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ALGEBRA COCHAINS AND CYCLIC COHOMOLOGY

by Daniel QUILLEN

Dedicated to René Thom.

INTRODUCTION

Historically cyclic cohomology was approached from two directions. In one, Connes developed cyclic cohomology as a noncommutative analogue of de Rham cohomology, which was suggested by the K-theory and index theory in his noncommutative geometry [C1]. In the other, cyclic homology appeared as a Lie analogue of algebraic K-theory defined using the Lie algebra homology of matrices [LQ, T].

In both of these approaches to the subject one first encounters the complex of cyclic cochains, or equivalently by duality, the cyclic complex of the algebra under consideration. However, in order to establish the fundamental properties of cyclic cohomology, one brings in a remarkable resolution of the cyclic complex, the cyclic bicomplex. This double complex is periodic of period two, where the periodicity is closely related to the cohomology of cyclic groups. Using it, one establishes the long exact sequence relating cyclic cohomology and Hochschild cohomology in which the basic S-operation on cyclic cohomology appears.

The first goal of the present article is to offer an explanation of this cyclic formalism. Our starting point is the bar construction of the augmented algebra obtained by adjoining an identity to the algebra being studied. The bar construction is a differential graded coalgebra, and we show how the cyclic complex and cyclic bicomplex can be defined naturally in terms of this structure. For example, just as an algebra R has a commutator quotient space R/[R, R], a coalgebra has a cocommutator subspace. We prove that the cocommutator subspace of the bar construction can be identified with the cyclic complex up to a dimension shift.

The second goal is to show how this coalgebra structure can be used to construct cyclic cohomology classes. The point is that the coalgebra structure gives rise to a differential graded algebra structure on cochains, and the familiar connection and curvature calculations of Chern-Weil theory can be used to produce cyclic cocycles. The moral appears to be that all interesting cyclic classes, especially those related to K-theory, are Chern character forms or variants of these when suitably interpreted. For example, we show that the odd cyclic classes of Connes, which are associated to an algebra extension together with a trace defined on the power of the ideal, can be viewed as Chern character forms. The even analogues of these cyclic classes studied in [Q2] turn out to be Chern-Simons forms.

The contents of the paper are as follows. In the first two sections we use the coalgebra structure on the bar construction to set up the DG algebra of cochains and the trace on this algebra with values in the complex of cyclic cochains. We then show how to construct the odd cyclic cohomology classes of Connes and the even Chern-Simons classes by the standard connection-curvature methods.

The next three sections are devoted to developing the formalism needed to prove the result of Connes that his odd cyclic classes are all related by the S-operation, as well as to extend this result to the even classes. In § 3 we study the bimodule of non-commutative differentials Ω^1_R over an algebra R and the periodic complex

$$\to R \to \Omega^1_{R,\natural} \to R \to \Omega^1_{R,\natural} \to \qquad \Omega^1_{R,\natural} = \Omega^1_R/[R, \Omega^1_R].$$

The point is to construct these objects in such a way that the corresponding constructions for coalgebras are clear by formal duality, that is, just reversing the arrows. The coalgebra version is developed in § 4 and applied to the bar construction in § 5, where we identify the cyclic bicomplex with the analogue of the above periodic complex for the bar construction. In addition we describe the cochain formalism corresponding to this result which is then applied to the study of cyclic cohomology in the remaining sections.

The sixth section contains the proof that the cyclic classes constructed in § 2 are related by the S-operation. In § 7 we consider a vector bundle with connection and construct periodic cyclic cocycles on its algebra of endomorphisms. In the last section we interpret the Chern character of Jaffe, Lesniewski, and Osterwalder [JLO] as the analogue of superconnection character forms in our cochain theory.

The last section on the JLO construction is an elaboration of key ideas I learned from Ezra Getzler, and I am very grateful to him for sharing his insights.

1. The bar construction and cyclic cocycles

1.1. The cyclic bicomplex

In this paper we work over a field k of characteristic zero. Algebras are assumed to be unital unless specified otherwise. We write $(v_1, \ldots, v_n)$ for the element $v_1 \otimes v_2 \otimes \ldots \otimes v_n \in V^{\otimes n}$.

Let A be nonunital algebra. In the study of cyclic cohomology an important role is played by the remarkable double complex [LQ]

$$\begin{array}{ccccccccc}
 & \downarrow b & & \downarrow b' & & \downarrow b & \\
\xleftarrow{N} & A^{\otimes 3} & \xleftarrow{1-T} & A^{\otimes 3} & \xleftarrow{N} & A^{\otimes 3} & \xleftarrow{1-T} \\
 & \downarrow b & & \downarrow b' & & \downarrow b & \\
\xleftarrow{N} & A^{\otimes 2} & \xleftarrow{1-T} & A^{\otimes 2} & \xleftarrow{N} & A^{\otimes 2} & \xleftarrow{1-T} \\
 & \downarrow b & & \downarrow b' & & \downarrow b & \\
\xleftarrow{N} & A & \xleftarrow{1-T} & A & \xleftarrow{N} & A & \xleftarrow{1-T} \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 &
\end{array}$$

where the operators b, b', T, N on $A^{\otimes n}$ are given by the formulas

$$b'(a_1, \ldots, a_n) = \sum_{i=1}^{n-1} (-1)^{i-1} (a_1, \ldots, a_i a_{i+1}, \ldots, a_n),$$

$$b(a_1, \ldots, a_n) = b'(a_1, \ldots, a_n) + (-1)^{n-1} (a_n a_1, a_2, \ldots, a_{n-1}),$$

$$T(a_1, \ldots, a_n) = (-1)^{n-1} (a_n, a_1, \ldots, a_{n-1}),$$

$$N = \sum_{i=0}^{n-1} T^i.$$

This double complex is periodic of period two in the horizontal direction, and its rows are exact. The column with the differential b is the standard complex for computing the Hochschild homology $H_{\bullet}(A, A)$ when A is unital. The column with the differential b' is the bar construction of the augmented algebra $\tilde{A} = k \oplus A$ except for removing the field in degree zero and shifting degrees by one. When A is unital, the b'-complex is exact.

The cokernel of the map $1 - T$ from the b'-complex to the b-complex is by definition the cyclic complex CC(A), whose homology is the cyclic homology $HC_{\bullet}(A)$ of A. By exactness of the rows the cyclic complex is also isomorphic to the kernel of $1 - T$, and we have an exact sequence of complexes

$$0 \to CC(A) \to \{ b' - cx \} \xrightarrow{1-T} \{ b - cx \} \to CC(A) \to 0.$$

From this follows the Connes long exact sequence

$$\to H_n(A, A) \to HC_n(A) \xrightarrow{S} HC_{n-2}(A) \to H_{n-1}(A, A) \to$$

when A is unital, and in general one obtains a similar long exact sequence with the Hochschild homology replaced by the reduced Hochschild homology of $\tilde{A}$ [LQ, 4.2].

1.2. Coalgebra structure and the algebra of cochains

The starting point for the present paper is the fact that the bar construction $B = \overline{B}(\widetilde{A})$ is a DG coalgebra. As a coalgebra it is the tensor coalgebra $T(A[1])$ of the underlying vector space of A located in degree one. Thus $B_n = A^{\otimes n}$ for $n \geqslant 0$, the coproduct is

$$\Delta(a_1, \ldots, a_n) = \sum_{i=0}^{n} (a_1, \ldots, a_i) \otimes (a_{i+1}, \ldots, a_n)$$

and the counit η is the projection onto $A^{\otimes 0} = k$. Its differential is b', which is to be interpreted as zero for $n = 0, 1$. The coproduct and counit maps $\Delta : B \to B \otimes B$, $\eta : B \to k$ are morphisms of complexes, making it a DG coalgebra. The homology of the bar construction is the coalgebra $\mathscr{T}or_{*}^{\widetilde{A}}(k, k)$.

By an n-cochain or cochain of degree n on A we will mean a multilinear function $f(a_1, \ldots, a_n)$ with values in some vector space V, or equivalently a linear map from $B_n = A^{\otimes n}$ to V. These cochains form a complex $\mathrm{Hom}(B, V)$, where the differential is

$$\delta(f) = -(-1)^n f b', \qquad f \in \mathrm{Hom}^n(B, V) = \mathrm{Hom}(B_n, V).$$

If L is an algebra, then the complex of cochains $\mathrm{Hom}(B, L)$ has a product defined by

$$fg = m(f \otimes g)\, \Delta,$$

where $m : L \otimes L \to L$ is the multiplication in L. If f, g have degrees p, q, we have

$$(fg)\,(a_1, \ldots, a_{p+q}) = (-1)^{pq} f(a_1, \ldots, a_p)\, g(a_{p+1}, \ldots, a_{p+q}),$$

where the sign is due to the way $f \otimes g$ is defined for complexes. As B is a DG coalgebra, it follows that $\mathrm{Hom}(B, L)$ is a DG algebra.

As an example of these formulas which will be important in the sequel, let ρ be a 1-cochain, that is, a linear map from A to L. We can view ρ as a " connection " form and construct its " curvature " $\omega = \delta\rho + \rho^2$, which is a 2-cochain. Then

$$\omega(a_1, a_2) = (\delta\rho + \rho^2)\,(a_1, a_2) = \rho(a_1 a_2) - \rho(a_1)\,\rho(a_2),$$

showing that ρ is a ring homomorphism if and only if the curvature is zero.

One has the Bianchi identity

$$\delta\omega = -\rho\omega + \omega\rho = -[\rho, \omega],$$

which up to sign is

$$\text{(1.1)} \qquad \omega(a_1 a_2, a_3) - \omega(a_1, a_2 a_3) = \rho(a_1)\,\omega(a_2, a_3) - \omega(a_1, a_2)\,\rho(a_3).$$

Since δ and $\alpha \mapsto [\rho, \alpha]$ are derivations, this implies

$$\delta\omega^n = -[\rho, \omega^n].$$

1.3. Traces

To proceed further as in the Chern-Weil theory we need to have a trace on cochains. Let $\tau : L \to V$ be a trace on the algebra L with values in the vector space V, that is, a linear map satisfying

$$\tau([x, y]) = \tau(xy) - \tau(yx) = 0.$$

It is equivalent to say that τ vanishes on the commutator subspace [L, L], which is the image of the map $m - m\sigma : L \otimes L \to L$, where σ denotes the canonical automorphism permuting the factors in the tensor product. Thus a trace is really a linear map defined on the commutator quotient space:

$$L_\natural \overset{\text{def}}{=} L/[L, L] = \operatorname{Coker}\{ m\sigma - m : L \otimes L \to L \}.$$

Formulated in this way as a linear map defined on the commutator quotient space, it is clear how to extend the notion of trace to superalgebras and DG algebras. In these contexts an extra sign occurs in the trace identity when both elements are odd (and hence a trace is what is usually called a supertrace), because the permutation isomorphism σ involves signs. It is also clear how to define the dual notions of cotrace and cocommutator subspace for coalgebras, supercoalgebras, and DG coalgebras.

Thus naturally associated to the bar construction B is its cocommutator subspace

$$B^\natural \overset{\text{def}}{=} \operatorname{Ker}\{ \Delta - \sigma\Delta : B \to B \otimes B \}.$$

This is a subcomplex of B as Δ, σ are morphisms of complexes. We let $\natural : B^\natural \to B$ denote the inclusion map; it is the universal cotrace in the sense that a cotrace $V \to B$ is the same as a linear map with values in $B^\natural$.

Next we combine the trace τ with the universal cotrace $\natural$ to define a morphism of complexes

$$\tau^\natural : \operatorname{Hom}(B, L) \to \operatorname{Hom}(B^\natural, V), \qquad \tau^\natural(f) = \tau f \natural,$$

which is a trace on the DG algebra of cochains because

$$\begin{aligned} \tau^\natural(fg) &= \tau m(f \otimes g)\, \Delta\natural = \tau m\sigma(f \otimes g)\, \sigma\, \Delta\natural \\ &= (-1)^{|f||g|}\, \tau m(g \otimes f)\, \Delta\natural = (-1)^{|f||g|}\, \tau^\natural(gf). \end{aligned}$$

We need later a slight generalization of this discussion. The concept of trace makes sense for a bimodule M over the algebra L; it is a linear map $\tau : M \to V$ vanishing on [L, M]. In this situation the cochains with values in M form a DG bimodule over the DG algebra of L-valued cochains, and

$$\tau^\natural : \operatorname{Hom}(B, M) \to \operatorname{Hom}(B^\natural, V), \qquad \tau^\natural(g) = \tau g \natural,$$

is a trace on this bimodule which is closed in the sense that it commutes with the differentials.

Applying the trace $\tau^\natural$ on the algebra of cochains to ω^n yields

$$\delta\{\tau^\natural(\omega^n)\} = \tau^\natural\{\delta(\omega^n)\} = \tau^\natural\{-[\rho, \omega^n]\} = 0.$$

This is the usual proof that Chern character forms are closed.

Thus we obtain a closed element of degree $2n$ in the complex $\mathrm{Hom}(B^\natural, V)$. We are now going to show that we have in effect constructed a cyclic $(2n-1)$-cocycle on A.

We begin by relating $B^\natural$ to the cyclic complex $CC(A)$.

Lemma **1.2.** — *The space* $B_n^\natural$ *is the kernel of* $(1 - T)$ *acting on* $A^{\otimes n}$.

Proof. — Let $p_{ij}(x)$ denote the component of $x \in B \otimes B$ of bidegree i, j. Then

$$\begin{aligned} p_{i,n-i}\, \sigma\Delta(a_1, \ldots, a_n) &= (-1)^{(n-i)i}(a_{n-i+1}, \ldots, a_n) \otimes (a_1, \ldots, a_{n-i}) \\ &= p_{i,n-i}\, \Delta T^i(a_1, \ldots, a_n), \end{aligned}$$

hence $p_{i,n-i}(\Delta - \sigma\Delta)\, x = p_{i,n-i}\, \Delta(1 - T^i)\, x$ for all $x \in A^{\otimes n}$. Thus

$$x = Tx \Rightarrow \Delta x = \sigma\Delta x \Rightarrow x \in B_n^\natural.$$

The converse is also true because $p_{1,n-1}\Delta : A^{\otimes n} \to A \otimes A^{\otimes n-1}$ is an isomorphism, so the lemma follows.

From properties of the cyclic bicomplex we know that the cyclic sum map N from the b-complex to the b'-complex is a morphism of complexes (i.e. $b'N = Nb$), and that it induces an isomorphism

$$CC(A) = \mathrm{Coker}(1 - T) \xrightarrow{\sim} \mathrm{Ker}(1 - T).$$

Therefore the lemma shows that the cocommutator subspace of the bar construction is essentially the same as the cyclic complex, the difference being that the former has the field in degree zero, and the degrees are shifted by one.

Thus we see that $\tau^\natural(\omega^n) \in \mathrm{Hom}(B^\natural, V)$ can be identified with a certain cyclic cocycle of degree $2n-1$. We have

$$\omega^n(a_1, \ldots, a_{2n}) = \omega(a_1, a_2) \ldots \omega(a_{2n-1}, a_{2n}) \tag{1.3}$$

and to obtain the cyclic cocycle we apply τ to the values and N to the arguments:

$$\tau^\natural(\omega^n)(a_1, \ldots, a_{2n}) = \sum_{i=0}^{2n-1} (-1)^i \tau\{\omega^n(a_{i+1}, \ldots, a_{2n}, a_1, \ldots, a_i)\}. \tag{1.4}$$

Because τ is a trace, ω^n is fixed under cyclic shifts by an even number of steps, so this becomes

$$\begin{aligned} &\tau^\natural(\omega^n)(a_1, \ldots, a_{2n}) = \\ &\quad n\tau\{\omega(a_1, a_2) \ldots \omega(a_{2n-1}, a_{2n}) - \omega(a_{2n}, a_1) \ldots \omega(a_{2n-2}, a_{2n-1})\}. \end{aligned} \tag{1.5}$$

Remark **1.6.** — The fact that this formula gives a cyclic cocycle is due to Connes [C1, Th. 5]. What we have done in effect is to interpret his calculations as the usual proof that Chern character forms are closed.

2. Cyclic classes associated to extensions

In this section we consider a (unital) algebra L, an ideal I in L, and a homomorphism $u : A \to L/I$ from the nonunital algebra A to the quotient algebra. We can then construct cyclic cohomology classes on A in two ways starting from a suitable trace τ. In the first, we suppose τ is a trace on the ideal I^m considered as a bimodule over L, and we obtain the Connes odd degree cyclic cohomology classes as Chern character classes. In the second, we suppose τ is a trace on the quotient algebra L/I^{m+1}, and we obtain even degree cyclic classes, which are given by Chern-Simons forms.

2.1. The odd cyclic classes of Connes

Let $\rho : A \to L$ be a linear map reducing modulo I to u, and let $\tau : I^m \to V$ be a trace on the L-bimodule I^m, that is, τ is a linear map vanishing on $[L, I^m]$. Since the constructions to follow are natural in the vector space V, one might as well take τ to be the canonical surjection onto $I^m/[L, I^m]$.

Because ρ is a homomorphism modulo I, its curvature ω is a 2-cochain with values in I, and ω^n is a $2n$-cochain with values in I^n. Now as remarked in the preceding section the I^m-valued cochains form a DG ideal in the DG algebra of L-valued cochains, and

$$\tau^\natural : \mathrm{Hom}(B, I^m) \to \mathrm{Hom}(B^\natural, V)$$

is a closed trace on this DG ideal. Hence $\tau^\natural(\omega^n)$ for $n \geqslant m$ is a well-defined element of $\mathrm{Hom}^{2n}(B^\natural, V)$, which is closed by the standard argument showing the Chern character forms are closed. Thus we have defined a cyclic $(2n-1)$-cocycle for $n \geqslant m$. It is given by the formula (1.4), and by (1.5) when $\tau[I, I^n] = 0$, which is certainly the case when $n > m$.

The next step in Chern-Weil theory is the treatment of a homotopy between connections, and a nice way to do this is to utilize the product of the given manifold with the line. In order to carry this out in the context of cochains, we need to extend our cochain formalism in the standard way so that the cochains have values in complexes.

Let $V = \{ \to V^n \to V^{n+1} \to \}$ be a complex, and suppose to simplify the discussion that it is bounded below in the sense that $V^n = 0$ for $n \ll 0$. Then the usual Hom-complex Hom(B, V) has the structure of a double complex $\mathrm{Hom}(B_q, V^p)$ with the anticommuting differentials

$$d(f) = d_V f, \qquad \delta(f) = (-1)^{|f|+1} f b',$$

where $|f|$ denotes the total degree $p + q$ of $f \in \mathrm{Hom}(B_q, V^p)$. The total differential is $d_{\mathrm{tot}} = d + \delta$. Similarly if L is a DG algebra bounded below, then Hom(B, L) is a bigraded DG algebra with product $fg = m(f \otimes g)\,\Delta$:

$$(fg)(a_1, \ldots, a_n) = \sum_{i=0}^{n} (-1)^{i|g|} f(a_1, \ldots, a_i)\, g(a_{i+1}, \ldots, a_n).$$

By a trace $\tau : L \to V$ or, more generally, $\tau : M \to V$ where M is a DG bimodule over L, we mean a linear map between complexes, not necessarily compatible with differentials, which vanishes on [L, M]. When it commutes with differentials, we say that it is closed. Then $\tau^\natural(f) = \tau f \natural$ defines a trace on Hom(B, M) with values in $\mathrm{Hom}(B^\natural, V)$ in this generalized setting, which is closed when τ is.

This terminology established we discuss homotopy.

Let $k[t]$ be the polynomial ring over k in the indeterminate t, and write $W[t] = k[t] \otimes W$ when W is a vector space. We consider a one-parameter family of linear maps from A to L or, more precisely, a linear map ρ_t from A to the algebra $L[t]$. We suppose that this is a family of homomorphisms modulo I, which means that the 2-cochain $\omega_t = \delta\rho_t + \rho_t^2$ has values in $I[t]$. Let

$$\mu_{n,t} = \sum_{i=1}^{n} \omega_t^{i-1}\, \dot\rho_t\, \omega_t^{n-i}, \qquad \dot\rho_t = \partial_t\, \rho_t.$$

This is a $(2n-1)$-cochain with values in $I^{n-1}[t]$, and it even has values in $I^n[t]$ when $\dot\rho$ has values in I, that is, when the family of algebra homomorphisms from A to L/I is constant in t. We extend the trace τ given on I^m to a $k[t]$-linear map

$$\tau : I^m[t] \to V[t].$$

We then have the following infinitesimal homotopy formula.

Proposition **2.1.** — *Assume that either $n > m$, or that $n = m$ and the family of homomorphisms from* A *to* L/I *induced by ρ_t is constant. Then $\tau^\natural(\mu_{n,t}) \in \mathrm{Hom}^{2n-1}(B^\natural, V[t])$ is a cyclic $(2n-2)$-cochain such that*

$$\partial_t\, \tau^\natural(\omega_t^n) = \delta\{\tau^\natural(\mu_{n,t})\}.$$

Proof. — Let $W[t, dt]$ denote $k[t, dt] \otimes W$, where

$$k[t, dt] = k[t] \oplus dt k[t], \qquad d = dt\, \partial_t,$$

is the ordinary de Rham complex of $k[t]$. Then $L[t, dt]$ with $d = dt\, \partial_t$ is a DG algebra containing $I[t, dt]$ as a DG ideal, and because $k[t, dt]$ is a commutative DG algebra, one sees easily that the $k[t, dt]$-linear extension of τ

$$\tau : I^m[t, dt] \to V[t, dt]$$

is a closed trace on this DG ideal. Consequently $\mathrm{Hom}(B, L[t, dt])$ with differential $dt\, \partial_t + \delta$ is a DG algebra containing the DG ideal $\mathrm{Hom}(B, I^m[t, dt])$, and

$$\tau^\natural : \mathrm{Hom}(B, I^m[t, dt]) \to \mathrm{Hom}(B^\natural, V[t, dt])$$

is a closed trace.

We let

$$\tilde\omega = (dt\, \partial_t + \delta)\, \rho_t + \rho_t^2 = \omega_t + dt\dot\rho_t$$

be the total curvature of the family. We have

$$\tilde{\omega}^n = \omega_t^n + \sum_1^n \omega_t^{i-1}\, dt\dot{\rho}_t\, \omega_t^{n-i} = \omega_t^n + dt\mu_{nt}.$$

Under our hypotheses ω_t^n, μ_{nt} belong to $\mathrm{Hom}(\mathrm{B}, \mathrm{I}^m[t])$ so

$$\tau^\natural(\tilde{\omega}^n) = \tau^\natural(\omega_t^n) + dt\tau^\natural(\mu_{nt}) \in \mathrm{Hom}(\mathrm{B}^\natural, \mathrm{V}[t, dt])$$

is defined. But this is closed by the standard argument:

$$(dt\, \partial_t + \delta)\, \tau^\natural(\tilde{\omega}^n) = \tau^\natural\{(dt\, \partial_t + \delta)\, \tilde{\omega}^n\} = -\, \tau^\natural([\rho_t, \tilde{\omega}^n]) = 0$$

and taking the coefficient of dt yields the proposition.

Remark **2.2.** — When $\tau[\mathrm{I}, \mathrm{I}^{n-1}] = 0$, the homotopy formula simplifies to

$$\partial_t\, \tau^\natural(\omega_t^n) = \delta\{\, n\tau^\natural(\dot{\rho}_t\, \omega_t^{n-1})\}.$$

We now apply this homotopy formula to prove the following theorem of Connes [C1, I.7, Th. 5].

Theorem 1. — Given the homomorphism $u : \mathrm{A} \to \mathrm{L/I}$ and the trace $\tau : \mathrm{I}^m \to \mathrm{V}$, let $\rho : \mathrm{A} \to \mathrm{L}$ be a linear lifting of u and let ω be its curvature. Then for $n \geqslant m$, the cochain $\tau^\natural(\omega^n)$ given by 1.4, *or by* 1.5 *when $\tau[\mathrm{I}, \mathrm{I}^{n-1}] = 0$, is a cyclic $(2n - 1)$-cocycle whose class in cyclic cohomology depends only on u and τ. Furthermore for $n > m$ this class is a homotopy invariant of u.*

Proof. — Given two linear liftings ρ_0 and ρ_1 of u into L, then we join them by the family $\rho_t = (1 - t)\, \rho_0 + t\rho_1$. We then apply the above homotopy formula which is valid for $n = m$, because the family is constant modulo I. Integrating this formula between 0 and 1 shows the two cyclic cocycles differ by a coboundary. The last assertion follows by the same argument once one notes that any one parameter family of homomorphisms $u_t : \mathrm{A} \to (\mathrm{L/I})\, [t]$ can be lifted to a linear map $\rho_t : \mathrm{A} \to \mathrm{L}[t]$.

2.2. Chern-Simons classes

We consider a homomorphism $u : \mathrm{A} \to \mathrm{L/I}$ as before, but now we suppose given a trace on the quotient algebra

$$\tau : \mathrm{L/I}^{m+1} \to \mathrm{V}.$$

Let $\rho : \mathrm{A} \to \mathrm{L}$ be a linear lifting of u and let ω be its curvature. We consider the one parameter family

$$\rho_t = t\rho, \qquad \omega_t = t\,\delta\rho + t^2\,\rho^2 = t\omega + (t^2 - t)\,\rho^2,$$

and note that the curvature has values in I when $t = 0, 1$. Hence the homotopy formula gives

$$\delta \int_0^1 \tau^\natural(\mu_{n+1,\,t})\, dt = \tau^\natural(\omega_t^{n+1})\Big|_0^1 = 0$$

for $n \geqslant m$. Here

$$\tau^{\natural}(\mu_{n+1,t}) = \tau^{\natural}(\sum_{i=0}^{n} \omega_t^i \, \rho\omega_t^{n-i}) = (n+1)\, \tau^{\natural}(\rho\omega_t^n).$$

Thus

$$\int_0^1 \tau^{\natural}\{\mu_{n+1,t}/(n+1)!\}\, dt = \int_0^1 \tau^{\natural}\{\rho(t\,\delta\rho + t^2\rho^2)^n/n!\} \in \mathrm{Hom}^{2n+1}(\mathrm{B}^{\natural}, \mathrm{V})$$

is a cyclic $2n$-cocycle for $n \geqslant m$. It is the analogue for cochains of the Chern-Simons transgression form associated to the $(n+1)$-st Chern character form.

Theorem 2. — The cyclic cohomology class of the above Chern-Simons cocycle is independent of the choice of the linear lifting ρ, *and for* $n > m$ *this class is a homotopy invariant of the homomorphism* u.

Proof. — We first derive a higher homotopy formula for a two-parameter family $\tilde{\rho} : \mathrm{A} \to \mathrm{L}[s, t]$. We work in the DG algebra

$$\mathrm{Hom}(\mathrm{B}, k[s, ds] \otimes k[t, dt] \otimes \mathrm{L}), \qquad d_{\mathrm{tot}} = ds\, \partial_s + dt\, \partial_t + \delta,$$

and let

$$\tilde{\omega} = (ds\,\partial_s + dt\,\partial_t + \delta)\,\tilde{\rho} + \tilde{\rho}^2 = \omega + ds\,\partial_s\tilde{\rho} + dt\,\partial_t\tilde{\rho}$$

be the total curvature of the family. We have

$$\tilde{\omega}^{n+1} = \omega^{n+1} + ds\nu + dt\mu + ds\,dt\lambda,$$

where

$$\mu = \sum_{i=0}^{n} \omega^i\, \partial_s\tilde{\rho}\omega^{n-i}, \qquad \nu = \sum_{i=0}^{n} \omega^i\, \partial_s\tilde{\rho}\omega^{n-i}.$$

Now

$$\tau^{\natural}(\tilde{\omega}^{n+1}) = \tau^{\natural}(\omega^{n+1}) + ds\tau^{\natural}(\nu) + dt\tau^{\natural}(\mu) + ds\,dt\tau^{\natural}(\lambda)$$

is killed by $ds\,\partial_s + dt\,\partial_t + \delta$ by the usual argument that the Chern character forms are closed. The coefficient of $ds\,dt$ gives the relation

$$\partial_s\tau^{\natural}(\mu) - \partial_t\tau^{\natural}(\nu) + \delta\tau^{\natural}(\lambda) = 0.$$

Integrating we obtain the formula

$$\partial_s \int_0^1 \tau^{\natural}(\mu)\, dt = \tau^{\natural}(\nu)\Big|_{t=0}^{1} - \delta\int_0^1 \tau^{\natural}(\lambda)\, dt.$$

Now suppose we have a one-parameter family of homomorphisms and linear liftings

$$u_s : \mathrm{A} \to (\mathrm{L}/\mathrm{I})\,[s], \qquad \rho_s : \mathrm{A} \to \mathrm{L}[s],$$

and let $\omega_s = \delta\rho_s + \rho_s^2$ be the family of curvatures. Applying the preceding formula to the two-parameter family $\tilde{\rho} = t\rho_s$, we obtain

$$\partial_s \int_0^1 \tau^{\natural}(\mu_{n+1,st})\, dt = (n+1)\, \tau^{\natural}(\partial_s\rho_s\,\omega_s^n) - \delta\int_0^1 \tau^{\natural}(\lambda)\, dt.$$

As τ vanishes on I^{m+1}, the first term on the right vanishes when $n > m$, showing that the derivative of the Chern-Simons cocycle is a coboundary. The same conclusion holds when $m = n$ provided $\partial_s \rho$ has values in I, that is, when the family of homomorphisms u_s is constant.

The rest of the proof is the same as for Theorem 1.

Remark **2.3.** — Simple functorial considerations show that the cyclic $(2n-1)$-cohomology class constructed in Theorem 1 is induced from a universal cyclic class on L/I with values in $I^m/[L, I^m]$. Such a class may be interpreted as a canonical map

$$HC_{2n-1}(L/I) \to I^m/[L, I^m].$$

Similarly from Theorem 2 we obtain a canonical map

$$HC_{2n}(L/I) \to (L/I^{m+1})_\natural = L/(I^{m+1} + [L, L]).$$

These maps are studied in [Q2].

3. Differentials over algebras

In the preceding sections a cochain formalism based on the coalgebra structure of the bar construction was developed and used to construct certain families of cyclic cohomology classes. We next want to show that the members of a family are related by the S-operation on cyclic cohomology. In order to do this, we need to extend the formalism to the whole cyclic bicomplex. This extension involves the analogue for coalgebras of the bimodule of noncommutative differentials Ω^1_R over an algebra R. In this section we study this bimodule, and we give a simple example of the calculation which will be used later to establish the S-relations between cyclic cocycles. The arguments will be dualized to coalgebras and applied to the bar construction in succeeding sections.

3.1. On Ω^1_R and $\Omega^1_{R,\natural}$

Let R be an algebra, and let

$$m : R \otimes R \to R, \qquad \varepsilon : k \to R,$$

be its product and unit maps. We describe our constructions in terms of these maps in order to make it obvious how to extend them to DG algebras and coalgebras. We write 1 for the identity map, and 1_R for the identity element of R. If $V_1, \ldots, V_n$ are vector spaces, we denote by

$$\sigma : V_1 \otimes V_2 \otimes \ldots \otimes V_n \xrightarrow{\sim} V_n \otimes V_1 \otimes \ldots \otimes V_{n-1}$$

the canonical isomorphism corresponding to the forward shift cyclic permutation.

Let M be an R-bimodule, that is, a vector space with left and right product maps

$$m_\ell : R \otimes M \to M, \qquad m_r : M \otimes R \to M,$$

defining left and right module structures which commute. We consider $R \otimes V \otimes R$, where V is a vector space, as a bimodule with $m_\ell = m \otimes 1 \otimes 1$, $m_r = 1 \otimes 1 \otimes m$. It is the free bimodule generated by V in the following sense.

Proposition **3.1.** — *There is a one-one correspondence between linear maps* $h : V \to M$ *and bimodule morphisms* $\tilde{h} : R \otimes V \otimes R \to M$ *given by*

$$\tilde{h} = m_r(m_\ell \otimes 1)\,(1 \otimes h \otimes 1), \qquad h = \tilde{h}(\varepsilon \otimes 1 \otimes \varepsilon).$$

Remark **3.2.** — The proof of this proposition is a routine verification. Usually it is done using elements of R and M, however, we note that it can be carried out entirely in terms of the product and unit maps m, ε, m_ℓ, m_r and the various associativity and unity identities satisfied by these maps, e.g. $m_\ell(1 \otimes m_r) = m_r(m_\ell \otimes 1)$. The same is true for all of the results in this section. An example is given below in 3.6, and one may supply similarly opaque demonstrations of the other assertions. This observation has the consequence that the whole discussion extends immediately to related contexts such as superalgebras and DG algebras. Also by reversing the arrows there are analogues for coalgebras and DG coalgebras, which will be needed later.

We now consider the b'-complex in the cyclic bicomplex of § 1 for R, which in low degrees has the form

$$\to R^{\otimes 4} \xrightarrow{\square} R^{\otimes 3} \xrightarrow{m \otimes 1 - 1 \otimes m} R^{\otimes 2} \xrightarrow{m} R \to 0,$$

where $\square = m \otimes 1 \otimes 1 - 1 \otimes m \otimes 1 + 1 \otimes 1 \otimes m$. This is a sequence of R-bimodule morphisms. It is exact because of the contracting homotopy $\varepsilon \otimes 1^{\otimes n} : R^{\otimes n} \to R^{\otimes n+1}$.

The bimodule Ω^1_R of (noncommutative) differentials over R is defined to be the kernel of m. By exactness, it is also the cokernel of $\square$, so we have bimodule exact sequences

$$0 \to \Omega^1_R \xrightarrow{I} R^{\otimes 2} \xrightarrow{m} R \to 0$$

$$R^{\otimes 4} \xrightarrow{\square} R^{\otimes 3} \xrightarrow{J} \Omega^1_R \to 0$$

such that $IJ = m \otimes 1 - 1 \otimes m$.

By the above proposition the map J is the bimodule morphism $\tilde{\partial}$ extending the linear map

$$\partial : R \to \Omega^1_R, \qquad \partial = J(\varepsilon \otimes 1 \otimes \varepsilon) \quad \text{or} \quad \partial x = J(1_R, x, 1_R).$$

Proposition **3.3.** — *The map* ∂ *is a derivation with values in the bimodule* Ω^1_R. *It is a universal derivation in the sense that any derivation* $D : R \to M$ *with values in a bimodule is induced from* ∂ *by a unique bimodule morphism from* Ω^1_R *to* M.

Proof. — Given a linear map $D : R \to M$, where M is a bimodule, the corresponding bimodule morphism $\tilde{D} : R^{\otimes 3} \to M$ satisfies

$$\begin{aligned}\tilde{D}\square(1_R, x, y, 1_R) &= \tilde{D}(x, y, 1_R) - \tilde{D}(1_R, xy, 1_R) + \tilde{D}(1_R, x, y) \\ &= x\,Dy - D(xy) + (Dx)\,y\end{aligned}$$

and as $\tilde{D}\square$ is a bimodule morphism, this shows that D is a derivation if and only if $\tilde{D}\square = 0$. As $\tilde{\partial} = J$ satisfies $J\square = 0$, it follows that ∂ is a derivation, and the universal property results from the fact that (Ω^1_R, J) is the cokernel of $\square$.

The following proposition gives a convenient characterization of Ω^1_R.

Proposition **3.4.** — *Up to isomorphism there is a unique triple* $(\Omega^1_R, I, \partial)$ *consisting of a* R-*bimodule* Ω^1_R, *a bimodule morphism* $I : \Omega^1_R \to R \otimes R$, *and a linear map* $\partial : R \to \Omega^1_R$, *such that* ∂ *is a derivation,* $\widetilde{\partial}$ *is surjective, and such that* $I\,\partial = 1 \otimes \varepsilon - \varepsilon \otimes 1$.

Proof. — Clearly the triple defined above has these properties. Let (Ω', I', ∂') be another triple with these properties. Because ∂' is a derivation, we have $\widetilde{\partial}' \,\square = 0$, hence we have a unique map $u : \Omega^1_R \to \Omega'$ carrying $\widetilde{\partial}$ to $\widetilde{\partial}'$, which is surjective as $\widetilde{\partial}'$ is assumed surjective. On the other hand, we have $I'\,\widetilde{\partial}' = m \otimes 1 - 1 \otimes m$, because both are bimodule morphisms extending $I'\,\partial' = 1 \otimes \varepsilon - \varepsilon \otimes 1$. It then follows from the exactness of the b'-sequence that u is also injective, proving the proposition.

If M is an R-bimodule, we let

$$M_\natural \overset{\text{def}}{=} M/[R, M] = \operatorname{Coker}\{\, m_l - m_r\,\sigma : R \otimes M \to M \,\}$$

be its commutator quotient space, and we let $\natural : M \to M_\natural$ denote the canonical surjection. In the case of a free bimodule we have the following identification.

Proposition **3.5.** — *There is a canonical isomorphism* $(R \otimes V \otimes R)_\natural \cong V \otimes R$ *relative to which the canonical surjection becomes*

$$\natural = (1 \otimes m)\,\sigma^{-1} : R \otimes V \otimes R \to V \otimes R.$$

Proof. — It suffices to show that the following sequence is exact

$$R \otimes R \otimes V \otimes R \xrightarrow{p} R \otimes V \otimes R \xrightarrow{q} V \otimes R \to 0$$

$$p = m_l - m_r\,\sigma^{-1} = m \otimes 1 \otimes 1 - (1 \otimes 1 \otimes m)\,\sigma^{-1}, \qquad q = (1 \otimes m)\,\sigma^{-1}.$$

Putting $r = 1 \otimes \varepsilon \otimes 1 \otimes 1$, $s = \varepsilon \otimes 1 \otimes 1$, one verifies that $qp = 0$, $qs = 1$, and $rp + sq = 1$, and the exactness follows.

We now apply the commutator quotient space functor to Ω^1_R and the canonical bimodule maps I, $\widetilde{\partial}$, using this proposition to identify the commutator quotient space for free bimodules. This gives a commutative diagram

$$\begin{array}{ccccc}
R^{\otimes 2} & \xleftarrow{\ I\ } & \Omega^1_R & \xleftarrow{\ \widetilde{\partial}\ } & R^{\otimes 3} \\
{\scriptstyle m\sigma}\downarrow & & \downarrow{\scriptstyle \natural} & & \downarrow{\scriptstyle (1 \otimes m)\,\sigma^{-1}} \\
R & \xleftarrow{\ \beta\ } & \Omega^1_{R,\natural} & \xleftarrow{\ \alpha\ } & R^{\otimes 2}
\end{array}$$

where the vertical arrows are the canonical surjections $\natural$ onto the commutator quotient space with respect to the identification of the above proposition, and α, β are the unique maps such that the diagram is commutative. We now derive formulas for α, β.

Lemma **3.6.** — *One has* $\alpha = \natural m_r(\partial \otimes 1)$, *that is*, $\alpha(x \otimes y) = \natural(\partial xy)$.

Proof. — This is a straightforward verification, which we write out in terms of the product and unit maps of the algebra in order to illustrate how the generalization to DG algebras and coalgebras proceeds. There are six steps:

$$\begin{aligned} \alpha(x \otimes y) &= \alpha(1 \otimes m)\,\sigma^{-1}(1_R \otimes x \otimes y) \\ &= \natural\,\tilde{\partial}(1_R \otimes x \otimes y) \\ &= \natural\,\tilde{\partial}(1_R \otimes x \otimes 1_R y) \\ &= \natural\,\tilde{\partial}\{(1_R \otimes x \otimes 1_R)\,y\} \\ &= \natural\{(\tilde{\partial}(1_R \otimes x \otimes 1_R))\,y\} \\ &= \natural(\partial xy) \end{aligned} \qquad \begin{aligned} \alpha &= \alpha(1 \otimes m)\,\sigma^{-1}(\varepsilon \otimes 1 \otimes 1) \\ &= \natural\,\tilde{\partial}(\varepsilon \otimes 1 \otimes 1) \\ &= \natural\,\tilde{\partial}(1 \otimes 1 \otimes m)\,(\varepsilon \otimes 1 \otimes \varepsilon \otimes 1) \\ &= \natural\,\tilde{\partial} m_r((\varepsilon \otimes 1 \otimes \varepsilon) \otimes 1) \\ &= \natural m_r(\tilde{\partial} \otimes 1)\,((\varepsilon \otimes 1 \otimes \varepsilon) \otimes 1) \\ &= \natural m_r(\partial \otimes 1) \end{aligned}$$

which use respectively the right identity property of ε, the definition of α, the left identity property of ε, the definition of right multiplication for $R^{\otimes 3}$, the fact that $\tilde{\partial}$ is a right R-module map, and the relation of ∂ and $\tilde{\partial}$.

Lemma **3.7.** — *One has* $\beta\alpha = m\sigma - m$, *that is*, $\beta\alpha(x \otimes y) = yx - xy = -[x, y]$.

This follows from the computation

$$\begin{aligned} \beta\alpha(x \otimes y) = \beta\natural(\partial xy) = m\sigma I(\partial xy) = m\sigma\{I(\partial x)y\} = m\sigma\{(x \otimes 1_R - 1_R \otimes x)\,y\} \\ = m\sigma(x \otimes y - 1_R \otimes xy) = m(y \otimes x - xy \otimes 1_R) = yx - xy. \end{aligned}$$

We define $\bar{\partial} : R \to \Omega^1_{R,\natural}$ to be the composition $\natural\,\partial$.

Proposition **3.8.** — *One has* $\beta\,\bar{\partial} = \bar{\partial}\beta = 0$, *hence a complex of period two*

$$\xrightarrow{\bar{\partial}} \Omega^1_{R,\natural} \xrightarrow{\beta} R \xrightarrow{\bar{\partial}} \Omega^1_{R,\natural} \xrightarrow{\beta}$$

Proof. — $\beta\,\bar{\partial}x = \beta\natural(\partial x 1_R) = \beta\alpha(x \otimes 1_R = -[x, 1_R] = 0$. Also

$$\partial\beta\alpha(x \otimes y) = \partial(yx - xy) = [\partial y, x] + [y, \partial x]$$

is killed by $\natural$, so $\bar{\partial}\beta\alpha = 0$. However α is surjective, since the maps $\natural$, $\tilde{\partial}$ are surjective, so we see that $\bar{\partial}\beta = 0$.

Remark **3.9.** — It can be shown that the chain complex of length one

$$\Omega^1_R \xrightarrow{\beta} R$$

is the quotient of the Hochschild complex having the same homology in degrees 0 and 1. By Hochschild complex we mean the b-complex in the cyclic bicomplex. Furthermore $\tilde{\partial}$ agrees with the map induced by the B operator of Connes on the Hochschild complex. Thus the periodic complex above can be viewed as a first order approximation to the cyclic theory for the algebra R in the following way. Consider the b, B bicomplex of Connes. The zero-th order approximation is the bottom edge consisting of copies of

R/[R, R] viewed as a quotient of the Hochschild complex. The first order approximation is the strip of height one along the bottom edge, and this is the periodic complex. This approximation is exact when the Hochschild homology vanishes in degrees $\geqslant 2$.

Example **3.10.** — In the case where R is the tensor algebra $T(V) = \bigoplus_{n\geqslant 0} V^{\otimes n}$, we have a canonical isomorphism of Ω^1_R with the free bimodule $R \otimes V \otimes R$. One way to see this is to explicitly check that any linear map from V to an R-bimodule extends uniquely to a derivation, and to use 3.3. A closely related method is to use 3.4 as follows. We consider the linear map $\partial : R \to R \otimes V \otimes R$ given by

$$\partial(v_1, \ldots, v_n) = \sum_{i=1}^{n} (v_1, \ldots, v_{i-1}) \otimes v_i \otimes (v_{i+1}, \ldots, v_n)$$

and the bimodule morphism $I = (m \otimes 1 - 1 \otimes m)\,(1 \otimes \iota \otimes 1)$:

$$I\{(v_1, \ldots, v_{p-1}) \otimes v_p \otimes (v_{p+1}, \ldots, v_n)\} = (v_1, \ldots, v_p) \otimes (v_{p+1}, \ldots, v_n) - (v_1, \ldots, v_{p-1}) \otimes (v_p, \ldots, v_n),$$

where $\iota : V \to R$ is the inclusion. One can verify that ∂ is a derivation, that $\widetilde{\partial}$ is surjective because it has the section $1 \otimes \iota \otimes 1$, and that $I\,\widetilde{\partial} = 1 \otimes \varepsilon - \varepsilon \otimes 1$. Thus by 3.4 we have an isomorphism of Ω^1_R with $R \otimes V \otimes R$ such that the canonical maps are given by the above formulas.

Combining this isomorphism with 3.5, we can identify $\Omega^1_{R,\natural}$ with $V \otimes R$ in such a way that the canonical maps $\bar{\partial}$, β are given by the formulas

$$\bar{\partial}(v_1, \ldots, v_n) = \sum_{i=1}^{n} v_i \otimes (v_{i+1}, \ldots, v_n, v_1, \ldots, v_i),$$

$$\beta\{v_1 \otimes (v_2, \ldots, v_n)\} = -[v_1, (v_2, \ldots, v_n)] = (v_2, \ldots, v_n, v_1) - (v_1, \ldots, v_n).$$

These maps are essentially $\Sigma_1^n \sigma^i$ and $\sigma^{-1} - 1$ on $V^{\otimes n}$, so the periodic complex is exact provided one replaces R by $\bar{R} = R/k$. (See [K, § 3] for a closely related discussion.)

3.2. DG algebras

Let $R = \{ \to R^n \to R^{n+1} \to \}$ be a DG algebra. As remarked above the various constructions we have given for algebras extend immediately to DG algebras. In this case, because the product and unit maps are morphisms of complexes, the bimodule of differentials Ω^1_R and its commutator quotient $\Omega^1_{R,\natural}$ are complexes in a natural way, and the various canonical maps between them are morphisms of complexes. Thus the periodic complex 3.8 for R is a double complex.

There are some additional signs due to the fact that the permutation isomorphism σ for complexes involves signs when odd degree elements are interchanged. Thus we have

$$\beta\natural(\partial x y) = (m\sigma - m)\,(x \otimes y) = (-1)^{|x||y|} yx - xy = -[x, y].$$

The following theorem is a simple illustration of the method we will use later to prove the S-relations among the family of cyclic cocycles obtained by connection-curvature methods as in § 2.

Let $\rho \in R^1$, and let $\omega = \delta\rho + \rho^2 \in R^2$ be its " curvature ", where δ is the differential in R. We also use δ to denote the differential in the associated complexes Ω^1_R, $\Omega^1_{R,\natural}$.

Theorem 3. — *The elements* $\omega^n \in R^{2n}$, $\natural(\partial\rho\omega^n) \in (\Omega^1_{R,\natural})^{2n+1}$ *for* $n \geq 0$ *satisfy the relations*

$$\delta(\omega^n/n!) = \beta\{\natural(\partial\rho\omega^n/n!)\}, \qquad \delta\{\natural(\partial\rho\omega^n/n!)\} = \bar{\partial}(\omega^{n+1}/(n+1)!).$$

Proof. — We have $\delta(\omega^n) = -[\rho, \omega^n] = \beta\{\natural(\partial\rho\omega^n)\}$, proving the first formula. Because $\alpha : (x \otimes y) \mapsto \natural(\partial xy)$ is a morphism of complexes, one has

$$\begin{aligned}\delta\{\natural(\partial\rho\omega^n)\} &= \natural\{\partial(\delta\rho)\,\omega^n - \partial\rho\,\delta(\omega^n)\} = \natural\{\partial(\delta\rho)\,\omega^n + \partial\rho(\rho\omega^n - \omega^n\rho)\} \\ &= \natural\{(\partial\,\delta\rho + \partial\rho\rho + \rho\,\partial\rho)\,\omega^n\} = \natural\{\partial\omega\omega^n\}.\end{aligned}$$

On the other hand

$$\bar{\partial}(\omega^{n+1}) = \natural\,\partial(\omega^{n+1}) = \natural\{\sum_0^n \omega^i\,\partial\omega\omega^{n-i}\} = (n+1)\,\natural(\partial\omega\omega^n),$$

proving the second formula.

4. Differentials over coalgebras

In this section we describe the analogue for coalgebras of the bimodule of differentials and the periodic complex discussed for algebras in the previous section. The theory is formally dual to that for algebras in the sense that the arrows are reversed. We study carefully the case of a tensor coalgebra, and derive formulas for the canonical maps, since these will be needed for the bar construction.

4.1. General properties of Ω^C

Let C be a coalgebra with coproduct $\Delta : C \to C \otimes C$ and counit $\eta : C \to k$. By a bicomodule M over C we mean a vector space equipped with left and right coproducts $\Delta_l : M \to C \otimes M$, $\Delta_r : M \to M \otimes C$ defining left and right comodule structures which commute: $(\Delta_l \otimes 1)\,\Delta_r = (1 \otimes \Delta_r)\,\Delta_l$. Its cocommutator subspace is

$$M^\natural \stackrel{\text{def}}{=} \operatorname{Ker}\{\Delta_l - \sigma\,\Delta_r : M \to C \otimes M\}$$

and we let $\natural : M^\natural \to M$ denote the inclusion map.

A free bicomodule is one of the form $C \otimes V \otimes C$, with $\Delta_l = \Delta \otimes 1 \otimes 1$, and $\Delta_r = 1 \otimes 1 \otimes \Delta$. The following two propositions are dual to 3.1 and 3.5.

Proposition 4.1. — *There is a one-one correspondence between linear maps* $h : M \to V$ *and bicomodule morphisms* $\tilde{h} : M \to C \otimes V \otimes C$ *given by*

$$\tilde{h} = (1 \otimes h \otimes 1)\,(\Delta_l \otimes 1)\,\Delta_r, \qquad h = (\eta \otimes 1 \otimes \eta)\,\tilde{h}.$$

Proposition **4.2.** — *There is a canonical isomorphism* $V \otimes C \simeq (C \otimes V \otimes C)^{\natural}$ *relative to which the canonical injection* $\natural$ *becomes*

$$\natural = \sigma(1 \otimes \Delta) : V \otimes C \to C \otimes V \otimes C.$$

The dual of the b'-sequence

$$0 \to C \xrightarrow{\Delta} C^{\otimes 2} \xrightarrow{\Delta \otimes 1 - 1 \otimes \Delta} C^{\otimes 3} \xrightarrow{\square} C^{\otimes 4} \to,$$

where $\square = \Delta \otimes 1 \otimes 1 - 1 \otimes \Delta \otimes 1 + 1 \otimes 1 \otimes \Delta$, is a sequence of bicomodule morphisms, which is exact because C has a counit. We let $\Omega^{C,1}$, or simply Ω^C, denote the bicomodule Coker Δ. One has exact sequences

$$0 \to C \xrightarrow{\Delta} C^{\otimes 2} \xrightarrow{I} \Omega^C \to 0,$$

$$0 \to \Omega^C \xrightarrow{J} C^{\otimes 3} \xrightarrow{\square} C^{\otimes 4},$$

where I, J are bicomodule maps such that $JI = \Delta \otimes 1 - 1 \otimes \Delta$.

By 4.1 we have $J = \widetilde{\partial}$, where $\partial = (\eta \otimes 1 \otimes \eta) J : \Omega^C \to C$. The dual of 3.3 shows that ∂ is a universal coderivation, where a coderivation $D : M \to C$, with M a bicomodule, is a linear map satisfying $\Delta D = (1 \otimes D) \Delta_l + (D \otimes 1) \Delta_r$.

We now apply the cocommutator subspace functor to Ω^C and the canonical bicomodule maps I, $\widetilde{\partial}$ using 4.2 to identify the cocommutator subspaces for free bicomodules. This gives the commutative diagram

$$\begin{array}{ccccc} C^{\otimes 2} & \xrightarrow{I} & \Omega^C & \xrightarrow{\widetilde{\partial}} & C^{\otimes 3} \\ \sigma\Delta \uparrow & & \uparrow \natural & & \uparrow \sigma(1 \otimes \Delta) \\ C & \xrightarrow{\beta} & \Omega^{C,\natural} & \xrightarrow{\alpha} & C^{\otimes 2} \end{array}$$

where the vertical arrows are the inclusions $\natural$ up to the identification 4.2, and α, β are the unique maps such that the diagram is commutative. Dualizing 3.6-3.8 we obtain the formulas

$$\alpha = (\partial \otimes 1) \Delta_r \natural, \qquad \alpha\beta = \sigma \Delta - \Delta$$

and the complex of period two

$$\xrightarrow{\bar{\partial}} C \xrightarrow{\beta} \Omega^{C,\natural} \xrightarrow{\bar{\partial}} C \xrightarrow{\beta},$$

where $\bar{\partial}$ is the composition $\partial\natural$.

4.2. Ω^C for a free coalgebra

Let C now be the tensor coalgebra $T(V) = \bigoplus_{n \geqslant 0} V^{\otimes n}$ with

$$\Delta(v_1, \ldots, v_n) = \sum_{0 \leqslant i \leqslant n} (v_1, \ldots, v_i) \otimes (v_{i+1}, \ldots, v_n)$$

and η equal the projection onto $V^{\otimes 0} = k$. We consider the free bicomodule $C \otimes V \otimes C$ and let $\partial : C \otimes V \otimes C \to C$ be the linear map given by

(4.3) $$\partial\xi = (v_1, \ldots, v_n) \quad \text{if } \xi = (v_1, \ldots, v_{p-1}) \otimes v_p \otimes (v_{p+1}, \ldots, v_n).$$

We claim ∂ is a coderivation. In effect, one has

$$(1\otimes\partial)\,\Delta_l\,\xi = (1\otimes\partial)\sum_{0\leqslant i<p}(v_1,\ldots,v_i)\otimes(v_{i+1},\ldots,v_{p-1})\otimes v_p\otimes(v_{p+1},\ldots,v_n)$$
$$= \sum_{0\leqslant i<p}(v_1,\ldots,v_i)\otimes(v_{i+1},\ldots,v_n).$$

Similarly

$$(\partial\otimes 1)\,\Delta_r\,\xi = \sum_{p\leqslant i\leqslant n}(v_1,\ldots,v_i)\otimes(v_{i+1},\ldots,v_n)$$

and these add up to $\Delta\,\partial\xi$.

The bicomodule morphism extending ∂ is

$$\widetilde{\partial} = (1\otimes\partial\otimes 1)\,(\Delta\otimes 1\otimes\Delta) : \mathrm{C}\otimes\mathrm{V}\otimes\mathrm{C}\to\mathrm{C}^{\otimes 3}$$

and is given by the formula

$$\textbf{(4.4)}\qquad \widetilde{\partial}\xi = \sum_{\substack{0\leqslant j<p\\ p\leqslant k\leqslant n}}(v_1,\ldots,v_j)\otimes(v_{j+1},\ldots,v_k)\otimes(v_{k+1},\ldots,v_n)$$

with ξ as above. It is injective, because it has the left inverse $1\otimes\pi\otimes 1$ where $\pi:\mathrm{C}\to\mathrm{V}$ is the projection onto the tensors of degree one.

Let I be the bicomodule morphism

$$\mathrm{I} = (1\otimes\pi\otimes 1)\,(\Delta\otimes 1 - 1\otimes\Delta) : \mathrm{C}\otimes\mathrm{C}\to\mathrm{C}\otimes\mathrm{V}\otimes\mathrm{C}.$$

We have

$$\textbf{(4.5)}\qquad \mathrm{I}\{(v_1,\ldots,v_p)\otimes(v_{p+1},\ldots,v_n)\} = (v_1,\ldots,v_{p-1})\otimes v_p\otimes(v_{p+1},\ldots,v_n) - (v_1,\ldots,v_p)\otimes v_{p+1}\otimes(v_{p+2},\ldots,v_n)$$

where the symbol $(v_i,\ldots,v_j) = v_i\otimes\ldots\otimes v_j$ is to be interpreted as zero when $i > j+1$. Hence we have

$$\partial\mathrm{I}\{(v_1,\ldots,v_p)\otimes(v_{p+1},\ldots,v_n)\} = \begin{cases}(v_1,\ldots,v_n) & 0<p=n\\ -(v_1,\ldots,v_n) & 0=p<n\\ 0 & 0<p<n \text{ or } 0=p=n\end{cases}$$

showing that $\partial\mathrm{I} = 1\otimes\eta - \eta\otimes 1$.

Summarizing, we have shown that the bicomodule $\mathrm{C}\otimes\mathrm{V}\otimes\mathrm{C}$ together with the maps ∂, I satisfy the conditions dual to those of Proposition 3.4, namely, ∂ is a coderivation such that $\widetilde{\partial}$ is injective, and I is a bimodule morphism such that $\partial\mathrm{I} = 1\otimes\eta - \eta\otimes 1$. Thus from the dual of this proposition we obtain the following.

Proposition **4.6.** — *There is a canonical isomorphism of* $\Omega^{\mathrm{C}}_{\mathrm{d}}$ *with the free bicomodule* $\mathrm{C}\otimes\mathrm{V}\otimes\mathrm{C}$ *such that the canonical maps* ∂, $\widetilde{\partial}$, I *are given by the formulas* 4.3-4.5.

Combining this isomorphism with 4.2 we obtain an induced isomorphism of $\Omega^{C,\natural}$ with $V \otimes C$ such that

$$\begin{array}{ccc} \Omega^{C} & \simeq & C \otimes V \otimes C \\ \natural \uparrow & & \uparrow \sigma(1 \otimes \Delta) \\ \Omega^{C,\natural} & \simeq & V \otimes C \end{array}$$

commutes.

Proposition **4.7.** — *With respect to this isomorphism* $\Omega^{C,\natural} \simeq V \otimes C$, *the inclusion* $\natural : \Omega^{C,\natural} \to \Omega^{C}$ *and the maps* $\bar\partial$, β *of the periodic complex are given by the formulas*

$$\natural\{v_1 \otimes (v_2, \ldots, v_n)\} = \sum_{1 \leqslant i \leqslant n} (v_{i+1}, \ldots, v_n) \otimes v_1 \otimes (v_2, \ldots, v_i),$$

$$\bar\partial\{v_1 \otimes (v_2, \ldots, v_n)\} = \sum_{1 \leqslant i \leqslant n} (v_{i+1}, \ldots, v_n, v_1, \ldots, v_i),$$

$$\beta\{v_1 \otimes (v_2, \ldots, v_n)\} = v_n \otimes (v_1, \ldots, v_{n-1}) - v_1 \otimes (v_2, \ldots, v_n).$$

Proof. — The first formula results from

$$\begin{aligned} \sigma(1 \otimes \Delta)\{v_1 \otimes (v_2, \ldots, v_n)\} &= \sigma \sum_{1 \leqslant i \leqslant n} v_1 \otimes (v_2, \ldots, v_i) \otimes (v_{i+1}, \ldots, v_n) \\ &= \sum_{1 \leqslant i \leqslant n} (v_{i+1}, \ldots, v_n) \otimes v_1 \otimes (v_2, \ldots, v_i) \end{aligned}$$

and the fact that $\natural = \sigma(1 \otimes \Delta)$ with respect to our identifications. The formula for $\bar\partial = \partial\natural$ follows by combining the formulas for ∂ and $\natural$. Finally we have

$$\begin{aligned} \natural\beta(v_1, \ldots, v_n) &= I\sigma\,\Delta(v_1, \ldots, v_n) \\ &= \sum_{1 \leqslant i \leqslant n} I\{(v_{i+1}, \ldots, v_n) \otimes (v_1, \ldots, v_i)\} \\ &= \sum_{0 \leqslant i < n} (v_{i+1}, \ldots, v_{n-1}) \otimes v_n \otimes (v_1, \ldots, v_i) \\ &\qquad - \sum_{0 < i \leqslant n} (v_{i+1}, \ldots, v_n) \otimes v_1 \otimes (v_2, \ldots, v_i) \\ &= \natural\{v_n \otimes (v_1, \ldots, v_{n-1}) - v_1 \otimes (v_2, \ldots, v_n)\} \end{aligned}$$

which proves the formula for β and completes the proof.

We leave to the interested reader to check the following formula for α

$$\alpha\{v_1 \otimes (v_2, \ldots, v_n)\} = \sum_{1 \leqslant j \leqslant k \leqslant n} (v_{k+1}, \ldots, v_n, v_1, \ldots, v_j) \otimes (v_{j+1}, \ldots, v_k).$$

5. Applications to the bar construction

In this section we return to the bar construction B of the augmented algebra $\tilde{A}$. Associated to its coalgebra structure are the bicomodule Ω^{B} and its cocommutator subspace $\Omega^{B,\natural}$, and these are complexes with differentials induced by the differential b' of the bar construction. From these complexes we obtain two new kinds of cochains related to the cochains and cyclic cochains which were discussed in § 1.

We begin by discussing the general behavior of these complexes and cochains. Using the fact that the bar construction is a tensor coalgebra, we identify these complexes and the various canonical maps between them. We show that up to a dimension shift $\Omega^{B,\natural}$ is the Hochschild complex, that is, the b-complex in the cyclic bicomplex. We also show that the cyclic bicomplex is essentially the periodic complex for coalgebras, when the coalgebra is the bar construction. Finally we derive various cochain formulas.

5.1. General discussion

We recall from the previous section that Ω^B is defined to be the cokernel of the coproduct map Δ for the bar construction, and that

$$I : B \otimes B \to \Omega^B, \qquad \partial : \Omega^B \to B$$

are respectively the canonical surjection onto this cokernel and the linear map such that $\partial I = 1 \otimes \eta - \eta \otimes 1$. On Ω^B there is a bicomodule structure Δ_l, Δ_r such that I is a bicomodule morphism, and such that ∂ is a coderivation. We have a commutative diagram

$$\begin{array}{ccccc} B^{\otimes 2} & \xrightarrow{I} & \Omega^B & \xrightarrow{\tilde\partial} & B^{\otimes 3} \\ {\scriptstyle \sigma\Delta}\uparrow & & \uparrow{\scriptstyle \natural} & & \uparrow{\scriptstyle \sigma(1\otimes\Delta)} \\ B & \xrightarrow{\beta} & \Omega^{B,\natural} & \xrightarrow{\alpha} & B^{\otimes 2} \end{array}$$

where $\tilde\partial = (1 \otimes \partial \otimes 1)(\Delta_l \otimes 1)\Delta_r$ is the bicomodule morphism extending ∂, and where $\alpha = (\partial \otimes 1)\Delta_r \natural$, $\alpha\beta = \sigma\Delta - \Delta$. We also have the periodic complex

$$\xrightarrow{\bar\partial} B \xrightarrow{\beta} \Omega^{B,\natural} \xrightarrow{\bar\partial} B \xrightarrow{\beta},$$

where $\bar\partial = \partial\natural$.

Moreover because B is a DG coalgebra, Ω^B and $\Omega^{B,\natural}$ are naturally complexes with differential induced from the differential in B, and the above canonical maps are morphisms of complexes.

We next turn to cochains. We have four complexes in play: B, $B^\natural$, Ω^B, $\Omega^{B,\natural}$, hence four kinds of cochains when we consider linear maps from one of them to another complex V. It is convenient to introduce some terminology to distinguish these cochains. We will usually refer to elements of Hom(B, V) simply as *cochains*, but as *bar cochains* if we want to be precise. Elements of Hom($B^\natural$, V) will be called *cyclic cochains*, since $\natural B$ is essentially the cyclic complex as we saw in § 1. We call elements of Hom($\Omega^{B,\natural}$, V) *Hochschild cochains*, since we will show below that $\Omega^{B,\natural}$ is essentially the b-complex which gives the Hochschild homology when the algebra is unital. Finally elements of Hom(Ω^B, V) will be called Ω-*cochains*.

Each of these Hom-complexes is bigraded, and a homogeneous element ξ has three degrees, a V-degree, an A-degree which is its degree as a multilinear functional on A, and a total degree $|\xi|$ which is their sum. The total differential is the sum of two partial differentials d and δ defined as in § 2.1.

We now describe various operations on cochains induced by the canonical maps associated to Ω^{B}.

Let L be a DG algebra. The bicomodule structure on Ω^{B} over B gives rise to a bimodule structure on $\mathrm{Hom}(\Omega^{B}, L)$ over the algebra $\mathrm{Hom}(\Omega^{B}, L)$. If f, γ are bar and Ω-cochains respectively, then left multiplication is given by $f\gamma = m(f\otimes\gamma)\,\Delta_{l}$, where m is the product in L, and similarly for right multiplication. Because Δ_{l}, m are maps of complexes one has

$$(5.1)\qquad \begin{aligned} \delta(f\gamma) &= \delta f\gamma + (-1)^{|f|} f\,\delta\gamma \\ d(f\gamma) &= df\gamma + (-1)^{|f|} f\,d\gamma \end{aligned}$$

and similar derivation formulas hold for right multiplication.

The canonical map ∂ induces a map $\partial(f) = f\partial$ from bar cochains to Ω-cochains which is compatible with δ and d. As ∂ is a coderivation, the induced map is a derivation:

$$\begin{aligned} \partial(fg) &= fg\,\partial = m(f\otimes g)\,\Delta\,\partial \\ &= m(f\otimes g)\,((\partial\otimes 1)\,\Delta_{r} + (1\otimes\partial)\,\Delta_{l}) \\ &= m(f\,\partial\otimes g)\,\Delta_{r} + m(f\otimes g\,\partial)\,\Delta_{l} \\ &= \partial(f)\,g + f\,\partial(g). \end{aligned}$$

The effect of $\tilde{\partial}$ on cochains combines this derivation with multiplication:

$$(5.2)\qquad \begin{aligned} f\,\partial gh &= m'(f\otimes\partial g\otimes h)\,(\Delta_{l}\otimes 1)\,\Delta_{r} \\ &= m'(f\otimes g\otimes h)\,(1\otimes\partial\otimes 1)\,(\Delta_{l}\otimes 1)\,\Delta_{r} \\ &= m'(f\otimes g\otimes h)\,\tilde{\partial}, \end{aligned}$$

where $m' = m(m\otimes 1)$ is the triple product map for L.

The canonical injection $\natural : \Omega^{B,\natural} \to \Omega^{B}$ induces a map from bar cochains to Hochschild cochains. If we combine it with a trace $\tau : L \to V$, we obtain a map

$$\tau^{\natural} : \mathrm{Hom}(\Omega^{B}, L) \to \mathrm{Hom}(\Omega^{B,\natural}, V), \qquad \tau^{\natural}(\gamma) = \tau\gamma\natural,$$

which is compatible with δ, and also compatible with d when τ is a closed trace. This map is a trace on the bimodule of Ω-cochains:

$$(5.3)\qquad \begin{aligned} \tau^{\natural}(f\gamma) &= \tau m(f\otimes\gamma)\,\Delta_{l}\,\natural \\ &= \tau m\sigma(f\otimes\gamma)\,\sigma\,\Delta_{r}\,\natural \\ &= (-1)^{|f|\,|\gamma|}\,\tau m(\gamma\otimes f)\,\Delta_{r}\,\natural \\ &= (-1)^{|f|\,|\gamma|}\,\tau^{\natural}(\gamma f). \end{aligned}$$

The effect of α is to associate to a pair of bar cochains f, g the Hochschild cochain $(\partial fg)\,\natural$. The maps $\bar{\partial}$, β of the periodic complex induce maps $\bar{\partial}(f) = f\,\partial\natural$,

$\beta(\gamma) = \gamma\beta$ from bar cochains to Hochschild cochains and the other way round. All of these maps are compatible with δ and d. We have

$$\bar{\partial}\{\tau(f)\} = \tau f\,\partial\natural = \tau^{\natural}(\partial f). \tag{5.4}$$

Using $\alpha\beta = \sigma\,\Delta - \Delta$ we deduce

$$\begin{aligned}\beta\{\tau^{\natural}(\partial fg)\} &= \tau m(f\partial\otimes g)\,\Delta_r\,\natural\beta = \tau m(f\otimes g)\,(\partial\otimes 1)\,\Delta_r\,\natural\beta = \tau m(f\otimes g)\,\alpha\beta\\ &= \tau m(f\otimes g)\,(\sigma\,\Delta - \Delta) = (-1)^{|f||g|}\,\tau m\sigma(g\otimes f)\,\Delta - \tau(fg)\\ &= -\,\tau([f,g]).\end{aligned} \tag{5.5}$$

5.2. Identification of the complexes Ω^{B}, $\Omega^{B,\natural}$

Because $B = T(A[1])$ is a free coalgebra, we know from § 4.2 that Ω^{B} can be identified with the free bicomodule $B\otimes A[1]\otimes B$ in such a way that the canonical morphisms ∂, $\widetilde{\partial}$, I are given by the formulas

$$\partial\{(a_1,\dots,a_{p-1})\otimes a_p\otimes(a_{p+1},\dots,a_n)\} = (a_1,\dots,a_n), \tag{5.6}$$

$$\widetilde{\partial}\{(a_1,\dots,a_{p-1})\otimes a_p\otimes(a_{p+1},\dots,a_n)\} = \sum_{\substack{0\leqslant j<p\\ p\leqslant k\leqslant n}} (a_1,\dots,a_j)\otimes(a_{j+1},\dots,a_k)\otimes(a_{k+1},\dots,a_n), \tag{5.7}$$

$$\begin{aligned}I\{(a_1,\dots,a_p)\otimes(a_{p+1},\dots,a_n)\} &= (a_1,\dots,a_{p-1})\otimes a_p\otimes(a_{p+1},\dots,a_n)\\ &\quad - (a_1,\dots,a_p)\otimes a_{p+1}\otimes(a_{p+2},\dots,a_n).\end{aligned} \tag{5.8}$$

We also know that $\Omega^{B,\natural}$ can be identified with $A[1]\otimes B$ in such a way that the canonical morphisms $\natural:\Omega^{B,\natural}\to\Omega^{B}$, $\bar{\partial}:\Omega^{B,\natural}\to B$, $\beta: B\to\Omega^{B,\natural}$ are given by the formulas

$$\natural\{a_1\otimes(a_2,\dots,a_n)\} = \sum_{1\leqslant i\leqslant n}(-1)^{i(n-1)}(a_{i+1},\dots,a_n)\otimes a_1\otimes(a_2,\dots,a_i), \tag{5.9}$$

$$\bar{\partial}\{a_1\otimes(a_2,\dots,a_n)\} = \sum_{1\leqslant i\leqslant n}(-1)^{i(n-1)}(a_{i+1},\dots,a_n,a_1,\dots,a_i), \tag{5.10}$$

$$\beta(a_1,\dots,a_n) = (-1)^{n-1}a_n\otimes(a_1,\dots,a_{n-1}) - a_1\otimes(a_2,\dots,a_n). \tag{5.11}$$

The new signs not appearing in § 4.2 are due to the fact that the permutations isomorphism σ for complexes involves signs when odd elements are moved past each other, and the elements of $A[1]$ are of odd degree.

The following propositions give formulas for the differentials in Ω^{B}, $\Omega^{B,\natural}$ induced by the differential b' of the bar construction.

Let b'' denote the induced differential in Ω^{B}.

Proposition **5.12.** — *With respect to the identification* $\Omega^{B} = B\otimes A[1]\otimes B$, *we have*

$$\begin{aligned}b''\{(a_1,\dots,a_{p-1})\otimes a_p\otimes(a_{p+1},\dots,a_n)\}\\ = b'(a_1,\dots,a_{p-1})\otimes a_p\otimes(a_{p+1},\dots,a_n)\\ + (-1)^{p-2}(a_1,\dots,a_{p-2})\otimes a_{p-1}a_p\otimes(a_{p+1},\dots,a_n)\\ + (-1)^{p-1}(a_1,\dots,a_{p-1})\otimes a_p a_{p+1}\otimes(a_{p+2},\dots,a_n)\\ + (-1)^{p}(a_1,\dots,a_{p-1})\otimes a_p\otimes b'(a_{p+1},\dots,a_n).\end{aligned}$$

Proof. — The differential b'' on $B \otimes A[1] \otimes B$ corresponding to the differential on Ω^B has the property that the map $\tilde{\partial}$ given by 5.7 is a morphism of complexes. Thus if $\xi = (a_1, \ldots, a_{p-1}) \otimes a_p \otimes (a_{p+1}, \ldots, a_n)$, we have

$$\begin{aligned}\tilde{\partial} b'' \xi = \sum_{\substack{0 \leqslant j < p \\ p \leqslant k \leqslant n}} \{ & b'(a_1, \ldots, a_j) \otimes (a_{j+1}, \ldots, a_k) \otimes (a_{k+1}, \ldots, a_n) \\ & + (-1)^j (a_1, \ldots, a_j) \otimes b'(a_{j+1}, \ldots, a_k) \otimes (a_{k+1}, \ldots, a_n) \\ & + (-1)^k (a_1, \ldots, a_j) \otimes (a_{j+1}, \ldots, a_k) \otimes b'(a_{k+1}, \ldots, a_n) \}.\end{aligned}$$

To find $b'' \xi$, we apply the left inverse to I given by $1 \otimes \pi \otimes 1 : B^{\otimes 3} \to B \otimes A[1] \otimes B$, where π is the projection of B onto A[1]. The only terms in the sum on the right contributing to $b'' \xi$ are when $j = p - 1$, $k = p$ in the upper term, when $j = p - 2$, $k = p$ and $j = p - 1$, $k = p + 1$ in the middle term, and when $j = p - 1$, $k = p$ in the lower term. These give the four terms in the formula for b''.

Let $\tilde{b}$ denote the induced differential in $\Omega^{B,\natural}$.

Proposition **5.13.** — *With respect to the identification* $A[1] \otimes B = \Omega^{B,\natural}$, *we have*

$$\begin{aligned}\tilde{b}\{ a_1 \otimes (a_2, \ldots, a_n)\} = & (-1)^n a_n a_0 \otimes (a_1, \ldots, a_{n-1}) \\ & + a_0 a_1 \otimes (a_2, \ldots, a_n) - a_0 \otimes b'(a_1, \ldots, a_n).\end{aligned}$$

Proof. — Since the map $\natural$ given by 5.9 is a morphism of complexes, we have, on setting $\xi = a_1 \otimes (a_2, \ldots, a_n)$,

$$\begin{aligned}\natural\tilde{b}\xi = b''\natural\xi = \sum_{0 \leqslant i \leqslant n} (-1)^{i(n-1)} \{ & b'(a_{i+1}, \ldots, a_n) \otimes a_1 \otimes (a_1, \ldots, a_i) \\ & + (-1)^{n-i-1} (a_{i+1}, \ldots, a_{n-1}) \otimes a_n a_1 \otimes (a_1, \ldots, a_i) \\ & + (-1)^{n-i} (a_{i+1}, \ldots, a_n) \otimes a_1 a_2 \otimes (a_2, \ldots, a_i) \\ & + (-1)^{n-i+1} (a_{i+1}, \ldots, a_n) \otimes a_1 \otimes b'(a_2, \ldots, a_i) \}.\end{aligned}$$

We next apply the left inverse $\eta \otimes 1 \otimes 1 : B \otimes A[1] \otimes B \to A[1] \otimes B$ to $\natural$. There are contributions only when the first factor in the triple tensor product is of degree zero. Of the four terms in the sum on the right, the first gives nothing since b' is zero in degree one, and the other three contribute the three terms in the formula for $\tilde{b}$, concluding the proof.

If we now make the identification $a_1 \otimes (a_2, \ldots, a_n) = (a_1, \ldots, a_n)$, then the differential $\tilde{b}$ becomes the Hochschild differential b in the cyclic bicomplex of § 1. Also from 5.10 and 5.11 we see that $\bar{\partial}$ and β become the operators N and T $-$ 1 of the cyclic bicomplex. Thus we have proved the following.

Theorem **4.** — *The complex* $\Omega^{B,\natural}$ *is canonically isomorphic to the b-complex in the cyclic bicomplex with degrees shifted by one. Relative to this isomorphism one has* $\beta = T - 1$, $\bar{\partial} = N$.

Remark **5.14.** — This result shows that the cyclic bicomplex can essentially be identified with the periodic sequence of complexes

$$\xrightarrow{\bar\partial} \overline{\mathrm{B}} \xrightarrow{-\beta} \Omega^{\mathrm{B},\natural} \xrightarrow{\bar\partial} \overline{\mathrm{B}} \xrightarrow{-\beta},$$

where $\overline{\mathrm{B}} = \mathrm{Ker}\{\eta : \mathrm{B} \to k\}$.

5.3. Cochain formulas

Let us call an Ω-cochain of A-degree n simply an Ω-n-cochain. It is a linear map γ defined on

$$\Omega^{\mathrm{B}}_n = (\mathrm{B} \otimes \mathrm{A}[1] \otimes \mathrm{B})_n = \bigoplus_{1 \leqslant i \leqslant n} \mathrm{A}^{\otimes(i-1)} \otimes \mathrm{A} \otimes \mathrm{A}^{\otimes(n-i)}$$

and hence it can be viewed as a family of multilinear maps $\gamma(a_1, \ldots, a_{i-1} \mid a_i \mid a_{i+1}, \ldots, a_n)$ for $1 \leqslant i \leqslant n$.

If f is a bar n-cochain, then $\partial f = f\partial$ is an Ω-n-cochain. Using 5.6, we see it is given by

$$\partial f(a_1, \ldots, a_{i-1} \mid a_i \mid a_{i+1}, \ldots, a_n) = f(a_1, \ldots, a_n). \tag{5.15}$$

Let f, g, h be cochains of A-degrees p, q, r, respectively, with values in L. Using 5.2, 5.7, one has, with $n = p + q + r$,

$$\begin{aligned}(h\,\partial fg)\,&(a_1, \ldots, a_{i-1} \mid a_i \mid a_{i+1}, \ldots, a_n)\\ &= m'(h \otimes f \otimes g)\widetilde{\omega}\{(a_1, \ldots, a_{i-1}) \otimes a_i \otimes a_{i+1}, \ldots, a_n\}\\ &= \sum_{\substack{1 \leqslant j < i\\ i \leqslant k \leqslant n}} m'(h \otimes f \otimes g)\{(a_1, \ldots, a_j) \otimes (a_{j+1}, \ldots, a_k) \otimes (a_{k+1}, \ldots, a_n)\}.\end{aligned}$$

All the terms in this sum are zero except when $j = r$, $k = r + p$, in which case we have $r < i \leqslant r + p$, and the term is

$$m(h \otimes f \otimes g)\{(a_1, \ldots, a_r) \otimes (a_{r+1}, \ldots, a_{r+p}) \otimes (a_{r+p+1}, \ldots, a_n)\} = (hfg)\,(a_1, \ldots, a_n),$$

where hfg is the product in Hom(B, L):

$$\begin{aligned}(hfg)\,&(a_1, \ldots, a_n)\\ &= (-1)^{|g|(r+p)+|f|r}\, h(a_1, \ldots, a_r) f(a_{r+1}, \ldots, a_{r+p})\, g(a_{r+p+1}, \ldots, a_n).\end{aligned}$$

Thus we have

$$(h\,\partial fg)\,(a_1, \ldots, a_{i-1} \mid a_i \mid a_{i+1}, \ldots, a_n) = \begin{cases}(hfg)\,(a_1, \ldots, a_n) & \text{if } r+1 \leqslant i \leqslant r+p,\\ 0 & \text{otherwise.}\end{cases} \tag{5.16}$$

Given an Ω-n-cochain γ, we compose it with the map $\natural : \Omega^{\mathrm{B},\natural} \to \Omega^{\mathrm{B}}$ and obtain a Hochschild cochain of A-degree n. From 5.9 we have the formula

$$\gamma\natural(a_1, \ldots, a_n) = \sum_{1 \leqslant i \leqslant n} (-1)^{i(n-1)}\, \gamma(a_{i+1}, \ldots, a_n \mid a_1 \mid a_2, \ldots, a_i). \tag{5.17}$$

For example, let f, g have A-degrees p, $n-p$ respectively. Taking h to be the identity cochain in 5.16, i.e. the 0-cochain $\eta 1_{\mathrm{L}}$, we obtain the formula

$$(5.18)\qquad (\partial fg)\natural(a_1,\ldots,a_n)=\sum_{n-p<i\leqslant n}(-1)^{i(n-1)}(fg)(a_{i+1},\ldots,a_n,a_1,\ldots,a_i),$$

that is

$$(5.19)\qquad (\partial fg)\natural=\sum_{i=0}^{p-1}(fg)\,\mathrm{T}^i.$$

This gives the effect of the canonical map α on cochains. In particular when $p=1,2$ we have

$$(5.20)\qquad (\partial fg)\natural(a_1,\ldots,a_n)=(-1)^{|g|}f(a_1)\,g(a_2,\ldots,a_n),$$

$$(5.21)\qquad (\partial fg)\natural(a_1,\ldots,a_n)=f(a_1,a_2)\,g(a_3,\ldots,a_n)+(-1)^{n-1}f(a_n,a_1)\,g(a_2,\ldots,a_{n-1}).$$

Finally we note that the effect of the canonical maps β and $\bar\partial$ on the cochain level is given explicitly by applying the operators $\mathrm{T}-1$ and N respectively to the arguments.

6. S-relations

In this section we show that the cyclic cohomology classes constructed in § 2 are related by the S-operation on cyclic cohomology.

6.1. The cyclic classes of Connes

We return to the situation of § 2.1, where $\rho:\mathrm{A}\to\mathrm{L}$ is a linear lifting of a homomorphism $u:\mathrm{A}\to\mathrm{L/I}$ and $\tau:\mathrm{I}^m\to\mathrm{V}$ is linear map vanishing on $[\mathrm{L},\mathrm{I}^m]$. We regard ρ as an element of degree one in the DG algebra $\mathrm{Hom(B, L)}$ of cochains. Its curvature $\omega=\delta\rho+\rho^2$ lies in $\mathrm{Hom}^2(\mathrm{B},\mathrm{I})$, and $\omega^n\in\mathrm{Hom}^{2n}(\mathrm{B},\mathrm{I}^n)$ satisfies $\delta\omega^n=-[\rho,\omega^n]$.

We consider also the complex $\mathrm{Hom}(\Omega^{\mathrm{B},\natural},\mathrm{L})$ of Ω-cochains discussed in § 5.1. It is a DG bimodule over the DG algebra of cochains, and we have a derivation ∂ from the algebra to this bimodule, which is compatible with the differentials δ in these complexes.

For $n\geqslant m$ both ω^n and $\partial\rho\omega^n=m(\rho\otimes\omega^n)\,\Delta_r$ have values in I^m, so we can apply the trace τ and define a cochain and Hochschild cochain

$$\tau(\omega^n)=\tau\omega^n\in\mathrm{Hom(B, V)},\qquad \tau^\natural(\partial\rho\omega^n)=\tau(\partial\rho\omega^n)\,\natural\in\mathrm{Hom}(\Omega^{\mathrm{B},\natural},\mathrm{V}).$$

Proposition **6.1.** — *We have*

$$\delta\{\tau(\omega^n/n!)\}=\beta\{\tau^\natural(\partial\rho\omega^n/n!)\},\qquad \delta\{\tau^\natural(\partial\rho\omega^n/n!)\}=\bar\partial\{\tau(\omega^{n+1}/(n+1)!)\}.$$

Proof. — Using 5.5 one has

$$\delta\{\tau(\omega^n)\}=\tau(\delta(\omega^n))=\tau(-[\rho,\omega^n])=\beta\{\tau^\natural(\partial\rho\omega^n)\},$$

whence the first formula. Also

$$\begin{aligned}\delta\{\tau^{\natural}(\partial\rho\omega^n)\} &= \tau^{\natural}\{\delta(\partial\rho)\,\omega^n - \partial\rho\,\delta(\omega^n)\}\\ &= \tau^{\natural}\{\partial(\delta\rho)\,\omega^n - \partial\rho(\rho\omega^n - \omega^n\rho)\} = \tau^{\natural}\{(\partial(\delta\rho) + \partial\rho\rho + \rho\,\partial\rho)\,\omega^n\}\\ &= \tau^{\natural}\{\partial(\delta\rho + \rho^2)\,\omega^n\} = \tau^{\natural}(\partial\omega\omega^n),\end{aligned}$$

where we have used the trace property 5.3 of $\tau^{\natural}$ and the fact that ρ, $\partial\rho$ are both odd. On the other hand by 5.4 and the fact that ∂ is a derivation, we have

$$\bar{\partial}\{\tau(\omega^{n+1})\} = \tau^{\natural}(\partial(\omega^{n+1})) = \sum_{i=0}^{n} \tau^{\natural}(\omega^i\,\partial\omega\omega^{n-i}) = (n+1)\,\tau^{\natural}(\partial\omega\omega^n)$$

completing the proof.

Remark **6.2.** — Since δ is $\pm\, b'$ or b, and $\bar{\partial} = \mathrm{N}$, $\beta = \mathrm{T} - 1$, on cochains and Hochschild cochains, where the sign is opposite to the parity, we see that this proposition is equivalent to the identities

$$\begin{aligned} b'\,\tau(\omega^n/n!) &= (1-\mathrm{T})\,\tau^{\natural}(\partial\rho\omega^n/n!),\\ b\tau^{\natural}(\partial\rho\omega^n/n!) &= \mathrm{N}\tau(\omega^{n+1}/(n+1)!).\end{aligned}$$

It is not hard and rather instructive to check these identities directly using the following formulas which result from 1.3 and 5.20:

$$\begin{aligned}\tau(\omega^n)\,(a_1, \ldots, a_{2n}) &= \tau\{\omega(a_1, a_2)\,\ldots\,\omega(a_{2n-1}, a_{2n})\},\\ \tau^{\natural}(\partial\rho\omega^n)\,(a_0, \ldots, a_{2n}) &= \tau\{\rho(a_0)\,\omega(a_1, a_2)\,\ldots\,\omega(a_{2n-1}, a_{2n})\}.\end{aligned}$$

We observe that the first identity of 6.1 implies immediately that $\tau^{\natural}(\omega^n/n!) = \mathrm{N}\tau(\omega^n/n!)$ is a cyclic cocycle. Thus this proposition can be viewed as a refinement of the fact that Chern character forms are closed. We are next going to see that the extra information it contains is just what is needed to prove the result of Connes that the classes of these cyclic cocycles for different n are related by the S-operation.

6.2. The S-operation and periodic cyclic cocycles

The S-operation on cyclic cohomology is most easily understood from the periodicity of the cyclic bicomplex. However this approach leaves the sign of the S-operation subject to certain choices, e.g whether to use $1 - \mathrm{T}$ or $\mathrm{T} - 1$ in the cyclic bicomplex. We now fix our sign conventions to be consistent with Connes paper [C1], where the S-operation is defined by an explicit formula.

Let $\varphi_j \in \mathrm{Hom}^j(\mathrm{B}, \mathrm{V})$, $\psi_j \in \mathrm{Hom}^j(\Omega^{\mathrm{B},\natural}, \mathrm{V})$ denote bar and Hochschild cochains of A-degree j. We observe that the formulas

(6.3) $$\delta\varphi_n = \beta\{(-1)^n\,\psi_{n+1}\}, \qquad \delta\{(-1)^n\,\psi_{n+1}\} = \bar{\partial}\varphi_{n+2},$$

of the type encountered in the above proposition are the same as the formulas

(6.4) $$b'\,\varphi_n = (1-\mathrm{T})\,\psi_{n+1}, \qquad b\psi_{n+1} = \mathrm{N}\varphi_{n+2},$$

using the operators on cochains which are the transposes of the operators in the cyclic bicomplex of § 1.

We consider the bicomplex of cochains obtained by applying Hom(?, V) to the cyclic bicomplex and making the following choice of signs for the arrows:

$$
(6.5)\qquad
\begin{array}{ccccccccc}
 & \uparrow{\scriptstyle -b} & & \uparrow{\scriptstyle b'} & & & & \\
\xrightarrow{N} & C^{0,2} & \xrightarrow{T-1} & C^{1,2} & \xrightarrow{N} & & & \\
 & \uparrow{\scriptstyle -b} & & \uparrow{\scriptstyle b'} & & \uparrow{\scriptstyle -b} & & \\
\xrightarrow{N} & C^{0,1} & \xrightarrow{T-1} & C^{1,1} & \xrightarrow{N} & C^{2,1} & \xrightarrow{T-1} \\
 & \uparrow{\scriptstyle -b} & & \uparrow{\scriptstyle b'} & & \uparrow{\scriptstyle -b} & & \\
\xrightarrow{N} & C^{0,0} & \xrightarrow{T-1} & C^{1,0} & \xrightarrow{N} & C^{2,0} & \xrightarrow{T-1} \\
 & \uparrow & & \uparrow & & \uparrow & & \\
 & 0 & & 0 & & 0 & &
\end{array}
$$

Here $C^{pq} = \mathrm{Hom}(A^{\otimes q+1}, V)$, and the $-b$ ensures that the squares anticommute, so that the sum of the horizontal and vertical arrows has square zero. This bicomplex has been arranged so that a string of cochains

$$\ldots\ \varphi_n \in C^{2i+1,\,n-1},\ \psi_{n+1} \in C^{2i,\,n},\ \varphi_{n+2} \in C^{2i-1,\,n+1}\ \ldots$$

satisfies the identities 6.4 if and only if it is killed by the total differential except for contributions at the ends of the string.

Let C_+ be the subcomplex where the columns with $p < 0$ have been made zero. The homology of C_+ in the horizontal direction is zero except on the line $p = 0$, where it is the complex $CC^\bullet(A, V)$ of cyclic cochains with values in V. By standard arguments the inclusion of the cyclic cochains into the total complex of C_+ is a quasi-isomorphism. On the other hand, there is the obvious embedding $S : C_+^{p,q} \xrightarrow{\sim} C_+^{p,q+2}$ of degree two of the total complex into itself, and this induces the S-operation on the cyclic cohomology.

With this definition of S we have

Lemma **6.6.** — *Assume that* φ_n, ψ_{n+1}, φ_{n+2} *satisfy the formulas* 6.3-6.4. *Then* $N\varphi_n$ *and* $N\varphi_{n+2}$ *are cyclic cocycles, and* $S[N\varphi_n] = [\varphi_{n+2}]$.

Proof. — The point is that if we apply the total differential to $\varphi_n + \psi_{n+1}$ sitting in $C^{1,n-1} \oplus C^{0,n}$, we obtain $N\varphi_n - N\varphi_{n+2} \in C^{2,n-1} \oplus C^{0,n+1}$, so the latter two terms represent the same cyclic cohomology class. But the first term represents the transform by S of the class of $N\varphi_n \in C^{0,n-1}$, so the lemma follows.

From this discussion we see that the following result of Connes is an immediate consequence of 6.1.

Theorem **5.** — *The odd cyclic cohomology classes of Theorem* 1 *satisfy*

$$S[\tau^{\natural}(\omega^n/n!)] = [\tau^{\natural}(\omega^{n+1}/(n+1)!)].$$

Before taking up the corresponding result for the Chern-Simons classes, we add a few comments about the bicomplex 6.5 and periodic cyclic cohomology which will be needed later.

There are two ways of making a total complex C from 6.5, depending on whether we let C^n be the direct sum or the direct product of $C^{p,q}$ for $p+q=n$. If we use the direct product, then because the rows are exact the resulting total complex has trivial homology. If we use the direct sum, then the resulting complex of " finite " cochains gives the periodic cyclic cohomology $S^{-1}\,HC^{\bullet}(A)$. This is **Z**-graded, but because of the periodicity, there are only two different periodic cyclic cohomology groups, even and odd. We can compute these by taking the quotient of the total complex by the action of the periodicity S. This gives the following **Z**/2-graded complex.

Definition **6.7.** — The complex of periodic cochains is the **Z**/2-graded complex whose even elements are alternating sequences $(\psi_1, \varphi_2, \psi_3, \ldots)$ of bar and Hochschild cochains with only finitely many nonzero terms, and whose odd elements are similar sequences of the form $(\varphi_1, \psi_2, \varphi_3, \ldots)$. The differential is

$$\begin{aligned} &d(\ldots, \varphi_n, \psi_{n+1}, \varphi_{n+2}, \ldots) \\ &\quad = (\ldots, -b\psi_{n-1} + N\varphi_n, b'\varphi_n + (T-1)\,\psi_{n+1}, -b\psi_{n+1} + N\varphi_{n+2}, \ldots). \end{aligned}$$

Thus a periodic cocycle is just a sequence satisfying 6.3 or 6.4.

Remarks **6.8.** — Suppose now that A is a unital algebra and that we are given a periodic cocycle as above satisfying

$$\psi_{n+1}(a_0, \ldots, a_n) = 0 \quad \text{if } a_i = 1 \text{ for some } i \geqslant 1,$$
$$\varphi_n(a_1, \ldots, a_n) = \psi_{n+1}(1, a_1, \ldots, a_n).$$

The first condition means that ψ_{n+1} is a normalized Hochschild cochain. Then for the B-operator of Connes we have

$$B\psi_{n+1} = Ns(1-T)\,\psi_{n+1} = N\varphi_n = b\psi_{n-1},$$

where s denotes the contracting homotopy of the b' complex which inserts 1 in the first argument. Thus a periodic cocycle satisfying these conditions gives rise to a cocycle in the Connes b, B bicomplex.

6.9. — The periodic cocycle described in Proposition 6.1 satisfies these conditions provided that the lifting ρ is chosen to preserve the units, as one easily sees from the formulas of 6.2.

6.10. — The cochain theory developed in this paper can be viewed as pertaining to the reduced cyclic theory of the augmented unital algebra $\widetilde{A}$. It is an interesting problem to find a good generalization to arbitrary unital algebras, which for example would explain the Connes b, B bicomplex made from the reduced Hochschild complex.

6.3. Homotopy

We discuss next the homotopy behavior of the cochains in proposition 6.1. We consider the situation of 2.1, where ρ_t is a one-parameter family of linear maps from A to L which are homomorphisms modulo I.

Proposition **6.11.** — *With the notations and hypotheses of* 2.1 *we have*

$$\partial_t\{\tau(\omega_t^n/n!)\} = \delta\{\tau(\mu_{n,t}/n!)\} - \beta\{\tau^\natural(\partial\rho_t\,\mu_{nt}/n!)\},$$
$$\partial_t\{\tau^\natural(\partial\rho_t\,\omega_t^n/n!)\} = -\delta\{\tau^\natural(\partial\rho_t\,\mu_{nt}/n!)\} + \bar\partial\{\tau(\mu_{n+1,t}/(n+1)!)\}.$$

Proof. — As in the proof of 2.1, we work with cochains having values in the DG algebra $L[t, dt]$. If $\tilde\omega = (\delta + dt\,\partial_t)\,\rho_t + \rho_t^2 = \omega_t + dt\dot\rho_t$ is the total curvature of the family, we have

$$\tau(\tilde\omega^n) = \tau(\omega_t^n + dt\mu_{nt}) = \tau(\omega_t^n) + dt\tau(\mu_{nt}),$$
$$\tau^\natural(\partial\rho_t\,\tilde\omega^n) = \tau^\natural(\partial\rho_t\,\omega_t^n) - dt\tau^\natural(\partial\rho_t\,\mu_{nt}).$$

An obvious extension of the proof of 6.1 gives

$$(\delta + dt\,\partial_t)\,\tau(\tilde\omega^n) = \beta\tau^\natural(\partial\rho_t\,\tilde\omega^n),$$
$$(n+1)\,(\delta + dt\,\partial_t)\,\tau^\natural(\partial\rho_t\,\tilde\omega^n) = \bar\partial\tau(\tilde\omega^{n+1}).$$

The desired formulas then result by comparing coefficients of *dt*.

6.4. Chern-Simons classes

We next consider the even dimensional cyclic cohomology classes constructed in § 2.2.

We recall that ρ, ω have the same meaning as above, but now $\tau : L/I^{m+1} \to V$ is a trace defined on this quotient algebra, where $m \geq 0$. We consider as in § 2.2 the one parameter family $\rho_t = t\rho$ with

$$\omega_t = t\,\delta\rho + t^2\,\rho^2 = t\omega + (t^2 - t)\,\rho^2, \qquad \mu_{nt} = \sum_1^n \omega_t^{i-1}\,\rho\omega_t^{n-i}.$$

We define cochains and Hochschild cochains

$$\varphi_{2n-1} = \int_0^1 \tau(\mu_{nt}/n!)\,dt \in \mathrm{Hom}^{2n-1}(B, V),$$
$$\psi_{2n} = \int_0^1 \tau^\natural(\partial\rho\mu_{nt}/n!)\,t\,dt \in \mathrm{Hom}^{2n}(\Omega^{B,\natural}, V).$$

Since the trace τ is defined on all of L, we can use the homotopy formula above in the case of the family $t\rho$, where the ideal is taken to be the whole algebra L. Integrating this formula from 0 to 1 gives

$$\delta\varphi_{2n-1} - \beta\psi_{2n} = \tau(\omega_t^n)\Big|_0^1,$$
$$-\,\delta\psi_{2n} + \bar\partial\varphi_{2n+1} = \tau^\natural(\partial\rho\omega_t^n)\,t\Big|_0^1.$$

Since ω has values in I and τ vanishes on I^{m+1}, the right sides are zero for $n > m$, so we obtain the relations

$$\delta\varphi_{2n-1} = \beta\psi_{2n}, \qquad \delta\psi_{2n} = \bar{\partial}\varphi_{2n+1}$$

for $n > m$.

We have seen that these relations imply that we have a family of cyclic cohomology classes which are linked by the S-operation. However $N\varphi_{2n+1}$ is just the Chern-Simons cyclic $2n$-cocycle of § 2, so we have proved the following.

Theorem **6.** — *The Chern-Simons cyclic cohomology classes* $c_{2n} = [N\varphi_{2n+1}]$ *of Theorem* 2 *satisfy* $Sc_{2n} = c_{2n+2}$ *for* $n \geqslant m$.

7. Vector bundles with connection

Let E be a vector bundle over the smooth manifold M, and let A be the algebra of its endomorphisms. We suppose given a connection on E and a closed current on M. To this data we are going to associate a periodic cocycle on A.

Let $\Omega(M)$ be the de Rham complex of M, and let $\Omega(M, E)$ be the space of differential forms with values in E. We can view the connection as an operator ∇ on $\Omega(M, E)$ of degree one satisfying a derivation formula with respect to multiplication by differential forms. Let $L = \Omega(M, \mathrm{End}\, E)$ be the graded algebra of forms with values in the endomorphism bundle. It operates on $\Omega(M, E)$ by multiplication, and we can identify the curvature ∇^2 with an element of degree two in $\Omega(M, \mathrm{End}\, E)$.

The induced connection in the endomorphism bundle is the degree one derivation $\tilde{\nabla} = \mathrm{ad}\, \nabla$ on $\Omega(M, \mathrm{End}\, E)$ such that $\tilde{\nabla}w = [\nabla, w]$, where the bracket means the commutator (with the appropriate signs) of operators on $\Omega(M, E)$. One has $\tilde{\nabla}(\tilde{\nabla}w) = [\nabla^2, w]$.

We now consider the space $\mathrm{Hom}(B, L)$ of cochains on A with values in L. This is a bigraded algebra having the anticommuting derivations δ and $\tilde{\nabla}$, where $\delta f = -(-1)^{|f|} fb'$, and where $\tilde{\nabla}f$ is $\tilde{\nabla}$ applied to the values of f. Let $\theta \in \mathrm{Hom}(B_1, L^0)$ be the obvious inclusion of A as the endomorphism valued forms of degree zero. As this is a homomorphism, we have $\delta\theta + \theta^2 = 0$. We set

$$K = \nabla^2 + \tilde{\nabla}\theta \in \mathrm{Hom}(B_0, L^2) \oplus \mathrm{Hom}(B_1, L^1)$$

and define the cochain e^K using the exponential series. It is a finite sum, since $K^n = 0$ for n above the dimension of the manifold. We write $\mathrm{ad}\, \theta$ for the derivation $f \mapsto [\theta, f]$.

Lemma **7.1.** — *One has* $(\delta + \mathrm{ad}\, \theta + \tilde{\nabla})\, K = (\delta + \mathrm{ad}\, \theta + \tilde{\nabla})\, e^K = 0$.

Proof. — We have the general differentiation formula

$$D(e^K) = \int_0^1 e^{(1-s)K}\, D(K)\, e^{sK}\, ds \tag{7.2}$$

where D is a derivation. Taking D to be $\delta + \mathrm{ad}\,\theta + \widetilde{\nabla}$, we see that it suffices to prove $\mathrm{D(K)} = 0$. We note that $\widetilde{\nabla}(\nabla^2) = [\nabla, \nabla^2] = 0$ and that $\delta\,\nabla^2 = 0$, because δ vanishes on 0-cochains. Hence

$$\begin{aligned}\mathrm{D(K)} &= (\delta + \mathrm{ad}\,\theta + \widetilde{\nabla})\,(\nabla^2 + \widetilde{\nabla}\theta) = \delta\,\widetilde{\nabla}\theta + [\theta, \widetilde{\nabla}\theta] + [\theta, \nabla^2] + \widetilde{\nabla}(\widetilde{\nabla}\theta)\\ &= \widetilde{\nabla}(\theta^2) + \theta\,\widetilde{\nabla}\theta - (\widetilde{\nabla}\theta)\,\theta + [\theta, \nabla^2] + [\nabla^2, \theta] = 0,\end{aligned}$$

completing the proof.

Remark **7.3.** — This lemma can be proved in a more conceptual way by introducing the "connection" $\delta + \theta + \nabla$ acting on $\mathrm{Hom(B}, \Omega(\mathrm{M, E}))$. Its curvature is K, and the identity $\mathrm{D(K)} = 0$ is the associated Bianchi identity.

In addition to the bigraded algebra of bar cochains, we also consider the bigraded module $\mathrm{Hom}(\Omega^{\mathrm{B}}, \mathrm{L})$ of Ω-cochains, which also has anticommuting operators δ, $\widetilde{\nabla}$. Given bar cochains f, g, we can form the Ω-cochain ∂fg as in § 5.1; this operation is compatible with δ as before and with $\widetilde{\nabla}$:

$$\begin{aligned}\widetilde{\nabla}(\partial fg) &= \widetilde{\nabla} m(f\otimes g)\,(\partial\otimes 1)\,\Delta_r\\ &= m(\widetilde{\nabla}\otimes 1 + 1\otimes\widetilde{\nabla})\,(f\otimes g)\,(\partial\otimes 1)\,\Delta_r\\ &= \partial(\widetilde{\nabla} f)\,g + (-1)^{|f|}\,\partial f\,\widetilde{\nabla} g.\end{aligned}$$

The trace map from endomorphisms to functions extends to a trace

$$\mathrm{tr_E} : \mathrm{L} = \Omega(\mathrm{M, End\,E}) \to \Omega(\mathrm{M})$$

such that $d\,\mathrm{tr} = \mathrm{tr}\,\widetilde{\nabla}$. Applying this trace to the values of cochains we define

$$\begin{aligned}\mathrm{tr_E}(e^{\mathrm{K}}) &= \mathrm{tr_E}\, e^{\mathrm{K}} \in \mathrm{Hom(B}, \Omega(\mathrm{M})),\\ \mathrm{tr}^{\natural}_{\mathrm{E}}(\partial\theta e^{\mathrm{K}}) &= \mathrm{tr_E}(\partial\theta e^{\mathrm{K}})\,\natural \in \mathrm{Hom}(\Omega^{\mathrm{B},\natural}, \Omega(\mathrm{M})).\end{aligned}$$

Proposition **7.4.** — *One has*

$$\begin{aligned}(\delta + d)\,\mathrm{tr_E}(e^{\mathrm{K}}) &= \beta\{\mathrm{tr}^{\natural}_{\mathrm{E}}(\partial\theta e^{\mathrm{K}})\},\\ (\delta + d)\,\mathrm{tr}^{\natural}_{\mathrm{E}}(\partial\theta e^{\mathrm{K}}) &= \bar{\partial}\{\mathrm{tr_E}(e^{\mathrm{K}})\}.\end{aligned}$$

Proof. — Setting $\tau = \mathrm{tr_E}$, we have, using the lemma,

$$(\delta + d)\,\tau(e^{\mathrm{K}}) = \tau((\delta + \widetilde{\nabla})\,e^{\mathrm{K}}) = -\,\tau([\theta, e^{\mathrm{K}}]) = \beta\{\tau^{\natural}(\partial\theta e^{\mathrm{K}})\},$$

whence the first formula. We also have

$$\begin{aligned}\delta\tau^{\natural}(\partial\theta e^{\mathrm{K}}) &= \tau^{\natural}\{\delta(\partial\theta e^{\mathrm{K}})\} = \tau^{\natural}\{\partial(\delta\theta)\,e^{\mathrm{K}} - \partial\theta\,\delta(e^{\mathrm{K}})\},\\ d\tau^{\natural}(\partial\theta e^{\mathrm{K}}) &= \tau^{\natural}\{\widetilde{\nabla}(\partial\theta e^{\mathrm{K}})\} = \tau^{\natural}\{\partial(\widetilde{\nabla}\theta)\,e^{\mathrm{K}} - \partial\theta\,\widetilde{\nabla}(e^{\mathrm{K}})\},\\ 0 = \tau^{\natural}([\theta, \partial\theta e^{\mathrm{K}}]) &= \tau^{\natural}\{(\theta\,\partial\theta + \partial\theta\theta)\,e^{\mathrm{K}} - \partial\theta[\theta, e^{\mathrm{K}}]\}.\end{aligned}$$

Adding these gives, using $\delta\theta = -\,\theta^2$,

$$(\delta + d)\,\tau^{\natural}(\partial\theta) = \tau^{\natural}(\partial(\widetilde{\nabla}\theta)\,e^{\mathrm{K}}) = \mathrm{tr}(\partial\mathrm{K}e^{\mathrm{K}}).$$

On the other hand we have

$$\bar{\partial}\tau(e^{K}) = \tau^{\natural}(\partial e^{K}) = \int_0^1 \tau^{\natural}\{e^{(1-s)K}\,\partial K e^{sK}\}\,ds$$
$$= \int_0^1 \tau^{\natural}(\partial K e^{K})\,ds = \tau^{\natural}(\partial K e^{K}),$$

which completes the proof.

Now let z be a closed current on M of dimension r, and let $\varepsilon = (-1)^r$ be its parity. Integrating over z we obtain cochains

$$\varphi = \int_z \mathrm{tr}_E(e^{K}) \in \mathrm{Hom}^{\varepsilon}(B, \mathbf{C}),$$
$$\psi = \int_z \mathrm{tr}_E^{\natural}(\partial\theta e^{K}) \in \mathrm{Hom}^{-\varepsilon}(\Omega^{B,\natural}, \mathbf{C}),$$

where $\mathrm{Hom}^{\varepsilon}$ is the space of maps of parity ε. Using the expansion for the exponential

$$e^{K} = \sum_{n\geqslant 0}\ \sum_{i_0,\ldots,i_n\geqslant 0} \nabla^{2i_0}[\nabla,\theta]\,\nabla^{2i_1}\ldots[\nabla,\theta]\,\nabla^{2i_n}/(n+i_0+\ldots+i_n)!$$

and 5.20, we see that φ, ψ have the components

$$\varphi_n(a_1,\ldots,a_n) = \int_z \sum \mathrm{tr}_E(\nabla^{2i_0}[\nabla,a_1]\,\nabla^{2i_1}\ldots[\nabla,a_n]\,\nabla^{2i_n})/(n+i_0+\ldots+i_n)!,$$
$$\psi_{n+1}(a_0,\ldots,a_n)$$
$$= \int_z \sum \mathrm{tr}_E(a_0\,\nabla^{2i_0}[\nabla,a_1]\,\nabla^{2i_1}\ldots[\nabla,a_n]\,\nabla^{2i_n})/(n+i_0+\ldots+i_n)!,$$

where the sum is over $i_0,\ldots,i_n \geqslant 0$ such that $n + \Sigma 2i_j = r$. Hence φ_n, ψ_{n+1} vanish unless $n \leqslant r$ and n, r have the same parity.

We now apply the integral over z map to the formulas of 7.4. Using the fact that z is a closed current and that the map has degree $-r$, we obtain

$$\delta\varphi = \beta\{(-1)^r\psi\}, \qquad \delta\{(-1)^r\psi\} = \bar{\partial}\varphi.$$

Thus we have proved the following.

Theorem **7.** — *The pair* (φ, ψ) *is a periodic cocycle of parity* $(-1)^r$ *in the sense of* 6.7.

Remarks **7.5.** — In the case of the trivial bundle with $\nabla = d$ this periodic cocycle reduces to the single cyclic r-cocycle

$$\psi_{r+1}(a_0,\ldots,a_r) = \int_z a_0\,da_1\ldots da_r/r!$$

and all the other components of φ, ψ vanish.

7.6. — The periodic cyclic cohomology class represented by the cocycle in the theorem is independent of the connection and it depends only on the de Rham class of the current z, as the reader may easily verify. The class can be described as follows using the Morita invariance of cyclic cohomology [C1, II, Cor. 24]. The algebra A is Morita equivalent to the algebra of smooth functions on the manifold. With respect to this equivalence the periodic class in question corresponds to the class described by the cyclic cocycle on functions described in the previous remark.

7.7. — The above periodic cocycle is closely related to the entire cyclic cocycle attached to Dirac operators, as we shall indicate in the next section.

8. The JLO cocycle

Our aim in this section is to interpret the Chern character of Jaffe, Lesniewski, Osterwalder [JLO] in terms of our cochain theory. We show how their construction appears naturally in our framework when connections are replaced by superconnections.

Let H be a Hilbert space, let L be the ring of bounded operators on it, and let X be an unbounded skew-adjoint operator such that the " heat " operator e^{tX^2} is of trace class for $t > 0$. Let A be an algebra acting on H, and assume that $[X, a]$ is densely-defined and bounded for any $a \in A$.

We will be working with inhomogeneous cochains on A with values in L which have infinitely many homogeneous components, that is, which lie in

$$\mathrm{Hom}(B, L) = \prod_{n \geqslant 0} \mathrm{Hom}(B_n, L).$$

Actually, in order to obtain an interesting theory, one has to consider cochains satisfying a certain growth condition on their homogeneous components; these are the entire cochains of Connes [C2]. In the following we discuss only formal aspects of the theory, and the reader interested in the real story should look at papers on entire cyclic cohomology [C2, JLO, GS].

The action of A on H gives us a homomorphism $A \to L$, which can be viewed as a 1-cochain θ such that $\delta\theta + \theta^2 = 0$. We now propose to combine the " connection " $\delta + \theta$ with X in analogy with the theory of superconnections. We follow the method of [Q1, § 1], since it enables us to simultaneously handle the two cases corresponding to even and odd K-theory.

We extend our algebra of cochains by adjoining an element σ

$$\mathrm{Hom}(B, L)\,[\sigma] = \mathrm{Hom}(B, L) \oplus \sigma\,\mathrm{Hom}(B, L)$$

such that $\sigma^2 = 1$ and $f\sigma = (-1)^{|f|}\sigma f$. This is naturally a superalgebra where σ is odd and Hom(B, L) has its usual even-odd grading. We extend the differential by setting $\delta\sigma = 0$.

Next we treat $\theta + \sigma X$ as a " superconnection form " and consider its curvature

$$R = \delta(\theta + \sigma X) + (\theta + \sigma X)^2 = X^2 + \sigma[X, \theta].$$

The exponential of the curvature is given by the perturbation series

$$e^R = \sum_{n \geqslant 0} \int_{\Delta(n)} e^{t_0 X^2}\,\sigma[X, \theta]\,e^{t_1 X^2} \ldots \sigma[X, \theta]\,e^{t_n X^2}\,dt_1 \ldots dt_n, \tag{8.1}$$

where $\Delta(n)$ is the n-simplex $\{(t_0, \ldots, t_n) \mid t_i \geqslant 0, \Sigma t_i = 1\}$. Because X is unbounded, neither the superconnection form nor the curvature are in the extended cochain algebra.

However e^{tX^2} and $\sigma[X, \theta]$ belong to this algebra, since $[X, \theta]$ is the map $a \mapsto [X, a]$, which by our assumption has values in L. Thus e^R belongs to Hom(B, L) $[\sigma]$.

Lemma **8.2.** — *One has* $\delta(e^R) + [\sigma X + \theta, e^R] = 0$.

Proof. — The Bianchi identity for $\theta + \sigma X$ and its curvature R is $D(R) = 0$, where D is the derivation $\delta + \operatorname{ad}\theta + \operatorname{ad}(\sigma X)$. So the lemma follows from the derivation formula 7.2.

We next need a trace to apply to e^R. Before taking this up however, it will be useful to give another interpretation of the extended algebra of cochains, and to discuss the graded case.

Up to now we have been considering B and $\Omega^{B,\natural}$ as **Z**-graded complexes, but in the present context it is natural to retain only their odd-even grading. So for example B is naturally a supercoalgebra and the space of maps from B to a superalgebra is a superalgebra.

Let $\mathbf{C}[\sigma] = \mathbf{C} \oplus \sigma\mathbf{C}$ be the Clifford algebra of degree one, where $\sigma^2 = 1$. It is a superalgebra where σ is odd, so the tensor product $L[\sigma] = \mathbf{C}[\sigma] \otimes L = L \oplus \sigma L$ is naturally a superalgebra with L even and σ odd. Then Hom(B, $L[\sigma]$) is a superalgebra with product given by the usual formula $\xi\eta = m(\xi \otimes \eta)\,\Delta$, but where this tensor product is defined in the manner appropriate to the super category:

$$(\xi \otimes \eta)\,(x \otimes y) = (-1)^{|\eta||x|}\,\xi(x) \otimes \eta(y).$$

Here $|\ |$ denotes the total degree modulo two. It is clear that σ commutes with even L-valued cochains in this algebra and anticommutes with the odd ones. Hence Hom(B, $L[\sigma]$) can be identified with the extended cochain algebra considered above.

So far we have been discussing the situation where the Hilbert space is ungraded, and this case is appropriate for handling odd K-classes of A. In the graded case needed for even K-classes, one supposes given a $(\mathbf{Z}/2)$-grading $H = H^+ \oplus H^-$ such that X is odd and the operators from A are even. Then L has a superalgebra structure with the grading $L = L^+ \oplus L^-$ into even and odd operators. We need to distinguish the algebra L from the superalgebra L and a convenient way to do this is identify the latter with the super subalgebra

$$\widetilde{L} \overset{\text{def}}{=} L^+ \oplus \sigma L^- \subset L[\sigma].$$

Thus we obtain a super subalgebra

$$\operatorname{Hom}(B, \widetilde{L}) \subset \operatorname{Hom}(B, L[\sigma]).$$

We note that although Hom(B, L) and Hom(B, $\widetilde{L}$) are essentially the same as vector spaces, their products are different. We observe that e^R belongs to the subalgebra Hom(B, $\widetilde{L}$) in the graded case, since $\sigma[X, \theta]\,(a) = \sigma[X, a]$ has values in σL^- by our assumptions.

We now consider traces, by which we mean traces in the super context, i.e. what are usually called supertraces. Let I be the ideal in L consisting of trace class operators, and let $\mathrm{tr}: \mathrm{I} \to \mathbf{C}$ denote the ordinary honest operator trace. In the even (graded) case we let τ be the ordinary operator supertrace

$$\tau : \widetilde{\mathrm{I}} = \mathrm{I}^+ \oplus \sigma \mathrm{I}^- \to \mathbf{C}, \qquad \tau(x^+ + \sigma x^-) = \mathrm{tr}(\varepsilon x^+),$$

where ε denotes the involution which is ± 1 on $\mathrm{H}^\pm$. This is a trace on $\widetilde{\mathrm{I}}$ considered as a bimodule over $\widetilde{\mathrm{L}}$. In the odd (ungraded) case we combine the canonical trace on the Clifford algebra $\mathbf{C}[\sigma] \to \mathbf{C}[\sigma]_\natural \simeq \sigma\mathbf{C} \simeq \mathbf{C}$ with the operator trace to obtain a trace of odd degree

$$\tau : \mathrm{I}[\sigma] \to \sigma\mathbf{C}, \qquad \tau(x + \sigma y) = \mathrm{tr}(y),$$

defined on the ideal $\mathrm{I}[\sigma] = \mathbf{C}[\sigma] \otimes \mathrm{I}$ in $\mathrm{L}[\sigma]$.

Applying τ to the values of cochains we define cochains

$$\varphi = \tau(e^{\mathrm{R}}) \in \mathrm{Hom}^s(\mathrm{B}, \mathbf{C}), \qquad \psi = \tau^\natural(\partial\theta e^{\mathrm{R}}) \in \mathrm{Hom}^{-s}(\Omega^{\mathrm{B},\natural}, \mathbf{C}),$$

where Hom^s denotes the space of cochains of parity s, and $s = +$ in the even case and $-$ in the odd case. Using the expansion of the exponential 8.1, the fact that $\sigma[\mathrm{X}, \theta]$ is even and 5.20, we obtain the following formula in the even case

$$\text{(8.3)} \qquad \begin{aligned} \psi_{n+1}(a_0, \ldots, a_n) &= \int_{\Delta(n)} \mathrm{tr}\{\varepsilon a_0\, e^{t_0 \mathrm{X}^2}[\mathrm{X}, a_1]\, e^{t_1 \mathrm{X}^2} \ldots [\mathrm{X}, a_n]\, e^{t_n \mathrm{X}^2}\}\, dt_1 \ldots dt_n, \\ \varphi_n(a_1, \ldots, a_n) &= \psi(1, a_1, \ldots, a_n). \end{aligned}$$

Here n is even and the components for odd n are zero. In the odd case ψ_{n+1} and φ_n are zero for n even, and for n odd they are given by the same formulas with the ε deleted.

Theorem **8.** — *One has*

$$\delta\{\tau(e^{\mathrm{R}})\} = \beta\{\pm \tau^\natural(\partial\theta e^{\mathrm{R}})\}, \qquad \delta\{\pm \tau^\natural(\partial\theta e^{\mathrm{R}})\} = \bar{\partial}\{\tau(e^{\mathrm{R}})\},$$

with $+$ in the even case and $-$ in the odd. Hence the cochains φ_n, ψ_{n+1} satisfy the formulas 6.4.

Proof. — The difference in sign in the two cases is due to the degree of τ, which leads to $\delta\tau = \pm\tau\,\delta$. Hence it suffices to prove the even case. We note that $\tau([\sigma\mathrm{X}, e^{\mathrm{R}}]) = 0$ because τ is a trace on $\mathrm{L}[\sigma]$, and the operations τ, $\mathrm{ad}(\sigma\mathrm{X})$ on cochains are defined by applying these operators to the values of a cochain. So

$$\delta\{\tau(e^{\mathrm{R}})\} = \tau(\delta e^{\mathrm{R}}) = \tau\{\delta e^{\mathrm{R}} + [\sigma\mathrm{X}, e^{\mathrm{R}}]\} = -\tau([\theta, e^{\mathrm{R}}]) = \beta\{\tau^\natural(\partial\theta e^{\mathrm{R}})\}$$

proving the first formula. Similarly using the fact that $\tau^\natural$ is a morphism of complexes with the trace property we have

$$\begin{aligned} \delta\{\tau^\natural(\partial\theta e^{\mathrm{R}})\} &= \tau^\natural\{\partial(-\theta^2)\, e^{\mathrm{R}} - \partial\theta\, \delta e^{\mathrm{R}}\}, \\ 0 = \tau^\natural([\theta, \partial\theta]) &= \tau^\natural\{(\theta\,\partial\theta + \partial\theta\theta)\, e^{\mathrm{R}} - \partial\theta[\theta, e^{\mathrm{R}}]\}, \\ 0 = \tau^\natural([\sigma\mathrm{X}, \partial\theta e^{\mathrm{R}}]) &= \tau^\natural\{\partial[\sigma\mathrm{X}, \theta]\, e^{\mathrm{R}} - \partial\theta[\sigma\mathrm{X}, e^{\mathrm{R}}]\}. \end{aligned}$$

Adding these and using the above lemma we obtain

$$\delta\{\tau^\natural(\partial\theta e^{\mathrm{R}})\} = \tau^\natural(\partial[\sigma\mathrm{X}, \theta]\, e^{\mathrm{R}}) = \tau^\natural(\partial\mathrm{R} e^{\mathrm{R}}).$$

On the other hand

$$\bar{\partial}\tau(e^{\mathrm{R}}) = \tau^{\natural}(\partial e^{\mathrm{R}}) = \int_0^1 \tau^{\natural}(e^{(1-s)\mathrm{R}}\,\partial\mathrm{R}e^{s\mathrm{R}})\,ds = \tau^{\natural}(\partial\mathrm{R}e^{\mathrm{R}}),$$

concluding the proof.

Remarks **8.4.** — The ψ_{n+1} are normalized Hochschild cochains so that as remarked in § 6.2, one has $\mathrm{B}\psi = b\psi$. This is the cocycle condition proved in [JLO], but written using a skew-adjoint operator.

8.5. — The relation between the JLO cocycle and the periodic cocycle discussed in the previous section can be briefly described as follows. Let X be the twisted Dirac operator on a compact Riemannian spin manifold of dimension $2m$ with twisting given by the vector bundle E and connection ∇, and let A to be the algebra of endomorphisms of E. We form the JLO cocycle using the operator $h\mathrm{X}$ with $h > 0$ and let h go to zero. Then the heat kernel method, e.g. in the form of Getzler's symbolic calculus [G], [CM, § 3], gives

$$\lim_{h\to 0} \tau^{\natural}\{\,\partial\theta e^{h^2\mathrm{X}^2+\sigma[h\mathrm{X},\,\theta]}\,\} = (i/2\pi)^m \int_{\mathrm{M}} \hat{\mathrm{A}}(\mathrm{M})\ \mathrm{tr}^{\natural}_{\mathrm{E}}\{\,\partial\theta e^{\nabla^2+[\nabla,\,\theta]}\,\}.$$

REFERENCES

[C1] A. Connes, Non-commutative differential geometry, *Publ. Math. I.H.E.S.*, **62** (1985), 41-144.

[C2] A. Connes, Entire cyclic cohomology of Banach algebras and characters of θ-summable Fredholm modules, *K-Theory*, **1** (1988), 519-548.

[CM] A. Connes and H. Moscovici, Cyclic cohomology, the Novikov conjecture and hyperbolic groups, *Topology*, to appear.

[G] E. Getzler, Pseudodifferential operators on supermanifolds and the Atiyah-Singer index theorem, *Commun. Math. Phys.*, **92** (1983), 163-178.

[GS] E. Getzler and A. Szenes, On the Chern character of a theta-summable Fredholm module, *J. Funct. Anal.*, to appear.

[JLO] A. Jaffe, A. Lesniewski and K. Osterwalder, Quantum K-theory, I. The Chern character, *Commun. Math. Phys.*, **118** (1988), 1-14.

[K] C. Kassel, L'homologie cyclique des algèbres enveloppantes, *Invent. Math.*, **91** (1988), 221-251.

[LQ] J.-L. Loday and D. Quillen, Cyclic homology and the Lie algebra homology of matrices, *Comment. Math. Helvetici*, **59** (1984), 565-591.

[Q1] D. Quillen, Superconnection character forms and the Cayley transform, *Topology*, **27** (1988), 211-238.

[Q2] D. Quillen, Cyclic cohomology and extensions of algebras, *K-Theory*, to appear.

[T] B. L. Tsygan, Homology of matrix algebras over rings and Hochschild homology, *Uspehi Mat. Nauk*, **38:2** (1983), 217-218; *Russian Math. Surveys*, **38** (1983), 198-199.

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