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CUSPIDAL LOCAL SYSTEMS AND GRADED HECKE ALGEBRAS, I.

by GEORGE LUSZTIG (1)

Introduction

The Hecke algebra attached to a finite (resp. affine) Weyl group plays a very important role in the representation theory of a reductive group over a finite (resp. p-adic) field. It is also of considerable interest to consider Hecke algebras in which the parameters attached to the simple reflections s_i are powers q^{n_i} of q (depending on s_i) where n_i are integers $\geqslant 1$ subject only to the condition that $n_i = n_j$ if s_i, s_j are conjugate. These more general Hecke algebras arise typically as endomorphism algebras of representations induced by cuspidal representations of parabolic (resp. parahoric) subgroups, trivial on the " unipotent radical ". (See [7, p. 34, 35].)

We would like to understand such Hecke algebras with unequal parameters from a geometric (rather than arithmetic) point of view and to classify their simple modules (in the affine case), extending the known results [5] for equal parameters. I believe that the proper setting for these questions is in equivariant K-homology (as in [3], [5]), mixed with the cuspidal local systems of [9]. This is made very plausible by the results of this paper, in which we replace the affine Hecke algebra by a certain graded version.

The connection between an affine Hecke algebra and its graded version is analogous to the connection between a reductive group and its Lie algebra or the connection between K-theory and homology. (In fact this is more than an analogy.)

0.1. We shall now define this graded version **H** of an affine Hecke algebra.

Let $\mathfrak{t}$ be a **C**-vector space of finite dimension and let $R \subset \mathfrak{t}^*$ be a root system, with a set of simple roots $\Pi = \{ \alpha_1, \ldots, \alpha_m \}$ and Weyl group W with corresponding simple reflections $\{ s_1, \ldots, s_m \}$. (Thus $\mathfrak{t}^*$ has a direct sum decomposition, one summand consisting of the W-invariants, the other having Π as basis.) Let **S** be the symmetric algebra of $\mathfrak{t}^* \oplus \mathbf{C}$; we denote $r = (0, 1) \in \mathfrak{t}^* \oplus \mathbf{C} \subset \mathbf{S}$. Let $c_1, \ldots, c_m$ be integers $\geqslant 2$ such that $c_i = c_j$ whenever s_i, s_j are conjugate in W. Let $\xi \mapsto {}^w\xi$ be the natural action of W on **S** and let e be the neutral element of W.

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By definition, $\mathbf{H}$ is the $\mathbf{C}$-vector space $\mathbf{S} \otimes \mathbf{C}[W]$ with a structure of associative $\mathbf{C}$-algebra with unit $1 \otimes e$, defined by the rules:

a) $\mathbf{S} \to \mathbf{H}$, $\xi \mapsto \xi \otimes e$, is an algebra homomorphism

b) $\mathbf{C}[W] \to \mathbf{H}$, $w \mapsto 1 \otimes w$, is an algebra homomorphism

c) $(\xi \otimes e).(1 \otimes w) = \xi \otimes w, \quad (\xi \in \mathbf{S}, w \in W)$.

d) $(1 \otimes s_i)(\xi \otimes e) - ({}^{s_i}\xi \otimes e)(1 \otimes s_i) = c_i \, r \dfrac{\xi - {}^{s_i}\xi}{\alpha_i} \otimes e, \quad (\xi \in \mathbf{S}, 1 \leqslant i \leqslant m)$.

The algebra $\mathbf{H}$ arises in nature as the graded algebra associated to a certain natural filtration of an affine Hecke algebra (with unequal parameters); this can be used to show that the multiplication given by *a)-d)* is well defined.

The variable r appearing in $\mathbf{H}$ should be thought of as related to q of the Hecke algebra by $q = e^{2r}$. Thus, just as the Hecke algebra specializes for $q \to 1$ to the group algebra of the affine Weyl group, the algebra $\mathbf{H}$ specializes for $r \to 0$ to the " semidirect product " of $S(\mathfrak{t}^*)$ and $\mathbf{C}[W]$. (This semidirect product has been considered in recent work of Kostant and Kumar.)

0.2. The main observation of this paper is that $\mathbf{H}$ can be realized geometrically for many choices of the c_i in terms of equivariant homology. (The experience of [5] has shown that equivariant K-homology is much better behaved than equivariant K-cohomology; for this reason we use equivariant homology instead of the more familiar equivariant cohomology. See § 1 for the definitions.)

Let G be a reductive connected algebraic group over $\mathbf{C}$, with Lie algebra $\mathfrak{g}$. We fix a parabolic subgroup P with Levi subgroup L, and unipotent radical U; let $\mathfrak{p}$, $\mathfrak{l}$, $\mathfrak{n}$ be the Lie algebras of P, L, U. We also fix a nilpotent L-orbit $\mathscr{C}$ in $\mathfrak{l}$ carrying an irreducible cuspidal L-equivariant local system $\mathscr{L}$ (in the sense of [9].)

Let $\mathfrak{t}$ be the centre of $\mathfrak{l}$. Let $R \subset \mathfrak{t}^*$ be the set of non zero linear forms on $\mathfrak{t}$ which appear as eigenvalues in the ad-action of $\mathfrak{t}$ on $\mathfrak{g}$. Then R is a root system with a canonical basis Π and with Weyl group $W = N(L)/L$.

We consider the varieties

a) $\dot{\mathfrak{g}}_N = \{(x, gP) \in \mathfrak{g} \times G/P \mid \mathrm{Ad}(g^{-1})\, x \in \mathscr{C} + \mathfrak{n}\}$

b) $\ddot{\mathfrak{g}}_N = \{(x, gP, g'P) \in \mathfrak{g} \times G/P \times G/P \mid (x, gP) \in \dot{\mathfrak{g}}_N, (x, g'P) \in \dot{\mathfrak{g}}_N\}$.

We have a natural $G \times \mathbf{C}^*$-action on $\dot{\mathfrak{g}}_N$:

c) $(g_1, \lambda) : (x, gP) \mapsto (\lambda^{-2}\, \mathrm{Ad}(g_1)\, x, g_1 gP)$

and this induces a $G \times \mathbf{C}^*$-action on $\ddot{\mathfrak{g}}_N$.

The local system $\mathscr{L}$ on $\mathscr{C}$ gives rise *via* the function $\mathrm{pr}_{\mathscr{C}}(\mathrm{Ad}(g^{-1})\, x)$ to a local system $\dot{\mathscr{L}}$ on $\dot{\mathfrak{g}}_N$. Let $\ddot{\mathscr{L}}$ be the pull back of $\dot{\mathscr{L}} \boxtimes \dot{\mathscr{L}}^*$ under $\ddot{\mathfrak{g}}_N \hookrightarrow \dot{\mathfrak{g}}_N \times \dot{\mathfrak{g}}_N$. This is a $G \times \mathbf{C}^*$-equivariant local system on $\ddot{\mathfrak{g}}_N$.

We consider the vector space

d) $H_{\bullet}^{G \times \mathbf{C}^*}(\ddot{\mathfrak{g}}_N, \ddot{\mathscr{L}})$

(equivariant homology). Using the two projection $\ddot{\mathfrak{g}}_N \to \dot{\mathfrak{g}}_N$ and the cup-product one can regard *d)* as a module over $H^{\bullet}_{G \times \mathbf{C}^*}(\dot{\mathfrak{g}}_N) \cong S(\mathfrak{t}^* \oplus \mathbf{C}) = \mathbf{S}$ in two different ways. We can also define on *d)* a $W \times W$-action using the method of [8], [9] of constructing Springer representations in terms of intersection cohomology.

It turns out that these module structures allow us to regard *d)* as the two sided regular representation of an algebra **H** as above. This gives a topological realization of the algebra **H** on the vector space *d)*; the constants c_i $(1 \leqslant i \leqslant m)$ can be determined explicitly.

0.3. The case with equal parameters $c_i = 2$ is obtained by taking in the previous setting P to be a Borel subgroup, $\mathscr{C} = 0$, $\mathscr{L} = \mathbf{C}$.

This is the case studied in [5] in K-theoretic terms; even in this special case, the present construction of the W-action is quite different from that of [5] where no intersection cohomology was used. It is likely that in general case, one can realize the affine Hecke algebra (with unequal parameters) as an equivariant K-homology $K_0^{G \times \mathbf{C}^*}(\ddot{\mathfrak{g}}_N, \ddot{\mathscr{L}})$.

0.4. Our method leads also to a parametrization of all simple **H**-modules on which r acts as $r_0 \in \mathbf{C}^*$, in terms of parameters (x, σ, ρ) (up to G-conjugacy) where x is a nilpotent element of $\mathfrak{g}$, σ is a semisimple element of $\mathfrak{g}$ such that $[\sigma, x] = 2r_0 x$ and ρ is an irreducible representation of a certain finite group. (The equation $[\sigma, x] = 2r_0 x$ is the Lie algebra analogue of the equation $\mathrm{Ad}(s)\, x = q_0\, x$ appearing in the parametrization of [5].) The parametrization of simple **H**-modules will be established in a sequel to this paper.

0.5. *Notation.* — All algebraic varieties are assumed to be over **C** and all algebraic groups are assumed to be affine. The stalks $\mathscr{L}_x$ of a constructible sheaf $\mathscr{L}$ (in particular a local system) are assumed to be finite dimensional **C**-vector spaces. If X is an algebraic variety, we denote by $\mathscr{D}X$ or $\mathscr{D}^b_c(X)$ the bounded derived category of complexes K of **C**-sheaves on X whose cohomology sheaves $\mathscr{H}^i K$ are constructible. If $f: X' \to X$ is a morphism, then $f_*: \mathscr{D}X' \to \mathscr{D}X$, $f_!: \mathscr{D}X' \to \mathscr{D}X$, $f^*: \mathscr{D}X \to \mathscr{D}X'$ are the usual functors. If $K \in \mathscr{D}X$, we denote by $H^i(X, K)$, $H^i_c(X, K)$ the hypercohomology (resp. hypercohomology with compact support) of X with coefficients in K. If $\mathscr{L}$ is a local system on X, we identify $\mathscr{L}$ with the complex $K \in \mathscr{D}X$ such that $\mathscr{H}^0 K = L$, $\mathscr{H}^i K = 0$ for $i \neq 0$. In particular, $H^{\bullet}(X, \mathscr{L}) = \bigoplus_i H^i(X, \mathscr{L})$ and $H^{\bullet}_c(X, \mathscr{L}) = \bigoplus_i H^i_c(X, \mathscr{L})$ are well defined.

We shall often denote the inverse image $f^* \mathscr{L}$ of $\mathscr{L}$ under a morphism $f: X' \to X$ again by $\mathscr{L}$.

We have induced homomorphisms:

$f^* : H^j(X, \mathscr{L}) \to H^j(X', \mathscr{L})$ (inverse image)

$f^* : H^j_c(X, \mathscr{L}) \to H^j_c(X', \mathscr{L})$ (inverse image, if f is proper)

$f_\natural : H^j_c(X', \mathscr{L}) \to H^{j-2\delta}_c(X, \mathscr{L})$ (integration along fibres, if f is a locally trivial fibration with all fibres of pure dimension δ)

$f_! : H^j_c(X', \mathscr{L}) \to H^j_c(X, \mathscr{L})$ (extension by zero on $X - X'$, if f is an open embedding).

When $\mathscr{L} = \mathbf{C}$, we write $H^j(X)$, $H^j_c(X)$ instead of $H^j(X, \mathbf{C})$, $H^j_c(X, \mathbf{C})$.

We shall denote by an upper-script* the dual of a vector space or of a local system. If G is a group acting on a vector space V, we denote the space of G-invariant vectors by V^G.

We shall denote $S(V) = \bigoplus_j S^j(V)$ the symmetric algebra of V; in particular $S^1 V = V$; we regard S(V) as a graded algebra: we assign degree $2j$ to the elements of $S^j V$.

If G is an algebraic group, we denote by G^0 its identity component. If H is a subgroup of G we denote by $\mathscr{N}H$ or $\mathscr{N}_G H$ the normalizer of H in G.

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1. Equivariant homology

1.1. Let G be an algebraic group and let X be a G-variety, that is an algebraic variety with an algebraic action. Let $\mathscr{L}$ be a G-equivariant local system (or G-local system, for short) on X.

We want to define for an integer j, the *equivariant cohomology* $H^j_G(X, \mathscr{L})$ and the *equivariant homology* $H^G_j(X, \mathscr{L})$.

When $G = \{e\}$, then

$$H^j_G(X, \mathscr{L}) = H^j(X, \mathscr{L}) \quad \text{and} \quad H^G_j(X, \mathscr{L}) = H^{2\dim X - j}_c(X, \mathscr{L}^*)^*.$$

In the general case, we follow Borel's procedure [2] to define $H^j_G(\ )$. For this we choose an integer $m \geqslant 1$ such that $m \geqslant j$ and

a) a smooth irreducible free G-variety Γ such that $H^i(\Gamma) = 0$ for $i = 1, \ldots, m$. (" Free " means that there exists a locally trivial principal G-fibration $\Gamma \to G\backslash\Gamma$.)

In Borel's definition of $\mathrm{H}^j_{\mathrm{G}}(\ )$, Γ is not required to be smooth; but the smoothness is important for our definition of $\mathrm{H}^{\mathrm{G}}_j(\ )$; it implies that $\mathrm{H}_c^{2\dim\Gamma - i}(\Gamma) = 0$ for $i = 1, \dots, m$.

(In any case, a Γ as in *a)* exists, as it is well known: embed G as a closed subgroup of $\mathrm{GL}_r(\mathbf{C})$, then $\mathrm{G} \subset \mathrm{GL}_r(\mathbf{C}) \times \{e\} \subset \mathrm{GL}_r(\mathbf{C}) \times \mathrm{GL}_{r'}(\mathbf{C}) \subset \mathrm{GL}_{r+r'}(\mathbf{C})$ hence G acts freely by left translation on $\Gamma = \{e\} \times \mathrm{GL}_{r'}(\mathbf{C}) \backslash \mathrm{GL}_{r+r'}(\mathbf{C})$; this Γ has the required properties as soon as $2r' \geqslant m + 2$.)

If G acts freely on a variety Y and $\mathscr{E}$ is a G-local system on Y, then $\mathscr{E}$ gives rise to a local system $\bar{\mathscr{E}}$ on $\overline{\mathrm{Y}} = \mathrm{G}\backslash\mathrm{Y}$; the stalk $\bar{\mathscr{E}}_{\bar{y}}$ at $\bar{y} \in \overline{\mathrm{Y}}$ is the space of G-invariant vectors in $\prod_{y \in \pi^{-1}(\bar{y})} \mathscr{E}_y$ (where $\pi : \mathrm{Y} \to \overline{\mathrm{Y}}$ is the canonical map); G acts naturally on this direct product by the G-equivariance of $\mathscr{E}$.

Applying this to $\mathrm{Y} = \Gamma \times \mathrm{X}$ with the diagonal free action of G and to $\mathscr{E}$, the inverse image of $\mathscr{L}$ under $\mathrm{pr}_2 : \Gamma \times \mathrm{X} \to \mathrm{X}$, we find a local system $\bar{\mathscr{E}} = {}_{\Gamma}\mathscr{L}$ on ${}_{\Gamma}\mathrm{X} \overset{\mathrm{def}}{=} \mathrm{G}\backslash(\Gamma \times \mathrm{X})$.

By definition,

$$\mathrm{H}^j_{\mathrm{G}}(\mathrm{X}, \mathscr{L}) = \mathrm{H}^j({}_{\Gamma}\mathrm{X}, {}_{\Gamma}\mathscr{L}), \qquad \mathrm{H}^{\mathrm{G}}_j(\mathrm{X}, \mathscr{L}) = \mathrm{H}_c^{2d-j}({}_{\Gamma}\mathrm{X}, {}_{\Gamma}\mathscr{L}^*)^*$$

where $d = \dim({}_{\Gamma}\mathrm{X})$. (When X is empty, d is not defined and we set $\mathrm{H}^{\mathrm{G}}_j(\mathrm{X}, \mathscr{L}) = 0$.) Hence $\mathrm{H}^j_{\mathrm{G}}(\ )$, $\mathrm{H}^{\mathrm{G}}_j(\ )$ are zero for $j < 0$. We shall write $\mathrm{H}^j_{\mathrm{G}}(\mathrm{X})$, $\mathrm{H}^{\mathrm{G}}_j(\mathrm{X})$, instead of $\mathrm{H}^j_{\mathrm{G}}(\mathrm{X}, \mathbf{C})$, $\mathrm{H}^{\mathrm{G}}_j(\mathrm{X}, \mathbf{C})$.

One has to verify independence of the choice of m, Γ. Let (m', Γ') be another choice for (m, Γ); then $(m + m', \Gamma \times \Gamma')$ is also such a choice.

We have diagrams

$$\mathrm{H}^j({}_{\Gamma}\mathrm{X}, {}_{\Gamma}\mathscr{L}) \xrightarrow{f^*} \mathrm{H}^j({}_{\Gamma \times \Gamma'}\mathrm{X}, {}_{\Gamma \times \Gamma'}\mathscr{L}) \xleftarrow{f'^*} \mathrm{H}^j({}_{\Gamma'}\mathrm{X}, {}_{\Gamma'}\mathscr{L})$$

$$\mathrm{H}_c^{2d-j}({}_{\Gamma}\mathrm{X}, {}_{\Gamma}\mathscr{L}^*) \xleftarrow{f_\natural} \mathrm{H}_c^{2\mathrm{D}-j}({}_{\Gamma \times \Gamma'}\mathrm{X}, {}_{\Gamma \times \Gamma'}\mathscr{L}^*) \xrightarrow{f'_\natural} \mathrm{H}_c^{2d'-j}({}_{\Gamma'}\mathrm{X}, {}_{\Gamma'}\mathscr{L}^*)$$

where $d' = \dim_{\Gamma'} \mathrm{X}$, $\mathrm{D} = \dim_{\Gamma \times \Gamma'} \mathrm{X}$ and $f : {}_{\Gamma \times \Gamma'}\mathrm{X} \to {}_{\Gamma}\mathrm{X}$, $f' : {}_{\Gamma \times \Gamma'}\mathrm{X} \to {}_{\Gamma'}\mathrm{X}$ are the canonical fibrations with fibres isomorphic to Γ', Γ respectively.

The maps $f^*, f'^*, f_\natural, f'_\natural$ are isomorphisms; for f^*, f'^* this is well known and for $f_\natural, f'_\natural$ it follows from the lemma below.

Then $(f'^*)^{-1} \circ f^*$ and the transpose of $f'_\natural \circ (f_\natural)^{-1}$ establish the independence of the definitions of the choices made.

Lemma **1.2.** — *Let $f : \mathrm{X}' \to \mathrm{X}''$ be a locally trivial fibration such that all fibres $f^{-1}(x'')$ are irreducible of dimension δ' and satisfy $\mathrm{H}_c^{2\delta'-1}(f^{-1}(x''), \mathbf{C}) = 0$ for $i = 1, \dots, m'$. Let $\mathscr{E}$ be a local system on X'. Let $a' = \dim \mathrm{X}'$, $a'' = \dim \mathrm{X}''$. Then*

$$f_\natural : \mathrm{H}_c^{2a'-i}(\mathrm{X}', f^* \mathscr{E}) \to \mathrm{H}_c^{2a''-i}(\mathrm{X}'', \mathscr{E})$$

is an isomorphism for $0 \leqslant i \leqslant m'$.

Proof. — We have a canonical spectral sequence

$$\mathrm{E}_2^{p,q} = \mathrm{H}_c^p(\mathrm{X}'', \mathscr{H}^q f_! f^* \mathscr{E}) \Rightarrow \mathrm{H}_c^{p+q}(\mathrm{X}'', f_! f^* \mathscr{E}) = \mathrm{H}_c^{p+q}(\mathrm{X}', f^* \mathscr{E}).$$

We have

$$H_c^j(X', f^* \mathscr{E}) \xrightarrow{\text{onto}} E_\infty^{j-2\delta', 2\delta'} \subset \ldots \subset E_3^{j-2\delta', 2\delta'} \subset E_2^{j-2\delta', 2\delta'}$$
$$= H_c^{j-2\delta'}(X'', \mathscr{E}).$$

The composition of these maps is by definition $f_\natural$. It remains to show that in our case, we have

a) $E_2^{2a''-i, 2\delta'} = E_3^{2a''-i, 2\delta'} = \ldots = E_\infty^{2a''-i, 2\delta'} = H_c^{2a'-i}(X', f^* \mathscr{E})$ for $0 \leqslant i \leqslant m'$.

We have $E_2^{p, 2\delta'-i} = 0$ if $1 \leqslant i \leqslant m'$ and $E_2^{p,q} = 0$ if $p > 2a''$. From this *a)* follows.

1.3. Consider the product

a) $$H_G^j(X) \otimes H_G^{j'}(X, \mathscr{L}) \to H_G^{j+j'}(X, \mathscr{L})$$

defined by the cup-product

$$H^j({}_\Gamma X) \otimes H^{j'}({}_\Gamma X, {}_\Gamma\mathscr{L}) \to H^{j+j'}({}_\Gamma X, {}_\Gamma\mathscr{L}).$$

where Γ is as in 1.1 *a)* with large *m*.

Similarly, the cup-product

$$H^j({}_\Gamma X) \otimes H_c^{2d-(j+j')}({}_\Gamma X, {}_\Gamma\mathscr{L}^*) \to H_c^{2d-j'}({}_\Gamma X, {}_\Gamma\mathscr{L}^*)$$

gives rise to a pairing

$$H^j({}_\Gamma X) \otimes H_c^{2d-j'}({}_\Gamma X, {}_\Gamma\mathscr{L}^*)^* \to H_c^{2d-(j+j')}({}_\Gamma X, {}_\Gamma\mathscr{L}^*)^*$$

hence to a product

b) $$H_G^j(X) \otimes H_{j'}^G(X, \mathscr{L}) \to H_{j+j'}^G(X, \mathscr{L}).$$

One verifies that *a)*, *b)* are independent of (m, Γ).

This makes $H_G^\bullet(X) = \bigoplus_j H_G^j(X)$ into a graded **C**-algebra (commutative in the graded sense) with 1 and

$$H_G^\bullet(X, \mathscr{L}) = \bigoplus_j H_G^j(X, \mathscr{L}), \qquad H_\bullet^G(X, \mathscr{L}) = \bigoplus_j H_j^G(X, \mathscr{L})$$

into graded $H_G^\bullet(X)$-modules.

1.4. We discuss the *functorial properties* of $H_G^\bullet(\ )$, $H_\bullet^G(\ )$. Let $f: X' \to X$ be a G-equivariant morphism between two G-varieties X, X', let $\mathscr{L}$ be a G-local system on X' and let $\mathscr{L}' = f^* \mathscr{L}$. We have natural homomorphisms:

a) $f^* : H_G^j(X, \mathscr{L}) \to H_G^j(X', \mathscr{L}')$ (in general)

b) $f_! : H_{j+2(\dim X' - \dim X)}^G(X', \mathscr{L}') \to H_j^G(X, \mathscr{L})$ (if f is proper)

c) $f^* : H_j^G(X, \mathscr{L}) \to H_j^G(X', \mathscr{L}')$ (if f is a locally trivial fibration with irreducible fibres of fixed dimension)

d) $f^* : H_j^G(X, \mathscr{L}) \to H_{j+2(\dim X' - \dim X)}^G(X', \mathscr{L}')$ (if f is an open embedding).

In terms of a Γ as in 1.1 *a)* with *m* large these maps are defined respectively by (or by transposes of):

a') $({}_{\Gamma}f)^* : \mathrm{H}^j({}_{\Gamma}\mathrm{X}, {}_{\Gamma}\mathscr{L}) \to \mathrm{H}^j({}_{\Gamma}\mathrm{X}', {}_{\Gamma}\mathscr{L}')$

b') $({}_{\Gamma}f)^* : \mathrm{H}_c^{2d-j}({}_{\Gamma}\mathrm{X}, {}_{\Gamma}\mathscr{L}^*) \to \mathrm{H}_c^{2d-j}({}_{\Gamma}\mathrm{X}', {}_{\Gamma}\mathscr{L}'^*)$

c') $({}_{\Gamma}f)_{\natural} : \mathrm{H}_c^{2d'-j}({}_{\Gamma}\mathrm{X}', {}_{\Gamma}\mathscr{L}'^*) \to \mathrm{H}_c^{2d-j}({}_{\Gamma}\mathrm{X}, {}_{\Gamma}\mathscr{L}^*)$

d') $({}_{\Gamma}f)_! : \mathrm{H}_c^{2d-j}({}_{\Gamma}\mathrm{X}', {}_{\Gamma}\mathscr{L}'^*) \to \mathrm{H}_c^{2d-j}({}_{\Gamma}\mathrm{X}, {}_{\Gamma}\mathscr{L}^*)$

(see 0.5) where ${}_{\Gamma}f : {}_{\Gamma}\mathrm{X} \to {}_{\Gamma}\mathrm{X}'$ is the map induced by $\mathrm{Id} \times f : \Gamma \times \mathrm{X} \to \Gamma \times \mathrm{X}$, $d = \dim {}_{\Gamma}\mathrm{X}$, $d' = \dim {}_{\Gamma}\mathrm{X}'$.

e) If f is a G-equivariant vector bundle with fibres of fixed dimension then f^* in *a)* and *c)* are isomorphisms.

This follows from the definitions.

Now let $\mathrm{G}' \hookrightarrow \mathrm{G}$ be a closed subgroup of G. If X, $\mathscr{L}$ are as in 1.1, then X is also a G'-variety and we have a natural homomorphism

f) $\mathrm{H}_j^{\mathrm{G}}(\mathrm{X}, \mathscr{L}) \to \mathrm{H}_j^{\mathrm{G}'}(\mathrm{X}, \mathscr{L})$.

It is defined as follows. Let Γ be as in 1.1 *a)* with *m* large. Then Γ is also a free G'-variety. Then (f) is the transpose of

$$\varphi_{\natural} : \mathrm{H}_c^{2d'-j}(\mathrm{G}'\backslash(\Gamma \times \mathrm{X}), \varphi^*({}_{\Gamma}\mathscr{L}^*)) \to \mathrm{H}_c^{2d-j}(\mathrm{G}\backslash(\Gamma \times \mathrm{X}), {}_{\Gamma}\mathscr{L}^*)$$

defined as integration along the fibres ($\approx \mathrm{G}/\mathrm{G}'$) of the canonical fibration

$$\varphi : \mathrm{G}'\backslash(\Gamma \times \mathrm{X}) \to \mathrm{G}\backslash(\Gamma \times \mathrm{X}).$$

($d = \dim(\mathrm{G}\backslash(\Gamma \times \mathrm{X}))$, $d' = \dim(\mathrm{G}'\backslash(\Gamma \times \mathrm{X}))$).

Similarly, we have a natural homomorphism

g) $\mathrm{H}_{\mathrm{G}}^j(\mathrm{X}, \mathscr{L}) \to \mathrm{H}_{\mathrm{G}'}^j(\mathrm{X}, \mathscr{L})$

defined as

$$\varphi^* : \mathrm{H}^j(\mathrm{G}\backslash(\Gamma \times \mathrm{X}), {}_{\Gamma}\mathscr{L}^*) \to \mathrm{H}^j(\mathrm{G}'\backslash(\Gamma \times \mathrm{X}), \varphi^*({}_{\Gamma}\mathscr{L}^*)).$$

From the definition of *f)* and *g)* we see that:

h) If G/G' is isomorphic as a variety to an affine space (for example, if G' is a maximal reductive subgroup of G) then the maps *f)*, *g)* are isomorphisms.

1.5. Now let F be a closed G-stable subvariety of X; let $\mathcal{O} = \mathrm{X} - \mathrm{F}$ and let $i : \mathrm{F} \hookrightarrow \mathrm{X}$, $i' : \mathcal{O} \hookrightarrow \mathrm{X}$ be the inclusions. We then have a natural long exact sequence

a) $$\ldots \to \mathrm{H}^{\mathrm{G}}_{j-2\dim \mathrm{X}+2\dim \mathrm{F}}(\mathrm{F}, \mathscr{L}) \xrightarrow{i_!} \mathrm{H}^{\mathrm{G}}_j(\mathrm{X}, \mathscr{L}) \xrightarrow{i'^*} \mathrm{H}^{\mathrm{G}}_{j-2\dim \mathrm{X}+2\dim \mathcal{O}}(\mathcal{O}, \mathscr{L})$$
$$\to \mathrm{H}^{\mathrm{G}}_{j+1-2\dim \mathrm{X}+2\dim \mathrm{F}}(\mathrm{F}, \mathscr{L}) \xrightarrow{i_!} \ldots$$

Indeed, let Γ be as in 1.1 *a)* with m large. The partition ${}_\Gamma X = {}_\Gamma F \cup {}_\Gamma \mathcal{O}$ with ${}_\Gamma F$ closed, gives rise to a long exact sequence

$$\dots \to H_c^{2d-j-1}({}_\Gamma F, {}_\Gamma \mathscr{L}^*) \to H_c^{2d-j}({}_\Gamma \mathcal{O}, {}_\Gamma \mathscr{L}^*) \to H_c^{2d-j}({}_\Gamma X, {}_\Gamma \mathscr{L}^*) \to H_c^{2d-j}({}_\Gamma F, {}_\Gamma \mathscr{L}^*) \to \dots$$

(where $d = \dim {}_\Gamma X$). Taking duals we find a portion of the exact sequence *a)*; increasing m, we find a larger and larger portion of *a)*; for $m \to \infty$ we find *a)*.

If, for example, we have $\dim F < \dim X$, then from *a)* we get an isomorphism

b) $H_0^G(X, \mathscr{L}) \xrightarrow[\approx]{i'^*} H_0^G(\mathcal{O}, \mathscr{L})$

since $H_j^G(F, \mathscr{L}) = 0$ for $j < 0$.

1.6. Let G' be a closed subgroup of G and let X' be a closed G'-stable subvariety of X such that the map $G' \backslash (G \times X') \to X$, $(g, x') \mapsto gx'$, is an isomorphism of G-varieties. (G' acts on $G \times X'$ by $g' : (g, x') \mapsto (gg'^{-1}, g'x')$ and G acts on $G' \backslash (G \times X')$ by $g_1 : (g, x') \mapsto (g_1 g, x')$).

We have natural isomorphisms

a) $H_G^j(X, \mathscr{L}) \cong H_{G'}^j(X', \mathscr{L})$, $H_j^G(X, \mathscr{L}) \cong H_j^G(X', \mathscr{L})$.

Indeed, choose Γ as in 1.1 *a)* with m large. Then the isomorphisms in *a)* are induced by the natural isomorphism $G' \backslash (\Gamma \times X') \xrightarrow{\approx} G \backslash (\Gamma \times X)$.

1.7. We write $H_G^\bullet$, $H_\bullet^G$ instead of $H_G^\bullet$ (point), $H_\bullet^G$ (point) where the point is regarded as a G-variety in the obvious way. The map $X \to$ point defines by 1.4 *a)* a $\mathbf{C}$-algebra homomorphism $\varepsilon : H_G^\bullet \to H_G^\bullet(X)$ preserving the grading. Since $H_G^\bullet(X, \mathscr{L})$, $H_\bullet^G(X, \mathscr{L})$ are $H_G^\bullet(X)$-modules (1.3) they can be also regarded as $H_G^\bullet$-modules, via ε.

1.8. Assume now that X has pure dimension. We have a natural homomorphism

a) $H_G^j(X, \mathscr{L}) \to H_j^G(X, \mathscr{L})$.

It is defined as follows. Choose Γ as in 1.1 *a)* with m large. We consider the composition

$$H^j({}_\Gamma X, {}_\Gamma \mathscr{L}) \otimes H_c^{2d-j}({}_\Gamma X, {}_\Gamma \mathscr{L}^*) \xrightarrow{\text{cup-product}} H_c^{2d}({}_\Gamma X) \xrightarrow{\pi_\natural} \mathbf{C}$$

where $\pi : {}_\Gamma X \to$ point, and $d = \dim {}_\Gamma X$.

This defines $H^j({}_\Gamma X, {}_\Gamma \mathscr{L}) \to H^{2d-j}({}_\Gamma X, {}_\Gamma \mathscr{L}^*)^*$, hence *a)*.

b) If X is smooth of pure dimension, then *a)* is an isomorphism.

This follows from Poincaré duality for the smooth variety ${}_\Gamma X$. In particular, we have

c) $H_G^\bullet \xrightarrow[(a)]{\approx} H_\bullet^G$ (isomorphism of $H_G^\bullet$-modules).

1.9. Let G' be a closed normal subgroup of G containing G^0.

a) The finite group G/G' acts naturally on $H_j^{G'}(X, \mathscr{L})$ and $H_{G'}^j(X, \mathscr{L})$ and we have $H_j^G(X, \mathscr{L}) \xrightarrow{\sim} H_j^{G'}(X, \mathscr{L})^{G/G'}$, $H_G^j(X, \mathscr{L}) \xrightarrow{\sim} H_{G'}^j(X, \mathscr{L})^{G/G'}$. (The maps are given by 1.6 *a)*.)

Indeed choose Γ as in 1.1 *a)* with m large. Then Γ is also a free G'-variety. Let $p : G'\backslash(\Gamma \times X) \to G\backslash(\Gamma \times X)$ be the natural map, a finite principal covering with group $G'\backslash G$. Then $p^*({}_\Gamma\mathscr{L})$ is a $G'\backslash G$-local system hence $G'\backslash G$ acts naturally on $H^j(G'\backslash(\Gamma \times X), p^*({}_\Gamma\mathscr{L}))$ and $H_c^{2d'-j}(G'\backslash(\Gamma \times X), p^*({}_\Gamma\mathscr{L}^*))$ ($d' = \dim G'\backslash(\Gamma \times X)$), and

$$p^* : H^j(G\backslash(\Gamma \times X), {}_\Gamma\mathscr{L}) \xrightarrow{\sim} H^j(G'\backslash(\Gamma \times X), p^*({}_\Gamma\mathscr{L}))^{G/G'}$$

$$p_\natural : H^{2d'-j}(G'\backslash(\Gamma \times X), p^*({}_\Gamma\mathscr{L}^*)) \xrightarrow{\sim} H^{2d'-j}(G\backslash(\Gamma \times X), p^*({}_\Gamma\mathscr{L}^*)).$$

Thus, we have *a)*.

1.10. Let T be a torus with Lie algebra $\mathfrak{t}$ and let $X(T)$ be its character group. For $\chi \in X(T)$ let $\mathbf{C}_\chi$ be $\mathbf{C}$ with the T-action defined by $(t, z) \to \chi(t)\, z$. Let $i : \{0\} \hookrightarrow \mathbf{C}$ and $\pi : \mathbf{C} \to \{0\}$ be the obvious map. The composition

$$H_\bullet^T(\{0\}) \xrightarrow{i!} H_\bullet^T(\mathbf{C}_\chi) \xrightarrow[(1.4(g))]{(\pi^*)^{-1}} H_\bullet^T(\{0\})$$

is $H_\bullet^T$-linear and of degree 2, hence it must be given by multiplication by an element $c(\chi) \in H_T^2$. (See 1.8 *c)*.) Then $c : X(T) \to H_T^2(\chi \to c(\chi))$ is a group homomorphism.

There is a unique isomorphism

a) $\Psi : \mathfrak{t}^* \xrightarrow{\approx} H_T^2$

such that the diagram

$$\begin{tikzcd}
& X(T) \arrow[dl, "d"'] \arrow[dr, "c"] & \\
\mathfrak{t}^* \arrow[rr, "\Psi"] & & H_T^2
\end{tikzcd}$$

is commutative, where $d\chi : \mathfrak{t} \to \mathbf{C}$ is the differential of $\chi : T \to \mathbf{C}^*$ at the identity.

More generally, let E be a finite dimensional $\mathbf{C}$-vector space with a given linear representation of T. Then $E \cong \mathbf{C}_{\chi_1} \oplus \ldots \oplus \mathbf{C}_{\chi_n}$ as a T-module. Let $i : \{0\} \hookrightarrow E$, $\pi : E \to \{0\}$ be the obvious maps. Then

b) the composition $H_\bullet^T\{0\} \xrightarrow{i!} H_\bullet^T(E) \xrightarrow{(\pi^*)^{-1}} H_\bullet^T\{0\}$ is the multiplication by

$$\Psi(d\chi_1) . \Psi(d\chi_2) \ldots \Psi(d\chi_n) = c(\chi_1) \ldots c(\chi_n).$$

1.11. Let $R_u G$ be the unipotent radical of G and $G_r = G/R_u G$.

Let $\mathfrak{g}_r$ be the Lie algebra of G_r. Then G acts naturally on $\mathfrak{g}_r$ via the adjoint action. Hence it acts on $S^j(\mathfrak{g}_r^*)$.

It is well known that we have natural isomorphisms

a) $S^j(g_r^*)^G \xrightarrow{\approx} H_G^{2j}$

b) $H_G^{2j+1} = 0$.

The map *a)* is characterized by properties *c)*, *d)*, *e)* below.

c) If T is a maximal torus of G, with Lie algebra $\mathfrak{t} \hookrightarrow \mathfrak{g}$, then the diagram

$$\begin{tikzcd}
S^j(\mathfrak{g}_r^*)^G \arrow[r, "(a)"] \arrow[d] & H_G^{2j} \arrow[d, "1.4\ (g)"] \\
S^j(\mathfrak{t}^*) \arrow[r, "(a)"] & H_T^{2j}
\end{tikzcd}$$

is commutative. (The left vertical map is induced by the natural map $\mathfrak{t} \hookrightarrow \mathfrak{g}$.)

d) If $G = T$ and $j = 1$ then the map *a)* coincides with Ψ in 1.10 *a)*.

e) The product 1.3 *a)* in $H_G^{\bullet}$ corresponds under *a)* to the natural algebra structure of $S(\mathfrak{g}_r^*)^G$.

1.12. Assume that G^0 is a central torus in G and that E is an irreducible algebraic representation of G (over **C**), trivial on G^0. Let $\mathfrak{g}$ be the Lie algebra of G. We have natural isomorphisms

a) $H_G^{\bullet} \xrightarrow[1.4(g)]{\approx} H_{G^0}^{\bullet}$

b) $H_{\bullet}^{G}(\text{point}, E \otimes E^*) \overset{\approx}{\rightarrow} H_{\bullet}^{G^0}$.

This is shown as follows. By our assumption, the adjoint action of G on $\mathfrak{g}$ is trivial. By 1.11 *a)*, we have

$$H_G^{\bullet} \cong S(\mathfrak{g}^*)^G = S(\mathfrak{g}^*) \cong H_{G^0}^{\bullet}$$

hence *a)*. By *a)*, we have $\dim H_G^j = \dim H_{G^0}^j$ for all j. Using 1.8 *c)* for G and G^0, we deduce that $\dim H_j^G = \dim H_j^{G^0}$ for all j. Hence from the isomorphism $H_j^G \overset{\approx}{\rightarrow} (H_j^{G^0})^{G/G^0}$ we can conclude that G/G^0 acts trivially on $H_j^{G^0}$. Using 1.9 *a)* we have

$$\begin{aligned}
H_{\bullet}^{G}(\text{point}, E \otimes E^*) &\cong (H_{\bullet}^{G^0}(\text{point}, E \otimes E^*))^G \\
&\cong (H_{\bullet}^{G^0} \otimes E \otimes E^*)^G \ \ (\text{since } G^0 \text{ acts trivally on } E \otimes E^*) \\
&\cong H_{\bullet}^{G^0} \otimes (E \otimes E^*)^G \ \ (\text{since } G \text{ acts trivially on } H_{\bullet}^{G^0}) \\
&\cong H_{\bullet}^{G^0} \ \ (\text{since } (E \otimes E^*)^G = \mathbf{C})
\end{aligned}$$

hence *b)*.

1.13. Let X, $\mathscr{L}$ be as in 1.1.

a) If $H_c^{\bullet}(X, \mathscr{L}) = 0$ then $H_{\bullet}^{G}(X, \mathscr{L}) = 0$.

Indeed let Γ be as in 1.1 *a)* with m large. We consider the natural map $f : G\backslash(\Gamma \times X) \rightarrow G\backslash\Gamma$ with fibres X. Our hypothesis implies that $\mathscr{H}^i f_!({}_\Gamma\mathscr{L}) = 0$ for all i. Hence $f_!({}_\Gamma\mathscr{L}) = 0$, so that $H_c^{\bullet}({}_\Gamma X, {}_\Gamma\mathscr{L}) = 0$, and *a)* follows.

2. Cuspidal local systems

2.1. In the remainder of this paper, G denotes a connected reductive algebraic group with Lie algebra $\mathfrak{g}$. Then G acts on $\mathfrak{g}$ by the adjoint action and $G \times \mathbf{C}^*$ acts on $\mathfrak{g}$ by $(g_1, \lambda) : x \to \lambda^{-2} \operatorname{Ad}(g_1)\, x$. If $x \in \mathfrak{g}$, we denote by $Z_G(x)$ the stabilizer of x in G and by $M_G(x)$ the stabilizer of x in $G \times \mathbf{C}^*$. Thus

a) $M_G(x) = \{(g_1, \lambda) \in G \times \mathbf{C}^* \mid \operatorname{Ad}(g_1)\, x = \lambda^2 x\}$.

Assume now that x is nilpotent.

b) By Jacobson-Morozov we can find a homomorphism of algebraic groups $\varphi : SL_2(\mathbf{C}) \to L$ such that $d\varphi \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = x$. We set

$$c)\ \left\{\begin{array}{l} Z_G(\varphi) = \{ g_1 \in G \mid g_1\, \varphi(A)\, g_1^{-1} = \varphi(A),\ \forall\, A \in SL_2(\mathbf{C})\} \\ M_G(\varphi) = \left\{(g_1, \lambda) \in G \times \mathbf{C}^* \,\middle|\, g_1 \varphi(A)\, g_1^{-1} = \varphi\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} A \begin{bmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{bmatrix}\right), \forall\, A \in SL_2(\mathbf{C}) \right\}. \end{array}\right.$$

d) It is known that $Z_G(\varphi)$ (resp. $M_G(\varphi)$) is a maximal reductive subgroup of $Z_G(x)$ (resp. $M_G(x)$).

It is clear that

$$e)\ (g_1, \lambda) \mapsto \left(g_1\, \varphi \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}, \lambda\right)$$

defines an isomorphism of algebraic groups $Z_G(\varphi) \times \mathbf{C}^* \xrightarrow{\approx} M_G(\varphi)$.

From *d)* and *e)* it follows that

f) the embedding $Z_G(x) \hookrightarrow M_G(x)$, $g_1 \mapsto (g_1, 1)$ induces $Z_G(x)/Z_G^0(x) \xrightarrow{\approx} M_G(x)/M_G^0(x)$.

From the existence of φ it follows that the G-orbit of x is also a $G \times \mathbf{C}^*$-orbit; from *f)* we see that a G-local system on a nilpotent G-orbit in $\mathfrak{g}$ is automatically $G \times \mathbf{C}^*$-equivariant.

2.2. Let $\mathscr{E}$ be an irreducible G-local system on a nilpotent G-orbit $\mathcal{O}$ in $\mathfrak{g}$. We say that $\mathscr{E}$ is *cuspidal* if it satisfies the condition *a)* below.

a) For any proper parabolic subalgebra $\mathfrak{p}_1$ of $\mathfrak{g}$ with nil-radical $\mathfrak{n}_1$ and any $y \in \mathfrak{p}_1$ we have

$$H_c^i((y + \mathfrak{n}_1) \cap \mathcal{O}, \mathscr{E}) = 0 \text{ for all } i.$$

This implies that:

b) If $j : \mathcal{O} \hookrightarrow \bar{\mathcal{O}}$ is the inclusion of $\mathcal{O}$ in its closure then $j_* \mathscr{E} = j_! \mathscr{E} \in \mathscr{D}(\bar{\mathcal{O}})$.

The closely related concept of cuspidal local system on a unipotent class of G is defined and studied in [9], [10]. The two concepts are related to each other by the exponential map; thus the results of *loc. cit.* can be transferred to Lie algebras. (See [11].) We shall use freely those results (in particular, the classification) in the case of Lie algebras. The implication *a)* ⇒ *b)* follows from [10].

2.3. We now fix a proper parabolic subgroup P of G with a Levi subgroup L and unipotent radical U. Let $\mathfrak{p}$, $\mathfrak{l}$, $\mathfrak{n}$ be the Lie algebras of P, L, U so that $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$. The concepts in 2.1, 2.2 can be applied to L instead of G.

We fix a nilpotent L-orbit $\mathscr{C}$ in $\mathfrak{l}$ and an irreducible L-equivariant (hence $L \times \mathbf{C}^*$-equivariant) cuspidal local system $\mathscr{L}$ on $\mathscr{C}$. We fix an element $x_0 \in \mathscr{C}$ and a homomorphism of algebraic groups $\varphi_0 : SL_2(\mathbf{C}) \to L$ such that $d\varphi_0 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = x_0$. Let T be the identity component of the centre of L and let $\mathfrak{t}$ be the Lie algebra of T.

Let $W = N_G(T)/L$. This is a finite group. For $w \in W$, $\dot{w}$ will always denote a representative of w in N(L); it acts by conjugation on L, T, hence on $\mathfrak{t}$.

From [9, 9.2 *b)*] we have, for all $w \in W$,

a) $\dot{w} : L \to L$ leaves $\mathscr{C}$ stable and the inverse image of $\mathscr{L}$ under $\dot{w} : \mathscr{C} \to \mathscr{C}$ is isomorphic to $\mathscr{L}$.

From [9, 2.8] and 2.1 *d)* we have

b) $Z^0_L(\varphi_0) = T$.

Using 2.1 *e)*, we see that there is an isomorphism

c) $T \times \mathbf{C}^* \to M^0_L(\varphi_0)$, $(\tau, \lambda) \mapsto \left(\tau\varphi_0 \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}, \lambda\right)$.

Using *c)* and the definition of $M_L(\varphi_0)$ we see that

d) $M^0_L(\varphi_0)$ is contained in the centre of $M_L(\varphi_0)$.

2.4. For any linear form $\alpha : \mathfrak{t} \to \mathbf{C}$ we set $\mathfrak{g}_\alpha = \{ x \in \mathfrak{g} \mid [y, x] = \alpha(y)\, x, \forall y \in \mathfrak{t} \}$. Then $g = \bigoplus_{\alpha \in \mathfrak{t}^*} \mathfrak{g}_\alpha$ since T is a torus. Let $R = \{ \alpha \in \mathfrak{t}^* \mid \alpha \neq 0, \mathfrak{g}_\alpha \neq 0 \}$. It is easy to see that $\mathfrak{n}$ is a sum of $\mathfrak{g}_\alpha$'s ($\alpha \in R$). We define $R^+ = \{ \alpha \in R \mid \mathfrak{g}_\alpha \subset \mathfrak{n} \}$.

Let $P_1, P_2, \ldots, P_m$ be the parabolic subgroups of G which contain strictly P and are minimal with this property. Let L_i be the Levi subgroup of P_i which contains L; let $\mathfrak{p}_i$, $\mathfrak{l}_i$ be the Lie algebras of P_i, L_i. Let $R_i^+ = \{ \alpha \in R^+ \mid \alpha \mid \text{centre}(\mathfrak{l}_i) = 0 \}$; then $\mathfrak{l}_i \cap \mathfrak{n} = \bigoplus_{\alpha \in R_i^+} \mathfrak{g}_\alpha$.

Proposition **2.5.** — R *is a (not necessarily reduced) root system in* $\mathfrak{t}^*$; *it spans the space of linear forms on* $\mathfrak{t}$ *which are zero on the centre of* $\mathfrak{g}$. *Moreover,* W *acts faithfully on* $\mathfrak{t}$, $\mathfrak{t}^*$ *(trivially on the centre of* $\mathfrak{g}$*) as the Weyl group of* R. *The set* R_i^+ *contains a unique element* α_i *such that* $\alpha_i/2 \notin R$ $(1 \leqslant i \leqslant m)$. *The set* $\Pi = \{ \alpha_1, \alpha_2, \ldots, \alpha_m \}$ *is a set of simple roots for* R. *The group* W *is a Coxeter group on generators* $s_1, s_2, \ldots, s_m$ *where* s_i *is the unique non-trivial element of* $N_{L_i}(T)/L$.

This is easily checked, case by case, using the known classification of cuspidal local systems: this forces $\mathfrak{p}$ to be a very special parabolic subalgebra of $\mathfrak{g}$. A slightly weaker statement, namely that the set of indivisible elements of R is a (reduced) root system with Weyl group W is proved in [6, 5.9] and [9, 9.2]. Note that in the generality of [6, 5.9], R is not necessarily a root system.

Proposition **2.6.** — a) T *is a maximal torus of* $Z = Z_G^0(\varphi_0)$.

b) *The natural map* $N_Z(T)/T \to N_G(T)/L = W$ *is an isomorphism.* (*Thus,* W *can be interpreted as the Weyl group of* Z.)

Proof. — *a)* Let S be a torus in Z containing T. Then S is in the centralizer of T, hence $S \subset L$. Thus $S \subset Z_L^0(\varphi_0)$. But the last group is T by 2.3 *b)*. Hence $S = T$.

b) To prove injectivity, it is enough to show that $Z \cap L \subset T$. If $g \in Z \cap L$ then g commutes with all elements of T (since it is in L). But the centralizer of T in Z is T, by *a)*. Hence $g \in T$. To prove surjectivity, it is enough to show that the generators s_i are in the image of our map. This follows from the analogous statement in which G is replaced by L_i. So we assume that P is a maximal parabolic subgroup of G. We can also assume that G is simply connected and even almost simple. In that case, W is of order 2 and by a case by case check (using classification of cuspidal local systems) we see that Z is not a torus so $N_Z(T)/T$ has also order 2. Hence our map, being injective, is automatically surjective.

2.7. For any $\alpha \in R$, $\mathfrak{g}_\alpha$ is L-stable for the adjoint action of L on $\mathfrak{g}$, since T is central in L. In particular, $\mathfrak{g}_\alpha$ can be regarded as an $SL_2(\mathbf{C})$-module via $\varphi_0 : SL_2(\mathbf{C}) \to L$. Let M_d denote a simple $SL_2(\mathbf{C})$-module of dimension d.

Proposition **2.8.** — *Let* $\alpha \in R$. *Then the* $SL_2(\mathbf{C})$-*module* $\mathfrak{g}_\alpha$ *is isomorphic to*:

$$\begin{cases} M_1 \oplus M_3 \oplus \ldots \oplus M_{2p+1}, & \textit{for some } p, \textit{ if } 2\alpha \notin R, \\ M_1 & \textit{if } \alpha/2 \in R, \\ M_2 \oplus M_4 \oplus \ldots \oplus M_{2p}, & \textit{for some } p, \textit{ if } 2\alpha \in R. \end{cases}$$

Proof. — W permutes R; moreover if $\dot{w} \in N(L)$ represents w, then $\mathrm{Ad}(\dot{w}) : \mathfrak{g}_\alpha \mapsto \mathfrak{g}_{w\alpha}$. We can assume that $\dot{w} \in Z_G^0(\varphi_0)$ (see 2.6 *b)*) so that $\mathrm{Ad}(\dot{w})$ commutes with the $SL_2(\mathbf{C})$-action on $\mathfrak{g}$. Hence the $SL_2(\mathbf{C})$-modules $\mathfrak{g}_\alpha$, $\mathfrak{g}_{w\alpha}$ are isomorphic. Since the W-orbits of R_i^+ $(1 \leqslant i \leqslant m)$ cover R, we see that we can assume $\alpha \in R_i^+$. We can then replace G by L_i and assume that P is a maximal parabolic subgroup. We may also assume that G is simply connected and even almost simple. In that case we use the classification of cuspidal local systems. We are in one of the four cases below. (In the following discussion (as well as in 2.10, 2.13) we shall describe nilpotent elements of classical Lie algebras by specifying the sizes of their Jordan blocks; this will be always taken with respect to the standard representation of that Lie algebra: of dimension $2n$ for $\mathfrak{sp}_{2n}$, of dimension n for $\mathfrak{so}_n$.)

Case 1. — $\mathfrak{g} = \mathfrak{sl}_{2n}$, $\mathfrak{l} = \mathfrak{sl}_n \oplus \mathfrak{sl}_n \oplus \mathbf{C} \ni (x'_0, x''_0, 0) = x_0$ where x'_0, x''_0 are regular nilpotent in $\mathfrak{sl}_n$, $(n \geqslant 1)$. In this case, $R^+ = \{\alpha\}$ and $\mathfrak{g}_\alpha \cong M_1 \oplus M_3 \oplus \ldots \oplus M_{2n-1}$ as an $SL_2(\mathbf{C})$-module.

Case 2. — $\mathfrak{g} = \mathfrak{sp}_{2n+2}$, $\mathfrak{l} = \mathfrak{sp}_{2n} \oplus \mathbf{C} \ni (x'_0, 0) = x_0$, where x'_0 is a nilpotent element in $\mathfrak{sp}_{2n}$ with Jordan blocks of sizes $2, 4, 6, \ldots, 2p$ so that $2n = 2 + 4 + 6 + \ldots + 2p$, $(n \geqslant 1)$. In this case, $R^+ = \{\alpha, 2\alpha\}$ and $\mathfrak{g}_{2\alpha} \cong M_1$, $\mathfrak{g}_\alpha \cong M_2 \oplus M_4 \oplus \ldots \oplus M_{2p}$ as an $SL_2(\mathbf{C})$-module.

Case 3. — $\mathfrak{g} = \mathfrak{so}_{n+2}$, $\mathfrak{l} = \mathfrak{so}_n \oplus \mathbf{C} \ni (x'_0, 0) = x_0$ where x'_0 is a nilpotent element in $\mathfrak{so}_n$ with Jordan blocks of sizes $1, 3, 5, \ldots, 2p + 1$ so that $n = 1 + 3 + \ldots + (2p + 1)$ $(n \geqslant 4)$.

In this case, $R^+ = \{\alpha\}$ and $\mathfrak{g}_\alpha \cong M_1 \oplus M_3 \oplus \ldots \oplus M_{2p+1}$ as an $SL_2(\mathbf{C})$-module.

Case 4. — $\mathfrak{g} = \mathfrak{so}_{n+4}$, $\mathfrak{l} = \mathfrak{so}_n \oplus \mathfrak{sl}_2 \oplus \mathbf{C} \ni (x'_0, x''_0, 0)$, where x'_0 is a nilpotent element in $\mathfrak{so}_n$ with Jordan blocks of sizes $1, 5, 9, \ldots, 4p + 1$ or $3, 7, 11, \ldots, 4p + 3$ (so that $n = 1 + 5 + 9 + \ldots + (4p + 1)$ or $n = 3 + 7 + 11 + \ldots + (4p + 3)$) and x''_0 is a regular nilpotent element in $\mathfrak{sl}_2$, $(n \geqslant 3)$.

In this case, $R^+ = \{\alpha, 2\alpha\}$ and $\mathfrak{g}_{2\alpha} \cong M_1$, $g_\alpha = M_2 \oplus M_4 \oplus \ldots \oplus M_{2k}$, as an $SL_2(\mathbf{C})$-module, where $2k = \begin{cases} 4p + 2, \text{ if } n = 1 + 5 + 9 + \ldots + (4p + 1) \\ 4p + 4, \text{ if } n = 3 + 7 + 11 + \ldots + (4p + 3) \end{cases}$.

2.9. Let $R'' \subset \mathfrak{t}^*$ be the set of roots of the reductive Lie algebra $\mathfrak{z} = \mathfrak{g}^{SL_2(\mathbf{C})}$ of $Z^0_G(\varphi_0)$ with respect to its Cartan subalgebra $\mathfrak{t}$. From 2.8, we see that, for $\alpha \in R$,

$$\dim(\mathfrak{g}_\alpha \cap \mathfrak{z}) = \begin{cases} 1 \text{ if } 2\alpha \notin R, \\ 0 \text{ if } 2\alpha \in R. \end{cases}$$

a) It follows that R'' is precisely the set of non-multipliable roots of R.

We define

b) $\mathscr{V}_{RN}$ = unique nilpotent G-orbit in $\mathfrak{g}$ such that $\mathscr{V}_{RN} \cap (x_0 + \mathfrak{n})$ is open dense in $x_0 + \mathfrak{n}$.

Proposition **2.10.** — *Assume that* P *is a maximal parabolic subgroup of* G *and let* α *be the unique simple root of* R. *Let* $a : T \to C^*$ *be the character by which* T *acts on* $\mathfrak{g}_\alpha$ *by the adjoint action, so that* $\alpha = da$. *Let* c *be the integer* $\geqslant 2$ *defined by the conditions*

$$a\,d(x_0)^{c-2} : \mathfrak{n} \to \mathfrak{n} \textit{ is } \neq 0, \qquad a\,d(x_0)^{c-1} : \mathfrak{n} \to \mathfrak{n} \textit{ is } 0.$$

Then

a) $\mathrm{Ker}(a\,d(x_0)^{c-2} : \mathfrak{n} \to \mathfrak{n})$ *is a hyperplane* $\mathscr{H}$ *in* $\mathfrak{n}$.

b) $\mathscr{H}$ *and* $\mathfrak{n}$ *are stable under the action of* $M^0_L(\varphi_0)$ (*restriction of the* $G \times \mathbf{C}^*$*-action* 2.1 *on* $\mathfrak{g}$) *and the induced action of* $M^0_L(\varphi_0)$ *on* $\mathfrak{n}/\mathscr{H}$ *is via the character*

$$(t, \lambda) \longmapsto a(t)\,\lambda^{-c}, \qquad (t, \lambda) \in T \times \mathbf{C}^* \overset{2.3\,(c)}{=\!=\!=} M^0_L(\varphi_0).$$

c) $\mathscr{V}_{RN} \cap (x_0 + \mathfrak{n}) = x_0 + (\mathfrak{n} - \mathscr{H})$.

Proof. — *a)* follows from $x_0 = (d\varphi_0)\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$ and the fact the $\mathrm{SL}_2(\mathbf{C})$-module $\mathfrak{n}$ is multiplicity free (cf. 2.8).

b) Follows from 2.8 and the following property of the irreducible representation $\rho\colon \mathrm{SL}_2(\mathbf{C}) \to \mathrm{Aut}(\mathrm{M}_d)$ of dimension d: the largest non-vanishing power of

$$(d\rho)\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} : \mathrm{M}_d \to \mathrm{M}_d$$

is the $(d-1)$-th power, and $\rho\begin{bmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{bmatrix}$ acts on $\mathrm{M}_d/\ker\left((d\rho)\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\right)^{d-1}$ as multiplication by λ^{-d+1}.

To prove *c)* we again assume that G is simply connected, almost simple, so that we are in one of the cases 1-4 in the proof of 2.8. The following results can be verified in each of those cases by simple computations.

Case 1. — If $x \in x_0 + (\mathfrak{n} - \mathscr{H})$, then $x \in \mathfrak{sl}_{2n}$ is a regular nilpotent element; if $x \in x_0 + \mathscr{H}$, then x is a non-regular nilpotent element.

Case 2. — If $x \in x_0 + (\mathfrak{n} - \mathscr{H})$, then $x \in \mathfrak{sp}_{2n+2}$ has Jordan blocks of sizes $2, 4, 6, \ldots, 2p-2, 2p+2$; if $x \in x_0 + \mathscr{H}$, then x has Jordan blocks of sizes $\leqslant 2p+1$.

Case 3. — If $x \in x_0 + (\mathfrak{n} - \mathscr{H})$, then $x \in \mathfrak{so}_{n+2}$ has Jordan blocks of sizes $1, 3, 5, \ldots, 2p-1, 2p+3$; if $x \in x_0 + \mathscr{H}$, then x has Jordan blocks of sizes $\leqslant 2p+2$.

Case 4. — If $x \in x_0 + (\mathfrak{n} - \mathscr{H})$ then $x \in \mathfrak{so}_{n+4}$ has Jordan blocks of sizes $1, 5, 9, \ldots, 4p-3, 4p+5$ (resp. $3, 7, 11, \ldots, 4p-1, 4p+7$) if $n = 1+5+9+\ldots+(4p+1)$ (resp. $n = 3+7+11+\ldots+(4p+3)$); if $x \in x_0 + \mathscr{H}$, then x has Jordan blocks of sizes $\leqslant 4p+4$ (resp. $\leqslant 4p+6$).

These results imply *c)* immediately.

2.11. The proof of 2.8 shows that the integer c in 2.10 is given explicitly in the cases 1-4 of 2.8 as follows.

Case 1. — $c = 2n$.

Case 2. — $c = 2p+1$.

Case 3. — $c = 2p+2$.

Case 4. — $c = \begin{cases} 4p+3, & \text{if } n = 1+5+\ldots+(4p+1). \\ 4p+5, & \text{if } n = 3+7+\ldots+(4p+3). \end{cases}$

We now drop the hypothesis in 2.10 that P is maximal.

Proposition **2.12.** — *Define integers $c_i \geqslant 2$ $(1 \leqslant i \leqslant m)$ by the requirement*

$$ad(x_0)^{c_i-2} : \mathfrak{l}_i \cap \mathfrak{n} \to \mathfrak{l}_i \cap \mathfrak{n} \textit{ is } \neq 0, \qquad ad(x_0)^{c_i-1} : \mathfrak{l}_i \cap \mathfrak{n} \to \mathfrak{l}_i \cap \mathfrak{n} \textit{ is } 0.$$

Then $c_i = c_j$ whenever s_i, s_j are conjugate in W, $(1 \leqslant i, j \leqslant m)$.

Proof. — If $ws_i w^{-1} = s_j$ then $w(\alpha_i) = \alpha_j$. We have $\mathfrak{l}_i \cap \mathfrak{n} = \mathfrak{g}_{\alpha_i} \oplus \mathfrak{g}_{2\alpha_i}$, $\mathfrak{l}_j \cap \mathfrak{n} = \mathfrak{g}_{\alpha_j} \oplus \mathfrak{g}_{2\alpha_j}$ and if we choose the representative $\dot{w}$ for w in $Z_G^0(\varphi_0)$ (see 2.6 *b)*) then $\mathrm{Ad}(\dot{w}) : \mathfrak{l}_i \cap \mathfrak{n} \to \mathfrak{l}_j \cap \mathfrak{n}$ is an isomorphism of $SL_2(\mathbf{C})$-modules. Since c_i, c_j are determined by the $SL_2(\mathbf{C})$-module structures, they must coincide.

2.13. In this section we assume that G is almost simple, simply connected. We shall indicate in every case that can arise the type of $\mathfrak{g}$, $\mathfrak{l}$, x_0, the type of R, and the values of c_i $(1 \leqslant i \leqslant m)$ corresponding to the various simple roots of R. We also indicate the type of a nilpotent element $x_{RN} \in \mathcal{V}_{RN}$.

a) $\mathfrak{g}$-simple, $\mathfrak{l}$-Cartan subalgebra, $x_0 = 0$, R-usual root system, $c_i = 2$, for all i.

b) $\mathfrak{g} = \mathfrak{sl}_{kn}$, $\mathfrak{l} = \underbrace{\mathfrak{sl}_n \oplus \mathfrak{sl}_n \oplus \ldots \oplus \mathfrak{sl}_n}_{k \text{ copies}} \oplus \mathbf{C}^{k-1}$ $(k \geqslant 2, n \geqslant 2)$, x_0 is regular nilpotent in $\mathfrak{l}$,
x_{RN} is regular nilpotent in $\mathfrak{g}$, R of type A_{k-1}, $c_i = 2k$ for all i.

c) $\mathfrak{g} = \mathfrak{sp}_{2n+2k}$, $\mathfrak{l} = \mathfrak{sp}_{2n} \oplus \mathbf{C}^k$ $(k \geqslant 1, n \geqslant 1)$,
$x_0 = (x'_0, 0) \in \mathfrak{l}$ where x'_0 is a nilpotent element in $\mathfrak{sp}_{2n}$ with Jordan blocks of sizes 2, 4, 6, ..., $2p$ (so that $2n = 2 + 4 + 6 + \ldots + 2p$),
x_{RN} is nilpotent in $\mathfrak{sp}_{2n+2k}$ with Jordan blocks of sizes 2, 4, 6, ..., $2p - 2$, $2p + 2k$,
R of type BC_k, c_i : $\overset{2}{\circ}-\overset{2}{\circ}-\overset{2}{\circ}\cdots\overset{2}{\circ}-\overset{2}{\circ}=\overset{2p+1}{\circ}$

d) $\mathfrak{g} = \mathfrak{so}_{n+2k}$, $\mathfrak{l} = \mathfrak{so}_n \oplus \mathbf{C}^k$ $(k \geqslant 1, n \geqslant 4)$,
$x_0 = (x'_0, 0) \in \mathfrak{l}$ where x'_0 is a nilpotent element in $\mathfrak{so}_n$ with Jordan blocks of sizes 1, 3, 5, ..., $2p + 1$ (so that $n = 1 + 3 + \ldots + (2p + 1)$),
x_{RN} is nilpotent in $\mathfrak{so}_{n+2k}$ with Jordan blocks of sizes 1, 3, 5, ..., $2p - 1$, $2p + 2k + 1$,
R of type B_k, c_i : $\overset{2}{\circ}-\overset{2}{\circ}-\overset{2}{\circ}\cdots\overset{2}{\circ}-\overset{2}{\circ}=\overset{2p+2}{\circ}$

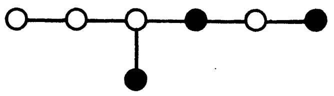
e) $\mathfrak{g} = \mathfrak{so}_{n+4k}$, $\mathfrak{l} = \mathfrak{so}_n \oplus \underbrace{\mathfrak{sl}_2 \oplus \ldots \oplus \mathfrak{sl}_2}_{k \text{ copies}} \oplus \mathbf{C}^k$,
$x_0 = (x'_0, x_0^{(1)}, x_0^{(2)}, \ldots, x_0^{(k)}, 0) \in \mathfrak{l}$ where $x_0^{(i)}$ are regular nilpotent in $\mathfrak{sl}_2$, x'_0 is a nilpotent element in $\mathfrak{so}_n$ with Jordan blocks of sizes 1, 5, 9, ..., $4p + 1$ (resp. 3, 7, 11, ..., $4p + 3$),
x_{RN} is nilpotent in $\mathfrak{so}_{n+4k}$ with Jordan blocks of sizes 1, 5, 9, ..., $4p - 3$, $4p + 4k + 1$ (resp. 3, 7, 11, ..., $4p - 1$, $4p + 4k + 3$),
R of type BC_k,

c_i : $\overset{4}{\circ}-\overset{4}{\circ}-\overset{4}{\circ}\cdots\overset{4}{\circ}-\overset{4}{\circ}=\overset{4p+3}{\circ}$ (resp. c_i : $\overset{4}{\circ}-\overset{4}{\circ}-\overset{4}{\circ}\cdots\overset{4}{\circ}-\overset{4}{\circ}=\overset{4p+5}{\circ}$)

f) g of type E_6, $\mathfrak{l} = \mathfrak{sl}_3 \oplus \mathfrak{sl}_3 \oplus \mathbf{C}^2$,
x_0 is regular nilpotent in $\mathfrak{l}$, x_{RN} is regular nilpotent in $\mathfrak{g}$,

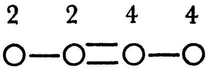
R of type G_2, c_i : $\overset{2}{\circ}\equiv\overset{6}{\circ}$

g) $\mathfrak{g}$ of type E_7, $\mathfrak{l} = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathbf{C}^4$ corresponds to the subgraph



marked with black nodes of the Coxeter graph of $\mathfrak{g}$, x_0 is regular nilpotent in $\mathfrak{l}$, x_{RN} is regular nilpotent in $\mathfrak{g}$,

R of type F_4, c_i :



We now return to the general case.

Proposition **2.14.** — *Let* $x \in \mathscr{V}_{\mathrm{RN}} \cap (x_0 + \mathfrak{n})$ *and* $g \in G$ *be such that* $\mathrm{Ad}(g)\, x \in \mathscr{C} + \mathfrak{n}$· *Then* $g \in P$.

Proof. — We can assume that G is simply connected, almost simple so that we are in one of the cases in 2.13. From 2.13 we see that

a) $Z_{\overline{G}}(x)/Z^0_{\overline{G}}(x) \cong Z_{\overline{L}}(x_0)/Z^0_{\overline{L}}(x_0)$

where $\overline{G}$ is the adjoint group of G and $\overline{L}$ is the image of L in $\overline{G}$. Let $\overline{P}$ be the image of P in $\overline{G}$. From [12] we have natural homomorphisms

$$Z_{\overline{L}}(x_0)/Z^0_{\overline{L}}(x_0) \leftarrow Z_{\overline{P}}(x)/Z^0_{\overline{P}}(x) \rightarrow Z_{\overline{G}}(x)/Z^0_{\overline{G}}(x)$$

the first of which is surjective and the second injective; from *a)* it then follows that both are isomorphisms. In particular, $Z_{\overline{G}}(x) \subset \overline{P}.Z^0_{\overline{G}}(x)$. By [12] we have $Z^0_{\overline{G}}(x) \subset \overline{P}$ so that $Z_{\overline{G}}(x) \subset \overline{P}$ hence $Z_G(x) \subset P$. Now let $g \in G$ be such that $\mathrm{Ad}(g)\, x \in \mathscr{C} + \mathfrak{n}$. Then $\mathrm{Ad}(g)\, x \in \mathscr{V}_{\mathrm{RN}} \cap (\mathscr{C} + \mathfrak{n})$. But P acts transitively on $\mathscr{V}_{\mathrm{RN}} \cap (\mathscr{C} + \mathfrak{n})$ ([12]) so, replacing g by pg ($p \in P$) if necessary, we can assume that $\mathrm{Ad}(g)\, x = x$, i.e. $g \in Z_G(x)$. Hence $g \in P$, as required.

3. The W × W-action

3.1. We shall need the following G × $\mathbf{C}^*$-stable subvarieties of $\mathfrak{g}$:

$\mathscr{V} = \bigcup_{g \in G} \mathrm{Ad}(g)\,(\overline{\mathscr{C}} + \mathfrak{t} + \mathfrak{n})$, where $\overline{\mathscr{C}}$ is the closure of $\mathscr{C}$,

$\mathscr{V}_{\mathrm{RS.}} = \bigcup_{g \in G} \mathrm{Ad}(g)\,(\mathscr{C} + \mathfrak{t}_{\mathrm{reg}} + \mathfrak{n})$, where $\mathfrak{t}_{\mathrm{reg}} = \{ x \in \mathfrak{t} \mid Z_G(x) = L \}$,

$\mathscr{V}_{\mathrm{N}} = \bigcup_{g \in G} \mathrm{Ad}(g)\,(\overline{\mathscr{C}} + \mathfrak{n})$,

$\mathscr{V}_{\mathrm{RN}}$ (see 2.9 *b)*),

$\mathfrak{g}_{\mathrm{N}} = \{ x \in \mathfrak{g} \mid x \text{ nilpotent} \}$.

If X is a closed subset of $\mathfrak{p}$, stable under the adjoint action of P, then $\bigcup_{g \in G} \mathrm{Ad}(g)\, X$ is a closed subset of $\mathfrak{g}$ (since G/P is complete).

Hence

a) $\mathscr{V}, \mathscr{V} - \mathscr{V}_{\mathrm{RS}}, \mathscr{V}_{\mathrm{N}}, \mathscr{V}_{\mathrm{N}} - \mathscr{V}_{\mathrm{RN}}, \mathfrak{g}_{\mathrm{N}}$ are closed subsets of $\mathfrak{g}$.

b) $\mathscr{V}_{\mathrm{RS}}$ is open dense in $\mathscr{V}$, $\mathscr{V}_{\mathrm{RN}}$ is open dense in $\mathscr{V}_{\mathrm{N}}$.

Clearly, $\mathscr{V}_{\mathrm{N}} = \mathfrak{g}_{\mathrm{N}} \cap \mathscr{V}$.

We define

c) $\dot{\mathfrak{g}} = \{(x, g\mathrm{P}) \in \mathfrak{g} \times \mathrm{G}/\mathrm{P} \mid \mathrm{Ad}(g^{-1})\, x \in \mathscr{C} + \mathfrak{t} + \mathfrak{n}\}$.

Note that $\mathrm{G} \times \mathbf{C}^*$ acts on $\dot{\mathfrak{g}}$ as in 0.2 *c)*.

For any $\mathrm{G} \times \mathbf{C}^*$-stable subvariety V of $\mathfrak{g}$ we define

d) $\dot{\mathrm{V}} = \{(x, g\mathrm{P}) \in \dot{\mathfrak{g}} \mid x \in \mathrm{V}\}$.

In particular, $\dot{\mathscr{V}}, \dot{\mathscr{V}}_{\mathrm{RS}}, \dot{\mathscr{V}}_{\mathrm{N}}, \dot{\mathscr{V}}_{\mathrm{RN}}, \dot{\mathfrak{g}}_{\mathrm{N}}$ are well defined $\mathrm{G} \times \mathbf{C}^*$-stable subvarieties of $\dot{\mathfrak{g}}$. We have

e) $\dot{\mathfrak{g}} = \dot{\mathscr{V}}$, $\dot{\mathfrak{g}}_{\mathrm{N}} = \dot{\mathscr{V}}_{\mathrm{N}}$.

Proposition **3.2.** — a) *The map* $\{(x, g\mathrm{L}) \in \mathscr{V}_{\mathrm{RS}} \times \mathrm{G}/\mathrm{L} \mid \mathrm{Ad}(g^{-1})\, x \in \mathscr{C} + \mathfrak{t}_{\mathrm{reg}}\} \xrightarrow{\psi} \dot{\mathscr{V}}_{\mathrm{RS}}$, $(x, g\mathrm{L}) \mapsto (x, g\mathrm{P})$, *is an isomorphism. Hence* $w : (x, g\mathrm{L}) \mapsto (x, g\dot{w}^{-1}\mathrm{L})$ *defines via* ψ *an action of* W *on* $\dot{\mathscr{V}}_{\mathrm{RS}}$. *This makes* $\mathrm{pr}_1 : \dot{\mathscr{V}}_{\mathrm{RS}} \to \mathscr{V}_{\mathrm{RS}}$ *into a finite principal* W*-covering.*

Both $\dot{\mathscr{V}}_{\mathrm{RS}}, \mathscr{V}_{\mathrm{RS}}$ *are smooth, irreducible of dimension*

$$\delta = \dim(\mathfrak{g}/\mathfrak{l}) + \dim(\mathscr{C} + \mathfrak{t}).$$

b) $\dot{\mathscr{V}}_{\mathrm{RN}}$ *is open dense in* $\dot{\mathscr{V}}_{\mathrm{N}}$. *Both* $\dot{\mathscr{V}}_{\mathrm{RN}}, \mathscr{V}_{\mathrm{RN}}$ *are smooth, irreducible of dimension*

$$\delta' = \dim(\mathfrak{g}/\mathfrak{l}) + \dim(\mathscr{C}).$$

c) $\dot{\mathscr{V}}$ *is smooth and is open dense in*

$$\hat{\mathscr{V}} = \{(x, g\mathrm{P}) \in \mathfrak{g} \times \mathrm{G}/\mathrm{P} \mid \mathrm{Ad}(g^{-1})\, x \in \bar{\mathscr{C}} + \mathfrak{t} + \mathfrak{n}\};$$

$\mathscr{V}$, $\dot{\mathscr{V}}$ *and* $\hat{\mathscr{V}}$ *are irreducible of dimension* δ.

d) $\mathrm{pr}_1 : \dot{\mathscr{V}}_{\mathrm{RN}} \to \mathscr{V}_{\mathrm{RN}}$ *is an isomorphism.*

Proof. — a) is a Lie algebra analogue of [9, 4.3 *c)*] and is proved in the same way. *b)* is obvious except for the formula for $\dim \mathscr{V}_{\mathrm{RN}}$. But it follows from [9] that $\mathrm{pr}_1 : \dot{\mathscr{V}}_{\mathrm{RN}} \to \mathscr{V}_{\mathrm{RN}}$ is a finite covering, hence $\dim \mathscr{V}_{\mathrm{RN}} = \dim \dot{\mathscr{V}}_{\mathrm{RN}}$. *c)* is obvious. By 2.14, the fibre of $\mathrm{pr}_1 : \dot{\mathscr{V}}_{\mathrm{RN}} \to \mathscr{V}_{\mathrm{RN}}$ at $x \in \mathscr{V}_{\mathrm{RN}} \cap (x_0 + \mathfrak{n})$ is (x, P). Since G acts transitively on $\mathscr{V}_{\mathrm{RN}}$, *d)* follows.

3.3. We now define

a) $\ddot{\mathfrak{g}} = \{(x, g\mathrm{P}, g'\mathrm{P}) \in \mathfrak{g} \times \mathrm{G}/\mathrm{P} \times \mathrm{G}/\mathrm{P} \mid (x, g\mathrm{P}) \in \dot{\mathfrak{g}}, (x, g'\mathrm{P}) \in \dot{\mathfrak{g}}\}$.

Then $\ddot{\mathfrak{g}}$ is a closed subvariety of $\dot{\mathfrak{g}} \times \dot{\mathfrak{g}}$, via $(x, g\mathrm{P}, g'\mathrm{P}) \to ((x, g\mathrm{P}), (x, g'\mathrm{P}))$, hence it inherits a $\mathrm{G} \times \mathbf{C}^*$-action (from the diagonal $\mathrm{G} \times \mathbf{C}^*$ action on $\dot{\mathfrak{g}} \times \dot{\mathfrak{g}}$) and two projections $\mathrm{pr}_{12}, \mathrm{pr}_{13} : \ddot{\mathfrak{g}} \to \dot{\mathfrak{g}}$.

If V is any $G \times \mathbf{C}^*$-stable subvariety of $\mathfrak{g}$, we define

b) $\ddot{V} = \{(x, gP, g'P) \in \ddot{\mathfrak{g}} \mid x \in V\}$.

This is a $G \times \mathbf{C}^*$-stable subvariety of $\ddot{\mathfrak{g}}$ and it has two projections $\mathrm{pr}_{12}, \mathrm{pr}_{13} : \ddot{V} \to \dot{V}$.

c) Let Ω be a locally closed subvariety of G which is a union of $P - P$ double cosets.

We define

d) $\ddot{V}^{\Omega} = \{(x, gP, g'P) \in \ddot{V} \mid g^{-1} g' \in \Omega\}$.

This is a $G \times \mathbf{C}^*$-stable subvariety of $\ddot{V}$.

A $P - P$ double coset in G is said to be *good* if it is of the form $\Omega(w) = P\dot{w}P$ for some $w \in W$. We have $\Omega(w) \neq \Omega(w')$ for $w \neq w'$. All other $P - P$ double cosets are said to be *bad*.

We have a finite partition

e) $\ddot{V} = \bigcup\limits_{\Omega} \ddot{V}^{\Omega}$

into locally coset subsets (Ω runs over the $P - P$ double cosets in G).

From 3.1 *e)* it follows that

f) $\ddot{\mathfrak{g}} = \ddot{\mathscr{V}}$, $\ddot{\mathfrak{g}}_{N} = \ddot{\mathscr{V}}_{N}$.

g) If $w \in W$, then $\ddot{\mathfrak{g}}^{\Omega(w)} = \ddot{\mathscr{V}}^{\Omega(w)}$ is smooth, irreducible of dimension δ (see 3.2 *a)*), and $\ddot{\mathfrak{g}}^{\Omega(w)} = \ddot{\mathscr{V}}^{\Omega(w)}$ is smooth, irreducible of dimension δ' (see 3.2 *b)*).

Indeed, we have a fibration $\ddot{\mathfrak{g}}^{\Omega(w)} \to G/P \cap \dot{w}P\dot{w}^{-1}$, $(x, gP, g'P) \mapsto g(P \cap \dot{w}P\dot{w}^{-1})$, with fibres

$\cong (\mathscr{C} + \mathfrak{t} + \mathfrak{n}) \cap (\mathrm{Ad}(\dot{w})\, \mathscr{C} + \mathfrak{t} + \mathrm{Ad}(\dot{w})\, \mathfrak{n})$
$= (\mathscr{C} + \mathfrak{t} + \mathfrak{n}) \cap (\mathscr{C} + \mathfrak{t} + \mathrm{Ad}(\dot{w})\, \mathfrak{n})$ (see 2.3 *a)*)
$= \mathscr{C} + \mathfrak{t} + (\mathfrak{n} \cap \mathrm{Ad}(\dot{w})\, \mathfrak{n})$.

(The same argument applies to $\ddot{\mathfrak{g}}_{N}^{\Omega(w)}$.)

h) If Ω is a bad $P - P$ double coset, then

$$\dim \ddot{\mathfrak{g}}^{\Omega} = \dim \ddot{\mathscr{V}}^{\Omega} < \delta \quad \text{and} \quad \dim \ddot{\mathfrak{g}}_{N}^{\Omega} = \dim \ddot{\mathscr{V}}_{N}^{\Omega} \leqslant \delta'.$$

(This can be deduced from [9, 1.2].)

i) If Ω is a bad $P - P$ double coset, then $\ddot{\mathscr{V}}_{RS}^{\Omega} = \emptyset$. If $w \in W$, then $\ddot{\mathscr{V}}_{RS}^{\Omega\, w)}$ is a smooth irreducible variety of dimension δ and $\ddot{\mathscr{V}}_{RS} = \bigcup\limits_{w} \ddot{\mathscr{V}}_{RS}^{\Omega(w)}$ is the decomposition of $\ddot{\mathscr{V}}_{RS}$ into connected components. Also, $\ddot{\mathscr{V}}_{RS}^{\Omega(w)}$ is open dense in $\ddot{\mathscr{V}}^{\Omega(w)}$.

(This follows from 3.2 *a)* and *g)*.)

j) If Ω is a $P - P$ double coset other than P then $\ddot{\mathscr{V}}_{RN}^{\Omega} = \emptyset$. Thus, $\ddot{\mathscr{V}}_{RN} = \ddot{\mathscr{V}}_{RN}^{P}$. This is open dense in $\ddot{\mathscr{V}}_{N}^{P}$.

(This follows from 3.2 *d)*, *b)*.)

From the results above, we deduce:

k) $\dim \ddot{\mathfrak g} = \dim \ddot{\mathscr V} = \dim \ddot{\mathscr V}_{\mathrm{RS}} = \delta$, $\dim \ddot{\mathfrak g}_{\mathrm N} = \dim \ddot{\mathscr V}_{\mathrm N} = \dim \ddot{\mathscr V}_{\mathrm{RN}} = \delta'$,

l) $\dim(\ddot{\mathscr V} - \ddot{\mathscr V}_{\mathrm{RS}}) < \delta$, $\dim(\ddot{\mathscr V}^{\mathrm P}_{\mathrm N} - \ddot{\mathscr V}_{\mathrm{RN}}) < \delta'$.

3.4. We define a local system $\dot{\mathscr L}$ on $\dot{\mathfrak g}$ by the requirement $f_2^* \dot{\mathscr L} = f_1^* \dot{\mathscr L}$ in the diagram

$$\mathscr C \xleftarrow{f_1} \{(x, g) \in \mathfrak g \times \mathrm G \mid \mathrm{Ad}(g^{-1})\, x \in \mathscr C + \mathfrak t + \mathfrak n\} \xrightarrow{f_2} \dot{\mathfrak g}$$

where $f_1(x, g) = \mathrm{pr}_{\mathscr C}(\mathrm{Ad}(g^{-1})\, x)$, $f_2(x, g) = (x, g\mathrm P)$.

Note that $\dot{\mathscr L}$ is well defined since $\mathscr L$ is L-equivariant. Moreover, $\dot{\mathscr L}$ is $\mathrm G \times \mathbf C^*$-equivariant since $\mathscr L$ is $\mathrm L \times \mathbf C^*$-equivariant and f_1, f_2 are $\mathrm G \times \mathbf C^*$-equivariant for the action of $\mathrm G \times \mathbf C^*$ on $\mathscr C$ given by $(g_1, \lambda) : x \mapsto \lambda^{-2} x$ and the action

$$(g_1, \lambda) : (x, g) \mapsto (\lambda^{-2}\, \mathrm{Ad}(g_1)\, x, g_1 g).$$

Similarly, replacing $\mathscr L$ by $\mathscr L^*$ in the definition of $\dot{\mathscr L}$, we get a local system $(\mathscr L^*)^\bullet$; it is the same as the dual $\dot{\mathscr L}^*$ of $\dot{\mathscr L}$.

Let $\quad \mathrm K = (\mathrm{pr}_1)_!\, \dot{\mathscr L}^* \in \mathscr D\mathfrak g, \quad (\mathrm{pr}_1 : \dot{\mathfrak g} \to \mathfrak g),$

$\quad\quad = i_!(\mathrm K_1) \quad (i : \mathscr V \hookrightarrow \mathfrak g)$

where $\mathrm K_1 = (\mathrm{pr}_1)_!\, \dot{\mathscr L}^* \in \mathscr D\mathscr V$, $(\mathrm{pr}_1 : \dot{\mathscr V} = \dot{\mathfrak g} \to \mathscr V)$. We have the following result.

a) $\mathrm K[\delta]$ *is a perverse sheaf on* $\mathfrak g$. *More precisely,* $\mathrm K_1$ *is the intersection cohomology complex on* $\mathscr V$ *defined by the local system* $(\mathrm{pr}_1)_!\, \dot{\mathscr L}^*$ *on* $\mathscr V_{\mathrm{RS}}$, $(\mathrm{pr}_1 : \dot{\mathscr V}_{\mathrm{RS}} \to \mathscr V_{\mathrm{RS}}$, *see* 3.2 *a)*).

To prove this, we need the following concept. A morphism $f : \mathrm Y \to \mathrm Y'$ is said to be *small* if Y, Y′ are irreducible varieties of the same dimension d, $\dim(\mathrm Y \times_{\mathrm Y'} \mathrm Y) = d$ and any irreducible component of dimension d of $\mathrm Y \times_{\mathrm Y'} \mathrm Y$ is mapped by f onto a dense subset of Y′.

This concept is inspired by (and is more general than) the concept of smallness of Goresky-MacPherson [4]: they require in addition that Y is smooth and f is proper and generically $1 - 1$.

In the case where $\mathrm P = \mathrm B$ and $\mathscr L = \mathbf C$, the proof of a) is based on the observation of [8] that $\mathrm{pr}_1 : \dot{\mathscr V} \to \mathscr V$ is small and proper. (In that case, $\mathscr V = \mathfrak g$.) In the general case, $\mathrm{pr}_1 : \dot{\mathscr V} \to \mathscr V$ is still small (by 3.2 *c)*, 3.3 *g)*, 3.3 *h)*, 3.3 *i)*) but is not necessarily proper; this defect will be compensated by the cuspidality of $\mathscr L$. From 2.2 *b)* we have $j_!\, \mathscr L = j_*\, \mathscr L \in \mathscr D\bar{\mathscr C}$, where $j : \mathscr C \hookrightarrow \bar{\mathscr C}$ is the inclusion. It follows immediately that $j_!\, \dot{\mathscr L} = \hat{j}_*\, \dot{\mathscr L} \in \mathscr D\hat{\mathscr V}$ where $\hat{j} : \dot{\mathscr V} \hookrightarrow \hat{\mathscr V}$ is the inclusion (see 3.2 *c)*). Let

$$\mathrm K' = (\mathrm{pr}_1)_!\, \dot{\mathscr L} \in \mathscr D(\mathfrak g), \quad (\mathrm{pr}_1 : \dot{\mathfrak g} \to \mathfrak g) \quad \text{i.e. } \mathrm K' = i_!\, \mathrm K'_1,$$

where $\quad \mathrm K'_1 = (\mathrm{pr}_1)_!\, \dot{\mathscr L} \in \mathscr D\mathscr V, \quad (\mathrm{pr}_1 : \dot{\mathscr V} \to \mathscr V).$

We shall denote the first projection $\hat{\mathscr V} \to \mathscr V$ by $\hat{\mathrm{pr}}_1$; it is a *proper* map, hence $(\hat{\mathrm{pr}}_1)_* = (\hat{\mathrm{pr}}_1)_!$. We have $\mathrm{pr}_1 = \hat{\mathrm{pr}}_1 \circ \hat{j} : \dot{\mathscr V} \to \mathscr V$ hence

$$\mathrm K'_1 = (\mathrm{pr}_1)_!\, \dot{\mathscr L} = (\hat{\mathrm{pr}}_1)_!\, (\check{j}_!\, \dot{\mathscr L}) = (\hat{\mathrm{pr}}_1)_*\, (\hat{j}_!\, \dot{\mathscr L})$$
$$= (\hat{\mathrm{pr}}_1)_*\, (\hat{j}_*\, \dot{\mathscr L}) = (\mathrm{pr}_1)_*\, \dot{\mathscr L}.$$

We denote by D the Verdier duality. We have $\mathrm{DK}_1 = \mathrm{D}((\mathrm{pr}_1)_! \,\dot{\mathscr{L}}) = (\mathrm{pr}_1)_* (\mathrm{D}\dot{\mathscr{L}})$.

Here $\dot{\mathscr{L}}$ is a local system on the smooth irreducible variety $\dot{\mathscr{V}}$ of dimension δ (3.2 *c)*) hence $\mathrm{D}\dot{\mathscr{L}} = \dot{\mathscr{L}}[2\,\delta]$. Thus

$$\mathrm{DK}_1 = (\mathrm{pr}_1)_* \,\dot{\mathscr{L}}[2\,\delta] = \mathrm{K}_1'[2\,\delta].$$

By the definition of an intersection cohomology complex, to verify *a)* it is enough to verify that

$$\dim \operatorname{supp} \mathscr{H}^i \,\mathrm{K}_1 < \delta - i \quad \text{for all } i > 0,$$

and

$$\dim \operatorname{supp} \mathscr{H}^i \,\mathrm{K}_1' < \delta - i \quad \text{for all } i > 0.$$

But these inequalities follow immediately from the fact that $\mathrm{pr}_1 : \dot{\mathscr{V}} \to \mathscr{V}$ is small. Thus, *a)* is proved.

Our present objective is to define an action of W on K.

The definition of such an action was given in [8] in the case where $\mathrm{P} = \mathrm{B}$, and in [9] in general; in these references instead of complexes on $\mathfrak{g}$, we considered complexes on G. The case of $\mathfrak{g}$ is entirely similar, but for the sake of completeness we shall explain it here.

Let $w \in \mathrm{W}$. The local system $\dot{\mathscr{L}}^*$ on $\dot{\mathscr{V}}_{\mathrm{RS}}$ is irreducible (since $\mathscr{L}$ is irreducible on $\mathscr{C}$) and is isomorphic to its inverse image $w^*(\dot{\mathscr{L}}^*)$ under $w : \dot{\mathscr{V}}_{\mathrm{RS}} \to \dot{\mathscr{V}}_{\mathrm{RS}}$ in 3.2 *a)* (due to 2.3 *a)*). Choose an isomorphism $\varphi_w^0 : \dot{\mathscr{L}}^* \xrightarrow{\sim} w^* \,\dot{\mathscr{L}}^*$ (of local systems on $\dot{\mathscr{V}}_{\mathrm{RS}}$). This gives isomorphisms on stalks $\varphi_{w,\varepsilon}^0 : \dot{\mathscr{L}}_\varepsilon^* \xrightarrow{\sim} \dot{\mathscr{L}}_{w\varepsilon}^*$ for all $\varepsilon \in \dot{\mathscr{V}}_{\mathrm{RS}}$. It is clear that for $w, w' \in \mathrm{W}$ we have

$$\varphi^0_{w,\, w'\varepsilon} \circ \varphi^0_{w',\varepsilon} = c_{w,\, w'}\, \varphi^0_{ww',\varepsilon}$$

where $c_{w,\,w'} \in \mathbf{C}^*$ is independent of ε. Taking direct sum over all $v \in \mathrm{W}$, we obtain isomorphisms

$$\bigoplus_{v \in \mathrm{W}} \dot{\mathscr{L}}^*_{v\varepsilon} \xrightarrow{\sim} \bigoplus_{v \in \mathrm{W}} \dot{\mathscr{L}}^*_{wv\varepsilon} = \bigoplus_{v' \in \mathrm{W}} \dot{\mathscr{L}}^*_{v'\varepsilon}$$

or, equivalently,

$$((\mathrm{pr}_1)_* \,\dot{\mathscr{L}}^*)_y \xrightarrow{\sim} ((\mathrm{pr}_1)_* \,\dot{\mathscr{L}}^*)_y, \qquad (\mathrm{pr}_1 : \dot{\mathscr{V}}_{\mathrm{RS}} \to \mathscr{V}_{\mathrm{RS}}, y \in \mathscr{V}_{\mathrm{RS}}).$$

(See 3.2 *a)*).

This is induced on stalks by an isomorphism $(\mathrm{pr}_1)_* \,\dot{\mathscr{L}}^* \xrightarrow{\sim} (\mathrm{pr}_1)_* \,\dot{\mathscr{L}}^*$ of local systems on $\mathscr{V}_{\mathrm{RS}}$.

Using *a)*, this extends uniquely to an isomorphism $\psi_w^0 : \mathrm{K} \xrightarrow{\sim} \mathrm{K}$ in $\mathscr{D}\mathfrak{g}$. We have $\psi^0_w \cdot \psi^0_{w'} = c_{w,\,w'}\, \psi^0_{ww'}$. To normalize φ^0_w and ψ^0_w, we shall use the following result:

b) $\mathscr{H}^0\,\mathrm{K} \mid \mathscr{V}_{\mathrm{RN}}$ is a non-zero, irreducible local system.

Now ψ^0_w induces an automorphism of the local system $\mathscr{H}^0\,\mathrm{K} \mid \mathscr{V}_{\mathrm{RN}}$ which by *b)* must be the multiplication by a scalar $\mu_w \in \mathbf{C}^*$.

Replacing φ^0_w, ψ^0_w by $\mu_w^{-1}\,\varphi^0_w = \varphi_w$, $\mu_w^{-1}\,\psi^0_w = \psi_w$, we can assume $\mu_w = 1$.

Thus there is a unique normalization φ^0_w, ψ^0_w (denoted φ_w, ψ_w) which induces the identity map on $\mathscr{H}^0 \mathrm{K} \mid \mathscr{V}_{\mathrm{RN}}$. We then have $\psi_w \psi_{w'} = \psi_{ww'}$, so that $w \mapsto \psi_w$ is a homomorphism

c) $\mathrm{W} \to \mathrm{Aut}_{\mathscr{D}\mathfrak{g}} \mathrm{K}$.

This is the W-action on K we wanted to construct.

3.5. We fix an integer $m \geqslant 1$ and a smooth irreducible variety Γ with a given free $\mathrm{G} \times \mathbf{C}^*$-action such that

$$\mathrm{H}^i(\Gamma) = 0 \quad \text{for all } i \in [1, m].$$

As in 1.1 (with $\mathrm{G} \times \mathbf{C}^*$ instead of G) for any $\mathrm{G} \times \mathbf{C}^*$-variety Y we shall write ${}_{\Gamma}\mathrm{Y}$ instead of $(\mathrm{G} \times \mathbf{C}^*)\backslash(\Gamma \times \mathrm{Y})$; if $\mathscr{E}$ is a $\mathrm{G} \times \mathbf{C}^*$-equivariant local system on Y, we write ${}_{\Gamma}\mathscr{E}$ for the corresponding local system on ${}_{\Gamma}\mathrm{Y}$; if $f : \mathrm{Y} \to \mathrm{Y}'$ is a $\mathrm{G} \times \mathbf{C}^*$-equivariant morphism between $\mathrm{G} \times \mathbf{C}^*$-varieties we shall write ${}_{\Gamma}f : {}_{\Gamma}\mathrm{Y} \to {}_{\Gamma}\mathrm{Y}'$ for the morphism induced by $\mathrm{Id} \times f : \Gamma \times \mathrm{Y} \to \Gamma \times \mathrm{Y}'$.

In particular, ${}_{\Gamma}\mathfrak{g}$, ${}_{\Gamma}\dot{\mathfrak{g}}$, ${}_{\Gamma}(\mathfrak{g} \times \mathfrak{g})$, ${}_{\Gamma}(\dot{\mathfrak{g}} \times \dot{\mathfrak{g}})$, ${}_{\Gamma}\dot{\mathscr{L}}$, ${}_{\Gamma}\dot{\mathscr{L}}^*$ are well defined. ($\mathrm{G} \times \mathbf{C}^*$ acts on $\mathfrak{g} \times \mathfrak{g}$, $\dot{\mathfrak{g}} \times \dot{\mathfrak{g}}$ diagonally.)

Consider the commutative diagram

a)
$$\begin{tikzcd}
{}_{\Gamma}\dot{\mathfrak{g}} \arrow[d, "{}_{\Gamma}\pi"'] & \Gamma \times \dot{\mathfrak{g}} \arrow[l, "\mu"'] \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{Id} \times \pi"'] & \dot{\mathfrak{g}} \arrow[d, "\pi"] \\
{}_{\Gamma}\mathfrak{g} & \Gamma \times \mathfrak{g} \arrow[l, "\mu'"'] \arrow[r, "\mathrm{pr}_2"] & \mathfrak{g}
\end{tikzcd}$$

where μ, μ' are the canonical maps (principal $\mathrm{G} \times \mathbf{C}^*$-bundles) and $\pi = \mathrm{pr}_1$. Let

$$\begin{aligned} {}_{\Gamma}\mathrm{K} &= ({}_{\Gamma}\pi)_! \, ({}_{\Gamma}\dot{\mathscr{L}}^*) \in \mathscr{D}({}_{\Gamma}\mathfrak{g}) \\ \widetilde{\mathrm{K}} &= (\mathrm{Id} \times \pi)_! \, \mathrm{pr}_2^*(\dot{\mathscr{L}}^*) \in \mathscr{D}(\Gamma \times \mathfrak{g}). \end{aligned}$$

It is clear that

$$\mu'^*({}_{\Gamma}\mathrm{K}) = \widetilde{\mathrm{K}} = \mathrm{pr}_2^*(\mathrm{K}).$$

Now μ' and $\mathrm{pr}_2 : \Gamma \times \mathfrak{g} \to \mathfrak{g}$ are smooth morphisms with connected fibres. Using [1, 4.2.5] and the fact that $\mathrm{K}[\delta]$ is perverse we deduce that

b) $\widetilde{\mathrm{K}} = \mathrm{pr}_2^*(\mathrm{K})$, suitably shifted, is perverse and pr_2^* defines

$$\mathrm{End}_{\mathscr{D}(\mathfrak{g})}(\mathrm{K}) \xrightarrow{\sim} \mathrm{End}_{\mathscr{D}(\Gamma \times \mathfrak{g})}(\mathrm{pr}_2^* \mathrm{K}) = \mathrm{End}_{\mathscr{D}(\Gamma \times \mathfrak{g})}(\widetilde{\mathrm{K}});$$

c) ${}_{\Gamma}\mathrm{K}$, suitably shifted, is perverse and μ'^* defines

$$\mathrm{End}_{\mathscr{D}({}_{\Gamma}\mathfrak{g})}({}_{\Gamma}\mathrm{K}) \xrightarrow{\sim} \mathrm{End}_{\mathscr{D}(\Gamma \times \mathfrak{g})}(\mu'^* \, {}_{\Gamma}\mathrm{K}) = \mathrm{End}_{\mathscr{D}(\Gamma \times \mathfrak{g})}(\widetilde{\mathrm{K}}).$$

Combining *b)* and *c)* we find a canonical isomorphism

$$\mathrm{End}_{\mathscr{D}\mathfrak{g}}(\mathrm{K}) \xrightarrow{\approx} \mathrm{End}_{\mathscr{D}({}_{\Gamma}\mathfrak{g})}({}_{\Gamma}\mathrm{K}).$$

Composing this with 3.4 *c)* we obtain a homomorphism

d) $W \to \mathrm{Aut}_{\mathscr{D}({}_\Gamma\mathfrak{g})}({}_\Gamma K)$.

We can perform the same construction, replacing $\mathscr{L}$ by $\mathscr{L}^*$; then ${}_\Gamma K$ is replaced by ${}_\Gamma K' = ({}_\Gamma\pi)_!\,({}_\Gamma\dot{\mathscr{L}})$ and *d)* becomes

e) $W \to \mathrm{Aut}_{\mathscr{D}({}_\Gamma\mathfrak{g})}({}_\Gamma K')$.

Let $w, w' \in W$. We denote by $w^{(1)} : {}_\Gamma K \to {}_\Gamma K$ the automorphism of ${}_\Gamma K$ corresponding to w under *d)* and by $w'^{(2)} : {}_\Gamma K' \to {}_\Gamma K'$ the automorphism of ${}_\Gamma K'$ corresponding to w' under *e)*. Then $w^{(1)}$, $w'^{(2)}$ define $w^{(1)} \boxtimes w'^{(2)} \in \mathrm{Aut}_{\mathscr{D}({}_\Gamma\mathfrak{g} \times {}_\Gamma\mathfrak{g})}({}_\Gamma K \boxtimes {}_\Gamma K')$.

It is clear that $(w, w') \mapsto w^{(1)} \boxtimes w'^{(2)}$ is a homomorphism

f) $W \times W \to \mathrm{Aut}_{\mathscr{D}({}_\Gamma\mathfrak{g} \times {}_\Gamma\mathfrak{g})}({}_\Gamma K \boxtimes {}_\Gamma K')$.

We have an embedding

$$i : {}_\Gamma(\mathfrak{g} \times \mathfrak{g}) \hookrightarrow {}_\Gamma\mathfrak{g} \times {}_\Gamma\mathfrak{g}$$

induced by $(\gamma, x, x') \mapsto ((\gamma, x), (\gamma, x'))$, $\gamma \in \Gamma$, $x \in \mathfrak{g}$, $x' \in \mathfrak{g}'$.

Let ${}_\Gamma K \,\tilde{\boxtimes}\, {}_\Gamma K' = i^*({}_\Gamma K \boxtimes {}_\Gamma K') \in \mathscr{D}({}_\Gamma(\mathfrak{g} \times \mathfrak{g}))$; i^* induces a homomorphism

$$\mathrm{End}_{\mathscr{D}({}_\Gamma\mathfrak{g} \times {}_\Gamma\mathfrak{g})}({}_\Gamma K \boxtimes {}_\Gamma K') \to \mathrm{End}_{\mathscr{D}({}_\Gamma(\mathfrak{g} \times \mathfrak{g}))}({}_\Gamma K \,\tilde{\boxtimes}\, {}_\Gamma K').$$

Composing this with *f)* we obtain a homomorphism

g) $W \times W \to \mathrm{Aut}_{(\mathscr{D}_{\Gamma}(\mathfrak{g} \times \mathfrak{g}))}({}_\Gamma K \,\tilde{\boxtimes}\, {}_\Gamma K')$.

Let

h) $\ddot{\mathscr{L}} = \mathscr{L} \boxtimes \dot{\mathscr{L}}$, $\ddot{\mathscr{L}}^* = \mathscr{L}^* \boxtimes \dot{\mathscr{L}}$.

These are $G \times \mathbf{C}^*$-equivariant local system on $\dot{\mathfrak{g}} \times \dot{\mathfrak{g}}$ hence they give rise to local systems ${}_\Gamma\ddot{\mathscr{L}}$, ${}_\Gamma\ddot{\mathscr{L}}^*$ on ${}_\Gamma(\dot{\mathfrak{g}} \times \dot{\mathfrak{g}})$.

From the definitions, it follows easily that

$${}_\Gamma K \,\tilde{\boxtimes}\, {}_\Gamma K' = {}_\Gamma(\pi \times \pi)_!\,({}_\Gamma\ddot{\mathscr{L}}^*)$$

so that *g)* can be regarded as a homomorphism

g') $W \times W \to \mathrm{Aut}_{\mathscr{D}({}_\Gamma(\mathfrak{g} \times \mathfrak{g}))}({}_\Gamma(\pi \times \pi)_!\,({}_\Gamma\ddot{\mathscr{L}}^*))$.

3.6. We shall denote the restrictions of $\ddot{\mathscr{L}}$, $\ddot{\mathscr{L}}^*$ (see 3.5 *h)*) to $\ddot{\mathfrak{g}}$, or more generally, to $\ddot{V}$ (for V a $G \times \mathbf{C}^*$-stable subvariety of $\mathfrak{g}$) again by $\ddot{\mathscr{L}}$, $\ddot{\mathscr{L}}^*$.

The diagonal inclusion $h : V \hookrightarrow \mathfrak{g} \times \mathfrak{g}$ induces ${}_\Gamma h : {}_\Gamma V \hookrightarrow {}_\Gamma(\mathfrak{g} \times \mathfrak{g})$ and $({}_\Gamma h)^*$ defines a homomorphism

$$\mathrm{End}_{\mathscr{D}({}_\Gamma(\mathfrak{g} \times \mathfrak{g}))}({}_\Gamma(\pi \times \pi)_!\,{}_\Gamma\ddot{\mathscr{L}}^*) \to \mathrm{End}_{\mathscr{D}({}_\Gamma V)}({}_\Gamma h)^*\,({}_\Gamma(\pi \times \pi)_!\,{}_\Gamma\ddot{\mathscr{L}}^*)$$
$$= \mathrm{End}_{\mathscr{D}({}_\Gamma V)}(({}_\Gamma \mathrm{pr}_1)_!\,{}_\Gamma\ddot{\mathscr{L}}^*)$$

where $\mathrm{pr}_1 : \ddot{V} \to V$ is the first projection.

Composing with 3.5 *g)* we find a homomorphism

a) $W \times W \to \mathrm{Aut}_{\mathscr{D}({}_\Gamma V)}(({}_\Gamma \mathrm{pr}_1)_!\,({}_\Gamma\ddot{\mathscr{L}}^*))$.

3.7. We want to give an alternative definition for the restrictions of the homomorphism 3.6 *a)* to $W \times \{e\}$ and $\{e\} \times W$. We have a cartesian diagram

$$\begin{tikzcd}
\ddot{V} \arrow[r, "p_{12}"] \arrow[d, "p_{13}"'] & \dot{V} \arrow[d, "p_1"] \\
\dot{V} \arrow[r, "p_1"'] & V
\end{tikzcd}$$

where p_1, p_{12}, p_{13} are the obvious projections. This induces a cartesian diagram

$$\begin{tikzcd}
{}_\Gamma\ddot{V} \arrow[r, "{}_\Gamma p_{12}"] \arrow[d, "{}_\Gamma p_{13}"'] & {}_\Gamma\dot{V} \arrow[d, "{}_\Gamma p_1"] \\
{}_\Gamma\dot{V} \arrow[r, "{}_\Gamma p_1"] & {}_\Gamma V
\end{tikzcd}$$

Let h' be the composition ${}_\Gamma\dot{V} \xrightarrow{{}_\Gamma p_1} {}_\Gamma V \hookrightarrow {}_\Gamma\mathfrak{g}$. Then $h'^*({}_\Gamma K) = ({}_\Gamma p_{13})_!\, ({}_\Gamma p_{12}^*({}_\Gamma\dot{\mathscr{L}}^*))$ and h'^* defines a homomorphism

a) $\mathrm{End}_{\mathscr{D}({}_\Gamma\mathfrak{g})}({}_\Gamma K) \to \mathrm{End}_{\mathscr{D}({}_\Gamma\dot{V})}(h'^*({}_\Gamma K)) = \mathrm{End}_{\mathscr{D}({}_\Gamma\dot{V})}(({}_\Gamma p_{13})_!\, ({}_\Gamma p_{12}^*({}_\Gamma\dot{\mathscr{L}}^*)))$.

Tensor product with the local system ${}_\Gamma\dot{\mathscr{L}}$ is a functor $\mathscr{D}({}_\Gamma\dot{V}) \to \mathscr{D}({}_\Gamma\dot{V})$ so it defines

b) $$\begin{aligned}\mathrm{End}_{\mathscr{D}({}_\Gamma\dot{V})}(({}_\Gamma p_{13})_!\, ({}_\Gamma p_{12}^*({}_\Gamma\dot{\mathscr{L}}^*))) \to \mathrm{End}_{\mathscr{D}({}_\Gamma\dot{V})}(({}_\Gamma p_{13})_!\, ({}_\Gamma p_{12}^*({}_\Gamma\dot{\mathscr{L}}^*)) \otimes {}_\Gamma\dot{\mathscr{L}}) \\ = \mathrm{End}_{\mathscr{D}({}_\Gamma\dot{V})}(({}_\Gamma p_{13})_!\, ({}_\Gamma p_{12}^*({}_\Gamma\dot{\mathscr{L}}^*) \otimes {}_\Gamma p_{13}^*({}_\Gamma\dot{\mathscr{L}}))) \\ = \mathrm{End}_{\mathscr{D}({}_\Gamma\dot{V})}(({}_\Gamma p_{13})_!\, ({}_\Gamma\ddot{\mathscr{L}}^*)).\end{aligned}$$

Composing *a)*, *b)* and 3.5 *d)*, we find a homomorphism

c) $W \to \mathrm{Aut}_{\mathscr{D}({}_\Gamma\dot{V})}(({}_\Gamma p_{13})_!\, ({}_\Gamma\ddot{\mathscr{L}}^*))$.

Now the functor $({}_\Gamma p_1)_!$ defines a homomorphism

d) $\mathrm{End}_{\mathscr{D}({}_\Gamma\dot{V})}(({}_\Gamma p_{13})_!\, ({}_\Gamma\ddot{\mathscr{L}}^*)) \to \mathrm{End}_{\mathscr{D}({}_\Gamma V)}(({}_\Gamma p_1)_!\, ({}_\Gamma p_{13})_!\, ({}_\Gamma\ddot{\mathscr{L}}^*)) = \mathrm{End}_{\mathscr{D}({}_\Gamma V)}(({}_\Gamma \mathrm{pr}_1)_!\, ({}_\Gamma\ddot{\mathscr{L}}^*))$.

(We have ${}_\Gamma p_1 \circ {}_\Gamma p_{13} = {}_\Gamma \mathrm{pr}_1 : {}_\Gamma\ddot{V} \to {}_\Gamma V$.)
Composing *c)* and *d)* we find a homomorphism

e) $W \to \mathrm{Aut}_{\mathscr{D}({}_\Gamma V)}(({}_\Gamma \mathrm{pr}_1)_!\, ({}_\Gamma\ddot{\mathscr{L}}^*))$.

A routine verification shows that *e)* coincides with the restriction of 3.6 *a)* to $W \times \{e\}$.

We have the following variants *a')* — *e')* of *a)* — *e)*.

We have $h''^*({}_\Gamma K') = ({}_\Gamma \mathrm{pr}_{12})_!\, ({}_\Gamma \mathrm{pr}_{13}^*({}_\Gamma\dot{\mathscr{L}}))$ and h''^* defines a homomorphism

a') $\mathrm{End}_{\mathscr{D}({}_\Gamma\mathfrak{g})}({}_\Gamma K') \to \mathrm{End}_{\mathscr{D}({}_\Gamma\dot{V})}(h''^*({}_\Gamma K')) = \mathrm{End}_{\mathscr{D}({}_\Gamma\dot{V})}(({}_\Gamma p_{12})_!\, ({}_\Gamma p_{13}^*({}_\Gamma\dot{\mathscr{L}})))$.

Tensor product with ${}_{\Gamma}\dot{\mathscr{L}}^*$ is a functor $\mathscr{D}({}_{\Gamma}\dot{V}) \rightarrow \mathscr{D}({}_{\Gamma}\dot{V})$ so it defines

b') $\mathrm{End}_{\mathscr{D}({}_{\Gamma}\dot{V})}(({}_{\Gamma}p_{12})_!\,({}_{\Gamma}p_{13}^*({}_{\Gamma}\dot{\mathscr{L}}))) \rightarrow \mathrm{End}_{\mathscr{D}({}_{\Gamma}\dot{V})}({}_{\Gamma}\dot{\mathscr{L}}^* \otimes (({}_{\Gamma}p_{12})_!\,{}_{\Gamma}p_{13}^*({}_{\Gamma}\dot{\mathscr{L}})))$

$$= \mathrm{End}_{\mathscr{D}({}_{\Gamma}\dot{V})}(({}_{\Gamma}p_{12})_!\,({}_{\Gamma}p_{12}^*({}_{\Gamma}\dot{\mathscr{L}}^*) \otimes {}_{\Gamma}p_{13}^*({}_{\Gamma}\dot{\mathscr{L}})))$$
$$= \mathrm{End}_{\mathscr{D}({}_{\Gamma}\dot{V})}(({}_{\Gamma}p_{12})_!\,({}_{\Gamma}\ddot{\mathscr{L}}^*)).$$

Composing *a')*, *b')* and 3.5 *e)* we find a homomorphism

c') $W \rightarrow \mathrm{Aut}_{\mathscr{D}({}_{\Gamma}\dot{V})}(({}_{\Gamma}p_{12})_!\,({}_{\Gamma}\ddot{\mathscr{L}}^*)).$

The functor $({}_{\Gamma}p_1)_!$ defines a homomorphism

d') $\mathrm{End}_{\mathscr{D}({}_{\Gamma}\dot{V})}(({}_{\Gamma}p_{12})_!\,({}_{\Gamma}\ddot{\mathscr{L}}^*)) \rightarrow \mathrm{End}_{\mathscr{D}({}_{\Gamma}V)}(({}_{\Gamma}p_1)_!\,({}_{\Gamma}p_{12})_!\,({}_{\Gamma}\ddot{\mathscr{L}}^*)) = \mathrm{End}_{\mathscr{D}({}_{\Gamma}V)}(({}_{\Gamma}\mathrm{pr}_1)_!\,({}_{\Gamma}\ddot{\mathscr{L}}^*)).$

Composing *c')* and *d')* we find a homomorphism

e') $W \rightarrow \mathrm{Aut}_{\mathscr{D}({}_{\Gamma}V)}(({}_{\Gamma}\mathrm{pr}_1)_!\,({}_{\Gamma}\ddot{\mathscr{L}}^*)).$

Again, a routine verification shows that *e')* coincides with the restriction of 3.6 *a)* to $\{e\} \times W$.

3.8. Our next objective is to define a homomorphism $W \times W \rightarrow \mathrm{Aut}\, H_j^{G \times \mathbf{C}^*}(\ddot{V}, \ddot{\mathscr{L}})$, for an integer $j \geqslant 0$.

We choose Γ and m as in 3.5 with $m \geqslant j$ and apply the functor

a) $H_c^{2d-j}({}_{\Gamma}V, \ ) : \mathscr{D}({}_{\Gamma}V) \rightarrow \mathbf{C}$-vector spaces

to 3.6 *a)* ($d = \dim {}_{\Gamma}\ddot{V}$). We obtain a homomorphism

$$W \times W \rightarrow \mathrm{Aut}\, H_c^{2d-j}({}_{\Gamma}V, ({}_{\Gamma}\mathrm{pr}_1)_!\,({}_{\Gamma}\ddot{\mathscr{L}}^*)) = \mathrm{Aut}\, H_c^{2d-j}({}_{\Gamma}\ddot{V}, {}_{\Gamma}\ddot{\mathscr{L}}^*).$$

Taking duals we obtain a homomorphism

$$W \times W \rightarrow \mathrm{Aut}\, H_c^{2d-j}({}_{\Gamma}\ddot{V}, \ddot{\mathscr{L}}^*)^*$$

or, equivalently, a homomorphism

b) $W \times W \rightarrow \mathrm{Aut}\, H_j^{G \times \mathbf{C}^*}(\ddot{V}, \ddot{\mathscr{L}})$

as desired.

This homomorphism is actually independent of the choice of Γ; the verification is routine and will be omitted.

Similarly applying the functor *a)* to 3.7 *e)* and 3.7 *e')* and then taking duals we find two homomorphisms

c) $W \rightrightarrows \mathrm{Aut}\, H_c^{2d-j}({}_{\Gamma}\ddot{V}, \ddot{\mathscr{L}}^*)^* = \mathrm{Aut}\, H_j^{G \times \mathbf{C}^*}(\ddot{V}, \ddot{\mathscr{L}})$

which coincide with the restriction of *b)* to $W \times \{e\}$, $\{e\} \times W$ respectively.

3.9. Let $i \in [1, m]$, P_i, L_i, $\mathfrak{p}_i$, $\mathfrak{l}_i$ be as in 2.4, let s_i be as in 2.5 and let $W_i = \{e, s_i\} \subset W$. Let $\overline{P} = L_i \cap P$, $\overline{\mathfrak{n}} = \mathfrak{l}_i \cap \mathfrak{n}$. Let Ω be a locally closed subset of G which is a union of $P_i - P$ double cosets. Our objective is to define a natural homomorphism

a) $W_i \rightarrow \mathrm{Aut}\, H_{\bullet}^{G \times \mathbf{C}^*}(\ddot{\mathfrak{g}}^{\Omega}, \ddot{\mathscr{L}})$

which, in the case where $\Omega = G$, should coincide with the restriction of 3.8 *b)* (for $V = \mathfrak{g}$) to the subgroup $W_i \times \{e\}$ of $W \times W$.

We introduce some notation. Let $\Lambda = \{(x, g\mathrm{P}_i) \in \mathfrak{g} \times \mathrm{G}/\mathrm{P}_i \mid \mathrm{Ad}(g^{-1})\, x \in \mathfrak{p}_i\}$,

$$\ddot{\mathfrak{g}}_1 = \{(y, h\mathrm{P}) \in \mathfrak{p}_i \times \mathrm{P}_i/\mathrm{P} \mid \mathrm{Ad}(h^{-1})\, y \in \mathscr{C} + \mathfrak{t} + \mathfrak{n}\}$$

$$\ddot{\mathfrak{g}}_2 = \{(\bar{y}, \bar{h}\bar{\mathrm{P}}) \in \mathfrak{l}_i \times \mathrm{L}_i/\bar{\mathrm{P}} \mid \mathrm{Ad}(\bar{h}^{-1})\, \bar{y} \in \mathscr{C} + \mathfrak{t} + \bar{\mathfrak{n}}\}.$$

Consider the commutative diagram

$$\begin{tikzcd}
\dot{\mathfrak{g}} \arrow[d, "\pi_0"'] & \dot{\mathfrak{g}}_1 \times \mathrm{G} \arrow[l] \arrow[r] \arrow[d] & \dot{\mathfrak{g}}_1 \arrow[r] \arrow[d] & \dot{\mathfrak{g}}_2 \arrow[d, "\pi_2"] \\
\Lambda & \mathfrak{p}_i \times \mathrm{G} \arrow[l] \arrow[r] & \mathfrak{p}_i \arrow[r] & \mathfrak{l}_i
\end{tikzcd}$$

in which all maps are obvious except for $\dot{\mathfrak{g}}_1 \times \mathrm{G} \to \dot{\mathfrak{g}}$ which is $(y, h\mathrm{P}, \gamma) \to (\mathrm{Ad}\,(\gamma)\, y, \gamma h\mathrm{P})$ and $\mathfrak{p}_i \times \mathrm{G} \to \Lambda$ which is $(y, \gamma) \to (\mathrm{Ad}(\gamma)\, y, \gamma\mathrm{P}_i)$.

Now $\dot{\mathfrak{g}}_2 \to \mathfrak{l}_i$ is exactly like $\dot{\mathfrak{g}} \to \mathfrak{g}$ when $(\mathrm{G}, \mathrm{P}, \mathrm{L}, \mathscr{C})$ is replaced by $(\mathrm{L}_i, \bar{\mathrm{P}}, \mathrm{L}, \mathscr{C})$. In particular, $\dot{\mathfrak{g}}_2$ carries a local system $\dot{\mathscr{L}}_2^*$ analogous to the local system $\dot{\mathscr{L}}^*$ on $\dot{\mathfrak{g}}$. Moreover the analogue of 3.4 *c)* (for L_i instead of G) gives a homomorphism

b) $\mathrm{W}_i \to \mathrm{Aut}_{\mathscr{D}(\mathfrak{l}_i)}((\pi_2)_!\, \dot{\mathscr{L}}_2^*)$

and the analogue of 3.4 *a)* shows that

c) $(\pi_2)_!\, (\dot{\mathscr{L}}_2^*)$, suitably shifted, is a perverse sheaf.

The inverse image of $\dot{\mathscr{L}}_2^*$ under the composition $\dot{\mathfrak{g}}_1 \times \mathrm{G} \to \dot{\mathfrak{g}}_1 \to \dot{\mathfrak{g}}_2$ is the same as the inverse image of $\dot{\mathscr{L}}^*$ under $\dot{\mathfrak{g}}_1 \times \mathrm{G} \to \dot{\mathfrak{g}}$. Since our diagram has cartesian squares and the horizontal arrows are smooth, we see as in 3.5 (using [1, 4.2.5] and *c)*) that

d) $\mathrm{End}_{\mathscr{D}(\mathfrak{l}_i)}((\pi_2)_!\, \dot{\mathscr{L}}_2^*) = \mathrm{End}_{\mathscr{D}(\Lambda)}((\pi_0)_!\, \dot{\mathscr{L}}^*)$, canonically.

e) $\mathrm{End}_{\mathscr{D}(\Lambda)}((\pi_0)_!\, \dot{\mathscr{L}}^*) = \mathrm{End}_{\mathscr{D}({}_\Gamma\Lambda)}(({}_\Gamma\pi_0)_!\, {}_\Gamma\dot{\mathscr{L}}^*)$, canonically
(for fixed Γ as in 3.5 with m large).

We now consider the cartesian diagram

$$\begin{tikzcd}
\ddot{\mathfrak{g}}^{\Omega} \arrow[r, "p_{12}"] \arrow[d, "\sigma"'] & \dot{\mathfrak{g}} \arrow[d, "\pi_0"] \\
\Sigma^{\Omega} \arrow[r, "p_{12}"'] & \Lambda
\end{tikzcd}$$

where $\Sigma^{\Omega} = \{(x, g\mathrm{P}_i, g'\,\mathrm{P}) \mid \mathrm{Ad}(g^{-1})\, x \in \mathfrak{p}_i, \mathrm{Ad}(g'^{-1})\, x \in \mathscr{C} + \mathfrak{t} + \mathfrak{n}, g^{-1}\, g' \in \Omega\}$, p_{12} are the obvious projections and σ is the obvious map. We have natural $\mathrm{G} \times \mathbf{C}^*$ actions on the varieties in this diagram, so we can form the cartesian diagram:

$$\begin{tikzcd}
{}_\Gamma\ddot{\mathfrak{g}}^\Omega \arrow[r, "{}_\Gamma p_{12}"] \arrow[d, "{}_\Gamma\sigma"'] & {}_\Gamma\dot{\mathfrak{g}} \arrow[d, "{}_\Gamma\pi_0"] \\
{}_\Gamma\Sigma^\Omega \arrow[r, "{}_\Gamma p_{12}"'] & {}_\Gamma\Lambda
\end{tikzcd}$$

We denote the inverse image of ${}_\Gamma\dot{\mathscr{L}}^*$ under ${}_\Gamma p_{12} : \ddot{\mathfrak{g}}^\Omega \to {}_\Gamma\dot{\mathfrak{g}}$ again by ${}_\Gamma\dot{\mathscr{L}}^*$. From the last cartesian diagram we obtain a homomorphism

f) $\mathrm{End}_{\mathscr{D}({}_\Gamma\Lambda)}(({}_\Gamma\pi_0)_!\,({}_\Gamma\dot{\mathscr{L}}^*)) \to \mathrm{End}_{\mathscr{D}({}_\Gamma\Sigma^\Omega)}({}_\Gamma\sigma_!({}_\Gamma\dot{\mathscr{L}}^*))$.

Consider $p_{13} : \Sigma^\Omega \to \dot{\mathfrak{g}}$. We denote the inverse image of ${}_\Gamma\dot{\mathscr{L}}$ under $({}_\Gamma p_{13})^*$ again by ${}_\Gamma\dot{\mathscr{L}}$, and its inverse image under $\sigma . p_{13}$ again by ${}_\Gamma\dot{\mathscr{L}}$. Tensor product by ${}_\Gamma\dot{\mathscr{L}}$ is a functor $\mathscr{D}({}_\Gamma\Sigma^\Omega) \to \mathscr{D}({}_\Gamma\Sigma^\Omega)$ hence it defines a homomorphism

g) $\mathrm{End}_{\mathscr{D}({}_\Gamma\Sigma^\Omega)}({}_\Gamma\sigma_!({}_\Gamma\dot{\mathscr{L}}^*)) \to \mathrm{End}_{\mathscr{D}({}_\Gamma\Sigma^\Omega)}({}_\Gamma\sigma_!({}_\Gamma\dot{\mathscr{L}}^*) \otimes {}_\Gamma\dot{\mathscr{L}})) = \mathrm{End}_{\mathscr{D}({}_\Gamma\Sigma^\Omega)}({}_\Gamma\sigma_!({}_\Gamma\dot{\mathscr{L}}^* \otimes {}_\Gamma\dot{\mathscr{L}}))$
$= \mathrm{End}_{\mathscr{D}({}_\Gamma\Sigma^\Omega)}({}_\Gamma\sigma_!({}_\Gamma\ddot{\mathscr{L}}^*))$.

Composing *b)*, *d)*, *e)*, *f)*, *g)* we find a homomorphism

h) $\mathrm{W}_i \to \mathrm{Aut}_{\mathscr{D}({}_\Gamma\Sigma^\Omega)}({}_\Gamma\sigma_!({}_\Gamma\ddot{\mathscr{L}}^*))$.

Applying the functor $\mathrm{H}_c^{2d-j}({}_\Gamma\Sigma^\Omega, \ )$ $(d = \dim {}_\Gamma\ddot{\mathfrak{g}}^\Omega)$, $j \leqslant m$, and taking duals we find a homomorphism

$$\mathrm{W}_i \to \mathrm{Aut}\ \mathrm{H}_c^{2d-j}({}_\Gamma\ddot{\mathfrak{g}}^\Omega, {}_\Gamma\ddot{\mathscr{L}}^*)^* = \mathrm{Aut}\ \mathrm{H}_j^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathfrak{g}}^\Omega, \ddot{\mathscr{L}})$$

which is the desired homomorphism *a)*; it is independent of the choice of Γ.

The following property follows easily from the definitions.

Let Ω' be a closed subset of Ω which is a union of $\mathrm{P}_i - \mathrm{P}$ double cosets. The natural map $\mathrm{H}_\bullet^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathfrak{g}}^{\Omega'}, \ddot{\mathscr{L}}) \to \mathrm{H}_\bullet^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathfrak{g}}^{\Omega}, \ddot{\mathscr{L}})$ induced by the closed embedding $\ddot{\mathfrak{g}}^{\Omega'} \hookrightarrow \ddot{\mathfrak{g}}^\Omega$ is compatible with the W_i actions *a)*.

In particular, taking $\Omega' = \mathrm{P}_i$, $\Omega = \mathrm{G}$ we see that

i) The natural map $\mathrm{H}_\bullet^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathfrak{g}}^{\mathrm{P}_i}, \ddot{\mathscr{L}}) \to \mathrm{H}_\bullet^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathfrak{g}}, \ddot{\mathscr{L}})$ induced by the closed embedding $\ddot{\mathfrak{g}}^{\mathrm{P}_i} \hookrightarrow \ddot{\mathfrak{g}}$ is compatible with the W_i actions *a)*.

Assume now that $\Omega = \mathrm{G}$; in this case we write Σ instead of Σ^Ω. We want to prove that in this case the action *a)* is the restriction to $\mathrm{W}_i \times \{e\}$ of the action 3.8 *b)* (for $\mathrm{V} = \mathfrak{g}$).

The key point to be verified is the following:

Consider the commutative diagram

$$\begin{tikzcd}
 & \dot{\mathfrak{g}} \arrow[dl, "\pi_0"'] \arrow[dr, "\pi"] & \\
\Lambda \arrow[rr, "\rho"] & & \mathfrak{g}
\end{tikzcd}$$

with $\rho(x, g\mathrm{P}_i) = x$;

then the action of s_i on $(\pi_0)_! \dot{\mathscr{L}}^*$ (composition of *b)*, *d)*) is mapped under

$$\rho_! : \mathrm{End}_{\mathscr{D}\Lambda}((\pi_0)_! \dot{\mathscr{L}}^*) \to \mathrm{End}_{\mathscr{D}\mathfrak{g}} (\rho_!(\pi_0)_! \dot{\mathscr{L}}^*) = \mathrm{End}_{\mathscr{D}\mathfrak{g}}(\pi_! \dot{\mathscr{L}}^*)$$

to the action of s_i on $\pi_! \dot{\mathscr{L}}^*$ given by 3.4 *c)*.

The verification of this fact is routine and is omitted.

Next, assume that $\Omega = \mathrm{P}_i$. We introduce the following notation

$$\begin{aligned} \tilde{\Delta} &= \{(x, g\mathrm{P}, g'\,\mathrm{P}) \in \ddot{\mathfrak{g}} \mid x \in \mathfrak{p}_i, g \in \mathrm{P}_i, g' \in \mathrm{P}_i\}, \\ \Delta &= \{(\bar{x}, \bar{g}\bar{\mathrm{P}}, \bar{g}'\,\bar{\mathrm{P}}) \in \mathfrak{l}_i \times \mathrm{L}_i/\bar{\mathrm{P}} \times \mathrm{L}_i/\bar{\mathrm{P}} \mid \mathrm{Ad}(\bar{g}^{-1})\, \bar{x} \in \mathscr{C} + \mathfrak{t} + \bar{\mathfrak{n}}, \\ &\qquad\qquad \mathrm{Ad}(\bar{g}'^{-1})\, \bar{x} \in \mathscr{C} + \mathfrak{t} + \bar{\mathfrak{n}}\}. \end{aligned}$$

Note that Δ is exactly like $\ddot{\mathfrak{g}}$, when $(\mathrm{G}, \mathrm{P}, \mathrm{L}, \mathscr{C})$ is replaced by $(\mathrm{L}_i, \bar{\mathrm{P}}, \mathrm{L}, \mathscr{C})$; in particular it is $\mathrm{L}_i \times \mathbf{C}^*$-equivariant and carries an $\mathrm{L}_i \times \mathbf{C}^*$ equivariant local system $\bar{\mathscr{L}}$ analogous to $\ddot{\mathscr{L}}$ on $\ddot{\mathfrak{g}}$. Now $\tilde{\Delta}$ is a $\mathrm{P}_i \times \mathbf{C}^*$-stable closed subvariety of $\ddot{\mathfrak{g}}^{\mathrm{P}_i}$ and the action of $\mathrm{G} \times \mathbf{C}^*$ on $\ddot{\mathfrak{g}}$ defines an isomorphism $((\mathrm{G} \times \mathbf{C}^*) \times \tilde{\Delta})/\mathrm{P}_i \times \mathbf{C}^* \xrightarrow{\sim} \ddot{\mathfrak{g}}^{\mathrm{P}_i}$.

Using 1.6 *a)*, 1.4 *h)*, it follows that

$$\mathrm{H}_\bullet^{\mathrm{G} \times \mathbf{C}^*}(\ddot{\mathfrak{g}}^{\mathrm{P}_i}, \ddot{\mathscr{L}}) \cong \mathrm{H}_\bullet^{\mathrm{P}_i \times \mathbf{C}^*}(\tilde{\Delta}, \ddot{\mathscr{L}}) \cong \mathrm{H}_\bullet^{\mathrm{L}_i \times \mathbf{C}^*}(\tilde{\Delta}, \ddot{\mathscr{L}}).$$

On the other hand we have an obvious $\mathrm{L}_i \times \mathbf{C}^*$-equivariant map $\tilde{\Delta} \to \Delta$ which is a vector bundle with fibres $\approx \mathfrak{p}_i/\mathfrak{l}_i$, hence, by 1.4 *e)* we have

$$\mathrm{H}_\bullet^{\mathrm{L}_i \times \mathbf{C}^*}(\tilde{\Delta}, \ddot{\mathscr{L}}) \cong \mathrm{H}_\bullet^{\mathrm{L}_i \times \mathbf{C}^*}(\Delta, \bar{\mathscr{L}}).$$

Combining this with the previous isomorphisms we obtain

j) $\mathrm{H}_\bullet^{\mathrm{L}_i \times \mathbf{C}^*}(\Delta, \bar{\mathscr{L}}) \cong \mathrm{H}_\bullet^{\mathrm{G} \times \mathbf{C}^*}(\ddot{\mathfrak{g}}^{\mathrm{P}_i}, \ddot{\mathscr{L}})$.

It is easy to see that the actions of s_i on these two vector spaces, one defined by *a)* for L_i (or equivalently, by the $s_i \times 1$ action in 3.8 *b)* for $\mathrm{G} = \mathrm{L}_i$, $\mathrm{V} = \mathfrak{g}$), the other defined by *a)* with $\Omega = \mathrm{P}_i$, correspond under *j)*.

Combining *i)* and *j)*, we obtain a map

k) $\mathrm{H}_\bullet^{\mathrm{L}_i \times \mathbf{C}^*}(\Delta, \bar{\mathscr{L}}) \to \mathrm{H}_\bullet^{\mathrm{G} \times \mathbf{C}^*}(\dot{\mathfrak{g}}, \ddot{\mathscr{L}})$

which is compatible with the actions of $s_i \times 1$ in 3.8 *b)* for L_i and for G.

3.10. Assume that we are given two locally closed $\mathrm{G} \times \mathbf{C}^*$-stable subvarieties V, V_1 of $\mathfrak{g}$ such that V_1 is closed in V. Let $\mathrm{V}_2 = \mathrm{V} - \mathrm{V}_1$. Then $\ddot{\mathrm{V}}_1$ (resp. $\ddot{\mathrm{V}}_2$) is a closed (resp. open) subvariety of $\ddot{\mathrm{V}}$ and the inclusions $i_1 : \ddot{\mathrm{V}}_1 \hookrightarrow \ddot{\mathrm{V}}$, $i_2 : \ddot{\mathrm{V}}_2 \hookrightarrow \ddot{\mathrm{V}}$ induce

a) $\mathrm{H}_\bullet^{\mathrm{G} \times \mathbf{C}^*}(\ddot{\mathrm{V}}_1, \ddot{\mathscr{L}}) \xrightarrow{(i_1)_!} \mathrm{H}_\bullet^{\mathrm{G} \times \mathbf{C}^*}(\ddot{\mathrm{V}}, \ddot{\mathscr{L}}) \xrightarrow{i_2^*} \mathrm{H}_\bullet^{\mathrm{G} \times \mathbf{C}^*}(\ddot{\mathrm{V}}_2, \ddot{\mathscr{L}})$.

The two maps in *a)* are compatible with the $\mathrm{W} \times \mathrm{W}$-actions in 3.8 *b)*. The verification is routine and will be omitted.

4. S-module structures

4.1. Let $\mathfrak{m}$ be the Lie algebra of $\mathrm{M}_{\mathrm{L}}^0(\varphi_0)$ (see 2.1, 2.3). The differential of the isomorphism $\mathrm{T} \times \mathbf{C}^* \xrightarrow{\sim} \mathrm{M}_{\mathrm{L}}^0(\varphi_0)$ (see 2.3 *c)*) is an isomorphism $\mathfrak{t} \oplus \mathbf{C} \to \mathfrak{m}$. We denote by r the linear form on $\mathfrak{m}$ corresponding to $\mathrm{pr}_2 : \mathfrak{t} \oplus \mathbf{C} \to \mathbf{C}$ under that isomorphism. Equivalently, r is the differential of the restriction of $\mathrm{pr}_2 : \mathrm{G} \times \mathbf{C}^* \to \mathbf{C}^*$ to $\mathrm{M}^0_{\mathrm{L}}(\varphi_0) \to \mathbf{C}^*$. We define

$$\mathbf{S} = \mathrm{S}(\mathfrak{m}^*).$$

From the decomposition $\mathfrak{m}^* = \mathfrak{t}^* \oplus \mathbf{C}$, we can write

$$\mathbf{S} = \mathrm{S}(\mathfrak{t}^*) \otimes \mathbf{C}[r].$$

The natural action of W on $\mathfrak{t}$ and $\mathfrak{t}^*$ extends to an action of W on $\mathbf{S}$ by $\mathbf{C}[r]$-algebra automorphisms; we denote it by $w : \xi \mapsto {}^w\xi$. In particular, ${}^w r = r$ for all w.

Proposition **4.2.** — *There is a natural isomorphism of graded algebras* $\mathrm{H}^{\bullet}_{\mathrm{G}\times\mathbf{C}^*}(\dot{\mathfrak{g}}) \cong \mathbf{S}$. *In particular,* $\mathrm{H}^{j}_{\mathrm{G}\times\mathbf{C}^*}(\dot{\mathfrak{g}}) = 0$ *for odd* j.

Proof. — We have an isomorphism

$$(\mathrm{P} \times \mathbf{C}^*)\backslash((\mathrm{G} \times \mathbf{C}^*) \times (\mathscr{C} + \mathfrak{t} + \mathfrak{n})) \xrightarrow{\sim} \dot{\mathscr{V}} = \dot{\mathfrak{g}}$$

defined by $((g_1, \lambda), x) \to (\lambda^{-2}\,\mathrm{Ad}(g_1)\,x, g_1\,\mathrm{P})$ hence, by 1.6 *a)*, we have

$$\mathrm{H}^{\bullet}_{\mathrm{G}\times\mathbf{C}^*}(\dot{\mathfrak{g}}) \cong \mathrm{H}^{\bullet}_{\mathrm{P}\times\mathbf{C}^*}(\mathscr{C} + \mathfrak{t} + \mathfrak{n}).$$

($\mathrm{P} \times \mathbf{C}^*$ acts on $\mathscr{C} + \mathfrak{t} + \mathfrak{n}$ as restriction of the $\mathrm{G} \times \mathbf{C}^*$-action on $\mathfrak{g}$. Since $\mathrm{pr}_1 : \mathscr{C} + \mathfrak{t} + \mathfrak{n} \to \mathscr{C}$ is a $\mathrm{P} \times \mathbf{C}^*$-equivariant vector bundle, we have (1.4 *e)*)

$$\mathrm{H}^{\bullet}_{\mathrm{P}\times\mathbf{C}^*}(\mathscr{C} + \mathfrak{t} + \mathfrak{n}) \cong \mathrm{H}^{\bullet}_{\mathrm{P}\times\mathbf{C}^*}(\mathscr{C}).$$

Here $\mathrm{P} \times \mathbf{C}^*$ acts on $\mathscr{C}$ via its quotient $\mathrm{L} \times \mathbf{C}^*$. Using 1.4 *h)* we have

$$\mathrm{H}^{\bullet}_{\mathrm{P}\times\mathbf{C}^*}(\mathscr{C}) \cong \mathrm{H}^{\bullet}_{\mathrm{L}\times\mathbf{C}^*}(\mathscr{C}).$$

Now $\mathrm{L} \times \mathbf{C}^*$ acts transitively on $\mathscr{C}$ and the stabilizer of x_0 is $\mathrm{M}_{\mathrm{L}}(x_0)$. Using 1.6 *a)* we have

$$\mathrm{H}^{\bullet}_{\mathrm{L}\times\mathbf{C}^*}(\mathscr{C}) \cong \mathrm{H}^{\bullet}_{\mathrm{M}_{\mathrm{L}}(x_0)}.$$

Using 2.1 *d)* and 1.4 *h)*, we have

$$\mathrm{H}^{\bullet}_{\mathrm{M}_{\mathrm{L}}(x_0)} \cong \mathrm{H}^{\bullet}_{\mathrm{M}_{\mathrm{L}}(\varphi_0)}.$$

Using 2.3 *d)*, 1.12 *a)* and 1.11 *a)*

$$\mathrm{H}^{\bullet}_{\mathrm{M}_{\mathrm{L}}(\varphi_0)} \cong \mathrm{H}^{\bullet}_{\mathrm{M}^0_{\mathrm{L}}(\varphi_0)} \cong \mathrm{H}^{\bullet}_{\mathrm{T}\times\mathbf{C}^*} = \mathbf{S}$$

and the proposition follows.

4.3. Let $f : \mathrm{X} \to \mathrm{Y}$ be a $\mathrm{G} \times \mathbf{C}^*$-equivariant morphism between two $\mathrm{G} \times \mathbf{C}^*$-varieties and let $\mathscr{L}_1$ be a $\mathrm{G} \times \mathbf{C}^*$-local system on X. Then $\mathrm{H}^{\mathrm{G}\times\mathbf{C}^*}_{\bullet}(\mathrm{X}, \mathscr{L}_1)$ can be regarded as a left $\mathrm{H}^{\bullet}_{\mathrm{G}\times\mathbf{C}^*}(\mathrm{Y})$-module via its $\mathrm{H}^{\bullet}_{\mathrm{G}\times\mathbf{C}^*}(\mathrm{X})$-module structure (1.3 *b)*) and the algebra homomorphism $f^* : \mathrm{H}^{\bullet}_{\mathrm{G}\times\mathbf{C}^*}(\mathrm{Y}) \to \mathrm{H}^{\bullet}_{\mathrm{G}\times\mathbf{C}^*}(\mathrm{X})$.

In particular, for any locally closed subvariety Ω of G which is a union of $\mathrm{P}-\mathrm{P}$ double cosets, and V as in 3.3, we consider

$$\mathrm{X} = \ddot{\mathrm{V}}^{\Omega} \quad (\text{see } 3.3\ b)), \qquad \mathscr{L}_1 = \ddot{\mathscr{L}}, \qquad \mathrm{Y} = \dot{\mathfrak{g}},$$

$f : \mathrm{X} \to \mathrm{Y}$ the map $\pi_{12} : (x, g\mathrm{P}, g'\,\mathrm{P}) \mapsto (x, g\mathrm{P})$ or $\pi_{13} : (x, g\mathrm{P}, g'\,\mathrm{P}) \mapsto (x, g'\,\mathrm{P})$, and we get two $\mathbf{S}$ $(= \mathrm{H}^{\bullet}_{\mathrm{G}\times\mathbf{C}^*}(\dot{\mathfrak{g}}))$-module structures on $\mathrm{H}^{\mathrm{G}\times\mathbf{C}^*}_{\bullet}(\ddot{\mathrm{V}}^{\Omega}, \ddot{\mathscr{L}})$. The two module structures are denoted as follows:

a) $\xi \in \mathbf{S},\ h \in \mathrm{H}^{\mathrm{G}\times\mathbf{C}^*}_{\bullet}(\ddot{\mathrm{V}}^{\Omega}, \ddot{\mathscr{L}}) \Rightarrow \begin{Bmatrix} \Delta(\xi)\ h \\ \Delta'(\xi)\ h \end{Bmatrix}$ in $\mathrm{H}^{\mathrm{G}\times\mathbf{C}^*}_{\bullet}(\ddot{\mathrm{V}}^{\Omega}, \ddot{\mathscr{L}})$;

by definition $\Delta(\xi)\ h = \pi^*_{12}(\xi)\,.h$, $\Delta'(\xi)\ h = \pi^*_{13}(\xi)\,.h$, where ξ is identified with an element of $\mathrm{H}^{\bullet}_{\mathrm{G}\times\mathbf{C}^*}(\dot{\mathfrak{g}})$ by 4.2, so that $\pi^*_{12}(\xi), \pi^*_{13}(\xi) \in \mathrm{H}^{\bullet}_{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathrm{V}}^{\Omega})$, and then the products are taken as in 1.3 *b)*.

It is easy to see that

b) $\Delta(\xi)\ \Delta'(\xi')\ h = \Delta'(\xi')\ \Delta(\xi)\ h \quad (\xi, \xi' \in \mathbf{S})$.

Now $\mathrm{H}^{\mathrm{G}\times\mathbf{C}^*}_{\bullet}(\ddot{\mathrm{V}}^{\Omega}, \ddot{\mathscr{L}})$ is an $\mathrm{H}^{\bullet}_{\mathrm{G}\times\mathbf{C}^*}$-module in a natural way (1.7) and it is clear that this is the restriction of either of the two $\mathbf{S}$-module structures above to $\mathrm{H}^{\bullet}_{\mathrm{G}\times\mathbf{C}^*}$ via the natural algebra homomorphism $\mathrm{H}^{\bullet}_{\mathrm{G}\times\mathbf{C}^*} \to \mathrm{H}^{\bullet}_{\mathrm{G}\times\mathbf{C}^*}(\dot{\mathfrak{g}}) = \mathbf{S}$ induced by the map $\dot{\mathfrak{g}} \to$ point, or equivalently by the map

c) $\mathrm{H}^{\bullet}_{\mathrm{G}\times\mathbf{C}^*} \to \mathrm{H}^{\bullet}_{\mathrm{M}^0_{\mathrm{L}}(\varphi_0)} = \mathbf{S}$

induced (1.4 *f)*) by the inclusion $\mathrm{M}^0_{\mathrm{L}}(\varphi_0) \hookrightarrow \mathrm{G} \times \mathbf{C}^*$.

The homomorphism *c)* has as image the algebra $\mathbf{S}^{\mathrm{W}}$ of W-invariants on $\mathbf{S}$. [An equivalent statement is the following. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{t}$. Then any W-invariant polynomial $\mathfrak{t} \to \mathbf{C}$ is the restriction of a polynomial $\mathfrak{h} \to \mathbf{C}$ invariant under the full Weyl group W′ of $\mathfrak{h}$; or, equivalently: the natural map $\mathfrak{t}/\mathrm{W} \to \mathfrak{h}/\mathrm{W}'$ is a (closed) embedding. The verification of this statement is omitted.]

It follows that

d) $\Delta(\xi) = \Delta'(\xi) \quad \text{for all } \xi \in \mathbf{S}^{\mathrm{W}}$

as operators on $\mathrm{H}^{\mathrm{G}\times\mathbf{C}^*}_{\bullet}(\ddot{\mathrm{V}}^{\Omega}, \ddot{\mathscr{L}})$.

We shall write r instead of $\Delta(r) = \Delta'(r)$.

e) $\Delta(\xi) = \Delta'(\xi)$ on $\mathrm{H}^{\mathrm{G}\times\mathbf{C}^*}_{\bullet}(\ddot{\mathrm{V}}^{\Omega}, \ddot{\mathscr{L}})$ for all $\xi \in \mathbf{S}^{\mathrm{W}}$.

Indeed, the two projections $\ddot{\mathrm{V}}^{\mathrm{P}} \to \dot{\mathfrak{g}}$ coincide.

We denote the operators defined in 3.8 *b)* by (w, e), (e, w) on $\mathrm{H}^{\mathrm{G}\times\mathbf{C}^*}_{\bullet}(\ddot{\mathrm{V}}, \ddot{\mathscr{L}})$ by $\Delta(w)$, $\Delta'(w)$ respectively $(w \in \mathrm{W})$.

Proposition **4.4.** — *Let* $w, w' \in \mathrm{W}$, $\xi \in \mathbf{S}$, $h \in \mathrm{H}^{\mathrm{G}\times\mathbf{C}^*}_{\bullet}(\ddot{\mathrm{V}}, \ddot{\mathscr{L}})$. *Then*

a) $\Delta(w)\ \Delta'(\xi)\ h = \Delta'(\xi)\ \Delta(w)\ h$,

b) $\Delta'(w)\ \Delta(\xi)\ h = \Delta(\xi)\ \Delta'(w)\ h$,

c) $\Delta(w)\ \Delta'(w')\ h = \Delta'(w')\ \Delta(w)\ h$.

Proof. — We use the following fact. If X is a variety and $\bar{K} \in \mathscr{D}X$, then we have the usual product

$$\mu : H^p(X) \otimes H_c^q(X, \bar{K}) \to H_c^{p+q}(X, \bar{K})$$

with the following property: if $\tau \in \mathrm{End}_{\mathscr{D}X} \bar{K}$, then τ induces $\tau^{(i)} : H_c^i(X, \bar{K}) \circlearrowleft$ for all i and

d) $\tau^{(p+q)}(\mu(A \otimes B)) = \mu(A \otimes \tau^{(q)} B)$

for $A \in H^p(X)$, $B \in H_c^q(X, \bar{K})$.

We apply this in the following situation. We can assume that ξ, h are homogeneous of degrees i_1, i_2 with $i_1 + i_2 = j$. Let Γ be as in 3.5 with m large. We take: $X = {}_\Gamma\dot{V}$, $\bar{K} = ({}_\Gamma p_{13})_! \, {}_\Gamma\ddot{\mathscr{L}}^*$, $\tau = w \in \mathrm{Aut}(\bar{K})$ (see 3.7 *c)*), $p = i_1$, $q = 2d - j$, $d = \dim({}_\Gamma\ddot{V})$. We have $H_c^\bullet({}_\Gamma\dot{V}, \bar{K}) = H_c^\bullet({}_\Gamma\ddot{V}, \ddot{\mathscr{L}}^*)$, so that μ becomes

$$H^{i_1}({}_\Gamma\dot{V}) \otimes H_c^{2d-j}({}_\Gamma\ddot{V}, {}_\Gamma\ddot{\mathscr{L}}) \to H_c^{2d-i_2}({}_\Gamma\ddot{V}, {}_\Gamma\ddot{\mathscr{L}}^*)$$

or, taking duals,

$$H^{i_1}({}_\Gamma\dot{V}) \otimes H_c^{2d-i_2}({}_\Gamma\ddot{V}, {}_\Gamma\ddot{\mathscr{L}}^*)^* \to H_c^{2d-j}({}_\Gamma\ddot{V}, {}_\Gamma\ddot{\mathscr{L}}^*)^*$$

or, equivalently,

$$\mu' : H^{i_1}_{G \times \mathbf{C}^*}(\dot{V}) \otimes H^{G \times \mathbf{C}^*}_{i_2}(\ddot{V}, \mathscr{L}) \to H^{G \times \mathbf{C}^*}_{j}(\ddot{V}, \ddot{\mathscr{L}}).$$

The identity *d)* implies in this case:

$$(w \times 1)\, \mu'(A' \otimes B') = \mu'(A' \otimes (w \times 1)\, B')$$

for $A' \in H^{i_1}_{G \times \mathbf{C}^*}(\dot{V})$, $B' \in H^{G \times \mathbf{C}^*}_{i_2}(\ddot{V}, \ddot{\mathscr{L}})$. We apply this for $A' =$ image of ξ under $H^{i_1}_{G \times \mathbf{C}^*}(\dot{\mathfrak{g}}) \to H^{i_1}_{G \times \mathbf{C}^*}(\dot{V})$ (induced by $\dot{V} \hookrightarrow \dot{\mathfrak{g}}$), $B' = h$ and we obtain *a)*. The proof of *b)* is completely similar, and *c)* is obvious from the definitions.

4.5. We shall regard $H^{G \times \mathbf{C}^*}_\bullet(\ddot{\mathfrak{g}}_N, \ddot{\mathscr{L}})$ and $H^{G \times \mathbf{C}^*}_\bullet(\ddot{\mathfrak{g}}, \ddot{\mathscr{L}})$ as **S**-modules in two different ways as in 4.3. These two **S**-module structures define two linear maps

a) $\mathbf{S} \otimes H^{G \times \mathbf{C}^*}_0(\ddot{\mathfrak{g}}_N, \ddot{\mathscr{L}}) \rightrightarrows H^{G \times \mathbf{C}^*}_\bullet(\ddot{\mathfrak{g}}_N, \ddot{\mathscr{L}})$

and two linear maps

b) $\mathbf{S} \otimes H^{G \times \mathbf{C}^*}_0(\mathfrak{g}, \ddot{\mathscr{L}}) \rightrightarrows H^{G \times \mathbf{C}^*}_\bullet(\ddot{\mathfrak{g}}, \ddot{\mathscr{L}})$.

Proposition **4.6.** — *The two maps* 4.5 *a) and the two maps* 4.5 *b) are isomorphisms. Moreover,* $\dim H^{G \times \mathbf{C}^*}_0(\ddot{\mathfrak{g}}_N, \ddot{\mathscr{L}}) = \dim H^{G \times \mathbf{C}^*}_0(\ddot{\mathfrak{g}}, \ddot{\mathscr{L}}) = \# W$.

The proposition is a special case of the following result.

Proposition **4.7.** — *Let* Ω *be a locally closed subvariety of* G, *which is a union of* P — P *double cosets. Let* $n(\Omega)$ *be the number of good double cosets contained in* Ω. *If* $V = \mathfrak{g}_N$ *or* $\mathfrak{g}$, *then the two* **S**-*module structures* (4.3) *on* $H^{G \times \mathbf{C}^*}_\bullet(\ddot{V}^\Omega, \mathscr{L})$ *define isomorphisms*

$$\mathbf{S} \otimes H^{G \times \mathbf{C}^*}_0(\ddot{V}^\Omega, \ddot{\mathscr{L}}) \rightrightarrows H^{G \times \mathbf{C}^*}_\bullet(\ddot{V}^\Omega, \ddot{\mathscr{L}}).$$

Moreover, $\dim H^{G \times \mathbf{C}^*}_0(\ddot{V}^\Omega, \ddot{\mathscr{L}}) = n(\Omega)$.

We shall consider only the **S**-module structure defined by the operators $\Delta(\xi)$ (see 4.3); the other **S**-module structure is treated in a similar way.

We assume first that Ω is a single $P - P$ double coset. We choose $g_0 \in \Omega$ such that L and $L' = g_0 L g_0^{-1}$ contain a common maximal torus T_0. Let $P' = g_0 P g_0^{-1}$, $U' = g_0 U g_0^{-1}$, $\mathfrak{p}'$, $\mathfrak{l}'$, $\mathfrak{n}'$ be the Lie algebras of P', L', U', $\mathscr{C}' = \mathrm{Ad}(g_0)\, \mathscr{C} \subset \mathfrak{l}'$, $\mathfrak{t} = \mathrm{Ad}(g_0)\, \mathfrak{t} \subset \mathfrak{l}'$. Using 1.6 *a)*, 1.4 *h)*, we have

$$H_\bullet^{G \times \mathbf{C}^*}(\ddot{V}^\Omega, \ddot{\mathscr{L}}) \cong H_\bullet^{(P \cap P') \times \mathbf{C}^*}(V', \ddot{\mathscr{L}}) \cong H_\bullet^{(L \cap L') \times \mathbf{C}^*}(V', \ddot{\mathscr{L}}),$$

where $V' = V \cap (\mathscr{C} + \mathfrak{t} + \mathfrak{n}) \cap (\mathscr{C}' + \mathfrak{t}' + \mathfrak{n}')$ and the inverse image of $\ddot{\mathscr{L}}$ under $V' \to \ddot{V}^\Omega$, $x \mapsto (x, P, g_0 P)$, is denoted again by $\ddot{\mathscr{L}}$. We have

a) $\mathfrak{p} \cap \mathfrak{p}' = (\mathfrak{l} \cap \mathfrak{p}') \oplus (\mathfrak{n} \cap \mathfrak{p}')$; $\mathfrak{l} \cap \mathfrak{p}' = (\mathfrak{l} \cap \mathfrak{l}') \oplus (\mathfrak{l} \cap \mathfrak{n}')$

b) $\mathfrak{p} \cap \mathfrak{p}' = (\mathfrak{p} \cap \mathfrak{l}') \oplus (\mathfrak{p} \cap \mathfrak{n}')$; $\mathfrak{p} \cap \mathfrak{l}' = (\mathfrak{l} \cap \mathfrak{l}') \oplus (\mathfrak{n} \cap \mathfrak{l}')$.

If $x \in V'$, we have in particular $x \in \mathfrak{p} \cap \mathfrak{p}'$ hence we can write uniquely $x = \gamma + \nu + \mu$, $\gamma \in \mathfrak{l} \cap \mathfrak{l}'$, $\nu = \mathfrak{l} \cap \mathfrak{n}'$, $\mu \in \mathfrak{n}$ (using *a)*) and $x = \gamma' + \nu' + \mu'$, $\gamma' \in \mathfrak{l} \cap \mathfrak{l}'$, $\nu' \in \mathfrak{n} \cap \mathfrak{l}'$, $\mu' \in \mathfrak{n}'$ (using *b)*). From $\gamma + \nu + \mu = \gamma' + \nu' + \mu'$ we deduce (using a root decomposition of $\mathfrak{g}$ with respect to T_0) that $\gamma = \gamma'$. For fixed γ, ν, ν' we have the equation $\mu + \nu = \mu' + \nu'$ for μ, μ'. Set $\tilde{\mu} = \mu - \nu'$, $\tilde{\mu}' = \mu' - \nu$. Then $\tilde{\mu} \in \mathfrak{n}$, $\tilde{\mu}' \in \mathfrak{n}'$ and $\tilde{\mu} = \tilde{\mu}' \in \mathfrak{n} \cap \mathfrak{n}'$. Thus we have an $(L \cap L') \times \mathbf{C}^*$-equivariant vector bundle

$$V' \to V'' = \{(\gamma, \nu, \nu') \in (\mathfrak{l} \cap \mathfrak{l}') \oplus (\mathfrak{l} \cap \mathfrak{n}') \oplus (\mathfrak{l}' \cap \mathfrak{n}) \mid \gamma + \nu \in (\mathscr{C} + \mathfrak{t}) \cap V,$$
$$\gamma + \nu' \in (\mathscr{C}' + \mathfrak{t}') \cap V\}$$

with fibres $\cong \mathfrak{n} \cap \mathfrak{n}'$. Using 1.4 *e)*, we have $H_\bullet^{(L \cap L') \times \mathbf{C}^*}(V', \ddot{\mathscr{L}}) = H_\bullet^{(L \cap L') \times \mathbf{C}^*}(V'', \tilde{\mathscr{L}})$ where $\tilde{\mathscr{L}}$ denotes the local system on V'' obtained as inverse image of $\mathscr{L} \boxtimes \mathscr{L}^*$ under the composition $V'' \xrightarrow{\alpha} \mathscr{C} \times \mathscr{C}' \xrightarrow{\beta} \mathscr{C} \times \mathscr{C}$, $\alpha(\gamma, \nu, \nu') = (\mathrm{pr}_{\mathscr{C}}(\gamma + \nu), \mathrm{pr}_{\mathscr{C}'}(\gamma + \nu'))$, $\beta(y, y') = (y, \mathrm{Ad}(g_0^{-1})\, y')$.

Assume that Ω is bad. We must show that $H_\bullet^{(L \cap L') \times \mathbf{C}^*}(V'', \tilde{\mathscr{L}}) = 0$. By 1.13 it is enough to show that $H_c^\bullet(V'', \tilde{\mathscr{L}}) = 0$. Let $\mathfrak{n}_1 = (\mathfrak{l} \cap \mathfrak{n}') \oplus (\mathfrak{l}' \cap \mathfrak{n})$, $\mathfrak{l}_1 = (\mathfrak{l} \cap \mathfrak{l}') \oplus (\mathfrak{l}' \cap \mathfrak{l})$; these are the nil-radical and Levi subalgebra of a proper parabolic subalgebra $\mathfrak{p}_1$ of $\mathfrak{l} \oplus \mathfrak{l}'$. Let $\mathscr{C}_1 = \mathscr{C} \times \mathscr{C}' \subset \mathfrak{l} \oplus \mathfrak{l}'$, $\mathfrak{t}_1 = \mathfrak{t} \oplus \mathfrak{t}' \subset \mathfrak{l} \oplus \mathfrak{l}'$. For $(\gamma, \nu, \nu') \in V''$ we define $\zeta \in \mathfrak{t}$, $\zeta' \in \mathfrak{t}'$ by $\gamma + \zeta + \nu \in \mathscr{C}$, $\gamma + \zeta' + \nu' \in \mathscr{C}'$, $\zeta_1 = (\zeta, \zeta') \in \mathfrak{t}_1$, $\nu_1 = (\nu, \nu') \in \mathfrak{n}_1$. If $V = \mathfrak{g}_N$, then ζ, ζ' are necessarily zero. Hence we can identify

$$V'' = \begin{cases} \{(\gamma, \nu_1, \zeta_1) \in (\mathfrak{l} \cap \mathfrak{l}') \times \mathfrak{n}_1 \times \mathfrak{t}_1 \mid (\gamma, \gamma) + \zeta_1 + \nu_1 \in \mathscr{C}_1\}, & \text{if } V = \mathfrak{g}, \\ \{(\gamma, \nu_1) \in (\mathfrak{l} \cap \mathfrak{l}') \times \mathfrak{n}_1 \mid (\gamma, \gamma) + \nu_1 \in \mathscr{C}_1\}, & \text{if } V = \mathfrak{g}_N. \end{cases}$$

We define

$$\tau : V'' \to (\mathfrak{l} \cap \mathfrak{l}') \times \mathfrak{t}_1, \quad \tau(\gamma, \nu_1, \zeta_1) = (\gamma, \zeta_1), \quad \text{if } V = \mathfrak{g}$$

$$\tau : V'' \to \mathfrak{l} \cap \mathfrak{l}', \quad \tau(\gamma, \nu_1) = \gamma, \quad \text{if } V = \mathfrak{g}_N.$$

From the Leray spectral sequence of τ, it suffices to show that

c) $H_c^\bullet(F, \tilde{\mathscr{L}}|_F) = 0$ for any fibre F of τ.

If $V = \mathfrak{g}$, the fibre at (γ, ζ_1) is $\{ \nu_1 \in \mathfrak{n}_1 \mid (\gamma, \gamma) + \zeta_1 + \nu_1 \in \mathscr{C}_1 \}$; if $V = \mathfrak{g}_N$, the fibre at γ is $\{ \nu_1 \in \mathfrak{n}_1 \mid (\gamma, \gamma) + \nu_1 \in \mathscr{C}_1 \}$. Since (γ, γ) and $(\gamma, \gamma) + \zeta_1$ are in $\mathfrak{p}_1$ we see that *c)* follows from the fact the local system $\beta^*(\mathscr{L} \boxtimes \mathscr{L}^*)$ on $\mathscr{C}_1 = \mathscr{C} \times \mathscr{C}'$ is cuspidal. (See 2.2 *a)*.)

Next, we assume that Ω is good. Then $g_0 L g_0^{-1} = L$, $\mathfrak{l} = \mathfrak{l}'$, $\mathfrak{t} = \mathfrak{t}'$, $\mathfrak{l} \cap \mathfrak{n}' = \mathfrak{l}' \cap \mathfrak{n} = 0$, hence we can identify V'' with $\{ \gamma \in \mathfrak{l} \mid \gamma \in (\mathscr{C} + \mathfrak{t}) \cap (\mathscr{C}' + \mathfrak{t}) \cap V \}$. From 2.3 *a)* we see that $\mathscr{C}' = \mathscr{C}$ and $\mathrm{Ad}(g_0) : \mathscr{C} \to \mathscr{C}$ carries $\mathscr{L}$ to $\mathscr{L}$. Hence $V'' = (\mathscr{C} + \mathfrak{t}) \cap V$. More precisely, we see that $V'' = \mathscr{C} + \mathfrak{t}$ if $V = \mathfrak{g}$ and $V'' = \mathscr{C}$ if $V = \mathfrak{g}_N$. We have in both cases $H_\bullet^{L \times \mathbf{C}^*}(V'', \tilde{\mathscr{L}}) = H_\bullet^{L \times \mathbf{C}^*}(\mathscr{C}, \mathscr{L} \otimes \mathscr{L}^*)$. (If $V = \mathfrak{g}$, we apply 1.4 *e)* to the vector bundle $\mathrm{pr}_1 : \mathscr{C} + \mathfrak{t} \to \mathscr{C}$.)

Using 1.6 *a)*, 1.4 *h)* we see that

$$H_\bullet^{L \times \mathbf{C}^*}(\mathscr{C}, \mathscr{L} \otimes \mathscr{L}^*) \cong H_\bullet^{M_L(x_0)}(\{x_0\}, (\mathscr{L} \otimes \mathscr{L}^*)_{x_0}) \cong H_\bullet^{M_L(\varphi_0)}(\{x_0\}, (\mathscr{L} \otimes \mathscr{L}^*)_{x_0}).$$

The last space is $\cong H_\bullet^{M_L(\varphi_0)}$ (by 1.12 *b)*, 2.3 *d)* and the irreducibility of $\mathscr{L}$) and hence it is $\cong H^\bullet_{M_L(\varphi_0)} = \mathbf{S}$ (by 1.8 *c)*).

Combining these isomorphisms we get an isomorphism $H_\bullet^{G \times \mathbf{C}^*}(\ddot{V}^\Omega, \ddot{\mathscr{L}}) \cong \mathbf{S}$. The proposition follows (in the case where Ω is a single P — P double coset).

We now prove the proposition for general Ω, by induction on the number $N(\Omega)$ of P — P double cosets contained in Ω. The case $N(\Omega) = 1$ is already settled. Assume now that $N(\Omega) > 1$. Let Ω_1 be a P — P double coset contained and open in Ω. We may assume that the proposition is already proved for $\Omega - \Omega_1$ and Ω_1. We apply 1.5 *a)* to the partition $\ddot{V}^\Omega = \ddot{V}^{\Omega_1} \cup \ddot{V}^{\Omega - \Omega_1}$. Assume first that Ω_1 is bad, so that $H_\bullet^{G \times \mathbf{C}^*}(\ddot{V}^{\Omega_1}, \ddot{\mathscr{L}}) = 0$. If $n(\Omega - \Omega_1) = 0$ then $H_\bullet^{G \times \mathbf{C}^*}(\ddot{V}^{\Omega_1}, \ddot{\mathscr{L}}) = H_\bullet^{G \times \mathbf{C}^*}(\ddot{V}^{\Omega - \Omega_1}, \ddot{\mathscr{L}}) = 0$ and 1.5 *a)* shows that $H_\bullet^{G \times \mathbf{C}^*}(\ddot{V}^\Omega, \ddot{\mathscr{L}}) = 0$, as required. If $n(\Omega - \Omega_1) \geqslant 1$ then from 3.3 *g)*, *h)* we see that $\dim \ddot{V}^\Omega = \dim \ddot{V}^{\Omega - \Omega_1}$ and 1.5 *a)* shows that $H_\bullet^{G \times \mathbf{C}^*}(\ddot{V}^{\Omega - \Omega_1}, \ddot{\mathscr{L}}) \xrightarrow{\sim} H_\bullet^{G \times \mathbf{C}^*}(\ddot{V}^\Omega, \ddot{\mathscr{L}})$. It follows that $H_\bullet^{G \times \mathbf{C}^*}(\ddot{V}^\Omega, \ddot{\mathscr{L}})$ has the required property.

Next, we assume that Ω_1 is good. Then by 3.3 *g)*, *h)*, $\dim \ddot{V}^{\Omega_1} = \dim \ddot{V}^\Omega$. If $n(\Omega - \Omega_1) = 0$ then $H_\bullet^{G \times \mathbf{C}^*}(\ddot{V}^{\Omega - \Omega_1}, \ddot{\mathscr{L}}) = 0$ and 1.5 *a)* shows that

$$H_\bullet^{G \times \mathbf{C}^*}(\ddot{V}^\Omega, \ddot{\mathscr{L}}) \cong H_\bullet^{G \times \mathbf{C}^*}(\ddot{V}^{\Omega_1}, \ddot{\mathscr{L}})$$

hence $H_\bullet^{G \times \mathbf{C}^*}(\ddot{V}^\Omega, \ddot{\mathscr{L}})$ has the required property. If $n(\Omega - \Omega_1) \geqslant 1$ then we also have $\dim \ddot{V}^{\Omega - \Omega_1} = \dim \ddot{V}^\Omega$ by 3.3 *g)*, *h)* and 1.5 *a)* gives the short exact sequence in the first row of the diagram

$$\begin{tikzcd}
0 \arrow[r] & H_\bullet^{G \times \mathbf{C}^*}(\ddot{V}^{\Omega - \Omega_1}, \ddot{\mathscr{L}}) \arrow[r] & H_\bullet^{G \times \mathbf{C}^*}(\ddot{V}^\Omega, \ddot{\mathscr{L}}) \arrow[r] & H_\bullet^{G \times \mathbf{C}^*}(\ddot{V}^{\Omega_1}, \ddot{\mathscr{L}}) \arrow[r] & 0 \\
0 \arrow[r] & \mathbf{S} \otimes H_0^{G \times \mathbf{C}^*}(\ddot{V}^{\Omega - \Omega_1}, \ddot{\mathscr{L}}) \arrow[r] \arrow[u] & \mathbf{S} \otimes H_0^{G \times \mathbf{C}^*}(\ddot{V}^\Omega, \ddot{\mathscr{L}}) \arrow[r] \arrow[u] & \mathbf{S} \otimes H_0^{G \times \mathbf{C}^*}(\ddot{V}^{\Omega_1}, \ddot{\mathscr{L}}) \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

We take that exact sequence in degree 0 and tensor it with **S**; we find the short exact sequence in the second row of the diagram. We map the second exact sequence to the first using the **S**-module structure. We obtain the commutative diagram above. By the induction hypothesis, the first and third vertical maps are isomorphism, hence so is the middle one. We also see that $\dim \mathrm{H}_0^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathrm{V}}^\Omega, \ddot{\mathscr{L}}) = n(\Omega_1) + n(\Omega - \Omega_1) = n(\Omega)$. The proposition is proved.

Corollary **4.8.** — *If* Ω *in* 4.7 *is closed in* G *and* $\mathrm{V} = \mathfrak{g}_{\mathrm{N}}$ *or* $\mathfrak{g}$, *then the inclusion* $i : \ddot{\mathrm{V}}^\Omega \hookrightarrow \ddot{\mathrm{V}}$ *induces an injective homomorphism* $i_! : \mathrm{H}_\bullet^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathrm{V}}^\Omega, \ddot{\mathscr{L}}) \to \mathrm{H}_\bullet^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathrm{V}}, \ddot{\mathscr{L}})$.

Proof. — This follows from the exact sequence 1.5 *a)* applied to $\ddot{\mathrm{V}} = \ddot{\mathrm{V}}^\Omega \cup \ddot{\mathrm{V}}^{\mathrm{G}-\Omega}$ together with the vanishing of $\mathrm{H}_i^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathrm{V}}^\Omega, \ddot{\mathscr{L}})$ and $\mathrm{H}_i^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathrm{V}}^{\mathrm{G}-\Omega}, \ddot{\mathscr{L}})$ for odd i (which can be seen from 4.7).

4.9. From 3.3 *l)* and 1.5 *b)* we see that the open embedding $j : \ddot{\mathscr{V}}_{\mathrm{RS}} \hookrightarrow \ddot{\mathscr{V}} = \ddot{\mathfrak{g}}$ induces an isomorphism

a) $j^* : \mathrm{H}_0^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathfrak{g}}, \ddot{\mathscr{L}}) \overset{\approx}{\to} \mathrm{H}_0^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathscr{V}}_{\mathrm{RS}}, \ddot{\mathscr{L}})$.

Similarly, for any good P — P double coset $\Omega(w)$, the open embedding

$$\ddot{\mathscr{V}}_{\mathrm{RS}}^{\Omega(w)} \hookrightarrow \ddot{\mathscr{V}}^{\Omega(w)} = \ddot{\mathfrak{g}}^{\Omega(w)}$$

induces an isomorphism

b) $\mathrm{H}_0^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathfrak{g}}^{\Omega(w)}, \ddot{\mathscr{L}}) \overset{\sim}{\to} \mathrm{H}_0^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathscr{V}}_{\mathrm{RS}}^{\Omega(w)}, \ddot{\mathscr{L}})$.

In particular, using 4.7 we have

$$\dim \mathrm{H}_0^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathscr{V}}_{\mathrm{RS}}^{\Omega(w)}, \ddot{\mathscr{L}}) = 1.$$

Using 3.3 *i)* we see that the open and closed embedding $\ddot{\mathscr{V}}_{\mathrm{RS}}^{\Omega(w)} \hookrightarrow \ddot{\mathscr{V}}_{\mathrm{RS}}$ defines an isomorphism

c) $\bigoplus_{w\in\mathrm{W}} \mathrm{H}_\bullet^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathscr{V}}_{\mathrm{RS}}^{\Omega(w)}, \ddot{\mathscr{L}}) \overset{\sim}{\to} \mathrm{H}_\bullet^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathscr{V}}_{\mathrm{RS}}, \ddot{\mathscr{L}})$.

This gives a direct sum decomposition

c') $\mathrm{H}_\bullet^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathscr{V}}_{\mathrm{RS}}, \ddot{\mathscr{L}}) = \bigoplus_{w\in\mathrm{W}} \mathrm{D}_w$,

where D_w is the image under *c)* of the summand corresponding to w (a graded space).

The $\mathrm{W} \times \mathrm{W}$-module structure on $\mathrm{H}_\bullet^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathscr{V}}_{\mathrm{RS}}, \ddot{\mathscr{L}})$ satisfies

d) $\Delta(w)\,\mathrm{D}_e = \mathrm{D}_w$, $\Delta'(w)\,\mathrm{D}_e = \mathrm{D}_{w^{-1}}$, $(w \in \mathrm{W})$;
this follows immediately from the definitions.

Taking components of degree 0 we obtain

e) $\mathrm{H}_0^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathscr{V}}_{\mathrm{RS}}, \ddot{\mathscr{L}}) = \bigoplus_{w\in\mathrm{W}} \Delta(w)\,\mathrm{D}_{e,0} = \bigoplus_{w\in\mathrm{W}} \Delta'(w)\,\mathrm{D}_{e,0}$

f) $\Delta(w)\,\mathrm{D}_{e,0} = \Delta'(w^{-1})\,\mathrm{D}_{e,0}$ ($\mathrm{D}_{e,0}$ is a line).

4.10. For $V = \mathfrak{g}$, $\mathfrak{g}_N$ or $\mathcal{V}_{RS}$ we defines a subspace $D_V \subset H_0^{G\times C^*}(\ddot{V}, \ddot{\mathcal{L}})$ as the image of the homomorphism $H_0^{G\times C^*}(\ddot{V}^P, \ddot{\mathcal{L}}) \to H_0^{G\times C^*}(\ddot{V}, \ddot{\mathcal{L}})$ induced by the closed embedding $\ddot{V}^P \hookrightarrow \ddot{V}$. (In each case, we have $\dim \ddot{V}^P = \dim \ddot{V}$, see 3.3.)

We shall denote D_V in the three cases $V = \mathfrak{g}, \mathfrak{g}_N, \mathcal{V}_{RS}$ as D, D_N, $D_{e,0}$, respectively. (Note that $D_{e,0}$ has already been introduced in 4.9. It is a line. Similarly, D and D_N are lines, by 4.8 and 4.7.)

These lines are related as follows:

a) Under the isomorphism 4.9 *a)*, D corresponds to $D_{e,0}$.

b) The homomorphism $H_0^{G\times C^*}(\ddot{\mathfrak{g}}_N, \ddot{\mathcal{L}}) \to H_{2b}^{G\times C^*}(\ddot{\mathfrak{g}}, \ddot{\mathcal{L}})$ induced by the closed embedding $\ddot{\mathfrak{g}}_N \subset \ddot{\mathfrak{g}}$ maps the line D_N onto the line $r^b.D$ ($b = \dim \mathfrak{t}$).

We prove *a)*. The cartesian diagram

$$\begin{tikzcd}
\ddot{\mathcal{V}}^P_{RS} \arrow[r, hook] \arrow[d] & \ddot{\mathcal{V}}_{RS} \arrow[d] \\
\ddot{\mathfrak{g}}^P \arrow[r, hook] & \ddot{\mathfrak{g}}
\end{tikzcd}$$

induces a commutative diagram

$$\begin{tikzcd}
H_0^{G\times C^*}(\ddot{\mathcal{V}}^P_{RS}, \ddot{\mathcal{L}}) \arrow[r] & H_0^{G\times C^*}(\ddot{\mathcal{V}}_{RS}, \ddot{\mathcal{L}}) \\
H_0^{G\times C^*}(\ddot{\mathfrak{g}}^P, \ddot{\mathcal{L}}) \arrow[r] \arrow[u, "\wr"'] & H_0^{G\times C^*}(\ddot{\mathfrak{g}}, \ddot{\mathcal{L}}) \arrow[u, "\wr"']
\end{tikzcd}$$

and *a)* follows.

We now prove *b)*. The commutative diagram

$$\begin{tikzcd}
\ddot{\mathfrak{g}}^P_N \arrow[r, hook] \arrow[d, hook, "f"'] & \ddot{\mathfrak{g}}_N \arrow[d, hook] \\
\ddot{\mathfrak{g}}^P \arrow[r, hook] & \ddot{\mathfrak{g}}
\end{tikzcd}$$

induces a commutative diagram

$$\begin{tikzcd}
H_0^{G\times C^*}(\ddot{\mathfrak{g}}^P_N, \ddot{\mathcal{L}}) \arrow[r, hook] \arrow[d, "f_!"'] & H_0^{G\times C^*}(\ddot{\mathfrak{g}}_N, \ddot{\mathcal{L}}) \arrow[d] \\
H_{2b}^{G\times C^*}(\ddot{\mathfrak{g}}^P, \ddot{\mathcal{L}}) \arrow[r, hook] & H_{2b}^{G\times C^*}(\ddot{\mathfrak{g}}, \ddot{\mathcal{L}})
\end{tikzcd}$$

and we are reduced to showing that image $(f_!) = r^b.\mathrm{H}_0^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathfrak{g}}^{\mathrm{P}}, \ddot{\mathscr{L}})$. As in the proof of 4.7 we have

$$\mathrm{H}_0^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathfrak{g}}_{\mathrm{N}}^{\mathrm{P}}, \ddot{\mathscr{L}}) = \mathrm{H}_0^{\mathrm{L}\times\mathbf{C}^*}(\mathscr{C}, \mathscr{L}\otimes\mathscr{L}^*) = \mathrm{H}_0^{\mathrm{M}\overset{0}{\mathfrak{L}}(\varphi_0)}(\{x_0\}, \mathscr{E})$$

$$\mathrm{H}_{2b}^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathfrak{g}}^{\mathrm{P}}, \ddot{\mathscr{L}}) = \mathrm{H}_{2b}^{\mathrm{L}\times\mathbf{C}^*}(\mathscr{C}+\mathfrak{t}, p_1^*(\mathscr{L}\otimes\mathscr{L}^*)) = \mathrm{H}_{2b}^{\mathrm{M}\overset{0}{\mathfrak{L}}(\varphi_0)}(x_0+\mathfrak{t}, \mathscr{E})$$

(where $p_1 : \mathscr{C}+\mathfrak{t}\to\mathscr{C}$ is the projection and $\mathscr{E}$ denotes $(\mathscr{L}\otimes\mathscr{L}^*)_{x_0}$ regarded as a local system over $\{x_0\}$ and also its inverse image under $x_0+\mathfrak{t}\to\{x_0\}$). We have only to show that the homomorphism

$$f'_! : \mathrm{H}_0^{\mathrm{M}\overset{0}{\mathfrak{L}}(\varphi_0)}(\{x_0\}, \mathscr{E}) \to \mathrm{H}_{2b}^{\mathrm{M}\overset{0}{\mathfrak{L}}(\varphi_0)}(x_0+\mathfrak{t}, \mathscr{E})$$

induced by the inclusion $f' : \{x_0\}\hookrightarrow x_0+t$ has as image $r^b\,\mathrm{H}_0^{\mathrm{M}\overset{0}{\mathfrak{L}}(\varphi_0)}(x_0+\mathfrak{t}, \mathscr{E})$. But this follows from 1.10 *b)*.

4.11. From 4.9 *e)*, *f)*, 4.10 *a)*, 4.9 *a)* and 3.10, it follows that

a) $\mathrm{H}_0^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathfrak{g}}, \ddot{\mathscr{L}}) = \bigoplus_{w\in\mathrm{W}} \Delta(w)\,\mathrm{D} = \bigoplus_{w\in\mathrm{W}} \Delta'(w)\,\mathrm{D}$

and

b) $\Delta(w)\,\mathrm{D} = \Delta'(w^{-1})\,\mathrm{D}$, $w\in\mathrm{W}$.

We now show that

c) the homomorphism $\gamma_! : \mathrm{H}_j^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathfrak{g}}_{\mathrm{N}}, \ddot{\mathscr{L}}) \to \mathrm{H}_{j+2b}^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathfrak{g}}, \ddot{\mathscr{L}})$ induced by the closed embedding $\gamma : \ddot{\mathfrak{g}}_{\mathrm{N}}\subset\ddot{\mathfrak{g}}$, is injective ($b = \dim\mathfrak{t}$).

Indeed, using 4.6 we are reduced to the case where $j = 0$. Let I be the image of γ (for $j = 0$). By 4.10 *b)*, I contains $r^b\,\mathrm{D}$. But I is clearly $\mathrm{W}\times\mathrm{W}$-stable (see 3.10) hence it contains $\sum_{w\in\mathrm{W}} r^b\,\Delta(w)\,\mathrm{D} = r^b\,\mathrm{H}_0^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathfrak{g}}, \ddot{\mathscr{L}})$ (see *a)*). This subspace has dimension equal to $\sharp\mathrm{W}$ (by 4.6). Thus $\dim\mathrm{I}\geqslant\sharp\mathrm{W}$. Using 4.6, we have $\dim\mathrm{H}_0^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathfrak{g}}_{\mathrm{N}}, \ddot{\mathscr{L}}) = \sharp\mathrm{W}$. It follows that $\dim\mathrm{I} = \sharp\mathrm{W}$ and γ is injective.

We have:

d) $\mathrm{H}_0^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathfrak{g}}_{\mathrm{N}}, \ddot{\mathscr{L}}) = \bigoplus_{w\in\mathrm{W}} \Delta(w)\,\mathrm{D}_{\mathrm{N}} = \bigoplus_{w\in\mathrm{W}} \Delta'(w)\,\mathrm{D}_{\mathrm{N}}$,

e) $\Delta(w)\,\mathrm{D}_{\mathrm{N}} = \Delta'(w^{-1})\,\mathrm{D}_{\mathrm{N}}$.

Indeed, by the previous argument, $h\to r^{-b}\gamma_!\,h$ defines an isomorphism

$$\mathrm{H}_0^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathfrak{g}}_{\mathrm{N}}, \ddot{\mathscr{L}}) \xrightarrow{\sim} \mathrm{H}_0^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathfrak{g}}, \ddot{\mathscr{L}})$$

compatible with the $\mathrm{W}\times\mathrm{W}$ actions and taking D_{N} to D; it remains to apply *a)*, *b)*.

4.12. We now want to study the **S**-module structures on $\mathrm{H}_\bullet^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathscr{V}}_{\mathrm{RS}}, \ddot{\mathscr{L}})$ in the case where P is a maximal parabolic subgroup. In this case, we have $\mathrm{W} = \{e, s\}$ and we have the following result.

Proposition **4.13.** — a) *The* **S**-*module structure on* $H_{\bullet}^{G \times \mathbf{C}^*}(\ddot{\mathscr{V}}_{RS}, \ddot{\mathscr{L}})$ *defined by the operators* $\Delta(\xi)$, $\xi \in \mathbf{S}$, *gives an isomorphism*

$$\mathbf{S}/(r) \otimes H_0^{G \times \mathbf{C}^*}(\ddot{\mathscr{V}}_{RS}, \ddot{\mathscr{L}}) \xrightarrow{\sim} H_{\bullet}^{G \times \mathbf{C}^*}(\ddot{\mathscr{V}}_{RS}, \ddot{\mathscr{L}})$$

where (r) *is the ideal of* **S** *generated by* r.

b) *On the image of* $H_{\bullet}^{G \times \mathbf{C}^*}(\ddot{\mathscr{V}}_{RS}^{P}, \ddot{\mathscr{L}}) \hookrightarrow H_{\bullet}^{G \times \mathbf{C}^*}(\ddot{\mathscr{V}}_{RS}, \ddot{\mathscr{L}})$ *we have* $\Delta(\xi) = \Delta'(\xi)$, $\xi \in \mathbf{S}$.

c) *On the image of* $H_{\bullet}^{G \times \mathbf{C}^*}(\ddot{\mathscr{V}}_{RS}^{\Omega(s)}, \ddot{\mathscr{L}}) \hookrightarrow H_{\bullet}^{G \times \mathbf{C}^*}(\ddot{\mathscr{V}}_{RS}, \ddot{\mathscr{L}})$ *we have* $\Delta(\xi) = \Delta'({}^s\xi)$, $\xi \in \mathbf{S}$.

d) $\Delta(s)\,\Delta(\xi)\,h = \Delta({}^s\xi)\,\Delta(s)\,h$ *for all* $h \in H_{\bullet}^{G \times \mathbf{C}^*}(\ddot{\mathscr{V}}_{RS}, \ddot{\mathscr{L}})$, $\xi \in \mathbf{S}$.

Proof. — We have $\mathfrak{t}_{reg} = \mathfrak{t} - \mathfrak{t}_0$ where $\mathfrak{t}_0$ is the centre of $\mathfrak{g}$ (a hyperplane in $\mathfrak{t}$).

In the following calculations we shall often omit writing local systems. Using the $P \times \mathbf{C}^*$-equivariant embedding $\mathscr{C} + (\mathfrak{t} - \mathfrak{t}_0) + \mathfrak{n} \hookrightarrow \ddot{\mathscr{V}}_{RS}^{P}$, $x \mapsto (x, P, P)$, we have

$$\begin{aligned} H_{\bullet}^{G \times \mathbf{C}^*}(\ddot{\mathscr{V}}_{RS}^{P}, \ddot{\mathscr{L}}) &\cong H_{\bullet}^{P \times \mathbf{C}^*}(\mathscr{C} + (\mathfrak{t} - \mathfrak{t}_0) + \mathfrak{n},\ ) \quad \text{(see 1.6 } a)) \\ &\cong H_{\bullet}^{L \times \mathbf{C}^*}(\mathscr{C} + (\mathfrak{t} - \mathfrak{t}_0),\ ) \quad \text{(see 1.4 } h), \text{ 1.4 } e)). \end{aligned}$$

Let $\mathfrak{t}_1$ be an affine hyperplane in $\mathfrak{t}$, parallel to $\mathfrak{t}_0$ but distinct from $\mathfrak{t}_0$. The stabilizer of $\mathscr{C} + \mathfrak{t}_1$ in $L \times \mathbf{C}^*$ is $L \times (\pm 1)$ and we have

$$\begin{aligned} H_{\bullet}^{G \times \mathbf{C}^*}(\ddot{\mathscr{V}}_{RS}^{P}, \mathscr{L}) &\cong H_{\bullet}^{L \times (\pm 1)}(\mathscr{C} + \mathfrak{t}_1,\ ) \quad \text{(by 1.6 } a)) \\ &\cong H_{\bullet}^{L \times (\pm 1)}(\mathscr{C}, \mathscr{L} \otimes \mathscr{L}^*) \quad \text{(by 1.4 } e)) \\ &\cong H_{\bullet}^{Z_L(x_0) \times (\pm 1)}(\{x_0\}, (\mathscr{L} \otimes \mathscr{L}^*)_{x_0}) \quad \text{(by 1.6 } a)) \\ &\cong H_{\bullet}^{Z_L(\varphi_0) \times (\pm 1)}(\{x_0\}, \mathscr{L} \otimes \mathscr{L}^*)_{x_0}) \quad \text{(by 1.4 } h)) \\ &\cong H_{\bullet}^{Z_L^0(\varphi_0)} = H^T. \quad \text{(by 1.12 } b), \text{ 2.3 } b), d)). \end{aligned}$$

Using the $L \times \mathbf{C}^*$ equivariant embedding $i : \mathscr{C} + (\mathfrak{t} - \mathfrak{t}_0) \hookrightarrow \ddot{\mathscr{V}}_{RS}^{\Omega(s)}$, $x \mapsto (x, P, \dot{s}P)$, we have

$$\begin{aligned} H_{\bullet}^{G \times \mathbf{C}^*}(\ddot{\mathscr{V}}_{RS}^{\Omega(s)}, \ddot{\mathscr{L}}) &\cong H_{\bullet}^{L \times \mathbf{C}^*}(\mathscr{C} + (\mathfrak{t} - \mathfrak{t}_0),\ ) \quad \text{(see 1.6 } a)) \\ &\cong H_{\bullet}^{T} \quad \text{(as above).} \end{aligned}$$

Using these isomorphisms and 4.9 *c)*, we see that *a)* holds.

Now *b)* follows from 4.3 *e)*.

We now prove *c)*. Note that using i above, we have

$$\begin{aligned} e)\ H^{\bullet}_{G \times \mathbf{C}^*}(\ddot{\mathscr{V}}_{RS}^{\Omega(s)}) &\cong H^{\bullet}_{L \times \mathbf{C}^*}(\mathscr{C} + (\mathfrak{t} - \mathfrak{t}_0)) \quad \text{(by 1.6 } a)) \\ &\cong H^{\bullet}_{L \times (\pm 1)}(\mathscr{C} + \mathfrak{t}_1) \cong H^{\bullet}_{Z_L(x_0) \times (\pm 1)} \cong H^{\bullet}_{T} \quad \text{(as above)} \end{aligned}$$

Consider the automorphisms φ, σ of $\ddot{\mathscr{V}}_{RS}^{\Omega(s)}$ defined by $\varphi(x, gP, g'P) = (x, g'P, gP)$ and $\sigma(x, gP, g'P) = (\mathrm{Ad}(\dot{s})\,x, \dot{s}gP, \dot{s}g'P)$.

We have a commutative diagram

$$\begin{tikzcd} \mathscr{C} + (\mathfrak{t} - \mathfrak{t}_0) \arrow[r, "i"] \arrow[d, "\mathrm{Ad}(\dot{s})"'] & \ddot{\mathscr{V}}_{RS}^{\Omega(s)} \arrow[d, "\varphi \circ \sigma"] \\ \mathscr{C} + (t - t_0) \arrow[r, "i"] & \ddot{\mathscr{V}}_{RS}^{\Omega(s)} \end{tikzcd}$$

This diagram shows that the automorphism of $\mathrm{H}^\bullet_{\mathrm{T}}$ induced by conjugation by $\dot{s}$ on T corresponds under *e)* to the automorphism φ^* of $\mathrm{H}^\bullet_{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathscr{V}}^{\Omega(s)}_{\mathrm{RS}})$ induced by φ. (We can ignore σ.) Hence, if $p_{12}, p_{13} : \ddot{\mathscr{V}}^{\Omega(s)}_{\mathrm{RS}} \to \dot{\mathfrak{g}}$ are the projections, we have $\varphi^*(p^*_{12}(\xi)) = p^*_{12}({}^s\xi)$ for $\xi \in \mathrm{H}^\bullet_{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathfrak{g}}) = \mathbf{S}$. We have $p_{13} = p_{12} \circ \varphi$ hence $p^*_{13}(\xi) = \varphi^*(p^*_{12}(\xi)) = p^*_{12}({}^s\xi)$. By definition, $\Delta'(\xi)\,\eta = p^*_{13}(\xi)\,.\,\eta = p^*_{12}({}^s\xi)\,.\,\eta = \Delta({}^s\xi)\,\eta$, $\eta \in \mathrm{H}^{\mathrm{G}\times\mathbf{C}^*}_\bullet(\ddot{\mathscr{V}}^{\Omega(s)}_{\mathrm{RS}}, \ddot{\mathscr{L}})$.

This completes the proof of *c)*.

We now prove *d)*. Assume first that $h \in \mathrm{D}_e$ (notation of 4.9). Then $\Delta(s)\,h \in \mathrm{D}_s$ by 4.9 *f)*, hence we have

$$\begin{aligned}\Delta(\xi)\,\Delta(s)\,h &= \Delta'({}^s\xi)\,\Delta(s)\,h \quad (\text{by } c))\\ &= \Delta(s)\,\Delta'({}^s\xi)\,h \quad (\text{by } 4.4\ a))\\ &= \Delta(s)\,\Delta({}^s\xi)\,h \quad (\text{by } 4.3\ e))\end{aligned}$$

as desired. If now $h \in \mathrm{D}_s$, then applying the identity *d)* to $\Delta(s)\,h \in \mathrm{D}_e$ we obtain $\Delta(s)\,\Delta(\xi)\,\Delta(s)\,h = \Delta({}^s\xi)\,h$. Multiplying both sides by $\Delta(s)$ gives $\Delta(\xi)\,\Delta(s)\,h = \Delta(s)\,\Delta({}^s\xi)\,h$.

We now use that $\mathrm{H}^{\mathrm{G}\times\mathbf{C}^*}_\bullet(\ddot{\mathscr{V}}_{\mathrm{RS}}, \ddot{\mathscr{L}}) = \mathrm{D}_e + \mathrm{D}_s$ and *d)* follows.

Corollary 4.14. — *In the setup of* 4.12, *there exists* $p \in \mathbf{C}$ *such that the identity*

$$\Delta(s)\,\Delta(\alpha) + \Delta(\alpha)\,\Delta(s) = p\ \mathrm{Id}$$

holds for all $h \in \mathrm{H}^{\mathrm{G}\times\mathbf{C}^*}_\bullet(\ddot{\mathfrak{g}}, \ddot{\mathscr{L}})$.

Proof. — From 4.6 and 4.13 *a)* we see that the open embedding $j : \ddot{\mathscr{V}}_{\mathrm{RS}} \hookrightarrow \ddot{\mathscr{V}} = \ddot{\mathfrak{g}}$ gives rise to an exact sequence

a) $0 \to \mathrm{H}^{\mathrm{G}\times\mathbf{C}^*}_\bullet(\ddot{\mathfrak{g}}, \ddot{\mathscr{L}}) \xrightarrow{r} \mathrm{H}^{\mathrm{G}\times\mathbf{C}^*}_\bullet(\ddot{\mathfrak{g}}, \ddot{\mathscr{L}}) \xrightarrow{j^*} \mathrm{H}^{\mathrm{G}\times\mathbf{C}^*}_\bullet(\ddot{\mathfrak{g}}, \ddot{\mathscr{L}}) \to 0.$

Since $\Delta(s)\,\Delta(\alpha) + \Delta(\alpha)\,\Delta(s) \equiv 0$ on $\mathrm{H}^{\mathrm{G}\times\mathbf{C}^*}_\bullet(\ddot{\mathfrak{g}}, \ddot{\mathscr{L}})$, we see from *a)* that there exists a $\mathbf{C}$-linear degree-preserving endomorphism Φ of $\mathrm{H}^{\mathrm{G}\times\mathbf{C}^*}_\bullet(\ddot{\mathfrak{g}}, \ddot{\mathscr{L}})$ such that

b) $\Delta(s)\,\Delta(\alpha)\,h + \Delta(\alpha)\,\Delta(s)\,h = r\Phi(h)$ for all $h \in \mathrm{H}^{\mathrm{G}\times\mathbf{C}^*}_\bullet(\ddot{\mathfrak{g}}, \ddot{\mathscr{L}})$.

We know that $\Delta(s)\,\Delta(\alpha^2)\,h = \Delta(\alpha^2)\,\Delta(s)\,h$, see 4.3 *d)*. Applying *b)* twice, we have

$$\begin{aligned}\Delta(\alpha^2)\,\Delta(s)\,h = \Delta(s)\,\Delta(\alpha)\,\Delta(\alpha)\,h &= -\,\Delta(\alpha)\,\Delta(s)\,\Delta(\alpha)\,h + r\Phi(\Delta(\alpha)\,h)\\ &= -\,\Delta(\alpha)\,(-\,\Delta(\alpha)\,\Delta(s)\,h + r\Phi(h)) + r\Phi(\Delta(\alpha)\,h)\\ &= \Delta(\alpha^2)\,\Delta(s)\,h - r\,\Delta(\alpha)\,\Phi(h) + r\Phi(\Delta(\alpha)\,h).\end{aligned}$$

Thus $r(\Phi(\Delta(\alpha)\,h) - \Delta(\alpha)\,\Phi(h)) = 0$. Cancelling r, we have

c) $\Phi(\Delta(\alpha)\,h) = \Delta(\alpha)\,\Phi(h)$.

Since $\Delta'(\alpha)$, $\Delta'(s)$ commute with $\Delta(s)$, $\Delta(\alpha)$ we also have

d) $\Phi(\Delta'(\alpha)\,h) = \Delta'(\alpha)\,\Phi(h)$,

e) $\Phi(\Delta'(s)\,h) = \Delta'(s)\,\Phi(h)$.

If $h_0 \in \mathrm{D}$, we have $\Delta(\alpha)\,h_0 = \Delta'(\alpha)\,h_0$ (see 4.3 *e)*).

If however, $h'_0 \in \Delta(s)\,\mathrm{D}$, $h'_0 \neq 0$, then $\Delta(\alpha)\,h'_0 \neq \Delta'(\alpha)\,h'_0$. Indeed, we have $h'_0 \in \Delta(s)\,h_0$, $h_0 \in \mathrm{D}$

$$\begin{aligned}\Delta'(\alpha)\,h'_0 = \Delta'(\alpha)\,\Delta(s)\,h_0 = \Delta(s)\,\Delta'(\alpha)\,h_0 &= \Delta(s)\,\Delta(\alpha)\,h_0\\ &= -\,\Delta(\alpha)\,\Delta(s)\,h_0 + r\Phi(h_0),\\ \Delta(\alpha)\,h'_0 = \Delta(\alpha)\,\Delta(s)\,h_0.&\end{aligned}$$

If we had $\Delta'(\alpha)\,h'_0 = \Delta(\alpha)\,h'_0$, then $2\Delta(\alpha)\,\Delta(s)\,h_0 = r\Phi(h_0)$. Using 4.6 it would follow that $h_0 = 0$ so $h'_0 = 0$, a contradiction.

Thus, $\mathrm{D} = \mathrm{Ker}(\Delta(\alpha) - \Delta'(\alpha) : \mathrm{H}_0^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathfrak{g}}, \ddot{\mathscr{L}}) \to \mathrm{H}_2^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathfrak{g}}, \ddot{\mathscr{L}}))$.

By *c)*, *d)*, Φ maps this kernel into itself, hence $\Phi\mathrm{D} \subset \mathrm{D}$. If h_0 is a non-zero element of D we have therefore $\Phi h_0 = p h_0$ for some $p \in \mathbf{C}$.

Using *e)* we have $\Phi(\Delta'(s)\,h_0) = p\,\Delta'(s)\,h_0$. Since h_0, $\Delta'(s)\,h_0$ form a basis of $\mathrm{H}_0^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathfrak{g}}, \ddot{\mathscr{L}})$ we must have $\Phi(h) = ph$ for all h of degree 0. Using 4.6 and *d)* we have $\Phi(h) = ph$ for all h.

Proposition **4.15.** — *The open embedding* $\mathscr{V}_{\mathrm{RN}} \hookrightarrow \mathscr{V}_{\mathrm{N}}$ *gives rise to an open embedding* $j : \ddot{\mathscr{V}}_{\mathrm{RN}} \hookrightarrow \ddot{\mathscr{V}}_{\mathrm{N}} = \ddot{\mathfrak{g}}_{\mathrm{N}}$ *and this induces a surjective homomorphism*

$$j^* : \mathrm{H}_0^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathfrak{g}}_{\mathrm{N}}, \ddot{\mathscr{L}}) \to \mathrm{H}_0^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathscr{V}}_{\mathrm{RN}}, \ddot{\mathscr{L}}).$$

Proof. — j^* preserves degrees by 3.3 *k)*. We have a cartesian diagram

$$\begin{tikzcd} \ddot{\mathscr{V}}^{\mathrm{P}}_{\mathrm{RN}} \arrow[r, hook, "j'"] \arrow[d, equal, "3.3\ (j)"'] & \ddot{\mathfrak{g}}^{\mathrm{P}}_{\mathrm{N}} \arrow[d] \\ \ddot{\mathscr{V}}_{\mathrm{RN}} \arrow[r, hook, "j"'] & \ddot{\mathfrak{g}}_{\mathrm{N}} \end{tikzcd}$$

It induces a commutative diagram

$$\begin{tikzcd} \mathrm{H}_0^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathscr{V}}^{\mathrm{P}}_{\mathrm{RN}}, \ddot{\mathscr{L}}) \arrow[d, equal] & \mathrm{H}_0^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathfrak{g}}^{\mathrm{P}}_{\mathrm{N}}, \ddot{\mathscr{L}}) \arrow[l, "j'^*"'] \arrow[d] \\ \mathrm{H}_0^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathscr{V}}_{\mathrm{RN}}, \ddot{\mathscr{L}}) & \mathrm{H}_0^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathfrak{g}}_{\mathrm{N}}, \ddot{\mathscr{L}}) \arrow[l, "j^*"] \end{tikzcd}$$

and it remains to show that j'^* is surjective. But j'^* is in fact an isomorphism, by 3.3 *l)* and 1.5 *b)*.

5. The commutation formula

Theorem **5.1.** — *Let* $s_i \in \mathrm{W}$, $\alpha_i \in \mathrm{R}$ *be as in* 2.5, *let* $c_i \geqslant 2$ *be as in* 2.12 $(1 \leqslant i \leqslant m)$, *let* $\xi \in \mathbf{S}$ *and let* $h \in \mathrm{H}_\bullet^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathrm{V}}, \ddot{\mathscr{L}})$, $\mathrm{V} = \mathfrak{g}$ *or* $\mathfrak{g}_{\mathrm{N}}$. *Then*

a) $\Delta(s_i)\,\Delta(\xi)\,h - \Delta({}^{s_i}\xi)\,\Delta(s_i)\,h = c_i\,\Delta(r(\xi - {}^{s_i}\xi)/\alpha_i)\,h$

b) $\Delta'(s_i)\,\Delta'(\xi)\,h - \Delta'({}^{s_i}\xi)\,\Delta'(s_i)\,h = c_i\,\Delta'(r(\xi - {}^{s_i}\xi)/\alpha_i)\,h.$

5.2. The proof will occupy most of this chapter. We shall first make some reductions. Using the injectivity of $\gamma_! : \mathrm{H}_\bullet^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathfrak{g}}_{\mathrm{N}}, \ddot{\mathscr{L}}) \to \mathrm{H}_\bullet^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathfrak{g}}, \ddot{\mathscr{L}})$ in 4.11 *c)* we see that if the theorem is known for $\mathrm{V} = \mathfrak{g}$, then it also follows for $\mathrm{V} = \mathfrak{g}_{\mathrm{N}}$. ($\gamma_!$ commutes with the operators $\Delta(s_i)$, $\Delta'(s_i)$, cf. 3.10, and with the operators $\Delta(\xi)$, $\Delta'(\xi)$.) Hence we can assume that $\mathrm{V} = \mathfrak{g}$. It is enough to prove 5.1 *a)*; the proof of 5.1 *b)* is the same.

Let P_i, Δ_i be as in 2.4, $\overline{\mathrm{P}}$, $\dot{\mathfrak{g}}_2$, $\overline{\mathscr{L}}$ as in 3.9. Assume that 5.1 *a)* holds on $\mathrm{H}_\bullet^{\mathrm{L}_i\times\mathbf{C}^*}(\dot{\mathfrak{g}}_2, \overline{\mathscr{L}})$ (i.e. when G, P, L, $\mathscr{C}$ are replaced by L_i, $\overline{\mathrm{P}}$, L, $\mathscr{C}$). The map 3.9 *k)* is compatible with the operators $\Delta(s_i)$ and the operators $\Delta(\xi)$; moreover, from its definition, its image is the same as the image I of

$$\mathrm{H}_\bullet^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathfrak{g}}^{\mathrm{P}_i}, \ddot{\mathscr{L}}) \hookrightarrow \mathrm{H}_\bullet^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathfrak{g}}, \ddot{\mathscr{L}})$$

(see 4.8 with $\Omega = \mathrm{P}_i$). Hence the identity 5.1 *a)* holds for $h \in \mathrm{I}$; in particular, it holds for $h \in \mathrm{D}$ (see 4.10). Hence it holds for h in $\sum_{w\in\mathrm{W}} \Delta'(w)\,\mathrm{D}$ (since $\Delta'(w)$ commutes with $\Delta(s_i)$, $\Delta(\eta)$ $(\eta \in \mathbf{S})$), hence, by 4.11 *a)* it holds for any $h \in \mathrm{H}_0^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathfrak{g}}, \ddot{\mathscr{L}})$. It must then hold for any element of form $h = \Sigma\,\Delta'(\eta_j)\,h_j$ $(\eta_j \in \mathbf{S}, h_j$ of degree 0), since $\Delta'(\eta_j)$ commutes with $\Delta(s_i)$, $\Delta(\eta)$ $(\eta \in \mathbf{S})$. But in this way one obtains the most general elements of $\mathrm{H}_\bullet^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathfrak{g}}, \ddot{\mathscr{L}})$ (by 4.6). Thus, we are reduced to proving 5.1 *a)* for L_i instead of G.

Therefore, *in the rest of this chapter we shall assume (as we may) that* $\mathrm{V} = \mathfrak{g}$ *and* P *is a maximal parabolic subgroup of* G. We shall write s, α, c instead of s_i, α_i, c_i.

Note that if the identity 5.1 *a)* holds for $\xi = \xi_1$ and $\xi = \xi_2$ (and any h) then it also holds for $\xi = \xi_1\,\xi_2$ and $\xi = a_1\,\xi_1 + a_2\,\xi_2$ $(a_1, a_2 \in \mathbf{C})$. Hence it is enough to prove 5.1 *a)* when ξ runs through a set of algebra generators of $\mathbf{S}$, for instance $\{\alpha\} \cup \mathbf{S}^{\mathrm{W}}$.

If $\xi \in \mathbf{S}^{\mathrm{W}}$ we have $\Delta(\xi) = \Delta'(\xi)$ (see 4.3 *d)*) and $\Delta(s_i)\,\Delta'(\xi) = \Delta'(\xi)\,\Delta(s_i)$ (see 4.4 *a)*) hence $\Delta(s_i)\,\Delta(\xi) = \Delta(\xi)\,\Delta(s_i)$ so that 5.1 *a)* holds in this case.

It remains to verify 5.1 *a)* for $\xi = \alpha$. We have ${}^{s}\alpha = -\alpha$ so that in this case, 5.1 *a)* is equivalent to

$$\Delta(s)\,\Delta(\alpha)\,h + \Delta(\alpha)\,\Delta(s)\,h = 2crh.$$

By 4.14, we only have to show that p in 4.14 is equal to $2c$.

Proposition **5.3.** — a) *The two* $\mathbf{S}$*-module structures on* $\mathrm{H}_\bullet^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathscr{V}}_{\mathrm{RN}}, \ddot{\mathscr{L}})$ *defined by* $\Delta(\xi)$, $\Delta'(\xi)$ *coincide*; *they give an isomorphism of* $\mathbf{S}$*-modules*

$$\mathbf{S}/(\alpha - cr) \otimes \mathrm{H}_0^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathscr{V}}_{\mathrm{RN}}, \ddot{\mathscr{L}}) \overset{\approx}{\to} \mathrm{H}_\bullet^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathscr{V}}_{\mathrm{RN}}, \ddot{\mathscr{L}})$$

where $(\alpha - cr)$ *is the ideal generated by* $\alpha - cr$. *In particular, on* $\mathrm{H}_\bullet^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathscr{V}}_{\mathrm{RN}}, \ddot{\mathscr{L}})$ *we have* $\Delta(\alpha) = \Delta'(\alpha) = cr$ *and multiplication by* r *has kernel* 0.

b) $\Delta(s) = \Delta'(s) = \mathrm{Id}$ *on* $\mathrm{H}_\bullet^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathscr{V}}_{\mathrm{RN}}, \ddot{\mathscr{L}})$.

c) $\dim \mathrm{H}_0^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathscr{V}}_{\mathrm{RN}}, \ddot{\mathscr{L}}) = 1$.

Proof. — b) is clear from the definitions of the actions 3.8 *b)*. The equality $\Delta(\xi) = \Delta'(\xi)$ on $\mathrm{H}_\bullet^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathscr{V}}_{\mathrm{RN}}, \ddot{\mathscr{L}})$ follows from 4.3 *e)* and 3.3 *j)*. In the following computations we shall sometimes omit writing local systems.

$$\begin{aligned}
\mathrm{H}_\bullet^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathscr{V}}_{\mathrm{RN}}, \ddot{\mathscr{L}}) &\cong \mathrm{H}_\bullet^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathscr{V}}^{\mathrm{P}}_{\mathrm{RN}}, \ddot{\mathscr{L}}) && (\text{see } 3.3\ j))\\
&\cong \mathrm{H}_\bullet^{\mathrm{P}\times\mathbf{C}^*}(\mathscr{V}_{\mathrm{RN}} \cap (\mathscr{C} + \mathfrak{n}),\ ) && (\text{by } 1.6\ a))\\
&\cong \mathrm{H}_\bullet^{\mathrm{L}\times\mathbf{C}^*}(\mathscr{V}_{\mathrm{RN}} \cap (\mathscr{C} + \mathfrak{n}),\ ) && (\text{by } 1.4\ h))\\
&\cong \mathrm{H}_\bullet^{\mathrm{M}_{\mathrm{L}}(x_0)}(\mathscr{V}_{\mathrm{RN}} \cap (x_0 + \mathfrak{n}),\ ) && (\text{by } 1.6\ a))\\
&\cong \mathrm{H}_\bullet^{\mathrm{M}_{\mathrm{L}}(x_0)}(x_0 + (\mathfrak{n} - \mathscr{H}),\ ) && (\text{see } 2.10\ c))\\
&\cong \mathrm{H}_\bullet^{\mathrm{M}_{\mathrm{L}}(\varphi_0)}(x_0 + (\mathfrak{n} - \mathscr{H}),\ ) && (\text{by } 1.4\ h)).
\end{aligned}$$

Let $\mathrm{M} = \mathrm{M}_{\mathrm{L}}(\varphi_0)$, M_1 the kernel of the character $\chi : \mathrm{M}^0 \to \mathbf{C}^*$ by which M^0 acts on $\mathfrak{n}/\mathscr{H}$, $\mathfrak{m} = \mathrm{Lie\ M}$, $d\chi : \mathfrak{m} \to \mathbf{C}$ the tangent map to χ, $\mathfrak{m}_1 = \mathrm{Lie\ M}_1 = \ker d\chi$. Let $\mathscr{H}_1$ be an affine hyperplane in $\mathfrak{n}$ parallel to $\mathscr{H}$ and distinct from $\mathscr{H}$.

Clearly $(\mathrm{M}_0 \times (x_0 + \mathscr{H}_1))/\mathrm{M}_1 \overset{\sim}{\to} x_0 + (\mathfrak{n} - \mathscr{H})$ as M^0-varieties.

We have

$$\begin{aligned}
\mathrm{H}_\bullet^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathfrak{g}}_{\mathrm{NR}}, \ddot{\mathscr{L}}) &\cong \mathrm{H}_\bullet^{\mathrm{M}_0}(x_0 + (\mathfrak{n} - \mathscr{H}),\ )^{\mathrm{M}} && (\text{by } 1.9\ a))\\
&\cong \mathrm{H}_\bullet^{\mathrm{M}_1}(x_0 + \mathscr{H}_1,\ )^{\mathrm{M}} && (\text{by } 1.6\ a))\\
&\cong \mathrm{H}_\bullet^{\mathrm{M}_1^0}(x_0, (\mathscr{L} \otimes \mathscr{L}^*)_{x_0})^{\mathrm{M}} && (\text{by } 1.4\ e),\ 1.9\ a))\\
&\cong \mathrm{H}_\bullet^{\mathrm{M}_1^0} && (\text{see } 1.12\ b)).
\end{aligned}$$

It remains to note that $\mathrm{H}^\bullet_{\mathrm{M}_1^0} \otimes \mathrm{H}_0^{\mathrm{M}_1^0} \overset{\sim}{\to} \mathrm{H}_\bullet^{\mathrm{M}_1^0}$, $\mathrm{H}^\bullet_{\mathrm{M}_1^0} = \mathrm{S}(\mathfrak{m}_1^*) = \mathbf{S}/(d\chi)$ and $d\chi = \alpha - cr$ (see 2.10 *b)*).

5.4. Using the injectivity of $\gamma_! : \mathrm{H}_\bullet^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathfrak{g}}_{\mathrm{N}}, \ddot{\mathscr{L}}) \to \mathrm{H}_\bullet^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathfrak{g}}, \ddot{\mathscr{L}})$ in 4.11 *c)*, we see that the identity in 4.14 which holds on $\mathrm{H}_\bullet^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathfrak{g}}, \ddot{\mathscr{L}})$ must also hold on $\mathrm{H}_\bullet^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathfrak{g}}_{\mathrm{N}}, \ddot{\mathscr{L}})$; using 4.15, it must also hold on arbitrary elements of $\mathrm{H}_0^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathscr{V}}_{\mathrm{RN}}, \ddot{\mathscr{L}})$. We now take such an element h with $h \neq 0$ (see 5.3 *c)*). By 5.3 *a)*, *b)*, we have $\Delta(\alpha)\ h = crh$, $\Delta(s)\ h = h$. Hence, if p is as in 4.14, we have

$$\begin{aligned}
0 &= \Delta(s)\ \Delta(\alpha)\ h + \Delta(\alpha)\ \Delta(s)\ h - prh\\
&= \Delta(s)\ crh + \Delta(\alpha)\ h - prh = 2crh - prh.
\end{aligned}$$

By 5.3 *a)*, we have $rh \neq 0$ in $\mathrm{H}_2^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathscr{V}}_{\mathrm{RN}}, \ddot{\mathscr{L}})$. It follows that $p = 2c$. This completes the proof of Theorem 5.1.

6. The algebra **H**

6.1. Let Ω be as in 3.3 *c)*; assume that it is closed in G. If $\Omega(w) \subset \Omega$, then the inclusion $i : \ddot{\mathfrak{g}}_N^{\overline{\Omega(w)}} \subset \ddot{\mathfrak{g}}_N^{\Omega}$ induces a homomorphism

$$i_! : H_0^{G \times \mathbf{C}^*}(\ddot{\mathfrak{g}}_N^{\overline{\Omega(w)}}, \ddot{\mathscr{L}}) \to H_0^{G \times \mathbf{C}^*}(\ddot{\mathfrak{g}}_N^{\Omega}, \ddot{\mathscr{L}})$$

(see 3.3 *g)*, *h)*).

Taking the direct sum, we get a homomorphism

$$\bigoplus_{\substack{w \in W \\ \Omega(w) \subset \Omega}} H_0^{G \times \mathbf{C}^*}(\ddot{\mathfrak{g}}_N^{\overline{\Omega(w)}}, \ddot{\mathscr{L}}) \to H_0^{G \times \mathbf{C}^*}(\ddot{\mathfrak{g}}_N^{\Omega}, \ddot{\mathscr{L}}).$$

This is an isomorphism; the proof is by induction on $N(\Omega)$ as in 4.7. In particular for $\Omega = G$, we have

$$\bigoplus_{w \in W} H_0^{G \times \mathbf{C}^*}(\ddot{\mathfrak{g}}_N^{\overline{\Omega(w)}}, \ddot{\mathscr{L}}) \overset{\approx}{\to} H_0^{G \times \mathbf{C}^*}(\ddot{\mathfrak{g}}_N, \ddot{\mathscr{L}}).$$

The image of $H_0^{G \times \mathbf{C}^*}(\ddot{\mathfrak{g}}_N^{\Omega(w)}, \ddot{\mathscr{L}}) \hookrightarrow H_0^{G \times \mathbf{C}^*}(\ddot{\mathfrak{g}}_N, \ddot{\mathscr{L}})$ is denoted by E_w. It is a line, since

$$H_0^{G \times \mathbf{C}^*}(\ddot{\mathfrak{g}}_N^{\overline{\Omega(w)}}, \ddot{\mathscr{L}}) \overset{\sim}{\to} H_0^{G \times \mathbf{C}^*}(\ddot{\mathfrak{g}}_N^{\Omega(w)}, \ddot{\mathscr{L}})$$

by the irreducibility of $\ddot{\mathfrak{g}}_N^{\Omega(w)}$ (3.3 *g)* and 1.5 *b)*), and the space $H_0^{G \times \mathbf{C}^*}(\ddot{\mathfrak{g}}_N^{\Omega(w)}, \ddot{\mathscr{L}})$ is one dimensional by 4.7.

Hence, we have a decomposition

a) $H_0^{G \times \mathbf{C}^*}(\ddot{\mathfrak{g}}_N, \ddot{\mathscr{L}}) = \bigoplus_{w \in W} E_w$.

We have clearly

b) $E_e = D_N$ (see 4.10).

We have isomorphisms

$$E_e \cong H_0^{G \times \mathbf{C}^*}(\ddot{\mathfrak{g}}_N^{P}, \ddot{\mathscr{L}}) \cong H_0^{M\mathfrak{L}(\varphi_0)} \cong H^0_{M\mathfrak{L}(\varphi_0)}$$

(see the proof of 4.7). We denote by **1** the element of E corresponding to the unit element of the algebra $H^0_{M\mathfrak{L}(\varphi_0)} \cong \mathbf{C}$.

Lemma **6.2.** — *For $w \in W$, we have $\Delta(w)\, \mathbf{1} = \Delta'(w^{-1})\, \mathbf{1} \in H_0^{G \times \mathbf{C}^*}(\ddot{\mathfrak{g}}_N, \ddot{\mathscr{L}})$.*

Proof. — Using 6.1 *b)* and 4.11 *e)*, we see that

a) $\Delta(w)\, \mathbf{1} = a\, \Delta'(w^{-1})\, \mathbf{1}$ for some $a \in \mathbf{C}^*$.

From 6.1 *a)*, we have

b) $\begin{cases} \Delta(w)\, \mathbf{1} - \beta.\mathbf{1} \in \bigoplus_{w' \neq e} E_{w'} \\ \Delta'(w^{-1})\, \mathbf{1} - \beta'\, \mathbf{1} \in \bigoplus_{w' \neq e} E_{w'} \end{cases}$

for some $\beta, \beta' \in \mathbf{C}$.

Under the homomorphism $j^* : H_0^{G\times C^*}(\ddot{\mathfrak{g}}_N, \ddot{\mathscr{L}}) \to H_0^{G\times C^*}(\ddot{\mathscr{V}}_{NR}, \ddot{\mathscr{L}})$ in 5.4, the lines $E_{w'}$ $(w' \neq e)$ are mapped to zero. Indeed, we have a cartesian diagram

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[d] & \ddot{\mathfrak{g}}_N^{\overline{\Omega(w)}} \arrow[d] \\
\ddot{\mathscr{V}}_{R\dot{N}} \arrow[r, hook] & \ddot{\mathfrak{g}}_N
\end{tikzcd}$$

where $X = \ddot{\mathscr{V}}_{RN} \cap \ddot{\mathfrak{g}}_N^{\overline{\Omega(w')}}$. By 3.3 *j)*, X is contained in $\ddot{\mathfrak{g}}_N^{\Omega(e)} \cap \mathfrak{g}_N^{\overline{\Omega(w')}}$ and hence $\dim X < \delta'$ (see 3.3 *g)*). On the other hand, $\ddot{\mathscr{V}}_{RN}$ is open in $\ddot{\mathfrak{g}}_N$ hence X is open in $\ddot{\mathfrak{g}}_N^{\overline{\Omega(w')}}$ which is irreducible of dimension δ'. It follows that X is empty. The cartesian diagram above gives rise to a commutative diagram

$$\begin{tikzcd}
0 \arrow[d] & H_0^{G\times C^*}(\ddot{\mathfrak{g}}_N^{\Omega(w')}, \ddot{\mathscr{L}}) \arrow[l] \arrow[d] \\
H_0^{G\times C^*}(\ddot{\mathscr{V}}_{RN}, \ddot{\mathscr{L}}) & H_0^{G\times C^*}(\ddot{\mathfrak{g}}_N, \ddot{\mathscr{L}}) \arrow[l]
\end{tikzcd}$$

so that $j^* E_{w'} = 0$, as asserted.

Applying j^* to *b)* we obtain therefore

$$\Delta(w)\, j^*(\mathbf{1}) = \beta j^*(\mathbf{1}), \qquad \Delta'(w^{-1})\, j^*\,\mathbf{1} = \beta' j^*(\mathbf{1}).$$

But $\Delta(w) = \Delta'(w^{-1}) = \mathrm{Id}$ on $H_0^{G\times C^*}(\ddot{\mathscr{V}}_{NR}, \ddot{\mathscr{L}})$ (by the definition of the actions 3.8 *b)*) hence $(\beta - 1)j^*(\mathbf{1}) = (\beta' - 1)j^*(\mathbf{1}) = 0$.

The surjectivity of j'^* in the proof of 4.15 shows that $j^*(\mathbf{1}) \neq 0$. It follows that $\beta = \beta' = 1$. Thus, $\Delta(w)\,\mathbf{1} - \mathbf{1}$ and $\Delta'(w^{-1})\,\mathbf{1} - \mathbf{1}$ are in $\bigoplus_{w' \neq e} E_{w'}$. Using *a)* we have

$$a\,\Delta'(w^{-1})\,\mathbf{1} - \mathbf{1} \in \bigoplus_{w' \neq e} E_{w'}.$$

Clearly we have

$$a\,\Delta'(w^{-1})\,\mathbf{1} - a\mathbf{1} \in \bigoplus_{w' \neq e} E_{w'}.$$

Subtracting we get $(a - 1)\,\mathbf{1} \in \bigoplus_{w' \neq e} E_{w'}$. But $\mathbf{1} \notin \bigoplus_{w' \neq e} E_{w'}$ so that $a = 1$. The lemma is proved.

Theorem **6.3.** — *Let* $\mathbf{H} = \mathbf{S} \otimes \mathbf{C}[W]$. *There is a unique structure of associative* **C**-*algebra with unit* $1 \otimes e$ *on the* **C**-*vector space* **H** *such that*

a) $\xi \mapsto \xi \otimes e$ *is an algebra homomorphism* $\mathbf{S} \to \mathbf{H}$,

b) $w \mapsto 1 \otimes w$ *is an algebra homomorphism* $\mathbf{C}[W] \to \mathbf{H}$,

c) $(\xi \otimes e).(1 \otimes w) = \xi \otimes w$, $(\xi \in \mathbf{S}, w \in W)$,

d) $(1 \otimes s_i)(\xi \otimes e) - ({}^{s_i}\xi \otimes e)(1 \otimes s_i) = c_i\, r((\xi - {}^{s_i}\xi)/\alpha_i) \otimes e$, $(\xi \in \mathbf{S}, 1 \leqslant i \leqslant m)$.

Proof. — Let $\mathscr{H} = \mathrm{H}_{\bullet}^{G \times \mathbf{C}^*}(\ddot{\mathfrak{g}}_N, \dot{\mathscr{L}})$. Let A be the subalgebra of $\mathrm{End}_{\mathbf{C}}(\mathscr{H})$ generated by the endomorphisms $\Delta(\xi)\,\Delta(w) : \mathscr{H} \to \mathscr{H}$, $(\xi \in \mathbf{S}, w \in \mathrm{W})$. Let A′ be the subalgebra of $\mathrm{End}_{\mathbf{C}}(\mathscr{H})$ generated by the endomorphisms $\Delta'(\xi)\,\Delta'(w) : \mathscr{H} \to \mathscr{H}$ $(\xi \in \mathbf{S}, w \in \mathrm{W})$.

We have **C**-linear maps

$$e)\ \begin{cases} \Delta\ : \mathbf{H} \to \mathrm{A}, & \Delta(\xi \otimes w) = \text{endomorphism } h \to \Delta(\xi)\,\Delta(w)\,h \quad \text{of } \mathscr{H} \\ \Delta' : \mathbf{H} \to \mathrm{A}', & \Delta'(\xi \otimes w) = \text{endomorphism } h \to \Delta'(\xi)\,\Delta'(w)\,h \text{ of } \mathscr{H} \end{cases}$$

$$\mu\ : \mathrm{A} \to \mathscr{H}, \quad \mu(f) = f(\mathbf{1})$$

$$\mu' : \mathrm{A} \to \mathscr{H}, \quad \mu'(f) = f'(\mathbf{1}).$$

From 4.11 *d)* and 4.6 it follows that

f) $\mu \circ \Delta$ and $\mu' \circ \Delta'$ are isomorphisms $\mathbf{H} \xrightarrow{\approx} \mathscr{H}$.

In particular, μ, μ' are surjective and Δ, Δ' are injective.

From 4.3 *b)*, 4.4 we see that

g) $\mathrm{A} \subset \mathrm{Z}(\mathrm{A}')$, $\mathrm{A}' \subset \mathrm{Z}(\mathrm{A})$

where Z(A) (resp. Z(A′)) is the set of all endomorphisms of $\mathscr{H}$ which commute with all endomorphisms in A (resp. A′).

Let $f \in \mathrm{Z}(\mathrm{A})$. Then by *f)*, $f(\mathbf{1}) = \Delta'(\chi')\,(\mathbf{1})$ for some $\chi' \in \mathbf{H}$. Now let $h \in \mathscr{H}$. We have (by *a)*) $h = \Delta(\chi)\,(\mathbf{1})$ for some $\chi \in \mathbf{H}$. Hence

$$\begin{aligned} f(h) &= f(\Delta(\xi)\,(\mathbf{1})) = \Delta(\chi)\,(f(\mathbf{1})) \quad (\text{since } f \in \mathrm{Z}(\mathrm{A})) \\ &= \Delta(\chi)\,(\Delta'(\chi')\,(\mathbf{1})) \\ &= \Delta'(\chi')\,(\Delta(\chi)\,\mathbf{1}) \quad \text{by } b) \\ &= \Delta'(\chi')\,h. \end{aligned}$$

Thus, $f = \Delta'(\chi')$. This shows that $\mathrm{Z}(\mathrm{A}) \subset \mathrm{image}(\Delta')$. We have obviously

$$\mathrm{image}(\Delta') \subset \mathrm{A}' \subset \mathrm{Z}(\mathrm{A})$$

(see *g)*) hence image $\Delta' = \mathrm{A}' = \mathrm{Z}(\mathrm{A})$. Similarly, image $\Delta = \mathrm{A} = \mathrm{Z}(\mathrm{A}')$.

It follows that Δ and Δ' are isomorphisms. Using *f)* it follows that μ, μ' are isomorphisms.

We now define an algebra structure on **H** by transporting to **H** via Δ the algebra structure of A. Then properties *a)-d)* are satisfied; for example property *d)* follows from 5.1 *a)*. The uniqueness statement in the theorem is clear since s_i $(1 \leqslant i \leqslant m)$ generate W. This completes the proof.

Corollary **6.4.** — a) $\mathscr{H} = \mathrm{H}_{\bullet}^{G \times \mathbf{C}^*}(\ddot{\mathfrak{g}}_N, \dot{\mathscr{L}})$ *can be regarded as a left* **H**-*module in two ways:*

$$\chi, h \mapsto \Delta(\chi)\,h$$

or

$$\chi, h \mapsto \Delta'(\chi)\,h.$$

b) *We have* $\Delta(\chi)\,\Delta'(\chi')\,h = \Delta'(\chi')\,\Delta(\chi)\,h$ *for all* $h \in \mathscr{H}$, $\chi, \chi' \in \mathbf{H}$.

c) *The two maps* $\mathbf{H} \to \mathscr{H}$, $\chi \mapsto \Delta(\chi)\,\mathbf{1}$ *and* $\chi \mapsto \Delta'(\chi)\,\mathbf{1}$ *are both isomorphisms.*

d) *There is a unique anti-automorphism* $\chi \to \hat{\chi}$ *of the algebra* $\mathbf{H}$ *such that* $\Delta(\chi)\,\mathbf{1} = \Delta'(\hat{\chi})\,\mathbf{1}$. *It satisfies* $(1 \otimes w)^{\wedge} = 1 \otimes w^{-1}$, $(\xi \otimes e)^{\wedge} = \xi \otimes e$, *for* $w \in \mathrm{W}$, $\xi \in \mathbf{S}$.

Proof. — *a)* is clear from the definitions of $\Delta(\chi)$, $\Delta'(\chi)$ (see 6.3 *e)*) and from 5.1; *b)* is just 6.3 *g)*; *c)* is just 6.3 *f)*.

From *c)* we see that there is a unique **C**-linear endomorphism $\chi \to \hat{\chi}$ of $\mathbf{H}$ such that $\Delta(\chi)\,\mathbf{1} = \Delta'(\hat{\chi})\,\mathbf{1}$. From *a), b), c)* we see that $\chi \to \hat{\chi}$ is an algebra anti-automorphism. We have $(\xi \otimes e)^{\wedge} = \xi \otimes e$ by 4.3 *e)* and $(1 \otimes w)^{\wedge} = 1 \otimes w^{-1}$ by 6.2.

Theorem **6.5.** — *The centre of* $\mathbf{H}$ *is* $\{\, \xi \otimes e \mid \xi \in \mathbf{S}^{\mathrm{W}} \,\}$.

Proof. — It is clear that $\xi \otimes e \in \text{centre}(\mathbf{H})$ for all $\xi \in \mathbf{S}^{\mathrm{W}}$. We now prove the converse. Let $h \in \text{centre}(\mathbf{H})$. We can write uniquely

$$h = \sum_{w \in \mathrm{W}} \xi_w \otimes w \quad (\xi_w \in \mathbf{S}).$$

Assume that $\xi_w \neq 0$ for some $w \neq e$. Then we can find an integer $j \geqslant 0$ such that $\xi_w \in r^j\,\mathbf{S}$ for all $w \neq e$ and $\xi_w \notin r^{j+1}\,\mathbf{S}$ for some $w \neq e$ (say $w = w_1$).

From 6.3 *d)* we see by induction on the length of w that, for any $\xi \in \mathbf{S}$,

$$(1 \otimes w)\,(\xi \otimes e) - ({}^{w}\xi \otimes e)\,(1 \otimes w) \in (r \otimes 1)\,\mathbf{H}, \quad (w \in \mathrm{W}).$$

We have therefore

$$(\xi_w \otimes w).(\xi \otimes e) - \xi_w . {}^{w}\xi \otimes w \begin{cases} \in (r^{j+1} \otimes 1)\,\mathbf{H} & \text{if } w \neq e. \\ = 0 & \text{if } w = e. \end{cases}$$

We use this in the following calculation

$$\begin{aligned} 0 &= (\xi \otimes e)\,h - h(\xi \otimes e) \\ &= \sum_w \xi\xi_w \otimes w - \sum_w \xi_w\, {}^{w}\xi \otimes w + (r^{j+1} \otimes 1)\,h' \\ &= \sum_w \xi_w(\xi - {}^{w}\xi) \otimes w + (r^{j+1} \otimes 1)\,h', \quad (h' \in \mathbf{H}). \end{aligned}$$

This equality shows that

$$\xi_{w_1}(\xi - {}^{w_1}\xi) \in r^{j+1}\,\mathbf{S}.$$

Since $\xi_{w_1} \notin r^{j+1}\,\mathbf{S}$, it follows that $\xi - {}^{w_1}\xi \in r\mathbf{S}$. But this clearly fails for ξ a generic element of $\mathfrak{t}^*$. This contradiction shows that $\xi_w = 0$ for all $w \neq e$. Thus $h = \xi_e \otimes e$. We now write the equation $(1 \otimes s_i)\,h = h(1 \otimes s_i)$. Using 6.3 *d)*, we see that $(\xi_e - {}^{s_i}\xi_e) \otimes s_i = c_i\,r((\xi_e - {}^{s_i}\xi_e)/\alpha_i) \otimes e$. It follows that $\xi_e = {}^{s_i}\xi_e$. Since i is arbitrary, it follows that $\xi_e \in \mathbf{S}^{\mathrm{W}}$. The theorem is proved.

7. Preparatory results

7.1. Let M be a *connected* algebraic group, let X be an M-variety and let $\mathcal{E}$ be an M-equivariant local system on X. We define a filtration of $H_\bullet^M(X, \mathcal{E})$ by

a) $F^i = F^i(H_\bullet^M(X, \mathcal{E})) =$ the $H_M^\bullet$-submodule of $H_\bullet^M(X, \mathcal{E})$ generated by $\bigoplus_{h \leqslant i} H_h^M(X, \mathcal{E})$.

Then $F^0 \subset F^1 \subset \ldots$ and $F^i = 0$ for $i < 0$. Let

b) $\Pi_i = H_i^M(X, \mathcal{E})/H_i^M(X, \mathcal{E}) \cap F_{i-1} =$ component of degree i of F_i/F_{i-1}.

We regard Π_i as a graded vector space which is zero in degrees $\neq i$.

We have a natural embedding $\Pi_i \hookrightarrow F_i/F_{i-1}$ of graded vector spaces (isomorphism in degree i). Since F_i/F_{i-1} is an $H_M^\bullet$-module, this extends to an $H_M^\bullet$-linear map

c) $H_M^\bullet \otimes_{\mathbf{C}} \Pi_i \to F_i/F_{i-1}$.

The homomorphism

d) $H_i^M(X, \mathcal{E}) \to H_i^{\{e\}}(X, \mathcal{E})$ (see 1.4 *f)*)

is zero on $H_i^M(X, \mathcal{E}) \cap F_{i-1}$ hence it factors through a $\mathbf{C}$-linear map

e) $\Pi_i \to H_i^{\{e\}}(X, \mathcal{E})$.

Proposition **7.2.** — *Recall that* M *is connected. Assume that* $H_c^{\mathrm{odd}}(X, \mathcal{E}) = 0$. *Then:*

a) *The maps* 7.1 *c), e) are isomorphisms. Hence* $H_M^\bullet \otimes_{\mathbf{C}} H_i^{\{e\}}(X, \mathcal{E}) \cong F_i/F_{i-1}$ *as* $H_M^\bullet$*-modules.*

b) $H_{\mathrm{odd}}^M(X, \mathcal{E}) = 0$.

c) *Each* F_i *is a finitely generated projective* $H_M^\bullet$*-module and* $F_{2\delta} = F_{2\delta+1} = \ldots = H_\bullet^M(X, \mathcal{E})$, *where* $\delta = \dim X$.

In particular, $H_\bullet^M(X, \mathcal{E})$ *is a finitely generated projective* $H_M^\bullet$*-module.*

d) *The* $\mathbf{C}$*-linear map* $\mathbf{C} \otimes_{H_M^\bullet} H_\bullet^M(X, \mathcal{E}) \to H_\bullet^{\{e\}}(X, \mathcal{E})$ *defined by* 7.1 *d) (where* $\mathbf{C}$ *is regarded as an* $H_M^\bullet$*-algebra via* $H_M^\bullet \xrightarrow{1.4(g)} H_{\{e\}}^\bullet = \mathbf{C}$) *is an isomorphism.*

Proof. — Let m be a large integer and let Γ be an irreducible, smooth variety with a free M-action such that $H^i(\Gamma) = 0$ for $1 \leqslant i \leqslant m$. We assume, as we may, that $M\backslash\Gamma$ is simply connected (since M is connected). Consider the fibration $f: M\backslash(\Gamma \times X) \to M\backslash\Gamma$ with fibres $\approx X$. We study the Leray-Serre spectral sequence for $f_!({}_\Gamma\mathcal{E}^*)$. Since $M\backslash\Gamma$ is simply connected, this spectral sequence has $E_2^{p,q} = H_c^p(M\backslash\Gamma) \otimes H_c^q(X, \mathcal{E}^*)$. This is zero if at least one of the following four conditions is satisfied:

p odd and $p \geqslant 2\delta' - m$ $(\delta' = \dim(M\backslash\Gamma))$,

q odd,

$q > 2\delta$,

$p > 2\delta'$.

(For the first condition, note that $H_M^i = 0$ for i odd, $H^i(M\backslash\Gamma) = H_M^i$ for $i \leqslant m$ and $M\backslash\Gamma$ is smooth so it satisfies Poincaré duality.)

If $E_2^{p,q}$ were zero for all odd p, then the spectral sequence would degenerate at E_2. As it is zero only for p odd in the indicated range, the spectral sequence is only degenerate in some range:

$$E_2^{p,q} = E_\infty^{p,q} \quad \text{if } p + q \geqslant 2(\delta + \delta') - (m - 2\,\delta - 1).$$

The spectral sequence converges to $H_c^\bullet(M\backslash(\Gamma \times X), {}_\Gamma\mathscr{E}^*)$.

We now define a new spectral sequence:

$$\overline{E}_r^{p,q} = (E_r^{2\delta' - p,\, 2\delta - q})^*$$

whose differentials are duals of the differentials of $\{ E_r^{p,q} \}$. We have

$$\overline{E}_2^{p,q} = H_p^M \otimes H_q^{\{e\}}(X, \mathscr{E}) \quad \text{if } p \leqslant m$$
$$\overline{E}_2^{p,q} = \overline{E}_\infty^{p,q} \qquad\qquad \text{if } p + q \leqslant m - 2\,\delta - 1$$

and the spectral sequence converges to $H_c^\bullet(M\backslash\Gamma \times X, {}_\Gamma\mathscr{E}^*)^*$, which is equal to $H_\bullet^M(X, \mathscr{E})$ in degrees $\leqslant m$.

a) follows from these statements, by taking m large. (Recall that $H_\bullet^M = H_M^\bullet$ by 1.7 *c)*.)

Now *b)*, *c)*, *d)* are immediate consequences of *a)*.

7.3. We want to state a form of Künneth's formula in equivariant homology. Let $\mathscr{E}$, $\mathscr{E}'$ be local systems on the algebraic varieties X, X′. We have a natural isomorphism (external cup-product)

a) $H_c^\bullet(X, \mathscr{E}) \otimes H_c^\bullet(X', \mathscr{E}') \xrightarrow{\approx} H_c^\bullet(X \times X', \mathscr{E} \boxtimes \mathscr{E}')$.

Assume now that X, X′ are M-varieties and $\mathscr{E}$, $\mathscr{E}'$ are M-equivariant. Let m, Γ be as in the proof of 7.2, with m large. Applying *a)* to $M\backslash(\Gamma \times X)$, $M\backslash(\Gamma \times X')$, ${}_\Gamma\mathscr{E}^*$, ${}_\Gamma\mathscr{E}'^*$ instead of X, X′, $\mathscr{E}$, $\mathscr{E}'$ we obtain an isomorphism

b)
$$\begin{tikzcd}
\bigoplus_{i+j=k} H_c^i(M\backslash\Gamma \times X, {}_\Gamma\mathscr{E}^*) \otimes (M\backslash\Gamma \times X', {}_\Gamma\mathscr{E}'^*) \arrow[d, "\wr"] \\
H_c^k((M \times M)\backslash\Gamma \times \Gamma \times X \times X', {}_\Gamma\mathscr{E}^* \boxtimes {}_\Gamma\mathscr{E}'^*);
\end{tikzcd}$$

taking duals and assuming $k \leqslant m$ we obtain an isomorphism

c) $H_k^{M \times M}(X \times X', \mathscr{E} \boxtimes \mathscr{E}') \cong \bigoplus_{i+j=k} (H_i^m(X, \mathscr{E}) \otimes_{\mathbf{C}} H_j^M(X', \mathscr{E}'))$.

We compose this with the homomorphism 1.4 *f)*

$$H_k^{M \times M}(X \times X', \mathscr{E} \boxtimes \mathscr{E}') \to H_k^M(X \times X, \mathscr{E} \boxtimes \mathscr{E}')$$

induced by the diagonal embedding $M \subset M \times M$ and we obtain a homomorphism

$$H_\bullet^M(X, \mathscr{E}) \otimes_{\mathbf{C}} H_\bullet^M(X, \mathscr{E}') \to H_\bullet^M(X \times X', \mathscr{E} \boxtimes \mathscr{E}').$$

One verifies easily that this homomorphism factors through a homomorphism of $H_M^\bullet$-modules:

d) $H_\bullet^M(X, \mathscr{E}) \otimes_{H_M^\bullet} H_\bullet^M(X, \mathscr{E}') \to H_\bullet^M(X \times X', \mathscr{E} \boxtimes \mathscr{E}')$.

Proposition **7.4.** — *Assume that* $H_c^{\text{odd}}(X, \mathscr{E}^*) = 0$, $H_c^{\text{odd}}(X', \mathscr{E}'^*) = 0$ *and that* M *is connected. Then the homomorphism* 7.3 *d) is an isomorphism.*

Proof. — Both factors in the left hand side of 7.3 *d)* have canonical filtrations F^i (see 7.1) hence their tensor product has a canonical product filtration; similarly, the right hand side of 7.3 *d)* has a canonical filtration F^i (by 7.1 for $X \times X'$). From the definitions we see easily that the map 7.3 *d)* is compatible with these filtrations hence it induces a homomorphism on the associated graded spaces for these filtrations:

a) $(H_M^\bullet \otimes_{\mathbf{C}} H_\bullet^{\{e\}}(X, \mathscr{E})) \otimes_{H_M^\bullet} (H_M^\bullet \otimes_{\mathbf{C}} H_\bullet^{\{e\}}(X', \mathscr{E}')) \to H_M^\bullet \otimes_{\mathbf{C}} H_\bullet^{\{e\}}(X \times X', \mathscr{E} \boxtimes \mathscr{E}')$.

(We have used 7.2 *a)* for $X, X', X \times X'$.) It is enough to show that *a)* is an isomorphism. But *a)* is the homomorphism 7.3 *d)* with M replaced by $\{e\}$ and with scalars extended to $H_M^\bullet$. Thus, we are reduced to the case where $M = \{e\}$. In that case, the result follows from 7.3 *a)*.

Proposition **7.5.** — *Let* M, X, $\mathscr{E}$ *be as in* 7.2, *and let* M′ *be a closed connected subgroup of* M. *Then the* $H_M^\bullet$*-linear map* $H_\bullet^M(X, \mathscr{E}) \to H_\bullet^{M'}(X, \mathscr{E})$ *(see* 1.4 *f)) extends to an* $H_M^\bullet$*-linear isomorphism*

a) $H_{M'}^\bullet \otimes_{H_M^\bullet} H_\bullet^M(X, \mathscr{E}) \xrightarrow{\sim} H_\bullet^{M'}(X, \mathscr{E})$.

($H_{M'}^\bullet$ *is regarded as an* $H_M^\bullet$*-algebra, via* $H_M^\bullet \to H_{M'}^\bullet$ *in* 1.4 *g).*)

Proof. — We consider the filtration F^i of $H_\bullet^M(X, \mathscr{E})$ in 7.1. It defines a filtration $\{H_{M'}^\bullet \otimes_{H_M^\bullet} F^i\}$ of the left hand side of *a)*. (We use 7.2 *c)*.) Similarly, $H_\bullet^{M'}(X, \mathscr{E})$ has a canonical filtration, by 7.1. It is clear that the map *a)* is compatible with these filtrations hence it induces a homomorphism on the associated graded spaces for these filtrations:

b) $H_{M'}^\bullet \otimes_{H_M^\bullet} (H_M^\bullet \otimes_{\mathbf{C}} H_\bullet^{\{e\}}(X, \mathscr{E})) \to H_{M'}^\bullet \otimes_{\mathbf{C}} H_\bullet^{\{e\}}(X, \mathscr{E})$.

(We have used 7.2 *a)* twice.) It is enough to show that *b)* is an isomorphism. But *b)* is the identity map. This completes the proof.

8. Standard H-modules

8.1. The results in this chapter bear some resemblance to results in chapter 5 of [5]. Let $y \in \mathfrak{g}$ be a nilpotent element, $\mathcal{O}$ its G-orbit in $\mathfrak{g}$, and

a) $\mathscr{P}_y = \{gP \in G/P \mid \mathrm{Ad}(g^{-1})y \in \mathscr{C} + \mathfrak{n}\}$.

Then $M(y) = M_G(y)$ (see 2.1 *a)*) acts on $\mathscr{P}_y$ by

b) $(g_1, \lambda) : gP \mapsto g_1 gP$.

Let $\widetilde{\mathcal{O}} = (G \times \mathbf{C}^*)/M^0(y) \xrightarrow{h} \mathfrak{g}$ be defined by $(g_1, \lambda) \to \lambda^{-2}\mathrm{Ad}(g_1)y$; this is a $G \times \mathbf{C}^*$-equivariant, finite principal covering of $\mathcal{O}$ with group

c) $\overline{M}(y) = M(y)/M^0(y)$

($G \times \mathbf{C}^*$ acts on $\mathcal{O}$ by left translations). Let

d) $\dot{\widetilde{\mathcal{O}}} = (G \times \mathbf{C}^* \times \mathscr{P}_y)/M^0(y)$

($M(y)$ acts on $G \times \mathbf{C}^* \times \mathscr{P}_y$ by $(g_1, \lambda) : (g'_1, \lambda', gP) \to (g'_1 g_1^{-1}, \lambda'\lambda^{-1}, g_1 gP)$).

We have a cartesian diagram

e)
$$\begin{tikzcd}
\dot{\tilde{\mathcal{O}}} \arrow[r, "\dot{h}"] \arrow[d, "\tau"'] & \dot{\mathfrak{g}} \arrow[d, "\pi"] \\
\tilde{\mathcal{O}} \arrow[r, "h"] & \mathfrak{g}
\end{tikzcd}$$

where τ is the projection to the first two factors and $\dot{h}$ is defined by

$$(g'_1, \lambda', gP) \to (\lambda'^{-2}\,\mathrm{Ad}(g'_1)y, g'_1\,gP).$$

We may regard $\mathcal{P}_y$ as a closed $M(y)$-stable subvariety of $\dot{\mathcal{O}}$ (see 3.1 *d)*) by identifying $gP \in \mathcal{P}_y$ with $(y, gP) \in \dot{\mathcal{O}}$. The restriction of $\dot{\mathcal{L}}$, $\dot{\mathcal{L}}^*$ from $\dot{\mathfrak{g}}_{\mathcal{O}}$ to $\mathcal{P}_y$ will be denoted again $\dot{\mathcal{L}}$, $\dot{\mathcal{L}}^*$.

We want to define operators $\Delta(w)$ on $H_j^{M^0(y)}(\mathcal{P}_y, \dot{\mathcal{L}})$ for any integer j and $w \in W$. Choose m, Γ as in 3.5 with $m \geqslant j$ and form the cartesian diagram associated to *e)*:

f)
$$\begin{tikzcd}
{}_\Gamma\dot{\tilde{\mathcal{O}}} \arrow[r, "{}_\Gamma\dot{h}"] \arrow[d, "{}_\Gamma\tau"'] & {}_\Gamma\dot{\mathfrak{g}} \arrow[d, "{}_\Gamma\pi"] \\
{}_\Gamma\tilde{\mathcal{O}} \arrow[r, "{}_\Gamma h"'] & {}_\Gamma\mathfrak{g}
\end{tikzcd}$$

Then ${}_\Gamma h^*({}_\Gamma K) = ({}_\Gamma\tau)_!\,({}_\Gamma\dot{h}^*)\,({}_\Gamma\dot{\mathcal{L}}^*)$ and ${}_\Gamma h^*$ defines a homomorphism

g) $\mathrm{End}_{\mathcal{D}({}_\Gamma\mathfrak{g})}({}_\Gamma K) \to \mathrm{End}_{\mathcal{D}({}_\Gamma\tilde{\mathcal{O}})}({}_\Gamma h^*({}_\Gamma K)) = \mathrm{End}_{\mathcal{D}({}_\Gamma\tilde{\mathcal{O}})}(({}_\Gamma\tau)_!\,({}_\Gamma\dot{h}^*)\,({}_\Gamma\dot{\mathcal{L}}^*))$.

Composing 3.5 *d)* with *g)* we find a homomorphism

$$W \to \mathrm{Aut}_{\mathcal{D}({}_\Gamma\tilde{\mathcal{O}})}(({}_\Gamma\tau)_!\,({}_\Gamma\dot{h}^*)\,({}_\Gamma\dot{\mathcal{L}}^*)).$$

This induces a homomorphism

$$W \to \mathrm{Aut}\,H_c^{2d-j}({}_\Gamma\tilde{\mathcal{O}}, ({}_\Gamma\tau)_!\,({}_\Gamma\dot{h}^*)\,({}_\Gamma\dot{\mathcal{L}}^*)) = \mathrm{Aut}\,H_c^{2d-j}({}_\Gamma\dot{\tilde{\mathcal{O}}}, {}_\Gamma\dot{h}^*\,{}_\Gamma\dot{\mathcal{L}}^*)$$

($d = \dim {}_\Gamma\dot{\tilde{\mathcal{O}}}$). Taking duals we find a homomorphism

h) $\Delta : W \to \mathrm{Aut}\,H_c^{2d-j}({}_\Gamma\dot{\tilde{\mathcal{O}}}, {}_\Gamma\dot{h}^*\,{}_\Gamma\dot{\mathcal{L}}^*)^*$
$= \mathrm{Aut}\,H_j^{G \times C^*}(\dot{\tilde{\mathcal{O}}}, \dot{h}^*\,\dot{\mathcal{L}}^*)$
$= \mathrm{Aut}\,H_j^{M^0(y)}(\mathcal{P}_y, \dot{\mathcal{L}})$. (See *d)* and 1.6 *a)*.)

This homomorphism defines the operators $\Delta(w)$ on $H_j^{M^0(y)}(\mathcal{P}_y, \dot{\mathcal{L}})$ ($w \in W$). Replacing in the previous construction $\dot{\mathcal{L}}$ by $\dot{\mathcal{L}}^*$ we obtain in the same way operators $\Delta'(w)$ on $H_j^{M^0(y)}(\mathcal{P}_y, \dot{\mathcal{L}}^*)$ ($w \in W$). Replacing in the previous construction $\dot{\mathcal{L}}$ by $\dot{\mathcal{L}} \boxtimes \dot{\mathcal{L}}^*$ and the diagram *f)* by

$$\begin{tikzcd}
\Gamma(\dot{\tilde{\mathcal{O}}} \times_{\mathcal{O}} \dot{\tilde{\mathcal{O}}}) \arrow[r, "\Gamma\ddot{h}"] \arrow[d] & \Gamma(\dot{\mathfrak{g}} \times \dot{\mathfrak{g}}) \arrow[d] \\
\Gamma\tilde{\mathcal{O}} \arrow[r] & \Gamma(\mathfrak{g} \times \mathfrak{g})
\end{tikzcd}$$

($\ddot{h}$ is defined by $(\dot{h}, \dot{h})$), we obtain a homomorphism

$$\begin{aligned} \mathrm{W} \times \mathrm{W} \to \mathrm{Aut}\, \mathrm{H}_j^{\mathrm{G} \times \mathbf{C}^*}(\dot{\tilde{\mathcal{O}}} \times_{\tilde{\mathcal{O}}} \dot{\tilde{\mathcal{O}}}, \ddot{h}^*(\dot{\mathscr{L}} \boxtimes \dot{\mathscr{L}}^*)) \\ = \mathrm{Aut}\, \mathrm{H}_j^{\mathrm{M}^0(v)}(\mathscr{P}_v \times \mathscr{P}_v, \dot{\mathscr{L}} \boxtimes \dot{\mathscr{L}}^*); \end{aligned}$$

the operator corresponding to $(w, w') \in \mathrm{W} \times \mathrm{W}'$ is denoted by $\Delta(w, w')$.

8.2. We now define operators $\Delta(\xi)$ on $\mathrm{H}_\bullet^{\mathrm{M}^0(v)}(\mathscr{P}_v, \dot{\mathscr{L}})$ for $\xi \in \mathbf{S}$. Let

$$\eta \in \mathrm{H}_\bullet^{\mathrm{M}^0(v)}(\mathscr{P}_v, \dot{\mathscr{L}}) = \mathrm{H}_\bullet^{\mathrm{G} \times \mathbf{C}^*}(\dot{\tilde{\mathcal{O}}}, \dot{h}^* \dot{\mathscr{L}}), \qquad \xi \in \mathbf{S} = \mathrm{H}^\bullet_{\mathrm{G} \times \mathbf{C}^*}(\dot{\mathfrak{g}}).$$

Then $h'^*(\xi) \in \mathrm{H}^\bullet_{\mathrm{G} \times \mathbf{C}^*}(\dot{\tilde{\mathcal{O}}})$ and we define $\Delta(\xi)\,\eta$ as the product (1.3 *b)*)

$$h'^*(\xi) \,.\, \eta \in \mathrm{H}_\bullet^{\mathrm{G} \times \mathbf{C}^*}(\dot{\tilde{\mathcal{O}}}, \dot{h}^* \dot{\mathscr{L}}) = \mathrm{H}_\bullet^{\mathrm{M}^0(v)}(\mathscr{P}_v, \dot{\mathscr{L}}).$$

Thus, $\mathrm{H}_\bullet^{\mathrm{M}^0(v)}(\mathscr{P}_v, \dot{\mathscr{L}})$ becomes an $\mathbf{S}$-module.

Similarly, replacing $\dot{\mathscr{L}}$ by $\dot{\mathscr{L}}^*$, we obtain an $\mathbf{S}$-module structure on $\mathrm{H}_\bullet^{\mathrm{M}^0(v)}(\mathscr{P}_v, \dot{\mathscr{L}}^*)$ with operators $\Delta'(\xi)$, $\xi \in \mathbf{S}$.

We now define operators $\Delta(\xi)$, $\Delta'(\xi)$ for $\xi \in \mathbf{S}$ on $\mathrm{H}_\bullet^{\mathrm{M}^0(v)}(\mathscr{P}_v \times \mathscr{P}_v, \dot{\mathscr{L}} \boxtimes \dot{\mathscr{L}}^*)$.

Let $\eta \in \mathrm{H}_\bullet^{\mathrm{M}^0(v)}(\mathscr{P}_v \times \mathscr{P}_v, \dot{\mathscr{L}} \boxtimes \dot{\mathscr{L}}^*) = \mathrm{H}_\bullet^{\mathrm{G} \times \mathbf{C}^*}(\dot{\tilde{\mathcal{O}}} \times_{\tilde{\mathcal{O}}} \dot{\tilde{\mathcal{O}}}, \ddot{h}^*(\dot{\mathscr{L}} \boxtimes \dot{\mathscr{L}}^*))$, $\xi \in \mathrm{H}^\bullet_{\mathrm{G} \times \mathbf{C}^*}(\dot{\mathfrak{g}})$, and let $p_i : \dot{\mathfrak{g}} \times \dot{\mathfrak{g}} \to \dot{\mathfrak{g}}$ be the two projections ($i = 1, 2$). Then $\ddot{h}^*\, p_i^*(\xi) \in \mathrm{H}^\bullet_{\mathrm{G} \times \mathbf{C}^*}(\dot{\tilde{\mathcal{O}}} \times_{\tilde{\mathcal{O}}} \dot{\tilde{\mathcal{O}}})$, ($\ddot{h}$ as in 8.1) and we define $\Delta(\xi)\,\eta$, $\Delta'(\xi)\,\eta$ respectively as the products (1.3 *b)*) $\ddot{h}^*\, p_1^*(\xi) \,.\, \eta$, $\ddot{h}^*\, p_2^*(\xi) \,.\, \eta \in \mathrm{H}_\bullet^{\mathrm{G} \times \mathbf{C}^*}(\dot{\tilde{\mathcal{O}}} \times_{\tilde{\mathcal{O}}} \dot{\tilde{\mathcal{O}}}, \ddot{h}^*(\dot{\mathscr{L}} \boxtimes \dot{\mathscr{L}}^*) = \mathrm{H}_\bullet^{\mathrm{M}^0(v)}(\mathscr{P}_v \times \mathscr{P}_v, \dot{\mathscr{L}} \boxtimes \dot{\mathscr{L}}^*)$.

Thus $\mathrm{H}_\bullet^{\mathrm{M}^0(v)}(\mathscr{P}_v \times \mathscr{P}_v, \dot{\mathscr{L}} \boxtimes \dot{\mathscr{L}}^*)$ becomes an $\mathbf{S}$-module in two different ways, via $\Delta(\xi)$ or $\Delta'(\xi)$.

8.3. The operators $\Delta(w)$, $\Delta'(w)$ in 8.1 are $\mathrm{H}^\bullet_{\mathrm{M}^0(v)}$-linear. (Same proof as for 4.4 *a)*, *b)*.) The operators $\Delta(\xi)$, $\Delta'(\xi)$ in 8.2 are also $\mathrm{H}^\bullet_{\mathrm{M}^0(v)}$-linear.

8.4. Consider the homomorphisms

$$\begin{tikzcd}
\mathrm{A}' = \mathrm{H}_\bullet^{\mathrm{M}^0(v)}(\mathscr{P}_v, \dot{\mathscr{L}}) \otimes_{\mathrm{H}^\bullet_{\mathrm{M}^0(y)}} \mathrm{H}_\bullet^{\mathrm{M}^0(v)}(\mathscr{P}_v, \dot{\mathscr{L}}^*) \arrow[d, "\alpha"] & \\
\mathrm{A}'' = \mathrm{H}_\bullet^{\mathrm{M}^0(v)}(\mathscr{P}_v \times \mathscr{P}_v, \dot{\mathscr{L}} \boxtimes \dot{\mathscr{L}}^*) & \\
\mathrm{A}''' = \mathrm{H}_\bullet^{\mathrm{M}(v)}(\mathscr{P}_v \times \mathscr{P}_v, \dot{\mathscr{L}} \boxtimes \dot{\mathscr{L}}^*) \arrow[u, "\beta"] \arrow[r, equal, "1.6\,(a)"] & \mathrm{H}_\bullet^{\mathrm{G} \times \mathbf{C}^*}(\ddot{\mathcal{O}}, \ddot{\mathscr{L}})
\end{tikzcd}$$

where α is given by 7.3 *d)*, β is given by 1.4 *f)*, and $\ddot{\mathcal{O}}$ is as in 3.3 *b)*. Let $w, w' \in \mathrm{W}$, $\xi, \xi' \in \mathbf{S}$. From the definitions one verifies that the operators

$$\begin{array}{llll} \Delta(w) \otimes \Delta'(w') & (8.1); & \Delta(\xi) \otimes \Delta'(\xi') & (8.2), \text{ on } \mathrm{A}' \\ \Delta(w)\, \Delta'(w') & (8.1); & \Delta(\xi)\, \Delta'(\xi') & (8.2), \text{ on } \mathrm{A}'' \\ \Delta(w)\, \Delta'(w') & (4.3); & \Delta(\xi)\, \Delta'(\xi') & (4.3), \text{ on } \mathrm{A}''' \end{array}$$

are compatible with the homomorphisms α, β.

8.5. The finite group $\overline{\mathrm{M}}(y)$ (see 8.1 *c)*) acts on $\mathrm{H}^{\bullet}_{\mathrm{M}^0(y)}$ (by 1.9 *a)*), by $\mathbf{C}$-algebra automorphisms and on $\mathrm{H}^{\mathrm{M}^0(y)}_{\bullet}(\mathscr{P}_y, \dot{\mathscr{L}})$, $\mathrm{H}^{\mathrm{M}^0(y)}_{\bullet}(\mathscr{P}_y, \dot{\mathscr{L}}^*)$, $\mathrm{H}^{\mathrm{M}^0(y)}_{\bullet}(\mathscr{P}_y \times \mathscr{P}_y, \dot{\mathscr{L}} \boxtimes \dot{\mathscr{L}}^*)$ (by 1.9 *a)*) by automorphisms which are compatible with the $\mathrm{H}^{\bullet}_{\mathrm{M}^0(y)}$-module structures (i.e. are semi-linear with respect to the automorphisms of $\mathrm{H}^{\bullet}_{\mathrm{M}^0(y)}$ defined by $\overline{\mathrm{M}}(y)$). The tensor product of the $\overline{\mathrm{M}}(y)$ actions on the factors defines an $\overline{\mathrm{M}}(y)$ action on $\mathrm{H}^{\mathrm{M}^0(y)}_{\bullet}(\mathscr{P}_y, \dot{\mathscr{L}}) \otimes_{\mathrm{H}^{\bullet}_{\mathrm{M}^0(y)}} \mathrm{H}^{\mathrm{M}^0(y)}_{\bullet}(\mathscr{P}_y, \dot{\mathscr{L}}^*)$. The map α in 8.4 is compatible with the $\overline{\mathrm{M}}(y)$-actions.

One verifies from the definitions that

a) the action of $\overline{\mathrm{M}}(y)$ commutes with the operators $\Delta(w)$, $\Delta(\xi)$ on $\mathrm{H}^{\mathrm{M}^0(y)}_{\bullet}(\mathscr{P}_y, \dot{\mathscr{L}})$, and with $\Delta'(w)$, $\Delta'(\xi)$ on $\mathrm{H}^{\mathrm{M}^0(y)}_{\bullet}(\mathscr{P}_y, \dot{\mathscr{L}}^*)$.

We have the following " vanishing " result.

Proposition **8.6.**

a) $\mathrm{H}^{\mathrm{odd}}_c(\mathscr{P}_y, \dot{\mathscr{L}}^*) = \mathrm{H}^{\mathrm{odd}}_c(\mathscr{P}_y, \dot{\mathscr{L}}) = 0$.

b) $\mathrm{H}^{\mathrm{M}^0(y)}_{\mathrm{odd}}(\mathscr{P}_y, \dot{\mathscr{L}}) = \mathrm{H}^{\mathrm{M}^0(y)}_{\mathrm{odd}}(\mathscr{P}_y, \dot{\mathscr{L}}^*) = \mathrm{H}^{\mathrm{M}^0(y)}_{\mathrm{odd}}(\mathscr{P}_y \times \mathscr{P}_y, \dot{\mathscr{L}} \boxtimes \dot{\mathscr{L}}^*) = 0$.

c) *α in 8.4 is an isomorphism.*

d) $\mathrm{H}^{\mathrm{M}^0(y)}_{\bullet}(\mathscr{P}_y, \dot{\mathscr{L}})$, $\mathrm{H}^{\mathrm{M}^0(y)}_{\bullet}(\mathscr{P}_y, \dot{\mathscr{L}}^*)$ *are finitely generated projective* $\mathrm{H}^{\bullet}_{\mathrm{M}^0(y)}$*-modules.*

e) $\mathrm{H}^{\mathrm{G} \times \mathrm{C}^*}_{\mathrm{odd}}(\ddot{\mathfrak{g}}_{\mathcal{O}}, \ddot{\mathscr{L}}) = 0$.

Proof. — *a)* is proved in [10, V, 24.8]. Now *b)*, *c)*, *d)* follow from *a)* in view of 7.2 *b)*, *c)* and 7.4; *e)* follows from *b)* and the injectivity of β in 8.4. (See 1.9 *a)*.)

8.7. $\mathrm{H}^{\bullet}_{\mathrm{M}^0(y)}$ is the coordinate ring of an affine algebraic variety V whose points are the semisimple $\mathrm{M}^0(y)$-orbits on

a) $\mathfrak{m}(y) = \mathrm{Lie}\,\mathrm{M}(y) = \mathrm{Lie}\,\mathrm{M}^0(y) = \{(x, r_0) \in \mathfrak{g} \oplus \mathbf{C} \mid [x, y] = 2r_0 y\}$.

(Let $v \in \mathrm{V}$. If $f \in \mathrm{H}^{\bullet}_{\mathrm{M}^0(y)}$, we may regard f as a polynomial on the reductive quotient $\mathfrak{m}(y)_r$, invariant under $\mathrm{M}^0(y)$, see 1.11 *a)*. Then $f(v) \in \mathbf{C}$ is by definition the value of f at any point in the image of v under the canonical map $\mathfrak{m}(y) \to \mathfrak{m}(y)_r$.) Then by 8.6 *d)*, *c)*, $\mathrm{H}^{\mathrm{M}^0(y)}_{\bullet}(\mathscr{P}_y, \dot{\mathscr{L}})$, $\mathrm{H}^{\mathrm{M}^0(y)}_{\bullet}(\mathscr{P}_y, \dot{\mathscr{L}}^*)$, $\mathrm{H}^{\mathrm{M}^0(y)}_{\bullet}(\mathscr{P}_y \times \mathscr{P}_y, \dot{\mathscr{L}} \boxtimes \dot{\mathscr{L}}^*)$ may be regarded as spaces of sections of the algebraic vector bundles E, E′, F = E ⊗ E′ (respectively) over V.

Thus, the fibre of E at $v \in \mathrm{V}$ is $\mathrm{E}_v = \mathbf{C}_v \otimes_{\mathrm{H}^{\bullet}_{\mathrm{M}^0(y)}} \mathrm{H}^{\mathrm{M}^0(y)}_{\bullet}(\mathscr{P}_y, \dot{\mathscr{L}})$ where $\mathbf{C}_v$ is $\mathbf{C}$ regarded as an $\mathrm{H}^{\bullet}_{\mathrm{M}^0(y)}$-algebra via the homomorphism $\mathrm{H}^{\bullet}_{\mathrm{M}^0(y)} \to \mathbf{C}$, $f \mapsto f(v)$ as above.

Similarly, $E'_v = \mathbf{C}_v \otimes_{H^\bullet_{M^0(y)}} H^{M^0(v)}_\bullet(\mathscr{P}_y, \mathscr{L}^*)$, $F_v = E_v \otimes_{\mathbf{C}} E'_v$.

By 8.3, the operators $\Delta(w)$, $\Delta(\xi)$ (resp. $\Delta'(w)$, $\Delta'(\xi)$) on $H^{M^0(v)}_\bullet(\mathscr{P}_y, \dot{\mathscr{L}})$ (resp. $H^{M^0(v)}_\bullet(\mathscr{P}_y, \dot{\mathscr{L}}^*)$) come from vector bundle maps $\Delta(w), \Delta(\xi) : E \to E$ (resp. $\Delta'(w), \Delta'(\xi) : E' \to E'$), inducing the identity on the base V. Hence these operators act on each fibre E_v, E'_v.

By 8.5, $\overline{M}(y)$ acts on $H^\bullet_{M^0(y)}$. This corresponds to an action of $\overline{M}(y)$ on V. (It is induced by the adjoint action of $M(y)$ on its Lie algebra.) Moreover, E, E′ are naturally $\overline{M}(y)$-equivariant vector bundles over V and the operators $\Delta(w)$, $\Delta(\xi)$, $\Delta'(w)$, $\Delta'(\xi)$ on them are $\overline{M}(y)$-invariant.

8.8. For each $v \in V$ we denote the stabilizer of v in $\overline{M}(y)$ by $\overline{M}(y, v)$. (We shall also write $\overline{M}(y, \sigma, r_0)$ instead of $\overline{M}(y, v)$ where (σ, r_0) is any element of the orbit v.) Then $\overline{M}(y, v)$ acts naturally on the fibres E_v, E'_v.

Let $\operatorname{rep}(\overline{M}(y, v))$ be a set of representatives for the isomorphism classes of irreducible $\overline{M}(y, v)$-modules.

For each $\rho \in \operatorname{rep}(\overline{M}(y, v))$ we define $E_{v,(\rho)}$ (resp. $E'_{v,(\rho^*)}$) to be the ρ-isotypical (resp. ρ^*-isotypical) component of the $\overline{M}(y, v)$-module E_v (resp. E'_v), and we define
a) $E_{v,\rho} = (\rho^* \otimes E_v)^{\overline{M}(y,v)}$, $E'_{v,\rho^*} = (\rho \otimes E'_v)^{\overline{M}(y,v)}$.

From 8.7 it is clear that $\Delta(w)$, $\Delta(\xi)$ (resp. $\Delta'(w)$, $\Delta'(\xi)$) act naturally on $E_{v,(\rho)}$, $E_{v,\rho}$ (resp. on $E'_{v,(\rho^*)}$, E'_{v,ρ^*}).

8.9. We have

a) $E_0 \overset{\approx}{\to} H^{\{e\}}_\bullet(\mathscr{P}_y, \dot{\mathscr{L}})$, $E'_0 \overset{\approx}{\to} H^{\{e\}}_\bullet(\mathscr{P}_y, \dot{\mathscr{L}}^*)$ (by 7.2 *d)*).

Now the action of $M(y)$ on $\mathscr{P}_y$, $\dot{\mathscr{L}}$, $\dot{\mathscr{L}}^*$ induces an action of $\overline{M}(y)$ on $H^\bullet_c(\mathscr{P}_y, \mathscr{L}^*)$, $H^\bullet_c(\mathscr{P}_y, \dot{\mathscr{L}})$ hence on $H^{\{e\}}_\bullet(\mathscr{P}_y, \dot{\mathscr{L}}^*)$, $H^{\{e\}}_\bullet(\mathscr{P}_y, \dot{\mathscr{L}})$; it is easy to see that these are compatible under *a)* with the actions of $\overline{M}(y)$ on E_0, E'_0 considered before. Let:

b) $\operatorname{rep}_0 \overline{M}(y, v)$ be the set of those $\rho \in \operatorname{rep} \overline{M}(y, v)$ which occur in the restriction of the $\overline{M}(y)$-module $H^{\{e\}}_\bullet(\mathscr{P}_y, \dot{\mathscr{L}})$ to $\overline{M}(y, v)$.

Proposition **8.10.** — *Let* $\rho \in \operatorname{rep} \overline{M}(y, v)$. *The following conditions are equivalent:*

a) $E_{v,\rho} \neq 0$;

b) $E'_{v,\rho^*} \neq 0$;

c) $\rho \in \operatorname{rep}_0 \overline{M}(y, v)$.

Proof. — We restrict the vector bundles E, E′ to the subset $V' = \{ tv \mid t \in \mathbf{C} \}$ of V. (If v is a semisimple $M^0(y)$-orbit in $\mathfrak{m}(y)$ and $t \in \mathbf{C}$ then $t.v$ is again a semisimple $M^0(y)$-orbit.) Now $\overline{M}(y, v)$ acts on these vector bundles (as identity on V′). Since V′ is connected and the representations of a finite group do not change by deformation, it follows that the $\overline{M}(y, v)$ modules E_v, E_0 (resp. E'_v, E'_0) are isomorphic. Using 8.9 *a)* we are reduced to showing that the $\overline{M}(y, v)$-module $H^{\{e\}}_\bullet(\mathscr{P}_y, \dot{\mathscr{L}})$ is isomorphic to the dual of the $\overline{\mathrm{M}}(y, v)$-module $\mathrm{H}^{\{e\}}_{\bullet}(\mathscr{P}_v, \dot{\mathscr{L}}^*)$. It is enough to show that the $\overline{\mathrm{M}}(y)$-module $\mathrm{H}^{\bullet}_{c}(\mathscr{P}_v, \dot{\mathscr{L}})$ is isomorphic to the dual of the $\overline{\mathrm{M}}(y)$-module $\mathrm{H}^{\bullet}_{c}(\mathscr{P}_v, \dot{\mathscr{L}}^*)$.

Since $\dot{\mathscr{L}}$ has finite monodromy, we can find a flat, positive definite hermitian form on $\dot{\mathscr{L}}$; we can assume, by averaging, that this form is invariant under the action of the maximal compact subgroup K of $\overline{\mathrm{M}}(y)$. This form gives an isomorphism $\dot{\mathscr{L}} \approx \dot{\mathscr{L}}^*$ of local systems/**R** which is semilinear with respect to complex conjugation and is K-invariant. This induces an isomorphism $\mathrm{H}^{\bullet}_{c}(\mathscr{P}_v, \dot{\mathscr{L}}) \overset{\approx}{\rightarrow} \mathrm{H}^{\bullet}_{c}(\mathscr{P}_v, \dot{\mathscr{L}}^*)$ of **R**-vector spaces which is again semilinear with respect to complex conjugation. It is compatible with the action of K hence with that of $\overline{\mathrm{M}}(y)$ (which is also the group of components of K). From this, the desired result follows immediately.

Proposition **8.11.** — *Let* Y *be a locally closed subvariety of* $\mathfrak{g}_{\mathrm{N}}$, *which is a union of nilpotent orbits.*

a) $\mathrm{H}^{\mathrm{G} \times \mathbf{C}^*}_{\mathrm{odd}}(\ddot{\mathrm{Y}}, \ddot{\mathscr{L}}) = 0$.

b) *The open embedding* $i : \mathrm{Y} \hookrightarrow \overline{\mathrm{Y}}$ *(closure of* Y*) induces a surjective homomorphism* $i^* : \mathrm{H}^{\mathrm{G} \times \mathbf{C}^*}_{\bullet}(\ddot{\overline{\mathrm{Y}}}, \ddot{\mathscr{L}}) \rightarrow \mathrm{H}^{\mathrm{G} \times \mathbf{C}^*}_{\bullet}(\ddot{\mathrm{Y}}, \ddot{\mathscr{L}})$.

c) *If* Y *is closed in* $\mathfrak{g}_{\mathrm{N}}$, *the closed embedding* $j : \mathrm{Y} \hookrightarrow \mathfrak{g}_{\mathrm{N}}$ *induces an injective homomorphism* $j_! : \mathrm{H}^{\mathrm{G} \times \mathbf{C}^*}_{\bullet}(\ddot{\mathrm{Y}}, \ddot{\mathscr{L}}) \rightarrow \mathrm{H}^{\mathrm{G} \times \mathbf{C}^*}_{\bullet}(\ddot{\mathfrak{g}}_{\mathrm{N}}, \ddot{\mathscr{L}})$.

Proof. — *a)* is proved by induction on the number $n(\mathrm{Y})$ of nilpotent orbits contained in Y. If $n(\mathrm{Y}) = 1$, we use 8.6 *e)*. If $n(\mathrm{Y}) > 1$, we write $\mathrm{Y} = \mathrm{Y}_1 \cup \mathrm{Y}_2$ where Y_1 is closed in Y, Y_2 is $\mathrm{Y} - \mathrm{Y}_1$ and $n(\mathrm{Y}_1) < n(\mathrm{Y})$, $n(\mathrm{Y}_2) < n(\mathrm{Y})$. We write the long exact sequence 1.5 *a)* for the partition $\ddot{\mathrm{Y}} = \ddot{\mathrm{Y}}_1 \cup \ddot{\mathrm{Y}}_2$. We may assume *a)* known for Y_1, Y_2 and we deduce that it is also true for Y. Now *b)* follows from the long exact sequence 1.5 *a)* for $\ddot{\overline{\mathrm{Y}}} = (\overline{\mathrm{Y}} - \mathrm{Y})^{\cdot\cdot} \cup \ddot{\mathrm{Y}}$ and from *a)* for Y, $\overline{\mathrm{Y}}$ and $\overline{\mathrm{Y}} - \mathrm{Y}$; *c)* follows from the long exact sequence 1.5 *a)* for $\ddot{\mathfrak{g}}_{\mathrm{N}} = \ddot{\mathrm{Y}} \cup (\mathfrak{g}_{\mathrm{N}} - \mathrm{Y})^{\cdot\cdot}$ and from *a)* for $\mathfrak{g}_{\mathrm{N}}$, Y and $\mathfrak{g}_{\mathrm{N}} - \mathrm{Y}$.

Corollary **8.12.** — *If* Y *is as in* 8.11 *then* $\xi \otimes 1 \rightarrow \Delta(\xi)$, $1 \otimes w \mapsto \Delta(w)$ (see 4.3) *define an* **H**-*module structure on* $\mathrm{H}^{\mathrm{G} \times \mathbf{C}^*}_{\bullet}(\ddot{\mathrm{Y}}, \ddot{\mathscr{L}})$. *Similarly,* $\xi \otimes 1 \rightarrow \Delta'(\xi)$, $1 \otimes w \mapsto \Delta'(w)$ *define another* **H**-*module structure on* $\mathrm{H}^{\mathrm{G} \times \mathbf{C}^*}_{\bullet}(\ddot{\mathrm{Y}}, \ddot{\mathscr{L}})$. *These two* **H**-*module structures commute with each other and hence define an* **H** ⊗ **H**-*module structure on* $\mathrm{H}^{\mathrm{G} \times \mathbf{C}^*}_{\bullet}(\ddot{\mathrm{Y}}, \ddot{\mathscr{L}})$.

Proof. — When $\mathrm{Y} = \mathfrak{g}_{\mathrm{N}}$, this follows from 5.1. When Y is closed in $\mathfrak{g}_{\mathrm{N}}$, one uses the case $\mathrm{Y} = \mathfrak{g}_{\mathrm{N}}$, together with 8.11 c) and 3.10. When Y is arbitrary, the corollary follows from the already known results for $\overline{\mathrm{Y}}$, using 8.11 b) and 3.10.

Theorem **8.13.** — *Let* $y \in \mathcal{O}$ *be as in* 8.1. *Then* $\xi \otimes 1 \rightarrow \Delta(\xi)$, $1 \otimes w \mapsto \Delta(w)$ (*see* 8.1, 8.2) *define an* **H**-*module structure on* $\mathrm{H}^{\mathrm{M}^0(v)}_{\bullet}(\mathscr{P}_y, \dot{\mathscr{L}})$.

Proof. — Let

$$\Pi_i = \Delta(s_i)\, \Delta(\xi) - \Delta({}^{s_i}\xi)\, \Delta(s_i) - c_i\, \Delta(r(\xi - {}^{s_i}\xi)/\alpha_i)$$

as an operator on $\mathrm{H}^{\mathrm{M}^0(v)}_{\bullet}(\mathscr{P}_y, \dot{\mathscr{L}})$ $(1 \leqslant i \leqslant m)$.

We may also regard Π_i as an endomorphism of the vector bundle $E \to V$ (see 8.7) mapping each fibre into itself. We only have to prove that this endomorphism is zero. Consider the endomorphism $\Pi_i \otimes 1$ of $E \otimes E' = F$ (see 8.7).

Let σ be a section of the vector bundle F which is $\overline{M}(y)$-invariant. Then σ is in the image of the homomorphism β in 8.4 (see 1.9 *a)*); now Π_i is zero on the source of β by 8.12 for $Y = \mathcal{O}$, and β is compatible with the operators $\Delta(w)$, $\Delta(\xi)$, hence $(\Pi_i \otimes 1) \circ \sigma = 0$ as a section of F.

Now let $v \in V$ and let f be an $\overline{M}(y, v)$ invariant element of the fibre F_v.

For each v' in the $\overline{M}(y)$-orbit of v, we define $f_{v'} \in F_{v'}$ by $f_{v'} = \gamma(f)$ where $\gamma : F_v \to F_{v'}$ is the action on F of an element $\gamma \in \overline{M}(y)$ such that $\gamma v = v'$. (Then $f_{v'}$ is independent of γ by the $\overline{M}(y, v)$ invariance of v.)

Since the $\overline{M}(y)$-orbit of v is finite, we see from the Chinese Remainder Theorem that there exists a section σ_1 of F such that $\sigma_1(v') = f_{v'}$ for all v' in that $\overline{M}(y)$-orbit.

Let $\sigma = | \overline{M}(y) |^{-1} \Sigma\gamma \circ \sigma_1$ (sum over all $\gamma \in \overline{M}(y)$).

This is then an $\overline{M}(y)$-invariant section, still satisfying $\sigma(v') = f_{v'}$ for all v' in the $\overline{M}(y)$-orbit in v. In particular, it satisfies $\sigma(v) = f_v = f$. As $(\Pi_i \otimes 1) \circ \sigma = 0$, we must have also $(\Pi_i \otimes 1)(f) = 0$.

Thus $\Pi_i \otimes 1$ is zero on $F_v^{\overline{M}(y, v)}$.

Consider the **C**-linear map

$$\Psi : E_v \otimes E'_v \otimes (E'_v)^* \to E_v$$

defined by $\varepsilon_1 \otimes \varepsilon_2 \otimes \varepsilon_3 \to \varepsilon_3(\varepsilon_2)\, \varepsilon_1$. It is clearly compatible with the endomorphisms $\Pi_i \otimes 1 \otimes 1$ of $E_v \otimes E'_v \otimes (E'_v)^*$ and Π_i of E_v.

Its restriction to the subspace $(E_v \otimes E'_v)^{\overline{M}(y, v)} \otimes (E'_v)^*$ is surjective. (This follows from the equivalence of *a)* and *b)* in 8.10.) But as we have seen above, $\Pi_i \otimes 1 \otimes 1$ is zero on this subspace. It follows that Π_i is zero on E_v. Since v is arbitrary, $\Pi_i : E \to E$ is zero and the theorem is proved.

8.14. The **H**-module structure on $H_\bullet^{M^0(y)}(\mathcal{P}_y, \dot{\mathcal{L}})$ in 8.13 is defined by operators which are $H^\bullet_{M^0(y)}$-linear and commute with the action of $\overline{M}(y)$, hence it defines an **H**-module structure on each fibre E_v ($v \in V$) of $E \to V$ (see 8.7) and on each $E_{v, \rho}$ ($\rho \in \text{rep}_0\, \overline{M}(y, v)$, see 8.9 *b)*). The **H**-modules $E_{v, \rho}$ ($\rho \in \text{rep}_0\, \overline{M}(y, v)$) are called *standard*. Let $(\sigma, r_0) \in v$. We shall write sometimes $E_{\sigma, r_0, y, \rho}$ instead of $E_{v, \rho}$. Thus, the standard **H**-module $E_{\sigma, r_0, y, \rho}$ is defined for any semisimple element $\sigma \in \mathfrak{g}$, any nilpotent element $y \in \mathfrak{g}$, any $r_0 \in \mathbf{C}$ such that $[\sigma, y] = 2r_0 y$ and any $\rho \in \text{rep}_0\, \overline{M}(y, \sigma, r_0)$ (see 8.8, 8.9). Clearly, the isomorphism class of the **H**-module $E_{\sigma, r_0, y, \rho}$ depends only on r_0 and the G-conjugacy class of σ, y, ρ.

Theorem **8.15.** — *Any simple* **H**-*module* $\mathcal{M}$ *is a quotient of a standard* **H**-*module.*

Proof. — We can clearly find a non-zero **H**-linear map $\mathbf{H} \to \mathcal{M}$, where **H** is regarded as a left **H**-module in the obvious way. By 6.3 we can find a non-zero **H**-linear map

$\mathrm{H}_\bullet^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathfrak{g}}_{\mathrm{N}}, \ddot{\mathscr{L}}) \to \mathscr{M}$ where $\mathrm{H}_\bullet^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathfrak{g}}_{\mathrm{N}}, \ddot{\mathscr{L}})$ is regarded as an **H**-module as in the first sentence of 8.12. Let Y be a closed $\mathrm{G} \times \mathbf{C}^*$ stable subvariety of $\mathfrak{g}_{\mathrm{N}}$ such that *a)* there exists a non-zero **H**-linear map $\varphi : \mathrm{H}_\bullet^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathrm{Y}}, \ddot{\mathscr{L}}) \to \mathscr{M}$ (for the **H**-module structure in the first sentence of 8.12) and *b)* Y is minimal subject to property *a)*. (" Minimal " refers to the partial order given by inclusion.) Such Y exists since $\mathrm{Y} = \mathfrak{g}_{\mathrm{N}}$ satisfies *a)* and the number of nilpotent orbits in $\mathfrak{g}$ is finite. Consider a nilpotent orbit which is open in Y. We may assume that it is $\mathscr{O}$ of 8.1. We have an exact sequence

$$0 \to \mathrm{H}_\bullet^{\mathrm{G}\times\mathbf{C}^*}((\mathrm{Y} - \mathscr{O})^{\ddot{}}, \ddot{\mathscr{L}}) \xrightarrow{\gamma} \mathrm{H}_\bullet^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathrm{Y}}, \ddot{\mathscr{L}}) \to \mathrm{H}_\bullet^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathscr{O}}, \ddot{\mathscr{L}}) \to 0$$

(by 1.5 *a)* and 8.11 *a)* applied to $\mathrm{Y}, \mathrm{Y} - \mathscr{O}, \mathscr{O}$). It is an exact sequence of **H**-modules (8.12, 3.10). By the minimality of Y, φ must be zero on the image of γ hence it factors through a non-zero **H**-linear map $\mathrm{H}_\bullet^{\mathrm{G}\times\mathbf{C}^*}(\ddot{\mathscr{O}}, \ddot{\mathscr{L}}) \to \mathscr{M}$.

Using 8.6 *a)* and the fact (1.9 *a)*) that β in 8.4 is an isomorphism onto the $\overline{\mathrm{M}}(y)$-invariants, we deduce that there exists a non-zero **H**-linear map $(\mathrm{A}')^{\overline{\mathrm{M}}(y)} \to \mathscr{M}$ where A' (see 8.4) is regarded as an **H**-module using the operators 8.13 on the first factor and the identity on the second factor of the tensor product. Composing this with the averaging map $\mathrm{A}' \to \mathrm{A}'^{\overline{\mathrm{M}}(y)}$ (a surjective **H**-linear map) we find a non-zero **H**-linear map $\mathrm{A}' \to \mathscr{M}$. It follows that there exists a non-zero **H**-linear map $\bar{\varphi} : \mathscr{H} \to \mathscr{M}$ where $\mathscr{H} = \mathrm{H}_\bullet^{\mathrm{M}^0(y)}(\mathscr{P}_y, \dot{\mathscr{L}})$.

A well-known argument of Dixmier (applicable since **H** has countable dimension over **C**) shows that the centre of **H** acts on $\mathscr{M}$ by scalar operators. Hence, by 6.5, there exists a maximal ideal I of $\mathbf{S}^{\mathrm{W}}$ such that I acts as zero on $\mathscr{M}$.

Let $\nu : \mathrm{H}^\bullet_{\mathrm{G}\times\mathbf{C}^*} \to \mathrm{H}^\bullet_{\mathrm{M}^0(y)}$ be the homomorphism induced (1.4 *g)*) by the inclusion $\mathrm{M}^0(y) \hookrightarrow \mathrm{G} \times \mathbf{C}^*$ and let $\nu' : \mathrm{H}^\bullet_{\mathrm{G}\times\mathbf{C}^*} \to \mathbf{S}^{\mathrm{W}}$ be the (surjective) homomorphism defined by 4.3 *c)*.

Then $\mathrm{I}' = \nu'^{-1}(\mathrm{I})$ is a maximal ideal of $\mathrm{H}^\bullet_{\mathrm{G}\times\mathbf{C}^*}$. From the definitions, it is easy to see that, for $h \in \mathrm{H}^\bullet_{\mathrm{G}\times\mathbf{C}^*}$, the action of $\nu(h)$ on $\mathscr{H}$ (by the $\mathrm{H}^\bullet_{\mathrm{M}^0(y)}$-module structure) coincides with the action of $\nu'(h)$ (by the **H**-module structure). It follows that $\bar{\varphi}(\mathrm{I}''.\mathscr{H}) = 0$ where I'' is the ideal of $\mathrm{H}^\bullet_{\mathrm{M}^0(y)}$ generated by $\nu(\mathrm{I}')$.

Now ν corresponds to a finite morphism between the corresponding affine varieties (a semisimple orbit of $\mathrm{G} \times \mathbf{C}^*$ on $\mathfrak{g} \oplus \mathbf{C}$ intersects $\mathfrak{m}(y)$ in a union of finitely many orbits of $\mathrm{M}^0(y)$). Hence $\mathrm{H}^\bullet_{\mathrm{M}^0(y)}$ is integral over the image of ν and I'' has finite **C**-codimension in $\mathrm{H}^\bullet_{\mathrm{M}^0(y)}$. Note also that I'' is a proper ideal (otherwise it would follow that $\bar{\varphi} = 0$). Since $\mathrm{H}^\bullet_{\mathrm{M}^0(y)}/\mathrm{I}''$ is an artinian **C**-algebra $\neq 0$, there exist maximal ideals $\mathrm{J}_1, \mathrm{J}_2, \ldots, \mathrm{J}_s$ of $\mathrm{H}^\bullet_{\mathrm{M}^0(y)}$ $(s \geqslant 1)$ and integers $n_1 \geqslant 1, n_2 \geqslant 1, \ldots, n_s \geqslant 1$ such that $\mathrm{I}'' \subset \mathrm{J}_1^{n_1}, \ldots, \mathrm{I}'' \subset \mathrm{J}_s^{n_s}$ and the natural map $\mathrm{H}^\bullet_{\mathrm{M}^0(y)}/\mathrm{I}'' \to \bigoplus_{1 \leqslant i \leqslant s} \mathrm{H}^\bullet_{\mathrm{M}^0(y)}/\mathrm{J}_i^{n_i}$ is an algebra isomorphism.

It follows that $\mathscr{H}/\mathrm{I}''\mathscr{H} \xrightarrow{\sim} \bigoplus_{1 \leqslant i \leqslant s} \mathscr{H}/\mathrm{J}_i^{n_i}\mathscr{H}$ hence there exists i $(1 \leqslant i \leqslant s)$ such that $\bar{\varphi}$ defines a non-zero **H**-linear map $\mathscr{H}/\mathrm{J}_i^{n_i}\mathscr{H} \to \mathscr{M}$.

Let $\mathscr{H}' = \mathscr{H}/\mathrm{J}^n\mathscr{H}$ $(\mathrm{J} = \mathrm{J}_i, n = n_i)$. Then $\mathscr{H}'$ is finite dimensional over **C**, is a module over the local algebra $H^{\bullet}_{M^0(y)}/J^n$ with maximal ideal J/J^n, is an **H**-module and there exists a non-zero **H**-linear map $\mathcal{H}' \to \mathcal{M}$. By a simple lemma [5, 5.14] it follows that there exists a non-zero **H**-linear map $\mathcal{H}'/J\mathcal{H}' = \mathcal{H}/J\mathcal{H} \to \mathcal{M}$. Hence with the notation of 8.7, there exists a non-zero **H**-linear map $E_v \to \mathcal{M}$ for some $v \in V$. Since E_v is a direct sum of standard **H**-modules, the theorem follows.

8.16. As in 8.7, we regard $H^{\bullet}_{G \times \mathbf{C}^*}$ as the coordinate ring of an affine variety U whose points are the semisimple $G \times \mathbf{C}^*$-orbits in $\mathfrak{g} \oplus \mathbf{C}$. If $\mathcal{M}$ is a finitely generated $H^{\bullet}_{G \times \mathbf{C}^*}$-module we define the support of $\mathcal{M}$ to be the set of all $u \in U$ such that the localization of $\mathcal{M}$ at the maximal ideal of $H^{\bullet}_{G \times \mathbf{C}^*}$ corresponding to u is non-zero. This is a closed subvariety of U. From the two descriptions of A‴ in 8.4, we see that the support of $H^{G \times \mathbf{C}^*}_{\bullet}(\ddot{\mathcal{O}}, \ddot{\mathcal{L}})$ is contained in the set of semisimple $G \times \mathbf{C}^*$-orbits on $\mathfrak{g} \oplus \mathbf{C}$ which meet $\mathfrak{m}(y)$. Using this, and the exact sequences 1.5 *a)* we see that:

a) if Y is as in 8.11 and $y_1, \ldots, y_s$ is a set of representatives for the G-orbits in Y, then the support of the $H^{\bullet}_{G \times \mathbf{C}^*}$-module $H^{G \times \mathbf{C}^*}_{\bullet}(\ddot{Y}, \ddot{\mathcal{L}})$ is contained in the set of semisimple $G \times \mathbf{C}^*$-orbits on $\mathfrak{g} \oplus \mathbf{C}$ which meet $\mathfrak{m}(y_1) \cup \mathfrak{m}(y_2) \cup \ldots \cup \mathfrak{m}(y_s)$.

Theorem **8.17.** — *Let* $(\sigma, r_0) \in \mathfrak{g} \oplus \mathbf{C}$ *be a semisimple element such that* $r_0 \neq 0$. *Then*

a) $Z_G(\sigma)$ *acts (by the adjoint action) on the vector space* $\{ x \in \mathfrak{g} \mid [\sigma, x] = 2r_0 x \}$ *with finitely many orbits. This vector space consists of nilpotent elements.*

b) *Let* y *be an element in the unique open orbit of the action in* a). *Then* $(\sigma, r_0) \in \mathfrak{m}(y)$. *Let* $\rho \in \mathrm{rep}_0\, \overline{M}(y, \sigma, r_0)$. *Then the standard* **H**-*module* $E_{\sigma, r_0, y, \rho}$ *(see* 8.14*) is simple.*

c) *Let* σ', r'_0, y', ρ' *be another set of data satisfying the same assumptions as* σ, r_0, y, ρ *above, and assume that the standard* **H**-*module* $E_{\sigma', r'_0, y', \rho'}$ *is isomorphic to* $E_{\sigma, r_0, y, \rho}$. *Then* $r_0 = r'_0$ *and there exists* $g \in G$ *which conjugates* (σ', y', ρ') *to* (σ, y, ρ).

In preparation for the proof, we state the following elementary result.

Lemma **8.18.** — *Let* A *be an algebra over* **C** *with an involutive anti-automorphism* $a \mapsto \tilde{a}$, *let* E_i, E'_i *be finite* **C**-*dimensional* A-*modules* $(1 \leqslant i \leqslant p)$. *We regard* $\overline{E} = \bigoplus_{1 \leqslant i \leqslant p} (E_i \otimes_{\mathbf{C}} E'_i)$ *as an* $A \otimes_{\mathbf{C}} A$-*module in a natural way. Assume that there exists* $\varepsilon \in \overline{E}$ *such that*

a) $(a \otimes 1)\, \varepsilon = (1 \otimes \tilde{a})\, \varepsilon$ *for all* $a \in A$,

b) $a \to (a \otimes 1)\, \varepsilon$ *is a surjective map* $A \to \overline{E}$.

Then E_i $(1 \leqslant i \leqslant p)$ *are simple, mutually non-isomorphic* A-*modules.*

The proof is left to the reader.

8.19. *Proof of* **8.17.** — The second statement of 8.17 *a)* is obvious. The first statement of 8.17 *a)* is a consequence of [5, 5.4 *c)*] and the finiteness of the number of nilpotent orbits in $\mathfrak{g}$.

We now prove 8.17 *b)*. As in 8.1, we denote by $\mathcal{O}$ the G-orbit of y in $\mathfrak{g}$. Let $\hat{\mathcal{O}}$ be the union of all nilpotent G-orbits in $\mathfrak{g}$ which contain $\mathcal{O}$ in their closure. Then $\mathcal{O}$ is closed in $\hat{\mathcal{O}}$

and $\hat{\mathscr{O}}$ is open in $\mathfrak{g}_N$. From our assumption it follows that the vector space in 8.17 *a)* has empty intersection with $\hat{\mathscr{O}} - \mathscr{O}$. Hence, by 8.16 *a)*,

a) the support of the $\mathrm{H}^{\bullet}_{G\times\mathbf{C}^*}$-module $\mathrm{H}_{\bullet}^{G\times\mathbf{C}^*}((\hat{\mathscr{O}} - \mathscr{O})^{\cdot\cdot}, \ddot{\mathscr{L}})$ does not contain the $G \times \mathbf{C}^*$-orbit of (σ, r_0) in $\mathfrak{g} \oplus \mathbf{C}$.

Using 1.5 *a)* for the partition $(\hat{\mathscr{O}})^{\cdot\cdot} = \ddot{\mathscr{O}} \cup (\hat{\mathscr{O}} - \mathscr{O})^{\cdot\cdot}$ and 8.11 *a)* we obtain an exact sequence of $\mathbf{H} \otimes \mathbf{H}$-modules

$$0 \rightarrow \mathrm{H}_{\bullet}^{G\times\mathbf{C}^*}(\ddot{\mathscr{O}}, \ddot{\mathscr{L}}) \rightarrow \mathrm{H}_{\bullet}^{G\times\mathbf{C}^*}((\hat{\mathscr{O}})^{\cdot\cdot}, \ddot{\mathscr{L}}) \rightarrow \mathrm{H}_{\bullet}^{G\times\mathbf{C}^*}((\hat{\mathscr{O}} - \mathscr{O})^{\cdot\cdot}, \ddot{\mathscr{L}}) \rightarrow 0$$

regarded as a $\mathrm{H}^{\bullet}_{G\times\mathbf{C}^*}$-algebra via the homomorphism $\mathrm{H}^{\bullet}_{G\times\mathbf{C}^*} \rightarrow \mathbf{C}$ defined by evaluation of an element of $S(\mathfrak{g} \oplus \mathbf{C})$ at (σ, r_0). We tensor the previous exact sequence with $\mathbf{C}_{\sigma, r}$ over $\mathrm{H}^{\bullet}_{G\times\mathbf{C}^*}$.

Using *a)*, we see that we obtain an isomorphism:

b) $\mathbf{C}_{\sigma, r_0} \otimes_{\mathrm{H}^{\bullet}_{G\times\mathbf{C}^*}} \mathrm{H}_{\bullet}^{G\times\mathbf{C}^*}(\ddot{\mathscr{O}}, \ddot{\mathscr{L}}) \overset{\approx}{\rightarrow} \mathbf{C}_{\sigma, r_0} \otimes_{\mathrm{H}^{\bullet}_{G\times\mathbf{C}^*}} \mathrm{H}_{\bullet}^{G\times\mathbf{C}^*}((\hat{\mathscr{O}})^{\cdot\cdot}, \ddot{\mathscr{L}})$.

From 8.11 *b)*, we have a *surjective* $\mathbf{H} \otimes \mathbf{H}$-linear map

c) $\mathrm{H}_{\bullet}^{G\times\mathbf{C}^*}(\ddot{\mathfrak{g}}_N, \ddot{\mathscr{L}}) \rightarrow \mathrm{H}_{\bullet}^{G\times\mathbf{C}^*}((\hat{\mathscr{O}})^{\cdot\cdot}, \ddot{\mathscr{L}})$.

Tensoring the algebra homomorphism $\mathrm{H}^{\bullet}_{G\times\mathbf{C}^*} \rightarrow \mathbf{C}_{\sigma, r_0}$ with $\mathrm{H}_{\bullet}^{G\times\mathbf{C}^*}((\hat{\mathscr{O}})^{\cdot\cdot}, \ddot{\mathscr{L}})$ gives a surjective $\mathbf{H} \otimes \mathbf{H}$-linear map

d) $\mathrm{H}_{\bullet}^{G\times\mathbf{C}^*}((\hat{\mathscr{O}})^{\cdot\cdot}, \ddot{\mathscr{L}}) \rightarrow \mathbf{C}_{\sigma, r_0} \otimes_{\mathrm{H}^{\bullet}_{G\times\mathbf{C}^*}} \mathrm{H}_{\bullet}^{G\times\mathbf{C}^*}((\hat{\mathscr{O}})^{\cdot\cdot}, \ddot{\mathscr{L}})$.

The composition of *c)*, *d)* and the inverse of *b)* is a surjective $\mathbf{H} \otimes \mathbf{H}$-linear map

e) $\mathrm{H}_{\bullet}^{G\times\mathbf{C}^*}(\ddot{\mathfrak{g}}_N, \ddot{\mathscr{L}}) \rightarrow \mathbf{C}_{\sigma, r_0} \otimes_{\mathrm{H}^{\bullet}_{G\times\mathbf{C}^*}} \mathrm{H}_{\bullet}^{G\times\mathbf{C}^*}(\ddot{\mathscr{O}}, \ddot{\mathscr{L}})$.

Using 8.4, 8.6 *a)* and 1.9 *a)* we have an isomorphism of $\mathbf{H} \otimes \mathbf{H}$-modules

f) $\mathrm{H}_{\bullet}^{G\times\mathbf{C}^*}(\ddot{\mathscr{O}}, \ddot{\mathscr{L}}) \cong (A')^{\overline{M}(y)}$.

(A' as in 8.4; we regard A' as an $\mathbf{H} \otimes \mathbf{H}$-module using the $\mathbf{H}$-module structure on $\mathrm{H}_{\bullet}^{M^0(y)}(\mathscr{P}_y, \dot{\mathscr{L}})$ in 8.13 and the analogous $\mathbf{H}$-module structure on $\mathrm{H}_{\bullet}^{M^0(y)}(\mathscr{P}_y, \dot{\mathscr{L}}^*)$.)

Let v be the $M^0(y)$-orbit of (σ, r_0) in $\mathfrak{m}(y)$. Taking the value of a section of F at v, defines an $\mathbf{H} \otimes \mathbf{H}$-linear $A' \rightarrow F_v$. This restricts to a *surjective* $\mathbf{H} \otimes \mathbf{H}$-linear map

g) $(A')^{\overline{M}(y)} \rightarrow F_v^{\overline{M}(y, v)}$

(see the proof of 8.13). We have

h) $F_v^{\overline{M}(y, v)} = (E_v \otimes E'_v)^{\overline{M}(y, v)} = \bigoplus_{\rho} (E_{v, \rho} \otimes E'_{v, \rho^*})$

where ρ runs over $\mathrm{rep}_0(\overline{M}(y, v))$ (see 8.10).

Composing *f)*, *g)*, *h)* we obtain a surjective $\mathbf{H} \otimes \mathbf{H}$-linear map

i) $\mathrm{H}_{\bullet}^{G\times\mathbf{C}^*}(\ddot{\mathscr{O}}, \ddot{\mathscr{L}}) \rightarrow \bigoplus_{\rho} (E_{v, \rho} \otimes E'_{v, \rho^*})$.

The maximal ideal I of $\mathrm{H}^{\bullet}_{G\times\mathbf{C}^*}$ corresponding to (σ, r_0) clearly acts as 0 on the right hand side of *i)* (via $\mathrm{H}^{\bullet}_{G\times\mathbf{C}^*} \rightarrow \mathbf{S}^W \subset \mathbf{H} \otimes \mathbf{C} \subset \mathbf{H} \otimes \mathbf{H}$) hence *i)* factors through a surjective $\mathbf{H} \otimes \mathbf{H}$-linear map

j) $\mathbf{C}_{\sigma, r_0} \otimes_{\mathrm{H}^{\bullet}_{G\times\mathbf{C}^*}} \mathrm{H}_{\bullet}^{G\times\mathbf{C}^*}(\ddot{\mathscr{O}}, \ddot{\mathscr{L}}) \rightarrow \bigoplus_{\rho} (E_{v, \rho} \otimes E'_{v, \rho^*})$.

Composing *e)* and *j)* gives a surjective $\mathbf{H} \otimes \mathbf{H}$-linear map

k) $H_\bullet^{G \times \mathbf{C}^*}(\ddot{\mathfrak{g}}_N, \ddot{\mathscr{L}}) \to \bigoplus_{\rho} (E_{v,\rho} \otimes E'_{v,\rho^*})$.

Using 8.4 we see that the assumptions of 8.18 are satisfied for $\bar{E}$ the right hand side of *k)*, $A = \mathbf{H}$, $\varepsilon =$ image of **1** under the map *k)*. We conclude that the **H**-modules $E_{v,\rho}$ are simple and the **H**-modules $E_{v,\rho}$, $E_{v,\rho'}$ are isomorphic if and only if $\rho = \rho'$. This proves 8.17 *b)*.

We now prove 8.17 *c)*. The $G \times \mathbf{C}^*$-orbit of $(\sigma, r_0) \in \mathfrak{g} \oplus \mathbf{C}$, or equivalently, the corresponding maximal ideal I of $H^\bullet_{G \times \mathbf{C}^*}$ can be reconstructed from the **H**-module $E_{v,\rho}$; indeed, I is the set of elements of $H^\bullet_{G \times \mathbf{C}^*}$ which act as 0 on $E_{v,\rho}$, via $H^\bullet_{G \times \mathbf{C}^*} \xrightarrow{\text{onto}} \mathbf{S}_W \hookrightarrow \mathbf{H}$. Hence (σ, r_0) and (σ', r'_0) in *c)* are $G \times \mathbf{C}^*$-conjugate. We can thus assume that $(\sigma, r_0) = (\sigma', r'_0)$. Now y, y' are in the same $Z_G(\sigma)$-orbit, by assumption. Hence we can assume $y' = y$. But then $\rho = \rho'$ by an earlier part of the proof. The theorem is proved.

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