0, 1/2).$$

By definition of $R_v(0, 1/2)$ this implies

$$\sup_{x \in B_\theta(0)} d(v(x), \hat{l}_2(x)) \leqslant c\,\theta^{1+\beta} \sup_{x \in B_{1/2}(0)} d(v(x), l_2(x)). \tag{5.5}$$

We now show that ${}^{\sigma}u$ is very close to v. To see this we note that (5.2) and (5.4) imply

$$\int_{\partial B_{\theta_1}(0)} d({}^{\sigma}u, v)\, d\Sigma \leqslant 4\,\varepsilon_1 D.$$

Since the function $x \mapsto d({}^{\sigma}u(x), v(x))$ is subharmonic by Lemma 5.3, we have

$$\sup_{B_{1/2}(0)} d({}^{\sigma}u, v) \leqslant c\,\varepsilon_1 D.$$

Combining this with (5.5) then gives us

$$\begin{aligned}\sup_{B_\theta(0)} d({}^{\sigma}u, \hat{l}_2) &\leqslant c\,\varepsilon_1 D + c\,\theta^{1+\beta} \sup_{B_{1/2}(0)} d(v, l_2)\\ &\leqslant (c\,\varepsilon_1 + c\,\theta^{1+\beta} + c\,\varepsilon_1\,\theta^{1+\beta}) D.\end{aligned}$$

This clearly implies

$$\sup_{B_\theta(0)} d({}^{\sigma}u, \hat{l}_2) \leqslant (c\varepsilon_1 + c\,\theta^{1+\beta}) \sup_{B_1(0)} d({}^{\sigma}u, l_2). \tag{5.6}$$

On the other hand we have, from (5.2) and (5.3) and the triangle inequality, that for $x \in \partial B_1(0)$

$$d({}^{\sigma}l_1(x), l_2(x)) \leqslant D + \delta_1.$$

Since ${}^{\sigma}l_1(0) = {}^{\sigma}u(0) = 0$, we have from (5.2)

$$d({}^{\sigma}l_1(0), l_2(0)) \leqslant D.$$

The fact that X has nonpositive curvature then implies that for any $x \in B_1(0)$ we have

$$d({}^{\sigma}l_1(x), l_2(x)) \leqslant D + |x| \delta_1,$$

so that, in particular,

$$\sup_{B_\theta(0)} d({}^{\sigma}l_1, l_2) \leqslant D + \delta_1 \theta.$$

Combining this again with (5.2), we finally have

$$\sup_{B_\theta(0)} d({}^{\sigma}u, {}^{\sigma}l_1) \leqslant \theta\delta_1 + 2D. \tag{5.7}$$

We now apply the previous argument for varying choices of σ. First take $\sigma = 1$ and $l_2 = l_1$. Set ${}_1l = \hat{l}_2$ and rewrite (5.6) as

$$\sup_{B_1(0)} d({}^{\theta}u, {}_1l) \leqslant (c\varepsilon_1 \theta^{-1} + c\theta^{\beta}) \sup_{B_1(0)} d(u, l_1). \tag{5.8}$$

On the other hand, (5.7) becomes

$$\sup_{B_1(0)} d({}^{\theta}u, {}^{\theta}l_1) \leqslant \delta_1 + 2\theta^{-1} \sup_{B_1(0)} d(u, l_1). \tag{5.9}$$

Set ${}_1\delta = \delta_1 + 2\theta^{-1} D_0$ where we set $D_0 = \sup_{B_1(0)} d(u, l_1)$. Assuming we have defined ${}_1l, {}_2l, \ldots, {}_il$, we set

$$D_i = \sup_{B_1(0)} d({}^{\theta^i}u, {}_il).$$

Assume by induction that for integers up to $i\,(i \geqslant 1)$ ${}_i\delta$ has been defined, and that we have the inequality

$$\sup_{B_1(0)} d({}^{\theta^i}u, {}^{\theta^i}l_1) \leqslant {}_i\delta. \tag{5.10}$$

Now apply the previous argument with $\sigma=\theta^i$, $l_2={}_il$. This can be done provided ${}_i\delta$ is sufficiently small depending on ε_1. Set ${}_{i+1}l=\hat{l}_2$ and observe that (5.6) then may be written

$$D_{i+1}\leqslant(c\,\varepsilon_1\,\theta^{-1}+c\,\theta^{\beta})\,D_i, \tag{5.11}$$

while (5.7) yields

$$\sup_{B_1(0)} d({}^{\theta^{i+1}}u,\ \theta^{i+1}\,l_1)\leqslant{}_i\delta+2\,\theta^{-1}\,D_i. \tag{5.12}$$

Therefore we may take ${}_{i+1}\delta$ to be

$${}_{i+1}\delta={}_i\delta 2\,\theta^{-1}\,D_i. \tag{5.13}$$

We now *fix* θ so small that $c\,\theta^{\beta}=1/4$ in (5.11), and we then *fix* ε_1 so small that $c\,\varepsilon_1\,\theta^{-1}=1/4$ in (5.11) so that we obtain $D_{i+1}\leqslant 1/2\,D_i$ provided ${}_i\delta$ is sufficiently small. Assuming ${}_1\delta,\ldots,{}_k\delta$ are small enough we then obtain

$$D_j\leqslant 2^{-j}\,D_0,\ j=1,\ldots,k+1.$$

Putting this into (5.13) then gives

$${}_{j+1}\delta\leqslant{}_j\delta+2\,\theta^{-1}\,2^{-j}\,D_0,\ j=1,\ldots,k+1.$$

In particular we have

$${}_{k+1}\delta\leqslant{}_1\delta+2\,\theta^{-1}\,D_0\sum_{j=1}^{k}2^{-j}\leqslant{}_1\delta+2\,\theta^{-1}\,D_0.$$

Recalling the definition of ${}_1\delta$ we have ${}_{k+1}\delta\leqslant\delta_1+4\,\theta^{-1}\,D_0$, but by (5.1) we have $D_0\leqslant 2\,\delta_0$, and hence finally

$${}_{k+1}\delta\leqslant\delta_1+8\,\theta^{-1}\,\delta_0.$$

Thus if δ_0, δ_1 are sufficiently small we may apply this argument for any k, and conclude from (5.12)

$$\sup_{B_1(0)} d({}^{\theta^i}u,\ {}^{\theta^i}l_1)\leqslant\delta+8\,\theta^{-1}\,\delta_0$$

for all nonnegative integers i. Now if $\sigma\in(0,1]$, we choose the nonnegative integer i so that $\sigma\in(\theta^{i+1},\theta^i]$, and we conclude

$$\sup_{B_\sigma(x_1)} d(u,\ l_1)\leqslant(\theta^{-1}\,\delta_1+8\,\theta^{-2}\,\delta_0)\,\sigma. \tag{5.14}$$

This then implies that most points in $B_\sigma(x_1)$ are mapped by u into the interior of X_0. Note that x_1 was an arbitrary point near x_0 such that $u(x_1)\in X_0$. If for $\sigma_0>0$, $u(B_{\sigma_0}(x_0)) \nsubseteq X_0$, then we can find a ball $B \subseteq u^{-1}(X\setminus X_0)\cap B_{\sigma_0}(x_0)$ such that at least one point x_1 in ∂B maps to X_0. By the choice of x_1 we have, for every $\sigma_1>0$, a substantial fraction (at least 1/2) of $B_\sigma(x_1)$ mapped into the closure of $X-X_0$. This contradiction shows that for some $\sigma_0>0$ we have $u(B_{\sigma_0}(x_0)) \subseteq X_0$, we have completed the proof of Theorem 5.1.

Proof of Lemma 5.2. — If we have some vector V at a point $P\in X$ such that the directional derivative $D_V\,\Pi$ exists, where Π is considered as a vector valued function, then we have $|D_V\Pi|\leqslant|V|$ since Π is distance decreasing. Therefore we may use Lemma 2.1 to argue that for almost every point $x\in\Omega$ we have

$$\left|\frac{\partial\Pi\circ v}{\partial x_j}\right|\leqslant\left|\frac{\partial v}{\partial xj}\right|,\qquad j=1,\ldots,n.$$

In particular we have $E(\Pi\circ v)\leqslant E(v)$. The last statement of Lemma 5.2 follows immediately.

Proof of Lemma 5.3. — First observe that if $C\subseteq X$ is a closed convex set, then the function $x\mapsto d(u(x), C)$ is weakly subharmonic . This was shown in Proposition 2.2 for the case that C is a point. (Note that $|\nabla u|^2\geqslant|\nabla d(u, P)|^2$, so Proposition 2.2 implies $\Delta d(u, P)\geqslant 0$.) The general case is an easy modification of this, and we omit the proof. Consider now the function $g: X\times X\to\mathbb{R}$ given by $g(P_1, P_2)=d(P_1, P_2)$. We take the product metric on $X\times X$, and we claim that g is a multiple of the distance function to the convex set $C\subseteq X\times X$ where $C=\{(P, P): P\in X\}$ is the diagonal in $X\times X$. To see this we observe that C is the fixed point set of the isometry F given by $F(P_1, P_2)=(P_2, P_1)$; so that if (Q, Q) is any point of C, then for any path γ from (Q, Q) to (P_1, P_2) the path $\gamma\cup F(\gamma)$ suitably oriented is a path from (P_1, P_2) to (P_2, P_1). In particular we have

$$L(\gamma)\geqslant\frac{1}{2}d((P_1, P_2), (P_2, P_1))=\frac{\sqrt{2}}{2}d(P_1, P_2),$$

and this is achieved when Q is the midpoint of the geodesic from P_1 to P_2. Therefore we have shown

$$g(P_1, P_2)=\sqrt{2}\,d((P_1, P_2), C).$$

It then follows that the function $x\mapsto d(u_1(x), u_2(x))$ is weakly subharmonic, as required. This completes the proof of Lemma 5.3.

We now present two applications of the previous theorem. We first show that a minimizing map is strongly differentiable in a neighborhood of a rank $k\,(=\dim \mathrm{X})$ point provided a mild regularity condition is satisfied for X. First recall that if $\mathrm{Ord}(x_0)=1$ for some $x_0\in\Omega$, then there exists a degree 1 homogeneous approximating map $u_*:\Omega_{x_0}\to \mathrm{X}_{u(x_0)}$. This map, by Proposition 3.1, is a linear map to a flat totally geodesic subcomplex of $\mathrm{X}_{u(x_0)}$. The rank of u_* is the dimension of this flat subspace. If a neighborhood of $u(x_0)$ is isometric to a neighborhood of the origin in the tangent cone $\mathrm{X}_{u(x_0)}$, we then have a flat totally geodesic subcomplex of X containing the image of u_* locally. This happens for example if the simplices in X are standard Euclidean simplices. In general we say that u_* is a *good* homogeneous approximating map if there exists a smooth Riemannian metric g_0 given in normal coordinates on the ball $\mathrm{B}_{r_1}(0)\subseteq\mathbb{R}^k$ and an isometric totally geodesic embedding $i:\mathrm{B}_{r_1}(0)\to\mathrm{X}$ with $i(0)=u(x_0)$ such that the image $i(\mathrm{B}_{\tau_1}(0))$ is contained in a totally geodesic subcomplex X_0 whose tangent cone at $u(x_0)$ is the image of u_*. We now state a theorem.

Theorem 5.4. — *If $u:\Omega\to\mathrm{X}$ is a minimizing map, and $x_0\in\Omega$ is a point at which u has a good homogeneous approximating map of rank $k=\dim\mathrm{X}$, then u is intrinsically differentiable in a neighborhood of x_0. In fact, the map u in a ball $\mathrm{B}_{\sigma_0}(x_0)$ for some $\sigma_0>0$ is given by $u=i\circ v$ where $v:\ \mathrm{B}_{\sigma_0}(x_0)\to\mathrm{B}_1(0)\subseteq\mathbb{R}^k$ is harmonic with respect to the metric g_0 described above.*

Proof. — Recall that $u_*=i_0\circ l_0$ where $i_0:\mathbb{R}^k\to\mathrm{X}_{u(x_0)}$ is an isometric totally geodesic embedding of the Euclidean space $\mathbb{R}^k$, and $l_0:\Omega_{x_0}\to\mathbb{R}^k$ is a linear map of rank k. On a small ball $\mathrm{B}_{\sigma_1}(x_0)$ we define $l:\mathrm{B}_{\sigma_1}(x_0)\to\mathrm{X}$ by $l=i\circ l_0$ where $\mathrm{B}_{\sigma_1}(x_0)\subseteq\Omega$ is identified with the ball of radius σ_1 centered at 0 in Ω_{x_0} via the exponential map. The map l is then essentially homogeneous of degree 1. We claim that l is effectively contained in X_0. To see this, let X_1 be the subcomplex of X_0 consisting of those simplices which are faces of a simplex of X which is *not* in X_0. Since no k-simplex can lie in X_1, it follows that X_1 is a subcomplex of codimension at least one in X_0. Thus $l^{-1}(\mathrm{X}_1)$ is a subset of $\mathrm{B}_{\sigma_1}(x_0)$ consisting of a finite number of compact smooth submanifolds with piecewise smooth boundary, each having codimension at least one. (Note that $n\geqslant k$.) It is then immediate that l is effectively contained in X_0.

We next observe that a smooth manifold is essentially regular. For suppose we have a harmonic map $v:\mathrm{B}_{\sigma_0}(x_0)\to\mathrm{N}^k$ with bounded energy. We may then assume that the image lies in a normal coordinate ball with coordinates $u^1,\dots,u^k$ centered at $u(x_0)$. We also assume that $x^1,\dots,x^n$ are normal coordinates centered at x_0. By Taylor's theorem we have $v(x)=l(x)+\mathrm{Q}(x)$, where c_0, given by

$$c_0=\sup_{\mathrm{B}_{\sigma_0}(0)}|x|^{-2}|\mathrm{Q}(x)|,$$

is bounded in terms of the second derivatives of v. Since the second derivatives of a harmonic map are bounded in terms of the energy (and the manifolds), we have

$$\sup_{B_\sigma(0)} |v-l| \leqslant c_0 \sigma^2 \sigma_0^{-2} \sup_{B_{\sigma_0}(0)} |v-l| \leqslant c_1 \sigma^2 \sup_{B_{\sigma_0}(0)} |v-l|,$$

with c_1 depending only on the energy of u, the manifolds, and σ_0. Since we have chosen normal coordinates in both domain and range, l is essentially homogeneous of degree 1, so we have

$$R(x_0, \sigma) \leqslant c_1 \sigma^2 \sup_{B_{\sigma_0}(0)} |v-l|.$$

Now if $l_1 : B_{\sigma_0}(x_0) \to N$ is any essentially homogeneous map of degree 1 with image near the image of v, then we have for $\sigma_1 < \sigma_0$

$$\sup_{B_{\sigma_0}(0)} d(l, l_1) \leqslant c\sigma_1^{-1} \sup_{B_{\sigma_1}(0)} d(l, l_1),$$

because both l, l_1 are essentially homogeneous of degree 1 and our metrics are nearly Euclidean. Now we have

$$\begin{aligned}\sup_{B_{\sigma_1}(0)} d(l, l_1) &\leqslant \sup_{B_{\sigma_0}(0)} d(v, l_1) + \sup_{B_{\sigma_1}(0)} d(v, l)\\ &\leqslant \sup_{B_{\sigma_0}(0)} d(v, l_1) + c\sigma_1^2 \sup_{B_{\sigma_0}(0)} d(v, l).\end{aligned}$$

Therefore we have

$$\begin{aligned}\sup_{B_{\sigma_0}(0)} d(v, l) &\leqslant \sup_{B_{\sigma_0}(0)} d(v, l_1) + \sup_{B_{\sigma_0}(0)} d(l, l_1)\\ &\leqslant (1 + c\sigma_1^{-1}) \sup_{B_{\sigma_0}(0)} d(v, l_1) + c\sigma_1 \sup_{B_{\sigma_0}(0)} d(v, l).\end{aligned}$$

Now, taking σ_1 so that $c\sigma_1 = 1/2$, we have

$$\sup_{B_{\sigma_0}(0)} d(v, l) \leqslant c \sup_{B_{\sigma_0}(0)} d(v, l_1).$$

Since l_1 was an arbitrary essentially homogeneous degree 1 approximation to v, we have finally shown

$$R(x_0, \sigma) \leqslant c\sigma^2 R(x_0, \sigma_0)$$

as required.

Since u_* is a homogeneous approximating map for u at x_0, for any given δ_0 (take the δ_0 determined in Theorem 5.1), there exists a small radius $\rho_0 > 0$ such that

$$\sup_{x \in B_1(0)} |\mu_0^{-1} u(\rho_0 x) - u_*(x)| \leqslant \delta_0.$$

This implies

$$\sup_{x \in B_{\rho_0}(0)} d(u(x), \bar{l}(x)) \leqslant \delta_0$$

where $\bar{l} = i \circ (\mu_0 l_0)$. By Theorem 5.1 we now conclude that u is intrinsically differentiable near x_0, and the image of a small ball $B_{\sigma_0}(x_0)$ under u lies in X_0. This completes the proof of Theorem 5.4.

The final result of this section will deal with the case dim $X = 1$. We now state it.

Theorem 5.5. — *If* dim $X = 1$, *then* X *is essentially regular. Moreover, if* $u : \Omega \to X$ *is a minimizing map, there exists a constant* $\hat{\varepsilon} > 0$ *depending only on* n *such that for all* $x_0 \in \Omega$ *we have* $\operatorname{Ord}(x_0) = 1$ *or* $\operatorname{Ord}(x_0) \geqslant 1 + \hat{\varepsilon}$.

Proof. — Since dim $X = 1$, any point $x_0 \in \Omega$ where $\operatorname{Ord}(x_0) = 1$ has a neighborhood in which u is intrinsically differentiable (in fact, defined by a smooth harmonic function to a geodesic of X). In particular we note that the set of points x where $\operatorname{Ord}(x) = 1$ is an open subset, denoted Ω_1, of Ω.

In this case one can see explicitly that if $\operatorname{Ord}(x_0) > 1$, then $\operatorname{Ord}(x_0) \geqslant 1 + \hat{\varepsilon}$. To see this, first observe that if $u(x_0)$ is not a vertex of X, then $\operatorname{Ord}(x_0) \geqslant 2$ since u is a smooth map near x_0. Let $\alpha = \operatorname{Ord}(x_0)$, and assume that $u(x_0)$ is a vertex of X with at least three edges emanating from $P_0 = u(x_0)$ (there cannot be one edge by the maximum principle, and if there are only two, then $\operatorname{Ord}(x_0) \geqslant 2$ as above). Consider any homogeneous approximating map $u_* : \mathbb{R}^n \to X_{P_0}$. If we choose an edge e emanating from P_0 and introduce an arc length parameter s along e which is zero at P_0, then on the open region $O_e = \{ x \in \mathbb{R}^n : u_*(x) \in e - \{ P_0 \} \}$ the function $h_e = s(u_*(x))$ is a harmonic function. Of course O_e is the cone over a domain $D_e \subseteq S^{n-1}$, and h_e is homogeneous of degree α in O_e. It follows that the restriction of h_e to D_e is a first Dirichlet eigenfunction of the domain D_e, and in particular we have $\lambda_1(D_e) = \alpha(\alpha + n - 2)$. Since D_e is non-empty for at least three distinct edges (otherwise $\operatorname{Ord}(x_0) \geqslant 2$ as above), there is an edge e for which D_e is nonempty and $\operatorname{Vol}(D_e) \leqslant 1/3 \operatorname{Vol}(S^{n-1})$. Standard results about eigenvalues then imply that there exists a number $\delta_n > 0$ such that $\lambda_1(D_e) \geqslant n - 1 + \delta_n$. It then follows that $\alpha \geqslant 1 + \hat{\varepsilon}$ for a fixed constant $\hat{\varepsilon}$ depending only on n.

To show that X is essentially regular, we must show that for any minimizing map $u : \Omega \to X$ and any compact subset $K \subseteq \Omega$, there exists r_0, c, β such that

$$R(x_0, \sigma) \leqslant c\,\sigma^{1+\beta} R(x_0, r_0)$$

for all $x_0 \in K$. Let r_0 be such that $B_{r_0}(x_0)$ is compactly contained in Ω. It clearly suffices to show that there exists $\theta \in (0, 1)$ such that for all $\sigma \in (0, r_0]$ and all $x_0 \in K$ we have

$$(\theta\sigma)^{-1} R(x_0, \theta\sigma) \leqslant (1/2)\,\sigma^{-1} R(x_0, \sigma).$$

This can be proved by contradiction. If this were not true there would be for any $\theta \in (0, 1)$ sequences $\{x_i\}$, $\{\sigma_i\}$ both of which converge to limits $\bar{x} \in K$, $\bar{\sigma} \in [0, r_0]$ such that

$$(\theta\sigma_i)^{-1} R(x_i, \theta\sigma_i) > (1/2)\,\sigma_i^{-1} R(x_i, \sigma_i).$$

We then rescale the maps in the usual way by setting u_i equal to

$$u_i(x) = \mu_i^{-1} u_i(\sigma_i x)$$

for $x \in B_1(0)$ so that $\sup_{B_1(0)} d(u_1(x), u_i(0)) = 1$. We then have

$$\theta^{-1} R^{u_i}(0, \theta) > (1/2)\, R^{u_i}(0, 1).$$

We may assume by taking a subsequence that $\{u_i\}$ converges uniformly to a minimizing map $\bar{u}: B_1(0) \to \bar{X}$ where the target complex is either a dilation of X or the tangent cone $X_{\bar{P}}$, where $\bar{P} = \lim u_i(x_i)$. Now $\mathrm{Ord}^{\bar{u}}(0) = 1$ or $\mathrm{Ord}^{\bar{u}}(0) \geqslant 1 + \hat{\varepsilon}$, and in the first case Theorem 5.1 implies that for i sufficiently large the map u_i is regular in a fixed neighborhood of 0. This contradicts the previous inequality if θ is small enough. In case $\mathrm{Ord}^{\bar{u}}(0) = \bar{\alpha} \geqslant 1 + \hat{\varepsilon}$, then we know from the proof of Theorem 2.3 that we have

$$\sup_{B_\sigma(0)} d(\bar{u}, \bar{u}(0)) \leqslant c\,\sigma^{\alpha} \sup_{B_1(0)} d(\bar{u}, \bar{u}(0)).$$

This implies, by an easy argument, that for θ small depending on c we have

$$\theta^{-1} R^{\bar{u}}(0, \theta) \leqslant 1/4\, R^{\bar{u}}(0, 1),$$

a contradiction. This completes the proof of Theorem 5.5.

6. Special structure of harmonic maps into building-like complexes

In this section we consider complexes X which satisfy a very special hypothesis which we describe shortly. In a certain sense these are higher dimensional generalizations of trees. For us the important property which a tree has is that two adjacent edges lie in a geodesic (the union of the two closed edges). We assume that the simplices of X are Euclidean simplices; that is, images under a linear transformation of the standard Euclidean simplex. We make the following definition.

Definition. — We say that a nonpositively curved complex X^k is F-*connected* if any two adjacent simplices are contained in a totally geodesic subcomplex X_0 which is isometric to a subset of the Euclidean space $\mathbb{R}^k$.

The most important F-connected complexes are the locally finite Euclidean buildings of Bruhat and Tits (see [BT]). We first want to show that F-connected complexes are essentially regular, thus generalizing Theorem 5.5. We first need two elementary lemmas.

Lemma 6.1. — *If* X_1, X_2 *are essentially regular complexes, then so is* $X_1 \times X_2$.

Proof. — This follows from the fact that a map $u=(u_1, u_2):\Omega \to X_1 \times X_2$ has energy $E(u)=E(u_1)+E(u_2)$. Thus u is minimizing if and only if both u_1, u_2 are minimizing. Thus u_1, u_2 are intrinsically differentiable on any compact subset of Ω, and hence so is u. This proves Lemma 6.1.

The next result enables us to find essentially regular totally geodesic subcomplexes which contain any given flat effectively. Since we are interested only in local constructions near a point $P_0 \in X$, and since a neighborhood of P_0 in X is isometric to a neighborhood of the origin in the tangent cone X_{P_0}, we may replace X by its tangent cone X_{P_0}. The fact that X is F-connected then implies that any two simplices (actually simplicial cones) in X_{P_0} are contained in a totally geodesic subcomplex isometric to the Euclidean space $\mathbb{R}^k$. Let $J:\mathbb{R}^m \to X_{P_0}$ be an isometric totally geodesic embedding for some m with $1 \leqslant m \leqslant k$. We may assume that $J(0)=0$. If we choose a point $x \neq 0$, $x \in \mathbb{R}^m$ such that neither x nor $-x$ lie in the $J^{-1}((m-1)$-skeleton of $X_{P_0})$, then any k-dimensional flat F which contains both x and $-x$ must contain the full image $J(\mathbb{R}^m)$. This is because $J^{-1}(F)$ contains cones over a neighborhood of both x and $-x$, and therefore contains the convex hull of these cones which is $\mathbb{R}^m$. It may happen that $J(\mathbb{R}^m)$ is contained in several distinct k-dimensional flats. We need the following result.

Lemma 6.2. — *Let* X_0 *be the union of all k-dimensional flats in* X_{P_0} *which contain* $J(\mathbf{R}^m)$. *The subcomplex* X_0 *is totally geodesic and is isometric to* $\mathbf{R}^m \times X_1^{k-m}$ *where* X_1 *is an* F-*connected complex of dimension* $k-m$. *If* $L:\mathbf{R}^n \to \mathbf{R}^m$ *is a linear map of rank m,* then $l:\mathbf{R}^n \to X_{P_0}$, *given by* $l=J\circ L$, *is effectively contained in* X_0.

Proof. — Let $F_1, \ldots, F_r$ denote the k-dimensional flats in X_{P_0} which contain $J(\mathbf{R}^m)$. Let $I_i:\mathbf{R}^k \to X_{P_0}$, $i=1, \ldots, r$, denote isometric embeddings to the F_i normalized so that $\mathrm{I}_i(0)=0$ and $\mathrm{I}_i(\mathbf{R}^m)=\mathrm{J}(\mathbf{R}^m)$, where we denote by $\mathbf{R}^m \subseteq \mathbf{R}^k$ the plane spanned by the first m standard basis vectors; i.e.

$$\mathbf{R}^m=\{(\bar{x}, 0): \bar{x}=(x^1, \ldots, x^m),\ x^j \in \mathbf{R} \text{ for } j=1, \ldots, m\}.$$

Similarly we denote by $\mathbf{R}^{k-m} \subseteq \mathbf{R}^k$ the orthogonal complement

$$\mathbf{R}^{k-m}=\{(0, \bar{\bar{x}}): \bar{\bar{x}}=(x^{m+1}, \ldots, x^k),\ x^j \in \mathbf{R} \text{ for } j=m+1, \ldots, k\}.$$

We may further normalize our isometries so that $\mathrm{I}_1, \ldots, \mathrm{I}_r$ and J are identical on $\mathbf{R}^m$ since this can be achieved by right composition of each I_i with an orthogonal transformation of $\mathbf{R}^m$. Each map I_i induces on $\mathbf{R}^k$ a cell decomposition by simplicial cones. We may describe X_0 as the disjoint union of r copies of $\mathbf{R}^k$ where cells Σ_i, Σ_j in the i-th, j-th decompositions are identified if $\mathrm{I}_i(\Sigma_i)=\mathrm{I}_j(\Sigma_j)$. By taking the intersection of cells with $\mathbf{R}^m$, $\mathbf{R}^{k-m}$ each cell decomposition of $\mathbf{R}^k$ induces a cell decomposition of both $\mathbf{R}^m$, $\mathbf{R}^{k-m}$ by simplicial cones. Since all of the embeddings agree on $\mathbf{R}^m$, the cell decompositions of $\mathbf{R}^m$ all coincide, so we may speak of the induced cell decomposition of $\mathbf{R}^m$.

Now suppose we have two points $(\bar{x}, \bar{\bar{x}})$, $(\bar{y}, \bar{\bar{y}})$ in $\mathbf{R}^k$ such that $\mathrm{I}_i(\bar{x}, \bar{\bar{x}})=\mathrm{I}_j(\bar{y}, \bar{\bar{y}})$. Since the point $(\bar{x}, 0) \in \mathbf{R}^m$ is the nearest point of $\mathbf{R}^m$ to $(\bar{x}, \bar{\bar{x}})$, and $(\bar{y}, 0)$ is nearest in $\mathbf{R}^m$ to $(\bar{y}, \bar{\bar{y}})$ and the maps I_i, I_j coincide on $\mathbf{R}^m$, we must have $\bar{x}=\bar{y}$. It also follows that the image of the closed convex hulls of $\mathbf{R}^m \cup \{(\bar{x}, \bar{\bar{x}})\}$ and $\mathbf{R}^m \cup \{(\bar{y}, \bar{\bar{y}})\}$ under I_i and I_j respectively must coincide in X. In particular we have $\mathrm{I}_i(0, \bar{\bar{x}})=\mathrm{I}_j(0, \bar{\bar{y}})$. We form a complex X_1 of dimension $k-m$ by taking the disjoint union of the r copies of $\mathbf{R}^{k-m}$ with their induced cell decompositions and identifying points in the i-th and j-th copy if their respective images under I_i and I_j coincide. The previous argument then shows that X_0 is isometric to $\mathbf{R}^m \times \mathrm{X}_1^{k-m}$.

We now show that any two cells in X_0 are contained in some F_j. This implies that X_0 is totally geodesic since the geodesic connecting two points P_1, $\mathrm{P}_2 \in \mathrm{X}_0$ lies in any F_j which contains P_1 and P_2, and hence lies in X_0. It also obviously implies that X_1 is F-connected. Suppose we have a pair of cells in X_0 which we may assume to be k-dimensional without loss of generality. We may think of these cells as Σ_i, Σ_j in two of the cell decompositions of $\mathbf{R}^k$ given by I_i, I_j respectively. Fix a nonzero point $(\bar{x}_0, 0) \in \mathbf{R}^m$ such that both $(\bar{x}_0, 0)$ and $(-\bar{x}_0, 0)$ are interior to m-cells in the cell decomposition of $\mathbf{R}^m$. Let C_i, C_j be the closed convex hulls of $\mathbf{R}^m \cup \Sigma_i$, $\mathbf{R}^m \cup \Sigma_j$ respectively. Choose points y_i, y_j in the interior of C_i, C_j respectively such that both y_i and y_j are interior to k-cells Σ_i^1, Σ_j^1 respectively. Assume also that y_i is close to $(\bar{x}_0, 0)$ and y_j is close to $(-\bar{x}_0, 0)$. It follows that the m-cell of $\mathbf{R}^m$ containing $(\bar{x}_0, 0)$ is a face of Σ_i^1, and the m-cell containing $(-\bar{x}_0, 0)$ is a face of Σ_j^1. Let F be a k-dimensional flat totally geodesic subcomplex of X which contains $\mathrm{I}_i(\Sigma_i^1)$ and $\mathrm{I}_j(\Sigma_j^1)$. Since F contains neighborhoods of a pair of antipodal points of $\mathrm{J}(\mathbf{R}^m)$ it follows that F contains $\mathrm{J}(\mathbf{R}^m)$, and hence $\mathrm{F}-\mathrm{F}_{i_0}$ for some i_0 with $1 \leqslant i_0 \leqslant r$. From the choices we have made, F_{i_0} must contain some interior point of both $I_i(\Sigma_i)$ and $I_j(\Sigma_j)$, and therefore F_{i_0} contains both cells. This proves the desired statement.

Finally, suppose $l = J \circ L : \mathbf{R}^n \to X_{P_0}$ where $L : \mathbf{R}^n \to \mathbf{R}^m$ is linear of rank m. We want to show that l is effectively contained in X_0. To see this, let X_2 be the subcomplex of $\mathbf{R}^m$ (with the above cell decomposition) consisting of those cells of $\mathbf{R}^m$ whose image under J is contained in the closure of some cell of X which is *not* in X_0. As in the proof of Theorem 5.4, if we can show that X_2 has codimension at least one in X_0 then the result follows. Let Σ be any m-cell of $\mathbf{R}^m$, and let Σ_1 be a cell of X which is not in X_0 such that $J(\Sigma) \subseteq \Sigma_1$. Without loss of generality we may assume Σ_1 is k-dimensional. Take an interior point $(\bar{x}_0, 0) \in \Sigma$ such that $(-\bar{x}_0, 0)$ is interior to some cell $\hat{\Sigma}$ of $\mathbf{R}^m$. Let Σ_2 be a cell of X which contains $J(\hat{\Sigma})$ in its closure. Then any k-flat of X_{P_0} which contains both Σ_1 and Σ_2 automatically contains all of $J(\mathbf{R}^m)$. This contradicts the fact that Σ_1 is not in X_0. We have completed the proof of Lemma 6.2.

Observe that if $P_0 \in X$, then a neighborhood of P_0 is isometric to a neighborhood of the origin in the tangent cone X_{P_0}. In fact, if we define $\mathrm{St}(P_0)$ to be the union of all simplices which contain P_0 in their closure, then $\mathrm{St}(P_0)$ is a totally geodesic subcomplex of X which is canonically isometric to a neighborhood of the origin in X_{P_0}. Note that if P_0 is a vertex, then $\mathrm{St}(P_0)$ is the star of P_0. Thus if X_0 is a totally geodesic subcomplex of X_{P_0}, then we get a totally geodesic subcomplex, which we will also refer to as X_0, of $\mathrm{St}(P_0)$ which is isometric to a neighborhood of 0 in X_0. We now prove an important result concerning minimizing maps into F-connected complexes.

Theorem 6.3. — *Let* X *be an* F-*connected complex. The following three properties hold:*

(i) *For any positive integer n and any compact subset* K_0 *of* X, *there exists* $\hat{\varepsilon} > 0$ *depending only on* K_0 *and n such that, for any minimizing map* $u : \Omega^n \to X$ *with* $u(\Omega) \subseteq K_0$, *we have, for all* $x_0 \in \Omega$, *either* $\mathrm{Ord}(x_0) = 1$ *or* $\mathrm{Ord}(x_0) \geqslant 1 + \hat{\varepsilon}$.

(ii) *Let* $u : \Omega \to X$ *be a minimizing map, and let* $x_0 \in \Omega$ *with* $\mathrm{Ord}(x_0) = 1$. *There exists a totally geodesic subcomplex* X_0 *of* $X_{u(x_0)}$ *which is isometric to* $\mathbf{R}^m \times X_1^{k-m}$ *for some integer m with* $1 \leqslant m \leqslant \min\{n, k\}$ *and some* F-*connected complex* X_1 *of dimension* $k-m$ *such that* $u(B_{\sigma_0}(x_0)) \subseteq X_0$ *for some* $\sigma_0 > 0$. *Moreover if we write* $u = (u_1, u_2) : B_{\sigma_0}(x) \to \mathbf{R}^m \times X_1$, *then* u_1 *is a harmonic map of rank m at each point of* $B_{\sigma_0}(x_0)$ *and* $\mathrm{Ord}^{u_2}(x_0) > 1$.

(iii) *The complex* X *is essentially regular.*

Proof. — To prove (i) we first observe that the tangent cone to X at a point P is the cone over the link of the open simplex which contains P. Therefore, there are only a finite number of cones which appear as tangent cones for points P in a compact subset K_0 of X. Thus to prove (i) we may restrict attention to a single cone X_{P_0} for some fixed $P_0 \in K_0$. Suppose we have a sequence of homogeneous harmonic maps $\{u_i\}$, $u_i : \mathbf{R}^n \to X_{P_0}$ such that u_i is homogeneous of degree α_i with $\alpha_i > 1$ and $\lim_{i\to\infty} \alpha_i = 1$. We may normalize u_i so that

$$\sup_{x \in B_1(0)} |u_i(x)| = 1.$$

With this normalization a subsequence of $\{u_i\}$ again denoted $\{u_i\}$ converges to a map $\bar{u} : \mathbf{R}^n \to X_{P_0}$ which is homogeneous of degree 1. Thus by Proposition 3.1 there exists an integer m with $1 \leqslant m \leqslant \min\{n, k\}$ such that $\bar{u} = J \circ L$ where $L : \mathbf{R}^n \to \mathbf{R}^m$ is a linear map of rank m and $J : \mathbf{R}^m \to X_{P_0}$ is an isometric embedding. By Lemma 6.2 the map $\bar{u}$ is effectively contained in a totally geodesic subcomplex X_0 of X_{P_0} which is isometric to $\mathbf{R}^m \times X_1$. By Theorem 5.1 we have the image of u_i contained in X_0 for i sufficiently large. It then follows that the projection of u_i to $\mathbf{R}^m$ is a harmonic map which is close to a linear map of rank m. Therefore u_i must be linear (since it is homogeneous harmonic of degree less than two), and we have a contradiction. This establishes property (i).

Property (ii) is an easy consequence of Lemma 6.2 and Theorem 5.1. To establish (iii) we work by induction on k. For $k = 1$ the result was established in Theorem 5.5. Assume that $k \geqslant 2$, and that all F-connected complexes of dimension less than k are essentially regular. We first prove the weaker result that for any $x_0 \in \Omega$, there exists $r_0 > 0$ such that for $\sigma \in (0, r_0]$

$$R(x_0, \sigma) \leqslant c\, \sigma^{1+\beta} R(x_0, r_0)$$

for constants C, r_0, β depending on the point x_0, the energy of u, and the spaces Ω, X. If we can prove this, then the same compactness argument used in Theorem 5.5 will imply that u is intrinsically differentiable on any compact subset of Ω. There are two cases to consider. First suppose $\mathrm{Ord}(x_0) > 1$; then from (i) we know that $\mathrm{Ord}(x_0) \geqslant 1 + \hat{\varepsilon}$. This implies, by the proof of Theorem 2.3, that

$$\sup_{x \in B_\sigma(x_0)} d(u(x), u(x_0)) \leqslant c\, \sigma^{1+\hat{\varepsilon}} \sup_{x \in B_{r_0}(x_0)} d(u(x), u(x_0))$$

for some constant c and $r_0 > 0$. This easily implies the desired decay on $R(x_0, \sigma)$. The other case we must consider is the case $\mathrm{Ord}(x_0) = 1$. In this case the result follows immediately from (ii), Lemma 6.1, and the inductive assumption. This completes the proof of Theorem 6.3.

It will be important for our application to do smooth differential geometric calculations for our harmonic maps. To make sense of this we need to refine the notion of intrinsic differentiability. This we now do.

Definition. — A point $x_0 \in \Omega$ is a *regular point* of u if there exists $\sigma_0 > 0$ and a k-flat $F \subseteq X_{u(x_0)}$ such that $u(B_{\sigma_0}(x_0)) \subseteq F$.

We see in particular that the map u is actually a real analytic map to the Euclidean space $\mathbf{R}^k$ in a neighborhood of a regular point. We then let $\mathscr{R}(u)$ denote the open subset of Ω consisting of all regular points of u, and we let $\mathscr{S}(u)$ denote the singular set, $\mathscr{S}(u) = \Omega - \mathscr{R}(u)$. We now prove the following result.

Theorem 6.4. — *Let u be a minimizing map from Ω to an* F-*connected complex* X. *The Hausdorff dimension of $\mathscr{S}(u)$ is at most $n-2$. For any compact subdomain Ω_1 of Ω there is a sequence of Lipschitz functions $\{\psi_i\}$ with $\psi_i \equiv 0$ in a neighborhood of $\mathscr{S} \cap \bar{\Omega}_1$, $0 \leqslant \psi_i \leqslant 1$, and $\psi_i(x) \to 1$ for all $x \in \Omega_1 - \mathscr{S}$* such that

$$\lim_{i \to \infty} \int_\Omega |\nabla\nabla u| \, |\nabla \psi_i| \, d\mu = 0.$$

Proof. — We first observe that the singular set $\mathscr{S}$ may be written as a union $\mathscr{S} = \mathscr{S}_0 \cup \ldots \cup \mathscr{S}_{k_0}$ where $k_0 = \min\{n, k-1\}$ and $\mathscr{S}_j$ consists of those singular points having rank j, where the rank of a point x_0 is the number m appearing in Theorem 6.3 if $\operatorname{Ord}(x_0) = 1$, and the rank is zero if $\operatorname{Ord}(x_0) > 1$. In other words, the rank of u at x_0 is the rank of the linear approximation to u near x_0 (which exists by Theorem 6.3). We will show first that $\dim \mathscr{S}_0 \leqslant n-2$, and the result for $\mathscr{S}$ will follow from an easy inductive argument based on Theorem 6.3. Let $\tilde{\mathscr{S}}_0(u) = \{x_0 \in \Omega : \operatorname{Ord}(x_0) > 1\}$, so that $\mathscr{S}_0(u) \subseteq \tilde{\mathscr{S}}_0(u)$. We deal with the larger set $\tilde{\mathscr{S}}_0$. The proof for $\tilde{\mathscr{S}}_0$ is an application of the basic argument of H. Federer [F2]. For any subset $E \subseteq \Omega$ and any real number $s \in [0, n]$ define an outer Hausdorff measure $\hat{\mathscr{H}}^s(.)$ by

$$\hat{\mathscr{H}}^s(E) = \inf\left\{\sum_{i=1}^{\infty} r_i^s : \text{all coverings } \{B_{r_i}(x_i)\}_{i=1}^{\infty} \text{ of E by open balls}\right\}.$$

The value of $\hat{\mathscr{H}}^s(E)$ is not important, and bears little resemblance to the s-dimensional measure of a set; however, it is clear that the Hausdorff dimension of a set E is given by

$$\dim E = \inf\{s : \hat{\mathscr{H}}^s(E) = 0\}.$$

We note the following result.

Lemma 6.5. — *If $\{u_i\}$ is a sequence of minimizing maps into a compact subset of* X *from $B_1(0) \subseteq \mathbf{R}^n$ equipped with metrics ${}^i g$ converging in the C^2 norm to a limit g,* *then a subsequence of* $\{u_i\}$ *converges uniformly on compact subsets of* $B_1(0)$ *to a minimizing map* $u:(B_1(0), g) \to X$, *and we have*

$$\hat{\mathscr{H}}^s(\tilde{\mathscr{S}}_0(u) \cap \overline{B_r(0)}) \geqslant \overline{\lim_{i \to \infty}} \ \hat{\mathscr{H}}^s(\tilde{\mathscr{S}}_0(u_i) \cap \overline{B_r(0)})$$

for all $r \in (0, 1)$. *In particular,* $\dim(\tilde{\mathscr{S}}_0(u)) \geqslant \overline{\lim_{i \to \infty}} \dim(\tilde{\mathscr{S}}_0(u_i))$.

Proof. – Since $\tilde{\mathscr{S}}_0(u) \cap \overline{B_r(0)}$ is compact we may consider finite coverings, $\tilde{S}_0(u) \cap \overline{B_r(0)} \subseteq \bigcup_{j=1}^{N} B_{r_j}(x_j)$. For any $\varepsilon > 0$ we have

$$\tilde{\mathscr{S}}_0(u_i) \cap \overline{B_r(0)} \subseteq \{x \in B_1(0) : \mathrm{Dist}(x, \tilde{\mathscr{S}}_0(u) \cap \overline{B_r(0)}) < \varepsilon\}$$

for i sufficiently large. This follows immediately from the fact that if $x_i \in \tilde{\mathscr{S}}_0(u_i) \cap \overline{B_r(0)}$, and $x_i \to x$, then

$$\overline{\lim_{i \to \infty}} \ \mathrm{Ord}^{u_i}(x_i) \leqslant \mathrm{Ord}^u(x)$$

and hence by part (i) of Theorem 6.3, $x \in \tilde{\mathscr{S}}_0(u) \cap \overline{B_r(0)}$. The conclusion of Lemma 6.5 now follows.

We now show that $\dim \tilde{\mathscr{S}}_0(u) \leqslant n-2$. Suppose $s \in [0, n]$ with $\mathscr{H}^s(\tilde{\mathscr{S}}_0(u)) > 0$. Then by [F1, 2.10.19] we may find a point $x_0 \in \Omega$ such that

$$\underline{\lim_{\sigma \to 0}} \ \sigma^{-s} \mathscr{H}^s(\tilde{\mathscr{S}}_0(u) \cap B_\sigma(x_0)) \geqslant 2^{-s}.$$

Let $u_*: \mathbf{R}^n \to X_{u(x_0)}$ be a homogeneous approximating map for u at x_0. Let $\alpha = \mathrm{Ord}^u(x_0)$, so that u_* is homogeneous of degree α, and since $x_0 \in \tilde{S}_0(u)$ we have $\alpha \geqslant 1 + \hat{\varepsilon}$ by Theorem 6.3. We may apply Lemma 6.5 to suitable rescalings $\{u_i\}$ of u near x_0, recalling that maps to a neighborhood of $u(x_0)$ may be thought of as maps to the *fixed* space $X_{u(x_0)}$, to conclude that $\mathscr{H}^s(\tilde{\mathscr{S}}_0(u_*)) > 0$. Since $\tilde{\mathscr{S}}_0(u_*)$ is a cone (u_* is homogeneous), it follows that there is a point $x_1 \in S^{n-1} \cap \tilde{\mathscr{S}}_0(u_*)$ such that

$$\underline{\lim_{\sigma \to 0}} \ \sigma^{-s} \mathscr{H}^s(\tilde{\mathscr{S}}_0(u_*) \cap B_\sigma(x_1)) \geqslant 2^{-s}.$$

Let u_1 be a homogeneous approximating map for u_* at x_1. Note that $\mathrm{Ord}^{u_1}(x_1) \geqslant 1 + \hat{\varepsilon}$, and hence $\mathrm{Ord}^{u_1}(tx_1) \geqslant 1 + \hat{\varepsilon}$ for $t \in [0, 1]$. Therefore the derivative of u_1 is zero along the ray $t \mapsto tx_1$, and hence $u_1(x_1) = 0 \in X_{u(x_0)}$. Therefore u_1 is a homogeneous map to the same cone $X_{u(x_0)}$. The homogeneity of u_* together with the fact that $\mathrm{Ord}^{u_*}(x_1) > 1$ translates to the statement that the map u_1 is independent of a direction. If we choose coordinates in which $x_1=(0,\ldots,0,1)$, then we have $\partial u_1/\partial x^n=0$. Therefore the restriction of u_1, denoted $\tilde{u}_1$, to the $\mathbf{R}^{n-1}$ spanned by the first $(n-1)$ standard basis vectors is a homogeneous map of degree $\alpha_1>1+\hat{\varepsilon}$. We then have

$$\tilde{\mathscr{S}}_0(u_1)=\tilde{\mathscr{S}}_0(\tilde{u}_1)\times\mathbf{R},$$

and thus $\mathscr{H}^{s-1}(\tilde{\mathscr{S}}_0(\tilde{u}_1))>0$. If $s>n-2$, we may repeat this argument inductively and produce finally an $\bar{\varepsilon}>0$ and a minimizing map $v:\mathbf{R}^2\to \mathrm{X}_{u(x_0)}$ homogeneous of degree $\bar{\alpha}\geqslant 1+\bar{\varepsilon}$ such that $\mathscr{H}^{s-(n-2)}(\tilde{\mathscr{S}}_0(v))>0$. In particular, there is a point of $\tilde{\mathscr{S}}_0(v)$ different from the origin. If we repeat the argument again we construct a geodesic. By the 1-dimensional analysis of Section 1 we know that for a geodesic we have $\operatorname{Ord}(x_0)=1$ for all x_0. This contradiction shows that $s\leqslant n-2$ for *any* $s\in[0,n]$ with $\mathscr{H}^s(\tilde{\mathscr{S}}_0(u))>0$. Therefore $\dim\tilde{\mathscr{S}}_0(u)\leqslant n-2$ as required.

We now show by induction on $k=\dim \mathrm{X}$ that we have $\dim\mathscr{S}(u)\leqslant n-2$. For $k=1$ we have $\mathscr{S}=\mathscr{S}_0$, and we have established this case. Assume that $k\geqslant 2$, and that for F-connected complexes $\tilde{\mathrm{X}}$ of dimension less than k we know that the singular sets of minimizing maps of any n-manifold into $\tilde{\mathrm{X}}$ have dimension at most $n-2$. Let x_0 be any point of $\mathscr{S}_m-\mathscr{S}_0$ for a minimizing map $u:\Omega\to\mathrm{X}^k$. We then have $\operatorname{Ord}(x_0)=1$, and by Theorem 6.3 there is a neighborhood $\mathrm{B}_{\sigma_0}(x_0)$ for some $\sigma_0>0$ such that the map u maps $\mathrm{B}_{\sigma_0}(x_0)$ into a totally geodesic subcomplex X_0 of X having the form $\mathbf{R}^m\times\mathrm{X}_1$. Thus we have $u=(u_1,u_2)$ where $u_1:\mathrm{B}_{\sigma_0}(x_0)\to\mathbf{R}^m$, $u_2:\mathrm{B}_{\sigma_0}(x_0)\to\mathrm{X}_1$ are both minimizing. The set $\mathscr{S}_m(u)\cap\mathrm{B}_{\sigma_0}(x_0)$ is then the same as the set $\mathscr{S}_0(u_2)\cap\mathrm{B}_{\sigma_0}(x_0)$ since u_1 has rank m at every point of $\mathrm{B}_{\sigma_0}(x_0)$. By the inductive hypothesis we then have $\dim(\mathscr{S}_m(u)\cap\mathrm{B}_{\sigma_0}(x_0))\leqslant n-2$. This shows that for each m, $\dim\mathscr{S}_m\leqslant n-2$ and therefore $\dim\mathscr{S}\leqslant n-2$ as required.

To prove the final statement of Theorem 6.4, that is, to construct the functions ψ_i, let $\varepsilon>0$ and $d>n-2$. Let Ω_2 be a fixed compact subdomain of Ω with $\Omega_1\Subset\Omega_2$, and choose a finite covering $\{\mathrm{B}_{r_j}(x_j):j=1,\ldots,l\}$ of the compact set $\mathscr{S}_0\cap\bar{\Omega}_1$ satisfying $\sum_{j=1}^{l} r_j^d\leqslant\varepsilon$. Assume also that $\mathrm{B}_{4r_j}(x_j)\subseteq\Omega_2$ as will be true if ε is small enough. Let φ_j be a Lipschitz function which is zero in $\mathrm{B}_{r_j}(x_j)$ and identically one on $\Omega-\mathrm{B}_{2r_j}(x_j)$ such that $|\nabla\varphi_j|\leqslant 2r_j^{-1}$. We assume also that $x_j\in\mathscr{S}_0\cap\bar{\Omega}_1$. Let φ be defined by

$$\varphi=\min\{\varphi_j:j=1,\ldots,l\}$$

and observe that φ vanishes in a neighborhood of $\mathscr{S}_0\cap\bar{\Omega}_1$ and $\varphi\equiv 1$ on $\Omega-\bigcup_{j=1}^{l}\mathrm{B}_{2r_j}(x_j)$. Now let $\psi_0=\varphi^2$ and observe

$$(6.1)\qquad \int_\Omega |\nabla\nabla u|\,|\nabla\psi_0|\,d\mu = 2\int_{\bigcup_{j=1}^l B_{2r_j}(x_j)} \varphi\,|\nabla\nabla u|\,|\nabla\varphi|\,d\mu$$

$$\leqslant 2\left(\int_{\bigcup_{j=1}^l B_{2r_j}(x_j)} |\nabla\nabla u|^2\,|\nabla u|^{-1}\varphi^2\,d\mu\right)^{1/2}\left(\int_{\bigcup_{j=1}^l B_{2r_j}(x_j)} |\nabla u|\,|\nabla\varphi|^2\,d\mu\right)^{1/2}$$

by the Schwarz inequality. On the other hand, an elementary result for harmonic maps (*see* [ES]) implies that on the regular set we have

$$(6.2)\qquad \frac{1}{2}\Delta|\nabla u|^2 \geqslant |\nabla\nabla u|^2 - c|\nabla u|^2.$$

For $j=1,\ldots,l$ let ρ_j be a Lipschitz function which is identically one on $B_{2r_j}(x_j)$ and identically zero on $\Omega - B_{3r_j}(x_j)$ with $|\nabla\rho_j| \leqslant 2r_j^{-1}$. Define ρ by

$$\rho = \max\{\rho_j : j=1,\ldots,l\}$$

and observe that $\rho\equiv 1$ on $\bigcup_{j=1}^l B_{2r_j}(x_j)$ while $\rho\equiv 0$ outside $\bigcup_{j=1}^l B_{3r_j}(x_j)$. We therefore have

$$(6.3)\qquad \int_{\bigcup_{j=1}^l B_{2r_j}(x_1)} |\nabla\nabla u|^2\,|\nabla u|^{-1}\varphi^2\,d\mu \leqslant \int_\Omega |\nabla\nabla u|^2\,|\nabla u|^{-1}\varphi^2\rho^2\,d\mu.$$

We now state the following result whose proof we give later.

Lemma 6.6. — *The conclusion of Theorem* 6.4 *implies that inequality* (6.2) *holds distributionally on all of* Ω.

Now we make the assumption either that $k=1$, or that Theorem 6.4 holds for maps into F-connected complexes of dimension less than k. In the first case we have $\mathscr{S}_0=\mathscr{S}$, so (6.2) holds away from $\mathscr{S}_0$. In the second case, for any point $x_0\in\mathscr{S}-\mathscr{S}_0$, the map is given locally as $u=(u_1, u_2)$ where u_1 is a map to a Euclidean space and u_2 is a map to a lower dimensional F-connected complex. Since we have

$$|\nabla u|^2 = |\nabla u_1|^2 + |\nabla u_2|^2,\quad |\nabla\nabla u|^2 = |\nabla\nabla u_1|^2 + |\nabla\nabla u_2|^2$$

by our inductive assumption and Lemma 6.6 we have inequality (6.2) distributionally on $\Omega-\mathscr{S}_0$. A result of [SY] implies that on the regular set we have $(1-\varepsilon_n)|\nabla\nabla u|^2 \geqslant |\nabla|\nabla u||^2$ for some $\varepsilon_n>0$ depending only on n. Therefore we have the distributional inequality

$$\Delta|\nabla u| \geqslant \varepsilon_n |\nabla\nabla u|^2\,|\nabla u|^{-1} - c|\nabla u|$$

on $\Omega-\mathscr{S}_0$. Using $\rho^2\varphi^2$ as a test function we find

$$\varepsilon_n \int_\Omega |\nabla\nabla u|^2 |\nabla u|^{-1} \rho^2 \varphi^2 \, d\mu \leqslant -2\int_\Omega \rho\varphi \langle \nabla |\nabla u|, \nabla(\rho\varphi)\rangle \, d\mu + c\int_\Omega |\nabla u| \rho^2 \varphi^2 \, d\mu.$$

This immediately implies

$$\int_\Omega |\nabla\nabla u|^2 |\nabla u|^{-1} \rho^2 \varphi^2 \, d\mu \leqslant c\int_\Omega |\nabla u| (\varphi^2 |\nabla\rho|^2 + \rho^2 |\nabla\varphi|^2 + \rho^2\varphi^2) \, d\mu.$$

Combining this with (6.1) and (6.3) we have

$$\int_\Omega |\nabla\nabla u| \, |\nabla\psi_0| \, d\mu \leqslant c\int_\Omega |\nabla u| (\varphi^2 |\nabla\rho|^2 + \rho^2 |\nabla\varphi|^2 + \rho^2\varphi^2) \, d\mu$$

Using the definition of φ and ρ we have the result

$$\int_\Omega |\nabla\nabla u| \, |\nabla\psi_0| \, d\mu \leqslant c \sum_{j=1}^{l} r_j^{-2} \int_{B_{3r_j}(x_j)} |\nabla u| \, d\mu.$$

On the other hand since $x_j \in \mathscr{S}_0$ we have $\operatorname{Ord}(x_j) \geqslant 1+\hat{\varepsilon}$, and therefore by Theorem 2.4

$$\sup_{B_{2r_j}(x_j)} |\nabla u| \leqslant c r_j^{\hat{\varepsilon}}.$$

Thus we have

$$\int_\Omega |\nabla\nabla u| \, |\nabla\psi_0| \, d\mu \leqslant c \sum_{j=1}^{l} r_j^{n-2+\hat{\varepsilon}} \leqslant \varepsilon$$

provided $d<n-2+\hat{\varepsilon}$ and ε is small. This gives the result for $k=1$ since $\mathscr{S}=\mathscr{S}_0$ in this case. For $k>1$, we cover the compact set $\left(\mathscr{S}-\bigcup_{j=1}^{l} B_{r_j}(x_j)\right)\cap\bar{\Omega}_1$ with balls

$$\{B_{r_p}(y_p) : p=1,\ldots,q\}$$

such that in each ball $B_{r_p}(y_p)$ the map can be written $u=(u_1, u_2)$ as in part (ii) of Theorem 6.3. By the inductive assumption there is a function ψ_p vanishing in a neighborhood of $\mathcal{S} \cap \overline{B_{r_p}(y_p)}$ and identically one outside a slightly larger neighborhood with

$$\int_{\Omega} |\nabla\nabla u| \, |\nabla \psi_p| \, d\mu \leqslant \varepsilon \, 2^{-p}.$$

We finally set $\psi = \min \{\psi_0, \psi_1, \ldots, \psi_q\}$ and conclude

$$\int_{\Omega} |\nabla\nabla u| \, |\nabla \psi| \, d\mu \leqslant \sum_{p=0}^{q} \int_{\Omega} |\nabla\nabla u| \, |\nabla \psi_p| \, d\mu \leqslant 2\,\varepsilon.$$

This completes the proof of Theorem 6.4 except for the proof of Lemma 6.6 which we now give.

Proof of Lemma 6.6. — Let ρ be any nonnegative function with compact support in Ω. Suppose Ω_1 is a domain in Ω with compact closure which contains the support of ρ. Let $\{\varepsilon_i\}$ be a sequence tending to zero and let $\{\psi_i\}$ be the corresponding sequence of functions coming from Theorem 6.4. We then have

$$-\frac{1}{2}\int_{\Omega} \langle \nabla |\nabla u|^2, \nabla (\psi_i \, \rho) \rangle \, d\mu \geqslant \int_{\Omega} |\nabla\nabla u|^2 \, \psi_i \, \rho \, d\mu - c \int_{\Omega} |\nabla u|^2 \, \psi_i \, \rho \, d\mu$$

since $\psi_i \, \rho$ has support in $\mathcal{R}(u)$. This implies

$$-\frac{1}{2}\int_{\Omega} \langle \nabla |\nabla u|^2, \nabla \rho \rangle \, \psi_i \, d\mu \geqslant \int_{\Omega} |\nabla\nabla u|^2 \, \psi_i \, \rho \, d\mu - c \int_{\Omega} |\nabla u|^2 \, \psi_i \, \rho \, d\mu$$
$$- \sup_{\Omega} \rho \int_{\Omega} |\nabla u| \, |\nabla\nabla u| \, |\nabla \psi_i| \, d\mu.$$

Since $|\nabla u|$ is bounded on compact subsets of Ω we conclude that the third term on the right tends to zero. The dominated convergence theorem then allows us to let $i \to \infty$ to conclude

$$-\frac{1}{2}\int_{\Omega} \langle \nabla |\nabla u|^2, \nabla \rho \rangle \, d\mu \geqslant \int_{\Omega} |\nabla\nabla u|^2 \, \rho \, d\mu - c \int_{\Omega} |\nabla u|^2 \, \rho \, d\mu$$

as required. This completes the proof of Lemma 6.6.

Part II:
Applications to discrete groups

In this second part of the paper we prove some rigidity theorems for discrete groups with the help of the theory developed in Part I. In particular, we prove our p-adic superrigidity results, and discuss fundamental groups of Kähler manifolds.

7. Equivariant harmonic maps to buildings and the Bochner method

We begin this section by extending the homotopy existence result of Section 4 to a more general setting for certain image complexes X. Suppose X is the Euclidean building associated to an almost simple p-adic algebraic group H (*see* [BT] for the construction of X). It will be important to know (*see* [B]) that X has a compactification $\bar{X} = X \cup \partial X$ such that any $h \in H$ acts continuously on $\bar{X}$ (of course h is an isometry of X). Moreover, if $\{P_i\}$ is a sequence from X with $\lim P_i = P^* \in \partial X$, and if $\{Q_i\}$ is another sequence from X with $d(P_i, Q_i) \leqslant C$ independent of i, then it follows that $\lim Q_i = P^*$. Finally, the isotropy subgroup of H fixing a point $P^* \in \partial X$ is a proper algebraic subgroup while the isotropy group of an interior point is a bounded subgroup. For a complete Riemannian manifold M, let $\tilde{M}$ denote the universal covering manifold, and suppose we have a homomorphism $\rho : \Gamma \to H$, where $\Gamma = \Pi_1(M)$. A Lipschitz map $v : \tilde{M} \to X$ is equivariant if $v \circ \gamma = \rho(\gamma) \circ v$ for all $\gamma \in \Gamma$. Note that the function $|\nabla v|^2$ is Γ invariant on $\tilde{M}$ and hence is well-defined on $M = \tilde{M}/\Gamma$. Thus we may say that $E(v)$ = total energy of v on M is well-defined even though v is not defined on M. We now prove the following existence result.

Theorem 7.1. — *Suppose* $\rho(\Gamma)$ *is Zariski dense in* H *and suppose there exists a Lipschitz equivariant map* $v : \tilde{M} \to X$ *with finite energy. Then there is a Lipschitz equivariant map u of least energy and the restriction of u to a small ball about any point is minimizing.*

Proof. — Let E_0 denote the infimum of the energy taken over all Lipschitz equivariant maps. Let $\{v_i\}$ be a sequence of Lipschitz equivariant maps with $E(v_i)$ tending to E_0. Let B be a ball in $\tilde{M}$ such that $\gamma(B) \cap B = \varnothing$ for all $\gamma \in \Gamma$. We may then construct a new minimizing sequence $\bar{v}_i$ by setting

$$\bar{v}_i = \begin{cases} v_i & \text{on } \tilde{M} - \bigcup\limits_{\gamma \in \Gamma} \gamma(B) \\ \rho(\gamma) \circ u_i \circ \gamma^{-1} & \text{on } \gamma(B) \text{ for any } \gamma \in \Gamma \end{cases}$$

where u_i is a minimizing map in B which agrees with v_i on ∂B. We have $E(\bar{v}_i) \leqslant E(v_i)$ and $\bar{v}_i$ is equivariant, so $\{\bar{v}_i\}$ is again a minimizing sequence. On a compact subset of B, the sequence $\{\bar{v}_i\}$ is uniformly Lipschitz by Theorem 2.4. It follows that a subsequence of $\{v_i\}$ converges uniformly on compact subsets of B to a map into $\bar{X}$ which either maps into X or maps to a single point $P^* \in \partial X$. We use the Zariski density of $\rho(\Gamma)$ to exclude the second possibility as follows. Let $\gamma \in \Gamma$, and let $x_0 \in \tilde{M}$ be the center of the chosen ball B. Let C be any smooth embedded curve from x_0 to γx_0. An elementary argument using Fubini's theorem shows that C may be chosen so that the energy of the restriction of each map u_i to C is uniformly bounded. Therefore the length of the curve $u_i(C)$ is uniformly bounded, and in particular $\{d(u_i(x_0), \rho(\gamma)(u_i(x_0)))\}$ is bounded. Therefore $\lim \rho(\gamma)(u_i(x_0)) = P^*$, and hence $\rho(\gamma)(P^*) = P^*$ for *every* $\gamma \in \Gamma$. This shows that $\rho(\Gamma)$ is contained in a proper algebraic subgroup of H contradicting the Zariski dense hypothesis.

Therefore we may assume that $\{\bar{v}_i\}$ converges uniformly on compact subsets of B. From Corollary 4.2 we have

$$\int_K |\nabla d(\bar{v}_i, v)|^2 \, d\mu \leqslant c$$

for any compact subset K of $\tilde{M}$. Since $\{\bar{v}_i\}$ converges uniformly on compact subsets of B, the function $d(\bar{v}_i, v)$ is uniformly bounded there. It then follows from Poincaré-type inequalities that

$$\int_K d^2(\bar{v}_i, v) \, d\mu \leqslant c$$

for any compact $K \subseteq \tilde{M}$. In particular, the sequence $\{\bar{v}_i\}$ converges weakly in $H^1(K, X)$ for any compact subset K to a map $u \in H^1(K, X)$. By the same argument as that given in Theorem 4.4 we conclude that u is a Lipschitz equivariant map of smallest energy. The local minimizing property of u follows. This completes the proof of Theorem 7.1.

We now show that the Bochner method can be applied for harmonic maps into F-connected complexes where by harmonic we mean locally minimizing; i. e. minimizing on a neighborhood of any point. Since Euclidean buildings are F-connected this will lead to strong conclusions concerning p-adic representations. We suppose u is an equivariant harmonic map from $\tilde{M}$ to X where X is an F-connected complex. We assume u is equivariant with respect to a homomorphism $\rho : \Pi_1(M) \to \mathrm{Isom}(X)$ from $\Gamma = \Pi_1(M)$ into the isometry group of the complex X. For example, if u is the lift of the solution of Theorem 4.4 of the homotopy problem, then u is equivariant with respect to the homomorphism φ_* induced on fundamental groups of the quotients. Generally equivariant maps need not be lifts from quotients, so this gives a means of studying more complicated representations.

If we take a regular point $x_0 \in \tilde{\mathrm{M}}$, then the image under u of a ball $\mathrm{B}_{\sigma_0}(x_0)$ is contained in at least one k-flat. Choose such a flat F, and let $u^*\mathrm{TF}$ denote the pullback of the tangent bundle of F under u. Let ∇ denote the pulled back connection, and let d_∇ denote the corresponding exterior derivative operator on p-forms with values in $u^*\mathrm{TF}$. Let δ_∇ denote its formal adjoint. The differential du of u then defines a 1-form with values in $u^*\mathrm{TF}$, and the harmonic map equation may be written $\delta_\nabla\, du = 0$. We now prove the following extension of the Corlette vanishing theorem [C].

Theorem 7.2. — *Let w be a parallel p-form on $\tilde{\mathrm{M}}$, and assume that u is a finite energy equivariant harmonic map into an* F*-connected complex* X. *In a neighborhood of any regular point of u the form $w \wedge du$ satisfies $\delta_\nabla(w \wedge du) \equiv 0$.*

Proof. — Consider any regular point $x_0 \in \tilde{\mathrm{M}}$, and any k-flat F containing $\mathrm{B}_{\sigma_0}(x_0)$ for some $\sigma_0 > 0$. The calculation of [C] then implies $d_\nabla \delta_\nabla (w \wedge du) \equiv 0$ near x_0. The sets $\mathscr{R}(u)$ and $\mathscr{S}(u)$ are invariant under Γ, and we define $\mathscr{R}_0 = \mathscr{R}(u)/\Gamma$ and $\mathscr{S}_0 = \mathscr{S}(u)/\Gamma$. We then have from Theorem 6.4 that $\dim \mathscr{S}_0 \leqslant n-2$, and for any compact subdomain Ω_1 in M, there is a sequence of nonnegative Lipschitz functions $\{\psi_i\}$ which vanish in a neighborhood of $\mathscr{S}_0 \cap \bar{\Omega}_1$ and tend to 1 on $\mathrm{M} - (\mathscr{S}_0 \cap \bar{\Omega}_1)$, such that

$$\lim_{i \to \infty} \int_{\mathrm{M}} |\nabla\nabla u|\, |\nabla \psi_i|\, d\mu = 0.$$

To see this we simply observe that we can prove Theorem 6.4 on the quotient by exactly the same argument. Now let ρ be a nonnegative Lipschitz function which is identically one on $\mathrm{B}_{\mathrm{R}}(x_0) \subseteq \mathrm{M}$ and identically zero outside $\mathrm{B}_{2\mathrm{R}}(x_0)$ with $|\nabla \rho| \leqslant 2\mathrm{R}^{-1}$. Let ψ be a nonnegative Lipschitz function vanishing in a neighborhood of $\mathscr{S}_0 \cap \overline{\mathrm{B}_{2\mathrm{R}}(x_0)}$. We then apply Stokes' Theorem on M using the identity $0 = \psi\rho^2 \langle w \wedge du, d_\nabla \delta_\nabla (w \wedge du) \rangle$. Thus we obtain

$$\int_{\mathrm{M}} \psi\rho^2 \, \| \delta_\nabla (w \wedge du) \|^2 \, d\mu$$
$$= \pm \int_{\mathrm{M}} \langle *(d(\psi\rho^2) \wedge *(w \wedge du), \delta_\nabla (w \wedge du) \rangle \, d\mu.$$

This implies

$$\int_{\mathrm{M}} \psi\rho^2 \, \| \delta_\nabla (w \wedge du) \|^2 \, d\mu$$
$$\leqslant c \int_{\mathrm{M}} (\psi\rho\, |\nabla \rho| + |\nabla \psi|\, \rho^2)\, |\nabla u|\, \| \delta_\nabla (w \wedge du) \| \, d\mu.$$

Using the inequality $2ab \leqslant \varepsilon a^2 + \varepsilon^{-1} b^2$ we then have

$$\int_M \psi\rho^2 \| \delta_\nabla (w \wedge du) \|^2 \, d\mu$$
$$\leqslant c \int_M \psi |\nabla \rho|^2 |\nabla u|^2 \, d\mu + c \int_M \rho^2 |\nabla \psi| \, |\nabla u| \, |\nabla\nabla u| \, d\mu$$

where we have used $\| \delta_\nabla (w \wedge du) \| \leqslant c |\nabla\nabla u|$. Therefore we have

$$\int_{B_R(x_0)} \psi \| \delta_\nabla (w \wedge du) \|^2 \, d\mu \leqslant c R^{-2} E(u) + c \int_M \rho^2 |\nabla\psi| \, |\nabla u| \, |\nabla\nabla u| \, d\mu.$$

We choose R so large that the first term is less than $\varepsilon/2$, and then we have, since u is Lipschitz on B_{2R},

$$\int_{B_R(x_0)} \psi \| \delta_\nabla (w \wedge du) \|^2 \, d\mu \leqslant \varepsilon/2 + c(R) \int_M |\nabla \psi| \, |\nabla\nabla u| \, d\mu.$$

We then choose ψ so that the second term is less than $\varepsilon/2$ and we find that for large R

$$\int_{B_R(x_0)} \psi \| \delta_\nabla (w \wedge du) \|^2 \, d\mu \leqslant \varepsilon.$$

It follows that $\delta_\nabla (w \wedge du) \equiv 0$ on $\mathcal{R}(u)$ as required. This completes the proof of Theorem 7.2.

We now derive two useful consequences of this result. First we consider the case in which M is a Kähler manifold and w is the Kähler two-form. We first make a definition.

Definition. — A harmonic map into an F-connected complex is *pluriharmonic* provided it is pluriharmonic in the usual sense, $\partial\bar{\partial} u = 0$, on the regular set.

Theorem 7.3. — *A finite energy equivariant harmonic map from a Kähler manifold into an* F-*connected complex is pluriharmonic.*

Proof. — By the previous result we have $\delta_\nabla (w \wedge du) = 0$. If we choose Euclidean coordinates $(u^1, \ldots, u^k)$ in a flat F which locally contains the image of u, then as in [C] this condition implies that the Hessian of each function $u^j(z)$ for $j = 1, \ldots, k$ is a symmetric matrix lying in the Lie algebra of the symplectic group. This implies that the trace of the restriction of this matrix to any complex line is zero, and hence each u^j is a pluriharmonic function.

Theorem 7.4. — *A finite energy equivariant harmonic map from Quaternionic hyperbolic space or the Cayley plane into an* F*-connected complex is constant.*

Proof. — By Corlette's result [C] the condition $\delta_\nabla(w \wedge du)=0$ where w is either the Quaternionic Kähler 4-form or the Cayley 8-form implies that the Hessian of u vanishes at each regular point. This implies that the forms du^j $j=1, \ldots, k$ are parallel 1-forms, and hence if they are nonzero we get a local isometric splitting of the domain as the product of $\mathbf{R}$ with a lower dimensional manifold. Since such a splitting does not exist for the Quaternionic hyperbolic space or the Cayley plane we conclude that $\nabla u \equiv 0$, and hence u is a constant map. This proves Theorem 7.4.

8. *p*-adic superrigidity for lattices in Sp (*n*, 1) and F_4

Let $\tilde{M}$ be the Quaternionic hyperbolic space (resp. the Cayley plane), so that the group $\mathrm{Sp}(n, 1)$ (resp. F_4) acts on $\tilde{M}$ by isometries. We denote the relevant group by G. A lattice Γ in G is a discrete subgroup with finite volume quotient. Let $\rho: \Gamma \to H$ be a representation of Γ in an almost simple *p*-adic algebraic group. Thus H acts by isometries on the associated Euclidean building X. By replacing Γ with a finite index subgroup we may assume Γ is a neat lattice (see [GR]), and we then have:

Lemma 8.1. — *There exists a finite energy Lipschitz equivariant map.*

Proof. — Let $M=\tilde{M}/\Gamma$ so that M is a finite volume rank 1 locally symmetric space. On each cusp $\hat{M}$ of M there exists a proper function $r: \hat{M} \to \mathbf{R}_+$ such that r is smooth with $|\nabla r| \equiv 1$, and r has compact level sets. The metric g on $\hat{M}$ may then be written $dr^2 + {}^r g$ where ${}^r g$ is a metric on $\Sigma_0 = r^{-1}(0)$. If we normalize the curvature of g to be between -1 and $-1/4$, then we have the inequalities on Σ for a constant $c>0$

$$(8.1) \qquad c^{-1} e^{-2r} ({}^0 g) \leqslant {}^r g \leqslant c e^{-r} ({}^0 g).$$

Consider first the compact manifold M_0 with boundary gotten by removing each of the finite number of cusps from M. Then $\tilde{M}_0$ is a Γ-invariant subregion of $\tilde{M}$, and we can choose a Lipschitz equivariant map v from $\tilde{M}_0$ to X. To extend v to all of $\tilde{M}$, we extend v to the cusps as a function independent of r. This produces a Lipschitz equivariant map, and we show that it has finite energy. The function $|\nabla v|^2$ is Γ-invariant and thus descends to M. From (8.1) we see that on $r^{-1}(a)$ we have $|\nabla v|^2 \leqslant c e^{2a}$ while the volume of $r^{-1}(a)$ is bounded above by a constant times $e^{-((n-1)/2)a}$. Since $n = \dim M \geqslant 8$ in the cases we are considering we have

$$\int_{\tilde{M}} |\nabla v|^2 \, d\mu = \int_0^\infty \left(\int_{r^{-1}(a)} |\nabla v|^2 \, d\Sigma \right) da < \infty.$$

This completes the proof of Lemma 8.1.

We are now in a position to prove the following main theorem.

Theorem 8.2. — *If* $\rho(\Gamma)$ *is Zariski dense in* H, *then* $\rho(\Gamma)$ *is contained in a bounded subgroup of* H.

Proof. — By Lemma 8.1 there is a finite energy equivariant map from $\tilde{M}$ to X. By Theorem 7.1 there exists a least energy minimizing map $u: \tilde{M} \to X$. By Theorem 7.4 the map u is constant, say $u(\tilde{M}) = \{P_0\}$. It follows that $\rho(\Gamma) \subseteq H_{P_0}$, the stabilizer of the point $P_0 \in X$. This is a bounded subgroup (*see* [B]), so we have completed the proof of Theorem 8.2.

As a corollary we obtain the following result.

Theorem 8.3 (p-adic superrigidity). — *Let* Γ *be as above and* ρ *be a* p*-adic representation of* Γ *i.e., a homomorphism* $\rho: \Gamma \to GL_N \mathbf{Q}_p$ *for some* $N = 1, 2, \ldots$, *and a prime* p. *Then the image* $\rho(\Gamma)$ *is precompact in* $GL_N \mathbf{Q}_p$.

Proof. — It suffices to prove that the image of a subgroup of finite index of Γ is precompact, hence we may assume as above that Γ is neat. Let H denote the Zariski closure of $\rho(\Gamma)$. If H is almost simple, then the claim follows from 8.2. Furthermore, if H is semi-simple the conclusion is obtained by applying 8.2 to the simple factors of H. Finally, in the general case, let R denote the radical of H and H_0 be the semi-simple factor-group, $H_0 = H/R$. The factor representation $\rho_0: \Gamma \to H_0$ has a precompact image $\rho_0(\Gamma) \subset H_0$ as follows from the above discussion and let $R_0 \subset H$ denote the pull-back of the closure $K_0 \subset H_0$ of $\rho_0(\Gamma)$ under the projection $H \to H_0$. Notice that R_0 is a locally compact group containing a solvable normal subgroup, namely $R \subset R_0$, and the quotient group K_0 is compact. Thus R is an *amenable* group and so $\rho: \Gamma \to R_0$ has bounded image because Γ satisfies Kazhdan's property T.

Remarks. — (a) Our super-rigidity complements the *Archimedian super-rigidity* theorem of Corlette:

Every representation $\rho: \Gamma \to GL_N \mathbf{R}$ *has either precompact image or it extends to a continuous representation of the ambient Lie group* (which contains Γ as a lattice).

(b) One could prove 8.3 directly without an appeal to the (elementary) structure theory of algebraic groups. First, using the T-property, one observes that $\rho(\Gamma)$ is contained in $SL_N \mathbf{Q}_p$ and then there is an action of Γ on the building X attached to SL_N. Then one applies our minimization process to the corresponding map $X \to M$

and if the minimizing sequence $\bar{v}_i : X \to M$ tends to infinity one brings it back using some isometries α_i of M_i, thus achieving a uniform convergence of $\alpha_i \circ \bar{v}_i$ on compact subsets in X. In fact, by our Bochner-Corlette formula, the limit of these maps is constant. Then one notices that applying α_i amounts to conjugating ρ by α_i and the above constancy of the limit map says that the representations $\alpha_i \rho \alpha_i^{-1}$ *converge* to a representation with a bounded image. Hence ρ itself has a bounded image as easily follows from the T-property of Γ.

Now we are in a position to conclude the following.

Theorem 8.4 (Arithmeticity). — *A lattice* Γ *in* $G = \mathrm{Sp}(n, 1)$, $n \geqslant 2$ (i.e., *the isometry group the quaternionic hyperbolic space*) *or n the isometry group* G *of the Cayley plane is arithmetic.*

In fact, Margulis has reduced the arithmeticity to the archimedian plus p-adic super-rigidity. (This reduction is clearly explained in Chapter 6 of [Zim].)

9. Pluriharmonic maps into trees and buildings

Let M be a Kähler manifold and u a harmonic map of M into a locally F-connected space X. We know that in certain cases (e.g., if M is a closed manifold) u is pluriharmonic (see section 7) and now we want to understand the local and global geometry of these pluriharmonic maps. Our main application of such maps, proven later on in this section (in the case where X is a tree), is the following

Theorem 9.1. — *Let* M *be a compact Kähler manifold without boundary and suppose the fundamental group* $\Gamma = \Pi_1(M)$ *admits an amalgamated product decomposition* $\Gamma = \Gamma_1 *_\Delta \Gamma_2$, *where the index of* Δ *in* Γ_1 *is at least* 2 *and the index of* Δ *in* Γ_2 *is at least* 3. *Then* M *admits a surjective holomorphic map onto a Riemann surface.*

Let us look at a map u of M into a tree X at a regular point $x_0 \in M$, such that (by the definition of regularity) the u-image of a small ball $B_{\sigma_0}(x_0) \subset M$ is contained in a geodesic (flat) $F \subset X$. Locally this F is isometric to $\mathbf{R}$ and so u near x_0 amounts to a real function $u : B_{\sigma_0}(x_0) \to \mathbf{R} = F \subset X$. Notice that this $F = \mathbf{R}$ carries no natural orientation and so the differential du is only defined up to $\pm$ sign. On the other hand, if u is pluriharmonic, then the complexified differential, say $d^{\mathrm{C}} u$, is *holomorphic*, if we choose the sign. Therefore, the square $(d^{\mathrm{C}} u)^2$ is a holomorphic quadratic differential (of rank one) defined on the set of regular points of u. Now, since u is Lipschitz, the differential du is bounded, and since the singular set has codimension $\geqslant 2$ in M, the differential $(d^{\mathrm{C}} u)^2$ extends to a holomorphic quadratic differential on all of M. Since this differential has rank $\leqslant 1$, there is a ramified double covering of M, say $M^\sim \to M$ (where $M^\sim$ may be singular), such that $(d^{\mathrm{C}} u)^2$ lifts to a holomorphic 1-form, denoted $d^\sim u$ on $M^\sim$. This form is closed and locally the differential of a holomorphic function on $M^{\sim}$, say $u^{\sim}: M^{\sim} \to C$. The levels of this function can be defined as maximal complex submanifolds (having complex codimension one in M) contained in the levels $u^{-1}(y) \subset M$, $y \in X$, of u lifted to $M^{\sim}$ (these have real coodimension one). The (Galois) involution on $M^{\sim}$ switches the sign of $d^{\sim} u$ and, hence, locally of $u^{\sim}$. It follows that the local partition into connected components of the levels of $(u^{\sim})^2$ descends back to M and by the (local version of) Stein factorization theorem the map u decomposes in a small neighborhood Ω of each (possibly singular) point of M into a composition of a holomorphic function $\Omega \to D \subset \mathbf{C}$ followed by a harmonic map $u_0: D \to X$ where D denotes the unit disk in C. Therefore, the singular locus of u appears as a holomorphic pull-back of that for the harmonic map $u_0: D \to X$ and the quadratic differential $(D^{C} u)^2$ comes from $(d^{C} u_0)^2$ on D. Since the singular locus of u_0 is contained in the (discrete) set of zeros of $(d^{C} u_0)^2$, it is discrete and hence, the singularity of u is a complex analytic subvariety in M locally given by a single function $M \to D$ whose level lifted to $M^{\sim}$ are connected components of levels of $u^{\sim}$.

Summing up, we arrive at the following

Corollary 9.2. — *Let* $u: M \to X$ *be a non-constant pluriharmonic map of a connected complex analytic manifold into a tree. The set* $\Sigma \subset M$ *of singular points of u is a complex analytic subvariety in* M *whose components have codimension* 1 *in* M. *Furthermore, the lift of* Σ *to* $M^{\sim}$ *equals the union of some leaves of the* (*holomorphic*) *foliation defined by the* (*closed holomorphic*) 1-*form* $(d^{\sim} u)^2$ *on* $M^{\sim}$.

Remark. — The pull-back $u^{-1}(v) \subset M$ of a vertex $v \in X$ is not, in general, a real analytic subset in M. For example, the standard harmonic map of the disk into the tripod (see § 0) has the pull-back $u^{-1}(v)$ homeomorphic to the tripod. This cannot be real analytic as it has an odd number (3 in this case) of branches. Yet this level is subanalytic. Moreover, all levels of an arbitrary pluriharmonic map u lifted to $M^{\sim}$ are real analytic subsets in $M^{\sim}$ as follows from 9.2. Every such subset at a *singular* point of u is necessarily reducible and the complement has more than two connected components sent by u to different branches of the tree.

Now we look at a slightly more general situation where we have a Γ-equivariant pluriharmonic map u of a connected Kähler manifold $\tilde{M}$ into a tree X. Here we assume that the group Γ is infinite and acts discretely, cocompactly, and isometrically on $\tilde{M}$, and that the action of Γ on the tree is also isometric for the given (piecewise linear) metric on the tree X.

We want to show that u decomposes into a holomorphic Γ-equivariant map of $\tilde{M}$ to a Riemann surface D with a discrete holomorphic action of Γ (typically, D is the unit disk in C) followed by a Γ-equivariant harmonic map $D \to X$. We divide our analysis of u into two cases depending on whether the map u is everywhere regular or has a non-empty singular set.

Decomposition Lemma in the singular case. — *If the singular locus* $\tilde{\Sigma} \subset \tilde{M}$ *of* u *is non-empty then the map* u *decomposes into a proper holomorphic* Γ*-equivariant map* $\tilde{M} \to D$ *followed by a* Γ*-equivariant harmonic map* $D \to X$.

Proof. — Let us look at the holomorphic foliation defined by the form $(d^C u)^2$. The singular locus $\tilde{\Sigma}$ of u is a union of leaves and, obviously, $\tilde{\Sigma} \subset \tilde{M}$ is Γ-closed i.e., the image of $\tilde{\Sigma}$ in $M = \tilde{M}/\Gamma$ is a closed subset. If follows that if $\tilde{\Sigma}$ is non-empty then all leaves of our foliation are Γ-closed (see explanations below) and so our foliation descends to a partition of $M = \tilde{M}/\Gamma$ into compact analytic subsets. We apply to this partition the Stein factorization theorem (*see* [Ste]) and thus obtain the desired factorization $\tilde{M} \to D$.

First explanation. — Let $l^{\sim}$ be a holomorphic (and hence closed) 1-form on a compact Kähler manifold $M^{\sim}$. Then there exists a complex torus N of real dimension k in the interval $2 \leqslant k \leqslant \operatorname{ran} H_1(M^{\sim})$, such that $l^{\sim}$ can be induced from some holomorphic 1-form l on N by a holomorphic map $\alpha : M^{\sim} \to N$ and where each leaf $L \subset N$ of the foliation defined by l satisfies the following

Minimality condition. — *The leaf* L *contains no complex subtorus of positive dimension.*

To construct N one starts with the Albanese variety N_0 of $M^{\sim}$ with the form l_0 corresponding to $l^{\sim}$ and then factorizes N_0 by the maximal complex subtori in the leaves of the corresponding foliation of N_0.

Notice that the minimality condition implies that no leaf $L \subset N$ contains a complex compact submanifold of positive dimension. (Indeed, each L is of the form $C^{k-1} \mathscr{L}$ for some non-cocompact lattice $\mathscr{L} \subset \mathbf{C}^{k-1}$. We turn L into an Abelian analytic Lie group by taking some point in L for the origin and observe that the additive span of a compact complex submanifold in L is a complex torus). It follows that if the pull-back $\alpha^{-1}(L)$ contains a compact component then $\alpha(M^{\sim}) \cap L$ is zero-dimensional and therefore the image $\alpha(M^{\sim}) \subset N$ is 1-dimensional.

The above provides an explanation needed for the global factorization theorem in the case where the action of Γ on $\tilde{M}$ is free: Here one takes the ramified double covering of $\tilde{M}/\Gamma$ for $M^{\sim}$ and observes that the lift of $\tilde{\Sigma}/\Gamma \subset \tilde{M}/\Gamma$ to M consists of a union of finitely many *compact* components of the pull-back of some leaves $L \subset N$.

In the general case, where Γ is non-free (but yet discrete), one notes that all *steps* of the above argument apply in the presence of fixed points if we work Γ-equivariantly on $\tilde{M}$ rather than on the quotient space $\tilde{M}/\Gamma$.

A judicious reader might have noticed that the above explanation, which relies on the theory of Albanese varieties, needs special care in the case where $M^{\sim}$ is singular. Instead of resolving this singularity we suggest below an elementary (albeit less elegant) argument where the singularity of $M^{\sim}$ does not enter the discussion at all.

Second explanation. — Let $\tilde{L}(\tilde{m})$, $\tilde{m} \in \tilde{M}$ denote the leaf of our foliation (defined by $\pm d^{C} u$) passing through m, and denote $\tilde{M}^* \subset \tilde{M}$ the union of the leaves $\tilde{L}(\tilde{m})$ whose projections $L(\tilde{m})$ to $M = \tilde{M}/\Gamma$ are compact. First we observe that the subset $\tilde{M}^* \subset \tilde{M}$ is *closed* in the following sense.

If $\tilde{m}(t) \in \tilde{M}$, $t \in [0, 1]$, *is a continuous path of points, such that* $\tilde{m}(t) \in \tilde{M}^*$ *for* $t > 0$, *then also* $\tilde{m}(0) \in \tilde{M}^*$.

It is easy to see (e.g. by looking at the function v below) that the homology class of $L(\tilde{m}(t))$ stays bounded for $t \in]0, 1]$. Therefore the Kählerian volume of $L(\tilde{m}(t)) \subset M = \tilde{M}/\Gamma$, being a homological invariant, is bounded for $t > 0$ and so the limit leaf $L(\tilde{m}(0))$ is compact. (Notice that Vol. $L(\tilde{m}(0)) = \text{Vol } L(\tilde{m}(t))$ for small $t > 0$ unless $L(\tilde{m}(0))$ is multiply covered by nearby leaves in $M = \tilde{M}/\Gamma$.)

Secondly, we show that *the subset* $\tilde{M}' \subset M$ *is open.*

Proof. — For each leaf $\tilde{L} \subset \tilde{M}$ there obviously exists a function $\tilde{v}$ on a small neighbourhood $\tilde{A} = \tilde{A}(\tilde{L}) \subset \tilde{M}$ of $\tilde{L}$, such that

(i) $(d\tilde{v})^2 = \tilde{v}(du)^2$; this is equivalent to $d\sqrt{\tilde{v}} = \pm d^{C} u$ and this makes the foliation into the connected components of the levels of $\tilde{v}$ equal to that defined by $\pm d^{C} u$ on $\tilde{A}$

(ii) $\tilde{v} \mid \tilde{L} = 0$.

Now, let the projection $L \subset M$ of $\tilde{L}$ be compact. Then $\tilde{v}$ descends to a function v on some neighbourhood $A \subset M$ of L, such that the pullback of v to $\tilde{M}$ equals $\tilde{v}$ on some neighbourhood of $\tilde{L}$ (where both functions are defined). Since $L \subset M$ is compact, the leaves $L(m)$, which equal the connected components of levels of v, stay close to L for m close to L (i.e. $L(m)$ Hausdorff converge to L for $m \to L$, compare the proof of 9.3 below). Therefore, these $L(m)$ are compact (as they do not reach the boundary of A) and they receive the leaves $\tilde{L}(\tilde{m})$ from $\tilde{M}$ for $\tilde{m}$ close to $\tilde{L}$. This yields the openess of $\tilde{M}^*$.

Finally we notice that every leaf contained in the singular locus $\tilde{\Sigma} \subset \tilde{M}$ has compact projection as $\tilde{\Sigma}/\Gamma$ is compact and so our $\tilde{M}^*$ is non-empty. Thus $\tilde{M}^* = M$, since M is connected.

Remark. — The above argument applies whenever the zero set of $d^{C} u$ contains a leaf $\tilde{L}$. Notice that the inclusion $\tilde{L} \subset \text{Zero}(d^{C} u)$ is equivalent to local disconnectedness of the levels of the above function $\sqrt{v}$ near each point $\tilde{m} \in \tilde{L}$.

Nonsingular case. — If $u : \tilde{M} \to X$ is a nonsingular pluriharmonic map then the complexified differential $d^{C} u$ is defined everywhere on $\tilde{M}$ up to $\pm$sign. It follows $d^{C} u$ becomes a honest holomorphic 1-form λ on a (non-ramified) double cover of $\tilde{M}$ which is invariant under a discrete group consisting of lifts of the transformations from Γ to this cover. To minimize notation we assume below that this double cover splits, and thus we have a Γ-invariant holomorphic form λ on $\tilde{M}$ itself, such that the map $u:\tilde{M}\to X$ is constant on the leaves of the foliations defined by λ on $\tilde{M}$. Furthermore, by passing to some Abelian covering of $\tilde{M}$ we can make λ exact as well as equivariant under a certain discrete cocompact group acting on this covering. Thus we may assume λ is exact to begin with. In this case the leaves, which are the connected components of the level sets of the holomorphic function z on $\tilde{M}$ with $dz=\lambda$, are *closed* analytic subvarieties in $\tilde{M}$. Moreover, we claim that the space of these leaves is Hausdorff (this is explained below) and therefore the Stein factorization theorem applies. Thus we arrive at the following

Factorization in the nonsingular case. — *Suppose u is non-singular and $\tilde{M}$ is simply connected (This takes care of the intermediate coverings). Then u decomposes into a holomorphic map $\tilde{M}\to D$ followed by a harmonic map $D\to X$, where these maps are equivariant with respect to Γ or a subgroup Γ' of index two in Γ (taking care of the $\pm$ sign of $d^C u$).*

Here, as earlier, D comes equipped with a holomorphic action of Γ (or Γ') but we do not claim at this stage that the action of Γ (or Γ') on D is discrete. In general it need not be discrete as the map $\tilde{M}\to D$ may be non-proper.

On the Hausdorff property of holomorphic foliations. — Let M and N be complex manifolds, $z:M\to N$ a holomorphic map and let $\bar{N}$ be the space of the leaves that are the connected components of the fibers $z^{-1}(n)\subset M$, $n\in N$. The topology of $\bar{N}$ is defined in the usual way: a subset in $\bar{N}$ is open if the union of the corresponding leaves is an open subset in M.

There is a simple criterion for $\bar{N}$ to be Hausdorff where z is a general (not necessarily holomorphic) smooth map. Namely, $\bar{N}$ is Hausdorff if z is a submersion (i. e. rank $z=\dim N$) which is, moreover, *infinitesimally enlarging* with respect to some Riemannian metric g_N on N and some *complete* metric g_M on M, where "infinitesimally enlarging" means that the pullback of the form g_N to the subbundle in T(M) normal to the kernel of dz dominates g_N, i. e. $z^*(g_N)\,|\,\mathrm{Ker}^{\perp} dz \geqslant g_M\,|\,\mathrm{Ker}^{\perp} dz$ (compare [Gro]). This applies in particular to the case where z is a submersion and the pull-back form $z^*(g_N)$ on M is invariant under some proper cocompact group Γ acting on M.

Now, we return to the holomorphic map $z:M\to N$, $\dim_C N=1$, and assume that there exists a Riemannian metric g_N on N, such that the pull-back $z^*(g_N)$ is invariant under some proper cocompact group action on M. (Notice that this condition is satisfied in the case of interest where we have an exact Γ-invariant 1-form on M.)

Lemma 9.3. — *The space* $\bar{\mathrm{N}}$ *is Hausdorff and hence admits a unique complex structure such that* z *decomposes into a composition of two holomorphic maps*

$$z: \mathrm{M} \to \bar{\mathrm{N}} \to \mathrm{N}$$

Proof. — Since z is holomorphic and $\dim \mathrm{N}=1$, the leaf $\mathrm{L}(m)$ passing through $m \in \mathrm{M}$ *continuously* depends on m with respect to the Hausdorff distance between compact subsets in the leaves. (Such local continuity is automatic for usual foliations, where the leaves are non-singular, while the foliation into the connected components of the levels of the map $\mathbf{R}^2 \to \mathbf{R}$, $(x, y) \mapsto xy$, indicates what might go wrong in the presence of a singularity). Now we chose a metric on M invariant under our group which preserves $z^*(g_{\mathrm{N}})$, and observe (looking at $z^*(g_{\mathrm{N}})$ and/or using Stein factorisation on compact subsets in M) that the above continuity on compact subsets is uniform for such subsets. Thus the correspondence $m \mapsto \mathrm{L}(m)$ is continuous for the Hausdorff metric on the set of leaves, and then the lemma easily follows.

On the discreteness of the action of Γ *on* D. — We return to the factorization in the non-singular case,

$$u: \tilde{\mathrm{M}} \to \mathrm{D} \to \mathrm{X},$$

and give a criterion for the implied action of Γ (or Γ') on D to be discrete.

Lemma 9.4. — *The action of* Γ *on* D *is discrete unless it factors through a virtually solvable group* (i.e. *the implied homomorphism* $\Gamma \to \mathrm{Aut}\,\mathrm{D}$ *has a virtually solvable image*).

Proof. — We may assume here that D is the unit disk and the image $\bar{\Gamma}$ of Γ in $\mathrm{Aut}\,\mathrm{D} = \mathrm{PSL}_2(\mathbf{R})$ is Zariski dense (otherwise it would be virtually solvable). Then either $\bar{\Gamma}$ is discrete in Aut D or it is topologically dense. In the latter case the commutator subgroup $[\bar{\Gamma}, \bar{\Gamma}]$ is also dense in Aut D and then every exact $\bar{\Gamma}$-invariant holomorphic form on D must be zero. But the form λ we started with obviously descends to D and thus the dense case is excluded.

Remarks. — (a) Let us indicate an alternative approach similar to that in [Sim]. First we notice that if the map $\bar{\mathrm{N}} \to \mathrm{N}$ has a ramification point, then by our "singular argument" $\tilde{\mathrm{M}}$ admits a *proper* Γ-equivariant map to a Riemann surface (with no use of Lemma 9.3). Then if no ramification point is present, our argument similar to that proving Lemma 9.3, shows that the map $\bar{\mathrm{N}} \to \mathrm{N}$ is (non-ramified) and so for $\mathrm{N}=\mathbf{C}$ we have $\bar{\mathrm{N}}=\mathbf{C}$ as well.

(*b*) Our original treatment of the "non-singular" case contained an error pointed out to us by K. Corlette.

(*c*) K. Corlette and C. Simpson suggested another approach to the factorization of the map $u: \tilde{M} \to X$ following the earlier work by Simpson (see [Sim]).

Harmonic maps into **Z**-*trees*. — We want to generalize the above discussion to the case where X is a **Z**-tree, i.e., where X may have infinitely many branches at the vertices. This is needed for the amalgamated product property stated at the beginning of section 9. First we notice that every singular point of u has a finite multiplicity, controlled by the total energy and so every harmonic map locally factors through a map into a finite tree X′ followed by an isometric embedding $X' \to X$. Hence u is pluriharmonic (even for **R**-trees) and so the above holomorphic discussion remains valid. What is left to do is to extend the existence theorem to the case where X is *not* locally compact. We shall state and prove the relevant property where X is a generalized tree i.e., **Z**-tree or an **R**-tree. Here one can not ensure the existence of a Γ-equivariant harmonic map $\tilde{M} \to X$ in a given homotopy class but one can obtain a harmonic map if one *modifies* the receiving space X by going to an appropriate limit. Namely, let $u_i: \tilde{M} \to X$ be a minimizing sequence. (In fact, we could allow maps with *variable* target X_i but this is not needed right now.) All these maps may be assumed uniformly Lipschitz and of uniform multiplicity. Thus they factor on compact subsets in $\tilde{M}$ through maps $\tilde{M} \to X_i \to X$ where X_i are finite trees. Then we pass to the Hausdorff (sub)-limit of the spaces X_i as $\tilde{M}$ is being exhausted by compact subsets and $i \to \infty$ and obtain the desired harmonic map $u: \tilde{M} \to X_\infty$.

Warning. — The trees X_i can be thought of as subsets of X but the limit X_∞ is not a part of X. For example, one may imagine X_i equal to a countable joint of the segments $I_i = [0, 1]$ at zero where $X_i = I_i$. The sequence $I_i \subset X$ diverges inside X but (identically) converges in the abstract sense to the unit interval.

A more invariant way to look at X_∞ is by concentrating on the function d_i on $\tilde{M} \times \tilde{M}$ induced by u_i from the distance function on X. All properties of the maps u_i relevant for us can be expressed in terms of d_i without ever referring to X and instead of the limit space X_∞ one can deal with a (sub)-limit d_∞ of d_i. This works perfectly well whenever the maps u_i are uniformly Lipschitz. Such an approach is especially attractive if X is an infinite dimensional symmetric space, e.g., the Hilbert space on the infinite-dimensional hyperbolic space, where one has a simple criterion for the existence of an isometric embedding of a given metric space $(\tilde{M}, d)$ into X.

Let us apply the above considerations to the amalgamated product problem. If the fundamental group $\Gamma = \Pi_1(M)$ decomposes as $\Gamma = \Gamma_1 *_\Delta \Gamma_2$ we have a Γ-equivariant map of $\tilde{M}$ to some tree X with a Γ-action where the degrees of the vertices are given by $\operatorname{ind}(\Delta \subset \Gamma_i)$, $i = 1, 2$ (*see* Serre [Ser]). Our assumption $\operatorname{ind}(\Delta \subset \Gamma_1) \geqslant 2$ and $\operatorname{ind}(\Delta \subset \Gamma_2) \geqslant 3$ ensures that our Γ is not virtually solvable and neither is the corresponding group $\bar{\Gamma} \subset \mathrm{Aut}\, \mathrm{D}$. Hence, we obtain a (non-constant!) Γ-equivariant map $\tilde{\mathrm{M}} \to \mathrm{X}_\infty$ which factors through a Riemann surface.

Let us briefly explain, without griving proofs, how the above results generalize to pluriharmonic maps $u: \mathrm{M} \to \mathrm{X}$ where X is a k-dimensional Euclidean building. Every such u gives rise to a holomorphic foliation on M, whose leaves are maximal connected complex submanifolds in the pull-backs of the points, $u^{-1}(x) \subset \mathrm{M}$, $x \in \mathrm{X}$. As earlier, these leaves may have singularities and the complex codimension of a generic leaf equals the real rank of u at a general regular point in M.

Next we invoke the (finite) Weyl group W associated with the building and we claim there exists a ramified Galois covering $\tilde{\mathrm{M}} \to \mathrm{M}$ with Galois group W, such that the lift of our foliation to $\tilde{\mathrm{M}}$ equals the zero set of a finite system of holomorphic 1-forms on $\tilde{\mathrm{M}}$.

Then, for Γ-equivariant maps one attempts to show that the leaves project onto *compact* submanifolds in M/Γ. For example, this can be proven if $\dim_{\mathbb{C}} \mathrm{M} \geqslant k+1$ and M/Γ is compact.

Finally, we notice that there are many instances of non-trivial p-adic representations of fundamental groups of algebraic manifolds M. These appear via the *p-adic uniformization theory* for *arithmetic* varieties M of the form

$$\mathrm{M} = \mathrm{U}(k) \diagdown \mathrm{U}(k, 1)/\Gamma,$$

as was pointed out to us by D. Kazhdan. We plan to study the pluriharmonic maps which appear here in another paper.

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