PUBLICATIONS MATHÉMATIQUES DE L'I.H.É.S.

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Publications mathématiques de l'I.H.É.S., tome 66 (1987), p. 93-103
<http://www.numdam.org/item?id=PMIHES_1987__66__93_0>

NON-ARITHMETIC GROUPS IN LOBACHEVSKY SPACES

by M. GROMOV and I. PIATETSKI-SHAPIRO

0. Introduction

In this paper we construct *non-arithmetic* lattices Γ (both cocompact and non-cocompact: see 1.3 for the definition) in the projective orthogonal group $PO(n, 1) = O(n, 1)/\{+1, -1\}$ for all $n = 2, 3, \ldots$. We obtain our Γ by "interbreeding" two *arithmetic* subgroups Γ_1 and Γ_2 in $PO(n, 1)$ as follows. Recall that $PO(n, 1)$ is the isometry group of the *Lobachevsky space* L^n and assume the subgroups $\Gamma_i \subset PO(n, 1)$, for $i = 1, 2$, have no torsion. Then the quotient spaces $V_i = \Gamma_i \backslash L^n$ are *hyperbolic manifolds* (i.e. complete Riemannian of constant curvature) and Γ_i is the fundamental group of V_i for $i = 1, 2$. Next, to make the interbreeding possible, we assume there exist connected submanifolds $V_1^+ \subset V_1$ and $V_2^+ \subset V_2$ of dimension n with boundaries $\partial V_1^+ \subset V_1$ and $\partial V_2^+ \subset V_2$, such that

a) The hypersurface $\partial V_i^+ \subset V_i$ for $i = 1, 2$ is totally geodesic in V_i. That is, the universal covering of ∂V_i^+ is a hyperplane in the universal covering L^n of V_i. In particular, ∂V_i^+ is an $(n-1)$-dimensional hyperbolic manifold.

b) The manifolds ∂V_1^+ and ∂V_2^+ are isometric.

Now we produce *the hybrid manifold* V by gluing together V_1^+ and V_2^+ according to an isometry between ∂V_1^+ and ∂V_2^+. This V carries a natural metric of constant negative curvature coming from those on V_1^+ and V_2^+ and this metric is complete apart from a few irrelevant exceptional cases (see 2.10). Then the universal covering of V equals L^n and the fundamental group Γ of V is a lattice in $PO(n, 1) = \text{Is}\, L^n$. Note that if the subgroups Γ_1 and Γ_2 are *cocompact* (i.e. if V_1 and V_2 are compact) then also Γ is cocompact.

Also note that the fundamental group Γ_i^+ of V_i^+ *injects* into Γ_i for $i = 1, 2$ (see 2.10) and that in the relevant cases Γ_i^+ satisfies the following.

0.1. Density property (see 1.7). — *The subgroup* $\Gamma_i^+ \subset PO(n, 1)$ *is Zariski dense in* $PO(n, 1)^0$ for $i = 1, 2$, where ^{0} stands for "the identity component of".

This density for $i = 1$ implies (see 1.2 and 1.6) the following

0.2. Commensurability property. — *If the group* Γ *(as well as* Γ_1*) is arithmetic then* Γ *and* Γ_1 *are commensurable. That is there exists a hyperbolic manifold admitting locally isometric finite covering maps onto* V *and onto* V_1.

Similarly, arithmeticity of Γ implies commensurability between Γ and Γ_2 and hence, commensurability between Γ_1 and Γ_2. Therefore, *one obtains a non-arithmetic* Γ *by taking* Γ_1 *and* Γ_2 *non-commensurable* (compare 2.6, 2.7 and 2.8).

0.3. *Historical remarks.* — *a)* Examples of non-arithmetic lattices Γ in L^3 (the existence of non-arithmetic lattices in L^2 is trivial) were first found by Makarov (see [M]) among *reflection groups* that are groups generated by reflections in some hyperplanes. Then non-arithmetic reflection lattices were constructed in L^4 and L^5. It is yet unknown for which n there exists a non-arithmetic reflection lattice in L^n, but one does know this n cannot be too large. In fact, no reflection lattice exists in L^n for $n \geqslant 995$ (see [V], [N] and references therein).

b) A famous theorem by Margulis asserts that every lattice in a simple Lie group G with $\mathrm{rank}_{\mathbf{R}}\, G \geqslant 2$ is arithmetic. The remaining non-compact groups (groups with $\mathrm{rank}_{\mathbf{R}} = 1$) are (up to local isomorphism): $O(n, 1)$, $U(n, 1)$, and their quaternion and Cayley analogues. Apart from $O(n, 1)$ where our interbreeding provides non-arithmetic lattices for all n, the existence of non-arithmetic lattices is only known for $SU(2, 1)$ and $SU(3, 1)$. Non-arithmetic lattices in these two groups were constructed by Mostow (see [Mo]) by using reflections in complex hyperplanes.

0.4. *Questions.* — Call a discrete subgroup $\Gamma_0 \subset PO(n, 1)$ *subarithmetic* if Γ_0 is Zariski dense and if there exists an arithmetic subgroup $\Gamma_1 \subset PO(n, 1)$ such that $\Gamma_0 \cap \Gamma_1$ has finite index in Γ_0. Does every lattice Γ in $PO(n, 1)$ (maybe for large n) contain a subarithmetic subgroup? Is Γ generated by (finitely many) such subgroups? If so, does $V = \Gamma \backslash L^n$ admit a "nice" partition into "subarithmetic pieces"?

Acknowledgements. — While preparing this paper the authors much benefited from discussions with Ofer Gabber, Ron Livney, John Morgan and George Mostow. We are especially thankful to Jacques Tits who read the first version of the manuscript and suggested a variety of improvements and corrections.

1. Rudiments of arithmetic groups

1.1. *Integral points in linear reductive groups.* — A connected Lie group G is called *reductive* if the center of G is compact and G/Center is semisimple. Such a G obviously contains a unique maximal compact normal subgroup $K \subset G$. The quotient group $G' = G/K$, clearly is of *adjoint type*. That is the adjoint representation $\mathrm{ad} : G' \to \mathrm{Aut}\, L'$ is injective, where L' denotes the Lie algebra of G' and Aut is the group of linear automorphisms of L'. Our basic example is $G' = PO(n, 1)^0$.

Sufficiently dense subgroups. — Call $\Gamma \subset G$ *sufficiently dense* if the image of Γ in $G' \subset \mathrm{Aut}\, L'$ is Zariski dense in G'.

Let $G \subset GL_N\, \mathbf{R}$ be a reductive subgroup and let $\Gamma \subset G$ be the subgroup of integral matrices in G with $\det = \pm 1$. That is

$$\Gamma = G \cap GL_N\, \mathbf{Z}.$$

Property A. — We say that G satisfies A if Γ is sufficiently dense in G.

1.2. Basic Theorem. — *A reductive subgroup* $G \subset GL_N\, \mathbf{R}$ *satisfies* A *if and only if* Γ *is a lattice in* G, *that is,* $\mathrm{Vol}\, G/\Gamma < \infty$.

Proof. — The implication

$$\mathrm{Vol}\, G/\Gamma < \infty \Rightarrow \text{Zariski density of } \Gamma' \text{ in } G'$$

holds true for all discrete subgroups $\Gamma \subset G$ and is called *Borel density theorem.* A short proof of this can be found in [Z] and $[G]_2$.

Let us indicate the (well-known, see $[B]_1$) proof of the implication $\mathrm{Vol}\, G/\Gamma < \infty \Leftarrow A$.

Step 1. — By elementary properties of reductive groups (see $[B]_2$), G equals the identity component of the Zariski closure $\overline{G} \subset GL_N\, \mathbf{R}$. Therefore, G contains the identity component $\overline{\Gamma}_0$ of the Zariski closure $\overline{\Gamma} \subset GL_N\, \mathbf{R}$.

Note that the inclusion $\overline{\Gamma}_0 \subset G$ is automatic in all our cases and so Step 1 can be omitted.

Step 2. — Property A immediately implies that the homomorphism $G \to G'$ maps $\overline{\Gamma}_0$ *onto* G'. It follows that $\overline{\Gamma}_0$ is reductive.

Step 3. — The Zariski density of integral points in $\overline{\Gamma}$ implies that $\overline{\Gamma}$ is defined over $\mathbf{Q}$. In fact one only needs Zariski density of *rational* points in $\overline{\Gamma}$. This easily follows from the very definition of the Zariski closure.

Step 4. — Since $\overline{\Gamma}$ is reductive, there exists a polynomial map $P : (\mathbf{R}^N)^k \to \mathbf{R}^\ell$ for some k and ℓ, such that

a) The set of linear transformations of $\mathbf{R}^N$ fixing P equals $\overline{\Gamma}$.

Furthermore, since $\overline{\Gamma}$ is defined over $\mathbf{Q}$ one can choose the above P integral. That is

b) $$P((\mathbf{Z}^N)^k) \subset \mathbf{Z}^\ell.$$

The existence of P is easy (see $[B]_1$) and follows directly from Step 2. (We included Step 3 only to bring our discussion nearer to the standard language.)

Step 5. — The orbit $\overline{\Gamma}(\mathbf{Z}^N)$ is *closed* in $GL_N\, \mathbf{R}/GL_N\, \mathbf{Z}$, where the quotient space $GL_N\, \mathbf{R}/GL_N\, \mathbf{Z}$ is identified in a natural way with the space of lattices in $\mathbf{R}^N$. (Note that this step brings us from algebra to geometry.)

Proof of step 5. — Observe that for each lattice $L \subset \mathbf{R}^N$ there exists a finite subset $F \subset L$, such that the values of P on F^k uniquely determine P among the polynomials of the same degree on $(\mathbf{R}^N)^k$. Thus the inequality $P \circ g = P$ on F^k implies $g \in \overline{\Gamma}$ for all $g \in GL_N\,\mathbf{R}$ and the diagonal action of $GL_N\,\mathbf{R}$ on $(\mathbf{R}^N)^k$.

If L lies in the closure of the orbit $\overline{\Gamma}(\mathbf{Z}^N)$, then there exists a sequence g_i converging to 1 in $GL_N\,\mathbf{R}$ and a sequence γ_i in $\overline{\Gamma}$ such that $g_i L = \gamma_i \mathbf{Z}^N$ for all $i = 1, 2, \ldots$. This follows from the very definition of the topology in the space of lattices, that is $GL_N\,\mathbf{R}/GL_N\,\mathbf{Z}$.

Since P is integer valued (i.e. $\mathbf{Z}^\ell$-valued) on $(\mathbf{Z}^N)^k$ and $\overline{\Gamma}$-invariant, the equality $g_i L = \gamma_i \mathbf{Z}^N$ shows that $P \circ g_i$ is integer valued on F^k.

Since $P \circ g$ is continuous in g and F is finite, we have $P \circ g_i = P$ on F^k for almost all i. This implies $P \circ g_i = P$ on all of $(\mathbf{R}^N)^k$ by our choice of F. Therefore, $g_i \in \overline{\Gamma}$ and $L = g_i^{-1} \gamma_i(\mathbf{Z}^N) \in \overline{\Gamma}(\mathbf{Z}^N)$. Q.E.D.

Step 6. — If the orbit $G(\mathbf{Z}^N)$ is *precompact* in $GL_N\,\mathbf{R}/GL_N\,\mathbf{Z}$, then by the previous step $G/\Gamma = G(\mathbf{Z}^N)$ is compact. That is, Γ is a *cocompact* lattice in G. Note that this case is sufficient for our examples of *compact* hybrids V.

If $G(\mathbf{Z}^N)$ is not precompact the proof of the lattice property

$$\operatorname{Vol} G(\mathbf{Z}^N) < \infty$$

is more complicated (see § 16 in $[B]_1$ and § 10 in [R]). Yet, in the cases needed for our purpose the proof is relatively simple (see § 2).

1.3. *Arithmetic groups.* — A discrete subgroup Γ in a reductive group G is called *arithmetic* if there exists a reductive subgroup $\overline{G} \subset GL_N\,\mathbf{R}$ for some $N = 1, 2, \ldots$ satisfying A and a continuous surjective homomorphism $\rho : \overline{G} \to G$ such that

(i) the kernel of ρ is a *compact* subgroup in $\overline{G}$;

(ii) the ρ-image of $\overline{G} \cap GL_N\,\mathbf{Z}$ is *commensurable* with Γ. That is, the intersection

$$\Gamma \cap \rho(\overline{G} \cap GL_N\,\mathbf{Z})$$

has finite index in Γ as well as in $\rho(\overline{G} \cap GL_N\,\mathbf{Z})$.

Remarks. — *a)* Since G is reductive and Ker ρ is compact, the group $\overline{G}$ is *necessarily* reductive.

b) Since $\overline{G} \cap GL_n\,\mathbf{Z} \subset \overline{G}$ is a lattice by 1.2, the subgroup $\Gamma \cap \rho(\overline{G} \cap GL_N\,\mathbf{Z})$ has finite index in Γ. Thus, it is enough to assume in (ii) that this subgroup has finite index in $\rho(\overline{G} \cap GL_N\,\mathbf{Z})$.

c) For our applications, we only need $G = PO(n, 1)$ and $PO(n, 1) \times PO(n, 1)$.

1.4. Criterion for non-arithmeticity. — *Let* $H \subset G$ *be a reductive subgroup. Then the intersection of* H *with an arithmetic subgroup* $\Gamma \subset G$ *is arithmetic in* H *if and only if this intersection* $H \cap \Gamma$ *is sufficiently dense in* H.

Proof. — Use $\overline{H} = \rho^{-1}(G) \subset \overline{G} \subset GL_N\,\mathbf{R}$ and 1.2.

1.4.A. Corollary. — *If* $\Gamma \subset G$ *is arithmetic and* $H \cap \Gamma$ *is sufficiently dense in* H *then* $H \cap \Gamma$ *is a lattice in* H. *That is,* $\mathrm{Vol}\ H/H \cap \Gamma < \infty$.

Proof. — Apply 1.2 again.

1.5. *Remarks.* — *a)* If Γ is cocompact in G, then 1.4.A obviously implies that $\Gamma \cap H$ is cocompact in H, provided Γ is arithmetic.

b) The above corollary can be used as a criterion of non-arithmeticity for Γ. For example, let H be isomorphic to $SL_2 \mathbf{R}$ or $PSL_2 \mathbf{R}$. Then an elementary argument shows that a discrete subgroup $\Gamma' \subset H$ is either sufficiently dense (here it is equivalent to Zariski dense) or virtually cyclic (i.e. contains a cyclic subgroup of finite index). Therefore, the intersection of an *arithmetic* subgroup $\Gamma \subset G$ with every H isomorphic to $SL_2 \mathbf{R}$ or $PSL_2 \mathbf{R}$ is either a lattice in H or a virtually cyclic group. (This observation is due to D. Toledo.)

1.6. Commensurability criterion. — *Let* Γ *and* Γ_1 *be arithmetic subgroups in* G *such that* $\Gamma \cap \Gamma_1$ *is sufficiently dense in* G. *Then* $\Gamma \cap \Gamma_1$ *has finite index in* Γ *as well as in* Γ_1.

Proof. — Observe that $\Gamma \times \Gamma_1$ is an arithmetic subgroup in $G \times G$ and that $\Gamma \cap \Gamma_1 \subset G$ equals $G \cap (\Gamma \times \Gamma_1)$ for the diagonal embedding $G \subset G \times G$. Hence, $\Gamma \cap \Gamma_1$ is a *lattice* in G by 1.4.A which implies the desired commensurability.

1.6.A. *Example : Commensurability of hyperbolic manifolds* (compare 0.2). — Let V and V_1 be n-dimensional hyperbolic manifolds whose fundamental groups Γ and Γ_1 are *arithmetic* subgroups in $PO(n, 1)$. Let $V^+ \subset V$ and $V_1^+ \subset V$ be connected mutually isometric submanifolds with sufficiently dense fundamental groups Γ^+ and Γ_1^+. That is, the images of Γ^+ and Γ_1^+ in Γ and Γ_1 respectively are Zariski dense in the ambient group $PO(n, 1)$. *Then there exists a hyperbolic manifold* V′ *which admits a finite locally isometric covering map onto* V *and onto* V_1.

Proof. — Since V^+ is isometric to V_1^+ the image of Γ^+ in $PO(n, 1)$ is conjugate to that of Γ_1^+. Therefore, we way assume that the intersection $\Gamma' = \Gamma \cap \Gamma_1$ in $PO(n, 1)$ contains the image of Γ^+. According to 1.6 this Γ' has finite index in Γ as well as in Γ_1. Hence, the manifold $V' = \Gamma' \backslash L^n$ *finitely* covers V and V_1.

1.7. *Density criterion for hyperbolic manifolds with boundary.* — Let V^+ be a connected n-dimensional manifold of constant negative curvature with non-empty totally geodesic boundary ∂V^+ having finitely many connected components. Assume V^+ is complete as a metric space and $\mathrm{Vol}\ V^+ < \infty$.

1.7.A. Lemma. — *Let the (image of the) fundamental group of every component of* ∂V^+ *have finite index in the fundamental group of* V^+. *Then* $n = 2$ *and* V^+ *is simply connected. It follows that* V^+ *is isometric to a* k*-gon in* L^2 *with vertices at infinity.*

Proof. — The finite index condition shows that the universal covering $\widetilde{V}^+$ also has finitely many boundary components. Then one may assume without loss of generality that the deck transformation group Γ maps every component into itself. Let ∂_0 be one of the components of $\partial\widetilde{V}^+$ and let $\bar{\partial}_i \subset \partial_0$ be the normal projections of the remaining components ∂_i, $i = 1, \ldots, k$, to ∂_0. The condition $\mathrm{Vol}\, V^+ < \infty$ implies that $\bigcup_{i=1}^{k} \bar{\partial}_i \subset \partial_0$ is a subset of full measure. Hence, $n = 2$, and the action of deck transformations is trivial. Q.E.D.

1.7.B. Corollary (compare 0.1). — *If* $\mathrm{Vol}\, \partial V^+ < \infty$, *then the fundamental group* Γ^+ *of* V^+ *is Zariski dense in* $\mathrm{PO}(n, 1)^0$.

Proof. — Since $\mathrm{Vol}\, \partial V^+ < \infty$ the Zariski closure $\bar{\Gamma}^+ \subset \mathrm{PO}(n, 1)$ of Γ^+ contains $\mathrm{PO}(n-1, 1)$ by Borel density theorem (see 1.2), where $\mathrm{PO}(n-1, 1) \subset \mathrm{PO}(n, 1)$ is identified with the isometry group of the space L^{n-1} serving as the universal covering of each component of ∂V^+. By the above lemma, $\dim \bar{\Gamma}^+ > \dim \mathrm{PO}(n, 1)$ because the (algebraic!) group $\bar{\Gamma}^+$ has at most finitely many connected components. It follows that $\bar{\Gamma} = \mathrm{SO}(n, 1)$, since $\mathrm{O}(n-1, 1)^0$ is a *maximal* connected subgroup in $\mathrm{SO}(n, 1)$.

2. Arithmetic subgroups in O(n, 1).

2.1. *Orthogonal groups.* — Let $K \subset \mathbf{R}$ be a number field and F be a non-singular quadratic form in $n+1$ variable with coefficient in K. Denote by $\Gamma(F) \subset GL_{n+1}\mathbf{R}$ the group of K-*integral* automorphisms of F. That is the group of F-orthogonal matrices with entries from the ring of integers in K. If the form F has real type (p, q), then $\Gamma(F)$ is contained in (some conjugate of) the orthogonal group $O(p, q)$. We are mainly interested in the case $p = n$ and $q = 1$.

Suppose K is totally real of degree $d+1$ and let $I_i : K \subset \mathbf{R}$, $i = 0, \ldots, d$ be the various embeddings where I_0 is the original embedding $K \subset \mathbf{R}$. For our applications we shall only need the fields $\mathbf{Q}$ and $\mathbf{Q}(\sqrt{2})$. Note that the embedding $I_1 : \mathbf{Q}(\sqrt{2}) \subset \mathbf{R}$ is obtained from I_0 by applying the automorphism $I : \alpha + \beta\sqrt{2} \mapsto \alpha - \beta\sqrt{2}$ to $\mathbf{Q}(\sqrt{2})$.

The following classical theorem (see $[B]_1$, for example) provides a variety of arithmetic subgroups in $O(n, 1)$.

2.2. Arithmeticity of $\Gamma(F)$. — *If the forms* $I_i F$ *are positive definite for* $i = 1, \ldots, d$, *then the subgroup* $\Gamma(F) \subset O(p, q)$ *is arithmetic. In particular,* $\Gamma(F)$ *is discrete and* $\mathrm{Vol}\, O(p, q)/\Gamma(F) < \infty$.

Proof. — The pertinent group $\bar{G}$ here (compare 1.3) for $G = O(p, q)$ is the Cartesian product of the real orthogonal groups $O(I_i F)$, $i = 0, 1, \ldots, d$ (where $O(I_0 F) = O(F) = O(p, q)$). Thus $\bar{G} \subset GL_N \mathbf{R}$ for $N = (n+1)(d+1)$, where $\mathbf{R}^N$ is given a K-*rational* basis, that is, a basic of vectors whose projections to the copies of

$\mathbf{R}^{d+1}$ lie in $K \subset \mathbf{R}^{d+1}$, where K embeds into $\mathbf{R}^{d+1}$ by $x \mapsto (I_0(x), \ldots, I_d(x))$ for all $x \in K$. Then the verification of the A-property of $\overline{G}$ and arithmeticity of $\Gamma(F)$ is straightforward (see $[B]_1$).

2.3. Cocompactness of $\Gamma(F)$. — *The above arithmetic group* $\Gamma(F)$ *is cocompact in* $O(p, q)$ *if and only if* F *has no non-trivial zero in* K.

This is a simple corollary of Mahler compactness theorem for lattices in $\mathbf{R}^N$ (see $[B]_1$, [R]).

2.3.A. *If* $d + 1 \geqslant 2$, *then* $\Gamma(F)$ *is cocompact.*

Proof. — If $F(x, x) = 0$, then also $I_i F(I_i(x), I_i(x)) = 0$ for $i > 0$, as I_i is an isomorphism. Since $I_i F$ is positive definite for $i > 0$, we have $I_i(x) = 0$ and thus $x = 0$.

2.4. *Remark.* — If $K = \mathbf{Q}$, then $\Gamma(F)$ may be both cocompact and non-cocompact for $n = 2, 3, 4$. But $\Gamma(F)$ is not cocompact for $n \geqslant 5$ as every indefinite rational quadratic form in five variables has a non-trivial rational zero by the Minkovski-Hasse theorem.

2.5. *Action of* $\Gamma(F)$ *on* L^n. — Let F be of signature $(n, 1)$ and consider the (pseudo)-sphere $S = S_F = \{x \in \mathbf{R}^{n+1} \mid F(x, x) = -1\} \subset \mathbf{R}^{n+1}$. This S has two connected components isometric to L^n for the metric induced from the pseudo-Euclidean metric F on $\mathbf{R}^{n+1}$. Thus $S/\{+1, -1\} = L^n$ and $PO(n, 1) = PO(F)$ acts isometrically on L^n. If $\Gamma \subset \Gamma(F)$ is a subgroup of finite index without torsion, then $\Gamma/\{+1, -1\}$ acts *freely* on L^n and the quotient space $\Gamma\backslash L^n$ is a hyperbolic manifold such that, according to 1.2,

$$\mathrm{Vol}(\Gamma\backslash L^n) < \infty.$$

2.5.A. *Congruence subgroups in* $\Gamma(F)$. — Take a prime ideal p in the ring of integers of K and define the *congruence subgroup* $\Gamma_p(F) \subset \Gamma(F)$ by

$$\Gamma_p(F) = \{\gamma \in \Gamma(F) \mid \gamma \equiv \mathrm{Id} \pmod{p}\}.$$

If $|p|$ is sufficiently large, then $\Gamma_p(F)$ has no torsion and the action of $\Gamma_p(F)$ on L^n is free (see $[B]_1$, [R]).

2.6. Commensurable manifolds. — *Let* F_1 *and* F_2 *be two forms over* K *of type* $(n, 1)$ *for* $n \geqslant 2$, *such that the corresponding groups* $\Gamma(F_1)$ *and* $\Gamma(F_2)$ *are commensurable (we stick to the assumptions in* 2.2 *so that these groups are arithmetic) in the following sense. There exists an isometry* α *of the (Lobachevsky) space* $L_1 = S_{F_1}/\{+1, -1\}$ *onto* $L_2 = S_{F_2}/\{+1, -1\}$ *which sends some subgroup of finite index* $\Gamma_1 \subset \Gamma(F_1)/\{+1, -1\}$ *(acting on* L_1*) into* $\Gamma(F_2)/\{+1, -1\}$ *(acting on* L_2*). Then the forms* F_1 *and* F_2 *are similar over* K. *That is there exists a linear* K*-isomorphism* $\mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ *sending* F_1 *to* λF_2 *for some* $\lambda \in K$.

Proof. — There obviously exists a unique (up to $\{+1, -1\}$) linear map $\bar{\alpha}: \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ sending F_1 to F_2 such that the induced map $L_1 \to L_2$ is α. Denote by $\bar{\Gamma}_1 \subset \Gamma(F_1)$ the $\{+1, -1\}$-extension of Γ_1. Since $\bar{\Gamma}_1$ is Zariski dense in $O(n, 1)$ and the action of $O(n, 1)$ on $\mathbf{R}^{n+1}$ is $\mathbf{C}$-irreducible for $n \geqslant 2$, the K-linear span of $\bar{\Gamma}_1$ in $\operatorname{End} \mathbf{R}^{n+1}$ equals $\operatorname{End} K^{n+1} \subset \operatorname{End} \mathbf{R}^{n+1}$. Since $\bar{\alpha}$ sends $\bar{\Gamma}_1$ in $\Gamma(F_2)$, the K-span of $\bar{\Gamma}_1$ goes to that of $\Gamma(F_2)$ and then the equality $\operatorname{Span}_K \bar{\Gamma}_1 = \operatorname{End} K^{n+1}$ implies that $\bar{\alpha} = \mu\bar{\alpha}_0$, where $\bar{\alpha}_0$ is defined over K and $\mu \in \mathbf{R}^x$. Now, α_0 sends F_1 to $\mu^{-2} F_2$ and since $F_1 \neq 0$, the factor μ^{-2} lies in K. Q.E.D.

2.7. Corollary. — *Let* F_1 *and* F_2 *be diagonal,*

$$F_1 = \sum_{i=1}^{n+1} a_i x_i^2 \quad \textit{and} \quad F_2 = \sum_{i=1}^{n+1} b_i x_i^2$$

for a_i *and* b_i *in* K. *Then for* $n + 1$ *even the ratio of the discriminants*

$$\prod_{i=1}^{n+1} a_i \,\Big|\, \prod_{i=1}^{n+1} b_i \quad \textit{lies in } (K^\times)^2.$$

Proof. — A linear transformation over K with determinant D multiplies discriminants by D^2 and similarity $F \mapsto \lambda F$ multiplies the discriminant of F by λ^{n+1}.

2.7.A. *Example.* — *a)* Let $K = \mathbf{Q}$ and

$$F_1 = x_0^2 + x_1^2 + \ldots + x_{n-1}^2 - x_n^2$$
$$F_2 = 2x_0^2 + x_1^2 + \ldots + x_{n-1}^2 - x_n^2.$$

Then for $n + 1$ even the groups $\Gamma(F_1)$ and $\Gamma(F_2)$ are not commensurable as 2 is not a square in $\mathbf{Q}$. Also note that these groups are not cocompact as $F_i(x, x) = 0$ for $x = (0, 0, \ldots, 0, 1, 1)$ and $i = 1, 2$ (compare 2.4).

b) Let $K = \mathbf{Q}(\sqrt{2})$ and

$$F_1 = x_0^2 + x_1^2 + \ldots + x_{n-1}^2 - \sqrt{2}\, x_n^2$$
$$F_2 = 3x_0^2 + x_1^2 + \ldots + x_{n-1}^2 - \sqrt{2}\, x_n^2.$$

Here again the corresponding groups are not commensurable for $n + 1$ even, but now these groups are cocompact (see 2.3.A).

2.8. *Totally geodesic submanifolds in hyperbolic manifolds.* Take a $(k + 1)$-dimensional linear subspace $R_0 \subset \mathbf{R}^{n+1}$ which meets the sphere $S = S(F) \subset \mathbf{R}^{n+1}$. Then the intersection $S_0 = S \cap R_0$ is a totally geodesic submanifold in S of dimension k. For a subgroup $\Gamma \subset \Gamma(F)$ denote by $\Gamma_0 \subset \Gamma$ the subgroup stabilizing R_0. If the subspace R_0 is K-rational and Γ_0 has finite index in Γ, then Γ_0 is arithmetic. That is, the image of Γ_0 in the full isometry group $\operatorname{Is} S_0 = O(k, 1)$ gives a *proper immersion* $\Gamma_0\backslash S_0 \to \Gamma\backslash S$ (by Step 5 in 1.2).

2.8.A. *Embedding criterion.* — Denote by $I_0 \in O(n, 1)$ the orthogonal reflection of $\mathbf{R}^{n+1}$ in R_0.

If I_0 *normalizes* Γ, *then the canonical map* $\Gamma_0\backslash S_0 \to \Gamma\backslash S$ *is a proper embedding, provided* Γ *has no torsion.*

Proof. — Suppose two distinct points s and s' from S_0 go to the same point in $\Gamma\backslash S$. That is $s' = \gamma(s)$ for some $\gamma \in \Gamma$. Since s and s' are fixed by I_0, the commutator $\delta = \gamma^{-1} I_0 \gamma I_0^{-1}$ fixes s. Since I_0 normalizes Γ this δ is contained in Γ and as Γ has no torsion and acts freely on S_0, we obtain $\delta = \mathrm{Id}$. Since S_0 equals the fixed point set of I_0, the equality $[\gamma, I_0] = \mathrm{Id}$ implies that $\gamma \in \Gamma_0$. Q.E.D.

2.8.B. *Remark.* — If $\Gamma_0\backslash S_0 \to \Gamma\backslash S$ is an embedding, then, obviously, the corresponding map $\Gamma'_0\backslash S_0 \to \Gamma'\backslash S$ also is an embedding for every subgroup $\Gamma' \subset \Gamma$.

Corollary. — *If the group generated by* Γ *and* $I_0 \Gamma I_0^{-1}$ *is discrete without torsion, then the map* $\Gamma_0\backslash S_0 \to \Gamma\backslash S$ *is an embedding.*

2.8.C. *Example.* — Let F_0 be a quadratic form in variables $x_1, \ldots, x_n$ over $K \subset \mathbf{R}$ of type $(n-1, 1)$ and $F = ax_0^2 + F_0$ for $a > 0$ in K. Then the reflection I_0 in the hyperplane $R_0 = \{ x_0 = 0 \} \subset \mathbf{R}^{n+1}$,

$$I_0 : (x_0, x_1, \ldots, x_n) \mapsto (-x_0, x_1, \ldots, x_n)$$

lies in $\Gamma(F)$ and the previous discussion applies to the congruence subgroups $\Gamma_p(F) \subset \Gamma(F)$ with $|p|$ sufficiently large. Therefore the hyperbolic manifold

$$V(F_0, p) = \Gamma_p(F_0)\backslash L^{n-1}$$

(where we identify L^{n-1} with $S_0/\{+1, -1\}$) isometrically embeds into $V(F, p) = \Gamma_p(F)\backslash L^n$.

Note that for p prime to 2 both manifolds $V(F, p)$ and $V(F_0, p)$ are orientable. In fact, if $-1 \neq 1 \pmod p$, then $\Gamma_p(F) \subset SO(n, 1)$ and $\Gamma_p(F_0) \subset SO(n-1, 1)$.

The hypersurface $V(F_0, p)$ does not necessarily bound in $V(F, p)$. (In fact for large $|p|$ it does not bound). However, there exists an obvious double covering $\widetilde{V}(F, p)$ of $V(F, p)$, such that the lift of $V(F_0, p)$ to $\widetilde{V}(F, p)$ consists of two disjoint copies of $V(F_0, p)$ which do *bound* some connected submanifold $V^+ \subset \widetilde{V}(F, p)$. That is the boundary ∂V^+ is the union of two copies of $V(F_0, p)$.

2.9. *Interbreeding hyperbolic manifolds.* — Take the forms $F_i = a_i x_0^2 + F_0$ as in the previous example for $i = 1, 2$, and assume for the uniformity of notation that $V(F_0, p)$ does not bound in either of the two manifolds $V(F_i, p)$. (As we mentioned earlier, this is the case for large $|p|$.) Then we take the corresponding manifolds $V_i^+ \subset \widetilde{V}(F_i, p)$ for $i = 1, 2$ and recall that V_1^+ and V_2^+ have isometric boundaries equal to $2V(F_0, p)$.

If $n + 1$ is even and a_1/a_2 is not a square in K then the forms F_1 and F_2 are not similar over K (compare 2.7) and the groups $\Gamma(F_1)$ and $\Gamma(F_2)$ are not commensurable (see 2.6). In this case the manifold V obtained by gluing V_1^+ to V_2^+ along the boundary is non-arithmetic (i.e. the fundamental group is not arithmetic: compare 0.2, 1.6.A).

If $(n+1)$ is odd, we consider a K-rational hyperplane $R' \subset \mathbf{R}^{n+1}$ normal to R_0. For example, let $F_0 = \sum_{i=1}^{n} b_i x_i^2$, where $b_1 > 0$ and take

$$R' = \{ x_1 = 0 \} \subset \mathbf{R}^{n+1}.$$

Then the corresponding hypersurfaces $V_i' \subset V(F_i, p)$ are normal to $V(F_0, p)$. Therefore, their "halfs" $V_1' \cap V_1^+$ and $V_2' \cap V_2^+$ glue together to a *totally geodesic* hypersurface $V' \subset V$. If V is arithmetic, then so is V′ (see 1.4). But V′ is non-arithmetic for $n-1 = \dim V' \geqslant 2$ by the previous argument and thus the non-arithmeticity of V (i.e. of the fundamental group Γ of V) is established for all $n \geqslant 3$. We leave the (trivial) case where $n = 2$ to the reader.

2.10. *Final hyperbolic remarks.* — To complete our discussion we need two simple facts from hyperbolic geometry.

2.10.A. *The fundamental group of* V^+ *injects into that of* V.

Proof. — The submanifold $V^+ \subset V$ has convex (in fact, totally geodesic) boundary and so every class in $\pi_1(V^+)$ is represented by a *geodesic* loop in V^+. Such a loop is not contractible in V, as V is complete of negative curvature. Q.E.D.

2.10.B. *The manifold* V *obtained by gluing* V_1^+ *and* V_2^+ (see § 0) *is complete provided these manifolds as well as their (totally geodesic) boundaries have finite volumes.*

Proof. — The claim is obvious if $V_1^+ = V_2^+$ is compact.

If V_1^+ is non-compact then the geometry at infinity is described with the following notion.

2.10.C. *Cusps.* — An n-dimensional *cusp with boundary* is a Riemannian manifold $C^+ = F^+ \times \mathbf{R}_+$, where F^+ is a compact flat manifold with totally geodesic boundary and where the metric in C^+ is $dt^2 + e^{-t} g$, where $t \in \mathbf{R}_+$ and g is the flat metric on F^+.

Observe that a compact connected flat manifold F^+ with a non-empty boundary either is isometric to a product $F_0 \times [-a, a]$ for some compact flat manifold F_0 without boundary, or has a double covering isometric to $F_0 \times [-a, a]$. In both cases the connected components of the levels of the distance function $\mathrm{dist}(x, \partial F^+)$ foliate F^+ into closed connected totally geodesic submanifolds F_θ for $\theta \in [0, a]$. It follows that a connected cusp with non-empty boundary is canonically foliated into leaves $C_\theta = F_\theta \times \mathbf{R}_+$. Note that this splitting of C_θ is unique. In fact, for each $x \in C_\theta$, there exists a unique closed connected $(n-2)$-dimensional hypersurface $F(x) \subset C_\theta$ passing through x, such that

a) the induced metric in $F(x)$ is flat;

b) also the induced metrics in the *parallel* hypersurfaces (which are defined as the level of the distance function to $F(x)$ in C_θ) are flat.

Since the hypersurfaces $F_\theta \times t \subset C_\theta$ have these properties, the hypersurface $F(x)$ for $x = (f, t)$ equals $F_\theta \times t$.

The $(n-2)$-dimensional volume of $F_\theta \times t$ is obviously const $\exp(n-2)\,t$. Hence, if $n \geqslant 3$, the parameter $t = t(x)$ for $x = (f, t)$ can be recaptured (up to an additive constant) by taking $\log \operatorname{Vol} F(x)$, for those x, for which the hypersurface is normally orientable and $\log 2 \operatorname{Vol} F(x)$ for the others.

Now it is clear that a manifold C, obtained by gluing together two cusps $C_i^+ = F_i^+ \times \mathbf{R}_+$ by isometries along their boundary cusps $\partial F_i^+ \times \mathbf{R}_+$, is again a cusp. In fact, the foliations on C_i^+ define a geodesic foliation of C into $(n-1)$-dimensional cusps C_θ without boundary and the cusp structure in C is seen with $t = \log \operatorname{Vol} F(x)$.

Finally, we conclude the proof of 2.10.B by invoking the following.

2.10.D. Proposition. — *Let* V^+ *be a complete hyperbolic manifold with totally geodesic boundary. If* $\operatorname{Vol} V^+ < \infty$, *then the complement to a compact subset in* V^+ *is isometric to a (possibly disconnected) cusp.*

Proof. — If V^+ has no boundary, this is standard (see $[B]_1$, [R], $[G]_1$), and the case with boundary follows by taking the double of V^+.

This proposition and the above discussion show that the glued manifold V is cuspidal at infinity. Since cusps are complete, V is complete. Q.E.D.

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Manuscrit reçu le 30 juin 1986.