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ON C^1 -STABILITY AND C^1 -DETERMINACY

by A. A. DU PLESSIS and C. T. C. WALL

Introduction

In Thom's 1959 lectures [20], he formulates some general problems about stability of smooth mappings. He sketches a proof that C^2 -stable maps are not dense in $C^\infty(N, P)$ for manifolds N and P of dimension 9, and conjectures that density does hold for any N and P for C^0 -stable maps ("weak conjecture") and even for C^1 -stable maps ("feeble conjecture").

Density of C^0 -stable maps (at least when N is compact) was established by Mather [9], [10], [6]. In contrast, the purpose of this paper is to show that the C^1 -stable maps are dense only if $(\dim N, \dim P)$ lies in the "nice dimensions" where (for N compact) the C^∞ -stable maps are already dense ([8], I-VI). The difference between these results rests essentially on the fact that, frequently, the intrinsic parts of higher derivatives are already invariant under C^1 -equivalence; the simplest case is shown in (4.3) below.

In order to deduce the main result from this, we need an elaboration of Mather's stability theory. A variant of the same arguments yields a parallel result where C^1 -stability is replaced by ∞ - C^1 -determinacy at all points, and the nice dimensions are replaced by the "semi-nice dimensions" [15], [26] (apart from six undecided cases).

Combining these arguments with some techniques of the first author [14] yields examples of manifolds N and P where C^1 -stable maps N and P not only fail to be dense but fail to exist at all; there are even examples where there are no maps $N \rightarrow P$ with all point-germs ∞ - C^1 -determined.

The plan of the paper is as follows. In § 0 we give a full discussion of our results, with arguments reducing the main theorem to more technical results which are proved in later sections. In § 1 we apply a lemma on algebraic group actions to contact classes in a jet space. In § 2 we introduce several notions of C^1 -invariance of submanifolds of jet space and show that stability and determinacy conditions imply at least a weak form of transversality here (the preimage is a C^1 -submanifold). In § 3 we use unfoldings and a perturbation lemma of May [12] to strengthen this to actual transversality. Finally in § 4 we prove the C^1 -invariance of Thom-Boardman varieties and, in some cases, of \mathcal{N}^r -orbits within them.

This paper is dedicated to René Thom, whose pioneering ideas and insights have been a continuing inspiration to both of us.

0. Statement of results

Our notation and terminology concerning germs, jets, and their equivalence relations will follow that of Mather [8, III]. In particular, we write \mathcal{K} for contact equivalence of germs, \mathcal{K}^r for that of r -jets, and \mathcal{A} for right-left-equivalence of germs. As in [27], $\mathcal{A}^{(\ell)}$ denotes the equivalence relation on germs (often called C^ℓ -equivalence) defined by composing on right and on left with C^ℓ -diffeomorphism germs. Here, and below, “ C^0 -diffeomorphism” is to be interpreted as “homeomorphism”.

A space of C^∞ -mappings has several topologies. We denote by τ_{C^r} the weak- or compact-open- C^r -topology and by τ_{W^r} the strong- or Whitney- C^r -topology of [8, II] for $0 \leq r \leq \infty$; and refer to τ_{C^∞} and τ_{W^∞} simply as the weak and strong topologies.

(0.1) Definitions.

A) (i) Two C^∞ maps $f_i: N_i \rightarrow P_i$ ($i = 1, 2$) are C^ℓ -equivalent ($0 \leq \ell \leq \infty$) if there exists C^ℓ -diffeomorphisms $h: N_1 \rightarrow N_2$, $k: P_1 \rightarrow P_2$ such that $f_2 = k \circ f_1 \circ h^{-1}$.

(ii) A C^∞ map $f: N \rightarrow P$ is C^ℓ -stable ($0 \leq \ell \leq \infty$) if there exists a τ_{W^∞} -neighbourhood \mathcal{W} of f in $C^\infty(N, P)$ such that every map in \mathcal{W} is C^ℓ -equivalent to f .

B) (i) Two C^∞ map-germs $f_i: (N_i, x_i) \rightarrow (P_i, y_i)$ ($i = 1, 2$) are $\mathcal{A}^{(\ell)}$ -equivalent ($0 \leq \ell \leq \infty$) if there exist C^ℓ -diffeomorphism-germs $h: (N_1, x_1) \rightarrow (N_2, x_2)$, $k: (P_1, y_1) \rightarrow (P_2, y_2)$ such that $f_2 = k \circ f_1 \circ h^{-1}$.

(ii) A C^∞ map-germ $f: (N, x) \rightarrow (P, y)$ is r - $\mathcal{A}^{(\ell)}$ -determined ($1 \leq r \leq \infty$, $0 \leq \ell \leq \infty$) if every C^∞ map-germ with the same r -jet as f is $\mathcal{A}^{(\ell)}$ -equivalent to f ; f is *finitely- $\mathcal{A}^{(\ell)}$ -determined* if it is r - $\mathcal{A}^{(\ell)}$ -determined for some $r < \infty$.

In a series of important papers ([8], I-VI) Mather showed that the set of C^∞ -stable mappings $f: N^n \rightarrow P^p$ (N compact) is τ_{W^∞} -dense in $C^\infty(N, P)$ only for certain “nice” dimensions (n, p) , but ([9], [10], see also [6]) that the C^0 -stable mappings are always dense.

There is a similar situation for determinacy. Finite \mathcal{A} -determinacy holds “in general” (i.e. off a subset of infinite codimension) only in certain “semi-nice” dimensions [15]. (The nice dimensions $n < {}^1\sigma(n, p)$, the semi-nice dimensions $n \leq {}^2\sigma(n, p)$, and the functions ${}^1\sigma$ and ${}^2\sigma$ that define them, are defined below and calculated in (0.7).) But, as Thom conjectured (in [21]) and Varchenko proved ([22], [23], see also [14] for a simpler proof, along the lines suggested by Thom), finite $\mathcal{A}^{(0)}$ -determinacy always holds in general.

In the C^1 case, much less is known. In [24] a proof was sketched that C^1 -stable maps are not dense in $C^\infty(N^n, P^p)$ for $n = 8$, $p = 6$. Mather proved in [11] that finite

$\mathcal{A}^{(1)}$ -determinacy does not hold in general for map-germs $(\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^{n-1}, 0)$ with $n \geq 15$. The main result of this paper is

(0.2) Theorem.

A) If $n \geq {}^1\sigma(n, p)$, then for any smooth manifolds N^n, P^p there is a non-empty, τ_{C^∞} -open subset $\mathcal{U} \subseteq C^\infty(N, P)$ containing no C^1 -stable mapping.

B) If $n - p \neq 0, 1, 2$ and $n > {}^2\sigma(n, p)$, or if $n - p = 0, 1, 2$ and $n > {}^2\sigma^2(n, p)$, then for any smooth manifolds N^n, P^p there is a non-empty τ_{C^∞} -open subset $\mathcal{U} \subseteq C^\infty(N, P)$ containing no mapping all of whose point-germs are ∞ - $\mathcal{A}^{(1)}$ -determined.

It follows from the calculations recalled in (0.7) that ${}^1\sigma(8, 6) = 8$ and ${}^2\sigma^2(n, n-1) = 12$ for $n \geq 4$, so this includes the results of [11], [24] just mentioned. It also follows that the result of B includes all the semi-nice dimensions except $(14, 14)$, $(15, 15)$, $(16, 16)$, $(12, 11)$, $(14, 12)$ and $(15, 13)$.

We will prove (0.2) by following the lines suggested by a (new) proof of the corresponding C^∞ result:

(0.3) Theorem.

A) If $n \geq {}^1\sigma(n, p)$, then, for any manifolds N, P of dimensions n, p respectively, there is a non-empty τ_{C^∞} -open subset $\mathcal{U} \subset C^\infty(N, P)$ containing no C^∞ -stable mapping.

B) If $n > {}^2\sigma(n, p)$, then, for any manifolds N, P of dimensions n, p respectively, there is a non-empty τ_{C^∞} -open subset $\mathcal{U} \subset C^\infty(N, P)$ containing no mapping all of whose point-germs are ∞ - \mathcal{A} -determined.

The proof uses the following two lemmas:

(0.4) Lemma.

A) If the smooth map $f: N \rightarrow P$ is C^∞ -stable, then $J^r f$ is transverse to all \mathcal{K}^r -classes in $J^r(N, P)$ for any $r < \infty$.

B) If the smooth map-germ $f: (N, x_0) \rightarrow (P, y_0)$ is ∞ - \mathcal{A} -determined, then it has a representative $\tilde{f}: U \rightarrow V$ such that $J^r \tilde{f}|_{U - \{x_0\}}$ is transverse to all \mathcal{K}^r -classes in $J^r(N, P)$ for any $r < \infty$.

(0.5) Lemma. — Let n, p, i be positive integers. Suppose that, for some $r > 1$, there exists a non-empty \mathcal{K}^r -invariant submanifold W of $J^r(n, p)$ of codimension $\leq n - i + 1$ supporting a smooth \mathcal{K}^r -invariant foliation \mathcal{F} of codimension $\geq i$.

Then, for any manifolds N, P of dimensions n, p respectively, there exists a non-empty τ_{C^∞} -open subset \mathcal{U} of $C^\infty(N, P)$ such that for any $f \in \mathcal{U}$ there exists a non-empty submanifold Z_f of N of dimension $\geq i - 1$ such that $J^r f$ is not transverse to the fibres of $\mathcal{F}(N, P)$ at any point of Z_f .

We prove (0.4) in (2.4): the result is essentially due to Mather (see [8, V] for A and [27] for B). (0.5) is more delicate, and is proved in (3.12).

We now give the definition of the numbers ${}^i\sigma(n, p)$ ($i = 1, 2, \dots$). For $d \geq 0$, write

$$W_d^r(n, p) = \{ z \in J^r(n, p) \mid \text{codim}(\mathcal{K}^r.z) \geq d \}.$$

This is an algebraic subvariety of $J^r(n, p)$. Let ${}^k W_d^r(n, p)$ be the union of its irreducible components with codimension (in $J^r(n, p)$) $\leq (d - k)$, and let ${}^k W^r(n, p) = \bigcup_{d \geq 0} {}^k W_d^r(n, p)$: “the set of r -jets with \mathcal{K}^r -modality $\geq k$ ”. This is a finite union, so ${}^k W^r(n, p)$ is an algebraic subvariety of $J^r(n, p)$: we denote its codimension by ${}^k \sigma^r(n, p)$ (and write ${}^k \sigma^r(n, p) = \infty$ if ${}^k W^r(n, p) = \emptyset$).

If $\pi^{s,r}: J^s(n, p) \rightarrow J^r(n, p)$ denotes the natural projection ($s > r$), we have $(\pi^{s,r})^{-1} {}^k W^r(n, p) \subseteq {}^k W^s(n, p)$, so the sequence $\{ {}^k \sigma^r(n, p) : r = 1, 2, \dots \}$ is non-increasing and non-negative, hence eventually constant. This constant value is denoted ${}^k \sigma(n, p)$.

We can now find foliated (indeed fibred) submanifolds of ${}^k W^r(n, p)$ of the type wanted for applications of (0.5):

(0.6) Lemma. — *Suppose that ${}^k W^r(n, p) \neq \emptyset$, and let C be a component of it of maximal dimension.*

Then C contains, as a Zariski-open and -dense subset, a \mathcal{K}^r -invariant submanifold W admitting a smooth, algebraic submersion $\pi: W \rightarrow L$ whose fibres are finite unions of \mathcal{K}^r -orbits. (Thus $\text{cod. } W = {}^k \sigma^r(n, p)$, while the fibres of π have constant codimension $\geq k$ in W .)

We prove this (not very surprising) result in § 1.

Proof of (0.3).

A) Since $n \geq {}^1\sigma(n, p)$, (0.6) shows that the hypotheses of (0.5) can be satisfied for the given n, p and $i = 1$. Let N, P be smooth manifolds of dimensions n, p respectively. It follows at once from (0.4) A that no mapping in the τ_{C^∞} -open subset \mathcal{U} of $C^\infty(N, P)$ promised by Lemma (0.5) is C^∞ -stable, and the proof is complete.

B) Since $n > {}^2\sigma(n, p)$, (0.6) shows that the hypotheses of (0.5) can be satisfied for the given n, p and $i = 2$. Let N, P be smooth manifolds of dimensions n, p respectively.

It follows at once from (0.4) B that no mapping in the τ_{C^∞} -open subset \mathcal{U} of $C^\infty(N, P)$ promised by Lemma (0.5) has all its point-germs ∞ - \mathcal{A} -determined, and the proof is complete. \square

All this is very abstract; but the results have a more concrete form arising from the calculations of ${}^1\sigma(n, p)$ (carried out in [8], VI) and ${}^2\sigma(n, p)$ (carried out in [25] for $n > p$, [4] for $n = p$, and [26] for $n < p$).

We have (as follows at once from the results of these articles):

(0.7) Proposition.

A) a) If $n < 4$ or $p < 2$, then $n < {}^1\sigma(n, p)$.

b) If $n \geq 4$ and $p \geq 2$, then for $n - p$ satisfying the conditions displayed, the \mathcal{K}^r -invariant submanifold ${}^1S^r$ given below is an open dense subset of a component of ${}^1W^r(n, p)$ of codimension ${}^1\sigma(n, p)$ in $J^r(n, p)$:

$n - p$	${}^1S^r$	${}^1\sigma(n, p)$
≥ 3	$(\pi^{3,2})^{-1} \Sigma^{n-p+1,3}$	$n - p + 7$
2	$(\pi^{2,1})^{-1} \Sigma^4$	8
1	$(\pi^{4,3})^{-1} \Sigma^{n-p+1,2,2}$	9
0, -1, -2, -3	$\Sigma^{3,(3)}$	$6(p - n) + 9$
≤ -4	$\Sigma^{4,(2)}$	$6(p - n) + 8$

B) a) If $n < 4$ or $p < 2$, then $n < {}^2\sigma(n, p)$.

b) If $n \geq 4$ and $p \geq 2$, then for $n - p$ satisfying the conditions displayed, the \mathcal{K}^r -invariant submanifold ${}^2S^r$ given below is an open dense subset of a component of ${}^2W^r(n, p)$ of codimension ${}^2\sigma(n, p)$ in $J^r(n, p)$:

$n - p$	${}^2S^r$	${}^2\sigma(n, p)$
≥ 7	$(\pi^{3,2})^{-1} \Sigma^{n-p+1,4}$	$n - p + 11$
7, 6, 5, 4, 3	$(\pi^{2,1})^{-1} \Sigma^{n-p+2}$	$2(n - p) + 4$
2	$(\pi^{3,2})^{-1} \Sigma^4\{(1, 1, 1), 1\}$	13
1	$(\pi^{3,2})^{-1} \Sigma^3\{(1, 1, 1)\}$	11
0	$(\pi^{3,2})^{-1} \Sigma^{3,(4)}$	13
-1, -2, -3	$(\pi^{2,1})^{-1} \Sigma^4$	$4(p - n) + 16$
≤ -4	$\Sigma^{4,(3)}$	$7(p - n) + 7$

B') a) If $n - p = 0, 1$ or 2 , and $n < 5$, then $n < {}^2\sigma^2(n, p)$.

b) If $n - p = 0, 1$ or 2 , and $n \geq 5$, then the \mathcal{K}^2 -invariant submanifold ${}^2S^2$ given below is an open dense subset of a component of ${}^2W^2(n, p)$ of codimension ${}^2\sigma^2(n, p)$ in $J^2(n, p)$:

$n - p$	${}^2S^2$	${}^2\sigma^2(n, p)$
2	$(\pi^{2,1})^{-1} \Sigma^5$	15
1	$(\pi^{2,1})^{-1} \Sigma^4$	12
0	$(\pi^{2,1})^{-1} \Sigma^4$	16

Here $\Sigma^{i,j,k,\dots}$ is a Thom-Boardman variety, $\Sigma^{(i)}$ is as defined in [16], p. 117, and the $\{ \}$ in the descriptions of ${}^2S^r$ in B refers to the Segre symbol of the pencil of quadratic forms defined by the restricted second intrinsic derivative for 2-jets with cokernel rank two.

Following the lines of the proof of (0.3), we see that proving (0.2) reduces to showing that C^1 -stability implies transversality, and ∞ - $\mathcal{A}^{(1)}$ -determinacy implies “transversality off the base point” to the fibres of the submersions π given in (0.6).

Let V be a submanifold of $J^r(n, p)$.

A germ $f: (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^p, 0)$ will be said to be V -transverse if $j^r f \in V$ and $J^r f \pitchfork V$ at 0.

Let $S \subset V$ be \mathcal{K}^r -invariant submanifolds of $J^r(n, p)$. We will say that S is an $\mathcal{A}^{(\ell)}$ -invariant for V -transverse germs, or, if $V = J^r(n, p)$, S is an $\mathcal{A}^{(\ell)}$ -invariant (where $\ell = 0, 1, 2, \dots$ or ∞) if

(i) for any V -transverse germs f, g with $j^r f \in S, j^r g \in V - S, f, g$ are not $\mathcal{A}^{(\ell)}$ -equivalent.

If, moreover, we also have

(ii) for V -transverse germs f, g with f S -transverse, $j^r g \in S$ but g not S -transverse, f, g are not $\mathcal{A}^{(\ell)}$ -equivalent

then we call S a strong $\mathcal{A}^{(\ell)}$ -invariant for V -transverse germs, or, if $V = J^r(n, p)$, a strong $\mathcal{A}^{(\ell)}$ -invariant.

Iterating this relative definition, we say that a \mathcal{K}^r -invariant submanifold S of $J^r(n, p)$ is a (strong) $\mathcal{A}^{(\ell)}$ -quasi-invariant if there exists a sequence

$$J^r(n, p) = S_0 \supset S_1 \supset \dots \supset S_k = S$$

of \mathcal{K}^r -invariant submanifolds of $J^r(n, p)$ such that, for $i = 1, 2, \dots, k, S_i$ is a (strong) $\mathcal{A}^{(\ell)}$ -invariant for S_{i-1} -transverse germs. We call such a sequence a defining sequence for S .

The definition makes sense for $\ell = 0, 1, 2, \dots$ or ∞ . However, it is useless if $\ell = 0$, for there are no $\mathcal{A}^{(0)}$ -invariants except $J^r(n, p)$ itself. Indeed any map germ $f: (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^p, 0)$ is $\mathcal{A}^{(0)}$ -equivalent to a map-germ with zero ∞ -jet: just compose f with a homeomorphism germ of either $(\mathbf{R}^n, 0)$ or $(\mathbf{R}^p, 0)$ which is flat at 0. The definition is also unnecessary if $\ell = \infty$, for all \mathcal{K}^r -invariant submanifolds of $J^r(n, p)$ are obviously strong $\mathcal{A}^{(\infty)}$ -invariants. Of course we plan to use the definition in the case $\ell = 1 \dots$

The transversality properties we have use for are:

(0.8) Lemma ($\ell = 0, 1, 2, \dots$ or ∞).

A) If the smooth map $f: N \rightarrow P$ is C^ℓ -stable, then f is transverse to all strong $\mathcal{A}^{(\ell)}$ -quasi-invariants.

B) If the smooth map-germ $f: (N, x_0) \rightarrow (P, y_0)$ is ∞ - $\mathcal{A}^{(\ell)}$ -determined, then f has a representative $f: U \rightarrow V$ such that $\tilde{f}|U - \{x_0\}$ is transverse to all strong $\mathcal{A}^{(\ell)}$ -quasi-invariants.

(0.9) Lemma ($\ell = 0, 1, 2, \dots$ or ∞).

Let $S \subset J^r(n, p)$ be a strong $\mathcal{A}^{(\ell)}$ -quasi-invariant satisfying the immersion condition, and let W be a \mathcal{K}^r -invariant submanifold of S and $\pi: W \rightarrow L$ a smooth submersion whose fibres are $\mathcal{A}^{(\ell)}$ -invariants for S -transverse germs.

Let N, P be manifolds of dimensions n, p respectively.

A) If the smooth map $f: N \rightarrow P$ is C^ℓ -stable, then f is transverse to the fibres of π .

B) If the smooth map-germ $f: (N, x_0) \rightarrow (P, y_0)$ is ∞ - $\mathcal{A}^{(\ell)}$ -determined, then f has a representative $\tilde{f}: U \rightarrow V$ such that $\tilde{f}: U - \{x_0\}$ is transverse to the fibres of π .

The proofs of (0.8), (0.9) are given as (2.3), (3.10) respectively. The “immersion condition” enters into the hypotheses because essential use is made of a result of May ([12]) that “topological transversality implies transversality” for \mathcal{K}^r -invariant submanifolds of jet-spaces satisfying this condition. Since this result of May is unpublished, we give a proof of it in (3.7). The definition of the “immersion condition” is given in (3.2); however, for the purpose of this introduction all that we need to know is that it is satisfied by the Thom-Boardman varieties of form $\Sigma^{1,0}$, and arbitrary \mathcal{K}^r -invariant submanifolds of them.

To apply this, we need a supply of $\mathcal{A}^{(1)}$ -invariants.

(0.10) Proposition.

1. The Thom-Boardman varieties Σ^I are strong $\mathcal{A}^{(1)}$ -quasi-invariants.

2. Let I be a non-increasing r -tuple of positive integers of form (k, \dots, k) or $(n - p + 1, k, \dots, k)$. Then each \mathcal{K}^{r+1} -orbit in $(\pi^{r+1, r})^{-1} \Sigma^I$ is an $\mathcal{A}^{(1)}$ -invariant for Σ^I -transverse germs.

We prove this in (4.11), via the intrinsic derivatives associated to the Thom-Boardman varieties (as described in [1], § 7). What is needed (beyond [1], § 7) is an elementary, but rather surprising, piece of differential calculus that establishes C^1 -invariance for the intrinsic derivative of a general C^1 -vector bundle homomorphism.

Putting all this together, we have:

Proof of (0.2).

A) It follows from (0.7), A, that there is a Thom-Boardman variety $\Sigma^I \subset J^{r-1}(n, p)$ (for some $r > 1$), where $I = (k, \dots, k)$ or $(n - p + 1, k, \dots, k)$, such that $(\pi^{r, r-1})^{-1} \Sigma^I \cap {}^1W^r(n, p)$ contains a \mathcal{K}^r -invariant submanifold ${}^1S^r$ of codimension ${}^1\sigma(n, p)$ in $J^r(n, p)$. Indeed, in all cases ${}^1S^r \cap \Sigma^{1,0}$ is open and dense in ${}^1S^r$. In the cases $n \geq p$, we have ${}^1S^r = (\pi^{r, r-1})^{-1} \Sigma^I$ for the appropriate r, I , so this is immediate from the fact that $\Sigma^{1,0}$ is open and dense in $(\pi^{r, r-1})^{-1} \Sigma^I$; while in the cases $n < p$, where we take $r = 2$, and $I = \{3\}$ (for $p - n \leq 3$) or $\{4\}$ (for $p - n \geq 4$), it follows from the facts that $\text{cod } \Sigma^{3,1} > \text{cod } \Sigma^{3,(3)}$ and $\text{cod } \Sigma^{4,1} > \text{cod } \Sigma^{4,(2)}$, as easy calculations show. According to (0.6), the Zariski closure of ${}^1S^r$ contains a \mathcal{K}^r -invariant sub-

manifold W of codimension ${}^1\sigma(n, p)$ in $J^r(n, p)$, admitting a submersion with \mathcal{K}^r -invariant fibres of codimension ≥ 1 . Moreover, W is Zariski-open and -dense in the Zariski-closure of ${}^1S^r$. So $W' = W \cap {}^1S^r \cap \Sigma^{\mathbf{I}, 0}$ is Zariski-open and -dense in ${}^1S^r$. Now let N, P be manifolds of dimensions n, p respectively. Since $n \geq {}^1\sigma(n, p) = \text{codim } W'$, (0.5) shows that there is a non-empty, τ_{C^∞} -open subset \mathcal{U} of $C^\infty(N, P)$ such that if $f \in \mathcal{U}$ then f is not transverse to the \mathcal{K}^r -orbits contained in W' . On the other hand, by (0.8) A, (0.9) A and (0.10), if $f: N \rightarrow P$ is C^1 -stable then it is transverse to all \mathcal{K}^r -orbits in $(\pi^{r, r-1})^{-1}\Sigma^{\mathbf{I}}$, so in particular to those in W' . Thus if $f \in \mathcal{U}$, f is not C^1 -stable, and the proof is complete.

B) It follows from (0.7), B and B', that there is a Thom-Boardman variety $\Sigma^{\mathbf{I}} \subset J^{r-1}(n, p)$ (for some $r > 1$; $r = 2$ if $n - p = 0, 1$ or 2), where $\mathbf{I} = (k, \dots, k)$ or $(n - p + 1, k, \dots, k)$, such that $(\pi^{r, r-1})^{-1}\Sigma^{\mathbf{I}} \cap {}^2W^r(n, p)$ contains a \mathcal{K}^r -invariant submanifold ${}^2S^r$ of codimension ${}^2\sigma(n, p)$ (if $n - p \neq 0, 1, 2$) or ${}^2\sigma^2(n, p)$ (if $n - p = 0, 1, 2$) in $J^r(n, p)$. Indeed, in all cases ${}^2S^r \cap \Sigma^{\mathbf{I}, 0}$ is open and dense in ${}^2S^r$. In the cases $n - p = 0, 1, 2$ we have ${}^2S^2 = (\pi^{2, 1})^{-1}\Sigma^{\mathbf{I}}$ (with $\mathbf{I} = \{4\}, \{4\}, \{5\}$, respectively), and in the cases $n - p \geq -3, n - p \neq 0, 1, 2$, we have ${}^2S^r = (\pi^{r, r-1})^{-1}\Sigma^{\mathbf{I}}$ for the appropriate r, \mathbf{I} , so this is immediate from the fact that $\Sigma^{\mathbf{I}, 0}$ is open and dense in $(\pi^{r, r-1})^{-1}\Sigma^{\mathbf{I}}$; while in the cases $n - p \leq -4$, where we take $r = 2$ and $\mathbf{I} = \{4\}$, it follows from the fact that $\text{cod } \Sigma^{4, 1} > \text{cod } \Sigma^{4, (3)}$, as an easy calculation shows. Arguing as in A above, we see that there is a Zariski-open and -dense submanifold W' of ${}^2S^r \cap \Sigma^{\mathbf{I}, 0}$ admitting a submersion with \mathcal{K}^r -invariant fibres of codimension ≥ 2 . Now let N, P be manifolds of dimensions n, p respectively. Since $n > {}^2\sigma(n, p)$ (if $n - p \neq 0, 1, 2$) or $n > {}^2\sigma^2(n, p)$ (if $n - p = 0, 1$ or 2), so that $n > \text{codim } W'$, (0.5) shows that there is a non-empty τ_{C^∞} -open subset \mathcal{U} of $C^\infty(N, P)$ such that if $f \in \mathcal{U}$ then f fails to be transverse to the \mathcal{K}^r -orbits contained in W' at every point of a non-empty submanifold $Z_f \subset N$ of dimension ≥ 1 . On the other hand, by (0.8) B, (0.9) B and (0.10), if $f: N \rightarrow P$ is such that all its point-germs are ∞ - $\mathcal{A}^{(1)}$ -determined, then there is a set of isolated points $S \subset N$ such that $f|N - S$ is transverse to the \mathcal{K}^r -orbits contained in W' . So if $f \in \mathcal{U}$, then f has point-germs which are not ∞ - $\mathcal{A}^{(1)}$ -determined, and the proof is complete. \square

Although most of our arguments are very geometric, they do involve many technicalities. A more direct approach is available in some cases, as we shall now see.

Let $n > p$, and let \mathbf{I} be the r -tuple (k, \dots, k) (with $k > n - p, r \geq 1$). Then $(\pi^{r+1, r})^{-1}\Sigma^{\mathbf{I}} \subset J^{r+1}(n, p)$ consists precisely of all \mathcal{K}^{r+1} -classes with truncated local algebra of the form

$$\mathbf{R}[x_1, \dots, x_k] / \langle \varphi_1, \dots, \varphi_{p-n+k} \rangle + m^{r+2}$$

where φ_i ($i = 1, \dots, p - n + k$) are homogeneous polynomials of order $r + 1$.

We define a \mathcal{K}^{r+1} -invariant, semi-algebraic open subset \mathcal{U} of $(\pi^{r+1, r})^{-1}\Sigma^{\mathbf{I}}$ by the requirements:

(i) The variety defined by the complexification of $\langle \varphi_1, \dots, \varphi_{p-n+k} \rangle$ is a complete intersection of codimension $p - n + k$ with an isolated singularity at the origin.

(Equivalently, $(\varphi_1, \dots, \varphi_{p-n+k})$ is a \mathcal{K} -finite germ. This is actually satisfied for a Zariski-open and ρ -dense subset of $(\pi^{r+1, r})^{-1} \Sigma^I$.)

(ii) The complexification of the set of zeroes of $\langle \varphi_1, \dots, \varphi_{p-n+k} \rangle$ is the set of zeroes of the complexification of $\langle \varphi_1, \dots, \varphi_{p-n+k} \rangle$.

It is well-known that, over \mathbf{C} , the ideal defining an affine complete intersection variety is reduced if (and only if) the set of singular points is of codimension ≤ 1 ; so that if this holds the ideal is determined by its set of zeroes (by the Hilbert Nullstellensatz). It follows immediately that when (i), (ii) hold, then the set of zeroes of $\langle \varphi_1, \dots, \varphi_{p-n+k} \rangle$ actually determines the ideal $\langle \varphi_1, \dots, \varphi_{p-n+k} \rangle$, and hence the \mathcal{K}^{r+1} -class. Moreover, if $f: (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^p, 0)$ has $(r+1)$ -jet in \mathcal{U} , the tangent cone to $f^{-1}(0)$ at 0 is the set of zeroes of $\langle \varphi_1, \dots, \varphi_{p-n+k} \rangle$, where the truncated local algebra of f at 0 is

$$\mathbf{R}[x_1, \dots, x_k] / \langle \varphi_1, \dots, \varphi_{p-n+k} \rangle + \mathfrak{m}^{r+2}.$$

Since, if $f, g: (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^p, 0)$ are $\mathcal{A}^{(1)}$ -equivalent, then the tangent-cones to $f^{-1}(0), g^{-1}(0)$ at 0 are linearly equivalent, it follows that if $f, g: (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^p, 0)$ have $(r+1)$ -jet in \mathcal{U} and are $\mathcal{A}^{(1)}$ -equivalent, then their $(r+1)$ -jets are \mathcal{K}^{r+1} -equivalent, and we have established a version of part of (0.10) by geometric means.

The results prior to ours alluded to immediately before (0.2) were obtained on the basis of special cases of the above:

If $n-p=2, k=4, r=1$, then we find $\mathcal{U} \subset (\pi^{2,1})^{-1} \Sigma^4$ (it is actually Zariski-open in this case), so that for $f: (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^{n-2}, 0)$ with $j^2 f \in \mathcal{U}$ the tangent cone to $f^{-1}(0)$ at 0 “is” the set of zeroes of a non-degenerate pencil of quadratic forms (whose classification involves a 1-dimensional parameter—essentially given by the cross-ratio of the eigenvalues of the pencil).

If $n-p=1, k=2, r=4$, then we find $\mathcal{U} \subset (\pi^{5,4})^{-1} \Sigma^{2,2,2,2}$ so that for $f: (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^{n-1}, 0)$ with $j^5 f \in \mathcal{U}$ the tangent cone to $f^{-1}(0)$ at 0 is five coplanar lines (whose classification involves a 2-dimensional parameter—essentially given by the cross-ratios of two distinct subsets of four out of the five lines).

It should perhaps be observed that, while the geometry described here certainly helps to explain what is going on, the invariance results obtained are weaker than those of (0.10) 2 in an important way (in that only \mathcal{K}^{r+1} -classes in \mathcal{U} , rather than in all $(\pi^{r+1, r})^{-1} \Sigma^I$ are shown to be distinguished by $\mathcal{A}^{(1)}$ -equivalence), so that proving the necessary transversality results to derive results like (0.2) from this geometry requires extra hypotheses. It turns out that non-density theorems as in (0.2) B, in cases where $n > p$ and the critical stratum is Σ^I , can still be proved; but it does not seem possible to derive non-density theorems for C^1 -stability per se as in (0.2) A, even in cases when $n > p$ and the critical stratum is Σ^I , except when $n = \text{codim } \Sigma^I$. Deriving results by these methods when $n > \text{codim } \Sigma^I$ seems to require stronger stability assumptions (the C^1 -analogue of the ε -stability notion of May [12] is sufficient here).

We observe that this discussion establishes invariants for $\mathcal{A}^{(1)}$ -equivalence without the necessity of any transversality conditions. Also, the idea of using the tangent cone

to other parts of the structure of a map-germ to establish $\mathcal{A}^{(1)}$ -invariance results seems promising. One example: the C^∞ -versal unfolding of the function-germs \tilde{E}_6 given by

$$(w_1, \dots, w_k, x, y, z) \rightarrow \sum_{i=1}^k \pm w_i^2 + x^3 + y^3 + z^3 + \lambda xyz \quad (\lambda \neq -3)$$

is such that the parameter λ is a linear invariant of the tangent cone to the locus of D_5 -singularities at points on the λ -axis, from which can be deduced that λ is an $\mathcal{A}^{(1)}$ -invariant for \mathcal{U} -transverse germs, where \mathcal{U} is the open dense subset of $(\pi^{3,2})^{-1} \Sigma^{n-p+1,3}$ ($n-p \geq 3$) consisting of appropriate unfoldings of the \tilde{E}_6 . This leads to an alternative (but weaker, in the same way as described above) approach to (0.2) A in the case $n-p \geq 3$.

The arguments of §§ 2, 3 yield further results, for example:

(0.11) *Lemma* ($\ell = 0, 1, 2, \dots$ or ∞).

Let N, P be manifolds of dimensions n, p respectively.

A) If the smooth map $f: N \rightarrow P$ is C^ℓ -stable, then f avoids all $\mathcal{A}^{(\ell)}$ -invariants of codimension $> n$.

B) If the smooth map-germ $f: (N, x_0) \rightarrow (P, y_0)$ is ∞ - $\mathcal{A}^{(\ell)}$ -determined, then f has a representative $\tilde{f}: U \rightarrow V$ such that $\tilde{f}|_{U - \{x_0\}}$ avoids all $\mathcal{A}^{(\ell)}$ -invariants of codimension $> n$.

(This is a special case of (2.7).)

Also, the arguments of § 4 provide further examples of invariants:

(0.12) *Lemma*. — Every $J^2 \mathcal{K}$ -orbit in $J^2(n, p)$ is an $\mathcal{A}^{(1)}$ -invariant.

(This is proved as (4.6).)

Putting these results together with some results from [16] yields:

(0.13) *Theorem*. — Let N, P be manifolds of dimensions n, p respectively, with N compact, and let $f: N \rightarrow P$ be a continuous map.

Suppose that, for some r , $\max(0, n-p) \leq r \leq n$, either (a) the $(p-n+r)$ -determinant

$$|\{w_{r+i-j} : 1 \leq i, j \leq p-n+r\}|$$

is non-zero (where $w_i = w_i(TN - f^* TP)$ are Stiefel-Whitney classes), or (b) $r, n-p$ are even,

$TN, f^* TP$ are orientable, and the $\frac{1}{2}(p-n+r)$ -determinant

$$\left| \left\{ p_{(\frac{r}{2}+i-j)} : 1 \leq i, j \leq \frac{1}{2}(p-n+r) \right\} \right|$$

is non-zero modulo two-torsion (where $p_j = p_j(TN - f^* TP)$ are Pontrjagin classes).

Then:

A) If $r(p - n + r) \leq n \leq (\frac{1}{2}r(r + 1) - p + n)(p - n + r) - r^2$, there are no C^1 -stable mappings $N \rightarrow P$ homotopic to f .

B) If $r(p - n + r) < n \leq (\frac{1}{2}r(r + 1) - p + n)(p - n + r) - r^2$, there are no smooth mappings $N \rightarrow P$ homotopic to f , all of whose point-germs are ∞ - $\mathcal{A}^{(1)}$ -determined.

Proof. — We first draw some conclusions from the results of [16]:

(i) It is shown in [16], (1.6) that the class in *a*) is, or is the mod 2 reduction of, the primary obstruction to sectioning $J^1(N, P) - \overline{\Sigma^r(N, P)}$ over graph f , and hence over graph g for any map g homotopic to f ; while, if the conditions in *b*) hold, then the class in *b*) is this primary obstruction modulo 2-torsion. So the hypotheses imply that this primary obstruction is non-zero. Since this class occurs in dimension $r(p - n + r)$, this means ([19], (35.4)) that $J^1(N, P) - \overline{\Sigma^r(N, P)}$ cannot be sectioned over the $r(p - n + r)$ -skeleton of any cellular decomposition of graph g , for any g homotopic to f . It follows that:

A) If $r(p - n + r) \leq n$, then $(J^1 g)^{-1} \overline{\Sigma^r(N, P)}$ is non-empty for any smooth map g homotopic to f .

B) If $r(p - n + r) < n$, then $(J^1 g)^{-1} \overline{\Sigma^r(N, P)}$ cannot be a finite set for any smooth map g homotopic to f .

(ii) The calculations of [16], (1.5) show that the codimension in $J^2(n, p)$ of any $J^2 \mathcal{K}$ -orbit in $(\pi^{2,1})^{-1} \Sigma^r(n, p)$ is $\geq (\frac{1}{2}r(r + 1) - p + n)(p - n + r) - (r^2 - 1)$. Since this number is increasing as a function of r , and since $\overline{\Sigma^r(n, p)} = \bigcup_{s \geq r} \Sigma^s(n, p)$, it follows that the codimension in $J^2(n, p)$ of any $J^2 \mathcal{K}$ -orbit in $(\pi^{2,1})^{-1} \overline{\Sigma^r(n, p)}$ is $\geq (\frac{1}{2}r(r + 1) - p + n)(p - n + r) - (r^2 - 1)$.

We now prove the theorem:

A) By (ii), the hypothesis of A implies that every $J^2 \mathcal{K}$ -orbit in $(\pi^{2,1})^{-1} \overline{\Sigma^r(n, p)}$ has codimension $> n$; so that by (0.11) A and (0.12) any C^1 -stable map $N \rightarrow P$ avoids these orbits, and thus avoids $\overline{\Sigma^r(N, P)}$. But by (i) A, no smooth map homotopic to f can avoid $\overline{\Sigma^r(N, P)}$.

B) By (ii), the hypothesis of B implies that every $J^2 \mathcal{K}$ -orbit in $(\pi^{2,1})^{-1} \overline{\Sigma^r(n, p)}$ has codimension $> n$; so that by (0.11) B and (0.12), $(J^1 g)^{-1} \overline{\Sigma^r(n, p)}$ is a set of isolated points for any smooth map $g : N \rightarrow P$ all of whose point-germs are ∞ - $\mathcal{A}^{(1)}$ -determined. Indeed, $(J^1 g)^{-1} \overline{\Sigma^r(n, p)}$ is finite, for it is closed in N , so compact. But, by (i) B, $(J^1 g)^{-1} \overline{\Sigma^r(N, P)}$ cannot be a finite set for any smooth map g homotopic to f . \square

Evaluating the characteristic classes in *a*), *b*) above shows, for example, that there are no C^1 -stable maps $\mathbf{RP}^{19} \rightarrow \mathbf{R}^{19}$ or $\mathbf{CP}^4 \rightarrow \mathbf{R}^6$; and that there are no maps $\mathbf{RP}^{27} \rightarrow \mathbf{R}^{27}$ or $\mathbf{CP}^9 \rightarrow \mathbf{R}^{18}$ all of whose point-germs are ∞ - $\mathcal{A}^{(1)}$ -determined.

The results we have just described raise further questions, for example:

Are $\mathcal{A}^{(1)}$ -equivalent germs of C^1 -stable maps \mathcal{A} -equivalent?

Does C^1 -stability imply C^∞ -stability?

Are ∞ - $\mathcal{A}^{(1)}$ -determined map-germs ∞ - \mathcal{A} -determined?

We conjecture a positive answer in all three cases. These are fairly delicate questions, however, for $\mathcal{A}^{(1)}$ -equivalence does not imply \mathcal{A} -equivalence, or even \mathcal{H} -equivalence in general. For example, [2], [27], [28] yield many examples of finitely $\mathcal{A}^{(1)}$ -determined map-germs which are not finitely \mathcal{H} -determined.

1. A fragment from the theory of algebraic groups

(1.1) *Proposition.* — Let G be an affine algebraic group over \mathbf{C} , and let X be an algebraic variety over \mathbf{C} on which G acts strictly.

Then there exists a Zariski-open and -dense G -invariant smooth subvariety $Y \subset X$, a smooth variety W defined over \mathbf{C} , and an algebraic submersion $\pi : Y \rightarrow W$ whose fibres are the G -orbits contained in Y .

Proof. — According to a result of Dixmier and Raynaud ([5], (1.6)), there exists a G -invariant, Zariski-open and -dense subvariety $Y' \subset X$ such that “ Y'/G exists”, i.e. such that there exists an algebraic variety W' over \mathbf{C} , and an algebraic morphism $\varphi : Y' \rightarrow W'$ such that

- (i) φ is open, constant on G -orbits, and induces a bijection of the set Y'/G onto W' ;
- (ii) if U is an open subset of W' then the morphism $\mathbf{C}[U] \rightarrow \mathbf{C}[\varphi^{-1}(U)]^G$ induced by φ is bijective.

Now set

$$W = W' - \text{sing } W' - \overline{\varphi(\Sigma(\varphi | Y' - \text{sing } Y'))},$$

$$Y = (\varphi | Y' - \text{sing } Y')^{-1} W,$$

and let $\pi : Y \rightarrow W$ be the restriction of φ . The set $Y' - \text{sing } Y'$ is G -invariant (for G acts by isomorphisms), so Y is G -invariant. The other properties claimed for Y and $\pi : Y \rightarrow W$ are immediate. \square

(1.2) *Proposition.* — Let G be an affine algebraic group defined over \mathbf{R} , and X' a real algebraic subset of an affine space $V(\mathbf{R})$ invariant under the action of $G(\mathbf{R})$ given by a strict action of G on V , defined over \mathbf{R} .

Let X be the variety defined by X' with the real reduced structure. Then there exist a G -invariant smooth subvariety Y of X , a smooth variety W and an algebraic submersion $\pi : Y \rightarrow W$, all defined over \mathbf{R} , such that $Y(\mathbf{R})$ is Zariski-open and -dense in $X(\mathbf{R})$ and the fibres of $\pi_{\mathbf{R}}$ are finite unions of $G(\mathbf{R})$ -orbits.

Proof. — We recall that the real reduced structure is defined by associating to X' the quotient of the polynomial ring $P(V)$ by the ideal of functions vanishing on X' . It follows that $X(\mathbf{R}) = X'$ is Zariski-dense in $X(\mathbf{C})$. Moreover, since the ideal is invariant by $G(\mathbf{R})$ we have an action of G on X , defined over \mathbf{R} .

In the proof of (1.1) we may suppose Y' —and hence Y —invariant under complex conjugation, for otherwise we may replace Y' by its intersection with its complex conjugate, which is still Zariski-open and -dense.

Let Y , W and π be as given in (1.1). The complex conjugation on $Y(\mathbf{C})$ induces an involution α on $W(\mathbf{C})$ such that $\alpha(\pi(x)) = \pi(\bar{x})$: this is well-defined since if $\pi(y) = \pi(x)$, then $y = g.x$ for some $g \in G(\mathbf{C})$, so $\bar{y} = \bar{g}.\bar{x}$ and $\pi(\bar{y}) = \pi(\bar{x})$. The involution α defines the desired real structure on W .

Since $Y(\mathbf{C})$ is Zariski-open and -dense in $X(\mathbf{C})$, its complement has strictly lower dimension. Hence this complement meets $X(\mathbf{R})$ in a set of dimension strictly lower than $\dim X$. Since by hypothesis $\dim_{\mathbf{R}}(X(\mathbf{R})) = \dim X$, it follows that $Y(\mathbf{R}) = Y(\mathbf{C}) \cap X(\mathbf{R})$ is Zariski-open and -dense in $X(\mathbf{R})$.

A fibre $F(\mathbf{R})$ of $\pi_{\mathbf{R}}$ is the intersection of $X(\mathbf{R})$ with the corresponding fibre $F(\mathbf{C})$ of $\pi_{\mathbf{C}}$, which is a $G(\mathbf{C})$ -orbit. Now $\dim_{\mathbf{R}}(F(\mathbf{R})) \leq \dim_{\mathbf{C}}(F(\mathbf{C}))$ and for any point $x \in X(\mathbf{R})$, $\dim_{\mathbf{R}}(G(\mathbf{R}).x) = \dim_{\mathbf{C}}(G(\mathbf{C}).x)$. Since $F(\mathbf{R})$ is $G(\mathbf{R})$ -invariant, it follows that the orbits of $G(\mathbf{R})$ are open in $F(\mathbf{R})$, so since this has only finitely many connected components, it is a finite union of $G(\mathbf{R})$ -orbits. ■

(1.3) (= (0.6)) Lemma. — Suppose ${}^k W^r(n, p) \neq \emptyset$, and let C be a component of it of maximal dimension. Then C contains, as a Zariski-open and -dense subset, a $J^r \mathcal{K}$ -invariant submanifold W of $J^r(n, p)$ which admits an algebraic submersion whose fibres are finite unions of $J^r \mathcal{K}$ -orbits.

Proof. — Since $J^r \mathcal{K}$ is the set of real points of a Zariski-connected affine algebraic group defined over \mathbf{R} which acts strictly on $J^r(n, p)$, and since C is a $J^r \mathcal{K}$ -invariant subvariety of $J^r(n, p)$, the result follows from (1.2). ■

2. Transversality, stability and determinacy : « easy » results

Our arguments are based on variants of the Thom transversality theorem:

(2.1) Theorem. — Let $Q_i \subset J^r_i(N, P)$ ($i = 1, 2, \dots$) be smooth submanifolds.

A) The set

$$\{g \in C^\infty(N, P) \mid g \text{ is transverse to each } Q_i (i = 1, 2, \dots)\}$$

is a countable intersection of τ_{W^∞} -open dense subsets of $C^\infty(N, P)$. In particular, it is dense in $C^\infty(N, P)$, for this is a Baire space (see e.g. [7], [13]).

B) Let $x_0 \in N$, let $f: (N, x_0) \rightarrow (P, y_0)$ be a smooth map-germ, and define

$$M = \{g \in C^\infty(N, P) \mid g_{x_0} \text{ has the same } \infty\text{-jet as } f\}.$$

Then the set

$$\{g \in M \mid g|_{N - \{x_0\}} \text{ is transverse to each } Q_i \ (i = 1, 2, \dots)\}$$

is a countable intersection of τ_{W^∞} -open dense subsets of M . In particular, it is dense in M , for this is a Baire space (see e.g. [7], [13]).

Proof.

A) See [8], V, (3.4).

B) See [28], (3.2).

(2.2) Lemma. — Let $S \subset J^r(n, p)$ be a strong $\mathcal{A}^{(\ell)}$ -quasi-invariant, and let $J^r(n, p) = S_0 \supset \dots \supset S_k = S$ be a defining sequence for it. Let N, P be smooth manifolds of dimensions n, p respectively, and let $f, g: N \rightarrow P$ be smooth C^ℓ -equivalent mappings. Suppose that g is transverse to S_i for $i = 0, \dots, k$. Then f is transverse to S_i for $i = 0, \dots, k$.

Proof. — Let h, k be C^ℓ -diffeomorphisms of N, P respectively such that

$$g \circ h = k \circ f.$$

We will prove by induction that, for $i = 0, \dots, k$,

$$(a)_i \quad h((J^r f)^{-1} S_i) = (J^r g)^{-1} S_i,$$

$$(b)_i \quad f, g \text{ are transverse to } S_i.$$

$(a)_0, (b)_0$ are trivially true. Suppose $(a)_{i-1}, (b)_{i-1}$ hold. Let $x \in (J^r f)^{-1} S_i$. Then $x \in (J^r f)^{-1} S_{i-1}$, so, by $(a)_{i-1}$, $h(x) \in (J^r g)^{-1} S_{i-1}$. The germs $f_x, g_{h(x)}$ are $\mathcal{A}^{(\ell)}$ -equivalent, so, since they are S_{i-1} -transverse (by $(b)_{i-1}$), and since $j^r f_x \in S_i$, it follows from the fact that S_i is an $\mathcal{A}^{(\ell)}$ -invariant for S_{i-1} -transverse germs that $j^r g_{h(x)} \in S_i$. Hence $h((J^r f)^{-1} S_i) \subset (J^r g)^{-1} S_i$.

Repeating the above argument with f, g interchanged (and with h replaced by h^{-1}) yields the opposite inclusion. So $(a)_i$ holds.

The map g is transverse to S_i , so the germ of g at each point of $(J^r g)^{-1} S_i$ is S_i -transverse. If $x \in (J^r f)^{-1} S_i$, then the germs $f_x, g_{h(x)}$ are $\mathcal{A}^{(\ell)}$ -equivalent, so, since $h(x) \in (J^r g)^{-1} S_i$ (by the above), $g_{h(x)}$ is S_i -transverse, and so, since S_i is a strong $\mathcal{A}^{(\ell)}$ -invariant for S_i -transverse germs, f_x is S_i -transverse. Thus f is transverse to S_i at each point of $(J^r f)^{-1} S_i$, and $(b)_i$ holds.

So the inductive step is established, and the proof is complete. \square

(2.3) (= (0.8)) Proposition.

A) Let $f: N \rightarrow P$ be C^ℓ -stable. Then f is transverse to all strong $\mathcal{A}^{(\ell)}$ -quasi-invariants.

B) Let $f: (N, x_0) \rightarrow (P, y_0)$ be ∞ - $\mathcal{A}^{(\ell)}$ -determined. Then f has a representative $\tilde{f}: U \rightarrow V$ such that $\tilde{f}|_{U - \{x_0\}}$ is transverse to all strong $\mathcal{A}^{(\ell)}$ -quasi-invariants.

Proof.

A) Let $S \subset J^r(n, p)$ be a strong $\mathcal{A}^{(\ell)}$ -quasi-invariant; and let

$$J^r(n, p) = S_0 \supset \dots \supset S_k = S$$

be a defining sequence for it. By (2.1) A, f can be arbitrarily closely approximated by maps $g : N \rightarrow P$ transverse to S_1, \dots, S_k ; and so, f being C^ℓ -stable, it is C^ℓ -equivalent to such a map. Then, by (2.2), f is transverse to S_1, \dots, S_k .

Since this holds for all strong $\mathcal{A}^{(\ell)}$ -quasi-invariants, the proof is complete.

B) Suppose that the result is false; so that there exists a representative $\tilde{f} : U' \rightarrow V'$ for f , a sequence $\{x_i\} \subset U'$, with $x_i \rightarrow x_0$, and strong $\mathcal{A}^{(\ell)}$ -quasi-invariants $S_i \subset J^{r_i}(n, p)$, with $j^{r_i} \tilde{f}_{x_i} \in S_i$, such that \tilde{f} is not transverse to S_i at x_i . Let $J^{r_i}(n, p) = S_{i0} \supset \dots \supset S_{ik_i} = S^i$ be a defining sequence for S_i . By (2.1) B, there exists a map $g : U' \rightarrow V'$ with $j^\infty g_{x_0} = j^\infty f$ such that $g|_{U' - \{x_0\}}$ is transverse to each $S_{i\ell}$ ($i = 1, 2, \dots; \ell = 1, \dots, k_i$). Since f is ∞ - $\mathcal{A}^{(\ell)}$ -determined, it is $\mathcal{A}^{(\ell)}$ -equivalent to g_{x_0} , so there exist open neighbourhoods U of x_0 in U' , V of y_0 in V' , with $\tilde{f}(U) \subset V$, and C^ℓ -embeddings $h : (U, x_0) \rightarrow (U', x_0)$, $k : (V, y_0) \rightarrow (V', y_0)$ such that

$$k \circ \tilde{f}|_U = g \circ h.$$

In particular, $\tilde{f}|_U : U - \{x_0\} \rightarrow V - \{y_0\}$ is C^ℓ -equivalent to $g|_U : h(U) - x_0 \rightarrow k(V) - y_0$. By (2.2), then, $\tilde{f}|_U - \{x_0\}$ is transverse to each $S_{i\ell}$ ($i = 1, 2, \dots; \ell = 1, \dots, k_i$). The set U being a neighbourhood of x_0 in U' , we have $x_i \in U$ for sufficiently large i , and thus a contradiction to the non-transversality of \tilde{f} to S_i at x_i for such i . Thus our supposition was false, and the result is proved. \square

(2.4) *Remark.* — (0.4) follows from (2.3), because any \mathcal{K}^r -invariant submanifold S of $J^r(n, p)$ is a strong $\mathcal{A}^{(\infty)}$ -invariant (for S is \mathcal{A}^r -invariant, and

$$\{z \in J^{r+1}(n, p) \mid (\pi^{r+1}, r) z \in S \text{ and any (hence all) representative of } z \text{ is transverse to } S \text{ at } 0\}$$

is \mathcal{A}^{r+1} -invariant).

It is a rather more delicate matter to obtain transversality statements when the “strong” in the hypotheses of (2.2) and (2.3) is dropped. However, a first step in this direction is easily made:

(2.5) *Definition.* — We will call a \mathcal{K}^r -invariant submanifold $S \subset J^r(n, p)$ an *almost-strong $\mathcal{A}^{(\ell)}$ -quasi-invariant* if S is an $\mathcal{A}^{(\ell)}$ -invariant for S' -transverse germs, where $S \subset S' \subset J^r(n, p)$, and S' is a strong $\mathcal{A}^{(\ell)}$ -quasi-invariant. We will call S' a *defining manifold* for S .

(2.6) *Lemma.* — Let $S \subset J^r(n, p)$ be an almost-strong $\mathcal{A}^{(\ell)}$ -quasi-invariant, with defining manifold S' . Let N, P be manifolds of dimensions n, p respectively, and let $f, g : N \rightarrow P$ be

C^ℓ -equivalent smooth mappings. If g is transverse to S, S' , then $(J^r f)^{-1} S$ is a C^ℓ -submanifold of N of dimension $n - \dim S$, or is empty.

Proof. — Let h, k be C^ℓ -diffeomorphisms of N, P respectively such that $g \circ h = k \circ f$. By (2.2) (and its proof), f is transverse to S' , and $h((J^r f)^{-1} S') = (J^r g)^{-1} S'$. Now, arguing exactly as in the first half of the proof of the inductive step in the proof of (2.2), we find $h((J^r f)^{-1} S) = (J^r g)^{-1} S$. Since g is transverse to S , $(J^r g)^{-1} S$ is a C^ℓ -submanifold of N of codimension $n - \text{cod } S$, or is empty; h being a C^ℓ -diffeomorphism, the same is true for $(J^r f)^{-1} S$. \square

(2.7) Proposition.

A) Let $f: N \rightarrow P$ be C^ℓ -stable. Then, for any almost-strong $\mathcal{A}^{(\ell)}$ -quasi-invariant $S \subset J^r(n, p)$, $(J^r f)^{-1} S$ is either a C^ℓ -submanifold of N of dimension $n - \text{cod } S$, or empty.

B) Let $f: (N, x_0) \rightarrow (P, y_0)$ be ∞ - $\mathcal{A}^{(\ell)}$ -determined. Then f has a representative $\tilde{f}: U \rightarrow V$ such that for any almost-strong $\mathcal{A}^{(\ell)}$ -quasi-invariant $S \subset J^r(n, p)$, $J^r(\tilde{f}|U - \{x_0\})^{-1} S$ is either a C^ℓ -submanifold of $U - \{x_0\}$ of dimension $n - \text{cod } S$, or empty.

Proof.

A) Let $S \subset J^r(n, p)$ be an almost-strong $\mathcal{A}^{(\ell)}$ -quasi-invariant, with defining manifold S' . By (2.1) A, f can be arbitrarily closely approximated by maps $N \rightarrow P$ transverse to S, S' ; being C^ℓ -stable, it is therefore C^ℓ -equivalent to such a map. Thus the result follows from (2.6).

B) This is derived from (2.1) B and (2.6) in exactly the same way as (2.3) B is derived from (2.1) B and (2.2). We suppose that the statement is false, and so find a representative $\tilde{f}: U \rightarrow V$ for f and a sequence $\{x_i\} \subset U$ tending to x_0 with $J^{r_i} \tilde{f}(x_i) \in S_i$, where S_i is an almost-strong $\mathcal{A}^{(\ell)}$ -quasi-invariant, but $(J^{r_i} \tilde{f})^{-1} S_i / \{x_0\}$ is not a C^ℓ -manifold of dimension $n - \text{cod } S_i$ at x_i . By (2.1) B, there exists a map $g: U \rightarrow V$ with the same ∞ -jet as x_0 as f , with $g|U - \{x_0\}$ transverse to all the S_i , as well as their defining manifolds. Since f is ∞ - $\mathcal{A}^{(\ell)}$ -determined, we can find neighbourhoods U', U'' of x_0 in U such that $f|U' - \{x_0\}, g|U'' - \{x_0\}$ are C^ℓ -equivalent. But (2.6) now yields a contradiction, completing the proof. \square

3. Transversality, stability and determinacy : delicate results

In this paragraph we will sharpen the results of § 2 to show that maps are transverse to the appropriate submanifolds (rather than just that the preimages of these are C^ℓ -submanifolds). The key to this is a result of May ([12], Ch. I), which we present in (3.7) since it has not yet been published, though an adaptation of it appeared in [3]. This leads, via (2.7), to a proof of (0.9). We then prove (0.5), using similar ideas.

The key to our presentation is the relation between transversality in jet spaces

and in unfoldings: we begin by recalling this. An unfolding $\{F; i, j\}$ (sometimes referred to simply as F) of a smooth map $f: N \rightarrow P$ is a commutative diagram

$$\begin{array}{ccc} N & \xrightarrow{f} & P \\ \downarrow i & & \downarrow j \\ N' & \xrightarrow{F} & P' \end{array}$$

such that i, j are embeddings and the diagram is cartesian in the sense that j is transverse to F , and (i, f) is a diffeomorphism onto the fibre product of F and j . The above refers to maps or (with trivial modification) to germs. The *dimension* of the unfolding is

$$\dim N' - \dim N = \dim P' - \dim P.$$

There is a close connection between the \mathcal{K} -classes of germs and of their unfoldings. Indeed, unfoldings F, F' of the same dimension of germs f, f' are \mathcal{K} -equivalent if and only if f, f' are; and correspondingly for their r -jets. For any $k \geq 0$, the map

$$S_k: J^r(n, p) \rightarrow J^r(n+k, p+k)$$

induced by sending $j^r f$ to $j^r(f \times 1_{(\mathbb{R}^k, 0)})$ is an embedding transverse to \mathcal{K}^r -orbits in $J^r(n+k, p+k)$. In particular, if V is a \mathcal{K}^r -invariant submanifold of $J^r(n, p)$, then the least \mathcal{K}^r -invariant subset of $J^r(n+k, p+k)$ containing $S_k(V)$ is a submanifold of the same codimension as V . We denote it V_k and call it the *k-suspension* of V .

(3.1) Proposition. — *Let V be a \mathcal{K}^r -invariant submanifold of $J^r(n, p)$. Let N, P be manifolds of dimensions n, p respectively and let $f: (N, x) \rightarrow (P, y)$ be a map-germ with r -jet in V and $\{F: (N', x') \rightarrow (P', y'); i, j\}$ an unfolding of f of dimension k transverse to V_k . Then the following are equivalent:*

- (i) f is transverse to V ;
- (ii) i is transverse to $(j^r F)^{-1} V_k$;
- (iii) j is transverse to $F((j^r F)^{-1} V_k)$.

Proof. — This follows by direct calculation: see [3, (1.3)]. ■

As in [12] and [3], to use this directly we need the following condition.

(3.2) Definition. — The \mathcal{K}^r -invariant submanifold $V \subset J^r(n, p)$ satisfies the *immersion condition* if, for any $z \in V$ and any C^∞ -stable germ f with r -jet $S^k(z)$, $f|_{(j^r f)^{-1} V_k}$ is an immersion germ.

(3.3) Lemma. — *If V satisfies the immersion condition, then for any $z \in V$ and any V_k -transverse germ f with r -jet $S^k(z)$, $f|_{(j^r f)^{-1} V_k}$ is an immersion germ.*

Proof. — First, we observe that the property in question depends only on the $(r+1)$ -jet of f . Thus by [27, (5.1)] we can suppose that f has finite singularity type, so possesses [6, III (2.8)] a C^∞ -stable unfolding

$$\{F : (\mathbf{R}^{n+k+\ell}, 0) \rightarrow (\mathbf{R}^{p+k+\ell}, 0); 1_{n+k} \times 0, 1_{p+k} \times 0\}.$$

Since f is V_k -transverse, $(\mathbf{R}^{n+k}, 0)$ is transverse to $(J^r F)^{-1} V_{k+\ell}$ (by (3.1)), so that $f | (J^r f)^{-1} V_k$ is the restriction of the immersion $F | (J^r F)^{-1} V_{k+\ell}$ to a submanifold, and so is itself an immersion. \square

The following addendum to (3.1) is now immediate.

(3.1.1) Addendum. — *Let $V, f, \{F; i, j\}$ be as in (3.1). If V satisfies the immersion condition, then f is V -transverse if and only if j is transverse to the manifold-germ $F((j^r F)^{-1} V_k)$.*

The advantage of this translation of transversality conditions is that we can reduce the problem of constructing deformations of f satisfying given transversality conditions on the r -jet to that of constructing deformations of j (in a suitable unfolding $\{F; i, j\}$) satisfying ordinary transversality conditions.

(3.4) Lemma. — *Let $f : (N, x_0) \rightarrow (P, y_0)$ be a smooth map-germ, and let $\{G : (N', x'_0) \rightarrow (P', y'_0); i, j\}$ be an unfolding of f .*

Let $J : (P \times \mathbf{R}, y_0 \times [-1, 1]) \rightarrow (P', y'_0)$ be a map germ with $J_0 = j$ (where $J_0(\mathcal{Y}) = J(\mathcal{Y}, 0)$). Then there exists $\eta, 0 < \eta \leq 1$, and map-germs

$$I : (N \times \mathbf{R}, x_0 \times [-\eta, \eta]) \rightarrow (N', x'_0),$$

$$F : (N \times \mathbf{R}, x_0 \times [-\eta, \eta]) \rightarrow (P, y_0),$$

with $I_0 = i, F_0 = f$, such that, for each $t \in (\mathbf{R}, [-\eta, \eta])$, $\{G; I_t, J_t\}$ is an unfolding of F_t .

Proof. — Since $\text{Im } TJ_{y'_0} + \text{Im } TG_{x'_0} = \text{TP}'_{y'_0}$ there exists $\eta > 0$ such that $\text{Im } TJ_t + \text{Im } TG_{x'_0} = \text{TP}'_{y'_0}$ for $t \in (\mathbf{R}, [-\eta, \eta])$; i.e. J_t is transverse to G for $t \in (\mathbf{R}, [-\eta, \eta])$.

Let $(Q, x'_0 \times y_0 \times [-\eta, \eta])$ be the fibre-product of G and $J | (P \times \mathbf{R}, y_0 \times [-\eta, \eta])$. Since G and J_t are transverse for $t \in (\mathbf{R}, [-\eta, \eta])$, the map-germ

$$\pi_{\mathbf{R}} : (Q, x'_0 \times y_0 \times [-\eta, \eta]) \rightarrow (\mathbf{R}, [-\eta, \eta])$$

induced by the projection of $N' \times P \times \mathbf{R}$ on \mathbf{R} is a submersion, with fibre $Q_t = \pi_{\mathbf{R}}^{-1}(t)$ the fibre product of G and J_t .

Thus the standard vector field $\frac{\partial}{\partial t}$ on $(\mathbf{R}, [-\eta, \eta])$ lifts to a vector field $\left(\xi, \frac{\partial}{\partial t}\right)$ on Q , and we can arrange that ξ is zero on $x'_0 \times y_0 \times (\mathbf{R}, [-\eta, \eta])$. Integrating this vector field gives an isotopy-germ

$$\theta : (Q \times \mathbf{R}, x'_0 \times y_0 \times [-\eta, \eta] \times [-1, 1]) \rightarrow (Q, x'_0 \times y_0 \times [-\eta, \eta])$$

and hence a germ of diffeomorphism

$$H : (Q, S' \times y_0 \times [-\eta, \eta]) \rightarrow (Q_0 \times 0 \times \mathbf{R}, x'_0 \times y_0 \times [-\eta, \eta])$$

given by

$$H(x', y, t) = (\theta(x', y, t, -t), t).$$

We can now define I, F by

$$I(x, t) = \pi_{N'}(H^{-1}(i(x), t)),$$

$$F(x, t) = \pi_P(H^{-1}(i(x), t)),$$

where $\pi_{N'}, \pi_P$ are given by projecting $N' \times P \times \mathbf{R}$ on N', P respectively.

It is immediate that I, F have the required property. \square

We now turn our attention to the theorem of May mentioned in the introduction.

(3.5) Definition (May [12]). — A smooth map $f: N \rightarrow P$ is *topologically transverse* to a submanifold Q of $J^k(N, P)$ at $x \in N$ if either $J^k f(x) \notin Q$ or there is an open neighbourhood U of x in N and an open neighbourhood \mathscr{W} of f in $C^\infty(N, P)$ such that for all $g \in \mathscr{W}$, one of the following holds:

- (i) $(J^k g)^{-1} Q \cap U$ is empty;
- (ii) $\text{cod } Q < \dim N$, and $(J^k g)^{-1} Q \cap U$ is a topological submanifold of U of codimension $\text{cod } Q$;
- (iii) $\text{cod } Q = \dim N$, and $(J^k g)^{-1} Q \cap U$ is a single point.

This is a local notion, in that if U_1, U_2 are any open neighbourhoods of x in N , then $f|U_1$ being topologically transverse to Q at x is equivalent to $f|U_2$ being topologically transverse to Q at x .

This is clear if $J^k f(x) \notin Q$. Otherwise, suppose that $f|U_1$ is topologically transverse to Q at x , and let U, \mathscr{W} be neighbourhoods as in the definition. We can shrink U as much as we like; so we suppose that it is a relatively compact open neighbourhood of x in $U_1 \cap U_2$. It is easy to construct a continuous map $\psi: C^\infty(U_2, P) \rightarrow C^\infty(U_1, P)$ carrying $f|U_2$ to $f|U_1$ so that $g, \psi(g)$ agree on U , while $f, \psi(g)$ agree off some relatively compact neighbourhood of \bar{U} in $U_1 \cap U_2$. Then $U, \psi^{-1}\mathscr{W}$ satisfy the condition of the definition for $f|U_2$; so this is topologically transverse to Q at x .

Note that in the above argument ψ is continuous from the weak topology on $C^\infty(U_2, P)$ to the strong topology on $C^\infty(U_1, P)$; in particular, taking $U_1 = U_2 = N$, we see that the definition of topological transversality is independent of the choice of topology on $C^\infty(N, P)$.

It is well-known that f transverse to Q implies f topologically transverse to Q . We aim to prove a converse for certain Q . The key to this is the following lemma:

(3.6) Lemma (May's "Perturbation Lemmas", [12]). — *Let Q be a smooth submanifold of a smooth manifold P . Let $y_0 \in Q$, and let $f: (N, x_0) \rightarrow (P, y_0)$ be a smooth map-germ not transverse to Q at y_0 .*

(i) Suppose $\dim N > \text{codim } Q$. Then there exists a smooth map-germ

$$F: (N \times \mathbf{R}, x_0 \times [-1, 1]) \rightarrow (P, y_0)$$

such that, for each $t \in [-1, 1] \setminus \{0\}$, $F_t^{-1}(Q)$ is not a germ of topological manifold of dimension $n - \text{codim } Q$. (Here $F_t: (N, x_0) \rightarrow (P, y_0)$ is defined by $F_t(x) = F(x, t)$.)

(ii) Suppose $\dim N = \text{codim } Q$. Then there exists a smooth map-germ

$$F: (N \times \mathbf{R}, x_0 \times [-1, 1]) \rightarrow (P, y_0)$$

such that $F^{-1}(Q)$ contains the germs of two curves through $x_0 \times 0$ which meet only at $x_0 \times 0$.

Proof. — Let $\pi: (P, y_0) \rightarrow (Z, z_0)$ be a germ of submersion such that $\pi^{-1}(z_0) = (Q, y_0)$. Choose coordinate systems $\{x_1, \dots, x_n\}$ for (N, x_0) , $\{z_1, \dots, z_k\}$ for (Z, z_0) linearly adapted for $\pi \circ f$, i.e. so that, for $r = rk.d(\pi \circ f)_{x_0}$,

$$z_i \circ \pi \circ f = x_i \quad (i = 1, \dots, r),$$

$$z_i \circ \pi \circ f \in m_N^2 \quad (i = r + 1, \dots, k).$$

Since f is not transverse to Q , $\pi \circ f$ is not a submersion, so $r < k$.

Now set $y_i = z_i \circ \pi$ ($i = 1, \dots, k$), and extend to a system of coordinates $\{y_1, \dots, y_p\}$ for (P, y_0) . For convenience we set $f_i = y_i \circ f$.

(i) By a linear change of source coordinates we can suppose that $z_k \circ \pi \circ f$ has the form

$$\sum_{i=k+1}^{\ell} \pm x_i^2 + R(\mathbf{x}),$$

where $R \in \{x_1, \dots, x_{k-1}\} \cdot m_N + m_N^3$, and $k \leq \ell \leq n$.

Define $F: (N \times \mathbf{R}, x_0 \times [-1, 1]) \rightarrow (P, y_0)$ by

$$F(x_1, \dots, x_n, t) = (x_1, \dots, x_r, f_{r+1} + tx_{r+1}, \dots, f_k + \sum_{i=\ell+1}^n tx_i^2, f_{k+1}, \dots, f_p).$$

We have, then,

$$\begin{aligned} \pi \circ F_t(x_1, \dots, x_n) &= (x_1, \dots, x_r, f_{r+1} + tx_{r+1}, \dots, \\ &f_{k-1} + tx_{k-1}, \sum_{i=k}^{\ell} \pm x_i^2 + \sum_{i=\ell+1}^n tx_i^2 + R(\mathbf{x})). \end{aligned}$$

It follows at once from [17], (3.1), that, for any $t \in [-1, 1] \setminus \{0\}$, coordinate systems $\{x'_1, \dots, x'_n\}$ for (N, x_0) , $\{z'_1, \dots, z'_k\}$ for (Z, z_0) can be chosen so that in these coordinates

$$\pi \circ F_t(x'_1, \dots, x'_n) = (x'_1, \dots, x'_{k-1}, \sum_{i=k}^n \pm x_i'^2),$$

hence

$$F_t^{-1}(Q) = (\pi \circ F_t)^{-1}(0) = \{x'_1 = \dots = x'_{k-1} = 0; \sum_{i=k}^n \pm x_i'^2 = 0\}$$

which is either a single point or a quadratic cone in $(n - k + 1)$ -space, and so is not a germ of topological submanifold of codimension $k (= \text{cod } Q)$ at x_0 .

(ii) Here $n = k$. Since, for $r + 1 \leq i \leq n$, $f_i \in m_{\mathbf{N}}^2$, there is a smooth function-germ $h_i : (\mathbf{R}, 0) \rightarrow (\mathbf{R}, 0)$ such that

$$f_i(0, \dots, 0, u) = u^2 h_i(u).$$

Define $F : (N \times \mathbf{R}, x_0 \times [-1, 1]) \rightarrow (P, y_0)$ by

$$F(x_1, \dots, x_n, t) = (x_1, \dots, x_r, f_{r+1} - x_n \text{th}_{r+1}(x_n), \dots, f_n - x_n \text{th}_n(x_n), f_{n+1}, \dots, f_p).$$

Then we have

$$\pi \circ F(x_1, \dots, x_n, t) = (x_1, \dots, x_r, f_{r+1} - x_n \text{th}_{r+1}(x_n), \dots, f_n - x_n \text{th}_n(x_n))$$

so that the curves

$$(0, \dots, 0, 0, t) \quad (-1 \leq t \leq 1)$$

and $(0, \dots, 0, t, t) \quad (-1 \leq t \leq 1)$

are contained in $(\pi \circ F)^{-1}(0) = F^{-1}(Q)$. \square

(3.7) Theorem (May [12]). *Let $V \subset J^r(n, p)$ be a \mathcal{H}^r -invariant submanifold which satisfies the immersion condition. Then a smooth map $f : N \rightarrow P$ is transverse to V if and only if $J^r f$ is topologically transverse to V .*

Proof. — “Only if”: is clear.

“If”: in the case $(J^r f)^{-1} V$ empty, there is nothing to prove. So we suppose $\text{cod } V \leq \dim N$ and that $(J^r f)^{-1} V$ is not empty; we will show that if f is not transverse to V , then f is not topologically transverse to V . Thus suppose f fails to be transverse to V at some point $x \in (J^r f)^{-1} V$. Since the notions involved are local, we can work in local coordinate charts at $x, f(x)$, and thus, indeed, suppose that f is a map $(N, 0) \rightarrow (P, 0)$, where N, P are open neighbourhoods of 0 in $\mathbf{R}^n, \mathbf{R}^p$ respectively.

Let $\{\tilde{G} : (\mathbf{R}^{n+k}, 0) \rightarrow (\mathbf{R}^{p+k}, 0); 1_N \times 0, 1_P \times 0\}$ be an unfolding of the germ f of f at 0 , transverse to V_k —it is easy to see that such \tilde{G} exist; for example, take the linear unfolding of f by a collection of map-germs $(\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^p, 0)$ whose r -jets span the normal space to V at $j^r f$.

Taking a representative G for \tilde{G} , we can, shrinking as necessary, suppose that

$$\{G : N \times W \rightarrow P \times W; 1_N \times 0, 1_P \times 0\}$$

(where W is an open neighbourhood of 0 in \mathbf{R}^k) is an unfolding of f , and that G is transverse to V_k .

Since V satisfies the immersion condition, we can, shrinking N, P, W further if necessary, suppose that $G | (J^r G)^{-1} V_k$ is an embedding, so

$$V_G = G((J^r G)^{-1} V_k)$$

is a submanifold of $P \times W$.

We have

$$\begin{aligned}\operatorname{cod} V_G &= p + k - (n + k - \operatorname{cod}(J^r G)^{-1} V_k) \\ &= p - n + \operatorname{cod} V\end{aligned}$$

i.e. $p - \operatorname{cod} V_G = n - \operatorname{cod} V.$

Thus, since $n \geq \operatorname{cod} V$, $p \geq \operatorname{cod} V_G$, with equality if and only if $n = \operatorname{cod} V$.

Since f is not transverse to V at 0 , it follows from (3.3) that $i_p \times 0$ is not transverse to V_G at 0 .

By the Perturbation Lemma (3.6), there exists a map-germ

$$J : (\mathbf{P} \times \mathbf{R}, 0 \times [-1, 1]) \rightarrow (\mathbf{P} \times W, 0)$$

with $J_0 = i_p \times 0$ such that:

- (i) if $p < \operatorname{cod} V_G$ (i.e. $n < \operatorname{cod} V$), then $J_t^{-1}(V_G)$ ($t \neq 0$) is not a germ of topological submanifold of dimension $p - \operatorname{cod} V_G$;
- (ii) if $p = \operatorname{cod} V_G$ (i.e. $n = \operatorname{cod} V$), then $J^{-1}(V_G)$ contains the germs of two curves meeting only at 0 .

By (3.4), and by taking representatives, we can, after possibly shrinking N, P, W , find $\eta > 0$ and maps

$$\begin{aligned}I : N \times [-\eta, \eta] &\rightarrow N \times W && \text{with } I_0 = 1_N, \\ F : N \times [-\eta, \eta] &\rightarrow P && \text{with } F_0 = f,\end{aligned}$$

such that, for all $t \in [-\eta, \eta]$, $\{G; I_t, J_t\}$ is an unfolding of F_t . We have, then,

$$\begin{aligned}J_t(J_t^{-1}(V_G)) &= J_t(P) \cap G((J^r G)^{-1} V_k) \\ &= G(I_t(N) \cap (J^r G)^{-1} V_k) \\ &= G \circ I_t((J^r F_t)^{-1} V),\end{aligned}$$

so that, since $J_t, I_t, G|_{(J^r G)^{-1} V_k}$ are embeddings, $J_t^{-1} V_G$ and $(J^r F_t)^{-1} V$ are homeomorphic.

It now follows from (i), (ii), above that

- (i) if $n > \operatorname{cod} V$, then $(J^r F_t)^{-1} V$ is not a topological manifold of dimension $n - \operatorname{cod} V$ for $t \in [-\eta, \eta] \setminus \{0\}$,
- (ii) if $n = \operatorname{cod} V$, then there exists ε , $0 < \varepsilon < \eta$, so that $(J^r F_t)^{-1} V$ contains at least two points for $t \in [-\varepsilon, \varepsilon] \setminus \{0\}$.

Since $t \rightarrow F_t$ is continuous (w.r.t. τ_{C^∞}), f is not topologically transverse to V at 0 . \square

It will also be convenient to have the following result (more a corollary of the proof than of the theorem itself) to hand:

(3.8) Corollary. — *Let $V \subset J^r(n, p)$ be a \mathcal{K}^r -invariant submanifold of codimension $< n$ satisfying the immersion condition. Let $f : (\mathbf{R}^p, 0) \rightarrow (\mathbf{R}^p, 0)$ be a map-germ such that $j^r f \in V$,*

but f is not V -transverse. Then there exists $s = s(n, p, r)$ such that $j^\infty f$ may be arbitrary closely approximated in $J^\infty f(n, p)$ by jets $z \in (\pi^{\infty, r})^{-1} V$ such that, for any germ $g : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^p, 0)$ with the same s -jet as z , $(J^r g)^{-1} V$ is not a topological manifold of dimension $n - \text{cod } V$.

Proof. — There exists $s_1 \geq r + 1$ such that

$$\text{cod } W^{s_1}(n, p) > \dim J^r(n, p);$$

s_1 clearly depends only on r, n, p .

(Here $W^r(n, p)$ is as defined in [27], p. 513; it is well-known that $\text{cod } W^r(n, p) \rightarrow \infty$, which establishes the claim.)

It follows at once that $j^\infty f$ can be arbitrarily closely approximated by $j^\infty f'$, for map-germs $f' : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^p, 0)$ with $j^{r+1} f' = j^{r+1} f$ and $j^{s_1} f' \notin W^{s_1}(n, p)$. Such f' are thus of finite singularity type, indeed, as is well-known, we have $m_n^{s_1} \cdot C_n^{\times p} \subset T\mathcal{K}f'$; so f' has a versal unfolding $\{G : (\mathbf{R}^{n+k}, 0) \rightarrow (\mathbf{R}^{p+k}, 0); i', j'\}$.

Since f' is not transverse to V , j' is not transverse to V_G and, by (3.6), (i), we can find a germ

$$J : (\mathbf{R}^p \times \mathbf{R}, 0 \times [-1, 1]) \rightarrow (\mathbf{R}^{p+k}, 0),$$

with $J_0 = j$, such that for $t \in (\mathbf{R}, [-1, 1]) \setminus \{0\}$, $(J_t)^{-1} V_G$ is either a quadratic cone or an isolated point. Indeed it is clear from the proof of (3.6), (i) that the same holds for any germ j^* with the same 2-jet as such a J_t .

By (3.4), there exists $\eta, 0 < \eta \leq 1$, and germs

$$\begin{aligned} I : (\mathbf{R}^n \times \mathbf{R}, 0 \times [-\eta, \eta]) &\rightarrow (\mathbf{R}^{n+k}, 0), & I_0 &= 1_n \times 0, \\ F : (\mathbf{R}^n \times \mathbf{R}, 0 \times [-\eta, \eta]) &\rightarrow (\mathbf{R}^p, 0), & F_0 &= f, \end{aligned}$$

such that $(G; I_t, J_t)$ is an unfolding of F_t for all $t \in (\mathbf{R}, [-\eta, \eta])$.

By [14], (3.5), for any map-germ $f^* : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^p, 0)$ with the same $3s_1$ -jet as F_t ($t \in (\mathbf{R}, [-\eta, \eta]) \setminus \{0\}$), there exist immersion-germs $i^* : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^{n+k}, 0)$, $j^* : (\mathbf{R}^p, 0) \rightarrow (\mathbf{R}^{p+k}, 0)$ with the same 2-jets as I_t, J_t respectively, such that $\{G; i^*, j^*\}$ is an unfolding of f^* . By the argument used in (3.7), $(J^r f^*)^{-1} V$ is homeomorphic to $(j^*)^{-1} V_G$, and so is not a topological manifold of dimension $n - \text{cod } V$.

Thus, taking $s = 3s_1$, we see that we can indeed approximate $j^\infty F$ by jets $z = j^\infty F_t$ with the required properties. \square

As a first application, we have:

(3.9) Lemma.

A) If the smooth map $f : N \rightarrow P$ is C^l -stable then it is transverse to all almost-strong $\mathcal{A}^{(l)}$ -quasi-invariants satisfying the immersion condition and of codimension $\neq n$.

B) If the smooth map-germ $f : (N, x_0) \rightarrow (P, y_0)$ is ∞ - $\mathcal{A}^{(l)}$ -determined, then f has a representative $\tilde{f} : U \rightarrow V$ such that $f|_{U - \{x_0\}}$ is transverse to all almost-strong $\mathcal{A}^{(l)}$ -quasi-invariants satisfying the immersion condition and of codimension $\neq n$.

Proof.

A) Let S be an almost-strong $\mathcal{A}^{(\ell)}$ -quasi-invariant satisfying the immersion condition and of codimension $\neq n$. Let \mathcal{U} be an open neighbourhood of f such that if $f' \in \mathcal{U}$, then f, f' are C^ℓ -equivalent. Then $f' \in \mathcal{U}$ implies that f' is C^ℓ -stable, and so by (2.7) A, $(J^r f')^{-1} S$ is a C^ℓ -manifold of dimension $n - \text{cod } S$, or is empty. Thus f is topologically transverse to S , and so, since S satisfies the immersion condition, f is transverse to S , as required.

B) Suppose the statement is false, so that there is a representative $\tilde{f}: U' \rightarrow V'$ of f , a sequence $\{x_i\} \subset U' - \{x_0\}$, $x_i \rightarrow x_0$, and almost-strong $\mathcal{A}^{(\ell)}$ -quasi-invariants $S_i \subset J^{r_i}(n, p)$ satisfying the immersion condition and of codimension $\neq n$ such that $J^{r_i} \tilde{f}(x_i) \in S_i$, but $J^{r_i} \tilde{f}$ is not transverse to S_i at x_i . By (2.7) B, $\text{cod } S_i < n$ (for $(J^{r_i} f)^{-1} S$ is not empty, so it is a C^ℓ -submanifold of dimension $n - \text{cod } S_i$, so $n \geq \text{cod } S_i$; but $n = \text{cod } S_i$ is ruled out by hypothesis).

According to (3.7), there exist $s_i = s(n, p, r_i)$ such that $J^\infty \tilde{f}(x_i)$ can be arbitrarily closely approximated by ∞ -jets $z_i \in (\pi^{\infty, r_i})^{-1} S_i$ with source x_i , target $\tilde{f}(x_i)$, such that, for any germ g_i with $j^{s_i} g_i = \pi^{\infty, s_i}(z_i)$, $(J^{r_i} g_i)^{-1} S_i$ is not a topological manifold of dimension $n - \text{cod } S_i$.

It follows from a result of Wilson ([27], (3.3)) that, after possibly passing to a subsequence of the x_i 's, we can find a map $g: U' \rightarrow V'$ with $j^\infty g_{x_0} = j^\infty f$ and $j^\infty g(x_i) = z_i$. Since f is ∞ - $\mathcal{A}^{(\ell)}$ -determined, so is g_{x_0} ; so that, by (2.7) B, there exist neighbourhoods U of x_0 in U' , V of y_0 in V' , with $g(U) \subset V$, such that, for any almost-strong $\mathcal{A}^{(\ell)}$ -quasi-invariant $S \subset J^r(n, p)$, $(J^r g | U - \{x_0\})^{-1} S$ is a C^ℓ -manifold of dimension $n - \text{cod } S$, or empty. This is a contradiction, for $(J^r g | U - \{x_0\})^{-1} S_i$ is neither empty nor a topological manifold of dimension $n - \text{cod } S_i$ at x_i , for all $x_i \in U$.

So our supposition was false, and the result is proved. \square

We can now give:

(3.10) (= (0.9)) Lemma ($\ell = 0, 1, 2, \dots$ or ∞). — Let $S \subset J^r(n, p)$ be a strong $\mathcal{A}^{(\ell)}$ -quasi-invariant satisfying the immersion condition, and let W be a \mathcal{K}^r -invariant submanifold of S furnished with a smooth submersion $\pi: W \rightarrow L$ (with $\dim L < \dim W$) whose fibres are $\mathcal{A}^{(\ell)}$ -invariants for S -transverse germs.

Let N, P be manifolds of dimensions n, p respectively.

A) If the smooth map $f: N \rightarrow P$ is C^ℓ -stable, then f is transverse to the fibres of π .

B) If the smooth map-germ $f: (N, x_0) \rightarrow (P, y_0)$ is ∞ - $\mathcal{A}^{(\ell)}$ -determined, then f has a representative $\tilde{f}: U \rightarrow V$ such that $\tilde{f}| U - \{x_0\}$ is transverse to the fibres of π .

Proof. — The result follows at once from (3.9) if the codimension of the fibres of π in $J^r(n, p)$ is not equal to n .

If this codimension is n :

A) By (2.3) A, f is transverse to W , and by (2.7) A $(J^r f)^{-1} \{ \text{a fibre of } \pi \}$ is a set of isolated points of $(J^r f)^{-1} W$. Let E_0 be a fibre of π . By the Thom transversality theorem (2.1) A, f can be arbitrarily closely approximated by, and so is C^l -equivalent to, a map $g : N \rightarrow P$ transverse to W and to E_0 . Arguing as in the proof of (2.6), the equivalence between f and g carries $(J^r f)^{-1} W$ to $(J^r g)^{-1} W$ and $(J^r f)^{-1} E$ to $(J^r g)^{-1} E$ for every fibre E of π ; so that there exists a C^l -diffeomorphism $\varphi : (J^r f)^{-1} W \rightarrow (J^r g)^{-1} W$ such that

$$\pi \circ (J^r f | (J^r f)^{-1} W) \circ \varphi^{-1} = \pi \circ (J^r g | (J^r g)^{-1} W).$$

Now g transverse to E_0 is equivalent to

$$\pi \circ (J^r g | (J^r g)^{-1} W) : (J^r g)^{-1} W \rightarrow L$$

being a submersion near $(J^r g)^{-1} E_0$; composing with φ , we see that $\pi \circ (J^r f | (J^r f)^{-1} W)$ is a submersion near $(J^r f)^{-1} E_0$, so f is transverse to E_0 .

Since E_0 was chosen arbitrarily, the result is proved.

B) By (2.3) B and (2.7) B, f has a representative $\tilde{f} : U \rightarrow V$ such that $\tilde{f} | U - \{x_0\}$ is transverse to W , and such that $(J^r \tilde{f} | U - \{x_0\})^{-1} \{ \text{a fibre of } \pi \}$ is a set of isolated points of $(J^r \tilde{f} | U - \{x_0\})^{-1} W$.

Suppose that the result is false, so that there exist fibres E_i of π ($i = 1, 2, \dots$) and points $x_i \in (J^r \tilde{f} | U - \{x_0\})^{-1} E_i$, with $x_i \rightarrow x_0$, such that f is not transverse to E_i at x_i . By the transversality theorem (2.1) B, \tilde{f} can be arbitrarily closely approximated by maps $g : U \rightarrow V$ transverse to W and to E_1, E_2, \dots off x_0 , with the same ∞ -jet at x_0 as \tilde{f} . Since f is ∞ - $\mathcal{A}^{(l)}$ -determined, we can find neighbourhoods \tilde{U} of x_0 in U , \tilde{V} of y_0 in V , with $\tilde{f}(\tilde{U}) \subset \tilde{V}$, and C^l -embeddings $h : (\tilde{U}, x_0) \rightarrow (U, x_0)$, $k : (\tilde{V}, y) \rightarrow (V, y_0)$ such that

$$k \circ \tilde{f} | \tilde{U} = g \circ h;$$

i.e. $\tilde{f} | \tilde{U} : \tilde{U} \rightarrow \tilde{V}$ and $g | h(\tilde{U}) : h(\tilde{U}) \rightarrow k(\tilde{V})$ are C^l -equivalent. Now, arguing as in (2.2) and (2.6), we see that the equivalence carries $(J^r \tilde{f} | \tilde{U} - \{x_0\})^{-1} W$ to $(J^r g | h(\tilde{U}) - \{x_0\})^{-1} W$ and $(J^r \tilde{f} | \tilde{U} - \{x_0\})^{-1} E$ to $(J^r g | h(\tilde{U}) - \{x_0\})^{-1} E$ for each fibre E of π ; so that the C^l -diffeomorphism

$$\varphi = h | : (J^r \tilde{f} | \tilde{U} - \{x_0\})^{-1} W \rightarrow (J^r g | h(\tilde{U}) - \{x_0\})^{-1} W$$

is such that

$$\begin{aligned} \pi \circ [J^r \tilde{f} | (J^r \tilde{f} | U - \{x_0\})^{-1} W] \circ \varphi^{-1} \\ = \pi \circ [J^r g | (J^r g | h(\tilde{U}) - \{x_0\})^{-1} W]. \end{aligned}$$

Now $g | h(\tilde{U}) - \{x_0\}$ being transverse to E_i ($i = 1, 2, \dots$) is equivalent to

$$\pi \circ J^r g | : J^r g | (h(\tilde{U}) - \{x_0\})^{-1} W \rightarrow L$$

being a submersion near $(J^r g | h(\tilde{U}) - \{x_0\})^{-1} E_i$ ($i = 1, 2, \dots$); composing with φ , we conclude that $f | \tilde{U} - \{x_0\}$ is transverse to E_i ($i = 1, 2, \dots$).

This contradicts our supposition, since $x_i \in \tilde{U}$ for sufficiently large i ; so the supposition is false, and the result is proved. \square

We conclude this section by proving (0.5).

We will need the following:

(3.11) Definition. — An unfolding $\{F : (N', x'_0) \rightarrow (P', y'_0); i, j\}$ of a map-germ $f : (N, x_0) \rightarrow (P, y_0)$ is C^∞ -versal if for any other unfolding $\{F' : (N'', x''_0) \rightarrow (P'', y''_0); i', j'\}$ of f there exist smooth map-germs $\phi : (N'', x''_0) \rightarrow (N', x'_0)$, $\psi : (P'', y''_0) \rightarrow (P', y'_0)$ such that the following diagram commutes:

$$\begin{array}{ccc}
 (N', x'_0) & \xrightarrow{F} & (P', y'_0) \\
 \uparrow \phi & \swarrow i & \nearrow j \\
 (N, x_0) & \xrightarrow{f} & (P, y_0) \\
 \downarrow i' & \searrow j' & \downarrow \psi \\
 (N'', x''_0) & \xrightarrow{F'} & (P'', y''_0)
 \end{array}$$

(A pair (ϕ, ψ) as above is called a *morphism* $\{F'; i', j'\} \rightarrow \{F; i, j\}$.)

The following are the key results on C^∞ -versal unfoldings:

- I) An unfolding $\{F; i, j\}$ of f is C^∞ -versal if and only if F is infinitesimally stable.
- II) A map-germ f has a C^∞ -versal unfolding if and only if it is of finite singularity type.

(These results are due to Mather; however, the only published proof known to us is in [6], III, § 3.)

(3.12) (= (0.5)) Lemma. — Let n, p, i be positive integers. Suppose that, for some $r > 1$, there exists a non-empty \mathcal{H}^r -invariant submanifold W of $J^r(n, p)$ of codimension $\leq n - i + 1$ supporting a smooth \mathcal{H}^r -invariant foliation \mathcal{F} of codimension $\geq i$.

Then, for any manifolds N, P of dimensions n, p respectively, there exists a non-empty, τ_{C^∞} -open subset \mathcal{U} of $C^\infty(N, P)$ such that for any $f \in \mathcal{U}$ there exists a non-empty submanifold Z_f of N of dimension $\geq i - 1$ such that $J^r f$ is not transverse to the fibres of $\mathcal{F}(N, P)$ at any point of Z_f .

Proof. — Let W be of codimension $n - \ell_1 + 1$ in $J^r(n, p)$, \mathcal{F} of codimension ℓ_2 in W . The hypotheses of (0.5) are thus that $\ell_1, \ell_2 \geq i$.

Let $z \in W$, let $f : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^p, 0)$ be a map-germ of finite singularity type with r -jet z (such exists by the argument of [27], (5.1)), and let $\{F : (\mathbf{R}^{n+k}, 0) \rightarrow (\mathbf{R}^{p+k}, 0); i, j\}$ be a C^∞ -versal unfolding of f . Since F is infinitesimally stable, it is transverse to all \mathcal{H} -classes, so to W_k , so that $(J^r F)^{-1} W_k$ is a submanifold of $(\mathbf{R}^{n+k}, 0)$.

Let $W_F = F((J^r F)^{-1} W_k)$. We can suppose that W_F is a submanifold of $(\mathbf{R}^{p+k}, 0)$.

For since f is of finite singularity type, so is F ; so that the restriction of F to its critical set, and so a fortiori to $(J^r F)^{-1} W_k$, is finite-to-one. Hence the set of immersive points for $F|_{(J^r F)^{-1} W_k}$ is dense in $(J^r F)^{-1} W_k$, so that, taking a representative \tilde{F} of F , we can find $\tilde{x} \in (J^r \tilde{F})^{-1} W_k$ arbitrarily close to 0 such that the germ of $\tilde{F}|_{(J^r \tilde{F})^{-1} W_k}$ at \tilde{x} is an immersion. Infinitesimal stability being an open condition, $\tilde{F}_{\tilde{x}}$ is infinitesimally stable. But it is easy to see that there exist map-germs $\tilde{f}: (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^p, 0)$, $\tilde{i}: (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^{n+k}, \tilde{x})$, $\tilde{j}: (\mathbf{R}^p, 0) \rightarrow (\mathbf{R}^{p+k}, \tilde{f}(\tilde{x}))$ such that $\{\tilde{F}_{\tilde{x}}; \tilde{i}, \tilde{j}\}$ is an unfolding of \tilde{f} . Now replace z, f, F by $j^r \tilde{f}, \tilde{f}, \tilde{F}_{\tilde{x}}$.

Since W has codimension $n - \ell_1 + 1$ in $J^r(n, p)$, W_F has codimension $p - \ell_1 + 1$ in \mathbf{R}^{p+k} .

The smooth foliation \mathcal{F} of W induces a smooth foliation-germ \mathcal{F}_F of W_F of codimension ℓ_2 (where, if L is a leaf of W , the corresponding leaf L_F of \mathcal{F}_F is $F((J^r F)^{-1} L)$); let $\zeta: (W_F, 0) \rightarrow (\mathbf{R}^{\ell_2}, 0)$ be a submersion-germ whose fibres are the leaves of \mathcal{F}_F . Choose a germ of embedding $j_1: (\mathbf{R}^p, 0) \rightarrow (\mathbf{R}^{p+k}, 0)$ which is transverse to W_F , and so to F (so that $\text{Im } j_1 \cap W_F$ has dimension $\ell_1 - 1$), and, if $\ell_1 - 1 \geq \ell_2$, such that $\zeta|_{\text{Im } j_1 \cap W_F}$ has an A_1 -singularity (in the notation of [8, VI]) at 0.

Now let N, P be smooth manifolds of dimensions n, p respectively; and let $x_0 \in N, y_0 \in P$. Let $(Q, 0 \times 0) \subset (\mathbf{R}^{n+k} \times \mathbf{R}^{p+k}, 0 \times 0)$ be the fibre-product of F and j_1 ; Q is an n -manifold. Let $h: (N, x_0) \rightarrow (Q, 0 \times 0), k: (P, y_0) \rightarrow (\mathbf{R}^p, 0)$ be arbitrary diffeomorphism-germs and define

$$j^* = j_1 \circ k,$$

$$i^* = \pi_1 \circ h,$$

$$f^* = \pi_2 \circ h,$$

where π_1, π_2 are projections onto the factors of $\mathbf{R}^{n+k} \times \mathbf{R}^{p+k}$.

Clearly $\{F; i^*, j^*\}$ is an unfolding of f^* ; indeed, F being infinitesimally stable, it is a C^∞ -versal unfolding. Moreover j^* is transverse to W_F , and, if $\ell_1 - 1 \geq \ell_2$, $\zeta|_{\text{Im } j^* \cap W_F}$ has an A_1 -singularity at x_0 .

Let $\tilde{f}: N \rightarrow P$ be any smooth map whose germ at x_0 is f^* . Let

$$d = \min(\ell_1, \ell_2) - 1$$

and define

$$\mathcal{U} = \{g \in C^\infty(N, P) \mid \text{There is a non-empty submanifold } Z_g \subset N \text{ of dimension } d \text{ such that } g \text{ is not transverse to } \mathcal{F} \text{ at any point of } Z_g.\}$$

To prove (0.5) it will now suffice to show that \mathcal{U} is a neighbourhood of \tilde{f} in τ_{C^∞} .

Since τ_{C^∞} has a countable basis, it is enough to show that every sequence of maps tending to f in τ_{C^∞} has an infinite subsequence contained in \mathcal{U} .

Let f_1, f_2, \dots be a sequence in $C^\infty(N, P)$ tending to f in τ_{C^∞} . Passing to a sub-

sequence we can, by a construction of Looijenga ([6], IV, (4.2)), suppose that there exists a smooth \mathbf{R} -level-preserving map $\tilde{G} : N \times \mathbf{R} \rightarrow P \times \mathbf{R}$ such that

$$\begin{aligned}\tilde{G}_{2-k} &= f_k \quad (k = 1, 2, \dots), \\ \tilde{G}_t &= f \quad \text{for } t \leq 0.\end{aligned}$$

Let G be the germ of \tilde{G} at $(x_0, 0)$. Then $\{G; i_N \times 0, i_P \times 0\}$ is an unfolding of f^* , so, $\{F; i^*, j^*\}$ being a versal unfolding of f^* , there exists a morphism

$$(\varphi, \psi) : \{G; i_N \times 0, i_P \times 0\} \rightarrow \{F; i^*, j^*\}.$$

Now take representatives of the map-germs involved:

$$\begin{array}{ccc} U' & \xrightarrow{\tilde{F}} & V' \\ \tilde{i} \swarrow & & \nearrow \tilde{j} \\ U & \xrightarrow{\tilde{f}|} & V \\ \tilde{\phi} \uparrow & & \uparrow \tilde{\psi} \\ U \times [-\varepsilon, \varepsilon] & \xrightarrow{\tilde{G}|} & V \times [-\varepsilon, \varepsilon] \end{array}$$

$1_U \times 0$ $1_V \times 0$

(where $\varepsilon > 0$, U, U', V, V' are neighbourhoods of $x_0, 0, y_0, 0$ in $N, \mathbf{R}^{n+k}, P, \mathbf{R}^{p+k}$ respectively, and $\tilde{F}, \tilde{i}, \tilde{j}, \tilde{\phi}, \tilde{\psi}$ are representatives for $F, i^*, j^*, \varphi, \psi$).

We can suppose, shrinking U, U', V, V' , and decreasing ε as necessary, that the above diagram commutes, all three squares are cartesian, \tilde{i}, \tilde{j} are embeddings, \tilde{F} is transverse to W_k , $\tilde{F}|(J^r \tilde{F})^{-1} W_k$ is an embedding whose image we call $W_{\tilde{F}}$, and \tilde{j} is transverse to $W_{\tilde{F}}$.

Define

$$\begin{aligned}\tilde{\varphi}_t : U &\rightarrow U' \quad \text{by } \tilde{\varphi}_t(x) = \tilde{\varphi}(x, t), \\ \tilde{\psi}_t : V &\rightarrow V' \quad \text{by } \tilde{\psi}_t(y) = \tilde{\psi}(y, t).\end{aligned}$$

Since $\tilde{\varphi}_0 = \tilde{i}$, $\tilde{\psi}_0 = \tilde{j}$, $\tilde{\varphi}_t, \tilde{\psi}_t$ are embeddings for sufficiently small t .

Since \tilde{j} is transverse to \tilde{F} and meets $W_{\tilde{F}}$ transversely at 0, $\tilde{\psi}_t$ is transverse to \tilde{F} and meets $W_{\tilde{F}}$ transversely at some point $y'_t \in V'$ near 0, for all sufficiently small t ; it follows by (3.1) that $\tilde{\varphi}_t$ meets $(J^r \tilde{F})^{-1} W_k$ transversely at $x'_t \in U'$ (with $\tilde{F}(x'_t) = y'_t$). Decreasing ε , we can suppose that all this holds for all $t \in [-\varepsilon, \varepsilon]$.

Suppose now that $\ell_1 \leq \ell_2$; and let $t \in [-\varepsilon, \varepsilon]$. Then $\tilde{\psi}_t$ fails to be transverse to $\mathcal{F}_{\tilde{F}}$ at any point of $\tilde{\psi}_t^{-1}(W_{\tilde{F}})$ (for the fibres of $\mathcal{F}_{\tilde{F}}$ are of codimension $(p - \ell_1 + 1) + \ell_2 > p$); and it follows from (3.1) that G_t is not transverse to \mathcal{F} at any point of

$$Z_t = \tilde{\varphi}_t^{-1}(\tilde{F}|(J^r \tilde{F})^{-1} W_k)^{-1} (\text{Im } \psi_t \cap W_{\tilde{F}}),$$

which is a non-empty (since it contains x'_t) $(\ell_1 - 1)$ -manifold; so that $G_t \in \mathcal{U}$.

Now suppose that $\ell_1 > \ell_2$. Shrinking U, U', V, V' again as necessary, we can

suppose that there is a representative $\tilde{\zeta}: W_{\tilde{F}} \rightarrow \mathbf{R}^{\ell_2}$ of ζ which is a submersion whose fibres are the leaves of $\mathcal{F}_{\tilde{F}}$.

Since $\tilde{\zeta}|_{\text{Im } \tilde{j} \cap W_{\tilde{F}}}$ has an A_1 -singularity at 0, $\tilde{\zeta}|_{\text{Im } \psi_t \cap W_{\tilde{F}}}$ has an A_1 -singularity at some point y'_t near 0 for all sufficiently small t ; decreasing ε , we can suppose this holds for all $t \in [-\varepsilon, \varepsilon]$.

The structure of A_1 -singularities shows at once that there is a neighbourhood N_t of y'_t in the critical set of $\tilde{\zeta}|_{\text{Im } \tilde{\psi}_t \cap W_{\tilde{F}}}$ which is a submanifold of $\text{Im } \tilde{\psi}_t \cap W_{\tilde{F}}$ of dimension $\ell_2 - 1$. Of course $\tilde{\psi}_t$ fails to be transverse to $\mathcal{F}_{\tilde{F}}$ at any point of $\tilde{\psi}_t^{-1}(N_t)$; and so it follows from (3.1) that G_t is not transverse to \mathcal{F} at any point of

$$Z_t = \varphi_t^{-1}(\tilde{F}|(J^r \tilde{F})^{-1}W_k)^{-1}(N_t),$$

which is a non-empty (since it contains $(\tilde{F}|(J^r \tilde{F})^{-1}W_k)^{-1}y'_t$) $(\ell_2 - 1)$ -manifold; so that $G_t \in \mathcal{U}$.

Since $\tilde{G}_{2^{-k}} = f_k$, we have shown that $f_k \in \mathcal{U}$ for $k \geq -\log \varepsilon / \log 2$, and the proof is complete. \square

4. Intrinsic derivatives, Thom-Boardman varieties, and $\mathcal{A}^{(1)}$ -equivalence

I. — Intrinsic second derivatives

Let $f: (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^p, 0)$ be a smooth map-germ, and let h, k be germs of smooth diffeomorphisms of $(\mathbf{R}^n, 0)$, $(\mathbf{R}^p, 0)$ respectively.

It follows from standard results on differentiating products that

$$\begin{aligned} d^2(k \circ f \circ h)_0(u, v) &= d^2 k_0 \cdot (df_0 \cdot dh_0, df_0 \cdot dh_0)(u, v) \\ &\quad + dk_0 \cdot d^2 f_0(dh_0, dh_0)(u, v) + dk_0 \cdot df_0 \cdot d^2 h_0(u, v) \end{aligned} \quad (1)$$

Let $\pi: \mathbf{R}^p \rightarrow \mathbf{R}^p / \text{Im } d(kfh)_0$ denote the projection; then (1) shows that, if $v \in \text{Ker } d(kfh)_0$, one has

$$\pi \cdot d^2(k \circ f \circ h)_0(u, v) = dk_0 \cdot d^2 f_0(dh_0, dh_0)(u, v) \quad (2)$$

Following Porteous [18], we define the *intrinsic second derivative*

$$\delta^2 f: \mathbf{R}^n \circ \text{Ker } df_0 \rightarrow \text{Coker } df_0$$

as the restriction and projection of the second derivative of f in some coordinates.

This is well-defined (i.e. independent of the coordinates chosen) by (2)—hence the name—and we have, again as an immediate consequence of (2):

(4.1) *Proposition* (Porteous [18]).

$$\delta^2(k \circ f \circ h) = \overline{dk_0} \cdot \delta^2 f \cdot (dh_0, dh_0 |)$$

(where $\overline{dk_0}: \text{Coker } df_0 \rightarrow \text{Coker } d(k \circ f \circ h)_0$ is the isomorphism induced by dk_0 , and $dh_0 | : \text{Ker } df_0 \rightarrow \text{Ker } d(k \circ f \circ h)_0$ is the isomorphism induced by restricting dh_0). \square

Let $f, f' : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^p, 0)$ be smooth map-germs. We will say that their second intrinsic derivatives are *equivalent* if there exist linear isomorphisms

$$\begin{cases} \alpha : \text{Coker } df_0 \rightarrow \text{Coker } df'_0 \\ \beta : \mathbf{R}^n \rightarrow \mathbf{R}^n \text{ such that } \beta(\text{Ker } df'_0) = \text{Ker } df_0 \end{cases}$$

such that

$$\delta^2 f' = \alpha \cdot \delta^2 f(\beta, \beta | \text{Ker } df'_0).$$

We have

(4.2) Proposition. — *If $f, f' : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^p, 0)$ have equivalent intrinsic second derivatives, then their 2-jets are $J^2 \mathcal{A}$ -equivalent.*

Proof. — After possibly composing f' with appropriate linear isomorphisms of $\mathbf{R}^n, \mathbf{R}^p$, we can suppose that

$$\begin{cases} df_0 = df'_0, \\ \delta^2 f = \delta^2 f' \end{cases} \quad (*)$$

Choose coordinates in $\mathbf{R}^n, \mathbf{R}^p$ which are linearly adapted for f , so that f has the form

$$(u_1, \dots, u_r, x_1, \dots, x_{n-r}) \mapsto (u_1, \dots, u_r, f_1(\mathbf{u}, \mathbf{x}), \dots, f_{p-r}(\mathbf{u}, \mathbf{x}))$$

where f_1, \dots, f_{p-r} have zero 1-jet.

By (*), f' has the form

$$(u_1, \dots, u_r, x_1, \dots, x_{n-r}) \mapsto (u_1 + a_1(\mathbf{u}, \mathbf{x}), \dots, u_r + a_r(\mathbf{u}, \mathbf{x}), f_1(\mathbf{u}, \mathbf{x}) + Q_1(\mathbf{u}) + b_1(\mathbf{u}, \mathbf{x}), \dots, f_{p-r}(\mathbf{u}, \mathbf{x}) + Q_{p-r}(\mathbf{u}) + b_{p-r}(\mathbf{u}, \mathbf{x}))$$

where $j^1 a_i = 0$ ($i = 1, \dots, r$), $j^2 b_j = 0$ ($j = 1, \dots, p - r$), and Q_i ($i = 1, \dots, p - r$) are quadratic forms.

Define germs of diffeomorphisms h of $(\mathbf{R}^n, 0)$, k of $(\mathbf{R}^p, 0)$ by

$$\begin{cases} u_i \circ h = u_i - a_i(\mathbf{u}, \mathbf{x}) & (i = 1, \dots, r) \\ x_j \circ h = x_j & (j = 1, \dots, n - r) \end{cases}$$

and

$$\begin{cases} U_i \circ k = U_i & (i = 1, \dots, r) \\ Y_j \circ k = Y_j + Q_j(\mathbf{U}) & (j = 1, \dots, p - r) \end{cases}$$

(where $\{\mathbf{U}, \mathbf{Y}\}$ is the coordinate system chosen in \mathbf{R}^p).

Then $k \circ f' \circ h$ has the same 2-jet as f , and the proof is complete. \square

All this is both easy and (presumably) well-known. The relation to $\mathcal{A}^{(1)}$ -equivalence is given by:

(4.3) Proposition. — *Let $f, f' : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^p, 0)$ be $\mathcal{A}^{(1)}$ -equivalent smooth map-germs. Then $\delta^2 f, \delta^2 f'$ are equivalent.*

Proof. — Let $f' = k \circ f \circ h$, where h, k are germs of C^1 -diffeomorphisms of $(\mathbf{R}^n, 0)$, $(\mathbf{R}^p, 0)$ respectively.

Taking representatives, we have, for $u, v \in \mathbf{R}^n$ and all sufficiently small $t \in \mathbf{R}$:

$$\begin{aligned} (d(k \circ f \circ h)_{tu} - d(k \circ f \circ h)_0)(v) &= (dk_{f \circ h(tu)} - dk_0) \cdot d(f \circ h)_0(v) \\ &+ dk_{f \circ h(tu)} \cdot (df_{h(tu)} - df_0) \cdot dh_{tu}(v) + dk_{f \circ h(tu)} \cdot df_0 \cdot (dh_{tu} - dh_0)(v). \end{aligned}$$

We will take $v \in \text{Ker } df'_0 = \text{Ker } d(f \circ h)_0$; so the first term on the RHS is zero.

Now divide both sides by t , and consider the limit as $t \rightarrow 0$. The LHS tends to $d^2(k \circ f \circ h)_0(u, v)$, while the second term on the RHS tends to $dk_0 \cdot d^2 f_0(dh_0, dh_0)(u, v)$. It follows that the third term on the RHS must tend to a limit too.

Since this term is contained in $dk_{f \circ h(tu)} \cdot \text{Im } df_0$, the limit is contained in the limit as $t \rightarrow 0$ of $dk_{f \circ h(tu)} \cdot \text{Im } df_0$ (considered as points in the appropriate Grassmannian), which is $dk_0 \cdot \text{Im } df_0$ (because dk_0 is an isomorphism) $= \text{Im } d(k \circ f)_0 = \text{Im } df'_0$.

Thus if $\pi : \mathbf{R}^p \rightarrow \mathbf{R}^p / \text{Im } df'_0$ is the projection, we have

$$\pi \cdot d^2 f'_0(u, v) = \pi \cdot dk_0 \cdot d^2 f_0(dh_0, dh_0)(u, v).$$

It follows that

$$\delta^2 f' = \overline{dk_0} \cdot \delta^2 f(dh_0, dh_0 |)$$

(with notation as in (4.1))

so that $\delta^2 f, \delta^2 f'$ are indeed equivalent. \square

Putting (4.2) and (4.3) together, we obtain:

(4.4) Proposition. — *If $f, f' : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^p, 0)$ are $\mathcal{A}^{(1)}$ -equivalent, then $j^2 f, j^2 f'$ are $J^2 \mathcal{A}$ -equivalent. \square*

This yields:

(4.5) Corollary. — *The first-order Thom-Boardman varieties $\Sigma^i \subset J^1(n, p)$ are strong $\mathcal{A}^{(1)}$ -invariants.*

Proof. — It is clear that the rank of a map-germ is invariant under $\mathcal{A}^{(1)}$ -equivalence; so the Σ^i are $\mathcal{A}^{(1)}$ -invariants.

It is also clear that whether or not a smooth map-germ of kernel-rank i is transverse to Σ^i , is determined by its 2-jet; this property being preserved under coordinate change, (4.4) implies that Σ^i is in fact a strong $\mathcal{A}^{(1)}$ -invariant. \square

(4.6) Corollary. — *Any $J^2 \mathcal{K}$ -orbit in $J^2(n, p)$ is an $\mathcal{A}^{(1)}$ -invariant.*

Proof. — Any $J^2 \mathcal{K}$ -orbit in $J^2(n, p)$ is $J^2 \mathcal{A}$ -invariant, so this is immediate from (4.4). \square

The assertions (4.5) and (4.6) are, together with §§ 1, 2, 3, already sufficient to prove (0.2) A where $n - p \leq 0$ or $n - p = 2$, and to prove (0.2) B when $n - p \leq 7$,

for in these cases the “critical submanifolds” ${}^4S^2$ are to be found in the 2-jet space, so the appeals to (1.10) in the proofs can be replaced by appeals to (4.5), (4.6).

For the remaining cases, some of Boardman’s deeper results are necessary. The basic analysis, however, is very similar.

II. — Intrinsic derivatives for vector-bundle homomorphisms

Let E, F be C^1 -vector bundles over a C^1 -manifold N , and let $\alpha : E \rightarrow F$ be a C^1 -vector bundle morphism; we will also identify α as a C^1 section of $\text{Hom}(E, F)$.

Let

$$\begin{aligned} (\varphi, \pi_E) : E|U &\rightarrow E_x \times U, \\ (\psi, \pi_F) : F|U &\rightarrow F_x \times U, \end{aligned}$$

be C^1 local trivialisations for E, F over a neighbourhood U of $x \in N$. Then a C^1 local trivialisation

$$(r, \pi) : \text{Hom}(E, F)|U \rightarrow \text{Hom}(E_x, F_x) \times U$$

is given by

$$r(\gamma) \circ \varphi|E_y = \psi \circ \gamma \quad (y \in U, \gamma \in \text{Hom}(E_y, F_y)).$$

Now let

$$q_x : \text{Hom}(E_x, F_x) \rightarrow \text{Hom}(\text{Ker } \alpha_x, \text{Coker } \alpha_x),$$

be the homomorphism induced by restriction and projection.

We define the *intrinsic derivative*

$$\delta\alpha_x : \text{TN}_x \rightarrow \text{Hom}(\text{Ker } \alpha_x, \text{Coker } \alpha_x)$$

by

$$\delta\alpha_x = q_x \circ T(r \circ (\alpha|U))_x.$$

This is intrinsic, in the sense that it is independent of the choice of C^1 local trivialisations made, as follows immediately from the following:

(4.7) Lemma. — *Let E, F, E', F' be real finite-dimensional vector spaces, and let $\alpha, \beta : (\mathbf{R}^n, 0) \rightarrow \text{Hom}(E, F)$ be C^1 map-germs. Suppose that there exist continuous map-germs*

$$\varphi : (\mathbf{R}^n, 0) \rightarrow \text{Iso}(E', E), \quad \psi : (\mathbf{R}^n, 0) \rightarrow \text{Iso}(F, F')$$

and a C^1 -diffeomorphism-germ h of $(\mathbf{R}^n, 0)$ such that the following diagram commutes:

$$\begin{array}{ccc} (\mathbf{R}^n, 0) & \xrightarrow{\alpha} & \text{Hom}(E, F) \\ \downarrow h & & \downarrow \text{Hom}(\varphi, \psi) \\ (\mathbf{R}^n, 0) & \xrightarrow{\beta} & \text{Hom}(E', F') \end{array} \quad (1)$$

Then the following diagram commutes:

$$\begin{array}{ccc} \mathbf{R}^n & \xrightarrow{\alpha(\alpha) \circ d\alpha} & \text{Hom}(\text{Ker } \alpha_0, \text{Coker } \alpha_0) \\ d\alpha_0 \downarrow & & \downarrow \text{Hom}(\varphi_0, \bar{\psi}_0) \\ \mathbf{R}^n & \xrightarrow{\alpha(\beta) \circ d\beta} & \text{Hom}(\text{Ker } \beta_0, \text{Coker } \beta_0) \end{array} \quad (2)$$

where $q(\alpha) : \text{Hom}(E, F) \longrightarrow \text{Hom}(\text{Ker } \alpha_0, \text{Coker } \alpha_0)$

and $q(\beta) : \text{Hom}(E', F') \longrightarrow \text{Hom}(\text{Ker } \beta_0, \text{Coker } \beta_0)$

are the obvious projections induced by restriction and projection, and $\varphi_0 | : \text{Ker } \beta_0 \rightarrow \text{Ker } \alpha_0$, $\bar{\psi}_0 : \text{Coker } \alpha_0 \rightarrow \text{Coker } \beta_0$ are the obvious isomorphisms induced by φ_0 , ψ_0 respectively.

Proof. — On taking representatives, (1) yields for $u \in \mathbf{R}^n$, $v \in E$, and all sufficiently small $t \in \mathbf{R}$:

$$\begin{aligned} \beta_{h(tu)} - \beta_0 &= (\psi_{tu} \cdot \alpha_{tu} \cdot \varphi_{tu} - \psi_0 \cdot \alpha_0 \cdot \varphi_0)(v) \\ &= (\psi_{tu} - \psi_0) \cdot \alpha_0 \cdot \varphi_0(v) \\ &\quad + \psi_{tu} \cdot (\alpha_{tu} - \alpha_0) \cdot \varphi_{tu}(v) \\ &\quad + \psi_{tu} \cdot \alpha_0 \cdot (\varphi_{tu} - \varphi_0)(v). \end{aligned}$$

Now take $v \in \text{Ker } \beta_0$. Since $\text{Ker } \beta_0 = \text{Ker}(\alpha_0 \cdot \varphi_0)$ (for $\beta_0 = \psi_0 \cdot \alpha_0 \cdot \varphi_0$), the first term on the RHS is zero.

Now divide both sides by t , and consider the limit as $t \rightarrow 0$. The LHS tends to $d\beta_0(dh_0(u))(v)$, while the second term on the RHS tends to $\psi_0 \cdot d\alpha_0(u) \cdot \varphi_0$. Thus the third term on the RHS must tend to a limit too. Since this term is contained in $\psi_{tu}(\text{Im } \alpha_0)$, the limit is contained in the limit as $t \rightarrow 0$ of $\psi_{tu}(\text{Im } \alpha_0)$ (considered as points in the appropriate Grassmannian), which is $\psi_0(\text{Im } \alpha_0)$ (because ψ_0 is an isomorphism) $= \text{Im}(\psi_0 \cdot \alpha_0) = \text{Im } \beta_0$.

Thus, if $\pi : F' \rightarrow F'/\text{Im } \beta_0$ is the obvious projection, we have

$$\pi \cdot d\beta_0(dh_0(u))(v) = \pi \cdot \psi_0 \cdot d\alpha_0(u) \cdot \varphi_0(v);$$

so that (2) commutes, as required. \square

(4.8) Corollary. — *Let $\alpha : E \rightarrow F$, $\alpha' : E' \rightarrow F'$ be C^1 vector-bundle homomorphisms over C^1 manifolds N, N' respectively. Suppose that there exist C^0 vector-bundle isomorphisms $\varphi : E \rightarrow E'$, $\psi : F \rightarrow F'$ over some C^1 -diffeomorphism $h : N \rightarrow N'$ such that the following diagram commutes:*

$$\begin{array}{ccc} E & \xrightarrow{\alpha} & F \\ \phi \downarrow & & \downarrow \psi \\ E' & \xrightarrow{\alpha'} & F' \end{array}$$

Then, for all $x \in N$, the following diagram commutes:

$$\begin{array}{ccc}
 \text{TN}_x & \xrightarrow{\delta\alpha_x} & \text{Hom}(\text{Ker } \alpha_x, \text{Coker } \alpha_x) \\
 \downarrow dh_x & & \downarrow \text{Hom}(\phi_{h(x)}^{-1} |, \bar{\psi}_x) \\
 \text{TN}'_{h(x)} & \xrightarrow{\delta\alpha'_{h(x)}} & \text{Hom}(\text{Ker } \alpha'_{h(x)}, \text{Coker } \alpha'_{h(x)})
 \end{array}$$

where $\phi_{h(x)}^{-1} |, \bar{\psi}_x$ are the obvious isomorphisms induced by $\phi_{h(x)}^{-1}, \psi_x$.

Proof. — Any choice of C^1 local trivialisations for E, F near x and E', F' near $h(x)$ reduces this at once to the statement of the previous lemma. \square

The notion of intrinsic derivative for *smooth* vector bundle homomorphisms is due to Porteous, and is described by Boardman in [1], (7.1)-(7.5). (The presentation is different from ours, but it is easy to see that the same intrinsic derivative is produced.)

Boardman goes on to show that the Thom-Boardman varieties can be defined via a sequence of intrinsic derivatives ([1], (7.6)-(7.14)), and, indeed, that the corresponding singularity subsets of mappings can, under appropriate transversality conditions, also be defined via a sequence of intrinsic derivatives, as follows.

III. — Thom-Boardman varieties and intrinsic derivatives

Given a map $f: N \rightarrow P$ and a sequence $I = (i_1, \dots, i_r)$ of integers such that $n \geq i_1 \geq \dots \geq i_r \geq 0$, we consider the following hypothetical construction ([1], p. 55).

We start from the data

$$(\alpha_1) \quad \Sigma_0(f) = N, \quad S_{-1} = \text{TN}, \quad K_0 = \text{TN}, \quad c_0: \mathbb{R} \equiv \mathbb{R}_0 (= f^* \text{TP}), \\
 d_1(f) = df: S_{-1} (= \text{TN}) \rightarrow (f^* \text{TP}) = \mathbb{R}_0$$

and, more generally, suppose that for some s ($0 \leq s \leq r$) we have data

- (α_s) a) a submanifold $\Sigma_{s-1}(f)$ of N ,
 b) a sub-bundle $S_{s-2} \subset \text{TN}$ defined over $\Sigma_{s-1}(f)$,
 c) sub-bundles $K_{s-1} \subset K_{s-2} \subset \dots \subset K_0$ defined over $\Sigma_{s-1}(f)$, such that $K_{s-1} \subset S_{s-2}$,
 d) a bundle map $c_{s-1}: \text{Hom}(K_{s-1} \circ \dots \circ K_1, \mathbb{R}) \rightarrow \mathbb{R}_{s-1}$ over $\Sigma_{s-1}(f)$,
 e) a bundle map $d_s(f): S_{s-2} \rightarrow \mathbb{R}_{s-1}$ over $\Sigma_{s-1}(f)$.

Then we continue the construction with:

$$(\beta_s) \text{ Define } \Sigma_s(f) = \{x \in \Sigma_{s-1}(f) \mid \dim \ker(d_s(f) | K_{s-1}) = i_s\}$$

(γ_s) Over $\Sigma_s(f)$, define

$$K_s = \text{Ker}(d_s(f) | K_{s-1})$$

and let

$$e_s : R_{s-1} \rightarrow Q_s$$

be the cokernel and associated projection of

$$d_s(f) | K_{s-1} : K_{s-1} \rightarrow R_{s-1}.$$

(δ_s) Define the bundle map

$$u_s : \text{Hom}(K_s \circ \dots \circ K_1, R) \rightarrow \text{Hom}(K_s, Q_s)$$

over $\Sigma_s(f)$ as the composition

$$\text{Hom}(K_s \circ \dots \circ K_1, R)$$

\cap

$$\text{Hom}(K_s \otimes (K_{s-1} \circ \dots \circ K_1), R) \cong \text{Hom}(K_s, \text{Hom}(K_{s-1} \circ \dots \circ K_1, R))$$

$$\downarrow \text{Hom}(1, e_{s-1})$$

$$\text{Hom}(K_s, R_{s-1})$$

$$\downarrow \text{Hom}(1, e_s)$$

$$\text{Hom}(K_s, Q_s).$$

(ε_s) Assuming that u_s has constant rank over $\Sigma_s(f)$, set $R_s = \text{Im } u_s$ and let

$$c_s : \text{Hom}(K_s \circ \dots \circ K, R) \rightarrow R_s$$

over Σ_s be the obvious homomorphism induced by u_s .

(ζ_s) Let $S_{s-1} = T(\Sigma_{s-1})$ and, assuming that the bundle map

$$\delta(d_s(f) | K_{s-1}) : S_{s-1} \rightarrow \text{Hom}(K_s, Q_s)$$

over $\Sigma_s(f)$ defined by taking intrinsic derivatives of $d_s(f) | K_{s-1}$ factors through R_s , call the first factor

$$d_{s+1}(f) : S_{s-1} \rightarrow R_s,$$

a bundle map over $\Sigma_s(f)$.

(η_s) Assume that $K_s \subset S_{s-1}$ over $\Sigma_{s-1}(f)$.

(θ_s) Assume that $\Sigma_s(f)$ is a submanifold of N .

The validity of the assumptions in (ε_s), (ζ_s), (η_s), (θ_s) can be expressed in terms of the conditions

(ι_t) $d_{t+1}(f) : S_{t-1} \rightarrow R_t$ is surjective in the fibres over points of $\Sigma_t(f)$

as follows:

(4.9) Theorem ([1], (7.15)),

(i) If (ι_t) holds for $0 < t < r$, then (ε_r) , (ζ_r) , (η_r) are valid, and

$$\Sigma_r(f) = (J^r f)^{-1} \Sigma^{i_1 \dots i_r}.$$

(ii) If (ι_r) also holds, then (θ_r) is valid.

(iii) (ι_t) holds for $0 < t \leq s$ if and only if f is transverse to $\Sigma^{i_1}, \dots, \Sigma^{i_1 \dots i_s}$.

Proof. — See [1], (7.15). \square

These constructions are “ $\mathcal{A}^{(1)}$ -invariant” in the following sense:

(4.10) Proposition. — Let $f, g: (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^p, 0)$ be $\Sigma^{i_1 \dots i_{r-1}}$ -transverse germs with r -jets in $\Sigma^{i_1 \dots i_r}$, and suppose that there exist C^1 germs of diffeomorphisms h of $(\mathbf{R}^n, 0)$, k of $(\mathbf{R}^p, 0)$ such that

$$g \circ h = k \circ f \tag{*}$$

Then, for $0 < s \leq r + 1$, the following hold:

(A_s) a) $h(\Sigma_{s-1}(f)) = \Sigma_{s-1}(g)$.

b) $dh_0(S_{s-2}(f)) = S_{s-2}(g)$.

c) $dh_0(K_t(f)) = K_t(g)$ for $0 \leq t \leq s - 1$.

d) There is a C^0 vector-bundle isomorphism

$$\psi_{s-1}: R_{s-1}(f) \rightarrow R_{s-1}(g)$$

such that

$$\begin{array}{ccc} \text{Hom}(K_{s-1}(f) \circ \dots \circ K_1(f), \mathbf{R}) & \xrightarrow{c_{s-1}(f)} & R_{s-1}(f) \\ \downarrow \text{Hom}(dh_0^{-1}| \circ \dots \circ dh_0^{-1}|, dk_0) & & \downarrow \psi_{s-1} \\ \text{Hom}(K_{s-1}(g) \circ \dots \circ K_1(g), \mathbf{R}) & \xrightarrow{c_{s-1}(g)} & R_{s-1}(g) \end{array} \tag{i}$$

commutes and such that

$$\begin{array}{ccc} S_{s-2}(f) & \xrightarrow{d_s(f)} & R_{s-1}(f) \\ \downarrow dh_0| & & \downarrow \psi_{s-1} \\ S_{s-2}(g) & \xrightarrow{d_s(g)} & R_{s-1}(g) \end{array} \tag{ii}$$

commutes.

Proof. — We proceed by induction. (A_0) holds trivially, so it is enough to show that if (A_s) holds, then (A_{s+1}) holds (for $0 \leq s \leq r$).

(A_{s+1}) *a)* and *c)* follow from the constructions (β_s) , (γ_s) for f and g , together with (A_s) *a)* and $(*)$, which imply

$$(g | \Sigma_{s-1}(g)) \circ (h | \Sigma_{s-1}(f)) = k \circ (f | \Sigma_{s-1}(f)).$$

It follows also from (A_s) *d)* (ii) that

$$\begin{array}{ccc} R_{s-1}(f) & \xrightarrow{e_s(f)} & Q_s(f) \\ \downarrow \Psi_{s-1} & & \downarrow \overline{\Psi_{s-1}} \\ R_{s-1}(g) & \xrightarrow{e_s(g)} & Q_s(g) \end{array} \quad (+)$$

commutes, where $\overline{\Psi_{s-1}}$ is the map of cokernels induced by Ψ_{s-1} .

Since f, g are $\Sigma^{i_1 \dots i_{s-1}}$ -transverse, (4.9) (iii) implies that (ι_t) holds for $0 < t \leq s-1$ for both f and g , so that (4.9) (i) implies that the assumptions of (ε_s) , (ζ_s) , (η_s) are valid for both f and g .

(A_{s+1}) *b)* now follows from the definitions,

$$(S_{s-1}(f) = T\Sigma_{s-1}(f), \quad S_{s-1}(g) = T\Sigma_{s-1}(g))$$

and (A_s) *a)*.

Furthermore, it follows from (+) and the constructions (δ_s) , (ε_s) for f and g that (A_{s+1}) *d)* (i) holds, with ψ_s the appropriate restriction of $\text{Hom}(dk_0^{-1} | K_s(g), \overline{\Psi_{s-1}})$.

Finally, (A_s) *d)* (ii) and (4.8) imply that

$$\begin{array}{ccc} S_{s-1}(f) & \xrightarrow{\delta(d_s(f) | K_{s-1}(f))} & \text{Hom}(K_s(f), Q_s(f)) \\ \downarrow dk_0 | & & \downarrow \text{Hom}(dk_0^{-1}, \overline{\Psi_{s-1}}) \\ S_{s-1}(g) & \xrightarrow{\delta(d_s(g) | K_{s-1}(g))} & \text{Hom}(K_s(g), Q_s(g)) \end{array}$$

commutes; and since, by (ζ_s) , $\delta(d_s(f) | K_{s-1}(f))$, $\delta(d_s(g) | K_{s-1}(g))$ have their images in $R_s(f)$, $R_s(g)$ respectively, (A_{s+1}) *d)* (ii) follows, completing the proof. \square

We can now give:

(4.11) (= (0.10)) Proposition.

1. The Thom-Boardman varieties Σ^I are strong $\mathcal{A}^{(1)}$ -quasi-invariants.
2. Let I be a non-increasing r -tuple of positive integers of the form (k, \dots, k) or $(n-p+1, k, \dots, k)$. Then each \mathcal{H}^{r+1} -orbit in $(\pi^{r+1}, r)^{-1} \Sigma^I$ is an $\mathcal{A}^{(1)}$ -invariant for Σ^I -transverse germs.

Proof.

1. Let $f, g : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^p, 0)$ be $\Sigma^{i_1 \cdots i_{r-1}}$ -transverse $\mathcal{A}^{(1)}$ -equivalent germs. Since (A_r) a) (of (4.10)) holds, $j^r f \in \Sigma^{i_1 \cdots i_r}$ if and only if $j^r g \in \Sigma^{i_1 \cdots i_r}$; so we have shown that $\Sigma^{i_1 \cdots i_r}$ is an $\mathcal{A}^{(1)}$ -invariant for $\Sigma^{i_1 \cdots i_{r-1}}$ -transverse germs. Moreover, (A_{r+1}) d) (ii) (of (4.10)) holds, so that $d_{r+1}(f)$ is fibre surjective if and only if $d_{r+1}(g)$ is fibre-surjective. But, by (4.9) (iii), this fibre-surjectivity is equivalent to transversality to $\Sigma^{i_1 \cdots i_r}$, so we have shown that $\Sigma^{i_1 \cdots i_r}$ is a strong $\mathcal{A}^{(1)}$ -invariant for $\Sigma^{i_1 \cdots i_{r-1}}$ -transverse germs.

2. a) Let \mathbf{I} be the r -tuple (k, \dots, k) , and let $f, g : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^p, 0)$ be $\Sigma^{\mathbf{I}}$ -transverse germs such that there exist \mathbf{C}^1 diffeomorphism-germs h of $(\mathbf{R}^n, 0)$, k of $(\mathbf{R}^p, 0)$ such that $g \circ h = k \circ f$. The definition of $\Sigma^{\mathbf{I}}$ implies that

$$\begin{cases} \mathbf{K}_s(f) = \text{Ker } df & \text{over } \Sigma_s(f) \\ \mathbf{K}_s(g) = \text{Ker } dg & \text{over } \Sigma_s(g) \end{cases}$$

for $1 \leq s \leq r$, and that

$$\begin{cases} d_{s+1}(f) | \mathbf{K}_s(f) = 0 \\ d_{s+1}(g) | \mathbf{K}_s(g) = 0 \end{cases}$$

for $0 < s < r$; it follows at once that

$$\begin{aligned} \mathbf{R}_s(f) &= \text{Hom}(\mathbf{O}^{s-1} \text{Ker } df, \text{Coker } df) & \text{over } \Sigma_s(f), \\ \mathbf{R}_s(g) &= \text{Hom}(\mathbf{O}^{s-1} \text{Ker } dg, \text{Coker } dg) & \text{over } \Sigma_s(g) \end{aligned}$$

for $0 \leq s \leq r+1$. Here \mathbf{O}^{s-1} is the $(s-1)$ -fold symmetric power. Since (A_{r+1}) holds, and since, as follows from the proof of (4.10), $\psi_s = \text{Hom}(dh^{-1} | \circ \dots \circ dh^{-1} |, \overline{dk}_0)$, we obtain a commutative diagram at 0:

$$\begin{array}{ccc} \text{Ker } df_0 & \xrightarrow{d_{r+1}(f) | \text{Ker } df_0} & \text{Hom}(\text{Ker } df_0^{\circ} \dots^{\circ} \text{Ker } df_0, \text{Coker } df_0) \\ \downarrow dk_0 & & \downarrow \text{Hom}(dh_0^{-1} | \circ \dots \circ dh_0^{-1} |, dk) \quad (*) \\ \text{Ker } dg_0 & \xrightarrow{d_{r+1}(g) | \text{Ker } dg_0} & \text{Hom}(\text{Ker } dg_0^{\circ} \dots^{\circ} \text{Ker } dg_0, \text{Coker } dg_0) \end{array}$$

Choosing linearly adapted coordinates for f puts it in the form

$$(u_1, \dots, u_{n-k}, x_1, \dots, x_k) \mapsto (u_1, \dots, u_{n-k}, \mathbf{F}_1(\mathbf{x}) + \mathbf{F}_2(\mathbf{u}, \mathbf{x})),$$

where $\mathbf{F}_2 \in m_{\mathbf{u}} \cdot (\mathbf{C}_{\mathbf{u}, \mathbf{x}})^{\times(p-n+k)}$, and an easy inductive argument (following the constructions previously described) shows that

- (i) $\mathbf{F}_1 \in (m_{\mathbf{x}}^{r+1})^{\times(p-n+k)}$ (i.e. \mathbf{F}_1 has zero r -jet),
- (ii) $d_{r+1}(f) | \text{Ker } df_0$ "is" the $(r+1)$ -st derivative of \mathbf{F}_1 in the chosen coordinates.

Similarly, we can put g into the form

$$(u_1, \dots, u_{n-k}, x_1, \dots, x_k) \mapsto (u_1, \dots, u_{n-k}, G_1(\mathbf{x}) + G_2(\mathbf{u}, \mathbf{x}))$$

where $G_2 \in m_{\mathbf{u}} \cdot (C_{\mathbf{u}, \mathbf{x}})^{\times(p-n+k)}$, G_1 has zero r -jet, and $d_{r+1}(g) | \text{Ker } dg_0$ "is" the $(r+1)$ -st derivative of G_1 . Since f, g are unfoldings of F_1, G_1 , respectively, $j^{r+1}f, j^{r+1}g$ are $J^{r+1}\mathcal{K}$ -equivalent if and only if $j^{r+1}F_1, j^{r+1}G_1$ are $J^{r+1}\mathcal{K}$ -equivalent, and it is easy to see that $j^{r+1}F_1, j^{r+1}G_1$ are $J^{r+1}\mathcal{K}$ -equivalent if and only if they are linearly equivalent. It now follows from (*) that, since f, g are $\mathcal{A}^{(1)}$ -equivalent, $j^{r+1}f, j^{r+1}g$ are $J^{r+1}\mathcal{K}$ -equivalent; so we have shown that the $J^{r+1}\mathcal{K}$ -orbits contained in Σ^I are indeed $\mathcal{A}^{(1)}$ -invariants for Σ^I -transverse germs.

b) Let $n > p$, and let I be the r -tuple $(n-p+1, k, \dots, k)$. Let

$$f, g: (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^p, 0)$$

be Σ^I -transverse germs for which there exist C^1 diffeomorphism-germs h of $(\mathbf{R}^n, 0)$, k of $(\mathbf{R}^p, 0)$ such that $g \circ h = k \circ f$. The definition of Σ^I implies that

$$\begin{cases} K_s(f) = K_s(g) & \text{over } \Sigma_s(f) \\ K_s(g) = K_s(f) & \text{over } \Sigma_s(g) \end{cases}$$

for $2 \leq s \leq r$; and that

$$\begin{cases} d_{s+1}(f) | K_s(f) = 0 \\ d_{s+1}(g) | K_s(g) = 0 \end{cases}$$

for $2 \leq s < r$. For an arbitrary smooth map $\tilde{f}: N \rightarrow P$, it follows from the definitions that

$$\delta(d\tilde{f})_x: \text{TN}_x \rightarrow \text{Hom}(\text{Ker } d\tilde{f}_x, \text{Coker } d\tilde{f}_x)$$

is restriction-and-projection of the second derivative of \tilde{f} at x with respect to any local coordinates at $x, \tilde{f}(x)$; in particular,

$$\delta(d\tilde{f})_x | \text{Ker } d\tilde{f}_x: \text{Ker } d\tilde{f}_x \rightarrow \text{Hom}(\text{Ker } d\tilde{f}_x, \text{Coker } d\tilde{f}_x)$$

is symmetric in $\text{Ker } d\tilde{f}_x$. If $\dim \text{Ker } d\tilde{f}_x = p - n + 1$, then $\dim \text{Coker } d\tilde{f}_x = 1$, and there is a natural isomorphism

$$\varphi_2(\tilde{f})_x: \text{Coker } (\delta(d\tilde{f})_x | \text{Ker } d\tilde{f}_x) \rightarrow \text{Hom}(\text{Ker } \delta(d\tilde{f})_x | \text{Ker } d\tilde{f}_x, \text{Coker } d\tilde{f}_x)$$

given by

$$\varphi_2(\tilde{f})_x([\alpha]) = \alpha | \text{Ker}(\delta(d\tilde{f})_x | \text{Ker } d\tilde{f}_x)$$

for any $\alpha \in \text{Hom}(\text{Ker } d\tilde{f}_x, \text{Coker } d\tilde{f}_x)$. Note that $\varphi_2(\tilde{f})_x$ is well-defined because if $v \in \text{Ker } d\tilde{f}_x$, then for any $v' \in \text{Ker}(\delta(d\tilde{f})_x | \text{Ker } d\tilde{f}_x)$,

$$\delta(d\tilde{f})_x(v)(v') = \delta(d\tilde{f})_x(v')(v) = 0.$$

Also, $\varphi_2(\tilde{f})_x$ is surjective, since any linear map $\text{Ker}(\delta(d\tilde{f})_x | \text{Ker } d\tilde{f}_x) \rightarrow \text{Coker } d\tilde{f}_x$ can obviously be extended to a linear map $\text{Ker } d\tilde{f}_x \rightarrow \text{Coker } d\tilde{f}_x$, so that, since source and target are of equal dimension, $\varphi_2(\tilde{f})_x$ is indeed an isomorphism.

In Boardman's notation this yields a natural isomorphism of vector bundles

$$\varphi_2(\tilde{f}) : R_2(\tilde{f}) \rightarrow \text{Hom}(K_2(\tilde{f}), Q_1(\tilde{f})) \quad \text{over } \Sigma^{p-n+1, k}(f).$$

Now, following the constructions of Boardman, it follows easily that $\varphi_2(f)$ induces natural isomorphisms

$$\begin{cases} \varphi_s(f) : R_s(f) \rightarrow \text{Hom}(\mathbf{O}^{s-1} K_2(f), Q_1(f)) & \text{over } \Sigma_s(f) \\ \varphi_s(g) : R_s(g) \rightarrow \text{Hom}(\mathbf{O}^{s-1} K_2(g), Q_1(g)) & \text{over } \Sigma_s(g) \end{cases}$$

for $s = 2, \dots, r$; as before \mathbf{O}^{s-1} is the $(s-1)$ -fold symmetric power. Since (A_{r+1}) holds, and since, as follows from the proof of (4.10) and the natural isomorphisms above, we have commutative diagrams

$$\begin{array}{ccc} R_s(f) & \xrightarrow{\phi_s(f)} & \text{Hom}(K_2(f) \circ \dots \circ K_2(f), Q_1(f)) \\ \downarrow \Psi_s & & \downarrow \text{Hom}(dh^{-1}| \circ \dots \circ dh^{-1}|, \overline{dk}) \\ R_s(g) & \xrightarrow{\phi_s(g)} & \text{Hom}(K_2(g) \circ \dots \circ K_2(g), Q_1(g)) \end{array}$$

over Σ_s , for $2 \leq s \leq r+1$, we obtain a commutative diagram at 0

$$\begin{array}{ccc} K_2(f)_0 & \xrightarrow{\phi_r(f)_0 | d_{r+1}(f)_0 | K_2(f)_0} & \text{Hom}(K_2(f)_0 \circ \dots \circ K_2(f)_0, Q_1(f)_0) \\ \downarrow dh_0| & & \downarrow \text{Hom}(dh_0^{-1}| \circ \dots \circ dh_0^{-1}|, \overline{dk}_0) \\ K_2(g)_0 & \xrightarrow{\phi_r(g)_0 | d_{r+1}(g)_0 | K_2(g)_0} & \text{Hom}(K_2(g)_0 \circ \dots \circ K_2(g)_0, Q_1(g)_0) \end{array} \quad (*)$$

It is easy to see (or consult [16], (3.1)) that an appropriate choice of coordinates puts f in the form

$$\begin{aligned} (u_1, \dots, u_{p-1}, y_1, \dots, y_{n-p+1-k}, x_1, \dots, x_k) \\ \mapsto (u_1, \dots, u_{p-1}, Q_f(\mathbf{y}) + R_f(\mathbf{u}, \mathbf{x}) + F(\mathbf{x})) \end{aligned}$$

where Q_f is a non-degenerate quadratic form, $R_f \in m_{\mathbf{u}} C_{\{\mathbf{u}, \mathbf{x}\}}$, and F has zero 2-jet; an easy inductive argument shows that

- (i) F has zero r -jet,
- (ii) $\phi_r(f)_0 | d_{r+1}(f)_0 | K_2(f)_0$ "is" the $(r+1)$ -st derivative of F in the chosen coordinates.

Similarly, we can put g in the form

$$\begin{aligned} (u_1, \dots, u_{p-1}, y_1, \dots, y_{n-p+1-k}, x_1, \dots, x_k) \\ \mapsto (u_1, \dots, u_{p-1}, Q_g(\mathbf{y}) + R_g(\mathbf{u}, \mathbf{x}) + G(\mathbf{x})) \end{aligned}$$

where Q_o is a non-degenerate quadratic form, $R_o \in m_u \cdot C_{\{u, x\}}$, and G has zero r -jet, and $\varphi_r(g)_o d_{r+1}(g)_o | K_2(g)_o$ "is" the $(r+1)$ -st derivative of G in the chosen coordinates.

It is well known that f, g have $J^{r+1} \mathcal{H}$ -equivalent $(r+1)$ -jets if and only if Q_f, Q_g are linearly equivalent and $J^{r+1} F, J^{r+1} G$ are $J^{r+1} \mathcal{H}$ -equivalent. But, since F, G have zero r -jets, $J^{r+1} F, J^{r+1} G$ are $J^{r+1} \mathcal{H}$ -equivalent if and only if they, and thus $\varphi_r(f)_o d_{r+1}(f)_o | K_2(f)_o$ and $\varphi_r(g)_o d_{r+1}(g)_o | K_2(g)_o$, are linearly equivalent; while clearly Q_f, Q_g are linearly equivalent if and only if $\delta(df)_o | \text{Ker } df_o, \delta(dg)_o | \text{Ker } dg_o$ (which "are" these quadratic forms) are linearly equivalent.

Since f, g are $\mathcal{A}^{(1)}$ -equivalent, (4.3) shows that $\delta(df)_o | \text{Ker } df_o, \delta(dg)_o | \text{Ker } dg_o$ are linearly equivalent, while (*) shows that

$$\varphi_r(f)_o d_{r+1}(f)_o | K_2(f)_o, \quad \varphi_r(g)_o d_{r+1}(g)_o | K_2(g)_o$$

are linearly equivalent; so $j^{r+1} f, j^{r+1} g$ are indeed $J^{r+1} \mathcal{H}$ -equivalent. Thus, we have indeed shown that the $J^{r+1} \mathcal{H}$ -orbits contained in Σ^I are $\mathcal{A}^{(1)}$ -invariants for Σ^I -transverse germs. \square

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