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ON THE CREATION OF HOMOCLINIC POINTS

by Ricardo MAÑÉ

INTRODUCTION

Let M be a compact boundaryless smooth manifold, $f: M \circlearrowleft$ a C^r diffeomorphism and p a hyperbolic fixed point of f (i.e. all the eigenvalues of $D_p f: T_p M \circlearrowleft$ have eigenvalues with modulus $\neq 1$). It is well known that the stable and unstable sets of p, denoted and defined respectively by

$$W^s(p) = \{ x \mid \lim_{n \to +\infty} d(f^n(x), p) = 0 \}$$

and

$$W^u(p) = \{ x \mid \lim_{n \to +\infty} d(f^{-n}(x), p) = 0 \},$$

are C^r injectively immersed submanifolds of M. The points of intersection of $W^s(p)$ with $W^u(p)$ different from p (i.e. the points in $W^s(p) \cap (W^u(p) - \{p\})$) are called *homoclinic points* associated to p. The points of intersection of the closure of $W^s(p)$ with $W^u(p)$ or of the closure of $W^u(p)$ with $W^s(p)$, different from p, will be called *almost homoclinic points* associated to p. In other words the set of almost homoclinic points associated to p is:

$$(\overline{W^s(p)} \cap W^u(p)) \cup (\overline{W^u(p)} \cap W^s(p)) - \{p\}.$$

The purpose of this paper is to study the well known problem of whether, when there exist almost homoclinic points, it is possible to create homoclinic points by a small perturbation of the diffeomorphism. To state this question more precisely, recall that if a C^r diffeomorphism $f: M \circlearrowleft$ has a hyperbolic fixed point p, there exist a neighborhood U of p and a C^r neighborhood $\mathcal{U}$ of f such that every $g \in \mathcal{U}$ has a unique fixed point $p(g)$ in U and moreover this fixed point is hyperbolic and depends continuously on g. Obviously $p(f) = p$.

Problem. — Suppose that a C^r diffeomorphism $f: M \circlearrowleft$ has almost homoclinic points associated to the hyperbolic fixed point p. Does there exist a C^r diffeormophism $g: M \circlearrowleft$, arbitrarily near to f in the C^r topology, having homoclinic points associated to $p(g)$?

A stronger form of this problem would be to specify an almost homoclinic point q of f and require g to have an homoclinic point nearby q. Observe that if this problem has an affirmative answer, then, by very soft arguments, it can be proved that for C^r generic diffeomorphisms, the set of transversal homoclinic points associated to a hyperbolic fixed point is dense in the set of almost homoclinic points. This conclusion has obvious interest since almost homoclinic points carry little further knowledge. On the other hand, every transversal homoclinic point is known to be an accumulation point of a sequence of invariant compact sets where the diffeomorphism acts as a transitive finite type subshift (Smale [4]). But we shall restrict our discussion of the problem to the version stated above because it seems idle to strenghten a question that even in its weaker form remains, in its full generality, completely unanswered.

However, positive answers have been obtained under supplementary hypotheses. Robinson [3] and Pixton [2] solved affirmatively the problem for diffeomorphisms of the two dimensional sphere. In fact they solve also the strong form of the problem. Takens [5] solved the problem for Hamiltonian diffeomorphisms, but only in the case $r = 1$. His solution too covers the strong version of the problem and as a corollary follows that for C^1 generic Hamiltonian diffeomorphisms the set of homoclinic points associated to a hyperbolic point p is dense in $W^s(p)$ and $W^u(p)$.

The purpose of this paper is to answer the problem under a different type of supplementary hypothesis. To explain it we need some preliminary definitions. Let $\mathcal{M}(M)$ denote the set of probabilities on the Borel σ-algebra of M endowed with its usual topology, i.e. the unique metrizable topology such that $\mu_n \to \mu$ if and only if $\int \varphi \, d\mu_n \to \int \varphi \, d\mu$ for every continuous function $\varphi : M \to \mathbf{R}$. Given $f \in \mathrm{Diff}^r(M)$, denote by $\mathcal{M}(f)$ the set of f-invariant elements of $\mathcal{M}(M)$. Associated to every $x \in M$ and $n \in \mathbf{Z}^+$, define a probability $\mu(x, n) \in \mathcal{M}(M)$ by

$$\mu(x, n) = \frac{1}{n} \sum_{j=1}^{n} \delta_{f^j(x)}.$$

Denote by $\mathcal{M}(x)$ the set of $\mu \in \mathcal{M}(M)$ such that there exists a sequence $n_1 < n_2 < \ldots$ of positive integers satisfying

$$\mu = \lim_{j \to +\infty} \mu(x, n_j).$$

Observe that $\mathcal{M}(x) \subset \mathcal{M}(f)$.

Theorem A. — Suppose that $f \in \mathrm{Diff}^r(M)$, $r = 1$ *or* 2, *has a hyperbolic fixed point* p *such that there exist* $x \in W^u(p) - \{p\}$ *and* $\mu \in \mathcal{M}(x)$ *satisfying* $\mu(\{p\}) > 0$. *Then in every* C^r *neighborhood of* f *there exists a diffeomorphism* g *that coincides with* f *in a neighborhood of* p *and has homoclinic points associated to* p.

This theorem will follow as a corollary of a stronger result that in particular implies that Theorem A also holds replacing the hyperbolic fixed point by an isolated hyperbolic set.

Let us recall some definitions that will appear in the statement of our main result. We say that $\Lambda \subset M$ is a *hyperbolic set* of $f \in \mathrm{Diff}^r(M)$ if it is compact, f-invariant and there exist a continuous splitting $TM/\Lambda = E^s \oplus E^u$, invariant under Df, and constants $C > 0$, $0 < \lambda < 1$ such that

$$||(D_x f^n)/E^s_x|| \leqslant C\lambda^n,$$
$$||(D_x f^{-n})/E^u_x|| \leqslant C\lambda^n$$

for all $x \in \Lambda$, $n > 0$. If there exists a compact neighborhood U of Λ such that

$$\bigcap_n f^n(U) = \Lambda,$$

we say that Λ is *isolated* and U is an *isolating block* of Λ. Observe that every compact neighborhood of Λ contained in an isolating block is also an isolating block. Define the stable and unstable manifolds of Λ by

$$W^s(\Lambda) = \{ x \mid \lim_{n\to+\infty} d(f^n(x), \Lambda) = 0 \},$$
$$W^u(\Lambda) = \{ x \mid \lim_{n\to+\infty} d(f^{-n}(x), \Lambda) = 0 \}.$$

When U is an isolating block of Λ, it is known ([1]) that

$$\bigcap_{n\geqslant 0} f^{-n}(U) = W^s(\Lambda) \cap U$$

and

$$\bigcap_{n\geqslant 0} f^n(U) = W^u(\Lambda) \cap U.$$

We say that p is an *homoclinic point* associated to Λ if

$$p \in W^s(\Lambda) \cap W^u(\Lambda) - \Lambda.$$

The next theorem is a generalization of Theorem A to hyperbolic sets and more general probabilities.

Theorem B. — Let f: M$\circlearrowleft$ be a C^r diffeomorphism ($r = 1$ or 2) and Λ an isolated hyperbolic set of f. Let $\{x_n\} \subset W^u(\Lambda)$ be a sequence converging to a point $x \notin \Lambda$ and $m_1 < m_2 < \ldots$ be a sequence of integers such that the probabilities $\mu(x_n, m_n)$ converge to a probability μ with $\mu(\Lambda) > 0$. Then every C^r neighborhood of f contains a diffeomorphism g coinciding with f in a neighborhood of Λ and having a homoclinic point associated to Λ.

Our next result says something about the case when the hypothesis $\{x_n\} \subset W^u(\Lambda)$ is dropped. We need another definition. Let us say that $\Lambda_0 \subset M$ is a *basic set* of $f \in \mathrm{Diff}^r(M)$ if it is hyperbolic, isolated and f/Λ_0 is transitive (i.e. there exists $p \in \Lambda_0$ with $\omega(p) = \Lambda_0$). Recall that if Λ is an isolated hyperbolic set of f: M$\circlearrowleft$ and $\Omega(f/\Lambda) = \Lambda$ (where $\Omega(f/\Lambda)$ denotes the set of nonwandering points of f/Λ) then Λ can be uniquely decomposed as a union $\Lambda = \Lambda_0 \cup \ldots \cup \Lambda_m$ of disjoint basic sets that we shall call *basic components* of Λ. This property is an obvious adaptation of Smale's Spectral Decomposition Theorem.

Theorem C. — Let $f: \mathrm{M} \circlearrowleft$ be a C^r diffeomorphism ($r = 1$ or 2) and Λ an isolated hyperbolic set of f such that $\Omega(f/\Lambda) = \Lambda$. Suppose that $\{x_n\} \subset \mathrm{M}$ is a sequence converging to a point $x \notin \Lambda$ and $m_1 < m_2 < \ldots$ is a sequence of integers such that the probabilities $\mu(x_n, m_n)$ converge to a probability μ with $\mu(\Lambda) > 0$. Then given a C^r neighborhood $\mathcal{U}$ of f one of the following properties holds:

I) *There exists $g \in \mathcal{U}$ coinciding with f in a neighborhood of Λ and having homoclinic points associated to a basic component of Λ.*

II) *For every neighborhood U of Λ there exist a neighborhood $\mathrm{V} \subset \mathrm{U}$, a diffeomorphism $g \in \mathcal{U}$ whose inverse coincides with f^{-1} in $\mathrm{V} \cup \mathrm{U}^c$, $k > 0$ and $0 < m \leqslant m_k$ such that*

$$g^j(x_k) = f^j(x_k)$$

for all $0 \leqslant j \leqslant m - 2$ and

$$g^m(x_k) \in \bigcap_{n \geqslant 0} g^{-n}(\mathrm{V}) = \bigcap_{n \geqslant 0} f^{-n}(\mathrm{V}).$$

Our final result shows a special case of Theorem C when property (I) always holds.

Theorem D. — Let $f: \mathrm{M} \circlearrowleft$ be a C^r diffeomorphism ($r = 1$ or 2) and Λ an isolated hyperbolic set of f with $\Omega(f/\Lambda) = \Lambda$. If there exists $x \notin \mathrm{W}^s(\Lambda)$ such that $\mu(\Lambda) > 0$ for all $\mu \in \mathcal{M}(x)$ then every C^r neighborhood of f contains a diffeomorphism that coincides with f in a neighborhood of Λ and has an homoclinic point associated to a basic component of Λ.

Theorems B and C can be stated in a unified (but less clear) form. In fact the next result (Theorem I.1) gives more information about the position of the homoclinic orbit that is created by a perturbation. Theorem D is proved through a variation of part of Theorem I.1.

We advise the reader to follow the proofs in the simpler case when Λ is a fixed point p, M is two dimensional and f is linear in a neighborhood of p. This simplifies the technicalities and exposes directly the idea of the proof.

I. — Proof of Theorems B and C

Here we shall prove the following result from which Theorems B and C follow immediately.

*Theorem **I.1**. — Let $f: \mathrm{M} \circlearrowleft$ be a C^r diffeomorphism ($r = 1$ or 2) and Λ an isolated hyperbolic set of f. Let $\bar{x} \notin \Lambda$ be a point such that there exist a sequence $\{x_n\} \subset \mathrm{M}$ converging to $\bar{x}$ and a sequence of integers $0 < m_1 < m_2 < \ldots$ such that the sequence of probabilities $\mu(x_n, m_n)$ converges to a probability μ that satisfies $\mu(\Lambda) > 0$. Then, for every neighborhood $\mathcal{U}$ of f in* $\mathrm{Diff}^r(\mathrm{M})$ *and every neighborhood U of Λ in M, there exist $g \in \mathcal{U}$ and a neighborhood $\mathrm{V} \subset \mathrm{U}$ of Λ such that*

$$f/(\mathrm{V} \cup \mathrm{U}^c) = g/(\mathrm{V} \cup \mathrm{U}^c),$$
$$f^{-1}/(\mathrm{V} \cup \mathrm{U}^c) = g^{-1}/(\mathrm{V} \cup \mathrm{U}^c)$$

and moreover satisfy one of the following properties:

I) *There exist* $k > 0$ *and* $0 < s < m_k$ *such that:*

$$g^j(x_k) = f^j(x_k)$$

for all $0 \leqslant j < s$ *and*

$$g^{s+1}(x_k) \in \bigcap_{n \geqslant 0} g^{-n}(\mathrm{V}) = \bigcap_{n \geqslant 0} f^{-n}(\mathrm{V}).$$

Moreover when there exists an isolating block U_0 *of* Λ *such that* $\{x_n\} \subset \bigcap_{n \geqslant 0} f^m(\mathrm{U}_0)$, *then g can be chosen also satisfying* $g^{-j}(x_n) = f^{-j}(x_n)$ *for all* $j \geqslant 0$, $n \geqslant 0$.

II) *There exist* $k > 0$ *and* $0 < s_0 < s_1 < m_k$ *such that*

(i) $$g^j(f^{s_0}(x_k)) = f^j(f^{s_0}(x_k))$$

for $0 \leqslant j < s_1 - s_0$ *and*

(ii) $$g^{-2}(f^{s_0}(x_k)) \in \bigcap_{n \geqslant 0} g^n(\mathrm{V}) = \bigcap_{n \geqslant 0} f^n(\mathrm{V}),$$

(iii) $$g^{(s_1 - s_0)+1}(f^{s_0}(x_k)) \in \bigcap_{n \geqslant 0} g^{-n}(\mathrm{V}) = \bigcap_{n \geqslant 0} f^{-n}(\mathrm{V}).$$

The proof of Theorem I.1 relies on four lemmas. Ths first one is quite simple and we leave its proof to the reader.

Lemma 1. — Given $c > 0$ *and a neighborhood* $\mathcal{N}$ *of the identity in* $\mathrm{Diff}^k(\mathrm{M})$, *there exists* $\mathrm{R} > 0$ *such that for every* $0 < r < \mathrm{R}$ *and every pair of points* $a \in \mathrm{M}$ *and* $b \in \mathrm{M}$ *satisfying*

$$d(a, b) \leqslant r^{k+c}$$

there exists $h \in \mathcal{N}$ *such that*

$$h(a) = b$$

and $$h(x) = x$$

for all x outside the ball $\mathrm{B}_r(a)$.

For the statement of the second lemma suppose that f, Λ and U satisfy the hypotheses of Theorem I.1. Suppose also M endowed with a Riemannian metric such that if $\mathrm{TM}/\Lambda = \mathrm{E}^s \oplus \mathrm{E}^u$ is the hyperbolic splitting of TM/Λ, there exists $0 < \lambda_0 < 1$ such that $\|(\mathrm{D}_x f)/\mathrm{E}^s_x\| < \lambda_0$, $\|(\mathrm{D}_x f^{-1})/\mathrm{E}^u_x\| < \lambda_0$ for all $x \in \Lambda$.

Lemma 2. — There exist compact sets $\mathrm{V}^+ \subset \mathrm{U}$, $\mathrm{V}^- \subset \mathrm{U}$, $0 < \varepsilon_0 < 1$ *and* $0 < \gamma < \lambda < 1$ *satisfying:*

a) $f(\mathrm{V}^+) \subset \mathrm{V}^+$, $f^{-1}(\mathrm{V}^-) \subset \mathrm{V}^-$;

b) $\Lambda = \mathrm{V}^+ \cap \mathrm{V}^-$;

c) $\gamma\, d(x, \mathrm{V}^-) \leqslant d(f(x), \mathrm{V}^-) \leqslant \lambda\, d(x, \mathrm{V}^-)$

and $$\gamma\, d(x, \mathrm{V}^+) \leqslant d(f^{-1}(x), \mathrm{V}^+) \leqslant \lambda\, d(x, \mathrm{V}^+),$$

for all x satisfying $d(x, \mathrm{V}^+) < \varepsilon_0$ *and* $d(x, \mathrm{V}^-) < \varepsilon_0$.

If the reader, following the suggestion given at the end of the Introduction, is keeping track of the proof in the case there described, then this lemma becomes completely trivial just taking a square W contained in U and defining V^+ and V^- as the connected components (that are segments) of $W^s(p) \cap W$ and $W^u(p) \cap W$, as illustrated in figure I.

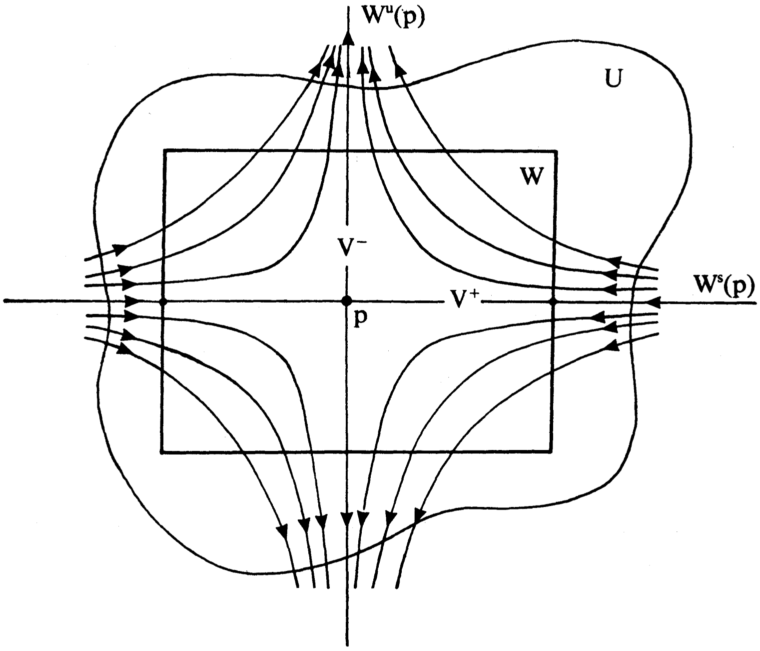

FIG. I

The proof of Lemma 2 will be given after the proof of Theorem I.1.

Let $\bar{x} \notin \Lambda$ be given as in the statement of the theorem and denote $\mu_k = \mu(x_k, m_k)$ for $k > 0$.

To prove Theorem I.1 we start by taking a compact neighborhood $W \subset U$ of Λ such that

$$\bigcap_n f^n(W) = \Lambda,$$

$$\bar{x} \notin W$$

and $\qquad f(\bar{x}) \notin W.$

Define $\qquad V(r) = \{ x \mid d(x, V^+) \leqslant r, d(x, V^-) \leqslant r \}.$

Choose $0 < \varepsilon_1 < \varepsilon_0$, where ε_0 is given by Lemma 2, such that

$$V(\varepsilon_1) \subset W.$$

Now take $0 < \delta < 1$ and a sequence $r_0, r_1, \ldots$ satisfying

(1) $\qquad 0 < r_0 < \varepsilon_1,$

(2) $\qquad r_{n+1} = r_n^{1+\delta}$

and

(3) $$\mu_k(\partial V(r_n)) = \mu(\partial V(r_n)) = 0$$

for all $n \geqslant 0$ and $k > 0$. Set $V_n = V(r_n)$ and define S_n as the set of points $x \in V_0$ that can be written as $x = f^m(y_n)$, $m \in \mathbf{Z}$, with $y_n \in V_n$ and $f^j(y_n) \in V_0$ for all $0 \leqslant j \leqslant m$ if $m \geqslant 0$, or for all $m \leqslant j \leqslant 0$ when $m \leqslant 0$. In the simplified case the sets V_n and S_n look as in figure II where V_n is a square of side $2r_n$ centered at p and S_n is the subset of V_0 bounded by hyperbolas passing through the vertices of V_n.

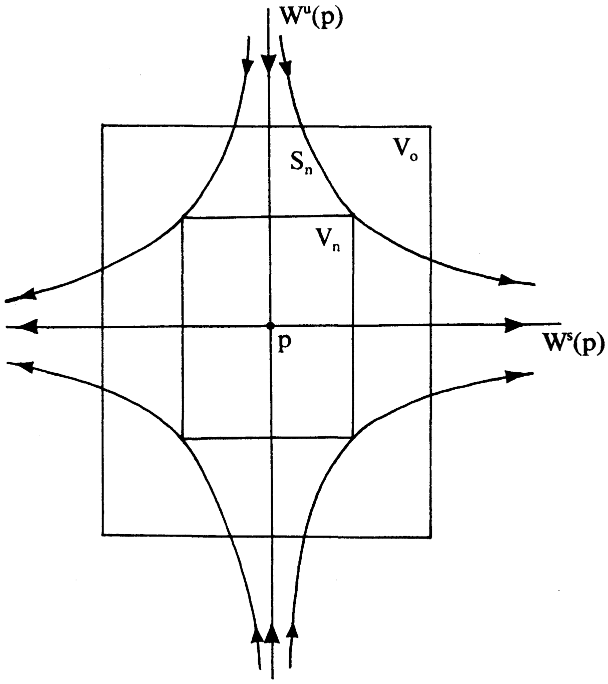

Fig. II

For $x \in V_0$ define:

$$T(x) = \sup\{j \mid f^i(x) \in V_0 \text{ for all } 0 \leqslant i \leqslant j\} + \sup\{j \mid f^{-i}(x) \in V_0 \text{ for all } 0 < i \leqslant j\}.$$

Lemma 3. — There exist $C_2 > C_1 > 0$ *such that, for* δ *as above and for all* $n > 0$,

$$T(x) \leqslant C_2(1+\delta)^n$$

if $x \in V_0 - S_n$ *and*

$$C_1(1+\delta)^n \leqslant T(x)$$

if $x \in S^n$.

Proof. — Observe that by (1), $V_0 = V(r_0) \subset V(\varepsilon_1) \subset V(\varepsilon_0)$. Then we can apply the inequalities of part c) of Lemma 2 to the points in V_0. Hence, if $f^j(x) \in V_0$ for all $-n_1 \leqslant j \leqslant n_2$, it follows that

$$r_0 \geqslant d(f^{n_2}(x), V^+) \geqslant \lambda^{-n_2} d(x, V^+),$$
$$r_0 \geqslant d(f^{-n_1}(x), V^-) \geqslant \lambda^{-n_1} d(x, V^-).$$

Since $x \notin V_n$ either $d(x, V^+) \geqslant r_n$ or $d(x, V^-) \geqslant r_n$. Then

$$r_n \leqslant \max\{\lambda^{n_2} r_0, \lambda^{n_1} r_0\}.$$

But by (2) we have $\log r_n = (1+\delta)^n \log r_0$, implying

$$(1+\delta)^n \log r_0 \leqslant \log r_0 + \log \lambda(\max\{n_1, n_2\}).$$

Hence
$$n_1 + n_2 \leqslant 2\,\frac{\log r_0}{\log \lambda}\,(1+\delta)^n.$$

From this inequality the upper bound of the lemma follows easily. The lower bound is obtained by the same method applying the lower bounds of part c) of Lemma 2.

Define a (n, k)-string as a set σ of the form $\sigma = \{f^j(x_k), \ldots, f^{j+\ell}(x_k)\} \subset V_n$, $1 \leqslant j \leqslant j + \ell \leqslant m_k$, such that $f^{j-1}(x_k) \notin V_n$ and $f^{j+\ell+1}(x_n) \notin V_n$. Observe that there is a natural order relation among (n, k)-strings, defined by $\sigma_1 < \sigma_2$ if the first element of σ_2 is a strictly positive iterate of the last member of σ_1. To simplify the notation, $(0, k)$-strings will be called k-strings. Most of the time we shall deal only with k-strings. More general strings will appear only once in our arguments.

Now we are ready to state the fundamental lemma of the proof of Theorem I.1.

Lemma 4. — *For all* $n_1 > 0$, *one of the following properties holds*:

a) *There exist* $n \geqslant n_1$, $k > 0$ *and two k-strings* $\sigma_1 \subset S_{n+1}$, $\sigma_1 < \sigma_2 \subset S_{n+1}$ *such that* $\sigma \cap (S_n - S_{n+1}) = \emptyset$ *for every k-string* $\sigma_1 < \sigma < \sigma_2$.

b) *There exist* $n \geqslant n_1$, $k > 0$ *and a k-string* $\sigma_1 \subset S_{n+1}$ *such that* $\sigma \cap (S_n - S_{n+1}) = \emptyset$ *for every k-string* $\sigma < \sigma_1$.

Proof. — Take $1 + \delta < \xi < 2$ and $s \in \mathbf{Z}^+$ such that $2s - 1 > \xi s$. Define $\nu_k(S_n)$ as the number of k-strings contained in S_n. Now we shall prove three claims.

Claim 1. — *Suppose that property* a) *of Lemma 4 does not hold for a certain* $n_1 > 0$. *Then*

$$\nu_k(S_n) \leqslant (1/\xi)\, \nu_k(S_{n-1})$$

for all $k > 0$ *and* $n > n_1$ *such that* $\nu_n(S_n) > s$.

Proof. — Suppose that $\sigma_1, \ldots, \sigma_p$, $p = \nu_k(S_n)$, are the k-strings contained in S_n. If $n > n_1$, the hypothesis of the claim implies that between any two consecutive k-strings σ_i, σ_{i+1} in S_n there exists a k-string $\hat{\sigma}_i$ such that $\hat{\sigma}_i \cap S_{n-1} \neq \emptyset$, therefore $\hat{\sigma}_i \subset S_{n+1}$ (by the definition of S_{n-1}) and then S_{n-1} contains the strings $\sigma_1, \ldots, \sigma_p$ and $\hat{\sigma}_1, \ldots, \hat{\sigma}_{p-1}$. Hence

$$\nu_k(S_{n-1}) \geqslant 2p - 1 = 2\nu_k(S_n) - 1 \geqslant \xi\nu_k(S_n),$$

because $\nu_k(S_n) > s$.

Claim 2. — Suppose that property a) *of Lemma 4 does not hold for a certain* $n_1 > 0$. *Then*

$$\mu_k(S_n - S_{n+1}) \leqslant C_2 \left(\frac{1+\delta}{\xi}\right)^n (1+\delta)\xi^{n_1},$$

for all $k > 0$ *and* $n > n_1$ *such that* $\nu_k(S_n) > s$.

Proof. — The first claim proves that

$$\nu_k(S_n) \leqslant (1/\xi)^{n-n_1} \nu_k(S_{n_1})$$

for all $k > 0$ and $n > n_1$ such that $\nu_k(S_n) > s$. Denote by T the supremum of the lenghts of the k-strings contained in $S_n - S_{n+1}$; then

$$\begin{aligned}\mu_k(S_n - S_{n+1}) &= (1/m_k)\, \#\{1 \leqslant j \leqslant m_k \mid f^j(x_k) \in S_n - S_{n+1}\} \\ &\leqslant (1/m_k)\, T(\nu_k(S_n) - \nu_k(S_{n+1})) \leqslant (1/m_k)\, T\nu_k(S_n).\end{aligned}$$

But by Lemma 3,

$$T \leqslant C_2(1+\delta)^{n+1}.$$

Hence
$$\begin{aligned}\mu_k(S_n - S_{n+1}) &\leqslant C_2(1+\delta)^{n+1} (1/m_k)\, \nu_k(S_n) \\ &\leqslant C_2(1+\delta)^{n+1} (1/m_k)\, (1/\xi)^{n-n_1} \nu_k(S_{n_1}).\end{aligned}$$

But
$$(1/m_k)\, \nu_k(S_{n_1}) \leqslant (1/m_k)\, \#\{1 \leqslant j \leqslant m_k \mid f^j(x_k) \in S_{n_1}\} \leqslant 1.$$

Then
$$\mu_k(S_n - S_{n+1}) \leqslant C_2 \left(\frac{1+\delta}{\xi}\right)^n (1+\delta)\, \xi^{n_1}.$$

Claim 3. — Suppose that there exists $n_1 > 0$ *such that property* b) *of Lemma 4 does not hold. Then*

$$\mu_k(S_n) \leqslant C_2\, C_1^{-1}(1+\delta)^{s+2} s\mu_k(S_{n-2} - S_n)$$

for all $n > n_1 + 2$ *and* $k > 0$ *such that* $\nu_k(S_n) \leqslant s$.

Proof. — If property b) of Lemma 4 does not hold for n_1, then, for all $m \geqslant n_1$, $k > 0$ and every S_j with $j > m$ that contains a k-string σ_1, it follows that $S_{j-1} - S_j$ also contains k-strings. This means that the family of sets $S_{j-1} - S_j, j > m$, that contain a k-string has the form $\{S_m - S_{m+1}, \ldots, S_{m+\ell} - S_{m+\ell+1}\}$. Now suppose that $n \geqslant n_1 + 3$. Put $m = n - 2$. Then $m > n_1$. Hence $S_m - S_{m+1} = S_{n-2} - S_{n-1}$ and $S_{m+1} - S_{m+2} = S_{n-1} - S_n$ contain k-strings. At least one of these two strings has the form $\sigma = \{f^j(x_k), \ldots, f^{j+t}(x_k)\}$ with $1 \leqslant j \leqslant j + t < m_k$ (remember that the definition of k-strings only requires $j \leqslant m_k$; but there exists at most *one* k-string whose last point is an iterate larger than m_k). By Lemma 3 the lenght of σ is larger than or equal to $C_1(1+\delta)^{n-2}$. Then

$$\mu_k(S_{n-2} - S_n) \geqslant (1/m_k)\, C_1(1+\delta)^{n-2}.$$

Moreover, denoting by T the supremum of the lenghts of the k-strings contained in S_n, we have

$$\mu_k(S_n) \leqslant (1/m_k)\, T\nu_k(S_n).$$

But since $n > n_1$, the above argument shows that there exists ℓ such that every set $S_j - S_{j+1}$, $n \leqslant j \leqslant n + \ell$, contains a k-string and conversely every k-string contained in S_n is contained in one of these sets. Therefore $\nu_k(S_n) \geqslant \ell + 1$ and $\nu_k(S_n) \leqslant s$ implies $\ell < s$. Now, by Lemma 3, we have

$$T \leqslant C_2(1+\delta)^{n+s},$$

hence $$\mu_k(S_n) \leqslant (1/m_k)\, C_2(1+\delta)^{n+s}\, s.$$

Therefore $$\mu_k(S_n) \leqslant C_2\, C_1^{-1}(1+\delta)^{s+2}\, s\mu_k(S_{n-2} - S_n).$$

Now we are ready to prove Lemma 4. Suppose it is false. Then there exists $n_1 > 0$ such that a) and b) do not hold for n_1. Define

$$r(k) = \min\{j \mid \nu_k(S_j) \leqslant s\}.$$

Then $$\mu_k(S_n) = \mu_k(S_{r(k)}) + \sum_{n \leqslant j < r(k)} \mu_k(S_j - S_{j+1}).$$

Moreover it is easy to see that

$$\lim_{k \to +\infty} r(k) = \infty.$$

Hence $r(k) > n_1 + 2$ if k is large enough. Applying Claim 3, we obtain:

$$\mu_k(S_{r(k)}) \leqslant C_2\, C_1^{-1}(1+\delta)^{s+2}\, s\mu_k(S_{r(k)-2} - S_{r(k)}).$$

Now $r(k) - 2 > n_1$ and $\nu_k(S_{r(k)-1}) > s$; therefore Claim 2 implies

$$\begin{aligned}\mu_k(S_{r(k)-2} - S_{r(k)}) &= \mu_k(S_{r(k)-2} - S_{r(k)-1}) + \mu_k(S_{r(k)-1} - S_{r(k)})\\ &\leqslant C_3\left(\frac{1+\delta}{\xi}\right)^{r(k)-2} + C_3\left(\frac{1+\delta}{\xi}\right)^{r(k)-1} \leqslant 2C_3\left(\frac{1+\delta}{\xi}\right)^{r(k)-2},\end{aligned}$$

where $C_3 = C_2(1+\delta)\,\xi^{n_1}$. Moreover, Claim 2 implies

$$\mu_k(S_j - S_{j+1}) \leqslant C_3\left(\frac{1+\delta}{\xi}\right)^{j}$$

for every $n_1 < j < r(k)$. Hence, setting $C_4 = 2sC_1^{-1}\, C_2\, C_3(1+\delta)^{s+2}$ we obtain

$$\mu_k(S_n) \leqslant C_4\left(\frac{1+\delta}{\xi}\right)^{r(k)-2} + C_3 \sum_{n \leqslant j < r(k)} \left(\frac{1+\delta}{\xi}\right)^{j}$$

for $n > n_1$ and k large. Taking limits when $k \to +\infty$, and recalling that $(1+\delta)/\xi < 1$ and $\lim_{k\to+\infty} r(k) = \infty$, we obtain

$$\lim_{n\to\infty} \lim_{k\to+\infty} \mu_k(S_n) = 0.$$

But (3) implies

$$\mu(S_n) = \lim_{k\to+\infty} \mu_k(S_n).$$

Hence $$\mu(\Lambda) = \lim_{n\to+\infty} \mu(S_n) = 0,$$

contradicting the hypothesis $\mu(\Lambda) > 0$ and proving Lemma 4.

Now let us complete the proof of Theorem I.1. By Lemma 4 there exist n, arbitrarily large, and $k > 0$ satisfying property a) or property b). Suppose that property a) is satisfied; in other words there exist consecutive k-strings $\sigma_1 < \sigma_2$ contained in S_{n+1}, such that $\sigma \cap S_n = \emptyset$ for all k-strings $\sigma_1 < \sigma < \sigma_2$. Now observe that by the definition of strings, the intersections $\sigma_1 \cap V_n$, $\sigma_2 \cap V_n$ are (n, k)-strings. Let q_1 be the last point of the (n, k)-string $\sigma_1 \cap V_n$ and q_2 the first point of the (n, k)-string $\sigma_2 \cap V_n$. In the simplified case, the situation is described in figure III.

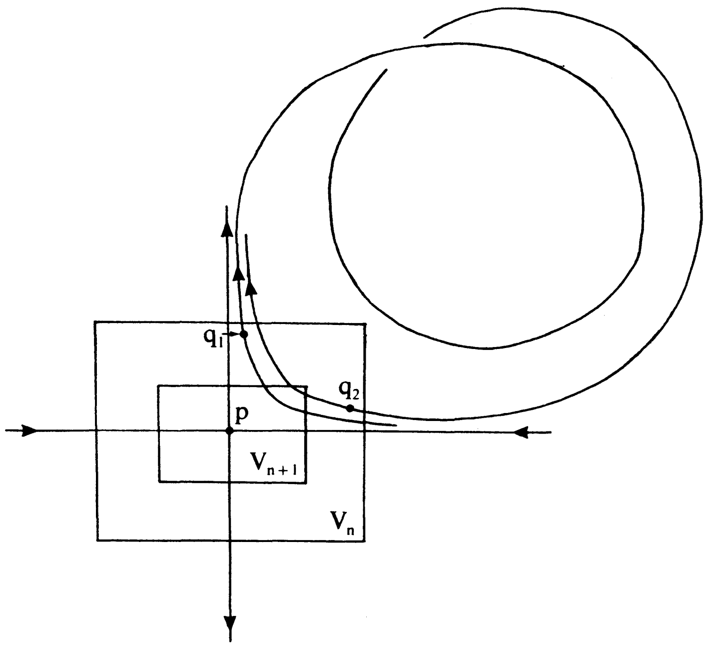

FIG. III

Since q_1 is the last point of $\sigma_1 \cap V_n$, $f(q_1)$ does not belong to V_n. Hence either $d(f(q_1), V^+) > r_n$ or $d(f(q_1), V^-) > r_n$. But Lemma 2 implies

$$d(f(q_1), V^-) \leqslant \lambda\, d(q_1, V^-) \leqslant \lambda r_n < r_n,$$

thus

(4) $$d(f(q_1), V^+) > r_n.$$

Take a point $p_1 \in \sigma_1 \cap V_{n+1}$ and write $q_1 = f^a(p_1)$. Then (4) and Lemma 2 imply

$$r_n < d(f^{a+1}(p_1), V^+) \leqslant \gamma^{-(a+1)} d(p_1, V^+) \leqslant \gamma^{-(a+1)} r_{n+1} = \gamma^{-(a+1)} r_n^{1+\delta},$$

and therefore

(5) $$r_n^\delta \geqslant \gamma^{a+1}.$$

Using again Lemma 2, we have

$$d(q_1, V^-) = d(f^a(p_1), V^-) \leqslant \lambda^a d(p_1, V^-) \leqslant \lambda^a r_{n+1} = \lambda^a r_n^{1+\delta}.$$

Write $\lambda = \gamma^\alpha$ with $0 < \alpha < 1$. From (5) it follows that

$$d(q_1, V^-) \leqslant \gamma^{\alpha a} r_n^{1+\delta} \leqslant (r_n^\delta/\gamma)^\alpha . r_n^{1+\delta} = \gamma^{-\alpha} r_n^{1+\delta+\alpha\delta}.$$

The only restriction we used about δ was $0 < \delta < 1$. Suppose we take it so near to 1 that $\delta(1+\alpha) > 1$ and take $0 < \beta < \delta(1+\alpha) - 1$. Then the last inequality, if n is large, yields

$$d(q_1, V^-) < r_n^{2+\beta}.$$

Then we can choose $y_1 \in V^-$ such that

(6) $$q_1 \in B(r_n^{2+\beta}, y_1).$$

Now we shall prove that if n is large enough, then

(7) $$f^{-j}(y_1) \notin B(r_n^{1+(\beta/3)}, y_1)$$

for all $j > 0$. From (4) and Lemma 2 it follows that

$$d(q_1, V^+) \geqslant \gamma\, d(f(q_1), V^+) > \gamma r_n,$$

and therefore (6) implies

(8) $$d(y_1, V^+) \geqslant d(q_1, V^+) - d(y_1, q_1) \geqslant \gamma r_n - r_n^{2+\beta}.$$

Also, for all $j > 0$, Lemma 2 implies that

$$d(f^{-j}(y_1), V^+) \leqslant \lambda^j d(y_1, V^+) < \lambda\, d(y_1, V^+),$$

hence from (8) it follows that

$$d(f^{-j}(y_1), y_1) \geqslant (1-\lambda)\, d(y_1, V^+) \geqslant (1-\lambda)\,(\gamma r_n - r_n^{2+\beta}).$$

But if n is large,

$$(1-\lambda)\,(\gamma r_n - r_n^{2+\beta}) > r_n^{1+(\beta/3)}.$$

Hence (7) holds for all $j > 0$, if n is large enough. Now write $q_1 = f^{t_1}(x_k)$, $q_2 = f^{t_2}(x_k)$ and define $N_0 = t_2 - t_1$. We claim that if n is large enough, then

(9) $$f^j(q_1) \notin B(r_n^{1+(\beta/3)}, y_1)$$

for all $0 < j \leqslant N_0$. First we prove (9) for $j = 1$. Take $A > 0$ such that $d(f(z), f(w)) \leqslant A\, d(z, w)$ for all $z, w \in M$. Then (6) implies

$$d(f(q_1), y_1) \geqslant d(y_1, f(y_1)) - d(f(q_1), f(y_1)) \geqslant d(y_1, f(y_1)) - A r_n^{2+\beta}.$$

Now from Lemma 2 and (8) it follows that

$$d(y_1,f(y_1)) \geqslant d(f(y_1),\mathrm{V}^+) - d(y_1,\mathrm{V}^+) \geqslant \left(\frac{1}{\lambda}-1\right)(\gamma r_n - r_n^{2+\beta}),$$

hence
$$d(f(q_1),y_1) \geqslant \left(\frac{1}{\lambda}-1\right)\gamma r_n - \left(\frac{1}{\lambda}-1+\mathrm{A}\right) r_n^{2+\beta}.$$

But if n is large, the right hand side of that inequality is larger than $r_n^{1+(\beta/3)}$, and (9) holds for $j=1$. Now suppose that

$$d(f^j(q_1),y_1) \leqslant r_n^{1+(\beta/3)}$$

for some $2 \leqslant j \leqslant \mathrm{N}_0$ and take $\mathrm{B} > 0$ such that $d(f^{-1}(z),f^{-1}(w)) \leqslant \mathrm{B}\, d(z,w)$ for all $z, w \in \mathrm{M}$. Since $y_1 \in \mathrm{V}^-$, Lemma 2 implies

$$d(f^{j-1}(q_1),\mathrm{V}^-) \leqslant d(f^{j-1}(q_1),f^{-1}(y_1)) \leqslant \mathrm{B} r_n^{1+(\beta/3)}$$

and therefore, if n is large,

(10) $$d(f^{j-1}(q_1^-),\mathrm{V}^-) \leqslant r_n.$$

On the other hand, (6) and Lemma 2 imply

$$d(f^{-1}(y_1),\mathrm{V}^+) \leqslant d(f^{-1}(y_1),f^{-1}(q_1)) + d(f^{-1}(q_1),\mathrm{V}^+)$$
$$\leqslant \mathrm{B}\, d(y_1,q_1) + \lambda\, d(q_1,\mathrm{V}^+) \leqslant \mathrm{B} r_n^{2+\beta} + \lambda r_n$$

because $q_1 \in \mathrm{V}_n$ and therefore

$$d(f^{j-1}(q_1),\mathrm{V}^+) \leqslant d(f^{j-1}(q_1),f^{-1}(y_1)) + d(f^{-1}(y_1),\mathrm{V}^+)$$
$$\leqslant (\mathrm{B} r_n^{\beta/3} + \mathrm{B} r_n^{1+\beta} + \lambda)\, r_n.$$

When n is large, this means

(11) $$d(f^{j-1}(q_1),\mathrm{V}^+) \leqslant r_n.$$

From (10) and (11) it follows that $f^{j-1}(q_1) \in \mathrm{V}_n$. Let σ be the k-string containing $f^{j-1}(q_1)$. Then $\sigma \cap \mathrm{S}_n \neq \varnothing$. Also $\sigma_1 \leqslant \sigma$ because $f^{j-1}(q_1) \in \sigma$, $j-1 \geqslant 1$ and $q_1 \in \sigma_1$. Moreover $\sigma \leqslant \sigma_2$ because

$$f^{\mathrm{N}_0-j+1}(f^{j-1}(q_1)) = f^{\mathrm{N}_0}(q_1) = q_2,$$

$\mathrm{N}_0 - j + 1 \geqslant 1$ and $f^{j-1}(q_1) \in \sigma$. Now $\sigma_1 \leqslant \sigma \leqslant \sigma_2$ and $\sigma \cap \mathrm{S}_n = \varnothing$ imply $\sigma_1 = \sigma$ or $\sigma = \sigma_2$. Suppose $\sigma_1 = \sigma$; then $f^{j-1}(q_1) \in \sigma_1 \cap \mathrm{V}_n$ and now $j-1 \geqslant 1$ means that q_1 is not the last point of the k-string σ_1 in V_n, which is a contradiction. Suppose $\sigma = \sigma_2$. Then $f^{-(\mathrm{N}_0-j+1)}(q_2) = f^{j-1}(q_1) \in \sigma_2 \cap \mathrm{V}_n$ and $\mathrm{N}_0 - j + 1 \geqslant 1$ imply that q_2 is not the first point of the k-string σ_2 in V_n, which also is a contradiction. This completes the proof of (9).

Similarly to (6) and (7) we prove that if n is large, there exists $y_2 \in \mathrm{V}^+$ such that

(12) $$q_2 \in \mathrm{B}(r_n^{2+\beta},y_2)$$

and

(13) $$f^j(y_2) \notin \mathrm{B}(r_n^{1+(\beta/3)},y_2)$$

for all $j > 0$. Moreover

(14) $$f^j(y_2) \notin B(r_n^{1+(\beta/3)}, y_1)$$

for all $j \geqslant 0$. To prove (14), observe that $f^j(y_2) \in V^+$ for all $j \geqslant 0$. Hence, using (8) we obtain

$$d(y_1, f^j(y_2)) \geqslant d(y_1, V^+) \geqslant \gamma r_n - r_n^{2+\beta}.$$

But if n is large, $\gamma r_n - r_n^{2+\beta} > r_n^{1+(\beta/3)}$, proving (14).

In a similar way we prove the property

(15) $$f^{-j}(y_1) \notin B(r_n^{1+(\beta/3)}, y_2)$$

for all $j \geqslant 0$. Moreover, the proof of property (9) can be adapted to show that

(16) $$f^j(q_1) \notin B(r_n^{1+(\beta/3)}, y_2)$$

for all $0 \leqslant j < N_0$. The case $j = N_0 - 1$ is treated separately and if (16) is false for some $0 \leqslant j \leqslant N_0 - 2$, proceeding similarly to the proof of (9), we find a string σ satisfying $\sigma_1 \leqslant \sigma \leqslant \sigma_2$ and $\sigma \cap S_n \neq \emptyset$ and then, by the way σ_1 and σ_2 were selected, it must satisfy $\sigma_1 = \sigma$ or $\sigma = \sigma_2$. As in the proof of (9), both possibilities immediately lead to contradictions.

Now apply Lemma 1 to $\mathcal{N} = \mathcal{U} \circ f^{-1}$ (where $\mathcal{U}$ is the neighborhood in the statement of Theorem I.1) and

$$c = \frac{2+\beta}{1+(\beta/3)} - 2$$

when $r = 2$, or

$$c = \frac{2+\beta}{1+(\beta/3)} - 1$$

when $r = 1$. This gives a number $R > 0$, and we can suppose, taking n large enough, that $r_n < R$. Then, by Lemma 1, there exists $h \in \mathcal{N}$ such that

$$h(q_2) = y_2,$$
$$h(y_1) = q_1$$

and $$h(z) = z \quad \text{if } z \notin B(r_n^{1+(\beta/3)}, y_1) \cup B(r_n^{1+(\beta/3)}, y_2).$$

Define $g = hf$ and

$$V = V_n - S - f^{-1}(S),$$

where $$S = B(r_n^{1+(\beta/3)}, y_1) \cup B(r_n^{1+(\beta/3)}, y_2).$$

Then $$g \in \mathcal{N} \circ f = \mathcal{U}.$$

Clearly $V \subset V_n$ and then, by (1),

$$V \subset U.$$

Moreover it is easy to check that

$$f/(V \cup U^c) = g/(V \cup U^c) \quad \text{and} \quad f^{-1}/(V \cup U^c) = g^{-1}/(V \cup U^c).$$

Now let us verify that V is a neighborhood of Λ if n is large. First we shall show that

(17) $$d(\Lambda, y_1) > r_n^{1+(\beta/3)},$$

(18) $$d(\Lambda, y_2) > r_n^{1+(\beta/3)}.$$

Using (8), we obtain

$$d(\Lambda, y_1) \geqslant d(V^+, y_1) \geqslant \gamma r_n - r_n^{2+\beta}.$$

When n is large, $\gamma r_n - r_n^{2+\beta} > r_n^{1+(\beta/3)}$, thus proving (17). Inequality (18) is proved similarly. Properties (17) and (18) imply that the closed set S does not intersect Λ. Since Λ is invariant, $\Lambda \cap f^{-1}(S) = \emptyset$. Then $V_n - (S \cup f^{-1}(S))$ is a neighborhood of Λ.

Now we shall check that option (II) of Theorem I.1 is satisfied for g, V and an appropriate choice of k and $0 < s_0 < s_1 < m_k$. As k we obviously take the k in the construction of σ_1 and σ_2. Choose $0 < s_0 < s_1 < m_k$ satisfying

$$f^{s_0}(x_k) = q_1, \quad f^{s_1}(x_k) = q_2.$$

Observe that s_0 and s_1 coincide with the numbers t_1 and t_2 used to define the number $N_0 = t_2 - t_1$ of property (9). Then $s_1 = s_0 + N_0$. We shall prove property (i) of Theorem I.1 by induction on j.

For $j = 1$,

$$g(f^{s_0}(x_k)) = g(q_1) = hf(q_1).$$

But by (9) and (16), $f(q_1) \notin S$. Then $g(q_1) = f(q_1)$ and

$$g(f^{s_0}(x_k)) = f(q_1) = f(f^{s_0}(x_k)).$$

Now assume that $f^i(q_1) = g^i(q_1)$, $1 \leqslant i < (s_1 - s_0) - 1$. Then

$$g^{i+1}(f^{s_0}(x_k)) = g^{i+1}(q_1) = g(g^i(q_1)) = g(f^i(q_1)).$$

But $f^{i+1}(q_1) \notin S$ because $i + 1 < s_1 - s_0 = N_0$ and, according to (9) and (16), $f^j(q_1) \notin S$ for all $0 < j < N_0$. Hence $g(f^i(q_1)) = hf(f^i(q_1)) = h(f^{i+1}(q_1)) = f^{i+1}(q_1)$ and then

(19) $$f^{i+1}(f^{s_0}(x_k)) = g^{i+1}(f^{s_0}(x_k)),$$

completing the proof of (i). To prove property (ii) of Theorem I.1, we first prove, by induction on j, that

(20) $$g^{-j}(f^{s_0}(x_k)) = f^{-j}(y_1)$$

for all $j \geqslant 1$. When $j = 1$,

$$g^{-1}(f^{s_0}(x_k)) = f^{-1}h^{-1}(q_1) = f^{-1}(y_1).$$

Assume that (20) holds for $j = i \geqslant 1$. Then

$$g^{-(i+1)}(f^{s_0}(x_k)) = g^{-1}g^{-i}(f^{s_0}(x_k)) = g^{-1}f^{-i}(y_1) = f^{-1}h^{-1}f^{-i}(y_1).$$

But by (7) and (15), $f^{-i}(y_1) \notin S$. Then

$$g^{-(i+1)}(f^{s_0}(x_k)) = f^{-1}f^{-i}(y_1) = f^{-(i+1)}(y_1).$$

This completes the proof of (20). It follows that

$$(21) \qquad g^{-j}(f^{s_0}(x_k)) \in V_n$$

for all $j \geq 1$ because, by Lemma 2, $f^{-j}(V_n \cap V^-) \subset V_n \cap V^-$ for all $j \geq 0$. Since $y_1 \in V_n \cap V^-$, this implies that $f^{-j}(y_1) \in V_n \cap V^- \subset V_n$ for all $j \geq 0$. This property together with (20) implies (21). Moreover, property (20), together with (7) and (15), implies

$$(22) \qquad g^{-j}(f^{s_0}(x_k)) = f^{-1}(y_1) \notin S$$

for all $j \geq 1$. Hence

$$(23) \qquad g^{-j}(f^{s_0}(x_k)) \notin f^{-1}(S)$$

for all $j \geq 2$. Clearly, property (ii) of Theorem I.1 follows from (21), (22) and (23). The equality

$$\bigcap_{n \geq 0} f^n(V) = \bigcap_{n \geq 0} g^n(V),$$

also contained in that property, is a trivial consequence of $g/V = f/V$. Property (iii) o Theorem I.1 is proved using similar methods, observing that (19) implies

$$\begin{aligned} g^{s_1-s_0}(f^{s_0}(x_k)) &= gg^{s_1-s_0-1}(f^{s_0}(x_k)) = hff^{s_1-s_0-1}(f^{s_0}(x_k)) \\ &= hf^{s_1-s_0}(f^{s_0}(x_k)) = hf^{s_1}(x_k) = h(q_2) = y_2. \end{aligned}$$

Then we have to prove that $g^j(y_2) = f^j(y_2) \in V$ for all $j > 0$. We shall do it by induction. For $j = 1$

$$g(y_2) = hf(y_2).$$

But by (13) and (14), $f(y_2) \notin S$. Then $g(y_2) = f(y_2)$. Moreover $y_2 \in V^+ \cap V_n$. Then $f(y_2) \in f(V^+ \cap V_n) \subset V^+ \cap V_n \subset V_n$. Finally observe that $f(y_2) \notin f^{-1}(S)$ because $f(y_2) \in f^{-1}(S)$ implies $f^2(y_2) \in S$ contradicting (13) or (14). Then

$$f(y_2) \in V_n - S - f^{-1}(S) = V.$$

Now suppose that $f^i(y_2) \in V$. Then $g^{i+1}(y_2) = g(g^i(y_2)) = g(f^i(y_2)) = hf^{i+1}(y_2)$. By (13) and (14), $f^{i+1}(y_2) \notin S$. Then $g^{i+1}(y_2) = f^{i+1}(y_2)$ and, by the argument already used in the case $j = 1$, $f^{i+1}(y_2) \in V_n$. Hence $g^{i+1}(y_2) \in V_n$. Moreover $g^{i+1}(y_2) \notin S \cup f^{-1}(S)$ because otherwise we have $g^{i+1}(y_2) = f^{i+1}(y_2) \in S \cup f^{-1}(S)$ and then either $f^{i+1}(y_2) \in S$ or $f^{i+2}(y_2) \in S$, violating properties (13) or (14). Hence $g^{i+1}(y_2) \in V_n - (S \cup f^{-1}(S)) = V$, thus concluding the proof or property (II) of Theorem I.1.

Finally, let us show that when in the application of Lemma 4 it is option b) that holds, then we can find $g \in \mathcal{U}$ and V satisfying property (I) of Theorem I.1. Let $\sigma_1 = \{f^{s_0}(x_k), \ldots, f^{s_0+\ell}(x_k)\}$ be the string given by option b) of Lemma 4. As in the previous case, we prove that, if n is large enough, there exists $y \in V^+$ such that, setting $q = f^{s_0}(x_k)$,

$$q \in B(r_n^{2+\beta}, y),$$
$$f^j(y) \notin B(r_n^{1+(\beta/3)}, y)$$

for all $j > 0$, and

$$f^j(x_k) \notin B(r_n^{1+(\beta/3)}, y)$$

for all $0 \leqslant j < s_0$. Now take $h \in \mathrm{Diff}^r(M)$ satisfying

$$h(q) = y$$

and $$h(z) = z \quad \text{if } z \notin B(r_n^{1+(\beta/3)}, y).$$

Using Lemma 1 as before, we show that if n is very large then $g = hf \in \mathcal{U}$. Define

$$S = B(r_n^{1+(\beta/3)}, y)$$

and $$V = V_n - S - f^{-1}(S).$$

Using the same methods as in the previous case it is easy to check that g, k, V and $f^{s_0}(x_k)$ satisfy property (I) of Theorem I.1. Moreover, in the particular case when there exists an isolating block U_0 of Λ such that $\{x_n\} \subset \Lambda^s$, where Λ^s is defined as $\Lambda^s = \bigcap_{m \geqslant 0} f^m(U_0)$, we have $g^{-j}(x_n) = f^{-j}(x_n)$ for all $j \geqslant 0$, $n \geqslant 0$, as property (I) of Theorem I.1 requires. To see this, observe that f^{-1} and g^{-1} differ only on S. Therefore we have only to show that $f^{-j}(x_n) \in S$ for all $j \geqslant 0$ and $n \geqslant 0$. But there exists $N_0 > 0$ such that $f^{-j}(x_n) \in V^-$ for all $j > N_0$, $n \geqslant 0$. Since $S \cap V^- = \varnothing$, it remains to show that $f^{-j}(x_n) \notin S$ for all $0 \leqslant j \leqslant N_0$, $n \geqslant 0$. But the set $\{f^{-j}(x_n) \mid n \geqslant 0, 0 \leqslant j \leqslant N_0\}$ is compact and disjoint from Λ; since S can be taken in an arbitrarily small neighborhood of Λ, it follows that S does not contain points of that set. This completes the proof of Theorem I.1 but for Lemma 2.

Finally, let us prove Lemma 2. For all $x \in \Lambda$ define

$$W_\varepsilon^u(x) = \{y \mid d(f^n(x), f^n(y)) \leqslant \varepsilon \text{ for all } n \geqslant 0\},$$
$$W_\varepsilon^s(x) = \{y \mid d(f^{-n}(x), f^{-n}(y)) \leqslant \varepsilon \text{ for all } n \geqslant 0\}.$$

When ε is small, these sets are C^1 disks tanget at x to the subspaces E_x^s, E_x^u. Take a compact neighborhood U_0 of Λ such that

$$\bigcap_n f^n(U_0) = \Lambda.$$

Choose $\varepsilon > 0$ so small that

(24) $$W_\varepsilon^u(x) \cup W_\varepsilon^s(x) \subset U_0$$

for all $x \in \Lambda$ and define $V^+ = \bigcup_{x \in \Lambda} W_\varepsilon^s(x)$, $V^- = \bigcup_{x \in \Lambda} W_\varepsilon^u(x)$. Clearly these sets are compact and satisfy

(25) $$f(V^+) \subset V^+$$

(26) $$f^{-1}(V^-) \subset V^-.$$

Moreover property (24) implies that $V^+ \subset U_0 \subset U$ and $V^- \subset U_0 \subset U$. From (25) and (26) follows that if $y \in V^+ \cap V^-$ then $f^n(y) \in f^n(V^+) \subset V^+ \subset U_0$, for all $n \geq 0$, and $f^{-n}(y) \in f^{-n}(V^-) \subset V^- \subset U_0$, for all $n \geq 0$. Hence $y \in \bigcap_n f^n(U_0) = \Lambda$. Then

$$V^+ \cap V^- = \Lambda. \tag{27}$$

To complete the proof of Lemma 2 it remains to show the inequalities of c). To prove them we take ε so small that

$$\begin{aligned} \overline{V^+ - f(V^+)} \cap \Lambda &= \emptyset \\ \overline{V^- - f^{-1}(V^-)} \cap \Lambda &= \emptyset. \end{aligned} \tag{28}$$

That such an ε exists is proved in [1]. Besides this non trivial property we shall need two easy lemmas.

Lemma 5. — There exist $\delta > 0$ and $0 < \gamma < \lambda < 1$ such that if $p \in \Lambda$, $x \in M$ and $d(x, p) \leq \delta$, then

$$\gamma\, d(x, W^u_\varepsilon(p)) \leq d(f(x), W^u_\varepsilon(f(p))) \leq \lambda\, d(x, W^u_\varepsilon(p))$$

and

$$\gamma\, d(x, W^s_\varepsilon(p)) \leq d(f^{-1}(x), W^s_\varepsilon(f^{-1}(p))) \leq \lambda\, d(x, W^s_\varepsilon(p)).$$

This lemma is an easy application of the basic properties of hyperbolic sets. We leave its proof to the reader.

Lemma 6. — For all $N > 0$ there exists $c(N) = c > 0$ such that if $d(x, \Lambda) \leq c$ and $p \in V^-$ satisfies $d(x, p) = d(x, V^-)$ then $p \in f^{-N}(V^-)$.

Proof. — If the lemma is false, there exist $N > 0$ and sequences $\{x_n\}$ and $\{p_n\} \subset V^-$ such that

$$d(x_n, \Lambda) \leq 1/n, \tag{29}$$

$$p_n \in V^-, \tag{30}$$

$$d(x_n, p_n) = d(x_n, V^-) \tag{31}$$

and

$$p_n \notin f^{-N}(V^-) \tag{32}$$

for all $n \geq 0$. We can suppose that the sequences $\{p_n\}$ and $\{x_n\}$ converge to p and x respectively. From (29) it follows that $x \in \Lambda$. Moreover, by (31),

$$d(x_n, p_n) = d(x_n, V^-) \leq d(x_n, \Lambda) \leq 1/n.$$

Hence $p = x$. But (30) and (32) imply that

$$p_n \in \overline{V^- - f^{-N}(V^-)}$$

for all $n \geqslant 0$, hence

$$x = p \in \overline{V^- - f^{-N}(V^-)}.$$

Then $$\overline{V^- - f^{-N}(V^-)} \cap \Lambda \neq \varnothing,$$

which is easily seen to contradict (28), thus proving Lemma 6.

To complete the proof of Lemma 2 take $0 < \delta_1 < \delta$ (where δ is given by Lemma 5) such that $d(a, b) \leqslant \delta_1$ implies $d(f^{-1}(a), f^{-1}(b)) \leqslant \delta$. Choose $N > 0$ such that

$$\operatorname{diam} f^{-N}(W^u_\varepsilon(p)) \leqslant \delta_1/2 \tag{33}$$

for all $p \in \Lambda$. By (27) there exists $\varepsilon_1 > 0$ that if $d(x, V^+) \leqslant \varepsilon_1$ and $d(x, V^-) \leqslant \varepsilon_1$ then $d(x, \Lambda) \leqslant c$ and $d(f(x), \Lambda) \leqslant c$, where $c = c(N)$ is given by Lemma 6. Define $\varepsilon_0 = \min\{\varepsilon_1, \delta_1/2\}$. Let us show that this choice of ε_0 satisfies property c) of Lemma 2. If $d(x, V^+) \leqslant \varepsilon_0$ and $d(x, V^-) \leqslant \varepsilon_0$, then

$$d(x, \Lambda) \leqslant c$$

and then, by Lemma 6, there exists $y \in f^{-N}(V^-)$ such that

$$d(x, y) = d(x, V^-).$$

Since $y \in f^{-N}(V^-)$ it follows that

$$y \in f^{-N}(W^u_\varepsilon(q)) \subset W^u_\varepsilon(f^{-N}(q))$$

for some $q \in \Lambda$. Define $p = f^{-N}(q)$. By (33), we have

$$d(y, p) \leqslant \delta_1/2$$

and then

$$d(x, p) \leqslant d(x, y) + d(y, p) \leqslant d(x, V^-) + \delta_1/2 \leqslant \varepsilon_0 + \delta_1/2 \leqslant \delta_1/2 + \delta_1/2 = \delta_1 \leqslant \delta.$$

Now we apply Lemma 5 to obtain

$$d(f(x), V^-) \leqslant d(f(x), W^u_\varepsilon(f(p))) \leqslant d(x, W^u_\varepsilon(p)) \leqslant \lambda\, d(x, y) = \lambda\, d(x, V^-),$$

thus proving one of the desired inequalities. Moreover, by the way we chose ε_0 and ε_1, we know that $d(x, V^-) \leqslant \varepsilon_0$ and $d(x, V^+) \leqslant \varepsilon_0$ imply

$$d(f(x), \Lambda) \leqslant c.$$

Then, by Lemma 6, there exists $z \in f^{-N}(V^-)$ such that

$$d(f(x), z) = d(f(x), V^-).$$

Since $z \in f^{-N}(V^-)$ we can write

$$z \in f^{-N}(W^u_\varepsilon(q)) \subset W^u_\varepsilon(f^{-N}(q))$$

for some $q \in \Lambda$. Define $p = f^{-N}(q)$. By (33), $d(z, p) \leqslant \delta_1/2$. Hence, arguing as above, we obtain

$$d(f(x), p) \leqslant d(f(x), z) + d(z, p) \leqslant d(f(x), V^-) + \delta_1/2 \leqslant \varepsilon_0 + \delta_1/2 \leqslant \delta_1$$

and, by the way we chose δ_1, this inequality implies

$$d(x, f^{-1}(p)) \leqslant \delta.$$

Applying Lemma 5 to x and $f^{-1}(p)$, we have

$$\gamma\, d(x, V^-) \leqslant \gamma\, d(x, W^u_\varepsilon(f^{-1}(p))) \leqslant d(f(x), W^u_\varepsilon(p)) \leqslant d(f(x), z) = d(f(x), V^-).$$

This concludes the proof of the inequalities of Lemma 2 involving $x, f(x)$ and V^-. The proof of those involving $x, f^{-1}(x)$ and V^+ is the same with some obvious modifications.

II. — Proof of Theorem D

To prove Theorem D we shall use all the auxiliary sets and notation of the proof of Theorem I.1. Suppose that $x \notin W^s(\Lambda)$ and $\mu(\Lambda) > 0$ for all $\mu \in \mathcal{M}(x)$. Since $x \notin W^s(\Lambda)$ there exists $N > 0$ such that $f^N(x) \notin V_0$. Obviously $\omega(x) \cap \Lambda \neq \emptyset$. Then there exists $N_1 > N$ such that $f^{N_1}(x) \in V_0$. If $j \geqslant N_1$ define

$$A(j) = \sup\{\, n \mid S_n \cap \{f^N(x), \ldots, f^j(x)\} \neq \emptyset \,\}.$$

Observe that $A(j) < +\infty$ because otherwise we would have:

$$\{f^N(x), \ldots, f^j(x)\} \cap (V^+ \cup V^-) = \bigcap_{n \geqslant 0} (\{f^N(x), \ldots, f^j(x)\} \cap S_n) \neq \emptyset$$

and then either $f^N(x) \in V^-$ (that is impossible since $f^N(x) \notin V_0$) or $f^j(x) \in V^+$ (that is impossible because $x \notin W^s(\Lambda)$). Hence $A(j) < +\infty$ and then in its definition the supremum is actually a maximum. Let $a_1 < a_2 < \ldots$ be the set of values of $A(j)$ for $j \geqslant N_1$. Define

$$\widetilde{m}_n = \max\{\, j \mid A(j) = a_n \,\}.$$

Then $\widetilde{m}_1 < \widetilde{m}_2 < \ldots$. Take a subsequence $\widetilde{m}_{n_1} < \widetilde{m}_{n_2} < \ldots$ such that, denoting $m_k = \widetilde{m}_{n_k}$, the probabilities $\mu(x, m_k)$ converge to $\mu \in \mathcal{M}(x)$. By hypothesis, $\mu(\Lambda) > 0$; hence $\mu(\Lambda_0) > 0$ for some basic component Λ_0 of Λ.

Let us apply Lemma 4 of Section I to $\bar{x} = x$, the isolated hyperbolic set Λ_0 and the sequences $x_k = x$ and $\mu(x, m_k)$, $k \in \mathbf{Z}^+$. If property a) of the conclusion of that lemma holds, then the proof of Theorem I.1 shows that it implies the validity of property I.1 (II). This means that in every neighborhood of f there exists a diffeomorphism g that coincides with f in a neighborhood of Λ and has an homoclinic point associated to Λ_0, thus proving Theorem D. Therefore the proof of Theorem D will be completed if we can show that when property b) of the conclusion of Lemma 4 holds, then property a) also holds. More specifically we shall prove that if for some $k > 0$ and $n > 0$ there exists a k-string $\sigma_1 \subset S_{n+1}$ such that $\sigma \cap (S_n - S_{n+1}) = \emptyset$ for every k-string, then there exist two $(k+1)$-strings $\sigma'_1 < \sigma'_2$ contained in S_{n+1} and such that $\sigma \cap (S_n - S_{n+1}) = \emptyset$ for every $(k+1)$-string $\sigma_1' < \sigma < \sigma_2'$. To prove this observe that since σ_1 is a k-string and is contained in S_{n+1}, we have

$$a_{n_k} = A(\tilde{m}_{n_k}) = A(m_k) \geqslant n+1.$$

Let σ_2 be the first $(k+1)$-string that is not a k-string. The definition of m_k implies that $\sigma_2 \subset S_\ell$ for some $\ell > a_{n_k}$, thus $\ell > n+1$. Then σ_1 and σ_2 are $(k+1)$-strings contained in S_{n+1} and $\sigma_2 > \sigma_1$. Moreover, if there exists a $(k+1)$-string σ such that $\sigma_1 < \sigma < \sigma_2$, then, by our choice of σ_2, necessarily σ is also a k-string and therefore $\sigma \cap (S_n - S_{n+1}) = \emptyset$ by hypothesis. Setting $\sigma_1' = \sigma_1$ and $\sigma_2' = \sigma_2$, the proof of Theorem D is concluded.

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