PUBLICATIONS MATHÉMATIQUES DE L'I.H.É.S.

A. RAMANATHAN

Equations defining Schubert varieties and Frobenius splittings of diagonals

Publications mathématiques de l'I.H.É.S., tome 65 (1987), p. 61-90
<http://www.numdam.org/item?id=PMIHES_1987__65__61_0>

EQUATIONS DEFINING SCHUBERT VARIETIES AND FROBENIUS SPLITTING OF DIAGONALS

by A. RAMANATHAN

Let $G_r(V)$ be the Grassmannian of r-dimensional linear subspaces of a vector space V. We can identify $G_r(V)$ with the decomposable vectors in $\mathbf{P}\wedge^r V$, the projective space of lines in the r-th exterior power of V. Thus $G_r(V)$ is a projective variety with a natural embedding in $\mathbf{P}\wedge^r V$. A basis $e_1, \ldots, e_n$ of V gives the basis $e_{i_1} \wedge \ldots \wedge e_{i_r}$, $i_1 < i_2 < \ldots < i_r$, of $\wedge^r V$. The coordinates of a vector in $\wedge^r V$ with respect to this basis are called its Plücker coordinates. For a vector in $\wedge^r V$ to be decomposable its Plücker coordinates should satisfy certain quadratic relations. For example when $n = 4$ and $r = 2$ (the case of projective lines in projective 3-space) the Grassmannian is actually a quadric surface in $\mathbf{P}^5$. Hodge gave certain natural quadratic relations in the Plücker coordinates which define the Grassmannian in $\mathbf{P}\wedge^r V$ (see [5]). In other words, if I is the homogeneous ideal of $G_r(V)$ in the homogeneous coordinate ring of $\mathbf{P}\wedge^r V$ then I is generated by certain quadrics.

In developing the theory of spinors, Cartan observed that there is a natural bijection between " pure " spinors and the maximal isotropic spaces of the basic quadratic form. He obtained quadratic relations defining pure spinors in the space of all spinors (see [20]). These results are analogous to the results of Hodge when one deals with the orthogonal group instead of the general linear group.

The Grassmannian has a natural decomposition into affine spaces. Let $0 = V_0 \subset V_1 \subset \ldots \subset V_n = V$ be a full flag of linear subspaces with $\dim V_i = i$. Let $1 \leqslant i_1 < i_2 < \ldots < i_r \leqslant n$ be a sequence of integers. Then

$$S = S(i_1, \ldots, i_r) = \{ W \in G_r(V) \mid \dim W \cap V_{i_j} = j \}$$

is isomorphic to an affine space and $G_r(V)$ is the disjoint union of these as $(i_1, \ldots, i_r)$ varies. The Zariski closure $\overline{S}$ of S is called a Schubert variety. Hodge also proved that the homogeneous ideal of $\overline{S}$ in the homogeneous coordinate ring of $G_r(V)$ (in the embedding in $\mathbf{P}\wedge^r V$) is generated by all those Plücker coordinates vanishing on S.

The cohomology group of $G_r(V)$ is generated by the cohomology classes of Schubert varieties and the problems of enumerative geometry studied by Schubert involve the computation of the structure of the cohomology ring of $G_r(V)$ (see [8]).

Hodge also computed the dimensions of the homogeneous components of the homogeneous coordinate rings of Schubert varieties. He proved that $G_r(V)$ and the Schubert varieties are arithmetically normal in the embedding $\mathbf{P}\wedge^r V$. These results suggested the vanishing of higher cohomology groups $H^i(\overline{S}, O(m))$ which was proved independently by several authors including Hochster, Kempf and Musili (see [9], [17]).

This theory for the Grassmannian naturally suggests analogous questions for other flag varieties and more generally for projective homogeneous varieties under algebraic groups.

Let G be a simply connected semisimple algebraic group over an algebraically closed field of arbitrary characteristic. Let B be a Borel subgroup and $Q \supset B$ a parabolic subgroup. A (generalized) Schubert variety in G/Q is the closure of a B-orbit in G/Q. Let L be an effective line bundle on G/Q. Kempf proved the vanishing theorem $H^i(G/Q, L) = 0$, $i > 0$. In his first paper he did this for the case $G = SL(n)$ by using a sequence of nonsingular Schubert varieties $S_0 \subset S_1 \subset \ldots \subset G/Q$, one in each dimension and proving $H^i(S_k, L \mid S_k) = 0$ for each S_k by induction on k. His method was generalized by Seshadri, Musili and Lakshmibai to the other classical groups $SO(n)$ and $Sp(n)$ and to the exceptional group G_2. Later Kempf succeeded in extending this method to prove the result for general G.

In [3] Demazure gave a proof for the vanishing $H^i(S, L \mid S) = 0$ for S any Schubert variety in G/Q, L an effective line bundle on G/Q and $i > 0$, when the base field is of characteristic zero. He also obtained a character formula for the action of the maximal torus on $H^0(S, L \mid S)$ and conjectured that the same results should hold for base fields of arbitrary characteristic. Unfortunately his proof contained a gap which was pointed out by V. Kac much later (in 1983).

Seshadri, Musili and Lakshmibai developed in a series of papers the theory of standard monomials aimed at giving explicit bases for $H^0(G/Q, L)$ and obtaining the vanishing theorem as part of this theory. They succeeded in doing this for all the classical groups G (or more generally when G is arbitrary but Q is restricted to be of classical type). For classical groups the theory of standard monomials also enables them to prove that the homogeneous ideal of G/Q in any embedding given by a very ample line bundle is generated by quadrics and that the homogeneous ideal of any Schubert variety in the homogeneous coordinate ring of G/Q is generated by linear homogeneous polynomials vanishing on it. They conjectured this result to be valid for general G/Q (see [9]). The main result of this paper proves their conjecture (over fields of arbitrary characteristic).

In another line of development Haboush [21] working over a base field of characteristic $p > 0$ gave a short and beautiful proof of the vanishing theorem $H^i(G/Q, L) = 0$ of Kempf, by exploiting the Frobenius morphism and Steinberg representations in characteristic p. Later Andersen also used characteristic p methods to prove Kempf's theorem. It should be noted that thanks to Chevalley the group G and the varieties G/Q can be defined over $\mathbf{Z}$ and one can use semicontinuity theorems to conclude vanishing theorems in characteristic 0 from the corresponding theorem in characteristic p. So proving these results in characteristic p in fact proves them for all fields.

Our method in this paper is also to work over fields of positive characteristic. We introduced in [12] the notion of Frobenius split varieties (see below). It turns out, greatly simplifying life with Schubert varieties, that Schubert varieties are compatibly Frobenius split in G/Q. A systematic use of this fact has given the cohomology vanishing theorems, projective normality and arithmetic Cohen-Macaulay property for Schubert varieties (see [12], [15], [16]). For our present purpose also Frobenius splitting proves no less useful.

For a variety X over a base field k of characteristic $p > 0$ we have the absolute Frobenius morphism $F : X \to X$ given by the ring homomorphism $a \mapsto a^p$ of any k-algebra A. The variety X is called *Frobenius split* if the p-th power map $\mathcal{O}_X \to F_* \mathcal{O}_X$ has a section (a $\mathcal{O}_X$-module homomorphism) $\varphi : F_* \mathcal{O}_X \to \mathcal{O}_X$. A closed subvariety $Y \subset X$ is called *compatibly Frobenius split* in X if $\varphi(F_* I) = I$ where I is the ideal sheaf of Y in X, The point of this definition is the trivially proved fact that if X is a projective variety. L an ample line bundle on X and $Y \subset X$ compatibly Frobenius split then $H^i(X, L) = H^i(Y, L) = 0$ for $i > 0$ and the restriction map $H^0(X, L) \to H^0(Y, L)$ is surjective.

Now suppose that the diagonal Δ in $X \times X$ is compatibly Frobenius split. Then for any ample line bundle L on X, $L^m \times L$ is ample on $X \times X$ and hence $H^0(X \times X, L^m \times L) \to H^0(\Delta, L^m \times L)$ is surjective. But this restriction map is isomorphic to the multiplication map $H^0(X, L^m) \otimes H^0(X, L) \to H^0(X, L^{m+1})$, by Kunneth. It follows that the map $H^0(X, L)^{\otimes m} \to H^0(X, L^m)$ is surjective. Thus if X is normal and Δ is split in $X \times X$ then X is projectively normal in the embedding given by any ample L.

Let $X_r = X \times \ldots \times X$, r factors, L_r the line bundle $L \times \ldots \times L$ on X_r and $\Delta_r \simeq X$ the diagonal in X_r. Towards proving that X is defined by quadrics in the embedding given by L one would like to have that the kernel K_3 of the restriction map $H^0(X, L)^{\otimes 3} \simeq H^0(X_3, L_3) \to H^0(\Delta_3, L^3)$ is the sum $K_2 \otimes H^0(X, L) + H^0(X, L) \otimes K_2$ where K_2 is the kernel of $H^0(X, L)^{\otimes 2} \simeq H^0(X_2, L_2) \to H^0(\Delta_2, L^2)$.

Now $K_j = H^0(X_j, I_j \otimes L_j)$ where I_j is the ideal sheaf of Δ_j in X_j, $j = 2, 3$. Let I_{12} be the ideal sheaf of $\Delta_{12} = \Delta_2 \times X$ in X_3 and I_{23} that of $X \times \Delta_2$. Then $K_2 \otimes H^0(X, L) = H^0(X_3, I_{12} \otimes L_3)$ and $H^0(X, L) \otimes K_2 = H^0(X_3, I_{23} \otimes L_3)$. Further $I_{12} + I_{23}$ is the ideal sheaf I_3 of Δ_3 in X_3, $I_{12} \cap I_{23}$ that of $\Delta_{12} \cup \Delta_{23}$ and we have the Mayer-Vietoris sequence

$$0 \to I_{12} \cap I_{23} \to I_{12} \oplus I_{23} \to I_{12} + I_{23} \to 0.$$

Tensoring this with L_3 and taking cohomology we see that what we wish to prove is the surjectivity of $H^0(X_3, I_{12} \otimes L_3) \oplus H^0(X_3, I_{23} \otimes L_3) \to H^0(X_3, I_3 \otimes L_3)$. The surjectivity follows if $H^1(X_3, I_{12} \cap I_{23} \otimes L_3) = 0$. Now using the exact sequence

$$0 \to I_{12} \cap I_{23} \to \mathcal{O}_{X_3} \to \mathcal{O}_{\Delta_{12} \cup \Delta_{23}} \to 0$$

we need only prove $H^1(X_3, L_3) = 0$ and the surjectivity of $H^0(X_3, L_3) \to H^0(\Delta_{12} \cup \Delta_{23}, L_3)$. But this will be so if $\Delta_{12} \cup \Delta_{23}$ is Frobenius split in X_3.

The above argument gives the essential point in the proof of the result that if X is a projective variety such that (a) the diagonal Δ_2 in X_2 is compatibly Frobenius split and (b) $\Delta_{12} \cup \Delta_{23}$ is compatibly Frobenius split in X_3 then in any projective embedding of X given by an ample line bundle the homogeneous ideal of X is generated by quadrics.

Now suppose that Y is a Cartier divisor in X. Then from

$$0 \to \mathcal{O}_X(-Y) \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$$

we see that K_m, the kernel of $H^0(X, L^m) \to H^0(Y, L^m)$, is $H^0(X, L^m(-Y))$. To prove that Y is defined by linear equations in X in the embedding given by L we should show that the multiplication map $H^0(X, L^m) \otimes H^0(X, L) \to H^0(X, L^{m+1})$ maps $K_m \otimes H^0(X, L)$ onto K_{m+1}. That is $H^0(X_2, L^m(-Y) \times L) \to H^0(\Delta_2, L^m(-Y) \times L)$ should be surjective. So we need $H^1(X_2, I_2 \otimes L^m(-Y) \times L) = 0$.

Since Y is a Cartier divisor $I_2 \otimes \mathcal{O}_{X_2}(-Y \times X)$ is isomorphic to the ideal sheaf of $\Delta_2 \cup Y \times X$. Tensoring the exact sequence

$$0 \to I_2 \otimes \mathcal{O}_{X_2}(-Y \times X) \to \mathcal{O}_{X \times X} \to \mathcal{O}_{\Delta_2 \cup Y \times X} \to 0$$

by $L^m \times L$ we see that the required surjectivity follows if $\Delta_2 \cup Y \times X$ is compatibly Frobenius split in X_2.

Thus to have generation by quadrics or linear functions one has criteria in terms of compatible Frobenius splitting of diagonal subvarieties of $X \times X$ and $X \times X \times X$. To check the conditions of these criteria for Schubert varieties one proceeds as follows.

One knows from [12] that G/B has a Frobenius splitting φ which compatibly splits all the Schubert varieties in G/B. If we take the product splitting $\varphi \times \varphi$ on $G/B \times G/B$ it will compatibly split the factor $G/B \times 0$ since the point Schubert variety 0 is compatibly split in G/B. If we could find an automorphism of $G/B \times G/B$ which pulled the factor into the diagonal we would be through. One cannot quite do this. However the unipotent radical $\tilde{U}$ of the opposite Borel subgroup is embedded as an open subset of G/B and we do have such an automorphism α of $\tilde{U} \times \tilde{U}$, given by $\alpha(x, y) = (x, y^{-1}x)$. One then shows that $\alpha^{-1}(\varphi \times \varphi)$, which is a splitting of the diagonal in $\tilde{U} \times \tilde{U}$, extends to the whole of $G/B \times G/B$. Similarly one uses the automorphism $(x, y, z) \mapsto (x, y^{-1}x, z^{-1}y)$ of $\tilde{U} \times \tilde{U} \times \tilde{U}$ to prove that $\Delta_{12} \cup \Delta_{23}$ is compatibly split in $G/B \times G/B \times G/B$.

One then uses semicontinuity to go from characteristic p to characteristic 0.

The result on the linear definition of Schubert varieties in G/Q together with the fact (proved in [16]) that the intersection of any set of Schubert varieties is reduced can be used to study the singularities of Schubert varieties. See [10], where for the classical groups standard monomial theory is invoked to give these results. I owe this remark to Seshadri.

The results of this paper have been announced in [15] and [16].

It seems that the methods of this paper could be applied to get the relations and higher syzygies too.

See the last section for further remarks.

1. Preliminaries on Frobenius splitting

Throughout this section the base field k will be an algebraically closed field of characteristic $p > 0$. By a variety we mean a reduced but *not necessarily irreducible* scheme over k.

Let X be a variety over k and $F : X \to X$ the absolute Frobenius morphism. F is the identity on the underlying set of X and on functions it is the p-th power map. We also use the same letter F to denote the p-th power map $F : \mathcal{O}_X \to F_* \mathcal{O}_X$. Note that for any coherent sheaf $\mathcal{G}$ on X the direct image $F_* \mathcal{G}$ is the same as $\mathcal{G}$ as an abelian group; only its $\mathcal{O}_X$-module structure changes to $f \circ g = f^p g$, $f \in \mathcal{O}_X$, $g \in \mathcal{G}$.

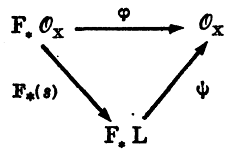
We give below some basic results on Frobenius splitting. Most of it is essentially contained in [12], [15] and [16]. We have given them here in a more explicit form.

1.1. *Definition* (Cf. [12], Definitions 2, 3). — *a)* We call X *Frobenius split* if the p-th power map $F : \mathcal{O}_X \to F_* \mathcal{O}_X$ has a splitting i.e. an $\mathcal{O}_X$-module morphism $\varphi : F_* \mathcal{O}_X \to \mathcal{O}_X$ such that the composite $\varphi F : \mathcal{O}_X \to F_* \mathcal{O}_X \to \mathcal{O}_X$ is the identity.

b) If Y is a closed subvariety of X with ideal sheaf I such that $\varphi(F_* I) = I$ then we say that Y is *compatibly split* in X.

c) If $Y_1, \ldots, Y_r$ are closed subvarieties of X which are all compatibly split by the same Frobenius splitting of X then we say that $Y_1, \ldots, Y_r$ are *simultaneously compatibly split* in X.

1.2. *Definition* (Cf. [15], Lemma 1). — *a)* Let L be a line bundle on X and $s : \mathcal{O}_X \to L$ be a nonzero section of L with zeroes precisely on the divisor D. We call X *Frobenius* D-*split* (or less precisely *Frobenius* L-*split*) if there exists $\psi : F_* L \to \mathcal{O}_X$ such that the composite $\varphi = \psi F_*(s)$



is a Frobenius splitting of X.

b) If Y is a closed subvariety of X such that (i) no irreducible component of Y is contained in supp D and (ii) φ gives a compatible splitting of Y in X then we say that Y is *compatibly* D-*split* (or *compatibly* L-*split*) in X by ψ.

c) If all the closed subvarieties $Y_1, \ldots, Y_r$ are compatibly D-split by the same D-splitting of X then we say that $Y_1, \ldots, Y_r$ are *simultaneously compatibly* D-*split* in X.

1.3. *Remarks.* — (i) If X is a scheme and $F : \mathcal{O}_X \to F_* \mathcal{O}_X$ has a splitting then X is necessarily reduced. For, if X is not reduced we can find a function f on an open subset of X such that for some $\nu \geqslant 1$, $f^{p^{\nu-1}} \neq 0$ and $f^{p^\nu} = 0$. This implies that $F(f^{p^{\nu-1}}) = 0$ so that F is not injective. Hence F then cannot have a splitting.

(ii) Suppose D′ is another (Cartier) divisor such that $0 \leqslant D' \leqslant D$. Then if X is D-split it is also D′-split. Let L′ be the line bundle $\mathcal{O}_X(D')$ and $s' : \mathcal{O}_X \to L'$ the section corresponding to D′. The divisor $D - D'$ gives a section $\mathcal{O}_X \to L \otimes L'^{-1}$ and hence a map $\eta : L' \to L$ and we have the commutative diagram

$$\begin{array}{ccc} \mathcal{O}_X & \xrightarrow{\ s\ } & L \\ & {\scriptstyle s'}\searrow \quad \nearrow{\scriptstyle \eta} & \\ & L' & \end{array}$$

It follows that $\psi F_* \eta$ gives a D′-splitting of X.

(iii) Consider the composite $F_*(s)\, F : \mathcal{O}_X \to F_* \mathcal{O}_X \to F_* L$. The L-splitting ψ gives a splitting of this since $\psi F_*(s)\, F = \varphi F$ which is the identity by assumption.

Since I is the ideal sheaf of Y we have an exact sequence $0 \to I \otimes L \to L \to L \mid Y \to 0$. Moreover, since $F_*(s)\, F$ maps I into $F_*(I \otimes L)$ we have the commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_Y & \longrightarrow & 0 \\ & & \downarrow & & {\scriptstyle \psi}\uparrow\downarrow{\scriptstyle F(s)F} & & \downarrow & & \\ 0 & \longrightarrow & F_*(I \otimes L) & \longrightarrow & F_* L & \longrightarrow & F_*(L \mid Y) & \longrightarrow & 0 \end{array}$$

Note that the last vertical map $\mathcal{O}_Y \to F_*(L \mid Y)$ induced by $F(s)\, F$ is also the map $F_Y(s \mid Y)\, F_Y$ where F_Y is the Frobenius of Y.

1.4. *Proposition.* — *If* Y *is compatibly* L*-split in* X *by* ψ *then in the above diagram* $\psi(F_*(I \otimes L)) = I$ *and* ψ *goes down to* $\bar{\psi} : F_*(L \mid Y) \to \mathcal{O}_Y$ *giving a* $(L \mid Y)$*-splitting of* Y.

Proof. — Let $J = \psi(F_*(I \otimes L))$. Then J is an ideal sheaf and since ψ is a splitting J contains I. We only have to prove that $J = I$; the other assertions of the proposition follow easily from this. Let U be the dense open set X — supp D. Then $s \mid U$ is an isomorphism and by the condition (ii) of Definition 1.2, $\psi F_* s \mid U$ is a compatible splitting of $Y \cap U$ in U. This implies that $J \mid U = I \mid U$. Since, by the condition (i) of Definition 1.2, U meets all the irreducible components of Y this implies that $J = I$ by the following lemma (which is merely an algebraic formulation of the fact that any function $g \in J$, since it vanishes on the dense open subset $Y \cap U$ of Y, should also vanish on the whole of Y and thus belong to I).

1.5. *Lemma.* — *Let* $J \supset I$ *be ideals in a noetherian ring* R. *Suppose that* $I_f = J_f$ *in the localisation* R_f *for some* f *not contained in any associated prime of* I. *Then* $J = I$.

Proof. — Since $I_f = J_f$, for any $x \in J$ there is an n such that $f^n x \in I$. Since f does not belong to any associated prime of I this implies that $x \in I$.

1.6. *Remark.* — Note that the condition (i) of Definition 1.2 is essential for the proof of Proposition 1.4.

1.7. *Proposition* (Cf. [12], Lemma 1). — *Let* U *be a dense open subset of the variety* X *which intersects nontrivially all the irreducible components of the closed subvariety* Y. *Let* $L = \mathcal{O}_X(D)$ *and* $\psi : F_* L \to \mathcal{O}_X$ *be such that* $\psi \mid U$ *is a compatible* $D \mid U$*-splitting of* $Y \cap U$ *in* U. *Then* ψ *is a compatible* D*-splitting of* Y *in* X.

Proof. — By assumption the composite $\psi F_*(s)\, F : \mathcal{O}_X \to \mathcal{O}_X$ is the identity on U and hence it must be the identity on the whole of X. Therefore ψ gives a D-splitting of X. Let I be the ideal sheaf of Y in X and $J = \psi F_*(s)(I)$. Then $J \supset I$ and $J \mid U = I \mid U$. By Lemma 1.5 this implies that $J = I$. Moreover since no irreducible component of $Y \cap U$ is contained in $\operatorname{supp}(D \mid U)$ and U intersects all the irreducible components of Y none of the latter can be contained in supp D. Therefore ψ gives a compatible D-splitting of Y in X.

1.8. *Proposition* (Cf. [12], Proposition 4). — *Let* $f : Z \to X$ *be a proper morphism of varieties such that* $f_* \mathcal{O}_Z = \mathcal{O}_X$. *Let* $D \geqslant 0$ *be a (Cartier) divisor in* X *and* Y *a closed subvariety of* Z *such that no irreducible component of* $f(Y)$ *is contained in* supp D. *If* $\psi : F_* f^* \mathcal{O}_X(D) \to \mathcal{O}_Z$ *is a compatible* f^* D*-splitting of* Y *in* Z, *then* $f_* \psi$ *gives a compatible* D*-splitting of* $f(Y)$ *in* X.

Proof. — Since the Frobenius morphism commutes with any morphism we have $f_* F_* f^* \mathcal{O}_X(D) = F_* f_* f^* \mathcal{O}_X(D)$. By the projection formula, since $f_* \mathcal{O}_Z = \mathcal{O}_X$, we have $f_* f^* \mathcal{O}_X(D) = \mathcal{O}_X(D)$. Therefore $f_* \psi$ is a map from $F_* \mathcal{O}_X(D)$ to $\mathcal{O}_X$.

Let I be the ideal sheaf of Y in X. Using $f_* \mathcal{O}_Z = \mathcal{O}_X$ it is easy to see that $f_* I$ is the ideal of $f(Y)$ in X (Cf. [12], Lemma 2). It is then straightforward to check that $f_* \psi$ gives a compatible D-splitting of $f(Y)$ in X.

1.9. *Proposition.* — *Let* Y_1, Y_2 *be closed subvarieties of* X. *Suppose that there is a Frobenius* D*-splitting* $\psi : F_* \mathcal{O}_X(D) \to \mathcal{O}_X$ *of* X *which gives a compatible* D*-splitting of both* Y_1 *and* Y_2. *Then* ψ *gives a compatible* D*-splitting of any irreducible component of* Y_1 *or* Y_2 *and the union* $Y_1 \cup Y_2$ *(taken with the reduced subscheme structure).*

Moreover, if $s : \mathcal{O}_X \to \mathcal{O}_X(D)$ *is the section corresponding to* D *then* $\psi F_*(s)$ *gives a compatible splitting of the scheme theoretic intersection* $Y_1 \cap Y_2$ *in* X *so that if no irreducible component of* $Y_1 \cap Y_2$ *is contained in* supp D *then* ψ *gives a compatible* D*-splitting of* $Y_1 \cap Y_2$ *in* X.

Proof. — Let Y′ be an irreducible component of Y_1 and U the complement in X of the union of all the other irreducible components of Y_1. Then clearly $\psi \mid U$ gives a compatible $D \mid U$-splitting of $Y' \cap U$. By Proposition 1.7 Y′ is compatibly D-split X.

Let I_1, I_2 be the ideal sheaves of Y_1, Y_2 respectively. Then the ideal sheaf of

$Y_1 \cup Y_2$ is $I_1 \cap I_2$ and that of $Y_1 \cap Y_2$ is $I_1 + I_2$. Since $\varphi = \psi F_*(s)$ is a compatible splitting of Y_1 and Y_2 we have $\varphi(F_* I_i) = I_i$, $i = 1, 2$. Therefore $\varphi(F_*(I_1 \cap I_2)) = I_1 \cap I_2$ and $\varphi(F_*(I_1 + I_2)) = I_1 + I_2$ (note that if we consider φ merely as a morphism of sheaves of abelian groups, ignoring the $\mathcal{O}_X$-module structure, then we can forget the F_* in front of $I_1 \cap I_2$ etc.). From this the other assertions of the proposition follow at once.

1.10. *Corollary (Cf.* [16], *Section* 2*). — If* Y_1 *and* Y_2 *are simultaneously compatibly split subvarieties of* X *then their scheme theoretic intersection is reduced.*

Proof. — This follows from the above proposition and Remark 1.3 (i).

1.11. *Corollary. — Suppose that* $Y_1, \ldots, Y_r$ *are simultaneously compatibly split subvarieties of* X. *Then, if* Y *is a subvariety obtained from the* Y_i*'s by repeatedly taking unions, intersections and irreducible components then* Y *is also compatibly split in* X, *simultaneously with the* Y_i*'s. If further the* Y_i*'s are simultaneously compatibly* D*-split in* X *and no irreducible component of* Y *is contained in* supp D *then* Y *is also compatibly* D*-split in* X, *simultaneously with the* Y_i*'s.*

1.12. *Proposition (Cf.* [12], *Proposition* 3; [15], *Lemma* 1*). — Let* X *be a Frobenius* L*-split projective variety and* M *a line bundle on* X. *Then*

(i) *If for some* $\nu \geqslant 1$, $H^i(X, L^{p^{\nu-1}+p^{\nu-2}+\cdots+1} \otimes M^{p^\nu}) = 0$ *then* $H^i(X, M) = 0$.

(ii) *If* Y *is a compatibly* L*-split subvariety of* X *and for some* $\nu \geqslant 1$ *the restriction map* $H^0(X, L^{p^{\nu-1}+p^{\nu-2}+\cdots+1} \otimes M^{p^\nu}) \to H^0(Y, L^{p^{\nu-1}+p^{\nu-2}+\cdots+1} \otimes M^{p^\nu})$ *is surjective then* $H^0(X, M) \to H^0(Y, M)$ *is surjective.*

Proof. — (i) We use the notation of Definition 1.2. The map $F_*(s)\, F : \mathcal{O}_X \to F_* L$ has a splitting (Remark 1.3 (iii)). Hence, by tensoring with M and taking cohomology, the map $H^i(X, M) \to H^i(X, M \otimes F_* L)$ is injective. Now by the projection formula $F_*(L \otimes F^* M) = M \otimes F_* L$. Moreover F being affine it commutes with cohomology. Therefore we have $H^i(X, M \otimes F_* L) = H^i(X, L \otimes F^* M)$. But $F^* M = M^p$ (Cf. [12], Section 1). Thus we have for any line bundle M an injection $H^i(X, M) \to H^i(X, L \otimes M^p)$. Replacing M by $L \otimes M^p$ we have an injection $H^i(X, L \otimes M^p) \to H^i(X, L^{p+1} \otimes M^{p^2})$. Iterating this process and composing all the resulting injections we get an injection $H^i(X, M) \to H^i(X, L^{p^{\nu-1}+\cdots+1} \otimes M^{p^\nu})$ for every $\nu \geqslant 1$. Therefore if the latter is zero for some $\nu \geqslant 1$ so is $H^i(X, M)$.

(ii) Since Y is compatibly L-split we have a commutative diagram (Cf. Remark 1.3 (iii) and Proposition 1.4):

$$\begin{array}{ccc} \mathcal{O}_X & \rightleftarrows & F_*(L) \\ \downarrow & & \downarrow \\ \mathcal{O}_Y & \rightleftarrows & F_*(L/Y) \end{array}$$

Tensoring with M and taking global sections we get

$$\begin{array}{ccc} H^0(X, M) & \rightleftarrows & H^0(X, L \otimes M^p) \\ \downarrow & & \downarrow \\ H^0(Y, M) & \rightleftarrows & H^0(Y, L \otimes M^p) \end{array}$$

It follows that if $H^0(X, L \otimes M^p) \to H^0(Y, L \otimes M^p)$ is surjective then so is $H^0(X, M) \to H^0(Y, M)$. Replacing M by $L \otimes M^p, \ldots, L^{p^{\nu-1}+\cdots+1} \otimes M^{p^\nu}$ in turn we see that if $H^0(X, L^{p^{\nu-1}+\cdots+1} \otimes M^{p^\nu}) \to H^0(Y, L^{p^{\nu-1}+\cdots+1} \otimes M^{p^\nu})$ is surjective for some $\nu \geqslant 1$ so is $H^0(X, M) \to H^0(Y, M)$.

1.13. *Proposition (Cf.* [12], *Section* 1; [15], *Section* 2*). — Let* X *be a projective variety,* Y *a closed subvariety and* M *a line bundle on* X. *Then*

(i) *If* X *is Frobenius split and* M *is an ample line bundle on* X *then* $H^i(X, M) = 0$ *for* $i > 0$. *If further* Y *is compatibly split then* $H^i(Y, M) = 0$ *for* $i > 0$ *and the restriction map* $H^0(X, M) \to H^0(Y, M)$ *is surjective.*

(ii) *If* X *is Frobenius* L*-split with* L *ample and* M *is a line bundle without base points (i.e. for every* $x \in M$ *there is an* $s \in H^0(X, M)$ *such that* $s(x) \neq 0$*) then* $H^i(X, M) = 0$ *for* $i > 0$. *If further* Y *is compatibly* L*-split then* $H^i(Y, M) = 0$ *for* $i > 0$ *and the restriction map* $H^0(X, M) \to H^0(Y, M)$ *is surjective.*

Proof. — This follows immediately from Proposition 1.12. To prove (i) take $\mathcal{O}_X$ for L in Proposition 1.12 and note that for an ample M, $H^i(X, M^r) = 0$ for $i > 0$ and $r \gg 0$ and $H^0(X, M^r) \to H^0(Y, M^r)$ is surjective for $r \gg 0$. To prove (ii) note that if L is ample and M is without base points then $L^r \otimes M^s$ is ample for all $r, s \geqslant 1$. Therefore by (i) just proved $H^i(X, L^r \otimes M^s) = 0$ for $i > 0$ and $H^0(X, L^r \otimes M^s) \to H^0(Y, L^r \otimes M^s)$ is surjective for $r, s \geqslant 1$. Now use part (ii) of Proposition 1.12.

1.14. Now we recall the duality theorem for finite morphisms ([4], Exercises III 6.10 and 7.2). Let $f: X \to Y$ be a finite morphism of schemes and $\mathcal{F}$, $\mathcal{G}$ be coherent sheaves on X, Y respectively. Since f is affine $Y = \operatorname{Spec} f_* \mathcal{O}_X$ and a coherent sheaf on X is equivalent to a coherent sheaf of $f_* \mathcal{O}_X$-modules on Y. Let $f^! \mathcal{G}$ be the coherent sheaf on X given by $\mathcal{H}om_Y(f_* \mathcal{O}_X, \mathcal{G})$ where $f_* \mathcal{O}_X$ acts on the first factor $f_* \mathcal{O}_X$ by multiplication. Then it follows easily that the natural map

$$f_* \mathcal{H}om_X(\mathcal{F}, f^! \mathcal{G}) \to \mathcal{H}om_Y(f_* \mathcal{F}, \mathcal{G})$$

sending $\eta \in \mathcal{H}om_X(\mathcal{F}, \mathcal{H}om_Y(f_* \mathcal{O}_X, \mathcal{G}))$ to the morphism $\tilde{\eta} : f_* \mathcal{F} \to \mathcal{G}$ defined by $\tilde{\eta}(s) = \eta(s)(1)$ where $s \in f_* \mathcal{F}$, is an isomorphism. This is the duality for f.

1.15. Suppose further that X and Y are smooth projective varieties of the same dimension n. Let K_X, K_Y be the canonical line bunles of X, Y respectively. Then by using Serre duality for Y and the relative duality for $f: X \to Y$ it follows that for any locally free sheaf V on X we have

$$\mathrm{Hom}(V, f^! K_Y) = \mathrm{Hom}(f_* V, K_Y) = H^n(Y, f_* V)^* = H^n(X, V)^*.$$

Thus $f^! K_Y$ satisfies the characterising property for the dualising sheaf of X and hence must be isomorphic to K_X ([4], Chapter III, § 7). By applying the duality for f to $\mathit{Hom}(f^* K_Y, f^! K_Y)$ this then gives $f^! \mathcal{O}_Y = K_X \otimes f^* K_Y^{-1}$.

1.16. Let Z be a smooth projective variety and K its canonical line bundle. Let L be a line bundle on X. Then for the Frobenius morphism $F: Z \to Z$ we have $F^! \mathcal{O}_Z = K \otimes F^* K^{-1} = K \otimes K^{-p} = K^{1-p}$. The duality for F gives

$$F_* \mathit{Hom}(L, F^! \mathcal{O}_Z) = \mathit{Hom}(F_* L, \mathcal{O}_Z);$$

using the above expression for $F^! \mathcal{O}_Z$ we get $F_*(L^* \otimes K^{1-p}) \approx (F_* L)^*$.

Therefore a section $\sigma \in H^0(Z, K^{1-p} \otimes \mathcal{O}_Z(-D))$ gives by duality $\tilde{\sigma}: F_* \mathcal{O}_Z(D) \to \mathcal{O}_Z$. For $\tilde{\sigma}$ to be a D-splitting of Z we further need that the composite $\tilde{\sigma} F_*(s)$, where $s: \mathcal{O}_Z \to \mathcal{O}_Z(D)$ is the section determined by D, is the identity. Note that Z being complete any map $\mathcal{O}_Z \to \mathcal{O}_Z$ is constant hence it is enough if $\tilde{\sigma} F(s)$ is nonzero at a single point of Z. Moreover using the inclusion $K^{1-p} \otimes \mathcal{O}(-D) \to K^{1-p}$ we can think of σ as a section of K^{1-p} vanishing on the divisor D.

Thus to give a D-splitting of Z we should look for a section σ of K^{1-p} vanishing on D and such that $\tilde{\sigma} F(s)$ is nonzero. Of course if Z is not Frobenius split $\tilde{\sigma} F$ will always be zero for any $\sigma \in H^0(Z, K^{1-p})$.

If Z is Frobenius split then it follows that it is K^{1-p}-split, since any section of K^{1-p} which gives a splitting vanishes on a divisor whose associated line bundle is K^{1-p}. However, since not all the sections of K^{1-p} give rise to a splitting, if D is a divisor belonging to K^{1-p} we cannot say that Z is D-split. This is true only for an open subset of such D.

We recall the following criterion for compatible splitting from [12].

1.17. *Proposition. — Let Z be a smooth projective variety of dimension n. Let $Z_1, \ldots, Z_n$ be smooth irreducible subvarieties of codimension* 1 *such that the scheme theoretic intersection $Z_{i_1} \cap \ldots \cap Z_{i_r}$ is smooth irreducible and of dimension $n - r$ for all $1 \leqslant i_j < \ldots < i_r \leqslant n$. If there exists a section $s \in H^0(Z, K^{-1})$ such that* div s, *the divisor of zeroes of s, is $Z_1 + \ldots + Z_n + D$ where D is an effective divisor not passing through the point $P = Z_1 \cap \ldots \cap Z_n$ then the section $\sigma = s^{p-1} \in H^0(Z, K^{1-p})$ gives, by duality* (§ 1.16), *a Frobenius $(p - 1)$ D-splitting of Z which makes all the intersections $Z_{i_1} \cap \ldots \cap Z_{i_r}$ compatibility $(p - 1)$ D-split in Z.*

We sketch below a proof somewhat different from the one given in [12]. We need he following two simple lemmas.

1.18. *Lemma.* — *Let* $X = \operatorname{Spec} R$ *be an affine scheme over* k. *Let* $a \in R$ *and* $\sigma \in F^! \mathcal{O}_X$. *Then* $a^{p-1}\sigma$, *considered as a map* $F_* \mathcal{O}_X \to \mathcal{O}_X$, *takes* $F_*(Ra)$ *into* Ra.

Proof. — Let $ab \in Ra$. Then $a^{p-1}\sigma(ab) = \sigma(a^p b)$ by the definition of the $\mathcal{O}_X$-multiplication in $F^! \mathcal{O}_X = \operatorname{Hom}(F_* \mathcal{O}_X, \mathcal{O}_X)$, see § 1.14. But $\sigma(a^p b)$ is $a(\sigma(b))$ by the definition of the $\mathcal{O}_X$-module structure on $F_* \mathcal{O}_X$.

1.19. *Lemma.* — *Let* $X = \operatorname{Spec} k[[t]]$ *where* $k[[t]]$ *is the power series ring in one variable. Then* $F^! \mathcal{O}_X$ *is locally free of rank* 1. *If* $\sigma \in F^! \mathcal{O}_X$ *is of order* $p - 1$ (*i.e.* σ *is in* $t^{p-1} F^! \mathcal{O}_X$ *but not in* $t^p F^! \mathcal{O}_X$), *considering* σ *as a map* $F_* \mathcal{O}_X \to \mathcal{O}_X$, *the composite* $\sigma F : \mathcal{O}_X \to F_* \mathcal{O}_X \to \mathcal{O}_X$ *is nonzero.*

Proof. — $F_* k[[t]]$ is free over $k[[t]]$ with basis $1, t, \ldots, t^{p-1}$. Let $\sigma_0 : F_* k[[t]] \to k[[t]]$ be the map defined by $\sigma_0(t^i) = 0$ for $0 \leqslant i \leqslant p - 2$ and $\sigma_0(t^{p-1}) = 1$. Then it is easy to see that $F^! \mathcal{O}_X$ is free with basis σ_0. Moreover if $n = ap + b$, $a, b \in \mathbf{N}$, $b \leqslant p - 1$ it follows that $\sigma_0(t^n) = 0$ if $b \neq p - 1$ and t^a if $b = p - 1$. Since by assumption $\sigma = t^{p-1} h\sigma_0$ where h is a unit it is then easy to see that the constant term of $\sigma(1)$ is $h(0)$. This prove the lemma.

To prove the above proposition let $\varphi : F_* \mathcal{O}_Z \to \mathcal{O}_Z$ be the morphism corresponding to σ under duality. Let $Q \in Z_1 \cap \ldots \cap Z_r$ and $Q \notin Z_i$ for $i > r$. To show that $\varphi(F_* I) = I$ where I is the ideal of $Z_1 \cap \ldots \cap Z_r$ in Z it is enough to show that $\hat{\varphi}(F_* \hat{I}) = \hat{I}$ where $\hat{}$ denotes the corresponding objects in the completion at Q. (Use Proposition 1.6 and faithful flatness of completion.) It is easy to see that the duality for the Frobenius morphism commutes with completion and hence $\hat{\varphi}$ is obtained from $\hat{\sigma}$ by duality. Choose local parameters $x_1, \ldots, x_r, \ldots, x_n$ in $\mathcal{O}_Q$ such that $\hat{Z}_i$ is defined by x_i, $i = 1, \ldots, r$. Then we can write $\hat{\sigma}$ as $(x_1 \ldots x_r)^{p-1} h/(dx_1 \ldots dx_n)^{p-1}$ where h is a unit in $\hat{\mathcal{O}}_Q$. It is now easy to deduce using Lemma 1.18 that $\hat{\varphi}(F_* \hat{I}) = \hat{I}$.

We will prove that $\varphi F : \mathcal{O}_Z \to F_* \mathcal{O}_Z \to \mathcal{O}_Z$ is nonzero at P. In $\hat{\mathcal{O}}_P$ we have as above $\hat{\sigma} = (x_1 \ldots x_n)^{p-1} h/(dx_1 \ldots dx_n)^{p-1}$. Since duality commutes with products we can reduce to the one variable case. Then apply Lemma 1.19.

1.20. *Proposition.* — *Let* Z *be a smooth projective variety and* $s \in H^0(X, K_Z^{-1})$ *be such that the section* $\sigma = s^{p-1}$ *of* K_Z^{1-p} *gives a Frobenius splitting of* Z (*under the duality of* § 1.16). *Then if* Y *is a prime divisor of* Z *on which* s *vanishes with multiplicity* 1 *then* σ *gives a compatible splitting of* Y *in* Z.

Proof. — As in the proof of Proposition 1.17 we can reduce to working in the completion $\hat{\mathcal{O}}_Q$ of the local ring at a general point of Y. Then Y is defined by a parameter and the compatible splitting follows from Lemma 1.18.

2. Consequences of Frobenius splitting of diagonals

Let X be a projective variety over the algebraically closed base field k of characteristic $p > 0$. Let $X \subset \mathbf{P}^n$ be a projective embedding given by a very ample line bundle on X. We shall show in this section how the compatible splitting of the diagonal Δ in $X \times X$ and the partial diagonals $\Delta \times X$ and $X \times \Delta$ in $X \times X \times X$ give a good hold on the homogeneous polynomials vanishing on X.

We shall prove some criteria in terms of Frobenius splitting for the projective normality of X, for X to be defined by quadrics and for subvarieties of X to be defined by linear equations.

The only application we make of these criteria will be to Schubert varieties.

2.1. Let M, N be line bundles on X. Then we have the natural multiplication map $\mu : H^0(X, M) \otimes H^0(X, N) \to H^0(X, M \otimes N)$ sending $s \otimes t$ to the section of $M \otimes N$ whose value at $x \in X$ is $s(x) \otimes t(x)$. The key fact for us is that this map can be obtained as a restriction map as follows. Let $M \times N$ be the line bundle $q_1^* M \otimes q_2^* N$ on $X \times X$ where q_i are the projections. Let Δ be the diagonal $\{ (x, x) \in X \times X \mid x \in X \}$ in $X \times X$. Then the map $H^0(X, M) \otimes H^0(X, N) \to H^0(X \times X, M \times N)$ sending $s \otimes t$ to the section of $M \times N$ whose value at (x, y) is $s(x) \otimes t(y)$ is an isomorphism. Further $M \times N \mid \Delta$ is naturally isomorphic to $M \otimes N$ on $X \simeq \Delta$ and we have the commutative diagram

$$\begin{array}{ccc} H^0(X, M) \otimes H^0(X, N) & \xrightarrow{\simeq} & H^0(X \times X, M \times N) \\ \mu \downarrow & & \downarrow \text{restrict} \\ H^0(X, M \otimes N) & \xrightarrow{\simeq} & H^0(\Delta, M \times N) \end{array}$$

We also have as above a natural multiplication map $H^0(X, M)^{\otimes m} \to H^0(X, M^m)$ for every integer $m \geqslant 0$. This map goes down to the symmetric power

$$\nu_m : S^m H^0(X, M) \to H^0(X, M^m).$$

2.2. *Proposition.* — *Let* X *be a projective variety and* M, N *line bundles on* X. *Then the natural multiplication maps* $\mu(M, N) : H^0(X, M) \otimes H^0(X, N) \to H^0(X, M \otimes N)$ *and* $\nu_m(M) : S^m H^0(X, M) \to H^0(X, M^m)$ *are surjective in the following two cases*:

(i) M, N *are ample and the diagonal* Δ *is compatibly split in* $X \times X$.

(ii) M, N *are without base points and* Δ *is compatibly* $X \times D$*-split in* $X \times X$ *where* D *is an ample divisor on* X.

Proof. — We will first deal with $\mu(M, N)$. As explained above $\mu(M, N)$ is isomorphic to the restriction map $H^0(X \times X, M \times N) \to H^0(\Delta, M \times N)$ with which we will identify it.

Case (i). — $\mu(M, N)$ is surjective by Proposition 1.13 part (i).

Case (ii). — Let $L = \mathcal{O}_X(D)$. By Proposition 1.12 part (ii) it is sufficient to prove that $\mu(M^r, N^r \otimes L^s) : H^0(X \times X, M^r \times (N^r \otimes L^s)) \to H^0(\Delta, M^r \times (N^r \otimes L^s))$ is surjective for $r, s \geqslant 1$. Since N is without base points and L is ample $N^r \otimes L^s$ is ample. Therefore if M is ample then $M^r \times (N^r \otimes L^s)$ is ample on $X \times X$. Thus if M is ample then $\mu(M^r, N^r \otimes L^s)$, and hence $\mu(M, N)$, are surjective. Interchanging M and N we see that $\mu(M, N)$ is surjective if N is ample. Applying this to M^r, $N^r \otimes L^s$ we see that $\mu(M^r, N^r \otimes L^s)$ is surjective since $N^r \otimes L^s$ is ample. But this we know implies that $\mu(M, N)$ is surjective.

To prove the surjectivity of $\nu_m(M)$ we use induction on M. We have the commutative diagram

$$\begin{array}{ccc} S^{m-1} H^0(X, M) \otimes H^0(X, M) & \xrightarrow{\nu_{m-1}(M) \otimes \mathrm{id}} & H^0(X, M^{m-1}) \otimes H^0(X, M) \\ \downarrow & & \downarrow \mu(M^{m-1}, M) \\ S^m H^0(X, M) & \xrightarrow[\nu_m(M)]{} & H^0(X, M^m) \end{array}$$

The top arrow is surjective by induction. $\mu(M^{m-1}, M)$ is surjective by what we have proved above. Hence it follows that $\nu_m(M)$ is surjective.

2.3. *Corollary.* — *Let* X *be a normal projective variety such that the diagonal in* $X \times X$ *is compatibly split. Then any very ample line bundle on* X *embeds* X *as a projectively normal variety.*

Proof. — For the projective embedding $X \subset \mathbf{P}H^0(X, M)^* = \mathbf{P}^N$, given by a very ample line bundle M, we have $S^m H^0(X, M) = H^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(m))$ and we have the commutative diagram

$$\begin{array}{ccc} S^m H^0(X, M) & \longrightarrow & H^0(X, M^m) \\ \| \downarrow & & \downarrow \| \\ H^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(m)) & \xrightarrow[\text{restrict}]{} & H^0(X, \mathcal{O}_{\mathbf{P}^N}(m)) \end{array}$$

Therefore, by the above proposition the restriction map $H^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(m)) \to H^0(X, \mathcal{O}_{\mathbf{P}^N}(m))$ is surjective for every $m \geqslant 0$. This together with the normality of X implies the projective normality of X in the projective embedding given by M ([4], Exercise II, 5.14 (d)).

2.4. *Remark.* — It follows from Proposition 2.2 that if the diagonal is compatibly split in $X \times X$ then any ample line bundle on X must be very ample. For, it is easy to see that $S^m H^0(X, M) \to H^0(X, M^m)$ surjective for large m already implies that M is very ample. See [14].

2.5. Let M_1, M_2, M_3 be line bundles on X. Consider the multiplication map $\mu : H^0(X, M_1) \otimes H^0(X, M_2) \otimes H^0(X, M_3) \to H^0(X, M_1 \otimes M_2 \otimes M_3)$.

As in § 2.1 this is the restriction map for the line bundle $M_1 \times M_2 \times M_3$ on $X \times X \times X$ to the diagonal. We also have the partial multiplication maps

$$\mu_{ij} : \bigotimes_{r=1}^{3} H^0(X, M_r) \to H^0(X, M_i \otimes M_j) \otimes H^0(X, M_k)$$

which multiplies the i-th and j-th factors and is the identity on the remaining k-th factor. It is clear that μ_{ij} is the restriction map for $M_1 \times M_2 \times M_3$ on $X \times X \times X$ to the partial diagonal $\Delta_{ij} = \{ (x_1, x_2, x_3) \mid x_i = x_j \} \simeq X \times X$. If

$$\mu_k : H^0(X, M_i \otimes M_j) \otimes H^0(X, M_k) \to H^0(X, M_1 \otimes M_2 \otimes M_3)$$

is the multiplication map then $\mu = \mu_k \, \mu_{ij}$.

We then have the following proposition regarding the kernel of μ. Note that the essential case of this proposition is when $X = Z$ (see below). We need the slight generalisation (obtained at the cost of a mild contortion in the proof) for applying to Schubert varieties embedded in G/B.

2.6. *Proposition.* — *Let* X *be embedded as a closed subvariety in a projective variety* Z. *Let* M_1, M_2, M_3 *be line bundles on* Z. *Then the kernel* K *of the multiplication map* $\mu : \bigotimes_{r=1}^{3} H^0(X, M_r) \to H^0(X, M_1 \otimes M_2 \otimes M_3)$ *is the sum* $K_{12} + K_{23}$ *where* K_{ij} *is the kernel of the partial multiplication* μ_{ij} *(see* § 2.5 *above) if* (a) *all the line bundles* M_i *are ample on* Z, (b) X *is compatibly split in* Z *and* (c) *the subvarieties* $\Delta'_{12} = \{ (x, x, z) \mid x \in X, z \in Z \}$ *and* $\Delta'_{23} = \{ (x, z, z) \mid x \in X, z \in Z \}$ *of* $Z \times Z \times Z$ *are simultaneously compatibly split in* $Z \times Z \times Z$.

Proof. — Let $Z_3 = Z \times Z \times Z$, $X_3 = X \times X \times X$ and $M = M_1 \times M_2 \times M_3$. We have the commutative diagram

$$\begin{array}{ccc} H^0(Z_3, M) & \longrightarrow & H^0(Z, M_1 \otimes M_2 \otimes M_3) \\ \downarrow & \searrow^{\mu'} & \downarrow \\ H^0(X_3, M) & \xrightarrow{\ \mu\ } & H^0(X, M_1 \otimes M_2 \otimes M_3) \end{array}$$

where the top horizontal arrow is the restriction from Z_3 to the diagonal Z in Z_3 and μ' is defined to be the composite of this with further restriction to X. Since X is split in Z and M_i are ample $H^0(Z, M_i) \to H^0(X, M_i)$ is surjective. Therefore under the restriction $H^0(Z_3, M) \to H^0(X_3, M)$ the kernel K' of μ' maps surjectively onto K.

Similarly let $\mu'_{12} : H^0(Z_3, M) \to H^0(X, M_1 \otimes M_2) \otimes H^0(Z, M_3)$ be the multiplication of the first two factors followed by restriction to X and the identity on $H^0(Z, M_3)$ and K'_{12} be its kernel. Then evidently K'_{12} maps into K_{12} under $H^0(Z_3, M) \to H^0(X_3, M)$. In the same way the kernel K'_{23} of the analogous map

$$\mu'_{23} : H^0(Z_3, M) \to H^0(X, M_1) \otimes H^0(Z, M_2 \otimes M_3)$$

maps into K_{23}. Therefore it is enough to show that $K' = K'_{12} + K'_{23}$.

Let I_{ij} be the ideal sheaf of Δ'_{ij} in Z_3. Tensoring the exact sequence $0 \to I_{ij} \to \mathcal{O}_{Z_3} \to \mathcal{O}_{\Delta'_{ij}} \to 0$ with M and taking global sections we see that

$$K'_{ij} = H^0(Z_3, I_{ij} \otimes M).$$

We have the Mayer-Vietoris sequence

$$0 \to I_{12} \cap I_{23} \to I_{12} \oplus I_{23} \to I_{12} + I_{23} \to 0$$

where $a \in I_{12} \cap I_{23}$ goes to $(a, -a)$ in $I_{12} \oplus I_{23}$ and $(a, b) \in I_{12} \oplus I_{23}$ goes to $a + b$ in $I_{12} + I_{23}$. The ideal sheaf $I = I_{12} + I_{23}$ is clearly the ideal sheaf of the diagonal $\Delta = \{ (x, x, x) \mid x \in X \}$ in Z_3 (use (c) and Corollary 1.10, if needed!) and $J = I_{12} \cap I_{23}$ that of $\Delta'_{12} \cup \Delta'_{23}$. Therefore $H^0(Z_3, I \otimes M) = K'$.

Tensoring the above Mayer-Vietoris sequence by M and taking cohomology we get

$$0 \to H^0(Z_3, J \otimes M) \to K'_{12} \oplus K'_{23} \to K' \to H^1(Z_3, J \otimes M).$$

Thus it is enough to prove $H^1(Z_3, J \otimes M) = 0$.

Now tensoring the exact sequence $0 \to J \to \mathcal{O}_{Z_3} \to \mathcal{O}_{\Delta'_{12} \cup \Delta'_{23}} \to 0$ with M and taking cohomology we get

$$H^0(Z_3, M) \to H^0(\Delta'_{12} \cup \Delta'_{23}, M) \to H^1(Z_3, J \otimes M) \to H^1(Z_3, M).$$

Since M is ample and $\Delta'_{12} \cup \Delta'_{23}$ is compatibly split in Z_3 (by (c) and Proposition 1.9) we have $H^1(Z_3, M) = 0$ and $H^0(Z_3, M) \to H^0(\Delta'_{12} \cup \Delta'_{23}, M)$ is surjective. Hence from the above exact sequence it follows that $H^1(Z_3, J \otimes M) = 0$ which proves $K' = K'_{12} + K'_{23}$. This completes the proof of the proposition.

2.7. *Proposition. — Let* X *be embedded as a closed subvariety of a projective variety* Z. *Let* L *be an ample line bundle on* Z. *Suppose that* (a) X *is compatibly split in* Z, (b) *the diagonal* Z *in* $Z \times Z$ *is compatibly split and* (c) *the subvarieties* $\Delta'_{12} = \{ (x, x, z) \mid x \in X, z \in Z \}$ *and* $\Delta'_{23} = \{ (x, z, z) \mid x \in X, z \in Z \}$ *of* $Z \times Z \times Z$ *are compatibility split in* $Z \times Z \times Z$. *Then the natural graded algebra homomorphism* $\bigoplus_{m=0}^{\infty} S^m H^0(X, L) \to \bigoplus_{m=0}^{\infty} H^0(X, L^m)$ *is surjective and its kernel is generated as an ideal by the kernel of* $S^2 H^0(X, L) \to H^0(X, L^2)$.

Proof. — By (a) $H^0(Z, L^m) \to H^0(X, L^m)$ is surjective (Proposition 1.13 (i)) and by (b) $S^m H^0(Z, L) \to H^0(Z, L^m)$ is surjective (Proposition 2.2 (i)). It follows that $S^m H^0(X, L) \to H^0(X, L^m)$ is surjective.

To simplify the writing out of the proof we introduce the following notation. Let $V = H^0(X, L)$, $V^m = V \otimes \ldots \otimes V$, m factors and $W_m = H^0(X, L^m)$. We have the natural multiplication $\mu_m : V^m \to W_m$. Let

$$\varphi_i = \mu_i \otimes \mathrm{id}_{V^{m-i}} : V^m = V^i \otimes V^{m-i} \to W_i \otimes V^{m-i}, \quad 0 \leqslant i \leqslant m.$$

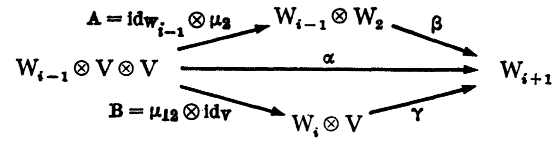
Note that $\varphi_0 = \mathrm{id}_{V^m}$ and $\varphi_m = \mu_m$. We claim

(*) $$\ker \varphi_{i+1} = \ker \varphi_i + V^{i-1} \otimes K \otimes V^{m-i-1}$$

where K is the kernel of $\mu_2 : V^2 \to W_2$. Since $\varphi_{i+1} = \mu_{i+1} \otimes \mathrm{id}_{V^{m-i-1}}$ we have $\ker \varphi_{i+1} = (\ker \mu_{i+1}) \otimes V^{m-i-1}$. Therefore to prove (*) it is enough to prove

$$(**) \qquad \ker \mu_{i+1} = (\ker \mu_i) \otimes V + V^{i-1} \otimes K.$$

Consider the commutative diagram



where the maps are all obvious multiplication maps. Note that $\mu_{i+1} = \gamma B(\mu_{i-1} \times \mathrm{id}_{V \times V})$. Therefore

$$(1) \qquad \ker \mu_{i+1} = (\mu_{i-1} \otimes \mathrm{id}_{V \otimes V})^{-1} B^{-1}(\ker \gamma).$$

We will now find $\ker \gamma$. Since B is surjective (by Proposition 2.2) and $\alpha = \gamma B$ we have $\ker \gamma = B(\ker \alpha)$. By Proposition 2.6 above $\ker \alpha = \ker A + \ker B$. Therefore $\ker \gamma = B(\ker A)$. Since $A = \mathrm{id}_{W_{i-1}} \otimes \mu_2$, $\ker A = W_{i-1} \otimes K$. Thus we have $\ker \gamma = B(W_{i-1} \otimes K)$. Using this in (1) we get

$$(2) \qquad \ker \mu_{i+1} = (\mu_{i-1} \otimes \mathrm{id}_{V \otimes V})^{-1} (\ker B + W_{i-1} \otimes K).$$

Since $B(\mu_{i-1} \otimes \mathrm{id}_{V \otimes V}) = \mu_i \otimes \mathrm{id}_V$ we have

$$(\mu_{i-1} \otimes \mathrm{id}_{V \otimes V})^{-1} (\ker B) = \ker(\mu_i \otimes \mathrm{id}_V).$$

Using this in (2) we get $\ker \mu_{i+1} = (\ker \mu_i) \otimes V + V^{i-1} \otimes K$ which proves (**) and hence (*).

Using (*) inductively it follows that $\ker \varphi_{i+1} = \sum_{j=1}^{i} V^{j-1} \otimes K \otimes V^{m-j-1}$. From this it is clear that K generates the kernel of $\bigoplus S^m H^0(X, L) \to \bigoplus H^0(X, L^m)$ as an ideal.

2.8. *Remark.* — Proposition 2.7 says that a projective variety X satisfying the conditions of the proposition is defined by quadrics in any projective embedding given by an ample line bundle (from Z). In fact it says that even the cone over X is defined scheme theoretically by quadrics (not as a complete intersection, of course). It can happen that the zero quadric is the only one vanishing on X. For e.g. $X = \mathbf{P}^n$ and $L = \mathcal{O}_{\mathbf{P}^n}(1)$.

Before stating the next result we give some definitions and lemmas.

2.9. *Definition* (Kempf [6], p. 567). — Let $f: X \to Y$ be a morphism of schemes. We call f *trivial* if the natural map $\mathcal{O}_Y \to f_* \mathcal{O}_X$ is surjective and the higher direct images $R^i f_* \mathcal{O}_X$ vanish for $i > 0$. The image of a trivial morphism f is the closed subscheme of Y having $f_* \mathcal{O}_X$ as structure sheaf. We denote the image scheme by $f(X)$. It is the scheme theoretic image of f.

2.10. *Lemma.* — *Let* $f: X \to Y$ *be a morphism of the projective varieties* X *and* Y. *If* f *is trivial then for any locally free sheaf* L *on* Y, $H^i(X, f^* L) = H^i(Y, L)$.

Proof. — This follows from the Leray spectral sequence for f and the projection formula.

The following basic lemma is due to Kempf. We will need it only in the next section.

2.11. *Lemma (Kempf).* — *Let* $f: X \to Y$ *be a proper morphism of algebraic schemes and* $X' \subset X$ *a closed subscheme. Let* $Y' \subset Y$ *be the scheme theoretic image of* X' *and* L *an ample line bundle on* Y. *Suppose that the following conditions hold:*

a) $f_* \mathcal{O}_X = \mathcal{O}_Y$,

b) $H^q(X, f^* L^n) = 0 = H^q(X', f^* L^n) = 0$ *for* $q > 0$, $n \gg 0$ *and*

c) $H^0(X, f^* L^n) \to H^0(X', f^* L^n)$ *is surjective for* $n \gg 0$. *Then* f *is trivial with image* Y *and the restriction* $f': X' \to Y$ *of* f *to* X' *is also trivial with image* Y'.

Proof. — This follows from the Leray spectral sequence for f (see [3], Proposition 2 in § 5).

2.12. *Lemma.* — *Let* $f: \tilde{X} \to X$ *be a proper, trivial morphism of the algebraic varieties* $\tilde{X}$, X *with image* X. *Let* $\tilde{Y}$ *be a closed subvariety of* $\tilde{X}$ *with ideal sheaf* $\tilde{I} \subset \mathcal{O}_{\tilde{X}}$. *Then* $f_* \tilde{I}$ *is the ideal sheaf* I *of all functions vanishing on the image* $f(\tilde{Y}) = Y$. *Further* $f \mid \tilde{Y}: \tilde{Y} \to X$ *is trivial with image* Y *if and only if* $R^i f_* \tilde{I} = 0$ *for* $i > 0$.

Proof. — $f_* \tilde{I} = I$ follows from $f_* \mathcal{O}_{\tilde{X}} = \mathcal{O}_X$ and the definition of direct image (Cf. [12], Lemma 2). The last assertion follows from the long exact sequence obtained by applying f_* to $0 \to \tilde{I} \to \mathcal{O}_{\tilde{X}} \to \mathcal{O}_{\tilde{Y}} \to 0$. (Cf. [16], Lemma 4.)

2.13. *Lemma.* — *Let* $f: \tilde{X} \to X$ *be a proper, trivial morphism with image* X. *Let* $\tilde{Y}_1$, $\tilde{Y}_2$ *be closed subvarieties of* $\tilde{X}$ *such that the restriction of* f *to* $\tilde{Y}_1$, $\tilde{Y}_2$ *and the scheme theoretic intersection* $\tilde{Y}_1 \cap \tilde{Y}_2$ *are also trivial with images, respectively,* Y_1, Y_2 *and* $Y_1 \cap Y_2$. *Then* $f \mid \tilde{Y}_1 \cup \tilde{Y}_2$ *is trivial with image* $Y_1 \cup Y_2$.

Proof. — Let $\tilde{I}_1$, $\tilde{I}_2$ be the ideal sheaves of $\tilde{Y}_1$, $\tilde{Y}_2$ respectively. We have the Mayer-Vietoris sequence

$$0 \to \tilde{I}_1 \cap \tilde{I}_2 \to \tilde{I}_1 \oplus \tilde{I}_2 \to \tilde{I}_1 + \tilde{I}_2 \to 0.$$

The lemma follows from the long exact sequence obtained by applying f_* to this and Lemma 2.12.

2.14. *Lemma.* — *Let* R *be a noetherian ring and* I *an ideal in* R. *Let* $J = Ra$ *be the principal ideal generated by a non-zero-divisor* $a \in R$ *such that* a *does not belong to any associated prime of* I. *Then the natural multiplication map* $I \otimes J \to I \cap J$ *is an isomorphism.*

Proof. — It follows easily that under these assumptions $IJ = I \cap J$. Since J is a free R-module tensoring $I \to R$ by J gives an injection $I \otimes J \to J$ with image IJ.

2.15. *Lemma.* — *Let* Y_1, Y_2, Y_3 *be closed subschemes of* X *such that* $Y_2 \subset Y_3$ *and the scheme theoretic intersections (given by the sum of the corresponding ideal sheaves)* $Y_1 \cap Y_2$ *and* $Y_1 \cap Y_3$ *are equal. Let* L *be a line bundle on* X. *If the restriction* $H^0(Y_3, L) \to H^0(Y_2, L)$ *is surjective then the restriction* $H^0(Y_1 \cup Y_3, L) \to H^0(Y_1 \cup Y_2, L)$ *is surjective (where the unions are given by the intersection of the corresponding ideals).*

Proof. — If I_1, I_2 are ideals in a ring R we have the Mayer-Vietoris exact sequence

$$0 \to R/I_1 \cap I_2 \to R/I_1 \oplus R/I_2 \to R/(I_1 + I_2) \to 0$$

where $a \to (a, -a)$ and $(a, b) \to a + b$.

This globalises to give an exact sequence

$$0 \to \mathcal{O}_{Y_1 \cup Y_2} \to \mathcal{O}_{Y_1} \oplus \mathcal{O}_{Y_2} \to \mathcal{O}_{Y_1 \cap Y_2} \to 0$$

and a similar one for Y_1 and Y_3. Tensoring these with L and taking sections we get the following commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(Y_1 \cup Y_3, L) & \longrightarrow & H^0(Y_1, L) \oplus H^0(Y_3, L) & \longrightarrow & H^0(Y_1 \cap Y_3, L) \\ & & \downarrow & & \downarrow & & \downarrow \| \\ 0 & \longrightarrow & H^0(Y_1 \cup Y_2, L) & \longrightarrow & H^0(Y_1, L) \oplus H^0(Y_2, L) & \longrightarrow & H^0(Y_1 \cap Y_2, L) \end{array}$$

where the vertical maps are given by restriction from Y_3 to Y_2. The lemma follows immediately from this diagram.

2.16. *Lemma.* — *Let* Y *be a closed subscheme of* X *with ideal sheaf* I *and* L *a line bundle on* X. *Then the restriction map* $H^0(X, L) \to H^0(Y, L)$ *is surjective if* $H^1(X, I \otimes L) = 0$. *If* $H^1(X, L) = 0$ *then conversely the surjectivity of* $H^0(X, L) \to H^0(Y, L)$ *implies that* $H^1(X, I \otimes L) = 0$.

Proof. — Tensoring $0 \to I \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$ with L and taking cohomology we get the exact sequence $H^0(X, L) \to H^0(Y, L) \to H^1(X, I \otimes L) \to H^1(X, L)$ from which the lemma follows.

2.17. *Definition.* — Let $Y \subset X$ be a codimension 1 closed subvariety. We say that $Y \subset X$ admits a *trivial resolution* if there exist a proper morphism $f : \tilde{X} \to X$ with $\tilde{X}$ smooth and a smooth closed subvariety $\tilde{Y} \subset \tilde{X}$ of codimension 1 such that *a)* f is trivial with image X and *b)* $f \mid \tilde{Y}$ is trivial with image Y.

2.18. *Definition.* — Let $Y \subset X$ be a closed subvariety and L a line bundle on X. We say that Y is *linearly defined* in X with respect to L if the natural restriction map $\bigoplus_{m=0}^{\infty} H^0(X, L^m) \to \bigoplus_{m=0}^{\infty} H^0(Y, L^m)$ is surjective and its kernel is generated as an ideal by the kernel of $H^0(X, L) \to H^0(Y, L)$.

2.19. *Proposition. — Let* X *be embedded as a closed subvariety in a projective variety* Z. *Let* Y *be a closed subvariety of codimension* 1 *in* X. *Let* L *be an ample line bundle on* Z. *Suppose that*

a) $Y \subset X$ *admits a trivial resolution (see Definition* 2.17*)*
b) Y *is compatibly split in* X *and* X *is compatibly split in* Z *and*
c) *the subvarieties* $\Delta = \{ (x, x) \in Z \times Z \mid x \in X \}$ *and* $Y \times Z$ *are simultaneously compatibly split in* $Z \times Z$.

Then Y *is linearly defined in* X *with respect to* L *(see Definition* 2.18*).*

Proof. — Since Y is compatibly split in X and L is ample on X the surjectivity of $\bigoplus H^0(X, L^m) \to \bigoplus H^0(Y, L^m)$ follows from Proposition 1.13 (i).

Let K_m be the kernel of $H^0(X, L^m) \to H^0(Y, L^m)$. By induction we assume that K_{m-1} is in the ideal generated by K_1. It is then enough to prove that the natural multiplication map $H^0(X, L^{m-1}) \otimes H^0(X, L) \to H^0(X, L^m)$ maps $K_{m-1} \otimes H^0(X, L)$ surjectively onto K_m.

Let $f: \widetilde{X} \to X$, $\widetilde{Y} \subset \widetilde{X}$ be a trivial resolution for $Y \subset X$. By Lemma 2.10 $H^0(X, L^m) = H^0(\widetilde{X}, L^m)$ and $H^0(Y, L^m) = H^0(\widetilde{Y}, L^m)$ where we have denoted $f^* L^m$ again by L^m. Therefore K_m is also the kernel of the restriction map $H^0(\widetilde{X}, L^m) \to H^0(\widetilde{Y}, L^m)$. Since $\widetilde{Y}$ is a divisor in the smooth variety $\widetilde{X}$ the ideal sheaf of $\widetilde{Y}$ in $\widetilde{X}$ is the line bundle $\mathcal{O}_{\widetilde{X}}(-\widetilde{Y})$ and we have the exact sequence $0 \to \mathcal{O}_{\widetilde{X}}(-\widetilde{Y}) \to \mathcal{O}_{\widetilde{X}} \to \mathcal{O}_{\widetilde{Y}} \to 0$. Tensoring this with L^m and taking global sections we see that $K_m = H^0(\widetilde{X}, L^m(-\widetilde{Y}))$.

As explained in § 2.1 the multiplication map $H^0(\widetilde{X}, L^{m-1}) \otimes H^0(\widetilde{X}, L) \to H^0(\widetilde{X}, L^m)$ is the same as the restriction map $H^0(\widetilde{X} \times \widetilde{X}, L^{m-1} \times L) \to H^0(\widetilde{\Delta}, L^{m-1} \times L)$ where $\Delta \subset \widetilde{X} \times \widetilde{X}$ is the diagonal. Hence we have a commutative diagram

$$\begin{array}{ccc} K_{m-1} \otimes H^0(\widetilde{X}, L) & \longrightarrow & K_m \\ \downarrow & & \downarrow \\ H^0(\widetilde{X} \times \widetilde{X}, L^{m-1}(-\widetilde{Y}) \times L) & \longrightarrow & H^0(\widetilde{\Delta}, L^m(-\widetilde{Y})) \end{array}$$

Therefore it is enough to show that the restriction map

$$H^0(\widetilde{X} \times \widetilde{X}, L^{m-1}(-\widetilde{Y}) \times L) \to H^0(\widetilde{\Delta}, L^m(-\widetilde{Y}))$$

is surjective. By Lemma 2.16, we need $H^1(\widetilde{X} \times \widetilde{X}, I_{\widetilde{\Delta}} \otimes (L^{m-1}(-\widetilde{Y}) \times L)) = 0$. Now $L^{m-1}(-\widetilde{Y}) \times L = (L^{m-1} \times L)(-\widetilde{Y} \times \widetilde{X})$ and by Lemma 2.14 $I_{\widetilde{\Delta}}(-\widetilde{Y} \times \widetilde{X})$ is the product $I_{\widetilde{\Delta}} . I_{\widetilde{Y} \times \widetilde{X}}$ which is the ideal sheaf J of $\widetilde{\Delta} \cup \widetilde{Y} \times \widetilde{X}$ in $\widetilde{X} \times \widetilde{X}$. Therefore we are reduced to proving $H^1(\widetilde{X} \times \widetilde{X}, J \otimes (L^{m-1} \times L)) = 0$. Since $L^{m-1} \times L$ is ample on the split variety $X \times X$ $H^1(\widetilde{X} \times \widetilde{X}, L^{m-1} \times L) = H^1(X \times X, L^{m-1} \times L) = 0$. Therefore by Lemma 2.16 it is enough to prove that

$$H^0(\widetilde{X} \times \widetilde{X}, L^{m-1} \times L) \to H^0(\widetilde{\Delta} \cup \widetilde{Y} \times \widetilde{X}, L^{m-1} \times L)$$

is surjective. By Lemma 2.13 $f \times f \mid \widetilde{\Delta} \cup \widetilde{Y} \times \widetilde{X}$ is trivial with image $\Delta \cup Y \times X$ where Δ is the diagonal in $X \times X$. (Note that since $\widetilde{Y}$, $\widetilde{X}$ are smooth the scheme theoretic intersection $\widetilde{\Delta} \cap \widetilde{Y} \times \widetilde{X}$ is the diagonal in $\widetilde{Y} \times \widetilde{Y} \subset \widetilde{X} \times \widetilde{X}$ with reduced structure.) Therefore by Lemma 2.10 we have only to prove that

$$H^0(X \times X, L^{m-1} \times L) \to H^0(\Delta \cup Y \times X, L^{m-1} \times L)$$

is surjective.

Having made this reduction we go over to Z. We have the commutative diagram of restriction maps

$$\begin{array}{ccc} H^0(Z \times Z, L^{m-1} \times L) & \twoheadrightarrow & H^0(\Delta \cup Y \times Z, L^{m-1} \times L) \\ \downarrow & & \downarrow \\ H^0(X \times X, L^{m-1} \times L) & \longrightarrow & H^0(\Delta \cup Y \times X, L^{m-1} \times L) \end{array}$$

Since by assumption (c) (and Proposition 1.9) $\Delta \cup Y \times Z$ is compatibly split in $Z \times Z$ the top horizontal arrow is surjective. Hence it is enough to prove that $H^0(\Delta \cup Y \times Z, L^{m-1} \times L) \to H^0(\Delta \cup Y \times X, L^{m-1} \times L)$ is surjective. But this last map is surjective by Lemma 2.15 since $\Delta \cap Y \times Z = \Delta \cap Y \times X = \Delta_Y$, the diagonal of Y. This completes the proof of the proposition.

2.20. *Lemma.* — *Let* $Y_1 \subset Y_2$ *be closed subvarieties of the projective variety* X *and* L *a line bundle on* X. *Then*

(i) *If* Y_1 *is linearly defined in* Y_2 *and* Y_2 *is linearly defined in* X *with respect to* L *then so is* Y_1 *in* X.

(ii) *If* Y_1 *and* Y_2 *are linearly defined in* X *with respect to* L *then* Y_1 *is linearly defined in* Y_2 *with respect to* L.

Proof. — We have a commutative diagram of restriction maps

$$\begin{array}{ccc} \bigoplus H^0(X, L^m) & \xrightarrow{\alpha} & \bigoplus H^0(Y_2, L^m) \\ & \beta \searrow \quad \swarrow \gamma & \\ & \bigoplus H^0(Y_1, L^m) & \end{array}$$

(i) Since α and γ are surjective β is surjective. Let $f \in \ker \beta$. Since Y_1 is linearly defined in Y_2 we can write $\alpha(f) = \Sigma \alpha(a_i)\, \alpha(\ell_i)$ where $\ell_i \in H^0(X, L)$ with $\alpha(\ell_i) \in \ker \beta$. Therefore $f = \Sigma a_i \ell_i + g$ with $g \in \ker \alpha$. Since Y_2 is linearly defined $g = \Sigma b_j m_j$ where $m_j \in H^0(X, L) \cap \ker \alpha$. Therefore $f = \Sigma a_i \ell_i + \Sigma b_j m_j$. Since $\ell_i \in H^0(X, L) \cap \ker \alpha$ also this shows Y_1 is linearly defined in X.

(ii) Since β is surjective so is γ. Let $\bar{f} \in \ker \gamma$. Since α is surjective we can find f such that $\alpha(f) = \bar{f}$. Then $f \in \ker \beta$. Since Y_1 is linearly defined in X we can write $f = \Sigma a_i \ell_i$, $\ell_i \in H^0(X, L) \cap \ker \beta$. Applying α gives $\bar{f} = \Sigma \alpha(a_i)\, \alpha(\ell_i)$. Since $\alpha(\ell_i) \in H^0(Y_2, L) \cap \ker \beta$ this shows that Y_1 is linearly defined in Y_2.

3. Frobenius splitting of diagonals for Schubert varieties

3.1. Let G be a connected simply connected semisimple algebraic group over an algebraically closed field of *arbitrary characteristic*. Let T be a maximal torus, $B \supset T$ a Borel subgroup, $Q \supseteq B$ a parabolic subgroup and $W = N(T)/T$ the Weyl group.

The homogeneous space G/Q is a projective variety. There are only finitely many B-orbits in G/Q. The closure of a B-orbit in G/Q is called a Schubert variety in G/Q. Note that G/Q itself is a Schubert variety since there is a dense B-orbit in G/Q. When $G = SL(n)$ and Q is a maximal parabolic subgroup this notion coincides with that of the classical Schubert varieties in Grassmannians.

We have a bijection of W with the set of Schubert varieties in G/B given by $\omega \mapsto X_\omega = \overline{B\omega B}$.

An excellent source for basic facts about the geometry of Schubert varieties is Kempf's paper [7].

3.2. We recall from Kempf [6] the construction of *standard resolutions* for X_ω (see also [3], [16]).

Let $\omega = s_{\alpha_1} \dots s_{\alpha_r}$ be a reduced expression for ω, where s_{α_i} is the reflection with respect to a simple root α_i. Let $\omega_i = s_{\alpha_1} \dots s_{\alpha_i}$, $1 \leqslant i \leqslant r$, and $X_i = X_{\omega_i}$. Since $\ell(\omega_i)$, the length of ω_i, is i we have $\dim X_i = i$. Since $\ell(\omega_i s_{\alpha_i}) = \ell(\omega_i) - 1$, X_{i-1} is of codimension 1 in X_i and under the map $\pi_i : G/B \to G/P_{\alpha_i}$ (where $P_{\alpha_i} = B \cup Bs_{\alpha_i}B$ is the minimal parabolic subgroup corresponding to α_i) X_i is saturated and X_{i-1} maps birationally onto $\pi_i(X_{i-1}) = \pi_i(X_i)$. (See [6] and [16], Section 1.)

The *standard resolution* $\psi_i : Z_i \to X_i$ is defined inductively. Let $Z_0 = X_0$, a point and $\psi_0 : Z_0 \to X_0$ the identity. Then ψ_i is defined by the pull back diagram

$$\begin{array}{ccccc} Z_i & & \xrightarrow{\quad\psi_i\quad} & & X_i \\ \sigma_i \uparrow\downarrow f_i & & & & \downarrow \pi_i \\ Z_{i-1} & \xrightarrow[\psi_{i-1}]{} & X_{i-1} & \xrightarrow[\pi_i]{} & \pi_i(X_{i-1}) \end{array}$$

Since $\pi_i : X_i \to \pi_i(X_{i-1}) = \pi_i(X_i)$ is a $\mathbf{P}^1$-bundle so is f_i. The section σ_i is defined by the inclusion $X_{i-1} \subset X_i$.

3.3. It is easy to see that the relative canonical bundle $K_{Z_i/Z_{i-1}}$ is $\mathcal{O}_{Z_i}(-\sigma_i(Z_{i-1})) \otimes (L^{-1} \otimes f_i^* \sigma_i^* L)$ where L is any line bundle on Z_i with degree 1 along the fibres of f_i ([16], Lemma 3, Section 1). Using this inductively it follows that the canonical bundle K_{Z_i} of Z_i is $\mathcal{O}_{Z_i}(-\partial Z_i) \otimes \psi_i^* L_\rho^{-1}$ where ∂Z_i is the divisor (with normal crossings) $\sum_{j=1}^{i} Z_{ij}$, $Z_{ij} = f_i^{-1} \dots f_{j+1}^{-1}(\sigma_j(Z_{j-1}))$ and L_ρ is the line bundle $\mathcal{O}_{G/B}(\partial(G/B))$ corresponding to the divisor $\partial(G/B)$ which is the sum of all Schubert varieties of codimension 1 in G/B ([16], Proposition 2, Section 1).

3.4. For any parabolic subgroup $Q \supset T$ we denote by $\tilde{Q}$ the opposite parabolic subgroup $\omega_N Q \omega_N^{-1}$, where ω_N is the element of maximal length in W. Let U_Q be the unipotent radical of Q and $\tilde{U}_Q$ that of $\tilde{Q}$. The orbit map $U_Q \to G/Q$ given by $u \mapsto u\omega_N Q$ identifies U_Q as an open subset of G/Q. The complement $G/Q - U_Q$ is an ample divisor D_Q whose components are the codimension 1 Schubert varieties in G/Q. Similarly $\tilde{U}_Q \to G/Q$, $\tilde{u} \mapsto \tilde{u}Q$, is an isomorphism onto its image and $G/Q - \tilde{U}_Q$ is the divisor $\tilde{D}_Q$ which is the translate $\omega_N D_Q$. Since G is a rational variety $\tilde{D}_Q$ is linearly equivalent to D_Q. We denote D_B, $\tilde{D}_B$, U_B and $\tilde{U}_B$ by D, $\tilde{D}$, U and $\tilde{U}$ for simplicity of notation.

Parts (ii) and (iii) of the following theorem are the main results on Frobenius splitting for Schubert varieties in this paper. Part (i) is proved in [12], [15] and [16].

3.5. *Theorem.* — *Let the base field k be of characteristic $p > 0$. Let the notation be as above* (§ 3.4).

(i) *The homogeneous space G/Q over k is Frobenius $(p-1)(D_Q + \tilde{D}_Q)$-split. All Schubert varieties in G/Q are simultaneously compatibly $(p-1)\tilde{D}_Q$-split in G/Q.*

(ii) *The diagonal Δ in $G/Q \times G/Q$ is compatibly $(p-1)(q_1^{-1} D_Q + q_2^{-1} \tilde{D}_Q)$-split where q_i are the projections. The diagonal Δ, all the subvarieties of the form $X \times G/Q$ where X is a Schubert variety in G/Q and $\Delta(X) = \{(x, x) \in G/Q \times G/Q \mid x \in X\}$ are simultaneously compatibly $(p-1)\, q_2^{-1} \tilde{D}_Q$-split in $G/Q \times G/Q$.*

(iii) *The partial diagonals $\Delta_{12} = \Delta \times G/Q$ and $\Delta_{23} = G/Q \times \Delta$ are simultaneously compatibly $(p-1)(q_1^{-1} D_Q + q_3^{-1} \tilde{D}_Q)$-split in $G/Q \times G/Q \times G/Q$. All the subvarieties of the form $X \times G/Q \times G/Q$, $X \subset G/Q$ a Schubert variety,*

$$\Delta_{12}(X) = \{(x, x, z) \in G/Q \times G/Q \times G/Q \mid x \in X, z \in G/Q\},$$
$$\Delta_{23}(X) = \{(x, z, z) \in G/Q \times G/Q \times G/Q \mid x \in X, z \in G/Q\},$$

and Δ_{12} and Δ_{23} are simultaneously compatibly $(p-1)\, q_3^{-1} \tilde{D}_Q$-split in $G/Q \times G/Q \times G/Q$.

Proof. — Consider the map $\pi : G/B \to G/Q$. If X is a Schubert variety in G/Q then $\pi^{-1}(X)$ is a Schubert variety in G/B. Moreover $\pi^{-1} D_Q < D$ and $\pi^{-1} \tilde{D}_Q < \tilde{D}$. Therefore by Remark 1.3 (ii) and Proposition 1.8 it follows that if we prove the theorem for B then by taking the direct image under π the theorem follows for Q. Hence we assume $Q = B$ in what follows.

Part (i) is proved in [12], [15] and [16]. We will give here a proof which will also be needed for the other parts. Let $\omega_N = s_{\alpha_1} \ldots s_{\alpha_N}$ be a reduced expression for the longest element of W. Then for the standard resolution $\psi : Z \to G/B$, where $Z = Z_N$ and $X_N = G/B$ in the notation of § 3.2, we have $K_Z^{-1} = \mathcal{O}_Z(\partial Z) \otimes \psi^* \mathcal{O}_{G/B}(\tilde{D})$. Therefore $\partial Z + \psi^{-1}(\tilde{D})$ is the divisor of a section s of K_Z^{-1}. Now $\partial Z = Z_{N1} + \ldots + Z_{NN}$ is a divisor with normal crossings and $\psi(Z_{N1} \cap \ldots \cap Z_{NN}) = X_0$, the point Schubert variety $B \in G/B$. Since $X_0 \notin \tilde{D}$ we have $Z_{N1} \cap \ldots \cap Z_{NN} \notin \operatorname{Supp} \psi^{-1}(\tilde{D})$. Therefore by Proposition 1.17 $\sigma = s^{p-1} \in H^0(Z, K_Z^{1-p})$ gives a simultaneous compatible $\psi^{-1}(\tilde{D})$-splitting of $Z_{N1}, \ldots, Z_{NN}$ in Z. Now the differential of ψ gives a map $K_Z^{-1} \to \psi^* K_{G/B}^{-1}$.

Composing this with s gives a section of $\psi^* K_{G/B}^{-1}$. By the projection formula $\psi_* \psi^* K_{G/B}^{-1} = K_{G/B}^{-1}$, since $\psi_* \mathcal{O}_Z = \mathcal{O}_{G/B}$, G/B being smooth. Thus the section s of $\psi^* K_{G/B}^{-1}$ gives a section $\bar{s}$ of $K_{G/B}^{-1}$. Since ψ is an isomorphism from $Z - \partial Z$ onto $G/B - D$ ([16], Proposition 1, Section 1) $\bar{s}$ has zeroes on $\tilde{D}$ and other possible zeroes only on the components of D. But $K_{G/B}^{-1} = \mathcal{O}_{G/B}(D + \tilde{D})$ and the components of D have linearly independent divisor classes. Therefore the divisor $\bar{s}$ has to be $D + \tilde{D}$. Therefore $\bar{\sigma} = \bar{s}^{p-1}$ gives a $(p-1)(D + \tilde{D})$-splitting of G/B by Proposition 1.20. It is easy to see that for $i > 1$ a Schubert variety of codimension i is an irreducible component of the set theoretic intersection of a suitable set of codimension $i-1$ Schubert varieties. It follows then by induction on codimension and Corollary 1.11 that $\bar{\sigma}$ gives a simultaneous $(p-1)\tilde{D}$-splitting of all Schubert varieties in G/B.

To prove (ii) we first remark that the product splitting $\bar{\sigma} \times \bar{\sigma}$ of $G/B \times G/B$ gives a compatible splitting of the factor $G/B \times 0$ where 0 is the point Schubert variety in G/B since $\bar{\sigma}$ splits 0 in G/B. If there existed an automorphism of $G/B \times G/B$ which pulls the factor into the diagonal we would be through. But such an automorphism may not exist. However on the open subset $\tilde{U} \subset G/B$ (§ 3.4 above) we can find such an automorphism by exploiting the group structure of $\tilde{U}$. Let $\alpha : \tilde{U} \times \tilde{U} \to \tilde{U} \times \tilde{U}$ be defined by $(x, y) = (x, y^{-1} x)$. Then clearly α maps the diagonal of $\tilde{U} \times \tilde{U}$ onto the factor $\tilde{U} \times 0$. Therefore the pull back section $\alpha^*(\bar{s} \times \bar{s})$ of $K_{\tilde{U} \times \tilde{U}}^{-1}$ gives rise to a compatible splitting of the diagonal in $\tilde{U} \times \tilde{U}$. Therefore if we can show that the rational section $\alpha^*(\bar{s} \times \bar{s})$ of $K_{G/B \times G/B}^{-1}$ defined on $\tilde{U} \times \tilde{U}$ is actually a regular section on the whole of $G/B \times G/B$ then it will follow by Proposition 1.8 that it gives rise to a splitting of the diagonal in $G/B \times G/B$.

For this purpose let us compute the zeroes and poles of the rational section $\alpha^*(\bar{s} \times \bar{s})$ in $G/B \times G/B$. Since the only zeroes of $\bar{s} \times \bar{s}$ in $\tilde{U} \times \tilde{U}$ are $D \cap \tilde{U} \times \tilde{U} + \tilde{U} \times D \cap \tilde{U}$ those of $\alpha^*(\bar{s} \times \bar{s})$ with support not contained in the complement of $\tilde{U} \times \tilde{U}$ are $D \times G/B + \bar{E}$ where $\bar{E}$ is the closure of $E = \{ (x, y) \mid x \in \tilde{U}, y^{-1} x \in D \cap \tilde{U} \}$.

We will now find the linear equivalence class of $\bar{E}$ in $G/B \times G/B$. Any line bundle on $G/B \times G/B$ is of the form $L_1 \times L_2$, L_i line bundle on G/B. Putting $y = e$, the identity in $\tilde{U}$, we see that the first factor of the line bundle corresponding to $\bar{E}$ must be $\mathcal{O}_{G/B}(D)$. Similarly putting $x = e$ the second factor must be $\mathcal{O}_{G/B}(D^{-1})$ where D^{-1} = closure of $\{ d^{-1} \mid d \in \tilde{U} \cap D \}$.

We now claim that $D^{-1} = D$. Since all the components of D interset $\tilde{U}$ nontrivially and $y \mapsto y^{-1}$ is an automorphism of $\tilde{U}$ it is enough to prove that this automorphism leaves $\tilde{U} - D$ invariant. Now $u \mapsto u\omega_N B$ identifies U with $G/B - D$ (see § 3.4). Thus $\tilde{u} \in \tilde{U}$ is in $\tilde{U} - D$ if and only if there are $u, u_1 \in U$ such that $un = \tilde{u} u_1$ where $n \in N(T)$ represents $\omega_N \in W$. Rearranging we get $u_1 n^{-1} = \tilde{u}^{-1} u$. Since $\omega_N^{-1} = \omega_N$ this shows that $\tilde{u}^{-1}$ also fulfils the condition for belonging to $\tilde{U} - D$. Thus we have proved $D^{-1} = D$.

It follows that the line bundle corresponding to $\bar{E}$ is $\mathcal{O}_{G/B}(D) \times \mathcal{O}_{G/B}(D)$. Therefore the zeroes of $\alpha^*(\bar{s} \times \bar{s})$ not supported on the components of

$$G/B \times G/B - \tilde{U} \times \tilde{U} = \tilde{D} \times G/B \cup G/B \times \tilde{D}$$

form the divisor $D \times G/B + \bar{E}$ linearly equivalent to $2(D \times G/B) + G/B \times D$. Now $K^{-1}_{G/B \times G/B}$ corresponds to the divisor class of $2(D \times G/B) + 2(G/B \times D)$, D is linearly equivalent to $\tilde{D}$ and the divisor classes of the components of $\tilde{D} \times G/B + G/B \times \tilde{D}$ are linearly independent. It follows that $\alpha^*(\bar{s} \times \bar{s})$ cannot have any poles and its zeroes outside $\tilde{U} \times \tilde{U}$ are precisely on $G/B \times \tilde{D}$.

Thus we have proved that $\alpha^*(\bar{s} \times \bar{s})$ is a regular section with divisor $D \times G/B + \bar{E} + G/B \times \tilde{D}$. This proves, as we have remarked earlier, that the diagonal Δ in $G/B \times G/B$ is compatibly $(p-1)\{D \times G/B + G/B \times \tilde{D}\}$-split by $\alpha^*(\bar{s} \times \bar{s})^{p-1}$ (see also the remark preceding Proposition 1.17). By Proposition 1.20 the zeroes of $\alpha^*(\bar{s} \times \bar{s})$ on $D \times G/B$ give that, for codimension 1 Schubert varieties X in G/B, $X \times G/B$ is split by $\alpha^*(\bar{s} \times \bar{s})^{p-1}$. By Corollary 1.11 and induction on codimension as done above in the proof of part (i) $\alpha^*(\bar{s} \times \bar{s})^{p-1}$ gives a compatible $G/B \times \tilde{D}$-splitting of $X \times G/B$ for any Schubert variety in X in G/B. The compatible splitting of $\Delta(X)$ follows since $\Delta(X) = \Delta \cap X \times G/B$. This completes the proof of part (ii).

Finally part (iii) can be proved by similar arguments using the automorphism $(x, y, z) \to (x, y^{-1}x, z^{-1}y)$ of $\tilde{U} \times \tilde{U} \times \tilde{U}$. We omit the details.

3.6. *Remark.* — Since $Z - \partial Z \simeq G/B - D \simeq U$

$$\alpha : U \times U \to U \times U, \quad \alpha(x, y) = (x, y^{-1}x)$$

gives a rational map of $Z \times Z$. But in this case $\alpha^*(s \times s)$ has poles as can be seen by an argument similar to the one used in the above proof. Thus this method does not give anything for the diagonal in $Z \times Z$. Regarding Schubert varieties, if one wants to prove results for line bundles on Schubert varieties which extend to G/B it is not necessary to know whether the diagonal of the Schubert variety X is split in $X \times X$ itself, thanks to the slight generalisation to the case of X embedded in Z in Propositions 2.7 and 2.19. So we have ignored, for now, the question whether for a Schubert variety X the diagonal is split in $X \times X$. However, it seems certain that the methods of this paper will give a solution to this.

We will put together the above theorem on Frobenius splitting and the results of the previous section to get results on the defining equations for Schubert varieties. Before doing that we need a few more results.

First we recall the following theorem which is proved in [12] for ample line bundles and in [15] in general.

3.7. *Theorem* ([15], *Theorem* 2). — *Let the base field be of arbitrary characteristic. Let* L *be a line bundle on* G/Q *such that* $H^0(G/Q, L) \neq 0$ *and* X *a Schubert variety in* G/Q. *Then* (i) $H^i(G/Q, L) = H^i(X, L) = 0$ *for* $i > 0$ *and* (ii) $H^0(G/Q, L) \to H^0(X, L)$ *is surjective.*

Proof. — If the base field is of characteristic $p > 0$ this follows at once from Theorem 3.6 (i) and Proposition 1.13 (ii). Since we can construct G/Q, X and L flat over **Z** (see [12], Lemma 3 and [17]), characteristic zero case follows by semicontinuity [13], § 5.

Remark. — For G/B we can prove slightly more by using the $(p-1)(\mathrm{D}+\widetilde{\mathrm{D}})$-splitting in Theorem 3.5 (i). Any line bundle on G/B is of the form $\bigotimes_{i=1}^{\ell} \pi_j^* \mathrm{L}_j^{n_j}$ where L_j is the ample line bundle on $\mathrm{G/P}_j$ generating Pic $\mathrm{G/P}_j$, $\pi_j : \mathrm{G/B} \to \mathrm{G/P}_j$, P_j maximal parabolic. Suppose $\mathrm{L} = \bigotimes_{j=1}^{\ell} \mathrm{L}_j^{n_j}$ with $n_j \geqslant -1$ then $\mathrm{H}^i(\mathrm{G/B}, \mathrm{L}) = 0$ for $i \geqslant 1$ and if any one $n_j = -1$ then $\mathrm{H}^i(\mathrm{G/B}, \mathrm{L}) = 0$ for $i \geqslant 0$. This follows from Proposition 1.12 (i) since G/B is $(p-1)(\mathrm{D}+\widetilde{\mathrm{D}})$-split and $\mathcal{O}_{\mathrm{G/B}}(\mathrm{D}+\widetilde{\mathrm{D}}) = \bigotimes_{j=1}^{\ell} \pi_j^* \mathrm{L}_j^2$.

3.8. *Theorem ([15], Section 3). — Let the base field be of arbitrary characteristic. Let* $\pi : \mathrm{G/B} \to \mathrm{G/Q}$ *be the natural projection and* X *a Schubert variety in* G/B. *Then* π *is trivial with image* G/Q *and* $\pi \mid \mathrm{X} : \mathrm{X} \to \mathrm{G/Q}$ *is trivial with image the Schubert variety* $\pi(\mathrm{X})$ *and we have for any locally free sheaf* L *on* G/Q, $\mathrm{H}^i(\mathrm{X}, \pi^* \mathrm{L}) = \mathrm{H}^i(\pi(\mathrm{X}), \mathrm{L})$.

Proof. — This is an immediate consequence of Theorem 3.7 above and Lemmas 2.11, 2.10.

3.9. *Theorem ([16], Theorem 4). — Any standard resolution* $\psi : \mathrm{Z} \to \mathrm{X}$ *of a Schubert variety* X *in* G/B *is a trivial morphism. In fact it is a rational resolution.*

Proof. — This is proved by induction on dimension using Theorem 3.7 and Lemma 2.11. See [16] for details. (Rational resolution means that ψ is proper birational, trivial and $\mathrm{R}^i \psi_* \mathrm{K}_\mathrm{Z} = 0$ for $i > 0$.)

3.10. *Lemma. — Let* $\mathrm{X} \subset \mathrm{G/Q}$ *be a Schubert variety. Then we can find a sequence of Schubert varieties* $\mathrm{X} = \mathrm{X}_0 \subset \mathrm{X}_1 \ldots \subset \mathrm{X}_r = \mathrm{G/Q}$ *such that the codimension of* X_i *in* X_{i+1} *is* 1 *and* $\mathrm{X}_i \subset \mathrm{X}_{i+1}$ *admits a trivial resolution (see Definition 2.17).*

Proof. — Given $\mathrm{X} \neq \mathrm{G/Q}$ it is enough to find such a X_1. We can then use induction. Let $\pi : \mathrm{G/B} \to \mathrm{G/Q}$ be the natural projection. Then $\widetilde{\mathrm{X}} = \pi^{-1}(\mathrm{X})$ is a Schubert variety in G/B.

Since $\mathrm{X} \neq \mathrm{G/Q}$, $\widetilde{\mathrm{X}} \neq \mathrm{G/B}$. If $\widetilde{\mathrm{X}} = \mathrm{X}_\omega$, $\omega \in \mathrm{W}$ we can then find a simple root α such that $\ell(\omega s_\alpha) = \ell(\omega) + 1$. Let $\widetilde{\mathrm{X}}_1 = \mathrm{X}_{\omega s_\alpha}$ and $\mathrm{X}_1 = \pi(\widetilde{\mathrm{X}}_1)$. As part of the ladder of standard resolutions for $\widetilde{\mathrm{X}}_1$ we then also get a trivial resolution

$$\begin{array}{ccc} \mathrm{Z} & \longrightarrow & \mathrm{Z}_1 \\ \downarrow & & \downarrow \\ \widetilde{\mathrm{X}} & \longrightarrow & \widetilde{\mathrm{X}}_1 \end{array}$$

of $\tilde{X} \subset \tilde{X}_1$ (§ 3.2 and Theorem 3.9). Composing with π then $Z \subset Z_1$ is also a trivial resolution for $X \subset X_1$ since by the Grothendick spectral sequence the composite of surjective trivial morphisms is a trivial morphism.

The following is the main theorem of this paper on the equations defining Schubert varieties.

3.11. *Theorem.* — *Let the base field k be of arbitrary characteristic. Let* $Y \subseteq X \subseteq G/Q$ *be Schubert varieties in* G/Q *and* L *a line bundle on* G/Q *with* $H^0(G/Q, L) \neq 0$. *Then*

(i) *The natural multiplication map of graded algebras* $\bigoplus_{m=0}^{\infty} S^m H^0(X, L) \to \bigoplus_{m=0}^{\infty} H^0(X, L^m)$ *is surjective and its kernel is generated as an ideal by the kernel of* $S^2 H^0(X, L) \to H^0(X, L^2)$. *(i.e.* L *is normally presented, in the terminology of* [14], *p.* 39, *Definition).*

(ii) *The natural restriction map of graded algebras* $\bigoplus_{m=0}^{\infty} H^0(X, L^m) \to \bigoplus_{m=0}^{\infty} H^0(Y, L^m)$ *is surjective and its kernel is generated as an ideal by the kernel of* $H^0(X, L) \to H^0(Y, L)$ *(i.e.* Y *is linearly defined in* X *with respect to* L, *see Definition* 2.18 *above).*

Proof. — First we show that it is enough to prove the theorem when the base field is of characteristic $p > 0$. We know that Y, X, G/Q and L come by base change from flat schemes $\mathfrak{X} \to \operatorname{Spec} \mathbf{Z}_f = U$ etc. (where $\mathbf{Z}_f$ is the localisation of $\mathbf{Z}$ with respect to a suitable $f \in \mathbf{Z}$. In fact one can take $f = 1$, see [17]). See [12], Lemma 3. Let $p \in \operatorname{Spec} \mathbf{Z}_f$. We will work with the discrete valuation ring $\mathbf{Z}_{(p)}$. Because of the vanishing theorem and the semicontinuity theorem [13], § 5 it follows that $\alpha_* \mathfrak{L}^m$ and $\beta_* \mathfrak{L}^m$, where $\alpha : \mathfrak{X} \to \operatorname{Spec} \mathbf{Z}_{(p)}$ and $\beta : \mathfrak{Y} \to \operatorname{Spec} \mathbf{Z}_{(p)}$, are free $\mathbf{Z}_{(p)}$-modules such that the base change to the generic and special fibres give the global sections on X and Y in characteristic zero and characteristic p respectively. Hence if we have proved the surjectivity of the restriction $\alpha_* \mathfrak{L} \to \beta_* \mathfrak{L}$ modulo p then the surjectivity in characteristic 0 follows. A similar argument gives the surjectivity in part (i) for characteristic 0 if one knows it for characteristic p.

For the kernel in part (i) let $a_1, \ldots, a_r \in \ker S^2 \alpha_* \mathfrak{L} \to \alpha_* \mathfrak{L}^2$ be such that modulo p they generate the kernel of $S^m \alpha_* \mathfrak{L} \to \alpha_* \mathfrak{L}^m$. Then clearly they generate it in characteristic 0 also. Again a similar argument takes care of part (ii).

We now make one more reduction to the case L ample. Since $H^0(G/Q, L) \neq 0$ there is a parabolic subgroup $Q' \supseteq Q$ and an ample line bundle L' on G/Q' such that $L = \pi^* L'$ where $\pi : G/Q \to G/Q'$ (Q' is the intersection of all the maximal parabolic subgroups corresponding to those fundamental weights which occur with a nonzero coefficient in the character of Q which gives L as the associated bundle of the Q-bundle $G \to G/Q$). Since $H^i(X, L) = H^i(\pi(X), L')$ and $H^i(Y, L) = H^i(\pi(Y), L')$ by Theorem 3.8 we can replace X, Y and G/Q with $\pi(X)$, $\pi(Y)$ and G/Q'. In other words we can assume that L is ample.

So from now on for the proof of the theorem we will assume that the characteristic of the base field k is $p > 0$ and that L is ample.

Consider the commutative diagram

$$\begin{array}{ccc} S^m H^0(G/Q, L) & \twoheadrightarrow & H^0(G/Q, L^m) \\ \downarrow & & \downarrow\!\!\downarrow \\ S^m H^0(X, L) & \longrightarrow & H^0(X, L^m) \end{array}$$

Since X is split in G/Q by Theorem 3.5 (i) $H^0(G/Q, L^m) \to H^0(X, L^m)$ is surjective. Hence if $S^m H^0(G/Q, L) \to H^0(G/Q, L^m)$ is surjective then so is $S^m H^0(X, L) \to H^0(X, L^m)$ by the commutativity of the diagram. The map for G/Q is surjective by Proposition 2.2 (i) since by Theorem 3.5 (ii) the diagonal is split in $G/Q \times G/Q$.

The kernel of the map in Part (i) is generated by degree 2 elements by Theorem 3.5 and Proposition 2.7.

Part (ii) says that Y is linearly defined in X with respect to L (Definition 2.18). We will first prove that X is linearly defined in G/Q. By Lemma 3.10 we can find a sequence of Schubert varieties $X = X_0 \subset X_1 \ldots \subset X_r = G/Q$ such that $X_i \subset X_{i+1}$ admits a trivial resolution (Definition 2.17). We will apply Proposition 2.19 with $X = X_{i+1}$, $Y = X_i$ and $Z = G/Q$ to prove that X_i is linearly defined in X_{i+1}. By Theorem 3.5 (i) X_i, X_{i+1} are simultaneously split G/Q. Hence clearly X_i is split in X_{i+1}.

This is condition (b) of Proposition 2.19. The condition (c) there is guaranteed by Theorem 3.5 (ii).

Hence we conclude that X_i is linearly defined in X_{i+1}. Now applying Lemma 2.20 (i) successively we conclude that X is linearly defined in G/Q. Similarly Y is linearly defined in G/Q. Lemma 2.20 (ii) then gives that Y is linearly defined in X. This completes the proof of Theorem 3.11.

3.12. *Remark.* — In informal terms Theorem 3.11 says that whenever a Schubert variety $X \subset G/Q$ (in particular the homogeneous space G/Q) is embedded in a projective space $\mathbf{P}^n$ by an ample line bundle on G/Q it is defined there by quadrics. In fact the affine cone over X is also defined scheme theoretically by quadrics. Moreover if Y is a Schubert subvariety of X then Y is the scheme theoretic intersection of X and all the hyperplanes of $\mathbf{P}^n$ passing through Y.

4. Remarks on the singularities of Schubert varieties and Demazure's work

The following theorem is proved in [15].

4.1. *Theorem (*[15], *Theorem* 3*).* — *Let the base field be of arbitrary characteristic. Let* L, M *be line bundles on* G/Q *and* $X \subset G/Q$ *a Schubert variety. Then*

(i) *Any Schubert variety is normal.*

(ii) *If* $H^0(G/Q, L) \neq 0$ *and* $H^0(G/Q, M) \neq 0$ *then* $H^0(X, L) \otimes H^0(X, M) \to H^0(X, L \otimes M)$ *is surjective.*

(iii) *If* L *is ample on* G/Q *then in the projective embedding given by* L, X *is projectively normal.*

Proof. — We sketch the proof from [15]. We can assume $X = X_\omega \subset G/B$. Find a simple root α such that $\ell(\omega s_\alpha) = \ell(\omega) - 1$. Then $X_{\omega s_\alpha} \subset X_\omega$ and $\pi : G/B \to G/P_\alpha$ maps $X_{\omega s_\alpha}$ birationally onto $\pi(X_\omega)$ over which X_ω is a $\mathbf{P}^1$-bundle. By induction $X_{\omega s_\alpha}$ is normal and since $\pi_* \mathcal{O}_{X_{\omega s_\alpha}} = \mathcal{O}_{\pi(X_\omega)}$ (Theorem 3.8), $\pi(X_\omega)$ is normal and hence so is X_ω.

(ii) and (iii) The proof of these in [15] uses some results on Steinberg modules. The methods of this paper on the Frobenius splitting of the diagonal (Theorem 3.5) give an alternative proof. See Theorem 3.11 (ii) and Corollary 2.3. (One goes to characteristic p by semicontinuity.)

In characteristic zero any proper birational trivial map $\psi : Z \to X$ with Z smooth automatically satisfies $R^i \psi_* K_Z = 0, i > 0$ by a theorem of Grauert-Riemenschneider. Hence X is then Cohen-Macaulay by a result of Kempf ([16], Proposition 4). Therefore since any Schubert variety $X \subset G/B$ admits a trivial resolution (Theorem 3.9) this gives a proof for the Cohen-Macaulayness of any Schubert variety in characteristic zero. This remark should be attributed to Kempf, see Demazure [3], § 5, Corollary 2. (Thus the first proof of the Cohen-Macaulayness for Schubert varieties in characteristic zero comes from the result of [12] and [18], see Remark 4.5 below.) For arbitrary characteristic we have the following result.

4.2. *Theorem (Cf.* [16], *Theorem* 5*).* — *Let the base field be of arbitrary characteristic.*

(i) *Any Schubert variety is Cohen-Macaulay.*

(ii) *In any projective embedding given by an ample line bundle on* G/Q *any Schubert variety is arithmetically Cohen-Macaulay.*

(iii) *The canonical sheaf of a Schubert variety* $X \subset G/B$ *is* $I_{\partial X} \otimes \mathcal{O}_X(-\widetilde{D} \cap X)$ *where* $I_{\partial X}$ *is the ideal sheaf in* X *of* $\partial X =$ *union of all the codimension* 1 *Schubert subvarieties of* X *and* $\widetilde{D} \cap X$ *is the divisor cut out on* X *by the divisor* $\widetilde{D}$ *in* G/B *(see* § 3.4*).*

Proof. — For the proof of (i) and (ii) see [16]. The assertion (iii) follows from the proof of Theorem 4 [16] by looking at the kernel of $\psi_* f^* \mathcal{O}_{Z_i}(-\partial Z_i) \to \psi_* \sigma_* \mathcal{O}_{Z_i}(-\partial Z_i)$ in the notation of that paper.

We take this opportunity to point out that for the proof the claim A_{i+1} in the course of the proof of Theorem 4 in [16] we use by induction not only A_i but also B_i. Unfortunately this is not made clear there, causing some obscurity.

The result (iii) will probably help one towards a combinatorial criterion for a Schubert variety to be Gorenstein.

4.3. *Remark.* — Instead of working with the split exact sequence

$$0 \to \mathcal{O}_{G/B} \to F_* \mathcal{O}_{G/B} \to C \to 0$$

one could tensor it with suitable line bundles and taking global sections one could work with the resulting sequences of G-modules (or B-modules). This way it is possible to work out proofs for all the results stated in this paper in a language which is less geometric and more suited to representation theory.

4.4. *Remarks.* — Using the inductive machinary of standard resolutions it is not difficult to see the following implications. These remarks are essentially due to Seshadri. See [17], [18], [19]. Let $X \subset G/B$ be a Schubert variety over a given field k.

(i) Normality of Schubert varieties is equivalent to the validity of the character formula of Demazure for large powers of an ample line bundle.

(ii) The result $H^i(X, L) = 0$, $i > 0$ and $H^0(G/B, L) \to H^0(X, L)$ surjective for ample L and the normality of Schubert varieties together imply the same results for effective L and Demazure's character formula.

(iii) $H^i(X, L) = 0$, $i > 0$, $H^0(G/B, L) \to H^0(X, L)$ surjective for effective $L \Rightarrow$ Normality of Schubert varieties and Demazure's character formula.

Joseph proved Demazure's character formula when char $k = 0$. (Demazure's proof in [3] contains an error (Proposition 11) as pointed out by V. Kac.) Therefore by (i) this implies the normality of Schubert varieties in characteristic 0.

Seshadri ([18], [19]) proved the normality of Schubert varieties in arbitrary characteristic. Hence by (ii) the results of [12] together with this give Demazure's character formula for arbitrary base fields and prove his conjecture. This is the first proof of Demazure's conjecture and justification of his work over arbitrary fields.

In [15] the result $H^i(X, L) = 0$, $H^0(G/B, L) \to H^0(X, L)$ surjective for effective L is proved. By (iii) this again gives a simple and complete justification of Demazure's work and his conjecture over arbitrary fields.

In addition the Frobenius method gives the projective normality [15] and arithmetic Cohen-Macaulay property [16] of Schubert varieties and the results of the present paper.

Andersen in his preprint [1], which is later than [18] but earlier than [15], claimed a proof of the normality of Schubert varieties and Demazure's character formula. But in his proof he assumed without proof (see page 9, line 3 of [1]), that $H^i(X, \pi^* L) = H^i(\pi(X), L)$ where $\pi : G/Q \to G/Q'$ and X a Schubert variety (Theorem 3.8 above). As is evident this is far from being an obvious fact; in fact one can quickly deduce normality etc. from this.

But this problem with [1] does not affect his paper [2] because the key point of [2], namely the splitting map θ of the lemma of § 2 in [2], is not in [1]: Andersen was influenced by the paper [15] of Ramanan and Ramanathan.

REFERENCES

[1] H. H. Andersen, *Schubert varieties and Demazure's character formula*, Aarhus Preprint Series No. 44, June 1984.
[2] H. H. Andersen, Schubert varieties and Demazure's character formula, *Invent. Math.*, **79** (1985), 611-618.
[3] M. Demazure, Désingularisations de variétés de Schubert généralisées, *Ann. Sci. E.N.S.*, **7** (1974), 53-88.
[4] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Math., Springer-Verlag, 1977.
[5] W. V. D. Hodge and D. Pedoe, *Methods of algebraic geometry*, Vol. II, Cambridge Univ. Press, 1952.
[6] G. Kempf, Linear Systems on homogeneous spaces, *Ann. of Math.*, **103** (1976), 557-591.

[7] G. KEMPF, The Grothendieck-Cousin complex of an induced representation, *Adv. in Math.*, **29** (1978), 310-396.
[8] S. L. KLEIMAN, Rigorous foundation for Schubert's enumerative calculus, *in* Mathematical developments arising from Hilbert problems, *A.M.S. Proc. of Symposia in Pure Math.*, Vol. XXVIII (1976), 445-482.
[9] LAKSHMIBAI and C. S. SESHADRI, Geometry of G/P-V, *J. of Algebra*, **100** (1986), 462-557.
[10] LAKSHMIBAI and C. S. SESHADRI, Singular locus of a Schubert variety, *Bull. A.M.S.*, **11** (1984), 363-366.
[11] G. LANCASTER and J. TOWBER, Representation functors and flag algebras for the classical groups I, *J. of Algebra*, **59** (1979), 16-38.
[12] V. B. MEHTA and A. RAMANATHAN, Frobenius splitting and cohomology vanishing for Schubert varieties, *Ann. of Math.*, **122** (1985), 27-40.
[13] D. MUMFORD, *Abelian Varieties*, Bombay, Oxford Univ. Press, 1974.
[14] D. MUMFORD, Varieties defined by quadratic equations, in *Questions on algebraic varieties*, Rome, C.I.M.E., 1970, 29-100.
[15] S. RAMANAN and A. RAMANATHAN, Projective normality of flag varieties and Schubert varieties, *Invent. Math.*, **79** (1985), 217-224.
[16] A. RAMANATHAN, Schubert varieties are arithmetically Cohen-Macaulay, *Invent. Math.*, **80** (1985), 283-294.
[17] C. S. SESHADRI, Standard monomial theory and the work of Demazure, in *Algebraic varieties and analytic varieties*, Tokyo, 1983, 355-384.
[18] C. S. SESHADRI, *Normality of Schubert varieties* (Preliminary version of [19] below), Manuscript, April 1984.
[19] C. S. SESHADRI, Line bundles on Schubert varieties, To appear in the *Proceedings of the Bombay colloquium on Vector Bundles on Algebraic varieties*, 1984.
[20] C. CHEVALLEY, *The algebraic theory of spinors*, New York, Columbia University Press, 1954.
[21] W. HABOUSH, Reductive groups are geometrically reductive, *Ann. of Math.* **102** (1975), 67-84.

School of Mathematics
Tata Institute of Fundamental Research
Bombay 400005.

Department of Mathematics
The Johns Hopkins University
Baltimore MD 21218.

Manuscrit reçu le 14 août 1985.