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ABSOLUTELY CONTINUOUS INVARIANT MEASURES FOR MAPS WITH FLAT TOPS

by Michael BENEDICKS *and* Michał MISIUREWICZ

0. Introduction

This paper is intended as a sequel to the paper [M2] by the second author. The aim of the paper is to study the problem of knowing what type of behavior near the critical points can replace the polynomial-like behavior in the proof of existence of absolutely continuous invariant measures.

The class of mappings treated is basically the same as the one in [M2], essentially piecewise monotone mappings with non-positive Schwartzian derivative, no sinks and trajectories of critical points staying far from the critical points. However there is one important difference: we will allow " flat " behavior at the critical points. It turns out that generically the following extra conditions on the map f from an interval to itself

(A) $$\int_I \log |f'(x)|\, dx > -\infty$$

or (equivalently for our class of functions)

(B) $$\int_I \log |f(x) - f(a)|\, dx > -\infty$$

for every critical point a are sufficient for the existence of absolutely continuous invariant measures.

It is important that the condition (A) (or (B)) is also necessary: If a unimodal map in our class satisfies

$$\int_I \log |f'(x)|\, dx = -\infty,$$

then f has no absolutely continuous invariant measure. The restriction to unimodal maps is for the following reason: in the presence of many critical points the dynamics may split into several independent parts, some of them involving only " good " critical points (e.g. with polynomial-like behaviour) and therefore having absolutely continuous invariant measure.

This answers the question posed by Collet and Eckmann [C-E, p. 159] as to which conditions should be imposed on " flat tops ".

The assumption that the Schwartzian derivative is non-negative may seem very strong. However it is not so. Surprisingly many commonly considered maps of an interval into itself have negative Schwartzian derivative (cf. [M1, Examples (2.8)]). Nevertheless one can ask whether our results still hold without this assumption (although one has to assume that the Schwartzian derivative is non-positive in some neighbourhood of each critical point—this property is absolutely necessary to all known proofs). Recently the results of [M2] were generalized in this manner by van Strien [S]. The methods of [S] are completely different from those of [M2], so one cannot generalize the results of this paper automatically. Moreover is seems that the polynomial-like behaviour near the critical points is more important in [S] than in [M2].

The proofs of this paper follow to a large extent those in Misiurewicz [M2]. We will state all lemmas needed but refer to [M2] for the proof if the lemma is unchanged. In certain cases, some simplifications of the proofs in [M2] are possible, mainly due to the fact that we do not allow the orbit of a critical point under iteration by f to hit another critical point. (This is the genericity mentioned above.)

This work was essentially done during a visit by the first author to Warsaw and he wants to express his gratitude to the Institute of Mathematics of the Polish Academy of Sciences for its hospitality.

1. Stretching far from the critical points

Let I be a closed interval, let U and V be relatively open subsets of I consisting of a finite number of intervals each, such that U contains the endpoints of I and $U \cup V = I$, and let $f: V \to I$ be a continuous mapping. We denote the n-th iterate of f by f^n, and the Lebesgue measure by λ.

An open interval $J \subset V$ is called a *homterval* if, for all n, f^n maps J homeomorphically into its image. We shall say that f has *no sinks* if there does not exist an interval $J \subset V$ and a positive integer n such that f^n maps J homeomorphically into J.

Theorem 1 [M2, *Theorem* (1.2)]. — *Let f have no sinks and be of class* C^1, *assume* $f'(x) \neq 0$ *for all* $x \in V$ *and let* $\log |f'(x)|$ *be Lipschitz continuous on components of* V. *Then for every homterval* J *there exists* $m \geq 0$ *such that*

$$\overline{f^m(J)} \subset U.$$

The Schwartzian derivative Sf of a function $f \in C^3$ is defined as

$$Sf = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2.$$

See [M2, p. 18] for a discussion.

Theorem **2** [M2, *Theorem* (1.3)]. — *Let* f *have no sinks and be of class* C^3, *assume* $f'(x) \neq 0$ *for all* $x \in V$ *and* $Sf \leqslant 0$. *Then there exists* $m \geqslant 1$ *such that, if* $f^j(x) \notin U$ *for* $j = 0, 1, \ldots, m-1$, *one has* $|(f^m)'(x)| > 1$.

2. Estimates I

In this section, U is a relatively open subset of I consisting of a finite number of intervals and such that the endpoints of I belong to U. Let $f: I\backslash U \to I$ be a map of class C^1 such that $|f'| \geqslant \alpha > 1$ and the function $\log|f'(x)|$ is Lipschitz continuous on the components of $I\backslash U$ and let B a subset of $I\backslash U$ such that $f(B) \subset B$ and $\mathrm{dist}(B, U) > 0$. Define $E_n = \{x \in I : f^k(x) \notin U \text{ for } k = 0, \ldots, n-1\}$. (Notice that since $I\backslash U$ is the domain of f, E_n is the domain of f^n.)

Proposition **1** [M2, *Proposition* (2.1)]. — *There exists a constant* η *such that* $0 < \eta < 1$ *and for every* $n \geqslant 0$

$$\lambda(E_n) \leqslant \eta^n \lambda(I).$$

We will now prove some technical lemmas on the integrability of functions, which are related to the conditions (A) and (B).

Lemma **1**. — *Assume that* ψ *is a* C^1 *function on* $[0, a]$ *such that*

a) $\psi(0) = \psi'(0) = 0$,

b) ψ' *is strictly increasing*.

Then the following conditions are equivalent:

(I) $\displaystyle\int_0^{\psi(a)} \frac{\psi^{-1}(t)}{t}\, dt < \infty$;

(II) $\displaystyle\int_0^a \log \psi(t)\, dt > -\infty$;

(III) $\displaystyle\int_0^a \log \psi'(t)\, dt > -\infty$.

Proof. — Note that *a)* and *b)* imply that $\psi(x) > 0$ and $\psi'(x) > 0$ for $0 < x \leqslant a$.

(I) $\Rightarrow$ (II): By the change of variables $t = \psi(u)$ we have

$$\int_0^{\psi(a)} \frac{\psi^{-1}(t)}{t}\, dt = \int_0^a \frac{u\psi'(u)}{\psi(u)}\, du.$$

Furthermore for every $\varepsilon \in (0, a)$

$$\int_0^a \frac{u\psi'(u)}{\psi(u)}\, du \geqslant \int_\varepsilon^a \frac{u\psi'(u)}{\psi(u)}\, du = [u \log \psi(u)]_\varepsilon^a - \int_\varepsilon^a \log \psi(u)\, du$$

and therefore

$$\int_\varepsilon^a \log \psi(u)\, du \geqslant a \log \psi(a) + \varepsilon \log \frac{1}{\psi(\varepsilon)} - \int_0^a \frac{u\psi'(u)}{\psi(u)}\, du.$$

If ε is small, then $\varepsilon \log(1/\psi(\varepsilon)) > 0$. Hence $\int_0^a \log \psi(u)\, du > -\infty$.

(II) $\Rightarrow$ (I): Since $\int_0^a \log \psi(u)\, du > -\infty$ and ψ is increasing,

$$\varepsilon \log \frac{1}{\psi(\varepsilon)} \leqslant \int_0^\varepsilon \log \frac{1}{\psi(u)}\, du \to 0 \quad \text{as } \varepsilon \to +0.$$

Hence $\lim_{\varepsilon \to +0} \varepsilon \log \psi(\varepsilon) = 0$ and the result follows again by integration by parts and change of variables.

(II) $\Rightarrow$ (III): Since ψ' is increasing,

$$\psi(t) = \int_0^t \psi'(x)\, dx \leqslant t\psi'(t).$$

Hence

$$\int_0^a \log \psi'(t)\, dt \geqslant \int_0^a \log \frac{1}{t}\, dt + \int_0^a \log \psi(t)\, dt > -\infty.$$

(III) $\Rightarrow$ (II): Since ψ' is increasing,

$$\psi'(t) \leqslant \frac{\psi(2t) - \psi(t)}{t} \leqslant \frac{\psi(2t)}{t}$$

and again the result follows by integration. □

Lemma **2.** — *Let* φ *be a positive increasing function on* $(0, b]$. *If for some* $\beta > 0$, $0 < \alpha < 1$ *and* n_0 *such that* $\beta\alpha^{n_0} \leqslant b$ *we have* $\Sigma_{n=n_0}^{\infty} \varphi(\beta\alpha^n) < \infty$, *then*

$$\int_0^b \frac{\varphi(t)}{t}\, dt < \infty.$$

Conversely if $\int_0^b (\varphi(t)/t)\, dt < \infty$, *then* $\Sigma_{n=n_0}^{\infty} \varphi(\beta\alpha^n) < \infty$ *for all* $\beta > 0$, $0 < \alpha < 1$, $\beta\alpha^{n_0} \leqslant b$.

Proof. — Since $v \to \varphi(\beta\alpha^v)$ is a decreasing function, this follows from the integral test for the summability of a series and the equality

$$\int_{n_0}^{\infty} \varphi(\beta\alpha^v)\, dv = \frac{1}{\log 1/\alpha} \int_0^{\beta\alpha^{n_0}} \frac{\varphi(t)}{t}\, dt$$

(obtained by the change of variables $t = \beta\alpha^v$). □

Lemma **3.** — *If* ψ *is a* C^3 *function on* $(0, a]$ *such that* $S\psi \leqslant 0$ *and* ψ' *vanishes at finitely many points, then there exists* $a' \in (0, a]$ *such that* ψ' *is either constant or strictly monotone on* $[0, a']$.

Proof. — Since the function $1/\sqrt{|\psi'|}$ is convex on some interval $[0, a'']$ (see [M2, (3.1)]) it is either constant or strictly monotone on some $[0, a'] \subset [0, a'']$. Hence $|\psi'|$ is also either constant or strictly monotone on this interval. □

3. Estimates II

In this section (and the next ones) I will be a closed interval, A a finite subset of I containing its endpoints, and $f: I \setminus A \to I$ a continuous map, strictly monotone on each component of $I \setminus A$. To avoid cumbersome notation we shall assume that f extends to a C^1 map of I (and in the sequel we shall consider this extended map $f: I \to I$).

However, the situation could be carried over in an obvious way to that in [M2] considering one-sided limits at the points of A.

Now we make further assumptions on f (as in [M2]):

(i) f is of class C^3 on $I \setminus A$;
(ii) $f' \neq 0$ on $I \setminus A$;
(iii) $Sf \leqslant 0$ on $I \setminus A$;
(iv) if $f^p(x) = x$ then $|(f^p)'(x)| > 1$;

(the condition (iv) implies that f has no sinks, see [M2, p. 25])

(v) *a)* there is a neighbourhood U of A such that for each $a \in A$ and $n \geqslant 0$, $f^n(a) \in A \cup (I \setminus U)$;
b) if $f'(a) = 0$ then $f'(f^n(a)) \neq 0$ for $n = 1, 2, \ldots$;
(vi) $\int_I \log |f'(x)| \, dx > -\infty$.

We also make two additional assumptions:

(vii) $|f'| > 1$ on $I \setminus U$;
(viii) if $a \in A$ is a periodic point for f it is a fixed point for f.

Note that (v) is different from the corresponding condition in [M2], in that *we do not allow critical points to be mapped onto critical points.*

Lemma 4. — *If f satisfies conditions* (i)-(vi) *then some iterate of f satisfies conditions* (i)-(viii) *(perhaps with a different set* A*).*

Proof. — Let $m \geqslant 1$, $\tilde{f} = f^m$, $\tilde{A} = \bigcup_{k=0}^{m-1} f^{-k}(A)$. It is easy to see that (i)-(iv) are satisfied by $\tilde{f}$, $\tilde{A}$ instead of f, A. In (v) we take $\tilde{U} = \bigcup_{k=0}^{m-1} f^{-k}(U)$. To prove (vi) use the chain rule. Here, it is essential that a critical point is not mapped onto another critical point.

Now it remains to show that if f satisfies (i)-(vi) then some iterate satisfies (vii) and (viii). The first fact follows from Theorem 2, and the other one is obvious. □

Set $A_1 = \{ a \in A : f(a) = a \}$, $A_2 = A \setminus A_1$, $A_2' = \{ a \in A_2 :$ there are $b \in A$ and $n \geqslant 1$ such that $f'(b) = 0$ and $f^n(b) = a \}$, $A_2'' = A_2 \setminus A_2'$, $C_n = \bigcup_{i=1}^{n} f^i(A)$, $C = \bigcup_{i=1}^{\infty} f^i(A)$, $B = \bar{C}$. For an L^1 function φ on I we denote by $\varphi\lambda$ the measure which is absolutely continuous with respect to λ and with density (i.e. Radon-Nikodym derivative) φ. For a measure μ and a map g, $g^*(\mu)$ denotes the image of μ under g, i.e. a measure such that for each measurable set E:

$$g^*(\mu)\,(E) = \mu(g^{-1}(E)).$$

For a map g we define the Perron-Frobenius operator g_* on an integrable function φ by

$$g_*(\varphi)\,\lambda = g^*(\varphi\cdot\lambda).$$

Notice that

$$\int_{g^{-1}(E)} \varphi\, d\lambda = \int_E g_*(\varphi)\, d\lambda.$$

Proposition **2** [M2, *Proposition* (3.2)]. — *If f satisfies* (i)-(iii) *then, for every $x \in I\backslash C_n$, one has*

$$f_*^n(1)\,(x) \leqslant \frac{\lambda(I)}{\operatorname{dist}(x, C_n)} \leqslant \frac{\lambda(I)}{\operatorname{dist}(x, B)}.$$

Now we assume that f statisfies (i)-(viii). By (v) $B\backslash A$ is disjoint from U. Since $Sf \leqslant 0$ and by the continuity of f' there exist open intervals U_a, $a \in A$, such that $\bigcup_{a\in A} \overline{U_a}$ is disjoint from $B\backslash A$, and $|f'| \geqslant \alpha > 1$ on $I\backslash \bigcup_{a\in A} \overline{U_a}$. It is easy to see that we can also have $|f'| \geqslant \alpha > 1$ on U_a if $a \in A_1$.

Let us introduce the class of functions $\mathscr{A}_a$ $(a \in I)$. A non-negative L^1-function ξ on a neighbourhood of $a \in I$ belongs to $\mathscr{A}_a$ if there exists a continuous function φ on a neighbourhood of a such that

a) $\varphi(a) = 0$,
b) φ is increasing,
c) φ is of class C^1 except at a,
d) $\dfrac{\varphi(t)}{t-a} \in L^1$ on some neighbourhood of a,
e) $\xi \leqslant \varphi'$ on a neighbourhood of a.

Remark. — Notice that if ξ is bounded in a neighbourhood of a then $\xi \in \mathscr{A}_a$.

Lemma **5.**

a) If $\xi_1, \xi_2 \in \mathscr{A}_a$ then $\xi_1 + \xi_2 \in \mathscr{A}_a$.
b) If $\xi \in \mathscr{A}_a$, and $K > 0$ then $K\xi \in \mathscr{A}_a$.
c) If $\xi \in \mathscr{A}_a$ and $f'(a) \neq 0$ then, for some neighbourhood V of a,

$$(f|_V)_*\,(\xi) \in \mathscr{A}_{f(a)}.$$

Proof. — *a)* and *b)* are obvious. To prove *c)* notice that for some neighbourhood V of a, $f|_V$ is one-to-one. Then

$$(f|_V)_*\,(\xi)\,(x) = \frac{\xi((f|_V)^{-1}\,(x))}{|f'((f|_V)^{-1}\,(x))|}.$$

If V is small, the denominator is bounded away from 0 by a constant $\delta > 0$. It is also bounded from above by a constant M. Therefore

$$(f|_V)_*(\xi)(x) \leqslant \frac{1}{\delta}\xi((f|_V)^{-1}(x))$$
$$\leqslant \frac{1}{\delta}\varphi'((f|_V)^{-1}(x)) \leqslant \frac{M}{\delta}|(\varphi\circ(f|_V)^{-1})'(x)|.$$

Consequently, to prove that $(f|_V)_*(\xi) \in \mathscr{A}_{f(a)}$ it remains to show that

$$\frac{\varphi\circ(f|_V)^{-1}(t)}{t-f(a)}$$

is integrable in a neighbourhood of $f(a)$. But this follows by the change of variables $t = f(u)$ and the fact that

$$\frac{u-a}{f(u)-f(a)}f'(u)$$

is bounded near a. □

Lemma **6**. — *If* $\xi : I \to \mathbf{R}$ *is non-negative and bounded in a neighbourhood* V *of* a, *then* $(f|_V)_*(\xi) \in \mathscr{A}_{f(a)}$.

Proof. — Take a small one-sided neighbourhood $\tilde{V}$ of a and a function given by

$$\tilde{\varphi}(x) = \begin{cases} |(f|_{\tilde{V}})^{-1}(x) - a| & \text{if } x \geqslant f(a) \\ -|(f|_{\tilde{V}})^{-1}(x) - a| & \text{if } x \leqslant f(a) \end{cases}$$

($\tilde{\varphi}$ is defined on a one-sided neighbourhood of $f(a)$). Clearly $\tilde{\varphi}(f(a)) = 0$, $\tilde{\varphi}$ is increasing and of class C^1. By Lemma 3 we can use Lemma 1 and the assumption (vi) to conclude that $\tilde{\varphi}(t)/(t-f(a))$ is integrable in a neighbourhood of $f(a)$ (we set $\psi(x) = f(x+a) - f(a)$ and get $|\tilde{\varphi}(x)| = |\psi^{-1}(x-f(a))|$). Since ξ is bounded by some constant K, we get $(f|_{\tilde{V}})_*(\xi) \leqslant K\tilde{\varphi}'$. Now the statement of the lemma follows from Lemma 5 *a)* and *b)*. □

Lemma 7 *(Cf.* [M2, *Lemma* (3.4)]*).*

1) *For all* $a \in A_2''$, $\sup_{n\geqslant 0} f_*^n(1)$ *is bounded on a neighbourhood of* a.

2) *For all* $a \in A_2'$, $\sup_{n\geqslant 0} f_*^n(1) \in \mathscr{A}_a$.

3) *For all* $a \in A_2$, $\sup_{n\geqslant 0}(f|_{U_a})_*(f_*^n(1)) \in \mathscr{A}_{f(a)}$.

Proof. — 1) follows immediately from the definition of A_2'', Proposition 2 and the form of the Perron-Frobenius operator.

2) and 3) follow from 1), assumption (v) *b)* and lemmata 5 and 6.

We can form the supremum with respect to n because in view of Proposition 2, we are interested only in what happens at a finite number of inverse images of a independently of n (cf. the proof of Lemma (3.4) of [M2]). □

Lemma **8** [M2, *Lemma* (3.5)]. — *Let* $H \subset I$ *and*

$$H_k = \{ x : f^i(x) \in H \quad for\ i = 0, 1, 2, \ldots, k-1 \}.$$

Then for every s, m we have

$$\int_{H_s} f_*^m(1)\, d\lambda \leqslant \sum_{k=s}^{\infty} \int_{H_k} \sup_{n \geqslant 0} (f|_{I \setminus H})_* (f_*^n(1))\, d\lambda + \lambda(H_{s+m}).$$

Lemma **9**. — *For every* $a \in A_1$, *and* $\varepsilon > 0$ *there is a neighbourhood* W *of* a *such that*

$$\int_{W \cap U_a} f_*^n(1)\, d\lambda < \varepsilon$$

for every $n \geqslant 0$.

Proof. — Let $a \in A_1$. There exists a constant $\beta > 1$ such that

$$|f(x) - f(a)| \geqslant \beta\, |x - a| \quad \text{for } x \in U_a.$$

The set $V_k = \{ x : f^i(x) \in U_a \text{ for } i = 0, \ldots, k-1 \}$ is a neighbourhood of a in U_a and we have $\lambda(V_k) \leqslant \lambda(I)/\beta^k$. Therefore it is enough to prove that $\sup_{m \geqslant 0} \int_{V_s} f_*^m(1)\, d\lambda \to 0$ as $s \to \infty$. First we write

$$(f|_{I \setminus U_a})_* (f_*^n(1)) = (f|_G)_* (f_*^n(1)) + (f|_{I \setminus (G \cup U_a)})_* (f_*^n(1)),$$

where $G = \bigcup_{b \in R} U_b$, $R = \{ b \in A \setminus \{ a \} : f(b) = a \}$.

By lemmata 5 and 7 and Proposition 2 we have

$$\xi = \sup_{n \geqslant 0} (f|_{I \setminus U_a})_* (f_*^n(1)) \in \mathscr{A}_a. \tag{3.1}$$

(Notice that $R \subset A_2$ so that we can use Lemma 7.) In view of Lemma 8 (for $H = U_a$; then $H_k = V_k$) and (3.1) we get for s large enough

$$\xi = \sup_{m \geqslant 0} \int_{V_s} f_*^m(1)\, d\lambda \leqslant \sum_{k=s}^{\infty} \int_{a - \frac{\lambda(I)}{\beta^k}}^{a + \frac{\lambda(I)}{\beta^k}} \varphi'(t)\, dt + \frac{\lambda(I)}{\beta^{s+m}},$$

where φ is a function as in the definition of $\mathscr{A}_a$. Set $\widetilde{\varphi}(t) = \varphi(t + a)$. We get

$$\sup_{m \geqslant 0} \int_{V_s} f_*^m(1)\, d\lambda \leqslant \sum_{k=s}^{\infty} \left[\widetilde{\varphi}\left(\frac{\lambda(I)}{\beta^k}\right) + \left| \widetilde{\varphi}\left(-\frac{\lambda(I)}{\beta^k}\right) \right| \right] + \frac{\lambda(I)}{\beta^s}.$$

This tends to 0 as $s \to \infty$ by Lemma 2 and condition *d)* in the definition of $\mathscr{A}_a$. □

Lemma **10**. — *For every* $\varepsilon > 0$ *there exists a neighbourhood* W *of* $B \setminus A$ *such that*

$$\int_W f_*^n(1)\, d\lambda < \varepsilon \quad \textit{for every } n \geqslant 0.$$

Proof. — If we put $\bigcup_{a \in A} U_a$ instead of U and B\A instead of B, then the hypotheses of Section 2 are satisfied. Let E_n be defined as in Section 2. Clearly, E_n is a neighbourhood of B\A. Hence it is enough to prove that

$$\sup_{m \geqslant 0} \int_{E_s} f_*^m(1)\, d\lambda \to 0 \quad \text{as } s \to \infty.$$

By Proposition 1 it follows that there exists a constant η, $0 < \eta < 1$, such that $\lambda(E_s) \leqslant \eta^s \lambda(I)$ for all $s \geqslant 0$. We obtain by (3.1) and Lemma 5 that for some $\widetilde{\varphi}$ positive and increasing in some $(0, \varepsilon]$ and with $\int_0^\varepsilon (\widetilde{\varphi}(t)/t)\, dt < \infty$,

$$\int_{E_s} \sup_{m \geqslant 0} (f|_{\bigcup_{a\in A} U_a})_* (f_*^m(1))\, d\lambda \leqslant \widetilde{\varphi}(\eta^s \lambda(I))$$

for all $s \geqslant 0$. Hence by Lemma 8 we obtain that with $H = I \backslash \bigcup_{a \in A} U_a$

$$\sup_{m \geqslant 0} \int_{E_s} f_*^m(1) \leqslant \sum_{k=s}^{\infty} \widetilde{\varphi}(\eta^k \lambda(I)) + \lambda(I)\, \eta^s \to 0 \quad \text{as } s \to \infty$$

by Lemma 2. □

Proposition **3.** — *If f satisfies* (i)-(vi), *then for every $\varepsilon > 0$ there exists $\delta > 0$ such that if* $G \subset I$ *and* $\lambda(G) < \delta$ *one has* $\int_G f_*^n(1)\, d\lambda < \varepsilon$ *for all n.*

Proof. — Suppose first that f satisfies (i)-(viii). From Lemma 7, 1) and 2), it follow that the statement of Lemma 9 also holds for all $a \in A_2$. This and lemmata 9 and 10 imply that for every $\varepsilon > 0$ there exists a neighbourhood W of B such that $\int_W f_*^n(1)\, d\lambda < \varepsilon$ for all n.

The rest of the proof proceeds as the proof of [M2, Proposition (3.8)]. □

4. The main results

We are now ready to state our first main result

Theorem **3.** — *If f satisfies* (i)-(vi) *then f has an invariant probability measure absolutely continuous with respect to Lebesgue's measure.*

Proof. — Take any weak-* limit of a subsequence of

$$\frac{1}{n} \sum_{k=0}^{n-1} f_*^n(\lambda).$$

It follows from Proposition 3 that it is absolutely continuous. □

Remark. — Notice that the rest of the results of [M2] also go through under our new assumptions (in particular Theorems (6.2) and (6.3)).

We next turn to the converse result.

Theorem **4.** — *Let* $f: \mathrm{I} \to \mathrm{I}$ *be a unimodal map satisfying conditions* (i)-(v). *If* f *has an absolutely continuous invariant measure then*

$$\int_{\mathrm{I}} \log |f'(x)|\, dx > -\infty.$$

For the proof we will need the theory of the functional spaces $\mathscr{D}_r(\mathrm{J})$ as developed in [M2], Section 4. Here we merely give the definition and refer to [M2] for results and proofs.

Definition. — Let J be an open interval. We denote by $\mathscr{D}_r(\mathrm{J})$ the set consisting of all positive C^r functions τ on J such that $1/\sqrt{\tau}$ is concave, and the function 0.

Lemma **11.** — *If a unimodal map* $f: \mathrm{I} \to \mathrm{I}$ *satisfying conditions* (i)-(v) *has an absolutely continuous invariant measure than it has an absolutely continuous invariant measure with density in* $\mathscr{D}_0$ *on each component of* $\mathrm{I} \backslash \mathrm{B}$, *positive at the critical point* c.

Proof. — Let ν be a weak-$*$ limit of some subsequence of

$$\left\{ \frac{1}{n} \sum_{k=0}^{n-1} (f^k)^* (\lambda) \right\}_{n=1}^{\infty}$$

(as before, λ is the Lebesgue measure on I). Then ν is invariant and since B is invariant then $\nu = \nu_1 + \nu_2$, where both ν_1 and ν_2 are invariant, ν_1 is supported by B and ν_2 by $\mathrm{I} \backslash \mathrm{B}$. On each compact subset $\mathrm{K} \subset \mathrm{I} \backslash \mathrm{B}$, $f_*^k(1)$ is bounded by $\lambda(\mathrm{I})/\mathrm{dist}(x, \mathrm{B})$, which is integrable on K. Thus, $\nu_2|_{\mathrm{K}}$ is absolutely continuous. Hence ν_2 is also absolutely continuous.

Set $\tau = d\nu_2/d\lambda$. By [M2, Proposition (4.5)], $\tau \in \mathscr{D}_0$ on components of $\mathrm{I} \backslash \mathrm{B}$. Let J be the component which contains c.

Suppose that $\tau|_{\mathrm{J}}$ is identically equal to 0. Since the inverse images of c are dense, we obtain $\nu_2 = 0$. Denote the absolutely continuons measure, which we assume to exist, by $\varkappa$. For every $\varepsilon > 0$ we have $\kappa = \kappa_1 + \kappa_2$, where $d\kappa_1/d\lambda$ is bounded by some constant M and $\kappa_2(\mathrm{I}) < \varepsilon$. Hence, on a set K as above

$$\frac{1}{n} \sum_{k=0}^{n-1} f_*^k \left(\frac{d\kappa_1}{d\lambda} \right) \leqslant \frac{1}{n} \sum_{k=0}^{n-1} f_*^k(\mathrm{M}) \to 0$$

on a subsequence which gives ν. Since for all n

$$\kappa(\mathrm{K}) = \frac{1}{n} \sum_{k=0}^{n-1} (f^*)^k (\kappa_1) (\mathrm{K}) + \frac{1}{n} \sum_{k=0}^{n-1} (f^*)^k (\kappa_2) (\mathrm{K}),$$

we have $\kappa(\mathrm{K}) \leqslant \kappa_2(\mathrm{I}) < \varepsilon$. Since ε and K were arbitrary, we obtain $\kappa(\mathrm{I} \backslash \mathrm{B}) = 0$. But $\lambda(\mathrm{B}) = 0$ and thus $\kappa(\mathrm{B}) = 0$. Consequently, $\kappa = 0$ — a contradiction.

This proves that $\tau|_{\mathrm{J}}$ is bounded away from 0. Therefore the measure ν_2 (after normalization) satisfies the assertion of the lemma. □

Proof of Theorem **4.** — For definiteness assume that f has a maximum. We now call our good invariant measure (i.e. ν_2) μ. The density of μ is bounded away from 0 by some constant $\theta > 0$ on some neighbourhood U of c. We can assume that dist(U, B) $= \gamma > 0$. Set

$$J_k = U \cap f^{-1}(I \setminus U) \cap \dots \cap f^{-k}(I \setminus U),$$
$$K_k = (I \setminus U) \cap f^{-1}(I \setminus U) \cap \dots \cap f^{-k}(I \setminus U).$$

Clearly $J_k \cap K_k = \emptyset$ and $J_{k+1} \cup K_{k+1} = f^{-1}(K_k)$. By induction

$$\mu(I) = \mu(J_0) + \mu(J_1) + \dots + \mu(J_k) + \mu(K_k).$$

We have $|f'| \leqslant \alpha$ for some $\alpha > 1$. Let

$$L_k = \left[f(c) - \frac{\gamma}{\alpha^k}, f(c) \right].$$

If k is sufficiently large then $f^{-1}(L_k) \subset J_k$. Hence

$$\mu(I) \geqslant \sum_{k=k_0}^{\infty} \mu(f^{-1}(L_k)) \geqslant \theta \sum_{k=k_0}^{\infty} \Theta\left(\frac{\gamma}{\alpha^k}\right).$$

Here $\Theta(x) = \varphi_l^{-1}(x) + \varphi_r^{-1}(x)$, where $\varphi(x) = f(c) - f(x - c)$, and φ_r^{-1} and φ_l^{-1} are the two branches of the inverse function of φ.

It follows from Lemma 1 and Lemma 2 that $\int_I \log |f'(x)| \, dx > -\infty$ and the proof of Theorem 4 is finished. □

REFERENCES

[C-E] P. Collet and J.-P. Eckmann, *Iterated Maps of the Interval as dynamical systems*, Birkhauser, 1980.

[M1] M. Misiurewicz, Structure of mappings of an interval with zero entropy, *Publ. Math. IHES*, **53** (1981), 5-16.

[M2] M. Misiurewicz, Absolutely continuous measures for certain maps of an interval, *Publ. Math. IHES*, **53** (1981), 17-51.

[S] S. van Strien, *Hyperbolicity and invariant measures for general C^2 interval maps satisfying the Misiurewicz condition*, preprint, Delft 1988.

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