PATRICK EBERLEIN
JENS HEBER

A differential geometric characterization of symmetric spaces of higher rank

Publications mathématiques de l'I.H.É.S., tome 71 (1990), p. 33-44
<http://www.numdam.org/item?id=PMIHES_1990__71__33_0>

A DIFFERENTIAL GEOMETRIC CHARACTERIZATION OF SYMMETRIC SPACES OF HIGHER RANK

by PATRICK EBERLEIN *and* JENS HEBER

Introduction

In this paper we consider simply connected Riemannian manifolds $\tilde{M}$ of nonpositive sectional curvature whose isometry group $I(\tilde{M})$ is large in an appropriate sense. We do not assume that $\tilde{M}$ has a lower bound for the sectional curvature, and we do not assume that $I(\tilde{M})$ contains a discrete lattice subgroup. The manifolds that we consider will have rank at least two in the sense of [BBE]; see (1.1) below for a definition.

We now define the *duality condition*, the basic hypothesis that we impose on the isometry group $I(\tilde{M})$ of $\tilde{M}$. Let $\Gamma \subseteq I(\tilde{M})$ be an arbitrary subgroup. A vector v in the unit tangent bundle $S\tilde{M}$ is said to be *nonwandering modulo* Γ if for every neighborhood $O \subseteq S\tilde{M}$ of v there exist sequences $\{ t_n \} \subseteq \mathscr{R}$ and $\{ \varphi_n \} \subseteq \Gamma$ such that $t_n \to \infty$ and $(d\varphi_n \circ g^{t_n})(O) \cap O$ is nonempty for every n, where $\{ g^t \}$ denotes the geodesic flow in $S\tilde{M}$ ([Ba1]). The group $\Gamma \subseteq I(\tilde{M})$ is said to satisfy the duality condition if every vector v in $S\tilde{M}$ is nonwandering modulo Γ. The duality condition is often defined in an equivalent way that we present below in (1.3).

The main result in this paper is the following

Theorem. — Let $\tilde{M}$ *be a complete, simply connected, irreducible Riemannian manifold of nonpositive sectional curvature with* $\operatorname{rank}(\tilde{M}) = k \geqslant 2$. *If* $I(\tilde{M})$ *satisfies the duality condition, then* $\tilde{M}$ *is isometric to a symmetric space of noncompact type and rank* k.

As corollaries we obtain the following results:

Corollary 1. — Let M *be a complete, Riemannian manifold of nonpositive sectional curvature and finite volume. If the universal Riemannian cover* $\tilde{M}$ *is irreducible and has* $\operatorname{rank}(\tilde{M}) = k \geqslant 2$, *then* $\tilde{M}$ *is isometric to a symmetric space of noncompact type and rank* k.

Supported in part by NSF Grant DMS-8601367.

From the theorem above and Proposition 4.1 of [E3] we also obtain

Corollary 2. — Let $\widetilde{M}$ *denote a complete, simply connected Riemannian manifold with nonpositive sectional curvature whose isometry group* $I(\widetilde{M})$ *satisfies the duality condition. Then* $\widetilde{M}$ *is the Riemannian product of a Euclidean space, a symmetric space of noncompact type and an arbitrary number* $N \geqslant 0$ *of spaces with rank 1 whose isometry groups are discrete and satisfy the duality condition.*

In Corollary 2 any of the factors listed could be absent. Corollary 1 is proved in [Ba2] and [BS] under the additional hypothesis that $\widetilde{M}$ admit a lower bound for the sectional curvature. If M has finite volume and nonpositive sectional curvature, then its fundamental group may be regarded as a discrete group Γ of isometries of the universal Riemannian covering space $\widetilde{M}$. The finite volume hypothesis then implies that Γ satisfies the duality condition by a standard recurrence argument due to Poincaré. Corollary 1 now follows from the main theorem.

The proof of the main result uses results from sections 1 and 2 of [BBE] and the following result of [E5], stated below as Theorem 2.3:

Theorem. — Let $\widetilde{M}$ *be irreducible and assume that* $I(\widetilde{M})$ *satisfies the duality condition. If* $I(\widetilde{M})$ *leaves invariant a proper closed subset of the boundary sphere* $\widetilde{M}(\infty)$, *then* $\widetilde{M}$ *is isometric to a symmetric space of noncompact type and* rank *at least 2.*

The main technical tool of this paper is the *Tits geometry* of the boundary sphere $\widetilde{M}(\infty)$ as developed by Gromov in section 4 of [BGS]. We summarize the features of the Tits geometry that we need in section 1. In section 2 we state relevant results from [BBE] and [E5].

We now outline the proof of the main theorem. For any number α, with $0 < \alpha \leqslant \pi$, we define $\widetilde{M}_\alpha(\infty) = \{ x \in \widetilde{M}(\infty)$: there exists $y \in \widetilde{M}(\infty)$ with $\angle_q(x,y) = \alpha$ for all $q \in \widetilde{M} \}$. Here $\angle_q(x,y)$ denotes the angle subtended at q by the points x and y. See section 1 for further discussion and appropriate definitions. The sets $\widetilde{M}_\alpha(\infty)$ are closed in $\widetilde{M}(\infty)$ and invariant under $I(\widetilde{M})$ for every α. In sections 3 and 4 we show that under the hypotheses of the main theorem one of the sets $\widetilde{M}_\alpha(\infty)$ is a nonempty proper subset of $\widetilde{M}(\infty)$. The main result now follows immediately from the result of [E5] stated above.

For a symmetric space $\widetilde{M}$ of noncompact type and rank at least 2 the sets $\widetilde{M}_\alpha(\infty)$ correspond in a natural way to Weyl chambers in the unit tangent bundle $S\widetilde{M}$ as defined in section 2 of [BBS]. If v is a regular vector of $S\widetilde{M}$ and if $C(v) \subseteq S\widetilde{M}$ denotes the Weyl chamber of v in the sense of [BBS], then for any vectors, v, w with $C(v) = C(w)$ the function $p \mapsto \angle_p(x,y)$ is constant in $\widetilde{M}$, where x and y are the points in $\widetilde{M}(\infty)$ determined by the geodesics γ_v, γ_w with initial velocities v and w. To prove this one shows that if $C(v) = C(w)$ then $G_x = G_y$, where $G = I_0(\widetilde{M})$ and $G_x = \{ g \in G : g(x) = x \}$. It then follows that the function $p \mapsto \angle_p(x,y)$ is constant in $\widetilde{M}$ since G_x acts transitively on $\widetilde{M}$.

This paper supersedes a preprint of the first author ([E1]), which gives a much longer proof of the main result presented here. The second author has also obtained an alternate proof of the main result based on showing the existence of singular vectors. See the remarks following the statement of Theorem 2.3.

1. Preliminaries

In general we shall assume the results of [BO] and [EO]. For a brief reference see also section 1 of [BBE].

Notation. — Let $\tilde{M}$ denote a complete, simply connected Riemannian manifold with nonpositive sectional curvature K. Let $S\tilde{M}$ denote the unit tangent bundle of $\tilde{M}$, and let $\pi : S\tilde{M} \to \tilde{M}$ denote the projection. All geodesics of $\tilde{M}$ are assumed to have unit speed, and $\{ g^t \}$ denotes the geodesic flow in $S\tilde{M}$. If $v \in S\tilde{M}$ is any vector, then γ_v denotes the geodesic with initial velocity v. We denote the isometry group of $\tilde{M}$ by $I(\tilde{M})$ and the Riemannian distance function of $\tilde{M}$ by $d(\ , \ )$.

We let $\tilde{M}(\infty)$ denote the boundary sphere of $\tilde{M}$ that consists of equivalence classes of asymptotic geodesics of $\tilde{M}$. If γ is any geodesic of $\tilde{M}$ then we let $\gamma(\infty)$ denote respectively the points in $\tilde{M}(\infty)$ that are determined by γ and $\gamma^{-1} : t \mapsto \gamma(-t)$. If $p \in \tilde{M}$ and $x \in \tilde{M}(\infty)$ are any points, then γ_{px} denotes the unique geodesic of $\tilde{M}$ that belongs to the asymptote class x and satisfies $\gamma_{px}(0) = p$. Let $V(p, x)$ denote the initial velocity of γ_{px}. If p is any point of $\tilde{M}$ and if x, y are any points of $\tilde{M}(\infty)$, then $\angle_p(x, y)$ denotes the angle subtended by $V(p, x)$ and $V(p, y)$. The space $\overline{M} = \tilde{M} \cup \tilde{M}(\infty)$ admits a topology such that $\tilde{M}$ is a dense open subset of $\overline{M}$; $\overline{M}$ is homeomorphic to a closed n-ball, and for any point p of $\tilde{M}$ the map $x \mapsto V(p, x)$ is a homeomorphism of $\tilde{M}(\infty)$ onto $S_p \tilde{M}$, the sphere of unit vectors at p.

Rank and regular vectors. — For each vector $v \in S\tilde{M}$ we define $r(v)$ to be the dimension of the vector space of all parallel Jacobi vector fields along γ_v.

(1.1) *Definition.* — $\operatorname{Rank}(\tilde{M}) = \min\{ r(v) : v \in S\tilde{M} \}$.

A vector $v \in S\tilde{M}$ is said to be *regular* if $r(v) = \operatorname{rank}(\tilde{M})$. We let $\mathscr{R}$ denote the set of regular vectors in $S\tilde{M}$. The set $\mathscr{R}$ is invariant under the geodesic flow. If $w \in S\tilde{M}$ is sufficiently close to a given vector $v \in S\tilde{M}$, then $r(w) \leqslant r(v)$. In particular the set $\mathscr{R}$ of regular vectors is open in $S\tilde{M}$.

Duality Condition. — Let $\Gamma \subseteq I(\tilde{M})$ be any group of isometries of $\tilde{M}$, not necessarily discrete. Following [Ba1] we define a subset $\Omega(\Gamma) \subseteq S\tilde{M}$, the nonwandering set of the geodesic flow mod Γ, as follows: a unit vector v lies in $\Omega(\Gamma)$ if and only if for every open set O in $S\tilde{M}$ that contains v and every positive number T there exist an isometry $\varphi \in \Gamma$ and a number $t > T$ such that $(d\varphi \circ g^t)(O) \cap O$ is nonempty.

From Proposition 3.7 of [E2] we obtain the following characterization of $\Omega(\Gamma)$.

(1.2) *Proposition.* — *Let* $\Gamma \subseteq I(\tilde{M})$ *be any group. Then* $v \in \Omega(\Gamma)$ *if and only if there exists a sequence* $\{\varphi_n\} \subseteq \Gamma$ *such that* $\varphi_n(p) \to \gamma_v(\infty)$ *and* $\varphi_n^{-1}(p) \to \gamma_v(-\infty)$ *as* $n \to \infty$ *for any point* p *of* $\tilde{M}$.

(1.3) *Definition.* — A group $\Gamma \subseteq I(\tilde{M})$ is said to satisfy the *duality condition* if $\Omega(\Gamma) = S\tilde{M}$.

If $\Gamma \subseteq I(\tilde{M})$ is a *lattice*, that is, if the quotient space $\tilde{M}/\Gamma$ is a smooth manifold of finite Riemannian volume, then a standard argument due to Poincaré shows that Γ satisfies the duality condition.

The duality condition has the merit that it is preserved under projection homomorphisms while discreteness, and hence the condition of being a lattice, is in general not preserved. More precisely, if $\tilde{M}$ is a Riemannian product $\tilde{M}_1 \times \tilde{M}_2$ and if $\Gamma \subseteq I(\tilde{M})$ is a group that preserves the product structure and satisfies the duality condition, then, for $i = 1, 2$, the group $p_i(\Gamma)$ satisfies the duality condition in $\tilde{M}_i$, where $p_i : \Gamma \to I(\tilde{M}_i)$ denotes the projection homomorphism.

Γ-recurrent vectors. — If $\Gamma \subseteq I(\tilde{M})$ is any group, then a vector $v \in S\tilde{M}$ is called *Γ-recurrent* if there exist a sequence $\{\varphi_n\} \subseteq \Gamma$ and a sequence $\{t_n\} \subseteq \mathscr{R}$ such that $t_n \to +\infty$ and $(d\varphi_n \circ g^{t_n})(v) \to v$ as $n \to \infty$.

(1.4) *Proposition.* — *Let* $\Gamma \subseteq I(\tilde{M})$ *be a group that satisfies the duality condition. Then the set of regular* Γ*-recurrent vectors is a dense* G_δ *subset of* $S\tilde{M}$.

Proof. — The set $\mathscr{R}$ of regular vectors is a dense open subset of $S\tilde{M}$ by the definition of $\mathscr{R}$ and Theorem 2.6 of [BBE]. It suffices to prove that the Γ-recurrent vectors of $S\tilde{M}$ form a dense G_δ. Let d^* be any metric on $S\tilde{M}$ whose topology is the same as the usual topology of $S\tilde{M}$. For every positive integer n, let $A_n = \{ v \in S\tilde{M} : d^*(v, (d\varphi \circ g^t)(v)) < 1/n$ for some element $\varphi \in \Gamma$ and some number $t > n\}$. The set A_n is clearly open in $S\tilde{M}$ and is dense in $S\tilde{M}$ since Γ satisfies the duality condition. The set of Γ-recurrent vectors in $S\tilde{M}$ is the intersection of the sets A_n and is dense by a Baire category argument. □

Tits geometry in $\tilde{M}(\infty)$. — We discuss briefly the Tits geometry in $\tilde{M}(\infty)$ as defined by Gromov in [BGS]. For further details see section 4 of [BGS].

One defines first a complete metric $\angle$ on $\tilde{M}(\infty)$ by setting

$$\angle(x, y) = \sup\{\angle_p(x, y) : p \in \tilde{M}\}.$$

If a curve $\sigma : [a, b] \to \tilde{M}(\infty)$ is continuous with respect to $\angle$, then we define the $\angle$-length $L(\sigma)$ to be

$$\sup\{\sum_{i=0}^{n-1} \angle(\sigma(t_i), \sigma(t_{i+1})) : a = t_0 < t_1 < \ldots < t_n = b \text{ is a partition of } [a, b]\}.$$

(1.5) *Definition.* — For any points x, y in $\tilde{M}(\infty)$ we define

$$T\,d(x,y) = \inf\{L(\sigma) : \sigma \text{ is an } \angle\text{-continuous curve in } \tilde{M}(\infty) \text{ from } x \text{ to } y\}.$$

If no $\angle$-continuous curve in $\tilde{M}(\infty)$ from x to y exists, then we define $T\,d(x,y) = +\infty$.

The pseudometric $T\,d(x,y)$ is called the *Tits (pseudo)metric* on $\tilde{M}(\infty)$. It is the inner metric determined by $\angle$. From the definition, it follows that $T\,d(x,y) \geqslant \angle(x,y)$ for all points $x, y \in \tilde{M}(\infty)$.

Remark. — If $T\,d(x,y) < \infty$, then, in a sense that can be made precise, the geodesic rays $\gamma_{px}[0,\infty]$ and $\gamma_{py}[0,\infty]$ bound an asymptotically flat triangular sector in $\tilde{M}$ for any point $p \in \tilde{M}$. If $\tilde{M}$ has sectional curvature $K \leqslant c < 0$, then $T\,d(x,y) = +\infty$ whenever $x \neq y$. In fact, this degeneration of $T\,d$ characterizes the visibility axiom ([BGS], p. 54). We shall not need these facts.

(1.6) *Definition.* — A curve $\sigma : [a,b] \to \tilde{M}(\infty)$ is called a *Tits geodesic* if the following two conditions are satisfied:

(1) There exists a constant $c \geqslant 0$ such that

$$L(\sigma \mid [a,t]) = c(t-a) \quad \text{for all } t \in [a,b].$$

(2) σ is locally distance minimizing, that is, for every $t \in (a,b)$ there exists $\varepsilon > 0$ such that $L(\sigma \mid [t-\varepsilon, t+\varepsilon]) = T\,d(\sigma(t-\varepsilon), \sigma(t+\varepsilon))$.

We say that the Tits geodesic σ has unit speed if $c = 1$ in (1) above, and σ is minimizing if $T\,d(\sigma(a), \sigma(b)) = L(\sigma[a,b])$.

We collect some useful facts from [BGS].

(1.7) *Proposition.* — *Let x, y be any points of $\tilde{M}(\infty)$. Then:*

(1) *$\angle(x,y) = T\,d(x,y)$ if $T\,d(x,y) \leqslant \pi$.*

(2) *If $T\,d(x,y) < \infty$, then there is a minimizing Tits geodesic joining x to y. This geodesic is unique if $T\,d(x,y) < \pi$.*

(3) *$\angle_p(x,y) = \angle(x,y)$ for some point p of $\tilde{M}$ if and only if the geodesic rays $\gamma_{px}[0,\infty)$ and $\gamma_{py}[0,\infty)$ bound a flat, convex surface $\Delta = \Delta(p,x,y)$ in $\tilde{M}$. In this case $\angle_p(x,y) = T\,d(x,y)$.*

(4) *If x, y are any points of $\tilde{M}(\infty)$ and if $\{x_n\}$, $\{y_n\}$ are sequences in $\tilde{M}(\infty)$ that converge to x, y in the sphere topology, then $T\,d(x,y) \leqslant \liminf_{n\to\infty} T\,d(x_n, y_n)$.*

In the sequel the surface Δ that arises in (3) will be called a *flat triangular sector.*

Proof. — Assertion (1) follows from Lemma 4.7 of [BGS, p. 40] and the fact that $\pi = \angle(x,y) \leqslant T\,d(x,y)$ if there exists a geodesic γ with $\gamma(\infty) = x$ and $\gamma(-\infty) = y$. Assertion (2) is proved in [BGS, p. 49]. Assertion (4) is a strengthened version of the assertion in [BGS, p. 46], but the proof is the same. We prove (3). If p, x and y determine a flat triangular sector in $\tilde{M}$ for some point p in $\tilde{M}$, then $\angle_p(x,y) = \angle(x,y)$ by Lemma 4.2 of [BGS, p. 33]. Conversely, suppose that $\angle_p(x,y) = \angle(x,y)$ for some point p of $\tilde{M}$. The angle $\varphi(s)$ subtended by x and y at $\gamma_{px}(s)$ (or $\gamma_{py}(s)$) is a nondecreasing function of s since the sum of the interior angles of a geodesic triangle in $\tilde{M}$ is at most π, even if one vertex is in $\tilde{M}(\infty)$. Hence $\varphi(s) \equiv \varphi(0) = \angle_p(x, y)$ and p, x and y determine a flat triangular sector by the following result from [E4, p. 78].

(1.8) *Lemma. — Let $\sigma : [0, c) \to \tilde{M}$ be a unit speed geodesic, where c is some positive number. Let $x \in \tilde{M}(\infty)$ be a point such that the function $t \mapsto \angle(\sigma'(t), V(\sigma t, x))$ is constant and equal to α, where $0 < \alpha < \pi$. Then $S = \cup \{\gamma_{px}[0, \infty) : p \in \sigma[0, c]\}$ is a flat, totally geodesic embedded surface in $\tilde{M}$.* □

The next result on flat triangular sectors will be useful in section 4.

(1.9) *Proposition. — Let $p \in \tilde{M}$ and $x, y \in \tilde{M}(\infty)$ be points such that p, x and y determine a flat triangular sector Δ with $\angle_p(x, y) = T\,d(x, y) = \beta < \pi$. Let $\sigma : [0, \beta] \to \tilde{M}(\infty)$ denote the unique unit speed Tits geodesic from $x = \sigma(0)$ to $y = \sigma(\beta)$. Then $V(p, \sigma t)$ is tangent to Δ for all $t \in [0, \beta]$.*

Proof. — Let $v : [0, \beta] \to T_p\,\Delta$ be the unit speed curve of unit vectors tangent to Δ at p such that $v(0) = V(p, x)$ and $v(\beta) = V(p, y)$. Define $\sigma^* : [0, \beta] \to \tilde{M}(\infty)$ by setting $\sigma^*(t) = \gamma_{v(t)}(\infty)$. Let s, t be any numbers in $[0, \beta]$ with $s < t$. The geodesic rays determined by $v(s)$ and $v(t)$ span a flat triangular sector $\Delta(s, t) \subseteq \Delta$. By (1.7), the definition of the $\angle$-length and the flat Euclidean geometry of Δ, it follows that $T\,d(\sigma^*(s), \sigma^*(t)) = \angle_p(\sigma^*(t), \sigma^*(s)) = L(\sigma^*[s, t])$. Hence σ^* is a minimizing unit speed Tits geodesic from $x = \sigma^*(0)$ to $y = \sigma^*(\beta)$. We conclude that $\sigma = \sigma^*$ by the uniqueness assertion (2) in (1.7). □

We conclude this section with a characterization of Euclidean spaces.

(1.10) *Proposition.* — Let $\tilde{M}$ have the property that for every $x \in \tilde{M}(\infty)$ *there exists a point $y \in \tilde{M}(\infty)$ such that $\angle_p(x, y) \equiv \pi$ for all $p \in \tilde{M}$. Then $\tilde{M}$ is isometric to a Euclidean space with $K \equiv 0$.*

Proof. — Let Π be a 2-plane in $T_p\,\tilde{M}$, where p is any point of $\tilde{M}$, and let v, w be a basis of Π. By hypothesis there exists a geodesic σ containing the point $\gamma_w(1)$ such that $\sigma(\infty) = \gamma_v(\infty)$ and $\sigma(-\infty) = \gamma_v(-\infty)$. The geodesics σ and γ_v bound a flat, totally geodesic strip S in $\tilde{M}$ by [EO], and the tangent space to S at p is Π. □

2. A characterization of symmetric spaces of higher rank

(2.1) *Definition.* — For an integer $k \geqslant 2$, a *k-flat* in $\tilde{M}$ is a complete, totally geodesic submanifold F of M that is isometric to a flat k-dimensional Euclidean space.

If $v \in S\tilde{M}$, we define $F(\gamma_v)$ to be the union of all geodesics σ in M such that $\sigma(\infty) = \gamma_v(\infty)$ and $\sigma(-\infty) = \gamma_v(-\infty)$. The next result is a restatement of Theorem 2.6 from [BBE].

(2.2) *Theorem.* — *Let* $\operatorname{rank}(\tilde{M}) = k \geqslant 2$ *and suppose that* $I(\tilde{M})$ *satisfies the duality condition. If* v *is a regular vector of* $S\tilde{M}$, *then* $F(\gamma_v)$ *is a* k*-flat, the unique* k*-flat of* $\tilde{M}$ *that contains* γ_v. *Every geodesic of* $\tilde{M}$ *is contained in at least one* k*-flat.*

The statements in [BBE] of this result and the supporting Lemma 2.5 require that $\tilde{M}$ admit a quotient manifold of finite volume. Nevertheless an inspection of the proof shows that it is enough to require that $I(\tilde{M})$ satisfy the duality condition.

The main tool needed for the proof of the main result stated in the introduction is the following result from [E5, Theorem 4.1].

(2.3) *Theorem.* — *Let* $\tilde{M}$ *be irreducible and suppose that* $I(\tilde{M})$ *satisfies the duality condition. If* $I(\tilde{M})$ *leaves invariant a proper, closed subset of* $\tilde{M}(\infty)$, *then* $\tilde{M}$ *is isometric to a symmetric space of noncompact type and* $\operatorname{rank} k \geqslant 2$.

Remarks. — 1) If $\tilde{M}$ is a symmetric space of noncompact type and $\operatorname{rank} k \geqslant 2$, then the closure in $\tilde{M}(\infty)$ of any orbit of $I(\tilde{M})$ is a proper subset of $\tilde{M}(\infty)$. We omit a proof of this fact since we do not use it.

2) Define a point $x \in \tilde{M}(\infty)$ to be *singular* if $V(p, x)$ is singular (i.e. not regular) for every point $p \in \tilde{M}$. The set of singular points at infinity is clearly a closed subset of $\tilde{M}(\infty)$ invariant under $I(\tilde{M})$. If $\tilde{M}$ is a symmetric space of noncompact type and $\operatorname{rank} k \geqslant 2$, then the set of singular points at infinity is nonempty, but this set might be empty if $\tilde{M}$ is arbitrary. The second author has proved that the set of singular points at infinity is nonempty if $I(\tilde{M})$ satisfies the duality condition and $\tilde{M}$ is irreducible with $\operatorname{rank} \geqslant 2$. Combining this result with Theorem 2.3 above one obtains an alternate proof of the main theorem.

3. Weyl chambers in $\tilde{M}(\infty)$

To motivate the title and the discussion of this section we consider the case that $\tilde{M}$ is a symmetric space of noncompact type and $\operatorname{rank} k \geqslant 2$. Let p be a point of $\tilde{M}$, and let v_1, v_2 be regular unit vectors in $T_p \tilde{M}$ that determine the same Weyl chamber in $S\tilde{M}$ as defined in section 2 of [BBS]. If $x_i = \gamma_{v_i}(\infty)$ for $i = 1, 2$, then one can show that $G_{x_1} = G_{x_2}$, where $G = I_0(\tilde{M})$ and $G_{x_i} = \{ g \in G : g(x_i) = x_i \}$ for $i = 1, 2$. It follows that $p \mapsto \angle_p(x_1, x_2)$ is a constant function in $\tilde{M}$ since $G_{x_1} = G_{x_2}$ acts transitively on $\tilde{M}$. In particular $\angle_p(x_1, x_2) = T\, d(x_1, x_2)$ for all points $p \in \tilde{M}$ by (1.7). We omit the proofs of these assertions. The purpose of our remarks is to point out that the ideas in this section are closely related to more standard notions of Weyl chamber.

The main result of this section is the following.

(3.1) *Proposition.* — *Let* $\widetilde{M}$ *be irreducible and suppose that* $I(\widetilde{M})$ *satisfies the duality condition. Then* $\widetilde{M}$ *is isometric to a symmetric space of noncompact type and* rank $k \geqslant 2$ *if there exist a constant* $\alpha^* > 0$ *and a point* $x \in \widetilde{M}(\infty)$ *that satisfy the following conditions*:

$$(*) \begin{cases} (1)\ B_{\alpha^*}(x) = \{ y \in \widetilde{M}(\infty) : T\,d(x,y) < \alpha^* \} \text{ contains a point other than } x; \\ (2)\ \text{if } y_1, y_2 \text{ are any two points of } B_{\alpha^*}(x), \text{ then } \angle_q(y_1, y_2) = T\,d(y_1, y_2) \text{ for all points } q \in \widetilde{M}. \end{cases}$$

Proof. — For any number α with $0 < \alpha \leqslant \pi$ we define

$$\widetilde{M}_\alpha(\infty) = \{ x \in \widetilde{M}(\infty) : \text{there exists } y \in \widetilde{M}(\infty) \text{ with } \angle_q(x,y) = \alpha \text{ for all } q \in \widetilde{M} \}.$$

Let $y \in B_{\alpha^*}(x) - \{ x \}$ and let $T\,d(x,y) = \rho \in (0, \alpha^*)$, where x and α^* are defined above in $(*)$. The set $\widetilde{M}_\rho(\infty)$ is nonempty since it contains x. From the definition one sees immediately that $\widetilde{M}_\alpha(\infty)$ is closed in $\widetilde{M}(\infty)$ and invariant under $I(\widetilde{M})$ for any positive number α. The assertion of the proposition is now a direct consequence of Theorem 2.3 and the following

(3.2) *Lemma.* — *Let* $\widetilde{M}$ *be any complete, simply connected Riemannian manifold with sectional curvature* $K \leqslant 0$. *Suppose there exist a positive constant* α^* *and a point* $x \in \widetilde{M}(\infty)$ *that satisfy the conditions* $(*)$. *Then either* $\widetilde{M}$ *is flat or* $\widetilde{M}_\beta(\infty)$ *is a nonempty, proper subset of* $\widetilde{M}(\infty)$ *for some* β *with* $0 < \beta \leqslant \pi$.

Proof of the lemma. — Let $\beta = \sup\{ \alpha : \widetilde{M}_\alpha(\infty) \text{ is nonempty} \}$. Note that $\beta \geqslant \rho$ since $\widetilde{M}_\rho(\infty)$ is nonempty, and it follows from the definition of β that $\widetilde{M}_\beta(\infty)$ is nonempty. We shall show that if $\widetilde{M}_\beta(\infty) = \widetilde{M}(\infty)$, then $\widetilde{M}$ is flat.

Suppose that $\widetilde{M}_\beta(\infty) = \widetilde{M}(\infty)$. Let $\alpha^* > 0$ and $x \in \widetilde{M}(\infty)$ be chosen to satisfy conditions $(*)$. Since $x \in \widetilde{M}_\beta(\infty)$ we may choose $y \in \widetilde{M}(\infty)$ so that $\angle_p(x,y) = \beta$ for all points $p \in \widetilde{M}$. Fix a point $p_0 \in \widetilde{M}$ and let $\gamma(t) = \gamma_{p_0 y}(t)$. If $\beta < \pi$, the fact that $t \mapsto \angle_{\gamma(t)}(x,y)$ is a constant function implies by (1.8) that γ bounds a flat half plane F^* such that $\gamma_{p_0 x}[0, \infty) \subseteq F^*$. Let $e_1 = \gamma'(0)$ and let $e_2 \in T_{p_0}\widetilde{M}$ be the unit vector that is orthogonal to e_1 and tangent to F^*. If $v(t) = (\cos t)\, e_1 + (\sin t)\, e_2$, then $v(t)$ is tangent to F^* for all $t \in [0, \pi]$, and $\sigma(t) = \gamma_{v(t)}(\infty)$ is a minimizing Tits geodesic on $[0, \pi]$ by the flat Euclidean geometry of F^* and the discussion in the proof of (1.9). Note that $y = \sigma(0)$ and $x = \sigma(\beta)$.

We prove that $\beta = \pi$. Since $\widetilde{M}_\beta(\infty) = \widetilde{M}(\infty)$ by hypothesis it will then follow from (1.10) that $\widetilde{M}$ is flat, completing the proof of the proposition. We suppose that $\beta < \pi$ and choose $\varepsilon > 0$ so that $\beta + \varepsilon < \pi$ and $\varepsilon < \alpha^*$, where α^* is the positive number occurring in the statement of the lemma. We define $z = \sigma(\beta + \varepsilon)$, where σ is the Tits geodesic in $\widetilde{M}(\infty)$ defined above. We shall prove that $\angle_q(y, z) = T\,d(y, z) = \beta + \varepsilon$ for all points $q \in \widetilde{M}$. This will show that $y \in \widetilde{M}_{\beta+\varepsilon}(\infty)$, contradicting the definition of β and proving that $\beta = \pi$.

Define $z^* = \sigma(\beta - \varepsilon)$. Let q be any point in $\tilde{M}$. By the definition of z and z^* we have $T\,d(x, z) = T\,d(x, z^*) = \varepsilon < \alpha^*$ since $x = \sigma(\beta)$. Hence

$$\angle_q(z, z^*) = T\,d(z, z^*) = 2\varepsilon$$

by the properties of x and α^* as defined above in condition $(*)$. By (1.7) it follows that q, z and z^* determine a flat triangular sector $\Delta_2 = \Delta(q, z, z^*)$. It follows from (1.9) that the unit vectors $V(q, \sigma t)$ are tangent to Δ_2 for all $t \in [\beta - \varepsilon, \beta + \varepsilon]$ since $z = \sigma(\beta + \varepsilon)$ and $z^* = \sigma(\beta - \varepsilon)$.

By the choice of x and y we know that $\angle_q(x, y) = T\,d(x, y) = \beta$, and hence q, x and y span a flat triangular sector $\Delta_1 = \Delta(q, x, y)$ in $\tilde{M}$. The unit vectors $V(q, \sigma t)$ are tangent to Δ_1 for $t \in [0, \beta]$ by (1.9) since $y = \sigma(0)$ and $x = \sigma(\beta)$. The unit vectors $V(q, \sigma t)$ are therefore tangent both to Δ_1 and to Δ_2 for $t \in [\beta - \varepsilon, \beta]$. We conclude that the flat triangular sectors Δ_1 and Δ_2 fit together to form a flat triangular sector $\Delta = \Delta_1 \cup \Delta_2$ such that the unit vectors $V(q, \sigma t)$ are tangent to Δ for $t \in [0, \beta + \varepsilon]$. Finally, $\angle_q(y, z) = \angle_q(\sigma(0), \sigma(\beta + \varepsilon)) = T\,d(\sigma(0), \sigma(\beta + \varepsilon)) = T\,d(y, z) = \beta + \varepsilon$, where the second equality follows from (1.7). Since $q \in \tilde{M}$ was arbitrary it follows that

$$q \to \angle_q(y, z) \equiv \beta + \varepsilon \quad \text{and} \quad y \in \tilde{M}_{\beta+\varepsilon}(\infty),$$

contradicting the definition of β. □

4. Proof of the main result

We restate the main result from the introduction.

(4.1) *Theorem.* — *Let $\tilde{M}$ be a complete, simply connected Riemannian manifold of nonpositive sectional curvature such that* $\operatorname{rank}(\tilde{M}) = k \geq 2$, *and* $I(\tilde{M})$ *satisfies the duality condition. If $\tilde{M}$ is irreducible, then $\tilde{M}$ is isometric to a symmetric space of noncompact type and* rank k.

We need two lemmas.

(4.2) *Lemma.* — *Let $v \in S\tilde{M}$ be a regular, $I(\tilde{M})$-recurrent vector, and let $x = \gamma_v(\infty)$. Let $F(\gamma_v)$ denote the unique k-flat that contains γ_v. There exists a positive number α such that if z is any point in $\tilde{M}(\infty) - F(\gamma_v)(\infty)$, then $T\,d(x, z) \geq \alpha$.*

(4.3) *Lemma.* — *Let v, x and α be as in Lemma 4.2. Let F be any k-flat such that $x \in F(\infty)$. Then $F(\infty) \supseteq B_\alpha(x) = \{y \in \tilde{M}(\infty) : T\,d(x, z) < \alpha\}$.*

Proof of the theorem. — For the moment we defer the proofs of the lemmas. Let x and α be as in Lemma 4.2, and let p be any point of $\tilde{M}$. By Theorem 2.2 the geodesic γ_{px} is contained in some k-flat F of $\tilde{M}$. If z_1, z_2 are any two points of $\tilde{M}(\infty)$ such that $T\,d(x, z_i) < \alpha$ for $i = 1, 2$, then $z_i \in F(\infty)$ for $i = 1, 2$ by Lemma 4.3. Hence $\angle_p(z_1, z_2) = T\,d(z_1, z_2)$ by (1.7) since $p \in F$, and it follows that the condition $(*)$ of Proposition 3.1 is satisfied by x and α. Theorem 4.1 now follows from Proposition 3.1. □

We now prove the two lemmas.

For the proof of Lemma 4.2 we need the following

Sublemma. — *Let* $y = \gamma_v(-\infty)$. *There exists a positive number* σ *such that if* F *is any* k*-flat with* $d(\pi v, \mathrm{F}) = 1$, *then*

$$\mathrm{T}\, d(x, \mathrm{F}(\infty)) + \mathrm{T}\, d(y, \mathrm{F}(\infty)) \geqslant \sigma.$$

Proof of the sublemma. — If the lemma were not true, then by the semicontinuity of $\mathrm{T}\,d$ expressed in (4) of (1.7) we could find a k-flat F with $d(\pi v, \mathrm{F}) = 1$ and $\mathrm{T}\, d(x, \mathrm{F}(\infty)) + \mathrm{T}\, d(y, \mathrm{F}(\infty)) = 0$. It would follow that $x, y \in \mathrm{F}(\infty)$, and hence from (1.7) we would conclude that $\angle_q(x, y) = \mathrm{T}\, d(x, y) = \pi$ for all points q in F since F is flat. Hence $\mathrm{F} \subseteq \mathrm{F}(\gamma_v)$ by (2.2) and equality would follow since both F and $\mathrm{F}(\gamma_v)$ are k-flats. This would contradict the fact that $d(\pi v, \mathrm{F}) = 1$. □

Proof of Lemma **4.2.** — Let σ be the constant from the sublemma. Choose a positive number δ so that if u is any unit vector at $p = \pi(v)$ with $\angle(u, v) < \delta$, then u is a regular vector. Let α be any positive number with $\alpha < \min\{\sigma, \delta, \pi\}$. We show that α satisfies the conditions of Lemma 4.2.

Let z be any point in $\widetilde{\mathrm{M}}(\infty) - \mathrm{F}(\gamma_v(\infty))$. It suffices to consider the case that $\mathrm{T}\, d(x, z) < \min\{\delta, \pi\}$. Let $\sigma^* : [0, 1] \to \widetilde{\mathrm{M}}(\infty)$ be the unique minimizing Tits geodesic with $\sigma^*(0) = x$ and $\sigma^*(1) = z$. If $v(s) = \mathrm{V}(p, \sigma^* s)$, then $v(s)$ is regular for all $s \in [0, 1]$ by the choice of δ since $\angle_p(v(s), v) = \angle_p(\sigma^* s, x) \leqslant \mathrm{T}\, d(\sigma^* s, x) \leqslant \mathrm{T}\, d(z, x) < \delta$.

Observe that $x \notin \mathrm{F}(\gamma_{pz})(\infty)$. If this were not the case, then we would have $\gamma_v = \gamma_{px} \subseteq \mathrm{F}(\gamma_{pz})$ and consequently the k-flat $\mathrm{F}(\gamma_{pz})$ would be contained in $\mathrm{F}(\gamma_v)$. It would follow that the k-flats $\mathrm{F}(\gamma_{pz})$ and $\mathrm{F}(\gamma_v)$ are equal, contradicting the hypothesis that $z \notin \mathrm{F}(\gamma_v)(\infty)$.

Since v is $\mathrm{I}(\widetilde{\mathrm{M}})$-recurrent there exist sequences $\{t_n\} \subseteq \mathscr{R}$ and $\{\varphi_n\} \subseteq \mathrm{I}(\widetilde{\mathrm{M}})$ such that $t_n \to +\infty$ and $(d\varphi_n \circ g^{t_n})(v) \to v$ as $n \to \infty$. Note that $d(\gamma_v\, t, \mathrm{F}(\gamma_{pz})) \to \infty$ as $t \to \infty$ since $x = \gamma_v(\infty) \notin \mathrm{F}(\gamma_{pz})(\infty)$. Thus we can choose a positive integer N such that $d(\gamma_v\, t_n, \mathrm{F}(\gamma_{pz})) \geqslant 1$ for all $n \geqslant \mathrm{N}$. For each positive integer n the function $f_n(s) = d(\gamma_v\, t_n, \mathrm{F}(\gamma_{v(s)}))$ is continuous and satisfies $f_n(0) = 0$ and $f_n(1) \geqslant 1$. Choose $s_n \in [0, 1]$ so that $d(\gamma_v\, t_n, \mathrm{F}(\gamma_{v_n})) = f_n(s_n) = 1$ for every n, where $v_n = v(s_n)$. Define F_n to be the k-flat $\varphi_n\, \mathrm{F}(\gamma_{v_n})$.

There exists a k-flat F in $\widetilde{\mathrm{M}}$ such that

1) a subsequence of F_n converges to F, uniformly on compact subsets,
2) $d(p, \mathrm{F}) = 1$, where $p = \pi(v)$, and
3) $y = \gamma_v(-\infty) \in \mathrm{F}(\infty)$.

The existence of F and assertions 1) and 2) follow from the facts that $(\varphi_n \circ \gamma_v)(t_n) \to p$ and $d((\varphi_n \circ \gamma_v)(t_n), \mathrm{F}_n) = d(\gamma_v(t_n), \mathrm{F}(\gamma_{v_n})) = 1$ for every $n \geqslant \mathrm{N}$. To prove 3) note that the convex function $t \mapsto d(\gamma_v\, t, \mathrm{F}(\gamma_{v_n}))$ increases from 0 to 1 on $[0, t_n]$ since $p = \pi(v_n) = \pi(v)$. If we define $\gamma_n(t) = (\varphi_n \circ \gamma_v)(t + t_n)$ for all $t \in \mathscr{R}$, then $t \mapsto d(\gamma_n(t), \mathrm{F}_n)$

increases from 0 to 1 on $[-t_n, 0]$. By the definition of $\{\varphi_n\}$ and $\{t_n\}$ it follows that $\gamma_n'(0) \to \gamma_v'(0) = v$, and hence $\gamma_n(t) \to \gamma_v(t)$ for all $t \in \mathscr{R}$. From 1) we conclude that $d(\gamma_v t, \mathrm{F}) \leqslant 1$ on $(-\infty, 0]$, which proves 3).

If $z_n = \gamma_{v_n}(\infty) = \sigma^*(s_n)$, then let $\varphi_n(z_n)$ converge to a point $z^* \in \widetilde{\mathrm{M}}(\infty)$ by passing to a subsequence if necessary. From 1) above we see that $z^* \in \mathrm{F}(\infty)$ since $\varphi_n(z_n) \in \mathrm{F}_n(\infty)$ for every n. Note that $\varphi_n(x) = \gamma_n(\infty) \to \gamma(\infty) = x$ as $n \to \infty$. We conclude that

$$\begin{aligned}
\alpha \leqslant \sigma &\leqslant \mathrm{T}\,d(x, \mathrm{F}(\infty)) + \mathrm{T}\,d(y, \mathrm{F}(\infty)) && \text{by 2) and the sublemma} \\
&= \mathrm{T}\,d(x, \mathrm{F}(\infty)) && \text{by 3) above} \\
&\leqslant \mathrm{T}\,d(x, z^*) && \text{since } z^* \in \mathrm{F}(\infty) \\
&\leqslant \liminf_{n\to\infty} \mathrm{T}\,d(\varphi_n(x), \varphi_n(z_n)) && \text{by (4) of (1.7)} \\
&= \liminf_{n\to\infty} \mathrm{T}\,d(x, z_n) = \liminf_{n\to\infty} [s_n\, \mathrm{T}\,d(x, z)] \\
&\leqslant \mathrm{T}\,d(x, z) && \text{since } s_n \in [0, 1]. \quad \square
\end{aligned}$$

Proof of Lemma **4.3**. — For any point $r \in \widetilde{\mathrm{M}}$ we let $\varphi_r : \widetilde{\mathrm{M}}(\infty) \to \mathrm{S}_r \widetilde{\mathrm{M}}$ denote the homeomorphism given by $\varphi_r(x) = \mathrm{V}(r, x) = \gamma_{rx}'(0)$. From Lemma 4.2 it follows immediately that $\mathrm{F}(\gamma_v)(\infty)$ contains $\mathrm{B}_\alpha(x) = \{y \in \widetilde{\mathrm{M}}(\infty) : \mathrm{T}\,d(x,y) < \alpha\}$. Now let F be any k-flat such that $x \in \mathrm{F}(\infty)$ and define $\mathrm{A} = \mathrm{B}_\alpha(x) \cap \mathrm{F}(\infty)$. If $p = \pi(v)$ and if q is any point of F, then by (4) of (1.7) the maps $\varphi_p : \mathrm{F}(\gamma_v)(\infty) \to \mathrm{S}_p\,\mathrm{F}(\gamma_v)$ and $\varphi_q : \mathrm{F}(\infty) \to \mathrm{S}_q\,\mathrm{F}$ are isometries with respect to the Tits metric on $\widetilde{\mathrm{M}}(\infty)$ and the usual angle metrics in $\mathrm{S}_p\,\mathrm{F}(\gamma_v)$ and $\mathrm{S}_q\,\mathrm{F}$. The map $\Phi = \varphi_p \circ \varphi_q^{-1} : \varphi_q(\mathrm{A}) \to \varphi_p(\mathrm{B}_\alpha(x))$ is an isometric embedding with respect to the angle metrics since $\mathrm{A} \subseteq \mathrm{B}_\alpha(x)$. Hence Φ is the restriction of a linear isometry of $\mathrm{S}_q\,\mathrm{F}$ onto $\mathrm{S}_p\,\mathrm{F}(\gamma_v)$ since $\varphi_q(\mathrm{A})$ is an open subset of $\mathrm{S}_q\,\mathrm{F}$. The sets $\varphi_q(\mathrm{A})$ and $\varphi_p(\mathrm{B}_\alpha(x))$ are open metric $(k-1)$-balls of radius α and centers $v^* = \mathrm{V}(q, x)$ and $v = \mathrm{V}(p, x)$ respectively. It follows that $\Phi : \varphi_q(\mathrm{A}) \to \varphi_p(\mathrm{B}_\alpha(x))$ is surjective since $\Phi(v^*) = v$, and this implies immediately that $\mathrm{A} = \mathrm{B}_\alpha(x)$, which is the assertion of Lemma 4.3. $\square$

REFERENCES

[Ba1] W. Ballmann, Axial isometries of manifolds of nonpositive curvature, *Math. Ann.*, **259** (1982), 131-144.

[Ba2] —, Nonpositively curved manifolds of higher rank, *Annals of Math.*, **122** (1985), 597-609.

[BBE] W. Ballmann, M. Brin and P. Eberlein, Structure of manifolds of nonpositive curvature, I, *Annals of Math.*, **122** (1985), 171-203.

[BBS] W. Ballmann, M. Brin and R. Spatzier, Structure of manifolds of nonpositive curvature, II, *Annals of Math.*, **122** (1985), 205-235.

[Be] M. Berger, Sur les groupes d'holonomie homogène des variétés à connexion affine et des variétés riemanniennes, *Bull. Soc. Math. France*, **83** (1953), 279-330.

[BGS] W. Ballmann, M. Gromov and V. Schroeder, *Manifolds of nonpositive curvature*, Basel, Birkhäuser, 1985.

[BO] R. Bishop and B. O'Neill, Manifolds of negative curvature, *Trans. Amer. Math. Soc.*, **145** (1969), 1-49.

[BS] K. Burns and R. Spatzier, Manifolds of nonpositive curvature and their buildings, *Publ. Math. IHES*, **65** (1987), 35-59.

[E1] P. EBERLEIN, A differential geometric characterization of symmetric spaces of noncompact type and rank at least two, *preprint*, 1987.

[E2] —, Geodesic flows on negatively curved manifolds, I, *Annals of Math.*, **95** (1972), 492-510.

[E3] —, Isometry groups of simply connected manifolds of nonpositive curvature, II, *Acta Math.*, **149** (1982), 41-69.

[E4] —, Rigidity of lattices of nonpositive curvature, *Erg. Th. Dynam. Sys.*, **3** (1983), 47-85.

[E5] —, Symmetry diffeomorphism group of a manifold of nonpositive curvature, II, *Indiana Univ. Math. Jour.*, **37** (1988), 735-752.

[EO] P. EBERLEIN and B. O'NEILL, Visibility manifolds, *Pacific J. Math.*, **46** (1973), 45-109.

[GKM] D. GROMOLL, W. KLINGENBERG and W. MEYER, *Riemannsche Geometrie im Grossen*, Lecture Notes in Math., vol. 55, Berlin, Springer, 1968.

Department of Mathematics
University of North Carolina
Chapel Hill, North Carolina 27514
U.S.A.

and

Mathematisches Institut
der Universität Augsburg
8900 Augsburg
Federal Republic of Germany

Manuscrit original du premier auteur (P. E.) reçu le 30 juin 1987.
Révisé le 31 octobre 1988.
Nouvelle version (par P. E. et J. H.) reçue le 2 août 1989.