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PERIODIC POINTS AND ROTATION NUMBERS FOR AREA PRESERVING DIFFEOMORPHISMS OF THE PLANE

by JOHN FRANKS

Abstract. — Let f be an orientation preserving diffeomorphism of $\mathbf{R}^2$ which preserves area. We prove the existence of infinitely many periodic points with distinct rotation numbers around a fixed poinf of f, provided only that f has two fixed points whose infinitesimal rotation numbers are not both 0.

We also show that if a fixed point z of f is enclosed in a "simple heteroclinic cycle" and has a non-zero infinitesimal rotation number r, then for every non-zero rational number p/q in an interval with endpoints 0 and r, there is a periodic orbit inside the heteroclinic cycle with rotation number p/q around z.

In this paper we investigate area preserving diffeomorphisms of $\mathbf{R}^2$ and the existence of periodic points with prescribed rotation number around a given fixed point. A motivating question for this investigation deals with a diffeomorphism $f: \mathbf{R}^2 \to \mathbf{R}^2$ which has two hyperbolic fixed points p_1, p_2, with a double saddle connection and an elliptic fixed point between the saddle connections (see Fig. 1*a*). The classical fixed point theorem of Poincaré and Birkhoff can be used to show that in this case for each rational p/q between 0 and the infinitesimal rotation at z there is a periodic orbit with rotation number p/q around z which lies inside the disk bounded by the saddle connections. This is done by "blowing up" the point z to obtain a homeomorphism of the annulus bounded by the saddle connections and the blow up of z, and applying the theorem of Poincaré and Birkhoff to this annulus homeomorphism.

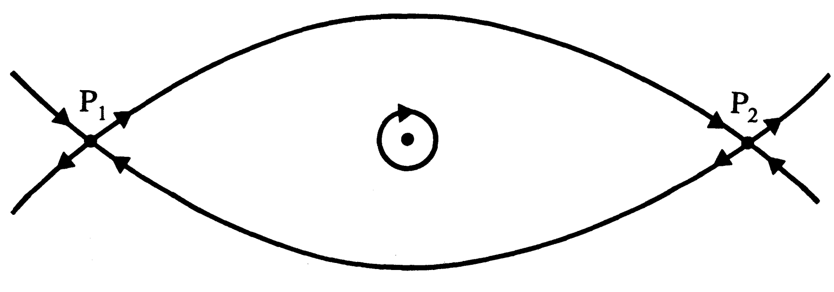

Fig. 1*a*

If the heteroclinic connections between the points p_1 and p_2 are more complicated (see Fig. 1*b*) this approach does not work, because there may be no invariant disk of finite area containing z, but one can ask if the result is still true. With rather modest assumptions on the hyperbolic points in this more complicated case (see the definition of " simple heteroclinic cycle " in (3.2)), we show that it remains true that there is a periodic point with rotation number p/q. This result is Theorem (3.4) below.

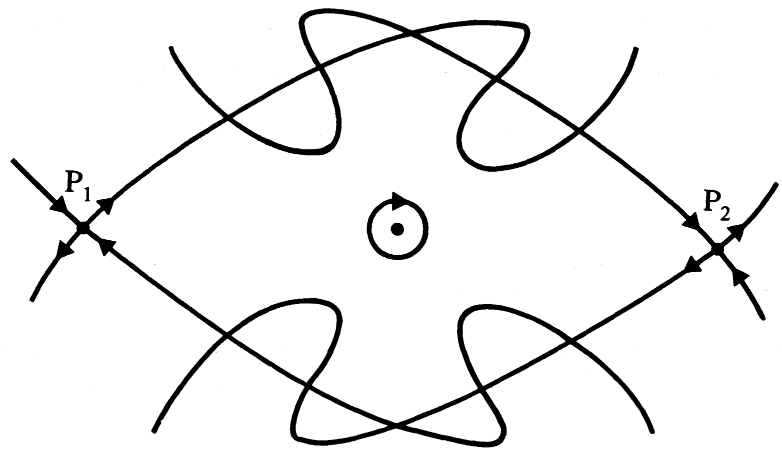

Fig. 1*b*

In § 2 we consider an even more general setting: any area preserving diffeomorphism $f: \mathbf{R}^2 \to \mathbf{R}^2$ which preserves orientation and has two fixed points z_0 and z_1. In (2.2) and (2.5) we show there are intervals with the property that for any non-zero p/q in their interior, there is a periodic point x such that p/q is the difference of the rotation numbers of x around z_1 and z_0. Such an interval can be chosen with endpoints the infinitesimal rotation number of z_1 and minus the infinitesimal rotation number of z_0 or with endpoints 0 and one of these numbers. In any case, unless the infinitesimal rotation numbers of z_0 and z_1 are both 0, there are infinitely many periodic orbits with distinct rotation numbers about at least one of the two fixed points.

I would like to thank Robert MacKay for posing the motivating question to me and for several valuable conversations on the possibility of its resolution.

1. Background and definitions

We are interested in investigating the existence of periodic orbits for area preserving diffeomorphisms of $\mathbf{R}^2$ and measuring their rotation around a given fixed point. We begin by recalling the definition of rotation number for a homeomorphism of the annulus. Suppose $f: \mathrm{B} \to \mathrm{B}$ is a homeomorphism of the annulus B which is homotopic to the identity map (we consider $\mathrm{B} = \mathbf{T}^1 \times \mathrm{I}$, where I is $[0, 1]$, or $(0, 1)$, or $[0, \infty)$). Let

$\pi : \widetilde{B} \to B$ be the universal cover and let $F : \widetilde{B} \to \widetilde{B}$ be a lift of f. If $x \in \widetilde{B}$, then the rotation number of x relative to F is defined to be

$$R_F(x) = \lim_{n\to\infty} \frac{(F^n(x) - x)_1}{n}$$

if this limit exists. Here $(\ )_1$ denotes projection on the first factor of $\widetilde{B} = \mathbf{R} \times I$. (The identification of $\widetilde{B}$ with $\mathbf{R} \times I$ requires the choice of a generator of $H_1(B)$ to specify the orientation of $\mathbf{R}$, or equivalently the choice of a generator T of the deck transformations of the cover π.)

If $R_F(x)$ exists and $\pi(x) = y \in B$ then the fractional part of $R_F(x)$ is independent of the choice of the lift F and is the same for all points of $\pi^{-1}(y)$. Hence it is referred to as the *rotation number* of $y \in B$ and will be denoted $R(y)$. The number $R(y)$ is well defined only mod 1 and is properly thought of as an element of $\mathbf{T}^1$ rather than a " number " as its name suggests.

We shall also want to consider an invariant which is a real number as opposed to an element of $\mathbf{T}^1$. To obtain a well defined element of $\mathbf{R}$ we need to specify a choice of the lift $F : \widetilde{B} \to \widetilde{B}$. A natural way to do this is by choosing a continuous path $\gamma : [0, 1] \to B$ such that $f(\gamma(0)) = \gamma(1)$ and using the lift F satisfying $F(\Gamma(0)) = \Gamma(1)$, for some (and hence any) lift Γ of γ to $\widetilde{B}$. To distinguish the two " numbers " we use the term *total rotation number* relative to γ for this invariant. More formally we have the following definition.

(1.1) *Definition. — Suppose $f : B \to B$ is a homeomorphism of the annulus B which is homotopic to the identity map. Let $\Gamma : [0, 1] \to \widetilde{B}$ be a lift of γ and let $F : \widetilde{B} \to \widetilde{B}$ be the unique lift of f such that $F(\Gamma(0)) = \Gamma(1)$. The total rotation number of $x \in B$ relative to γ, denoted $\mathcal{R}_\gamma(x, f)$, is defined to be $R_F(x_0)$, if it exists, where $\pi(x_0) = x$.*

It is easy to check that this value is independent of the choice of Γ and of the choice of $x_0 \in \widetilde{B}$.

Often we will be interested in the case where y is a fixed point of f and γ is a constant path with value y, i.e. when $\gamma(t) = y$ for some fixed point y and all $t \in [0, 1]$. In this case we will sometimes abuse notation and write y for γ so the total rotation number relative to the fixed point y will be denoted $\mathcal{R}_y(x, f)$.

We next recall the process of " blowing up " a fixed point z of a diffeomorphism f. Intuitively we remove the fixed point z and replace it with a " circle of directions " to which the diffeomorphism can be extended. More precisely, if $f : \mathbf{R}^2 \to \mathbf{R}^2$ is a diffeomorphism with fixed point z, let $A = \mathbf{T}^1 \times [0, \infty)$ and consider the homeomorphism $h : A - (\mathbf{T}^1 \times \{0\}) \to \mathbf{R}^2 - \{0\}$ given by $h(x, t) = tx$, where we identify $\mathbf{T}^1$ with the unit vectors of $\mathbf{R}^2$. Define $g : A \to A$ by

$$g(x, t) = \begin{cases} (f(tx)/\|f(tx)\|, \|f(tx)\|) & \text{if } t > 0, \\ (Df_0(x)/\|Df_0(x)\|, 0) & \text{otherwise.} \end{cases}$$

It is not difficult to see that this defines a continuous function since $\lim_{t\to 0} f(tx)/t = \mathrm{D}f_0(x)$, so

$$\lim_{t\to 0} \frac{f(tx)}{\|f(tx)\|} = \lim_{t\to 0} \frac{f(tx)}{t} \frac{t}{\|f(tx)\|} = \frac{\mathrm{D}f_0(x)}{\|\mathrm{D}f_0(x)\|}.$$

Clearly g is a homeomorphism of A and if $\mathrm{A} - (\mathbf{T}^1 \times \{0\})$ is identified with $\mathbf{R}^2 - \{0\}$ via the homeomorphism h then $g = f$ on this set. We shall refer to $g : \mathrm{A} \to \mathrm{A}$ as the homeomorphism obtained by *blowing up z* and generally identify appropriate points of A with points of $\mathbf{R}^2$ via the homeomorphism h. In a similar fashion one can define the homeomorphism obtained by blowing up a fixed point of a diffeomorphism of any surface.

Next we define the rotation number of a periodic point x around a fixed point z.

(1.2) *Definition. — Suppose $f: \mathbf{R}^2 \to \mathbf{R}^2$ is an orientation preserving diffeomorphism with fixed point z and a periodic point x, and suppose $\gamma : [0, 1] \to \mathbf{R}^2$ is a path with $f(\gamma(0)) = \gamma(1)$. Let $g : \mathrm{A} \to \mathrm{A}$ be the homeomorphism obtained by blowing up z. The total rotation number of x around z relative to γ, denoted $\mathcal{R}_\gamma(x, z, f)$, is defined to be the total rotation number $\mathcal{R}_\gamma(x, g)$, if it exists. The rotation number of x around z, denoted $\mathcal{R}(x, z, f)$, is the rotation number $\mathrm{R}(x, g)$, which equals the fractional part of $\mathcal{R}_\gamma(x, z, f)$. If y is a fixed point of f we will use $\mathcal{R}_y(x, z, f)$ to denote $\mathcal{R}_\gamma(x, z, f)$, where γ is the constant path with value y and refer to this value as the total rotation number of x around z relative to y.*

We need also to measure the rate at which a fixed point is rotating infinitesimally.

(1.3) *Definition. — Suppose $f: \mathbf{R}^2 \to \mathbf{R}^2$ is an orientation preserving diffeomorphism with fixed point z and suppose $\gamma : [0, 1] \to \mathbf{R}^2$ is a path satisfying $f(\gamma(0)) = \gamma(1)$. Let $g : \mathrm{A} \to \mathrm{A}$ be the homeomorphism obtained by blowing up the fixed point z. The total infinitesimal rotation number of the fixed point z relative to γ, denoted $\rho_\gamma(z, f)$, is defined to be the total rotation number of any point $x \in \mathbf{T}^1 \times \{0\}$ relative to γ, i.e., $\mathcal{R}_\gamma(x, g)$. The infinitesimal rotation number of z, denoted $\rho(z, f)$, is $\mathrm{R}(x, g)$ which equals the fractional part of $\rho_\gamma(z, f)$.*

The value of $\rho(z, f)$ is easily obtained from $\mathrm{D}f_z$. If, for example, z has complex eigenvalues λ and $\bar{\lambda}$, then $\rho(z, f)$ is $\pm \arg(\lambda)/2\pi$. If $\mathrm{D}f_z$ is hyperbolic, then $\rho(z, f)$ is 0 if its eigenvalues are positive and 1/2 if they are negative.

Next we briefly review the basic results about Lyapounov functions and chain recurrence developed by Charles Conley in [C]. In the following, $f: \mathrm{X} \to \mathrm{X}$ will denote a homeomorphism of a compact metric space X.

(1.4) *Definition. — An ε-chain for f from x to y is a sequence of points $x_1, x_2, \ldots, x_n$, in X such that*

$$d(f(x_i), x_{i+1}) < \varepsilon \quad \textit{for } 1 \leqslant i \leqslant n-1,$$

and

$$d(x, x_1) < \varepsilon/2, \qquad d(y, f(x_n)) < \varepsilon/2.$$

A point $x \in \mathrm{X}$ is called chain recurrent if for every $\varepsilon > 0$ there is an ε-chain from x to itself. The set $\mathrm{R}(f)$ of chain recurrent points is called the chain recurrent set of f.

The definition of ε-chain from x to y is often given requiring the sequence $x_1, x_2, \ldots, x_n$ to satisfy $x_1 = x$, and $d(y, f(x_n)) < \varepsilon$. The form we give here results in an equivalent notion of chain recurrence and chain transitivity (see below) and is symmetric with respect to f and f^{-1}. In particular with the definition given above, if $x_1, x_2, \ldots, x_n$ is an ε-chain from x to y for f then $f(x_n), f(x_{n-1}), \ldots f(x_1)$ is an ε-chain from y to x for f^{-1}. The $\varepsilon/2$ at the ends is necessary so that the concatenation of a ε-chain from x to y with one from y to z is one from x to z.

It is easily seen that $R(f)$ is compact and invariant under f. If we define a relation $\sim$ on $R(f)$ by $x \sim y$ if for every $\varepsilon > 0$ there is an ε-chain from x to y and another from y to x, then it is clear that $\sim$ is an equivalence relation.

(1.5) *Definition. — The equivalence classes in* $R(f)$ *for the equivalence relation* $\sim$ *above are called the chain transitive components of* $R(f)$. *A compact invariant set* Λ *is called chain transitive if it is a subset of a single chain transitive component.*

The following result is well known and easy. A proof can be found in (1.2) of [F1].

(1.6) *Proposition. — If* X *is connected and* $R(f) = X$ *then* X *is chain transitive.*

(1.7) *Definition. — A complete Lyapounov function for* $f: X \to X$ *is a continuous function* $g: X \to \mathbf{R}$ *satisfying:*

a) if $x \notin R(f)$, *then* $g(f(x)) < g(x)$;

b) if $x, y \in R(f)$, *then* $g(x) = g(y)$ *if and only if* $x \sim y$ *(i.e.,* x *and* y *are in the same chain transitive component)*;

c) $g(R(f))$ *is a compact nowhere dense subset of* $\mathbf{R}$.

The following theorem is a result of C. Conley [C]. We have changed the setting from flows to homeomorphisms. For a proof in this setting see [F2]. For the smoothness see [W].

(1.8) *Theorem* [C]. — *If* $f: X \to X$ *is a homeomorphism of a compact metric space, then there is a complete Lyapounov function* $g: X \to \mathbf{R}$ *for* f. *If* X *is a manifold then* g *can be chosen* C^∞.

The main tool we use in proving the existence of periodic points with prescribed rational rotation numbers is the following theorem (see (2.2) and (2.4) of [F1]).

(1.9) *Theorem* [F1]. — *Suppose* $f: B \to B$ *is a homeomorphism of the annulus* $B = \mathbf{T}^1 \times [0, 1]$ *which is homotopic to the identity map and* $\Lambda \subset B$ *is a chain transitive compact invariant set. Let* $\pi: \widetilde{B} \to B$ *be the universal cover and let* $F: \widetilde{B} \to \widetilde{B}$ *be a lift of* f. *If* $x, y \in \pi^{-1}(\Lambda)$ *and*

$$\liminf_{n\to\infty} \frac{(F^n(x) - x)_1}{n} \leqslant \frac{p}{q} \leqslant \limsup_{n\to\infty} \frac{(F^n(y) - y)_1}{n},$$

then f *has a periodic point* z *with* $R_F(z_0) = p/q$ *for any* $z_0 \in \pi^{-1}(z)$.

2. Periodic points and rotation numbers

In this section we prove results about rotation numbers for diffeomorphisms of the plane with two fixed points. Given such a diffeomorphism f we can construct a homeomorphism of the annulus by first extending f to S^2, the one-point compactification of $\mathbf{R}^2$, setting $f(\infty) = \infty$, and then blowing up the two fixed points. We can then apply (1.9) to this annulus homeomorphism to obtain results about the original f. The fact that the annulus homeomorphism satisfies the hypothesis of (1.9) follows from the surprising result that the one-point compactification of an area preserving homeomorphism gives a homeomorphism with every point chain recurrent.

(2.1) *Proposition. — Suppose* M *is a non-compact connected manifold and* $f : \mathrm{M} \to \mathrm{M}$ *is a homeomorphism leaving invariant a measure* μ *which is positive on open sets and finite on compact sets. Let* $\mathrm{X} = \mathrm{M} \cup \{\infty\}$ *be the one point compactification of* M *and extend* f *to* X *by letting* $f(\infty) = \infty$. *Then every point of* X *is in the chain recurrent set* $\mathrm{R}(f)$ *of* $f : \mathrm{X} \to \mathrm{X}$.

Proof. — Let x be a point of X. We must show that x is chain recurrent. Suppose ε is given and let U be a disk of radius $\varepsilon/2$ centered at x. If there is $n \in \mathbf{Z}$, $n \neq 0$, such that $f^n(\mathrm{U}) \cap \mathrm{U} \neq \emptyset$, then there is an orbit segment $x_1, x_2, \ldots, x_n$ with $d(x, x_1) < \varepsilon/2$ and $d(x, x_n) < \varepsilon/2$. Hence in this case we have the desired ε-chain. If there is no such n then the sets $f^n(\mathrm{U})$, $n \geqslant 0$, are mutually disjoint, so

$$\mu\Big(\bigcup_{n \geqslant 0} f^n(\mathrm{U})\Big) = \infty.$$

It follows that this union is not contained in any compact subset of M and hence there is a ε-chain from x to ∞ in X. In a similar manner, considering the homeomorphism f^{-1}, one shows that x is chain recurrent or there is a ε-chain from ∞ to x. Hence in all cases x is chain recurrent. ∎

We remark that the result above does not imply that points $x \in \mathrm{M}$ are chain recurrent for $f : \mathrm{M} \to \mathrm{M}$. Chain recurrence is not a topological property on non-compact spaces, as it depends on the particular metric. This problem does not occur on compact metric spaces and in this paper we will only speak of chain recurrence for homeomorphisms of such spaces.

(2.2) *Theorem. — Suppose* $f : \mathbf{R}^2 \to \mathbf{R}^2$ *is an orientation preserving diffeomorphism leaving invariant a measure* μ *which is positive on open sets and finite on bounded sets and suppose* z_0, z_1 *are fixed points of* f. *Let* $\gamma : [0, 1] \to \mathbf{R}^2 - \{z_0, z_1\}$ *be a path with* $f(\gamma(0)) = \gamma(1)$. *If* p/q *is a non-integer rational number in the interval with endpoints* $-\rho_\gamma(z_0)$ *and* $\rho_\gamma(z_1)$, *then* f *possesses a periodic point* x *such that*

$$\mathscr{R}_\gamma(x, z_1) - \mathscr{R}_\gamma(x, z_0) = p/q.$$

Alternatively, if for some $u_0, u_1 \in \mathbf{R}^2$, $\mathscr{R}_\gamma(u_i, z_i)$ *exists, then for any non-integer* p/q *in the interval with endpoints* $-\mathscr{R}_\gamma(u_0, z_0)$ *and* $\mathscr{R}_\gamma(u_1, z_1)$, *there exists a periodic point with*

$$\mathscr{R}_\gamma(x, z_1) - \mathscr{R}_\gamma(x, z_0) = p/q.$$

Proof. — We prove the theorem for the interval with endpoints $-\rho_\gamma(z_0)$ and $\rho_\gamma(z_1)$; the other case has a nearly identical but slightly easier proof. We obtain a homeomorphism $g : \mathrm{A} \to \mathrm{A}$ of the annulus $\mathrm{A} = \mathbf{T}^1 \times [0, 1]$ as follows. First blow up the two fixed points z_0 and z_1 to obtain a self diffeomorphism g of the plane with two open disks removed. The measure on $\mathbf{R}^2$ lifts to a measure invariant under g. Next observe that the one-point compactification of this space is A. We extend g to A by setting $g(\infty) = \infty$. By (2.1) the chain recurrent set of g is all of A. Hence, by (1.7), g is chain transitive on A. Let $\pi : \tilde{\mathrm{A}} \to \mathrm{A}$ be the universal cover of A and let $\mathrm{G} : \tilde{\mathrm{A}} \to \tilde{\mathrm{A}}$ be the lift of g for which $\mathrm{G}(\Gamma(0)) = \Gamma(1)$, where Γ is a lift of the path γ. If C_i is the circle added when z_i was blown up equipped with the orientation inherited from the plane (counterclockwise), then C_0 and C_1 both represent generators of $\mathrm{H}_1(\mathrm{A})$, but with opposite sign. We will use the orientation of C_1 to pick a generator of the deck transformations of π for the purpose of defining rotation numbers $\mathrm{R}_\mathrm{G}(y)$ for $y \in \tilde{\mathrm{A}}$. If y_i is a point of $\tilde{\mathrm{A}}$ with $\pi(y_i) \in \mathrm{C}_i$, then it follows from the definitions that

$$\rho_\gamma(z_1) = \mathrm{R}_\mathrm{G}(y_1),$$

and

$$\rho_\gamma(z_0) = -\mathrm{R}_\mathrm{G}(y_0),$$

because $\rho_\gamma(z_i)$ is defined relative to the orientation of C_i.

By (1.9) there is a periodic point $x \in \mathrm{A}$ such that $\mathrm{R}_\mathrm{G}(w) = p/q$ for any point $w \in \pi^{-1}(x)$. Equivalently, all points of $\pi^{-1}(x)$ are fixed points of $\mathrm{T}^{-p} \circ \mathrm{G}^q$, where T is the generator of the deck transformations of π consistent with the orientation of C_1. Since p/q is not an integer we know that $x \neq \infty \in \mathrm{A}$.

Choose $x_0 \in \pi^{-1}(x)$ and let $\alpha : [0, 1] \to \mathrm{A}$ be a path from x to $g(x)$ formed as follows. Choose a path φ in the interior of $\mathrm{A} - \{\infty\}$ from $\pi(x_0) = x$ to $\gamma(0)$; follow it by γ to $\gamma(1)$, and then follow that by $g \circ \varphi$ parameterized backwards from $\gamma(1)$ to $g(x)$. Reparameterize the resulting path and call it α. Let $\alpha_0 : [0, 1] \to \tilde{\mathrm{A}}$ be the lift of α satisfying $\alpha_0(0) = x_0$. It is easy to see from standard covering space theory that

$$\mathrm{G}(\alpha_0(0)) = \mathrm{G}(x_0) = \alpha_0(1).$$

Form a path $\beta_0 : [0, 1] \to \tilde{\mathrm{A}}$ by connecting the paths $\mathrm{G}^i \circ \alpha_0$, for $i = 0, 1, \ldots, q-1$ and reparameterizing. Then β_0 is a path from x_0 to $\mathrm{G}^q(x_0)$ and $\mathrm{T}^p(\beta_0(0)) = \beta_0(1)$. If $\beta : [0, 1] \to \mathrm{A}$ is defined to be $\pi \circ \beta_0$, then β is a closed loop in A and its homology class $[\beta]$ in $\mathrm{H}_1(\mathrm{A})$ is equal to $p[\mathrm{C}_1]$, where $[\mathrm{C}_1]$ is the homology class of C_1.

Let $\mathrm{A}_0 = \mathrm{A} - \{\infty\}$. Then $\mathrm{H}_1(\mathrm{A}_0) = \mathbf{Z}^2$ with generators $[\mathrm{C}_0]$ and $[\mathrm{C}_1]$. In $\mathrm{H}_1(\mathrm{A}_0)$ the class $[\beta]$ is equal to $r[\mathrm{C}_1] + s[\mathrm{C}_0]$ for some integers r and s satisfying $r - s = p$. If we form the space B_0 by starting with A_0 and collapsing C_0 to a point which we call z_0, then in $\mathrm{H}_1(\mathrm{B}_0)$ the class $[\beta]$ is equal to $r[\mathrm{C}_1]$. Clearly $g : \mathrm{B}_0 \to \mathrm{B}_0$ can be identified with the homeomorphism obtained by blowing up z_1 for $f: \mathbf{R}^2 \to \mathbf{R}^2$. Let β_1, Γ_1 and φ_1 be lifts of β, Γ and φ respectively to the universal covering space of B_0, which are chosen to satisfy $\varphi_1(0) = \beta_1(0)$ and $\varphi_1(1) = \Gamma_1(0)$. By the construction of β, if G_0 is the lift of $g: B_0 \to B_0$ to this universal covering space which satisfies $G_0(\Gamma_1(0)) = \Gamma_1(1)$, then $G_0^q(\beta_1(0)) = \beta_1(1)$.

Thus

$$\mathcal{R}_\gamma(x, z_1, f) = R_{G_0}(\beta_1(0)) = r/q.$$

In a similar fashion one shows that $\mathcal{R}_\gamma(x, z_0, f) = s/q$. So

$$\mathcal{R}_\gamma(x, z_1, f) - \mathcal{R}_\gamma(x, z_0, f) = (r - s)/q = p/q. \quad \blacksquare$$

(2.3) *Definition. — Let $f: \mathbf{R}^2 \to \mathbf{R}^2$ be a diffeomorphism with a hyperbolic fixed point z. A branch of the stable (or unstable) manifold of z is z together with one component of $W^s(z) - \{ z \}$ (or $W^u(z) - \{ z \}$). We say a branch is properly embedded provided for some parameterization $\varphi : [0, \infty) \to X$, $\varphi(0) = z$, we have $\lim_{t \to \infty} \varphi(t) = \infty$.*

(2.4) *Corollary. — Let $f: \mathbf{R}^2 \to \mathbf{R}^2$ be a diffeomorphism satisfying the hypothesis of (2.2) with fixed points z_i, $i = 0, 1$. Suppose z_0 is hyperbolic with positive eigenvalues and one branch of either its stable or unstable manifold is properly embedded. If p/q is any non-integer rational number between 0 and $\rho_{z_0}(z_1, f)$, the total infinitesimal rotation number of z_1 relative to z_0, then there is a periodic point x whose total rotation number $\mathcal{R}_{z_0}(x, z_1, f)$ equals p/q.*

Proof. — Suppose that a branch of $W^u(z_0)$ is properly embedded; the other case is similar. Choose a path $\gamma : [0, 1] \to W^s(z_0, f)$ such that $f(\gamma(0)) = \gamma(1)$. This guarantees that $\rho_\gamma(z_0, f) = 0$ and that $\rho_\gamma(z_1, f) = \rho_{z_0}(z_1, f)$. We now proceed as in the proof of (2.2) but choose the path $\varphi : [0, 1] \to A$ so that it misses the compact embedded interval $J \subset A$ made up of the properly embedded branch of $W^u(z_0)$ together with the point ∞. It follows from the construction of β that no point of the image of β is in the interval J. It is then clear that if $[\beta] = r[C_1] + s[C_0]$ in $H_1(A_0)$, it must be the case that $s = 0$ and $r = p$. The remainder of the proof of (2.2) then shows that

$$\mathcal{R}_{z_0}(x, z_1, f) = \mathcal{R}_\gamma(x, z_1, f) = p/q. \quad \blacksquare$$

(2.5) *Corollary. — Suppose $f: \mathbf{R}^2 \to \mathbf{R}^2$ is an orientation preserving diffeomorphism leaving invariant a measure μ which is positive on open sets and finite on bounded sets and suppose z_0, z_1 are fixed points of f. At least one of the two intervals of $\mathbf{T}^1$ with endpoints $-\rho(z_0)$ and $\rho(z_1)$ has the property that if p/q is a non-zero rational point in its interior, then f possesses a periodic point x such that*

$$\mathcal{R}(x, z_1) - \mathcal{R}(x, z_0) = p/q.$$

The same is true for at least one of the intervals with endpoints 0 and $-\rho(z_0)$ and at least one of the intervals with endpoints 0 and $\rho(z_1)$.

Proof. — The first statement (using the endpoints $-\rho(z_0)$ and $\rho(z_1)$) is an immediate corollary of (2.2) obtained by reducing modulo 1 all the rotation numbers in (2.2).

To deal with the other cases we proceed as follows. Construct $g: A \to A$ as in (2.2) by compactifying and blowing up z_0 and z_1. Let $G: \tilde{A} \to A$ be the lift of g which fixes the points of $\pi^{-1}(\infty)$. Choose a path $\Gamma: [0, 1] \to \tilde{A}$ whose image lies in the interior of $\tilde{A}$ and is disjoint from $\pi^{-1}(\infty)$, and which satisfies $G(\Gamma(0)) = \Gamma(1)$. Let $\gamma: [0, 1] \to A$ be $\pi \circ \Gamma$. Observe that if $w \in \pi^{-1}(\infty)$, then $G(w) = w$, so $R_G(w) = 0$. As in (2.2), if y_1 is a point of $\tilde{A}$ with $\pi(y_1) \in C_1$, then

$$\rho_\gamma(z_1) = R_G(y_1).$$

It then follows from (1.9) that if p_0/q_0 is in the interval with endpoints 0 and $\rho_\gamma(z_1)$, there is a point $x \in A$ with $R_G(x_0) = p_0/q_0$, if $\pi(x_0) = x$. We then proceed exactly as in (2.2) to show that

$$\mathscr{R}_\gamma(x, z_1, f) - \mathscr{R}_\gamma(x, z_0, f) = p_0/q_0.$$

Reducing this modulo 1 gives the desired result. The case of p/q in the interval with endpoints 0 and $-\rho_\gamma(z_0)$ is handled similarly. ∎

Remark. — In (2.2), (2.4) and (2.5), the hypothesis that f is area preserving is used only to prove that the fixed points z_1, z_2 are in the same chain transitive component. Or equivalently, that for any $\varepsilon > 0$ there are ε-chains in both directions between the two circles added when these fixed points are blown up. Hence the area preserving hypothesis can be replaced by the hypothesis that z_1 and z_2 are in the same chain transitive component. For example, if they are both part of the same heteroclinic cycle (see (3.1) below) that would suffice.

3. Heteroclinic cycles

(3.1) *Definition.* — *A heteroclinic cycle for a diffeomorphism $f: \mathbf{R}^2 \to \mathbf{R}^2$ is a set of hyperbolic fixed points $\{p_i\}_{i=1}^n$ for f together with heteroclinic points $\{x_i\}_{i=1}^n$ with $x_i \in W^u(p_i) \cap W^s(p_{i+1})$ (where $p_{n+1} = p_1$).*

We do not exclude the possibility that $n = 1$ in the above definition. While such a cycle might more properly be called homoclinic rather than heteroclinic, we will use the term heteroclinic in this case too, for simplicity of expression.

(3.2) *Definition.* — *A simple heteroclinic cycle for a diffeomorphism $f: \mathbf{R}^2 \to \mathbf{R}^2$ is a heteroclinic cycle as above such that*

a) the eigenvalues of Df at p_i are positive,

b) the segments of $W^u(p_i)$ bounded by p_i and x_i and the segments of $W^s(p_{i+1})$ bounded by x_i and p_{i+1} form a Jordan curve in $\mathbf{R}^2$,

c) if D is the disk bounded by this Jordan curve, then the points $\{p_i\}_{i=1}^n$ are in the closure of the unbounded component of the complement of $D \cup f(D)$.

Note that condition *c)* will always be satisfied if $D \cup F(D)$ is a topological disk. Also condition *a)* implies that f must be orientation preserving.

We next investigate the chain transitive components for the one-point compactification of a surface diffeomorphism on a non-compact surface M^2 (but which may have non-empty compact boundary). Suppose $f: M^2 \to M^2$ is a diffeomorphism which possesses a heteroclinic cycle with hyperbolic fixed points $\{ p_i \}_{i=1}^n$ and heteroclinic points $\{ x_i \}_{i=1}^n$. Let $X = M^2 \cup \{ \infty \}$ be the one-point compactification of M^2 and extend f to X by letting $f(\infty) = \infty$.

(3.3) *Lemma. — Let* U *be any component of the complement in* X *of the segments of* $W^u(p_i)$ *with endpoints* p_i *and* x_i *and the segments of* $W^s(p_{i+1})$ *with endpoints* x_i *and* p_{i+1}. *Suppose there is a measure* μ *on* U *which is positive on open sets, and satisfies* $\mu(V) = \mu(f(V))$ *whenever* $V \cup f(V) \subset U$. *Then, if* $x \in U$ *is a chain recurrent point of* $f: X \to X$, *it is in the same chain transitive component as each of the points* $\{ p_i \}_{i=1}^n$.

Proof. — Clearly the points $\{ p_i \}_{i=1}^n$ are all in the same chain transitive component Λ since they lie on a heteroclinic cycle. Suppose $x \notin \Lambda$. Then by (1.8) there is a smooth complete Lyapounov function $g: X \to \mathbf{R}$. We assume that $g(x) < g(\Lambda)$ (if $g(x) > g(\Lambda)$ replace f by f^{-1}).

Let $c \in \mathbf{R}$ be a regular value of g such that $g(x) < c < g(\Lambda)$ and let $C_0 = g^{-1}(c)$, so C_0 is the boundary of the surface $X_0 = g^{-1}((-\infty, c])$ and $x \in X_0$. Since points of C_0 are not in $R(f)$ it follows that $f(X_0)$ is in the interior of X_0. Since $p_i \notin C_0$, there is $n > 0$ such that $C = f^n(C_0)$ is disjoint from the segment of $W^u(p_i)$ with endpoints p_i and x_i for all i. For any $z_1 \in C$ we have $g(z_1) < c < g(\Lambda)$ and for any $z_2 \in W^s(p_i)$ we have $g(z_2) > g(\Lambda)$, so it is clear that C is disjoint from $W^s(p_i)$ for all i as well.

Since C is the boundary of the surface $X_1 = f^n(X_0)$ and $x \in X_1$, it follows that $X_1 \subset U$. But $f(X_1)$ is strictly contained in the interior of X_1, which is not possible since $\mu(f(X_1)) = \mu(X_1)$. This contradicts the assumption that $x \notin \Lambda$. ∎

We are now prepared for a result on the existence of periodic orbits with prescribed rotation number when one has a simple homoclinic cycle. Although the hypothesis of this result is somewhat stronger than the result of § 2, the periodic orbits obtained are entirely inside the disk bounded by the homoclinic cycle so we obtain much more information about their location.

(3.4) *Theorem. — Let* $f: \mathbf{R}^2 \to \mathbf{R}^2$ *be a diffeomorphism leaving invariant a measure* μ *which is positive on open sets and finite on bounded sets. Suppose f possesses a simple heteroclinic cycle bounding a disk* D *and* z *is a fixed point of f contained in* D. *Let* p/q *be a rational number between* 0 *and* $\rho_{p_i}(z, f)$, *the total infinitesimal rotation number relative to* p_i, *for some* i. *Then* D *contains a periodic point* x *whose total rotation number around* z, $\mathcal{R}_{p_i}(x, z, f)$, *is equal to* p/q. *The entire orbit of* x *lies in* D.

Proof. — Let X denote the one-point compactification $\mathbf{R}^2 \cup \{ \infty \}$ and extend f to it by setting $f(\infty) = \infty$. Our object is to restrict the diffeomorphism f to D and extend that to a homeomorphism h of X in such a way that the dynamics of all points outside D is easily understood. In particular we will arrange that for h every point outside D has either a forward orbit limiting on p_i for some i, or a backward orbit with limit ∞. We will then prove that h has a periodic orbit (which must necessarily lie in D) with the desired rotation number.

We proceed with this program. Property (3) of the definition of simple heteroclinic cycle implies that there is a point $q_i \in W^s_\varepsilon(p_i)$ which is in the component of ∞ in the complement of $D \cup f(D)$. Let J_i be the arc in $W^s(p_i)$ with endpoints q_i and x_{i-1}, and note that $p_i \in J_i$ (see Fig. 2). We can choose embeddings $\alpha_i : [0, 1] \to X$ such that $\alpha_i(0) = q_i$, $\alpha_i(1) = \infty$ and $\alpha_i(t)$ is disjoint from $D \cup \{\infty\}$ for other values of t. Clearly we can choose these embeddings to be disjoint except at ∞. The fact that p_i is on the boundary of the component of ∞ in the complement of $D \cup f(D)$ also implies that the same is true for x_i. Hence we can find embeddings $\beta_i : [0, 1] \to X$ with all the properties of α_i except $\beta_i(0) = x_i$ and we can choose them so that their images are disjoint from all the alphas, except at ∞.

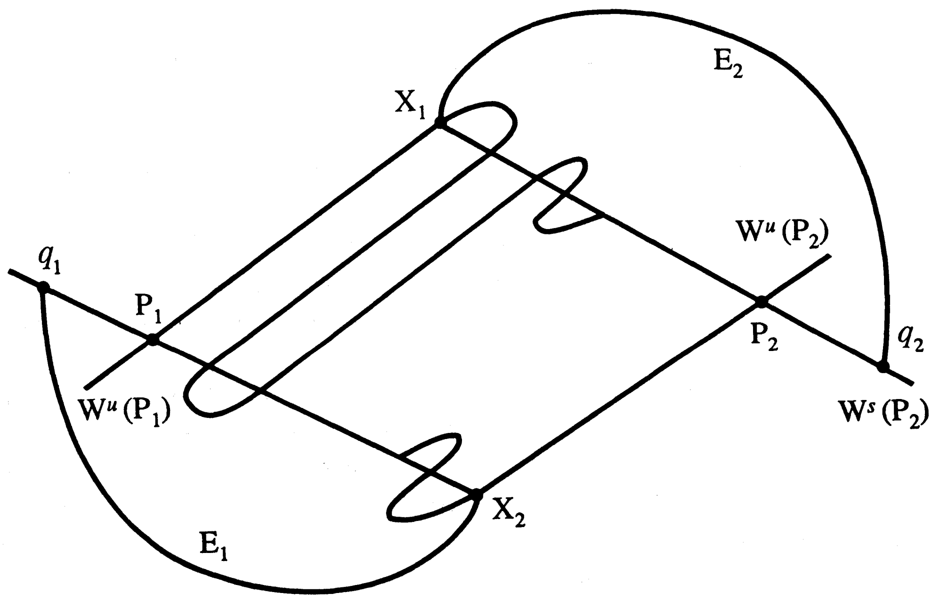

Fig. 2

The embedded arcs J_i, α_i and β_{i-1} form a Jordan curve bounding a disk whose intersection with $D \cup f(D)$ is the arc J_i. Shrinking this disk in along the parts of its boundary near ∞ we obtain a closed embedded disk E_i (see Fig. 2) with the following properties:

a) E_i is disjoint from E_j if $i \neq j$;

b) $E_i \cap (D \cup f(D)) = J_i$;

c) $D \cup (\bigcup_i E_i)$ is a closed topological disk disjoint from $\{\infty\}$.

We will apply the following lemma to this situation.

(3.5) *Lemma. — Suppose* E *is a topological disk in* $\mathbf{R}^2$ *whose boundary consists of two embedded arcs* I *and* J. *Let* $h_0 : \mathrm{J} \to \mathrm{J}$ *be a one-to-one contraction mapping of* J *to a subinterval of itself which preserves orientation and has fixed point* p. *Then* h_0 *extends to an embedding* h *of* E *to a subset of itself with the following properties:*

a) $h(\mathrm{E}) \subset (h(\mathrm{J}) \cup \mathrm{E}^0)$, *where* E^0 *denotes the interior of* E;

b) $\lim_{n \to \infty} h^n(x) = p$ *for all* x *in* E.

If h_0 *is also defined on neighborhoods of the endpoints of* I *and embeds the union of* J *and these subintervals in* E, *then we can arrange that* $h = h_0$ *on (perhaps smaller) neighborhoods of the endpoints of* I.

Proof. — Let $\mathrm{E}_0 = \{(x,y) \in \mathbf{R}^2 \mid y \geqslant 0,\ x^2 + y^2 \leqslant 1\}$ and let $\mathrm{J}_0 = \{(x,y) \in \mathrm{E}_0 \mid y = 0\}$. Define $g : \mathrm{E}_0 \to \mathrm{E}_0$ by $g(x,y) = (x/2, y/2)$. Choose a homeomorphism $\varphi : \mathrm{J} \to \mathrm{J}_0$ which conjugates g and h, i.e. choose φ so that $g(\varphi(x)) = \varphi(h(x))$. Extend φ to a homeomorphism of the arc I onto the arc $\{(x,y) \in \mathrm{E}_0 \mid x^2 + y^2 = 1\}$. We now have a homeomorphism from the boundary of E_0 to the boundary of E which we extend to a homeomorphism $\varphi : \mathrm{E}_0 \to \mathrm{E}$. Define $h : \mathrm{E} \to \mathrm{E}$ by $h(y) = \varphi(g(\varphi^{-1}(y)))$. Then $h = f$ on J and h on E is topologically conjugate to g on E_0.

If h_0 is defined on neighborhoods of the endpoints of I as well as on J, let Z be the union of two small intervals in I which contain the endpoints and on which h_0 is defined. Starting with φ defined on J as above we can extend it to $\mathrm{J} \cup \mathrm{Z} \cup h(\mathrm{Z})$ in such a way that $\varphi(\mathrm{Z})$ consists of two intervals in the semi-circular segment of the boundary of E_0 which contain the endpoints of this segment, and so that $g(\varphi(x)) = \varphi(h(x))$ for $x \in \mathrm{Z}$. If we then extend φ to all of E_0 as above and define $h : \mathrm{E} \to \mathrm{E}$ by $h(y) = \varphi(g(\varphi^{-1}(y)))$, we have the desired embedding. ∎

We now return to the proof of (3.4). Note that the disks E_i satisfy the hypothesis of (3.5) if we take J_i as J. Recall that our object is to restrict the diffeomorphism f to D and extend that to a homeomorphism h of X in such a way that every point outside D has either a forward orbit limiting on p_i for some i, or a backward orbit with limit ∞. To do this we define $h(x) = f(x)$ if $x \in \mathrm{D}$ and let $h : \mathrm{E}_i \to \mathrm{E}_i$ be the extension of h on J_i given by (3.5). By construction, for any x in J_i or in a neighborhood of x_{i-1} or p_i in the boundary of D_i, we have $h(x) = f(x)$. We have now defined h on $\mathrm{D} \cup (\bigcup_i \mathrm{E}_i)$ which is topologically a closed disk. We need to alter this disk slightly so that h maps its boundary into its interior.

To do this we choose a smooth arc γ_i from q_i to a point of the arc β_i near to x_i (see Fig. 3). We do this in such a way that γ_i is very close to the arcs made up of the segment of $\mathrm{W}^s(p_i)$ with endpoints q_i and p_i, and the segment of $\mathrm{W}^u(p_i)$ with endpoints p_i and x_i. These two segments together with γ_i and a short segment of β_i bound a thin strip we will denote by Y_i (see Fig. 3). If the segment of β_i is sufficiently short then $h = f$ on this segment. By choosing γ_i properly we can arrange that if x is a point of γ_i then $f(x)$ is in the interior of $\mathrm{Y}_i \cup \mathrm{D} \cup (\bigcup_i \mathrm{E}_i)$. Further we can arrange that if x is a point of γ_i or in the interior of Y_i then, for some n, $f^{-n}(x) \notin \mathrm{Y}_i$ (and is also outside of D).

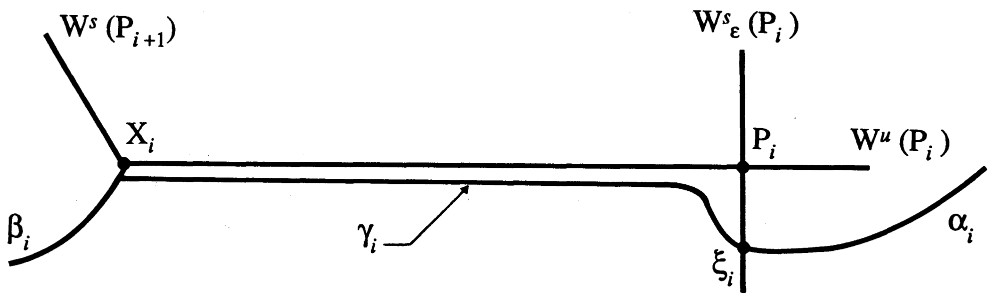

Fig. 3

Let Q denote the topological disk $U \cup (\bigcup_i E_i) \cup (\bigcup_i Y_i)$. Extend h to Q by setting $h = f$ on Y_i. Then by construction $h(Q)$ is contained in the interior of Q.

We complete the extension of h to all of X by letting P be the complement of $h(Q)$ in X and defining h^{-1} on P. Thus, P is a topological disk, h^{-1} is already defined on its boundary and, in fact, h^{-1} carries the boundary into the interior of P. It is well known and easy to prove by methods similar to those of (3.5) that under these circumstances h^{-1} can be extended to an embedding of P into its interior with a single attracting fixed point at ∞. We can also arrange that h be smooth in a neighborhood of ∞ and that the infinitesimal rotation number at ∞ be irrational. It is now the case that, if $x \in E_i$, then

$$\lim_{n\to\infty} h^n(x) = p_i,$$

and if $x \in P$ or $x \in Y_i - D$, then

$$\lim_{n\to\infty} h^{-n}(x) = \infty.$$

In any case there are no periodic points other than ∞ outside of D. Since $h = f$ on D, the measure μ is preserved there.

By lemma (3.3) if we blow up the point $z \in D$ and z_0 is a chain recurrent point of the circle added in the blow up, then z_0 and p_i are in the same chain transitive component. If we further blow up ∞ to form a homeomorphism $h : A \to A$, where $A = \mathbf{T}^1 \times [0, 1]$, then z_0 and p_i remain in the same chain transitive component. Let $\pi : \widetilde{A} \to A$ be the universal covering space and let $F : \widetilde{A} \to \widetilde{A}$ be the lift of h which fixes points of $\pi^{-1}(p_i)$. Then, for $w \in \pi^{-1}(p_i)$,

$$\lim_{n\to\infty} \frac{(F^n(w) - w)_1}{n} = 0,$$

and for $y_0 \in \pi^{-1}(z_0)$,

$$\lim_{n\to\infty} \frac{(F^n(y_0) - y_0)_1}{n} = R_G(y_0) = \rho_{p_i}(z, f).$$

Thus by (1.9) there is a periodic point $x \in A$ with $R_F(y) = p/q$ if $\pi(y) = x$. From the definition it is clear that

$$\mathscr{R}_{v_i}(x, z, f) = R_F(y) = p/q.$$

Since the points on the circle added when ∞ was blown up all have irrational rotation number, x is not one of them. Thus it must be the case that $x \in D$. Since the same argument applies to $f^i(x)$, the entire orbit of x must be in D. ∎

4. The rotation number at infinity

Some interesting applications of the results in § 2 and § 3 are obtained by their direct application to diffeomorphisms of the plane obtained as the lift to the universal covering space of an area preserving diffeomorphism of a surface (even a non-compact surface). In this section we deal with a similar setting but add one new ingredient—the rotation number at infinity.

Let M^2 be an oriented connected surface and suppose $\pi : U \to M^2$ is its universal covering space. We allow M^2 to have compact boundary components and to be non-compact. If the Euler characteristic of M^2 is negative or M^2 is $\mathbf{T}^2$ or $\mathbf{T}^1 \times [0, 1]$, there is a natural compactification of U which we will denote $\overline{U}$, and which is topologically a closed disk. If M^2 has no boundary, then the boundary circle of $\overline{U}$ consists of " ideal points " at infinity. Otherwise it contains some such points as well as points whose images under π lie in the boundary of M^2. If F is the lift to U of a homeomorphism of M^2, it extends to a homeomorphism of all of $\overline{U}$. If F_t is the lift of an isotopy on M^2 and M^2 has no boundary, then the extension to the circle at infinity is independent of t (see [Th] for some of these facts).

(4.1) *Definition. — Suppose* M^2 *is a surface as described above and* $f: M^2 \to M^2$ *is an orientation preserving diffeomorphism with fixed point* z *and suppose* $\pi : U \to M^2$ *is the universal covering space. Let* $F : U \to U$ *be a lift of* f *which fixes a point* $z_0 \in \pi^{-1}(z)$. *The rotation number at infinity with respect to the fixed point* z *is the rotation number of* F *restricted to the circle at infinity of* $F : \overline{U} \to \overline{U}$.

It is easy to see that the rotation number at infinity is independent of the choice of $z_0 \in \pi^{-1}(z)$ since any other choice would result in a different lift but one which is conjugate to F by a covering transformation. Hence the two lifts extend to conjugate homeomorphisms on the circle at infinity.

(4.2) *Proposition. — Suppose* $\chi(M^2) < 0$, *or* $M^2 = \mathbf{T}^2$, *or* $M^2 = \mathbf{T}^1 \times [0, 1]$, *and* $f: M^2 \to M^2$ *is an area preserving, orientation preserving diffeomorphism with fixed point* z *and suppose* $\pi : U \to M^2$ *is the universal covering space. Let* $F : U \to U$ *be a lift of* f *which fixes a* *point* $z_0 \in \pi^{-1}(z)$. *Then for any* p/q *in the interior of an interval of* $\mathbf{T}^1$ *with endpoints* $\rho(z_0)$, *the infinitesimal rotation number of* z_0, *and the rotation number at infinity of* f *with respect to* z, *there is a periodic point* $x \in \mathrm{U}$ *of* F *such that* $\mathscr{R}(x, z_0, \mathrm{F}) = p/q$.

Proof. — Consider $\mathrm{F} : \overline{\mathrm{U}} \to \overline{\mathrm{U}}$ and the invariant (infinite) measure μ on it obtained from lifting the measure on M^2. We first observe that for any x in the interior of $\overline{\mathrm{U}}$ either $x \in \mathrm{R}(\mathrm{F})$ or for any ε there are ε-chains from x to the circle at infinity and from the circle at infinity to x. The proof of this is almost identical to the proof of (2.1): because of the invariant measure, a small neighborhood of x must either have points which return under iteration or points which get outside of any compact set.

To apply (1.6) and conclude that F is chain transitive on $\overline{\mathrm{U}}$, we need only show that the points on the circle at infinity are in the chain recurrent set R(F). If the rotation number at infinity is irrational this is the case since the circle homeomorphism must be conjugate to an irrational rotation or a Denjoy type example. If the rotation number at infinity is rational there is a periodic point p_0 on the circle at infinity. In this case there is an isotopy on the circle, which starts with the given homeomorphism on the circle at infinity and ends with a finite order homeomorphism, and which has the property that at every stage of the isotopy the action on the orbit of p_0 is the same. We now attach a collar neighborhood $\mathrm{S}^1 \times [0, 1]$ to the boundary of $\overline{\mathrm{U}}$ and extend F to the union by having it preserve concentric circles of the collar and act on them in the way prescribed by the isotopy.

Denote this union by D and the extended homeomorphism by $\mathrm{F} : \mathrm{D} \to \mathrm{D}$. Note that for this homeomorphism all points in the collar neighborhood are chain recurrent and that they all have the same rotation number, namely the rotation number at infinity of f with respect to z. (In the irrational case there is no need to add a collar: we let $\mathrm{D} = \overline{\mathrm{U}}$.) We can now apply (1.6) and conclude that F is chain transitive on D.

Next we blow up the fixed point z_0 of F in D to obtain a homeomorphism of an annulus. All points of the circle added in blowing up are chain recurrent. This is because one can lift the measure on U to one on U with z_0 blown up and repeat the argument of (2.1) mentioned above which shows that every point y in this circle is either chain recurrent or for any ε there are ε-chains from y to the circle at infinity in $\overline{\mathrm{U}}$ and from the circle at infinity to y. But this also implies that y is chain recurrent on D with z_0 blown up. It is thus clear by (1.6) that the homeomorphism on the annulus is chain transitive.

Hence we can apply (1.9) to obtain a periodic point x with rotation number p/q. This point cannot be in the collar we may have added because all points in the collar have rotation number equal to an endpoint of the interval we are considering, and p/q is in the interior. It follows that x, considered as a point of U, is the desired point. ■

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