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# SPHERICAL ISOTROPY REPRESENTATIONS

*by* TED PETRIE *and* JOHN RANDALL

## 1. History and discussion of ideas

An old question of P. A. Smith asks [Sm]: If a finite group acts smoothly on a closed homotopy sphere $\Sigma$ with fixed set $\Sigma^G$ consisting of two points $p$ and $q$, are the isotropy representations $T_p \Sigma$ and $T_q \Sigma$ of G at $p$ and $q$ equal? Put another way: Describe the representations (V, W) of G which occur as $(T_p \Sigma, T_q \Sigma)$ for $\Sigma$ a sphere with smooth action of G and $\Sigma^G = p \cup q$. Under these conditions we say that V and W are *Smith equivalent* [P] and write $V \sim W$. Prior to the results of this paper all evidence suggested a positive answer to Smith's question. We show here that the answer to Smith's question is no. Specifically suppose G is an odd order abelian group with at least 4 non cyclic Sylow subgroups. Then (Theorem A') there is a nontrivial subgroup (1.7) $rI'(G)$ of the real representation ring RO(G) such that each $z \in rI'(G)$ occurs as a difference of two Smith equivalent representations. Even more generally there is an action of G on a homotopy sphere $\Sigma$ such that $\Sigma^G$ consists of an arbitrary number of points and the pairwise differences of the isotropy representations realize given elements of $rI'(G)$ (Theorem A). This paper presents proofs and elaboration of the results announced in [$P_1$] and [$P_2$].

The question of Smith was undoubtedly motivated by the observation that if $\Sigma = S(V)$ is the unit sphere of a representation V, then $T\Sigma|_{\Sigma^G}$, the G tangent bundle of $\Sigma$ restricted to $\Sigma^G$, is $\Sigma^G \times V'$. Here V' is $V/V^G$. In particular this means that the isotropy representations $T_x \Sigma$ for $x$ in $\Sigma^G$ are independent of $x$. One might then wonder about the case where $\Sigma$ is an arbitrary homotopy sphere with G action such that $\Sigma^G$ is a homotopy sphere. Smith's question deals with the case in which $\Sigma^G$ is the 0 dimensional sphere. For interesting results for the positive dimensional case see Schultz [$Sc_1$] and Ewing [E].

The problem of characterizing Smith equivalent representations has a rich history which we mention to motivate the material here. The first major breakthroughs on this topic are due to Atiyah-Bott [AB] and Milnor [M] and rest on the Atiyah-Bott fixed point theorem. The set of isotropy groups $\{G_x \mid x \in X\}$ of a G space X is denoted by Iso(X).

*Theorem* **1.1.** (Atiyah-Bott [AB]). — *Let* V *be a representation of a finite group* G *such that* $\mathrm{Iso}(V - 0) = \{1\}$. *If* W *is a representation of* G *with* $V \sim W$, *then* $V = W$.

One important consequence of their proof is this:

*Lemma* **1.2.** — *If* G *has odd order and* V *and* W *are Smith equivalent representations of* G, *then* $V - W \in \mathrm{Ker}(\mathrm{RO}(G)) \xrightarrow{\alpha} \prod_{P \in \mathscr{S}(G)} \mathrm{RO}(P))$. *(Here* $\mathscr{S}(G)$ *denotes the set of Sylow subgroups of* G *and* $\alpha$ *is a product homomorphism each of whose components is restriction to* P.)

For even order groups there is a weaker result:

*Theorem* **1.3** (Bredon [$B_1$]). — *If* G *is cyclic of* 2 *power order and* $V \sim W$, *then* $W - V$ *is contained in* $2^{f(V)}\,\mathrm{RO}(G)$ *where* $f(V)$ *is an explicit power of* 2 *which increases as dimension of* V *increases.*

*Corollary* **1.4.** — *Same hypothesis. If* dim V *is large compared with the order* $|G|$ *of* G, *then* $V \sim W$ *implies* $V = W$.

This corollary indicates some of the complexity of Smith equivalence. For example $V \sim W$ does not imply $V \oplus S \sim W \oplus S$. The next result when compared with the main results here shows the subtlety of the notion of Smith equivalence.

*Theorem* **1.5** (Sanchez [S]). — *If* $|G|$ *is odd and* V *and* W *are Smith equivalent representations realized as* $T_p \Sigma$ *and* $T_q \Sigma$ *for an action of* G *on a homotopy sphere* $\Sigma$ *such that* $\Sigma^H$ *is either connected or* $p \cup q$ *for all* $H \subset G$, *then* $V = W$.

In order to state the main results here which deal with Smith equivalence we fix some notation. In this paper

**1.6.** — G *is an odd order abelian group.*

Let R(G) (resp. RO(G)) denote the complex (resp. real) representation ring of G. If H is a subgroup of G, $\mathrm{res}_H : R(G) \to R(H)$ is the restriction homomorphism and $\mathrm{fix}_H : R(G) \to R(G/H)$ is the homomorphism defined by sending a representation V to $V^H$ its H fixed set. Realification defines a homomorphism $r : R(G) \to \mathrm{RO}(G)$. Let $\mathscr{P}$ denote the set of groups of prime power order.

Define subgroups

$$(1.7)\qquad I(G) = \mathrm{Ker}(R(G) \xrightarrow{\alpha} \prod_{P \in \mathscr{S}(G)} R(P))$$

$$I'(G) = \mathrm{Ker}(R(G) \xrightarrow{\beta} (\prod_{P \in \mathscr{S}(G)} R(P) \times \prod_{G/H \in \mathscr{P}} R(G/H)))$$

where $\alpha$ is the product of $\mathrm{res}_P$ for $P \in \mathscr{S}(G)$ and $\beta$ is the product of $\mathrm{res}_P$ for $P \in \mathscr{S}(G)$ and $\mathrm{fix}_H$ for $G/H \in \mathscr{P}$. The corresponding subgroups of RO(G) are denoted by IO(G) and IO′(G). One of the main results here about Smith equivalence is:

*Theorem A′. — Let* G *be an odd order abelian group having at least* 4 *non cyclic Sylow subgroups. Then every element of* $rI'(G)$ *occurs as the difference of Smith equivalent representations.* (*Compare with* 1.2.)

This theorem is a consequence of Theorem A and Lemma 2.6. In 2.5 we define a set $\mathscr{R}$ of complex representations of G.

*Theorem A. — Let* K *be any non empty finite set. Suppose for each* $u, v \in K$ *that* $R_u \in \mathscr{R}$ *and* $R_u - R_v \in I(G)$. *Then there is a closed homotopy sphere* $\Sigma$ *with smooth action of* G *such that* $\Sigma^G = K$ *and* $\{T_u \Sigma \mid u \in K\} = \{rR_u \mid u \in K\}$.

Theorem A′ gives a sufficient condition to realize an element of RO(G) as a difference of two Smith equivalent representations. This condition involves the subgroup $rI'(G)$ of IO′(G). The latter is naturally related to necessary conditions (2.7 and subsequent remarks). For a large family $\mathscr{S}$ of representations of G, Theorem A′ together with Lemmas 2.7 and 2.8 lead to a necessary and sufficient condition, Theorem B′, for representations in $\mathscr{S}$ to be stably Smith equivalent. (By definition, representations V and W are *stably Smith equivalent* if there is a representation S such that $V \oplus S$ and $W \oplus S$ are Smith equivalent.) Define $\mathscr{S}$ as follows: A representation V of G is in $\mathscr{S}$ if and only if $\dim V^H = 0$ for each subgroup H for which $G/H \in \mathscr{P}$.

*Theorem B′. — A necessary and sufficient condition that two representations* $V, W \in \mathscr{S}$ *be stably Smith equivalent is that* $V - W \in IO'(G)$.

*Theorem B. — Let* K *be any non empty finite set. Suppose for each* $u \in K$, $R_u$ *is a representation in* $\mathscr{S}$. *Then there exists a representation* S *of* G *and a smooth* G *action on a closed homotopy sphere* $\Sigma$ *such that* $\Sigma^G = K$ *and* $\{T_u \Sigma \mid u \in K\} = \{R_u \oplus S \mid u \in K\}$ *if* $R_u - R_v \in IO'(G)$ *for all* $u, v \in K$.

Using Theorems A′ and B′ it is easy to exhibit non isomorphic Smith equivalent representations. Here is a representative example: Let L be the cyclic group of order $pq$ where $p$ and $q$ are distinct odd primes. View L as the group of $pq$-th roots of unity in **C** and let $t^i$ denote the complex one dimensional representation of L defined by asserting that $g \in L$ acts on $v \in t^i$ by complex multiplication by $g^i$. The representations $V = t^{a(p+1)(q+1)} \oplus t^a$ and $W = t^{a(p+1)} \oplus t^{a(q+1)}$ of L where $(a, pq) = 1$ are not stably Smith equivalent as representations of L; however, if G is a group satisfying the conditions of Theorem A′ and $\varphi : G \to L$ is a surjective homomorphism, then $r\varphi^* V$ and $r\varphi^* W$ are stably Smith equivalent representations of G. This follows immediately from Theorem A′ because $\varphi^*(V - W) \in I'(G)$; so $r\varphi^*(V - W) \in IO'(G)$.

There is an interesting problem associated with Smith's questions which is not treated here: namely, to relate the differential structure of $\Sigma$ and the isotropy representations $T_x \Sigma$ for $x \in \Sigma^G$ when $\Sigma$ is a homotopy sphere. As noted by Schultz and others the results of this paper can be achieved on the standard sphere. See section 5.

In the long period since the discovery of Theorem A (see [$P_2$]) and the publication of its proof together with the other results here, there have been a number of interesting papers published on the topic of Smith equivalence. Treating the case of cyclic groups of even order are papers of Cappell-Shaneson, Petrie, Dovermann and Siegel: see [CS], [$P_7$], [Dov] and [Si]. For noncyclic groups of even order, there are papers of Suh and Cho. See [Suh] and [C]. A forthcoming paper of Dovermann-Petrie [$DP_3$] treats the case of cyclic groups of odd order using many of the geometric methods of this paper as well as methods particular to cyclic groups of odd order. A good deal of [PR] is devoted to Smith equivalence.

There are three general ingredients to the proof of Theorem A. These are: the one fixed point actions of G on homotopy spheres of [$P_4$], the Completion Theorem of Atiyah [$A_1$] and the Induction Theorem in equivariant surgery given in section four. The main result of [$P_4$] produces for each $R \in \mathscr{R}$ a smooth action of G on a homotopy sphere X with exactly one fixed point $u$ such that $T_u X = R$ and the equivariant tangent bundle TX is stably isomorphic to $X \times R$. Lemma 1.2 ties together with another result of Atiyah [$A_1$] which asserts that I(G) is the kernel of the completion homomorphism $R(G) \to \hat{R}(G)$ from R(G) to the completed representation ring $\hat{R}(G)$. This may be interpreted geometrically. Let E be an acyclic space on which G acts freely. Then any representation R of G gives a G vector bundle $E \times R$ over E. The above result of Atiyah geometrically interpreted means that if $R - R' \in I(G)$, then $E \times R$ and $E \times R'$ are stably isomorphic G vector bundles over E.

As an approximation to the G homotopy sphere $\Sigma$ of Theorem A consider the manifold

$$W = \coprod_{u \in K'} W_u$$

described as follows: Add a point $o$ to K to give the set K′. Select any point $z \in K$ and set $T = R_z$. For $u \in K$, $W_u$ is a G manifold $X_u$ with $X_u^G = u$, $T_u X_u = R_u$ and $TX_u = X_u \times R_u$ as a stable G vector bundle. For $u = o$, $W_u$ is a G manifold with $W_u^G = \emptyset$, $TW_u = W_u \times T$ as a stable G vector bundle and the Euler characteristics of fixed sets $W_o^H$ for $H \subset G$ are arranged so that condition 4.18 (i) of the Induction Theorem 4.19 is satisfied. Let $\mathbf{R}$ be the real line with trivial action of G and let

$Z = S(T \oplus \mathbf{R})$ be the unit sphere of $T \oplus \mathbf{R}$. There is an equivariant map $F : W \to Z$ which collapses the complement of an invariant disk about $z \in W$ to a point. It has degree 1. Since $R_u - R_v \in I(G)$ for $u, v \in K'$, Atiyah's result gives a stable G vector bundle isomorphism

$$\beta_\infty : E \times TW \xrightarrow{\cong} E \times (W \times T)$$

of G vector bundles over $E \times W$. Then $\mathscr{W} = (W, F, \beta_\infty)$ is roughly what is needed to produce the sphere $\Sigma$ in Theorem A. Observe that $W^G = K$ and $T_u W = R_u$ for $u \in K$. If F were a homotopy equivalence, Theorem A would be established. The goal is to convert W to a G homotopy sphere $\Sigma$ using $\mathscr{W}$ and without destroying these side conditions.

The obvious tool for converting F to a homotopy equivalence is equivariant surgery. The setting for this is a triple $\mathscr{W} = (W, F, \beta)$ where $F : W \to Z$ is an equivariant map of degree 1 and $\beta : TW \to F^* \xi$ is a " bundle " isomorphism for some equivariant vector bundle $\xi$ over Z. The triple $\mathscr{W}$ is called a G *normal map*. Equivariant surgery is a process designed to produce a G normal cobordism between $\mathscr{W}$ and $\mathscr{W}' = (W', F', \beta')$ where $F' : W' \to Z$ is a homotopy equivalence. This is not always possible as there are obstructions. One powerful tool which guarantees success is an Induction Theorem which asserts that if for each hyperelementary subgroup H of G $\mathrm{res}_H \mathscr{W}$ is the boundary of some H normal map $\mathscr{W}(H)$, then $\mathscr{W}$ is G normally cobordant to $\mathscr{W}' = (W', F', \beta')$ with $\beta'$ a homotopy equivalence.

The first such Induction Theorem, due to Dress, applies to *free actions*. Its chief geometric application was the construction of free actions on homotopy spheres [D]. Induction Theorems for non free actions were established in [$DP_1$] and [$P_3$] and were employed in the construction of one fixed point actions on homotopy spheres [$P_3$] [$P_4$]. These induction theorems deal with G normal maps and depend therefore on the definition of " bundle " isomorphism occurring in the definition of G normal map. In the free case there is little ambiguity. " Bundle isomorphism " means a stable G vector bundle isomorphism. In the case of non free actions the role of the " bundle " isomorphism is far more substantial and the definition depends on the contemplated application. (The idea of a flexible notion of " bundle " equivalence is dealt with in [PR, pp. 91-95].) In the cases cited above TW is stably G isomorphic to $F^*(\xi)$ for some G vector bundle $\xi$ over Z. This definition must be altered to treat Theorem A as the following result shows.

*Lemma C. — Let* $\Sigma$ *be given by Theorem A. Let* $z \in K$ *and* $F : \Sigma \to S(R \oplus \mathbf{R}) = Z$, $(R = T_z \Sigma)$, *be any* G *map. If there is a* G *vector bundle* $\xi$ *over* Z *such that* $T\Sigma = F^* \xi$ *as a stable* G *vector bundle, then* $T_u \Sigma = T_v \Sigma$ *for* $u, v \in \Sigma^G$.

*Proof.* — Since $R \in \mathscr{R}$, $\dim R^H > 0$ for each cyclic subgroup H of G by 2.3 and 2.5. This means that $Z^H$ is connected; so if $\xi$ is any G vector bundle over Z, then the representation of H on $\xi_x$ (the fiber at $x$) is independent of $x \in Z^H$. If $T\Sigma = F^* \xi$, then for $u, v \in K = \Sigma^G$, $\mathrm{res}_H(T_u \Sigma - T_v \Sigma) = \mathrm{res}_H(\xi_{u'} - \xi_{v'}) = 0$, where $u' = F(u)$ and $v' = F(v)$. Since this holds for each cyclic subgroup of G, $T_u \Sigma = T_v \Sigma$.

To apply these considerations then to prove Theorem A, we need a definition of " bundle isomorphism " for $\beta : TW \to F^* \xi$ which is weaker than those previously used in equivariant surgery. One notion which works is called a Smith framing. It incorporates the isomorphism $\beta_\infty$ constructed from Atiyah's Theorem and the P vector bundle isomorphisms $\beta_p : TW \to W \times R$ for $P \in \mathscr{S}(G)$ arising from the isomorphism $\mathrm{res}_P R_u = \mathrm{res}_P R$. The definition and formal properties of Smith framings are treated in section 3.

The proof of the Induction Theorem 4.19 separates into an algebraic part and a geometric part. The geometric part consists in establishing two fundamental lemmas (4.11 and 4.12) from equivariant surgery (see [PR, Ch. 3, § 10]). They assert that equivariant surgery is possible (with respect to the definition of G normal map using Smith framing). These two results are useful in their own right as they are the key geometric steps in producing actions on disks with isolated fixed points and distinct isotropy representations—the first such [$P_6$]. The algebraic part of the proof of the Induction Theorem 3.19 and [$DP_1$, 2.6] are identical; so we appeal to [$DP_1$, 2.6] to complete the proof.

In section 2 we treat some algebraic preliminaries concerning the representation ring and the Burnside ring. In section 3 we develop the notion of Smith framing. In section 4 we treat the Induction Theorem. In section 5 we prove the main results by constructing a G normal map from the input from Theorem A. This normal map satisfies the requirements of the Induction Theorem which is applied to prove Theorem A. Theorems B and B′ are also proved in section 5. Theorem A and Lemma 2.8 lead to Theorem B, while Theorem B′ follows from Theorem B and Lemma 2.7.

## 2. The representation ring and the Burnside ring

If H is a subgroup of G, $\mathrm{res}_H$ denotes restriction of G data to H data. E.g. if X is a G space $\mathrm{res}_H X$ means that X is viewed as an H space. The set of hyperelementary subgroups of G groups is denoted by $\widetilde{\mathscr{P}}$. Since G is abelian, these are the groups which are a product of a cyclic group and a $p$-group of relatively prime order. The *Burnside ring* of G, $\Omega(G)$, is the ring of equivalence classes of smooth G manifolds with X equivalent to Y if the Euler characteristics $\chi(X^H)$ and $\chi(Y^H)$ are equal for all $H \subset G$. The equivalence class of a manifold X is written [X]. Additively $\Omega(G)$ is the free abelian group generated by [G/H] as H runs over conjugacy classes of subgroups of G. When $\mathscr{H}$ is a family of subgroups of G, $\Omega(G, \mathscr{H})$ denotes the subgroup generated by $\{[G/H] \mid H \in \mathscr{H}\}$.

Let $\Delta(G) \subset \Omega(G)$ be the ideal defined in [O, p. 339]. By [O, Prop. 2]

$$\Delta(G) = \bigcap_{H \in \widetilde{\mathscr{P}}} \operatorname{Ker}(\operatorname{res}_H : \Omega(G) \to \Omega(H))$$

because G is abelian.

If $E \in \Omega(G)$, its character $\chi_E$ is the integral valued function defined on the set of conjugacy classes of subgroups defined by

$$\chi_E(H) = \chi(E^H).$$

The definition of the equivalence relation in $\Omega(G)$ means $\Delta(G) = \{E \mid \chi_E(P) = 0$ for $P \subset G$ and $P \in \widetilde{\mathscr{P}}\}$. Set

(2.1) $$\mathscr{H} = \{H \subset G \mid G/H \notin \mathscr{P}\} \quad \text{and} \quad \Omega_0 = \Omega(G, \mathscr{H}).$$

*Lemma* **2.2.** — *Suppose* G *is an abelian group with at least four non-cyclic Sylow subgroups. Then the unit* 1 *of* $\Omega(G)$ *lies in the subgroup* $\Delta(G) + \Omega_0$.

*Proof.* — For each Sylow subgroup S of G, let $E = E(S) \in \Omega(G/S)$ be a virtual finite G set with $\chi_E(P) = 1$ for each $P \subset G/S$ which is hyperelementary and $\chi_E(G/S) = 0$. Since G/S has at least 3 non-cyclic Sylow subgroups, the existence of E is provided in [$P_4$]. Since $\Omega(G)$ is contravariant in G, we may regard E(S) as being in $\Omega(G)$. Set

$$X = \prod_S E(S) \in \Omega(G).$$

Then $\chi_X(G) = 0$ and $\chi_X(P) = 1$ whenever P is a hyperelementary subgroup of G i.e. $P \in \widetilde{\mathscr{P}}$. Thus $X = 1 - U$ with $U \in \Delta(G)$.

Let $X = \Sigma a_H[G/H]$ and set $\operatorname{Iso}(X) = \{H \mid a_H \neq 0\}$. Observe that if A and B are G sets, the isotropy groups of $A \times B$ are intersections of isotropy groups of each factor. Viewing E(S) as an element of $\Omega(G)$, $\operatorname{Iso}(E(S)) = \{S \times H(S) \mid H(S)$ is a proper subgroup of $G/S\}$. This uses the fact that $\chi_E(G/S) = 0$. From the above comment, $H \in \operatorname{Iso}(X)$ implies $H = \bigcap_S (S \times H(S))$, where S runs through the set of all Sylow subgroups of H. Then the index of H(S) in G/S divides $[H : G]$, the index of H in G, for every Sylow subgroup S. Since there are at least four distinct Sylow subgroups, $[H : G]$ is divisible by at least 2 distinct primes so $G/H \notin \mathscr{P}$, i.e. $H \in \mathscr{H}$. This shows that $X \in \Omega_0$.

The following easy lemma is left to the reader.

*Lemma* **2.3.** — *Let* G *be as in Theorem A'. Then* $G \notin \mathscr{H}$ *and* $\mathscr{H} \supset \widetilde{\mathscr{P}} \supset \mathscr{P}$.

Let $I'(G)$ be the subgroup defined in 1.7.

*Lemma* **2.4.** — $z \in I'(G)$ *if and only if there are representations* V *and* W *of* G *such that* $V^H = W^H = 0$ *whenever* $G/H \in \mathscr{P}$, $V - W \in I(G)$ *and* $z = V - W$.

*Proof.* — Let $\chi$ be an irreducible representation of G whose kernel K has index $p_t$ for some prime $p$. If $\langle \chi, V \rangle$ denotes the multiplicity of $\chi$ in V and $V - W = z \in I'(G)$, then $\langle \chi, V \rangle = \langle \chi, V^K \rangle = \langle \chi, W^K \rangle = \langle \chi, W \rangle$. The middle inequality uses the assumption $V^K = W^K$ in G/K. Removing all irreducible representations with kernel of prime power index from V and W gives V′ and W′ with $V' - W' = z$. Observe that $V'^H = W'^H = 0$ whenever $G/H \in \mathscr{P}$.

If R is a complex (real) representation of G and $\chi$ is a complex (real) irreducible representation of G, then $n_\chi(R)$ $(m_\chi(R))$ is the multiplicity of $\chi$ in R. Define a set $\mathscr{R}$ of complex representations of G by asserting a representation R of G is in $\mathscr{R}$ if and only if

**2.5.** — (i) $\mathrm{Iso}(R - 0) = \mathscr{H}$, $\dim_C R^H \geq 2$ for $H \in \mathscr{H}$, $\dim_C R^P \geq 3$ $P \in \mathscr{P}$, $\dim_C R^K < \frac{1}{2} \dim_C R^H$ whenever K contains H and $H \in \mathscr{H}$, and

(ii) $\dim_C R^H < n_\chi(R)$ whenever $H \subset G$ and $\chi$ is a non trivial irreducible representation of H with $n_\chi(R) \neq 0$.

*Lemma* **2.6.** — *For each* $z \in I'(G)$ *there are representations* $R_0$ *and* $R_1$ *in* $\mathscr{R}$ *with* $R_0 - R_1 = z$.

*Proof.* — By 2.4 there exists V and W with $V - W = z$ and $V^H = W^H = 0$ whenever $G/H \in \mathscr{P}$. This means $\mathrm{Iso}(A - 0) \subset \mathscr{H}$ for $A = V, W$; so we can apply [$P_4$, 1.3] to see that there is a representation S of G such that $V \oplus S$ and $W \oplus S$ satisfies 2.5 (i). It is an elementary exercise using Frobenius Reciprocity to see that that proof also shows that 2.5 (ii) is also satisfied (see [PR, § 9]); so that $V \oplus S$ and $W \oplus S$ are in $\mathscr{R}$. Their difference is again $z$.

It seems appropriate to motivate the appearance of the groups I′(G) and IO′(G) in the study of Smith equivalence of representations. The motivation comes from the following necessary condition for Smith equivalence:

*Lemma* **2.7.** — *Necessary conditions that two representations* V *and* W *of* G *be Smith equivalence are*: a) $\mathrm{res}_P V \cong \mathrm{res}_P W$ *for all subgroups* P *of* G *in* $\mathscr{P}$. b) *Whenever* H *is a subgroup of* G *such that* G/H *is cyclic and in* $\mathscr{P}$, *either* (i) $\mathrm{res}_H V \cong \mathrm{res}_H W$ *or* (ii) $V^H = W^H = 0$.

*Remarks.* — Note that the condition that $V^H = 0$ whenever G/H is cyclic and in $\mathscr{P}$ is equivalent the condition that $V \in \mathscr{S}$ (section 1) i.e. $V^H = 0$ whenever G/H is in $\mathscr{P}$. This uses the fact that G is abelian. The real analog of 2.4 describes IO′(G) as the subgroup $\{V - W \mid V, W \in \mathscr{S}$ and $\mathrm{res}_P(V - W) = 0$ whenever $P \in \mathscr{P}\}$. This is the subgroup of RO(G) defined by 2.7 a) and b) (ii).

*Proof of* 2.7. — The proof is based on a theorem of Atiyah-Bott. This theorem [AB] asserts that a cyclic group C of odd prime power order cannot act smoothly on an oriented manifold M of positive dimension in such a way that $M^C$ is exactly one point.

We apply this theorem to this situation: Let H be a subgroup of G such that $G/H = C$ is cyclic of odd prime power order. Suppose that G acts smoothly on a homotopy sphere $\Sigma$, $\Sigma^G$ consists of two points $x$ and $y$ and $T_x\Sigma = V$, $T_y\Sigma = W$. If $\mathrm{res}_H V$ is not isomorphic to $\mathrm{res}_H W$, then $x$ and $y$ are not in the same component of $\Sigma^H$; so the connected component of $\Sigma^H$ which contains $x$ is a closed manifold M supporting an action of the cyclic group C such that $M^C$ is one point $x$. Thus $\dim M = 0 = \dim V^H = 0$ and similarly $\dim W^H = 0$ or equivalently $V^H = W^H = 0$. This verifies condition b). The necessity of condition a) has been noted in Lemma 1.2.

The next lemma enters into the proof (§ 5) of Theorems B and B'. For the proof of this lemma, we collect some facts and notation from representation theory. Let $\xi$ be a primitive $k$-th root of unity and $\mathbf{Z}[\xi]$ the subring of $\mathbf{C}$ additively generated by powers of $\xi$. If $g \in G$ is an element of order $k$ and $x \in R(G)$ is viewed as a complex-valued function, its value $x(g)$ at $g$ lies in $\mathbf{Z}[\xi]$. This leads to a ring homomorphism $\mathrm{ev}_g : R(G) \to \mathbf{Z}[\xi]$ with $\mathrm{ev}_g(x) = x(g)$. For each cyclic subgroup C of G, $\xi_C$ denotes a primitive $|C|$-th root of unity and $g(C)$ a generator of C. Set $\overline{R}(G) = \prod \mathbf{Z}[\xi_C]$ where the product ranges over the cyclic subgroups of G. Then there is a homomorphism $\mathrm{ev} : R(G) \to \overline{R}(G)$ whose C-th coordinate is $\mathrm{ev}_C = \mathrm{ev}_{g(C)}$. A similar situation applies for real representations. In this case $\mathbf{Z}[\xi_C]$ is replaced by $\mathbf{Z}[\xi_C \oplus \bar{\xi}_C]$ (where $^-$ denotes complex conjugate) and $\overline{RO}(G)$ is the product of these rings as C varies again over the cyclic subgroups of G. Comparing the real and complex setting leads to a commutative diagram:

$$\begin{array}{ccc} R(G) & \xrightarrow{\mathrm{ev}} & \overline{R}(G) \\ \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle \bar{r}} \\ RO(G) & \xrightarrow{\mathrm{ev}} & \overline{RO}(G) \end{array}$$

where $r$ is realification and $\bar{r}(x) = x + \bar{x}$. Recall that $I(G)$ is the ideal in $R(G)$ defined by $I(G) = \{x \in R(G) \mid \mathrm{res}_P(x) = 0 \text{ for all } P \in \mathscr{P}\}$ and $IO(G) \subset RO(G)$ is similarly defined.

*Lemma* **2.8.** — *The realification homomorphism* $r : I(G) \to IO(G)$ *is surjective.*

*Proof.* — First observe that $\mathrm{coker}\, r$ is annihilated by 2 because complexification followed by realification is multiplication by 2. We assert that $\mathrm{coker}\, r$ is annihilated by an odd integer too; so $\mathrm{coker}\, r = 0$. The assertion follows from a diagram chase in the above diagram and uses these facts:

(i) $\mathrm{Ker}\, r = \{x - \bar{x} \mid x \in R(G)\}$. (Here $^-$ denotes the involution in $R(G)$ induced by the automorphism of G sending $g$ to $g^{-1}$.) $\mathrm{Ker}\, \bar{r} = \{x - \bar{x} \mid x \in \overline{R}(G)\}$. Here $^-$ denotes complex conjugation in each factor of $\overline{R}(G)$.)

(ii) $|G|$ is odd.
(iii) If $x \in R(G)$ or $RO(G)$, then $res_P(x) = 0$ for all $P \in \mathscr{P}$ if and only if $ev_C(x) = 0$ whenever $C \in \mathscr{P}$ is cyclic.
(iv) $|G|.\bar{R}(G) \subset ev(R(G))$.

The last statement has an easy proof due to R. Lyons. It suffices to exhibit for each cyclic subgroup C an element $f_C \in R(G)$ such that $ev(f_C)$ has C coordinate $|G|$ and all other coordinates 0. View $R(G)$ as a ring of complex functions on G. Note that the function $f_C$ which sends $g \in G$ to $|G|$ if $g$ generates C and to 0 otherwise is in $R(G)$. To see that $f_C$ is in $R(G)$ note that its inner product with an irreducible character is an algebraic integer (in $\mathbf{Z}[\xi_C]$) which is Galois invariant; so it is in $\mathbf{Z}$. This ends the proof of the lemma.

In order to apply the results of this paper, one needs to produce elements in $I'(G)$ or $IO'(G)$. Here are two relevant points about this (which lie behind the example in section 1): (i) If $\varphi : G \to L$ is a surjective homomorphism, then $\varphi^* : F(L) \to F(G)$ where F is $I'$ or $IO'$. An especially good choice for L is a cyclic group which is not a $p$-group. (ii) The Adams operations $\psi^k$ can be used to construct elements in $I'(G)$. For each $p$-Sylow subgroup P of G choose an integer $a_p$ prime to the order of G and congruent to 1 mod the order of P. Then for any representation R in $\mathscr{R}$,

$$\Pi(\psi^{a_p} - 1)\, R \in I'(G).$$

Just note $(\psi^{a_p} - 1)\, \mathrm{Res}_p R = 0$ in $R(P)$ and $(\psi^{a} - 1)\, R^H = [(\psi^{a} - 1)\, R]^H = 0$ in $R(G/H)$.

## 3. Framings and equivariant cell attachment

*Background*

We begin with a few words of motivation and notation. Recall that G is abelian. For any group $\Gamma$, $E(\Gamma)$ is a contractible space with free $\Gamma$ action and $E = E(G)$. If $H \subset G$ and R is a representation of G, $res_H R$ is its restriction to H and $|H|$ is the order of H. We denote by $I(G)$ the ideal in $R(G)$ consisting of those elements which are in the kernel of $res_P$ for each Sylow subgroup P of G. Suppose W is a sphere or disk with a smooth action of G. Then Smith theory implies:

(i) $W^H$ is a mod $p$ sphere or disk if $H \subset G$, $|H| = p^n$, $p$ prime. In particular $\pi_k(W^H) \otimes Z_p = 0$ if $k < \dim W^H$.

Smith theory or the Atiyah-Singer Index Theorem implies:

(ii) If $R, R' \in \{T_x W \mid x \in W^G\}$, then $res_P R \cong res_P R'$ for each Sylow subgroup P of G. (If W is a sphere, $p$ must be odd.)

The Atiyah Completion Theorem implies:

(iii) If $R'$ and R are complex representations whose difference is in $I(G)$, then $E \times R'$ and $E \times R$ are stably isomorphic G vector bundles over E.

Equivariant surgery is a process for reducing the homotopy groups of $W^H$, $H \subset G$, where W is a smooth G manifold. In one setting it may be described like this: Given is a representation R of G and some sort of bundle isomorphism (framing) $b : TW \to W \times R$ and an element $a \in \text{image}(\pi_k(\partial W^H) \to \pi_k(W^H))$. There are two steps to surgery using this data:

(iv) Using $a$ and the framing $b$, an equivariant handle is attached to W giving $W' \supset W$ and $a$ is killed in $\pi_k(W'^H)$.
(v) The framing is extended over TW'.

If the definition of framing is too strong (e.g. if $b$ is a stable G vector bundle isomorphism), then $T_x W$ is isomorphic to R as a representation of G for all $x \in W^G$. If the definition is too weak, it may not be possible to achieve (iv) or (v).

It is natural to use surgery to convert a manifold W with framing into a sphere or disk by achieving (i) for all primes $p$. Since we do not want $T_x W = R$ for all $x \in W^G$, we must avoid too strong of a definition of framing. Point (ii) gives some guidance. Since we must have isomorphisms $\varepsilon_p : \text{res}_P R' \cong \text{res}_P R$ for each $p$-Sylow subgroup P of G, we might postulate stable P vector bundle isomorphisms $b_p : \text{res}_P TW \to \text{res}_P W \times R$ for each prime $p$ as part of the definition of framing. This is sufficient to achieve (iv) but not (v). For that we introduce the notion of an R-P framing and insist that $b_p$ be an R-P framing. This means $b_p$ is a compromise between a stable G vector bundle isomorphism and a stable P vector bundle isomorphism. With the collection $\{b_p\}$ we can achieve (iv) and (v) for all $H \neq 1$. We treat the group $H = 1$ using a stable G vector bundle isomorphism $b_\infty : E \times \eta \cong E \times W \times R$ manufactured from (iii). The collection $\{b_p, b_\infty\}$ provides the needed data for a "framing". This is called a *Smith framing* and its precise definition occurs in 3.24. We should emphasize the role of the assumption that G be abelian. For these groups complex representations are represented by diagonal matrices; so if R and R' are complex representations of G whose restrictions to $P \subset G$ are isomorphic, there is a P isomorphism $\varepsilon_p$ between them such that $\theta(\varepsilon_p) \in \Delta_P(R)$—a torus. For a general group this would only be an element of $\Delta'_P(R)$ which is not a torus. See the discussion after 3.33.

In this section we treat the part of surgery dealing with framings of equivariant bundles. This does not require restriction to manifolds or their tangent bundles. The main results are 3.12 and 3.13. We use these in the next section to treat the manifold case which needs some added input concerning framings and embeddings.

*Smith framing—desired formal properties*

One of the basic constructions in homotopy theory is to attach a cell $D^{k+1}$ to a topological space W via a map $i : S^k \to W$ representing an element $a \in \pi_k(W)$. This data produces a new space $O = W \cup_i D^{k+1}$ in which $a$ is null homotopic. Surgery can be viewed as an elaboration of this process in the smooth category. There the initial data includes also a vector bundle $\eta$ over W (which in practice is the tangent bundle TW) and a stable vector bundle isomorphism (framing)

$$(3.1) \qquad b: \eta \to W \times R \quad \text{where } R = \mathbf{R}^n.$$

This gives rise to a stable vector bundle isomorphism

$$(3.2) \qquad \ell(b): S^k \times R \to i^* \eta\big|_{S'} \qquad S' = iS^k$$

and if $k < n$ a vector bundle isomorphism

$$(3.3) \qquad \ell'(b): S^k \times R \to i^* \eta$$

whose stabilization is $\ell(b)$. Using this we obtain a vector bundle $\Gamma = \Gamma(\eta, a, b)$ over O defined by

$$(3.4) \qquad \Gamma = \eta \cup D^{k+1} \times R$$

and a framing $b': \Gamma \to O \times R$ extending $b$. Here $S^k \times R$ is identified with $i^* \eta$ via $\ell'(b)$.

Now we discuss an equivariant analog of this construction. We have noted there is ambiguity as to what the definition of framing should be; however, that is usually dictated by application. We develop one notion of framing called *Smith framing*, which reflects Smith theory and the Atiyah Completion Theorem mentioned earlier. First, let us specify what a Smith framing should provide. For this we need some notation: W is a G space, $\eta$ is a G vector bundle over W, R is a representation of G and $a \in \pi_k(W^H)$ for a given subgroup H of G. Let

$$(3.5) \qquad n + 1 = \dim R^H, \quad S = S^k \times D^{n-k} \quad \text{and} \quad D = D^{k+1} \times D^{n-k}.$$

View S and D as H spaces with trivial H action. Let

(3.6) $i: S \to W^H$ be a map such that $i\big|_{S^k}$ represents $a$. Briefly we say that $i$ *represents* $a$.

Let X be an H space. Set $\text{ind}_H^G X = G \times_H X$. This is a G space. If X′ is a G space and $f: X \to X'$ is an H map, $\text{ind}_H^G f: \text{ind}_H^G X \to X'$ is the G map induced by $f$ i.e. $\text{ind}_H^G f[g, x] = gf(x)$ for $[g, x] \in G \times_H X$. Set

$$(3.7) \qquad O = W \cup_f \text{ind}_H^G D \quad (f = \text{ind}_H^G i : \text{ind}_H^G S \to W^H).$$

Then O is a G space in which $a$ is null homotopic. When R is a representation of G, we let $\widetilde{R}$ denote the G vector bundle over W whose total space is $W \times R$. Since the base space W of $\widetilde{R}$ does not explicitly appear in the notation, it must be determined from the context. Let $A = \{p \mid p \text{ is prime and } p \text{ divides } |G|\}$ and P be the $p$-Sylow subgroup of G. Henceforth G space (resp. G subspace) means G c.w. complex (resp. G subcomplex).

A Smith framing of $\eta$ is defined relative to a *Smith decomposition* **W** of W—the base space of $\eta$. By definition $\mathbf{W} = \{W_p, i_p \mid p \in A\}$ where $W_p$ is a G subspace of W such that the isotropy group $G_x$ at $x$ satisfies

**(3.8)** $$|G_x| = p^{n(x)} \quad \text{with } n(x) > 0 \quad \text{for } x \in W_p.$$

The inclusion of $W_p$ in W is $i_p$. If we delete the requirement that $W_p$ be a subspace of W, we call **W** a *generalized Smith decomposition* of W. We write $\mathbf{W} = \{W_p \mid p \in A\}$ when each $i_p$ is an inclusion.

Here are two important examples of Smith decompositions:

**(3.9)** *W is a smooth G manifold.* Let N be an open G regular neighborhood of $\{x \in W \mid G_x \notin \mathscr{P}\}$ defined with respect to a G $C^1$ triangulation of W. See [I1], [R], [ST]. For $p \in A$, let

$$W_p = \{x \in W - N \mid |G_x| = p^{n(x)},\ n(x) > 0\}.$$

In this case $i_p$ is an inclusion. Then $\mathbf{W} = \{W_p, i_p \mid p \in A\}$.

**(3.10)** *W is a point* $x_0$. Let $x_{0p} = E(G/P)$ and let $i_p : x_{0p} \to x_0$ be the unique map. Then $\mathbf{x}_0 = \{x_{0p}, i_p \mid p \in A\}$ is a generalized Smith decomposition of $x_0$.

Throughout this section H shall be a subgroup of G whose order is $p^m$ for some prime $p \in A$. We distinguish two cases referred to as Case I and Case II. These are respectively $m > 0$ and $m = 0$ (so $H = 1$). In either case H is a subgroup of P—the $p$-Sylow subgroup of G.

Now suppose W is a G space with Smith decomposition **W** and $a \in \pi_k(W_p^H)$ in Case I or $a \in \pi_k(W)$ in Case II. In either case the G space O in (3.7) is defined. It has a natural Smith decomposition **O** defined in terms of **W** and $a$. The spaces $O_q$ for $q \in A$ are in the two cases:

*Case I*: $O_p = W_p \cup_f \operatorname{ind}_H^G D$, $O_q = W_q$ $q \neq p$, if $f = \operatorname{ind}_H^G i$.

*Case II*: $O_q = W_q$ for all $q$.

Theorem 3.12 provides one of the important tools in the main geometric construction in equivariant surgery. There it is applied with $\eta$ the tangent bundle of W. In 3.12 we use these notations and hypotheses:

**(3.11)**
(i) $\omega$ is the H vector bundle over D defined by $\omega = D \times R$ (3.5).
(ii) W is a G space with this property: each component of $W^H$ contains a point whose isotropy group is H.
(iii) **W** is a Smith decomposition of W.
(iv) $\eta$ is a stable vector bundle over W (3.34).
(v) $a \in \pi_k(W_p^H)$ in Case I or $a \in \pi_k(W)$ in Case II and $k < \dim R^H$.

*Theorem* **3.12.** — *Assume* (3.11). *Let* $\beta : \eta \to R$ *be a Smith framing of* $\eta$ rel **W** (3.23). *Then there is an* H *vector bundle isomorphism* $\ell'(\beta) : \omega\big|_S \to i^*(\eta\big|_{S'})$ *which defines the* G *vector bundle* $\Gamma = \Gamma(\eta, a, \beta) = \eta \cup_f \mathrm{ind}_H^G\, \omega$ *over* O, $(f = \mathrm{ind}_H^G\, \ell'(\beta))$. *There is a Smith framing* $\beta' : \Gamma \to \widetilde{R}$ *rel* **O** *extending* $\beta$.

Smith decompositions and Smith framings restrict to subgroups and subspaces. Here is a discussion: First we show how to restrict a Smith decomposition **W** of a G space W to a subgroup H of G and to a G subspace X of W. For the latter the notation is $\mathbf{W}\big|_X$. This is the Smith decomposition of X with $X_p = X \cap W_p$ for each prime $p$ which divides $|G|$. The H Smith decomposition $\mathrm{res}_H\, \mathbf{W}$ of $\mathrm{res}_H\, W$ is defined as follows: $\mathrm{res}_H\, \mathbf{W} = \{ W'_p \,|\, p \,|\, |H| \}$ where $W'_p = \{ x \in W_p \,|\, |G_x \cap H| = p^{n(x)}, n(x) > 0 \}$. Now we discuss restriction for Smith framings. Let $\beta$ be a Smith framing rel **W** (3.23) where W is a G space. Let H be a subgroup of G and X a G subspace of W. Restricting G data to H data gives a Smith framing $\mathrm{res}_H\, \beta$ rel $\mathrm{res}_H\, \mathbf{W}$ and restricting G data to X gives a Smith framing $\beta\big|_X$ rel $\mathbf{W}\big|_X$. Smith framings are related to the ideal I(G).

*Theorem* **3.13.** — *Let* R *and* R′ *be complex representations of* G *such that* $R' - R \in I(G)$. *Let* W *be any* G *space and* **W** *a generalized Smith decomposition of* W. *Then there is a Smith framing* $\beta_W(R', R) : \widetilde{R}' \to \widetilde{R}$ rel **W** *which is natural with respect to restriction to subgroups and subspaces.*

*Smith framings—definitions*

We proceed with the definition of Smith framing and the proofs of 3.12 and 3.13 by first providing some definitions and conventions. If M is a representation of G and $\eta$ is a G vector bundle over W, then $s_M(\eta) = \eta \oplus \widetilde{M}$ and $s(\eta)$ denotes $s_M(\eta)$ for some M. Similarly for the representation R, $s_M(R) = R \oplus M$ and $s(R) = s_M(R)$ for some M. If $\eta'$ is another G vector bundle over W and $b : \eta \to \eta'$ is a G vector bundle isomorphism, then $s_M(b) : b \oplus 1_{\widetilde{M}} : s_M(\eta) \to s_M(\eta')$ is a G vector bundle isomorphism. A stable G vector bundle isomorphism $b : \eta \to \eta'$ is by definition a G vector bundle isomorphism $b : s_M(\eta) \to s_M(\eta')$ for some M. Sometimes we write $b : s(\eta) \to s(\eta')$ to mean that $b$ is a stable G vector bundle isomorphism and if $b$ is a G vector bundle isomorphism, $s(b) : s(\eta) \to s(\eta')$ means $s(b) = s_M(b)$ for some M.

There is one point where we need to use $s_M$ where M is allowed to be a countable direct sum of finite dimensional representations. This is to apply the Atiyah Completion Theorem. In this case $s_M(\eta)$ is only used when G acts freely on the base space W. Then G vector bundles over W are equivalent to vector bundles over W/G and a G vector bundle isomorphism $b : s_M(\eta) \to s_M(\eta')$ means an isomorphism $s_M(\eta)/G \to s_M(\eta')/G$ of $U_\infty$ vector bundles over W/G. Here $U_\infty$ is the infinite unitary group or infinite orthogonal group depending on whether $\eta$ is a complex or real vector bundle.

We associate certain G spaces to a representation R of G and a $p$-Sylow subgroup P of G:

U(R) — This is the space of $n \times n$ unitary matrices when R is a complex representation of dimension $n$. When R is a real $n$ dimensional representation, this is the group $O_n$.

M(R) — This is the maximal torus in U(R).

$\Delta_P(R)$ — This is the space of maps of G/P to M(R) which carry the identity coset $\bar{1} \in G/P$ to the identity matrix $1 \in M(R)$. Denote by $1 \in \Delta_P(R)$ the map which carries G/P to $1 \in M(R)$. We make $\Delta_P(R)$ a G space via

$$(gf)(\bar{h}) = f(\bar{h}\bar{g})f(\bar{g})^{-1}.$$

Here $g \in G$, $\bar{g}$ is its coset in G/P, $\bar{h} \in G/P$ and $f \in \Delta_P(R)$. Note that $P \subset G$ acts trivially on $\Delta_P(R)$. It is a torus.

$\Delta'_P(P)$ — This is the space of maps of G/P to U(R) which carry $\bar{1}$ to $1 \in U(R)$, so $\Delta_P(R)$ is a subspace of $\Delta'_P(R)$.

If $\eta$ is a G vector bundle over W and $b : \eta \to \tilde{R}$ is a P vector bundle isomorphism, define $\theta(b) : W \to \Delta'_P(R)$ by

**(3.14)** $$b_{gx} = \theta(b)(x)(\bar{g})\, g b_x g^{-1}$$

for $g \in G$, $x \in W$. Here $b_x : \eta_x \to \tilde{R}_x$ is the map on the fiber over $x$. The proof of the next lemma is immediate from definitions. Compare $[B_2, 11.1]$ in the case $P = 1$.

*Lemma* **3.15.** — *If* $b : \eta \to R$ *is a* P *vector bundle isomorphism and if*

$$\theta(b)(x) \in \Delta_P(R) \subset \Delta'_P(R) \quad \textit{for all } x \in W,$$

*then* $\theta(b)$ *is a* G *map from* W *to* $\Delta_P(R)$.

There is an obvious inclusion $s : \Delta_P(R) \to \Delta_P(sR)$ and if $b : \eta \to R$ is a P vector bundle isomorphism, $s\theta(b) = \theta(sb)$.

By definition an R *framing* of $\eta$ is a stable G vector bundle isomorphism $b : s(\eta) \to s(\tilde{R})$ and $b$ is called the R framing. We usually write $b : \eta \to \tilde{R}$ deleting $s$. If $b$ is only equivariant with respect to $P \subset G$, we say $b$ is a $\mathrm{res}_P$ R *framing* of $\eta$. For such a $\mathrm{res}_P$ R framing we define

**(3.16)** $$\theta(b) : W \to \Delta'_P(sR)$$

using (3.14). Of course $\theta(b)$ maps W to 1 if $b$ is an R framing.

The cone $c(W)$ on W is a G space and the cone point $x_0 \in c(W)$ is fixed by G. The restriction of a map $f : c(W) \to W'$ to $W \subset c(W)$ is denoted by $f_0$.

*Definition* **3.17.** — An R-P *framing* $b$ of $\eta$ is (i) a $\mathrm{res}_P$ R framing $b$ such that $\theta(b) : W \to \Delta_P(sR)$ together with (ii) a G map $\theta(b) : c(W) \to \Delta_P(sR)$ such that $\theta_0(b) = \theta(b)$ and $\theta(b)(x_0) = 1$. It is worth noting that $\Delta_P(R)^G = \mathrm{Hom}(G/P, M(R))$

is discrete; so if $b$ is an R-P framing of $\eta$, $\eta_x$ is isomorphic to R as a representation of G for all $x \in W^G$. This is the reason for excluding the G fixed points of W in $W_p$ (3.8); so R-P framings of $\eta|_{W_p}$ impose no condition on $\eta_x$ for $x \in W^G$.

We observe that U(R) is a G space with

**(3.18)** $$gu \overset{\text{def}}{=} R(g)\, uR(g)^{-1} \quad \text{for } g \in G \quad \text{and} \quad u \in U(R).$$

Here $R(g)$ is the matrix in U(R) given by the representation R. We abbreviate the right side of this equality by $gug^{-1}$. When R and P are fixed, we abbreviate $\Delta_P(R)$ and U(R) by $\Delta$ and U. The next definition requires a G map $\theta : W \to \Delta$.

*Definition* **3.19.** — A G* *map* $f: W \to U$ is a map which satisfies

**(3.20)** $$f(gx) = gf(x)\, g^{-1}.\theta(x)\,(\bar{g})^{-1} \quad \text{for } x \in W,\ g \in G.$$

Here dot denotes multiplication in U. Sometimes to emphasize the role of $\theta$ we say that $f$ is a G* map (*wrt* $\theta$). Note that when $\theta = 1$ (the map which sends W to $1 \in \Delta$), $f$ is a G map with respect to the G action on U defined in (3.18).

The notion of a G* map is utilized as follows: Let $\widetilde{b}$ be an R framing of $\eta$ and let $b$ be an R-P framing of $\eta$. Then for each $x \in W$, we have $\widetilde{b}_x = \theta(\widetilde{b}, b)(x)\, b_x$ for some $\theta(\widetilde{b}, b)(x) \in U(sR)$. This defines a map $\theta(\widetilde{b}, b) : W \to U(sR)$ and

**(3.21)** $$\widetilde{b} = \theta(\widetilde{b}, b)\, b.$$

One easily verifies that $\theta(\widetilde{b}, b)$ is a G* map (*wrt* $\theta(b)$).

*Definition* **3.22.** — An R *framing* $\widetilde{b}$ of $\eta$ compatible with the R-P framing $b$ of $\eta$ is (i) an R framing $\widetilde{b}$ together with (ii) a G* map (*wrt* $\theta(b)$) $\theta(\widetilde{b}, b) : c(W) \to U(sR)$ such that $\theta_0(\widetilde{b}, b) = \theta(\widetilde{b}, b)$ and $\theta(\widetilde{b}, b)(x_0) = 1$.

Recall that $E = E(G)$. Let W be a G space and $\hat{W} = E \times W$. This is a G space. If $f: W \to W'$ is a G map, then $\hat{f}: \hat{W} \to \hat{W}'$ is the map $1_E \times f$.

Given a (generalized) Smith decomposition **W** of W and a G vector bundle $\eta$ over W, define

**(3.23)** $$\eta_p = i_p^* \eta \quad \text{(cf. (3.9))}.$$

*Definition* **3.24.** — A *Smith framing* $\beta : \eta \to \widetilde{R}$ rel **W** is a collection $\{b_p, b_\infty \mid p \in A\} = \beta$ where $b_p$ is an R-P framing of $\eta_p$ and $b_\infty$ is an R framing of $\hat{\eta}$ such that $b_{\infty p} = \hat{i}_p^* b_\infty : \hat{\eta}_p \to \widetilde{R}$ is compatible with $\hat{b}_p$.

*Remarks.* — An *implicit* part of the data of a Smith framing is the collection of maps $\{\theta(b_p), \theta(b_{\infty p}, \hat{b}_p) \mid p \in A\}$ involved with the R-P framing $b_p$ and the compatibility of $b_p$ and $\hat{b}_p$. Note that $\hat{b}_p = 1_E \times b_p$ is P equivariant! It is an R-P framing where $\theta(\hat{b}_p)$ is the composition $c(\hat{W}_p) \to c(W_p) \to \Delta_p(sR)$. The last map is $\theta(b_p)$.

*Proofs of* 3.12 *and* 3.13

Since the techniques of the proof of 3.13 are used in the proof of 3.12, we treat 3.13 first. In addition to the Atiyah Completion Theorem, it rests on these two elementary lemmas.

*Lemma* **3.25.** — *If* $\eta$ *is a* G *vector bundle over* W, G/P *acts freely on* W/P, $H^1(W, \mathbf{Z}) = 0$ *and* $b$ *is a* $\mathrm{res}_P$ R *framing of* $\eta$ *such that* $\theta(b)$ *has values in* $\Delta_P(sR)$, *then* $b$ *defines an* R-P *framing of* $\eta$.

*Lemma* **3.26.** — *Suppose* W *is a* G *space such that* G *acts freely on* W *and suppose* $K^1(W/G) = 0$. *If* $b$ *is an* R-P *framing of* $\eta$ *and* $\tilde{b}$ *is an* R *framing of* $\eta$, *then* $\tilde{b}$ *is compatible with* $b$.

What 3.25 asserts is the existence of $\theta(b) : c(W) \to \Delta_P(sR)$ satisfying 3.17 and 3.26 asserts the existence of $\theta(\tilde{b}, b) : c(W) \to U(sR)$ satisfying 3.22. We begin preparation for 3.25 and 3.26.

The space of $G^*$ maps (3.19) from W to $U = U(R)$ is denoted by $(W, U)^{G^*}$. This space is closely related to the space $(W, W \times U)^G$ of G maps from W to $W \times U$ where $W \times U$ is made into a G space via

$$g(x, u) = (gx, gug^{-1}\theta(x)(\bar{g})^{-1}), \quad x \in W, \ g \in G.$$

*Lemma* **3.27.** — *There are maps*

$$\psi : (W, U)^{G^*} \to (W, W \times U)^G \quad \textit{and} \quad \lambda : (W, W \times U)^G \to (W, U)^{G^*}$$

*such that* $\psi\lambda$ *is the identity.*

*Proof.* — Define $\psi(f)(x) = (x, f(x))$ and define $\lambda(h) = \pi h$ where $\pi : W \times U \to U$ is projection.

There is a $G^*$ homotopy extension theorem which goes like this. Let W be a G space, I the unit interval with trivial action, $S' = W \times I$, $S = A \times I \cup W \times 0$ where A is a closed G invariant subspace of W and $\theta : S' \to \Delta = \Delta_P(R)$.

*Lemma* **3.28.** — *Given any* $G^*$ *map* $f : S \to U$, *there is a* $G^*$ *extension* $F : S' \to U$.

*Proof.* — Let $H : S' \to S' \times U$ be a G map which extends $\psi(f)$. This exists by the G homotopy extension theorem. Define $F = \lambda(H)$. Then F is a $G^*$ map by Lemma 3.27 which extends $f$ because $\lambda\psi = 1$.

Decompose the cone $c$W as $W' \cup W''$ where $W' = \{(x, t) \mid x \in W, 0 \le t \le 1/2\}$ and $W' \cap W'' = W \times 1/2$. Call a G map $\theta : cW \to \Delta$ *special* if $\theta$ maps $W''$ to 1. Given any G map $\theta$ of $c(W)$ to $\Delta$, such that $\theta(x_0) = 1$, there is a special map $\theta'$ such that $\theta' = \theta$ on $W = W \times 0$. Just take any G map $T : cW \to cW$ such that $T|_W$ is the identity and $TW'' = x_0$. Then $\theta \circ T = \theta'$ is special. Let $[W, W']^G$ denote the set of G homotopy classes of maps from W to W′. Let $U = U(R)$ be the G space with action defined by (3.18). Write $[W, U]^G = 0$ if this set contains only the map of W to $1 \in U$.

*Theorem* **3.29.** — *If* $[W, U]^G = 0$ *and* $\theta : c(W) \to \Delta$ *is special, then any* $G^*$ *map* (*wrt* $\theta|_W$) $f : W \to U$ *extends to a* $G^*$ *map* $F : cW \to U$ *such that* $F(x_0) = 1 \in U$.

*Proof.* — By the $G^*$ homotopy extension lemma 3.28 with $A = \emptyset$, there is a $G^*$ extension $F'$ of $f$ to W′. Since $\theta$ is special, $\theta(W \times 1/2) = 1$; so $f' = F|_{W \times 1/2} \in [W, U]^G$. (See 3.19.) Since this set is zero by hypotheses, $f'$ extends to a G map $F'' : W'' \to U$ such that $F''(x_0) = 1$. Then $F' \cup F'' = F$ is the desired $G^*$ extension of $f$.

Let $\mathbf{C}^\infty$ denote the countable direct sum of copies of $\mathbf{C}$ with trivial action of G. Then the inclusion of $U(\mathbf{C}^\infty)$ into $U(\mathbf{C}^\infty \oplus R)$ which sends $u$ to $1_{\mathbf{C}^\infty} \oplus u$ is a G map which is a homotopy equivalence for any complex representation R of G; so $[W, U(\mathbf{C}^\infty)]^G \to [W, U(\mathbf{C}^\infty + R)]^G$ is bijective if G acts freely on W.

*Corollary* **3.30.** — *If* G *acts freely on* W, $K^1(W/G) = [W, U(sR)]^G$ *for any* R.

*Proof.* — The left side is $[W/G, U(\mathbf{C}^\infty)] = [W, U(\mathbf{C}^\infty)]^G$ by definition. The right side is $[W, U(\mathbf{C}^\infty + R)]^G$ for $sR = \mathbf{C}^\infty + R$.

*Theorem* **3.31** $[A_1]$. — *If* E′ *is a contractible space with free* G *action*, $K^1(E'/G) = 0$.

*Lemma* **3.32.** — *If* W *is a* G *space such that* $H^1(W, \mathbf{Z}) = 0$ *and* G/P *acts freely on* W/P, *then any* G *map* $\theta : W \to \Delta$ *extends to a (special)* G *map* $\theta : c(W) \to \Delta$ *with* $\theta(x_0) = 1$.

*Proof.* — As P acts trivially on $\Delta$, $\theta$ factors through W/P. Now, G/P acts freely on W/P and $\Delta$ is a torus; so $\pi_i(\Delta) = 0$ for $i > 1$, and G/P homotopy classes of maps of W/P are in $1 - 1$ correspondence with $H^1(W/G, \pi_1(\Delta)) = 0$. To see this, note that G maps of W to $\Delta$ are in $1 - 1$ correspondence with sections of the fibration $\Delta \to W \times_G \Delta \to W/G$ $[B_1]$. Since the fiber $\Delta$ is a torus, the statement follows. Thus $\theta$ is G homotopic to the map which takes W to 1. A homotopy between them produces $\theta$. By the discussion preceeding 3.31, we may suppose that $\theta$ is special.

*Proof of* 3.25. — Lemma 3.32 provides an extension $\theta : c(W) \to \Delta$ of $\theta(b)$. Set $\theta(b) = \theta$. Then $b$ and $\theta(b)$ define an R-P framing of $\eta$.

*Proof of* 3.26. — We have $[W, U(sR)]^G = K^1(W/G) = 0$ (3.30). Apply 3.29 to $\theta = \theta(b) : c(W) \to \Delta_P(sR)$ and $f = \theta(\tilde{b}, b) : W \to U(sR)$. This produces an extension $\theta(\tilde{b}, b) : c(W) \to U(sR)$ such that $\theta(\tilde{b}, b)(x_0) = 1$; so $\tilde{b}$ is compatible with $b$. (To use 3.29 we have implicitly assumed $\theta(b)$ to be special. As noted, this does not restrict generality.)

The next lemma needs the notion of a map $\mathbf{f}: \mathbf{W} \to \mathbf{W}'$ between generalized Smith decompositions of W and W'. By definition this means

$$\mathbf{f} = \{f_p, f_\infty \mid p \in A\}$$

where $f_p : W_p \to W'_p$ and $f_\infty : W \to W'$ are G maps such that $f_\infty i_p = i_p f_p$ for $p \in A$. Lemma 3.33 shows that $\mathbf{x}_0$ (3.10) is universal in the sense that any generalized Smith decomposition maps to it.

*Lemma* **3.33.** — *If* W *is any* G *space and* **W** *is a generalized Smith decomposition of* W, *then there is a map* $\mathbf{f}: \mathbf{W} \to \mathbf{x}_0$.

*Proof.* — Let $f_\infty : W \to x_0$ be the unique map. Since G/P acts freely on $W_p/P$, there is a G/P map $f'_p : W_p/P \to x_{0p}$ which classifies this free action. Let $f_p$ be the composition of this with projection of $W_p$ on $W_p/P$. Then $\mathbf{f} = \{f_p, f_\infty \mid p \in A\}$.

Let R and R' be two complex representations of G of the same dimension. Since G is abelian, we may suppose the matrices $R(g)$ and $R'(g)$ for $g \in G$ representing G via R and R' are diagonal. This means $R(g)$ and $R'(g)$ lie in $M(R) = M(R')$. The normalizer of M(R) in U(R) is the group of matrices permuting the complex coordinates of R. This means that if R and R' are isomorphic as representations of P, there is a P isomorphism $\varepsilon_p : R' \to R$ such that $\varepsilon_p$ normalizes M(R). Then for all $g \in G$

$$\theta(\varepsilon_p)(\bar{g}) = \varepsilon_p R'(g) \varepsilon_p^{-1} R(g)^{-1} \in M(R)$$

by 3.14; so

$$\theta(\varepsilon_p) \in \Delta_P(R).$$

If $R' - R \in I(G)$, then the P isomorphisms $\varepsilon_p$ are defined for all $p \in A$.

*Proof of* 3.13. — First we treat the case in which $W = x_0$ is a point and **W** is $\mathbf{x}_0$ (3.10) and produce a Smith framing $\beta_{x_0} : \widetilde{R}' \to \widetilde{R} \text{ rel } \mathbf{x}_0$. Of course in this case $\widetilde{R}'$ and $\widetilde{R}$ are just R' and R viewed as G vector bundles over $x_0$. To fit previous notation, set $\eta = \widetilde{R}'$. Let $b_p = i_p^* \varepsilon_p : \eta_p \to \widetilde{R}$. Here $\eta_p = i_p^* \eta$. This is a G vector bundle over E(G/P) which is contractible. In addition $\theta(b_p)(x) = \theta(\varepsilon_p) \in \Delta_P(sR)$ for all $x \in E(G/P)$; so 3.25 implies that $b_p$ defines an R-P framing of $\eta_p$. Since $R' - R \in I(G)$, the Atiyah Completion Theorem implies that there is an R framing $b_\infty : \hat{\eta} = E \times R' \to E \times R = \widetilde{R}$. Note that $\hat{\eta}_p$ is a G vector bundle over $E \times E(G/P)$ on which G acts freely and $K^1(E \times E(G/P)/G) = 0$ by 3.31. Then 3.26 implies that $b_{\infty p} = \hat{i}_p^* b_\infty$ is compatible with $\hat{b}_p$. Taken together these facts imply that $\beta_{x_0}(R', R) = \{b_p, b_\infty \mid p \in A\}$ is a Smith framing. This completes the case when $\mathbf{W} = \mathbf{x}_0$. Now let W be a G space and **W** a generalized Smith decomposition. Then there is a map $\mathbf{f}: \mathbf{W} \to \mathbf{x}_0$ (3.33). Set $\beta_W(R', R) = \mathbf{f}^* \beta_{x_0}(R', R)$.

Now we begin the treatment of 3.12. Let W be a G space and $\eta$ be a *stable* G vector bundle over W. This means

**(3.34)** For each $x \in W$ and each irreducible representation $\chi$ of $G_x$, either the multiplicity $m_\chi(\eta_x)$ of $\chi$ in the $G_x$ representation $\eta_x$ is 0 or $\dim \eta_x^{G_x} \leq d_\chi\, m_\chi(\eta_x)$. Here $d_\chi$ is 1, 2 or 4 (5.5) depending on $\chi$. See [PR, Ch. 3, § 9] or [$P_3$, 3.6].

As before $H \subset P$. We shall use one of the framings

**(3.35)** (i) $b$ is an R-P framing of $\eta$ or
(ii) $b$ is an R framing of $\hat{\eta}$.

and an element $a \in \pi_k(W^H)$ with $k < \dim R^H$ to define an H vector bundle isomorphism

**(3.36)** $$\ell'(b) : \omega\big|_S \to i^*(\eta\big|_{S'}), \quad \omega = D \times R = R$$

and the G vector bundle $\Gamma = \Gamma(\eta, a, b)$ over O

**(3.37)** $$\Gamma = \eta \cup_f \operatorname{ind}_H^G \omega, \quad f = \operatorname{ind}_H^G \ell'(b)$$

extending $\eta$. As before $i : S \to W^H$ represents $a$ and $S' = iS$. We also view $i$ as a map of S to $\hat{W}$ with image $S'$ as follows: Let $z \in F$ be any point. Then $S'$ is identified with $z \times S' \subset E \times W = \hat{W}$ and $i$ is the composition $S \to S' \subset \hat{W}$. Then $\hat{\eta}\big|_{S'} = \eta\big|_{S'}$ and in either case in 3.35 we have that $i^*(b\big|_{S'}) : i^* s(\eta\big|_{S'}) \to i^* sR = s(\omega\big|_S)$.

The definition of $\ell'(b)$ depends on the choice of $b$ in 3.35 which in turn depends on whether H falls in Case I ($|H| = p^m$, $m > 0$) or Case II ($H = 1$). The convention is to take 3.35 (i) in Case I or 3.35 (ii) in Case II. In either case define

**(3.38)** $$\ell(b) = (i^* b\big|_{S'})^{-1} : s\omega\big|_S \to i^* s\eta\big|_{S'}.$$

This is a *stable* H vector bundle isomorphism. Since $\eta$ is stable and $k < \dim R^H$ and W satisfies 3.11 (ii), there is an H vector bundle isomorphism

**(3.39)** $$\ell'(b) : \omega\big|_S \to i^*(\eta\big|_{S'})$$

such that $s\ell'(b) = \ell(b)$ up to regular H homotopy. See [PR, Ch. 3, § 9] or [$P_3$, 3.8].

By definition two H vector bundle isomorphisms are *regularly* H *homotopic* if there is a homotopy of H vector bundle isomorphisms connecting them.

*Lemma* **3.40.** — *Case I: Any* R-P *framing of* $\eta$ *extends to an* R-P *framing* $b'$ *of* $\Gamma$. *Case II: Any* R *framing of* $\hat{\eta}$ *extends to a* R *framing* $b'$ *of* $\hat{\Gamma}$.

*Proof.* — Since $s\ell'(b) = \ell(b)$ up to regular H homotopy, $\Gamma' = s\eta \cup_f \operatorname{ind}_H^G s\omega$, ($f = \operatorname{ind}_H^G \ell(b)$), is G isomorphic to $s\Gamma$; so we must produce a P vector bundle isomorphism $b' : \Gamma' \to s\tilde{R}$ together with $\theta(b') : c(O) \to \Delta_P(sR)$ in Case I and in Case II a G vector bundle isomorphism $b' : \hat{\Gamma}' \to s\tilde{R}$.

*First case.* — By the G homotopy extension theorem, there is a G map $\theta' : c(\mathrm{O}) \to \Delta_{\mathrm{P}}(s\mathrm{R})$ which extends $\theta(b)$. Define $b'$ on $s\eta$ to be $b$. On $(\mathrm{ind}_{\mathrm{H}}^{\mathrm{G}}\, s\omega)_{gx}$ define $b'_{gx} = \theta'_0(x)(\bar{g})$ for $x \in \mathrm{D}$ and $g \in \mathrm{G}$. If $v \in (\omega|_{\mathrm{S}})_x$, observe that $gv$ in $\mathrm{ind}_{\mathrm{H}}^{\mathrm{G}}\,\omega$ is identified with $gb^{-1}(v)$ by definition of $\ell(b)$, $\mathrm{ind}_{\mathrm{H}}^{\mathrm{G}}\,\ell(b)$ and $\Gamma'$. Then

$$b(gb^{-1}(v)) = \theta(b)(x)(\bar{g})\, gv$$

by definition of $\theta(b)$. Since this is $b'(gv)$, $b'$ is well-defined. It is a P vector bundle isomorphism. Define $\theta(b')$ to be $\theta'$. Then $b'$, $\theta(b')$ defines an R-P framing $b'$ of $\Gamma$ extending $b$.

*Second case.* — Note that $\hat{\ell}(b) = \hat{i}^*(b|_{\mathrm{S}'})^{-1} = \ell$ up to regular H homotopy. This follows from the fact that E is contractible and $(b|_{\mathrm{S}'}) = (b|_{\hat{\mathrm{S}}'})|_{\mathrm{S}'}$. This means that $\hat{\Gamma}' = s\hat{\eta} \cup_f \mathrm{ind}_{\mathrm{H}}^{\mathrm{G}}\, s\hat{\omega}$, $(f = \mathrm{ind}_{\mathrm{H}}^{\mathrm{G}}\,\ell)$, up to G vector bundle isomorphism; so we need an extension $b' : \hat{\Gamma}' \to s\widetilde{\mathrm{R}}$ of $b$. Set $b' = b$ on $s\hat{\eta}$. Let $b'$ be the identity on $\mathrm{ind}_{\mathrm{H}}^{\mathrm{G}}\, s\hat{\omega} = s\widetilde{\mathrm{R}}$. This is a well-defined G vector bundle isomorphism which extends $b$.

*Lemma* **3.41.** — *Let $\eta$ be a stable* G *vector bundle over* W. *Suppose $b$ is an* R-P *framing of $\eta$ and $\tilde{b}$ is an* R *framing of $\hat{\eta}$ compatible with $\hat{b}$. Let $a \in \pi_k(\mathrm{W}^{\mathrm{H}})$ where $|\mathrm{H}| = p^m$, $m > 0$. Suppose* W *satisfies* 3.11 (ii) *and $k < \dim \mathrm{R}^{\mathrm{H}}$. Then there is an* R-P *framing $b'$ of $\Gamma = \Gamma(\eta, a, b)$ extending $b$ and an* R *framing of $\hat{\Gamma}$ extending $\tilde{b}$ which is compatible with $\hat{b}'$.*

*Proof.* — The first statement is Case I of 3.40; so we treat the second claim. Use the $\mathrm{G}^*$ homotopy extension lemma (3.28) to produce a $\mathrm{G}^*$ map (*wrt* $\theta(\hat{b}')$) $\theta' : c(\hat{\mathrm{O}}) \to \mathrm{U}(s\mathrm{R})$ which extends $\theta(\tilde{b}, \hat{b})$. Define $\tilde{b}'$ to be $\theta'_0\, \hat{b}'$ on $\hat{\Gamma}$. Then $\tilde{b}'$ is an R framing which extends $\tilde{b}$, is compatible with $\hat{b}'$ and $\theta(\tilde{b}', \hat{b}') = \theta'$.

*Proof of Theorem* 3.12. — Case I ($|\mathrm{H}| = p^m$, $m > 0$). Let $i : \mathrm{S} \to \mathrm{W}_p^{\mathrm{H}}$ be a map which represents $a$, and set $\mathrm{S}' = i\mathrm{S}$. Note that $\eta|_{\mathrm{S}'} = (\eta_p)|_{\mathrm{S}'}$ by definition of $\eta_p$ as $i_p^*\,\eta$. Let $\beta = \{b_q, b_\infty \mid q \in \mathrm{A}\}$ be the given Smith framing of $\eta$. Define $\ell'(\beta)$ by

$$\ell'(\beta) = \ell'(b_p) : \omega|_{\mathrm{S}} \to i^*(\eta|_{\mathrm{S}'}) \quad \text{(cf. 3.39)}.$$

This is the H vector bundle isomorphism which defines $\Gamma(\eta, a, \beta) = \Gamma$ as a G vector bundle over O extending $\eta$. By definition $\Gamma_p = \Gamma|_{\mathrm{O}_p}$. This is $\Gamma(\eta_p, a, b_p)$ by inspection; moreover, $\Gamma_q = \eta_q$ for $q \neq p$. By Lemma 3.41 there is an R-P framing $b'_p : \Gamma_p \to \widetilde{\mathrm{R}}$ extending $b_p$ and a G framing $b'_{\infty p} : \hat{\Gamma}_p \to \widetilde{\mathrm{R}}$ extending $b_{\infty p} = \hat{i}_p^*\, b_\infty$ which is compatible with $\hat{b}'_p$. Since $\eta_p = \eta \cap \Gamma_p$ and $\Gamma = \eta \cup \Gamma_p$, we can define $b'_\infty : \hat{\Gamma} \to \widetilde{\mathrm{R}}$ by $b'_\infty = b_\infty$ on $\hat{\eta}$ and $b'_\infty = b'_{\infty p}$ on $\hat{\Gamma}_p$. Finally set $b'_q = b_q$ for $q \neq p$. Then $\beta' = \{b'_q, b'_\infty \mid q \in \mathrm{A}\}$ is a Smith framing of $\Gamma$ which extends $\beta$.

*Case II* (H = 1). — Define

$$\ell'(\beta) = \ell'(b_\infty) : \omega\Big|_S \to i^*(\eta\Big|_{S'}) \quad \text{(cf. 3.39)}.$$

This is an H = 1 vector bundle isomorphism. Again $\Gamma(\eta, a, \beta)$ is a G vector bundle over O. The Smith decomposition $\mathbf{O} = \{O_p \mid p \in A\}$ has $O_p = W_p$ for all $p \in A$; so $\eta_p = \Gamma_p$ for all $p \in A$; hence to define $\beta'$ we need only define $b'_\infty : \hat{\Gamma} \to \tilde{\mathbf{R}}$ extending $b_\infty$. This is provided by 3.40.

## 4. An Induction Theorem

In this section we prove the Induction Theorem 4.19. It is analogous in statement to the Induction Theorem 2.6 of [DP$_1$]. There is an essential geometric difference which appears in the definition of a normal map. See 4.9 and [DP$_1$, 2.4]. The proof of the Induction Theorem [DP$_1$, 2.6] is algebraic but appeals to two geometric results about normal maps [DP$_1$, 3.12 and 3.13] which give condition for doing surgery to kill a homotopy class. Their analogs here 4.11 and 4.12 are much deeper with the definition 4.9. Once they are established, the proofs of the two induction theorems are identical. The main bulk of this section is devoted to treating 4.11 and 4.12. These are the main geometric steps in any equivariant surgery procedure. See e.g. [W], [DP$_2$], [PR].

A *manifold triad* (W; $W_0$, $W_1$) is a triple of manifolds such that $\partial W = W_0 \cup W_1$ and $W_0 \cap W_1 = \partial W_0 = \partial W_1$. A G *manifold triad* is a manifold triad such that G acts on W and respects the triad structure. A G *map of triads* is a G map which respects the triad structure. Let W be a G manifold triad. Set $X = W_0$ and assume the following *gap hypotheses*:

**(4.1)** For each subgroup K of G in $\mathscr{P}$ and each subgroup $L > K$, every component of $W^L$ has dimension less than one half the dimension of the corresponding component of $W^K$.

If X is not empty, then 4.1 holds for X in place of W and this implies each component of $X^K$ has a point whose isotropy group is K provided $X^K \neq \emptyset$ and $K \in \mathscr{P}$.

In the preceding section we treated Smith framings rel **W** where **W** was a quite general Smith decomposition. We shall need more structure for its role in the process of equivariant surgery on a G manifold W. In this case $\mathbf{W} = \mathbf{W}(\mathscr{S})$ is defined in terms of a *set* $\mathscr{S}$ *(Smith set) of submanifolds of* W *which are untouched in the surgery process.*

Let W be a G manifold with boundary X and $\mathscr{S} = \mathscr{S}(W)$ be a set of invariant submanifolds, called a *Smith set* for W, with these properties:

**(4.2)** (i) If $M \in \mathscr{S}$ and $\partial M \neq \emptyset$ then $\partial M = M \cap X$ and if $M \subset W^H$ where H is a non trivial $p$ group for some prime $p$, then

$$\dim M < \frac{1}{2} \dim W^H \quad \text{and} \quad \dim M \cap X < \frac{1}{2} \dim X^H.$$

(ii) If $L \notin \mathscr{P}$, $W^L \in \mathscr{S}$.

(iii) W has an equivariant $C^1$ triangulation ([$I_1$]) such that each $M \in \mathscr{S}$ is a subcomplex.

The Smith set $\mathscr{S}(W)$ for W, determines the Smith sets

$$\mathscr{S}(W)\big|_X = \{M \cap X \mid M \in \mathscr{S}(W)\}$$

and $$\operatorname{res}_H \mathscr{S}(W) = \{\operatorname{res}_H M \mid M \in \mathscr{S}(W)\}$$

for $X = \partial W$ and $\operatorname{res}_H W$.

Let $N(\mathscr{S})$ be an open G regular neighborhood [St] with respect to the $C^1$ triangulation in (iii) of the union $W_{\mathscr{S}}$ of the A's in $\mathscr{S}(W)$. For example take interior of the second derived neighborhood of $W_{\mathscr{S}}$ in W [St, p. 44]. Then $N(\mathscr{S}) \cap X = N(\mathscr{S}\big|_X)$ is an open regular neighborhood in X of $X_{\mathscr{S}}$ where $X_{\mathscr{S}}$ is the union of the manifolds in $\mathscr{S}\big|_X$. With respect to this data define the Smith decomposition of W:

$$\mathbf{W}(\mathscr{S}) = \{W_p \mid p \mid |G|\}$$

where $$W_p = \{x \in W - N(\mathscr{S}) \mid |G_x| = p^{n(x)} \text{ and } n(x) > 0\}.$$

The Smith decomposition $\mathbf{W}(\mathscr{S})$ is determined by $\mathscr{S}$ alone by the uniqueness of G regular neighborhoods [R]. Since we shall work with a fixed G regular neighborhood $N(\mathscr{S})$ of $W_{\mathscr{S}}$, the full strength of this is not needed.

Whenever a Smith set $\mathscr{S} = \mathscr{S}(W)$ is given explicitly for W, we shall use the Smith decomposition $\mathbf{W}(\mathscr{S})$ of W implicitly and *abbreviate the phrase Smith framing* rel $\mathbf{W}(\mathscr{S})$ *by Smith framing*. This abbreviation works smoothly with respect to $\operatorname{res}_H$ and restriction to boundary because of these two identities:

$$\operatorname{res}_H \mathbf{W}(\mathscr{S}) = \mathbf{W}(\operatorname{res}_H \mathscr{S}) \quad \text{and} \quad \mathbf{W}(\mathscr{S})\big|_{\partial W} = \partial\mathbf{W}(\mathscr{S}\big|_{\partial W}).$$

Let S stand for either W or X from above and let H be a subgroup of G. Set $S_*^H = \{x \in S - N(\mathscr{S}\big|_S) \mid G_x = H\}$.

*Lemma* **4.3.** — *If* $H \in \mathscr{P}$, W *satisfies* 4.1 *and* $k \leq \frac{1}{2} \dim S^H$, *then any map of* $S^k$ *into* $S^H$ *is homotopic to one into* $S_*^{(H)}$.

*Proof.* — Let Q be the union of the M in $\mathscr{S}\big|_S$. By 4.2 (i) and general position, any map of $S^k$ into $S^H$ can be assumed to miss $Q^H$. Since there exist G regular neighborhoods of Q in S in any open neighborhood of Q and since G regular neighborhoods are G isotopy invariant, we may suppose $S^k$ misses $N(\mathscr{S}\big|_S)^H$. Similarly using 4.1 we may suppose $S^k$ misses $S^L$ whenever $L > H$. Then $S^k$ lies in $S_*^H$.

Now let Z be a G manifold triad, $F : W \to Z$ be a G map of triads and let $X = W_0$, $Y = Z_0$ and $f = F|_X : X \to Y$. In what follows we suppose that $\mathscr{S} = \mathscr{S}(W)$ is a Smith set (4.2) for W, $\mathbf{W} = \mathbf{W}(\mathscr{S})$ and $\mathbf{X} = \mathbf{W}|_X$.

Let H be a subgroup of G. If $C_{f^H}$ is the mapping cylinder of $f^H$, then by definition $\pi_{k+1}(f^H) = \pi_{k+1}(C_{f^H}, X^H)$. An element $\mu \in \pi_{k+1}(f^H)$ is represented by a homotopy class of diagrams

$$
\textbf{(4.4)} \qquad \mu : \begin{array}{ccc} S^k & \xrightarrow{\iota'} & X^H \\ \downarrow{\scriptstyle j'} & & \downarrow{\scriptstyle f^H} \\ D^{k+1} & \xrightarrow{\kappa'} & Y^H \end{array}
$$

Set $a = \partial\mu \in \pi_k(X^H)$. Let $n = \dim X^H$ and let

$$
\textbf{(4.5)} \qquad \mu' : \begin{array}{ccc} S & \xrightarrow{i} & X^H \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle f^H} \\ D & \xrightarrow{h} & Y^H \end{array} \qquad \begin{array}{l} D = D^{k+1} \times D^{n-k} \\ S = S^k \times D^{n-k} \\ i = \partial\mu' \text{ represents } a \text{ in the sense of } 3.6 \end{array}
$$

be a diagram which gives $\mu$ by restriction to $(D^{k+1}, S^k) \subset (D, S)$. We remark that by property 4.2, we may suppose $iS \subset X^H_* \subset X^H_p$; so in particular $\mu$ is actually represented by a class in $\pi_{k+1}(f^H_p)$. Set $D' = \operatorname{ind}_H^G D$ where H acts trivially on D. As in 3.7 define

$$
\textbf{(4.6)} \qquad O = W \cup_{\operatorname{ind} i} D' \quad \text{where } \operatorname{ind} = \operatorname{ind}_H^G
$$

and extend F to $F'' : O \to Z$ by setting $F''|_{D'} = \operatorname{ind}_H^G h$.

Fix this notation:

**(4.7)** $\beta : TW \to R$ is a Smith framing rel $\mathbf{W}$; $H \in \mathscr{P}$; $n = \dim R^H = \dim W^H$; $\Omega$ is the H representation $R/R^H$; $k < \dim R^H$, $\mu \in \pi_{k+1}(f^H)$; $D = D^{k+1} \times D^{n-k}$ and $S = S^k \times D^{n-k}$ viewed as H manifolds with trivial H action; $\mathbf{D} = \operatorname{ind}_H^G D(R)$ and $\mathbf{D}_0 = \operatorname{ind}_H^G S \times D(\Omega) \subset \mathbf{D}$.

*Definition* **4.8.** — An extension $\mathscr{U}$ of $\mu \in \pi_{k+1}(f^H)$ is a commutative diagram of G maps

$$
\mathscr{U} : \begin{array}{ccc} \mathbf{D}_0 & \xrightarrow{\iota} & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ \mathbf{D} & \xrightarrow{\kappa} & Y \end{array}
$$

such that the restriction to $(D^{k+1}, S^k) \subset (\mathbf{D}, \mathbf{D}_0)$ is $\mu$. If $\iota$ in 4.8 is an embedding, then

$$W' = W \cup_\iota \mathbf{D}$$

is a smooth G manifold. There is a G map $F' : W' \to Z$ extending F with $F'\big|_{\mathbf{D}} = \kappa$. Let $\mathscr{S}' = \mathscr{S}(W') = \mathscr{S}(W)$. Then $\mathscr{S}'$ satisfies 4.2 for W' and we take $N(\mathscr{S}(W')) = N(\mathscr{S}(W))$. Then $\mathbf{W}'(\mathscr{S}) = \{W'_p \mid p \in A\}$ is defined. Observe that O is a G deformation retract of W'. In fact there is a G retraction $r$ which carries each $W'_p$ to $O_p$ (defined in § 3) for each $p \in A$.

The aim is to give conditions for which there is an extension $\mathscr{U}$ with $\iota$ an embedding. This requires some added structure which we now discuss. A normal map $\mathscr{W} = (W, F, \beta)$ consists of

**(4.9)** (i) A G map $F : (W, \partial W) \to (Z, \partial Z)$ of degree 1 between smooth G manifolds of the same dimension. It is required that W satisfies the gap hypotheses 4.1, Iso(A) contains $\mathscr{P}$ for $A = Z$, W, and Iso($\partial$A) contains $\mathscr{P}$ if $\partial A \neq \emptyset$. In addition TW must be a stable G vector bundle.

(ii) A Smith framing $\beta : TW \to \widetilde{R} \text{ rel } \mathbf{W}$ (3.24) for some representation R of G. (Here $\widetilde{R} = W \times R$ and $\mathbf{W} = \mathbf{W}(\mathscr{S})$ for a given Smith set $\mathscr{S}$ for W (4.2).)

*Remarks* **4.9'**. — When we need to keep track of the group acting, we call this a G normal map. This definition of a normal map is a variation of those in [W], [DP$_1$] [DP$_2$] and [P$_3$]. For theorems 4.11 and 4.12 the degree 1 assumption is irrelevant (it is needed in 4.21). The Smith set $\mathscr{S}(W)$ for W and the G regular neighborhood $N(\mathscr{S})$ of $W_{\mathscr{S}}$ are an implicit part of the structure of the normal map $\mathscr{W}$. These are used in defining $\mathbf{W}(\mathscr{S})$ and the Smith framing $\beta$.

A *normal map of triads* is a normal map which is also a map of G manifold triads. Observe that if R is a representation of G, W is a smooth G manifold with boundary X and $\beta : TW \to \widetilde{R} \oplus \widetilde{\mathbf{R}}$ is a Smith framing, then $\beta\big|_X : \widetilde{TX} \oplus \widetilde{\mathbf{R}} \to \widetilde{R} \oplus \widetilde{\mathbf{R}}$ is a Smith framing. Because of the definition of Smith framing, we may express this as $\beta\big|_X : TX \to \widetilde{R}$. This means that a normal map of triads $\mathscr{W} = (W, F, \beta)$ defines two additional normal maps $\mathscr{W}_i = (W_i, F_i, \beta_i)$ for $i = 0, 1$ as follows:

$$\mathscr{S}(W_i) = \mathscr{S}(W)\big|_{W_i} = \{M \cap W_i \mid M \in \mathscr{S}(W)\},$$

$$\beta_i = \beta\big|_{W_i} \quad \text{and} \quad F_i = F\big|_{W_i} : W_i \to Z_i.$$

Suppose $\mathscr{W} = (W, F, \beta)$ and $\mathscr{W}' = (W', F', \beta')$ are normal maps of triads with $F : W \to Z$ and $F' : W' \to Z$. We say that $\mathscr{W}'$ *extends* $\mathscr{W}$ if $W \subset W'$, F' extends F and $\beta' : TW' \to \widetilde{R}$ extends $\beta : TW \to \widetilde{R}$. Denote $\mathcal{W}_0 = (W_0, F_0, \beta_0)$ by $(X, f, \beta_0)$ and let H be a $p$ subgroup of G for some $p \in A$.

*Definition* **4.10.** — We say that $\mathcal{W}'$ *arises from* $\mathcal{W}$ by *surgery on* $\mu \in \pi_{k+1}(f^H)$ if $\mathcal{W}'$ extends $\mathcal{W}$ and there is an extension $\mathcal{U}$ of $\mu$ (4.5) such that $\iota$ is an embedding, $W' = W \cup_\iota \mathbf{D}$ and $F'\big|_{\mathbf{D}} = \kappa$. (In particular $\beta' : TW' \to \widetilde{R}$ is a Smith framing rel $\mathbf{W}'$. Here $\widetilde{R} = W' \times R$.)

If there exists an $\mathcal{W}'$ which arises from $\mathcal{W}$ surgery on $\mu$, we say that *surgery on* $\mu$ *is possible*.

The input for 4.11 and 4.12 consists of a normal map $\mathcal{W} = (W, F, \beta)$ as above with $\mathcal{W}_0 = (X, f, \beta_0)$ and $\mu \in \pi_{k+1}(f^H)$. Choose any map $i : S = S^k \times D^{n-k} \to X^H$ representing $\partial\mu$ (3.6 and 4.4). Then $\mathcal{W}$ and $\beta$ define a stable vector bundle isomorphism $\ell'(B)^H : TS \to i^* TX^H$ (4.14).

Two equivariant bundle isomorphisms are called (stably) regularly homotopic if there is an equivariant homotopy of (stable) bundle isomorphisms connecting them. See [PR, pp. 92-93] for elaboration if needed.

We state 4.11 now and prove it later. Compare [$DP_1$, 3.12] and [PR, 10.2, p. 141].

*Theorem* **4.11.** — *Suppose* $\mathcal{W}$ *is a normal map of triads with* $\partial\mathcal{W} = \mathcal{W}_0 \cup \mathcal{W}_1$ *and* $\mathcal{W}_0 = (X, f, \beta_0)$. *Let* H *be a subgroup of* G *of* $p$ *power order,* $p \in A$, $\mu \in \pi_{k+1}(f^H)$, $k \leq n/2$, $n = \dim X^H \geq 5$. *Then surgery on* $\mu$ *is possible if there is an embedding* $i' : S \to X^H$ *representing* $\partial\mu$ *such that* $\mathrm{ind}_H^G\, i'$ $(\iota' = i\big|_{S^k} = \partial\mu)$ *is an embedding and the differential* $di'$ (*defined below*) *is stably regularly homotopic to* $\ell'(B)^H$.

*Corollary* **4.12.** — *Surgery on* $\mu \in \pi_{k+1}(f^H)$ *is possible if* $k < \frac{1}{2} n$ *and* $n = \dim X^H \geq 5$.

*Proof.* — This is an immediate consequence of 4.11 and 4.16 below. Take $i' = \iota\big|_S$ in 4.16.

Let S and X be two manifolds of the same dimension and let $\iota : S \to X$ be an immersion. Its differential $d\iota : TS \to TX$ induces an isomorphism between TS and $i^* TX$ also denoted by

$$d\iota : TS \to i^* TX$$

and also called the differential of $\iota$. Suppose S′ and X′ are two manifolds of the same dimension having S resp. X as submanifolds. Suppose $\iota : (S', S) \to (X', X)$ is an immersion of pairs. The composition

$$\nu(S, S') \to TS'\big|_S \xrightarrow{d\iota} TX'\big|_X \to \nu(X, X')$$

induces an isomorphism

$$n(\iota)_S : \nu(S, S') \to \iota\big|_{S^*} \nu(X, X')$$

called the normal differential of $\iota$ at S.

Here is a mild generalization of a lemma of Hirsch [H] which produces embeddings (immersions) from vector bundle isomorphisms. It is proved in [$DP_1$], [$P_3$], [PR, 10.4, p. 142].

*Lemma* **4.13**. — *Suppose* G *acts freely on the smooth manifold* X *of dimension* $n \geq 5$. *Let* $S = S^k \times D^{n-k}$, $k \leq n/2$ *and let* $i : S \to X$ *be a map. Any stable vector bundle isomorphism* $\ell : TS \to i^* TX$ *determines a* G *immersion of* $\text{ind}_1^G S$ *in* X, G *homotopic to* $\text{ind}_1^G i$, *whose differential is stably regularly* G *homotopic to* $\text{ind}_1^G \ell$. *If* $k < n/2$ *the immersion may be taken to be an embedding.*

Theorem 4.11 is the smooth analog of 3.12 and is partially reduced to that theorem. This we now explain. Let $\mathscr{W} = (W, F, \beta)$ be a normal map with Smith framing $\beta : TW \to \widetilde{R}$ rel **W**. The H vector bundle $\omega$ over D defined in 3.11 (i) is $\mathbf{TD}\big|_D$ by inspection (see 4.7). Equally clear are the equalities (i) $\mathbf{TD}\big|_S = \mathbf{TD}_0\big|_S \oplus \widetilde{\mathbf{R}}$ and (ii) $TW\big|_{S'} = TX\big|_{S'} \oplus \widetilde{\mathbf{R}}$. Here **R** denotes the space of real numbers and $S' = iS$ has trivial action. Then the H vector bundle isomorphism

$$\ell'(\beta) : \omega\big|_S \to i^* TW\big|_{S'}$$

in 3.12 becomes an isomorphism

$$\ell'(\beta) : \mathbf{TD}\big|_S \to i^* TW\big|_{S'}.$$

Let $\xi$ be an H vector bundle over an H space S. We suppose $\xi$ is given an H invariant inner product on fibers. Define an H vector bundle $\xi_H$ over $S^H$ via $\xi\big|_{S^H} = \xi^H \oplus \xi_H$, i.e. $\xi_H$ is the orthogonal complement of $\xi^H$ in $\xi\big|_{S^H}$. If $b : \xi \to \xi'$ is an H vector bundle isomorphism, then Schur's Lemma implies $b\big|_{S^H} = b^H \oplus b_H$ where $b^H : \xi^H \to \xi'^H$ and $b_H : \xi_H \to \xi'_H$. Apply this to $\mathbf{TD}\big|_S$, $TW\big|_{S'}$ and $b = \ell'(\beta)$ noting (i) and (ii) and $S^H = S$. This gives H vector bundle isomorphisms:

**(4.14)**
$$\ell'(\beta)^H : TS \oplus \widetilde{\mathbf{R}} \to i^*(TX^H \oplus \widetilde{\mathbf{R}})\big|_{S'}$$
$$\ell'(\beta)_H : \nu(S, \mathbf{D}_0) \to i^* \nu(X^H, X)\big|_{S'}$$

because $(\mathbf{TD}_0\big|_S)^H = TS$ and $(\mathbf{TD}_0\big|_S)_H = \nu(S, \mathbf{D}_0)$.

*Lemma* **4.15.** — *Let $\mathcal{U}$ be an extension of* $\mu$ *with* $\iota$ *an embedding. If*

$$d(\iota\big|_S) \oplus 1_{\mathbf{R}} : \mathrm{TS} \oplus \widetilde{\mathbf{R}} \to i^* \mathrm{TX}^{\mathrm{H}} \oplus \widetilde{\mathbf{R}} \quad (i = \iota\big|_S)$$

*is regularly homotopic to* $\ell'(\beta)^{\mathrm{H}}$ *and if* $n(\iota)_S = \ell'(\beta)_{\mathrm{H}}$, *then* $\Gamma = \Gamma(\mathrm{TW}, a, \beta) = \mathrm{TW}'\big|_O$. (*Here* O *is defined by* $\mu'$*—the restriction of* $\mathcal{U}$ *to* (D, S). *See* 4.5-4.7.)

*Proof.* — Both $\mathrm{TW}'\big|_O$ and $\Gamma$ are obtained from TW by gluing on $\mathrm{ind}_{\mathrm{H}}^{\mathrm{G}}(\mathbf{TD}\big|_D)$. The gluing isomorphism in the first case is $\mathrm{ind}_{\mathrm{H}}^{\mathrm{G}}(d(\iota)\big|_S \oplus 1_{\mathbf{R}})$ and in the second it is $\mathrm{ind}_{\mathrm{H}}^{\mathrm{G}} \ell'(\beta)$. Since $d(\iota)\big|_S \oplus 1_{\mathbf{R}} = d(\iota\big|_S) \oplus 1_{\mathbf{R}} \oplus n(\iota)_S = \ell'(\beta)$ up to regular H homotopy, these G vector bundles are isomorphic.

*Lemma* **4.16.** — *Let* $\mathcal{W} = (\mathrm{W}, \mathrm{F}, \beta)$ *be a normal map of triads*, H *a p-subgroup of* G *and* $\mu \in \pi_{k+1}(f^{\mathrm{H}})$ *with* $k \leq n/2$, $n \geq 5$ $(n+1 = \dim \mathrm{W}^{\mathrm{H}})$ *be given. Then there is an extension* $\mathcal{U}$ *of* $\mu$ *such that* $\iota$ *is an immersion* (*embedding if* $\mathrm{ind}_{\mathrm{H}}^{\mathrm{G}} \iota'$ $(\iota' = \iota\big|_{S^k})$ *is an embedding, in particular if* $k < n/2$). *The differential of* $\iota\big|_S$ *is stably regularly homotopic to* $\ell'(\beta)^{\mathrm{H}}$ *and the normal differential* $n(\iota)_S$ *of* $\iota : (\mathrm{S}, \mathbf{D}_0) \to (\mathrm{X}^{\mathrm{H}}, \mathrm{X})$ *is* $\ell(\beta)_{\mathrm{H}}$.

*Proof.* — Let $\mu'$ be the diagram 4.5 which gives $\mu$ by restriction to $(\mathrm{D}^{k+1}, \mathrm{S}^k)$. We may suppose that $i\mathrm{S} \subset \mathrm{X}_*^{\mathrm{H}} = \{x \in \mathrm{X}_p^{\mathrm{H}} \mid \mathrm{G}_x = \mathrm{H}\}$ by 4.1 and 4.3. Since G/H acts freely on $\mathrm{X}_*^{\mathrm{H}}$, Lemma 4.13 applied to $i$ and the group G/H gives a G immersion of $\mathrm{ind}_{\mathrm{H}}^{\mathrm{G}}\mathrm{S}$ into $\mathrm{X}_*^{\mathrm{H}}$ whose differential is stably regularly G homotopic to $\ell'(\beta)^{\mathrm{H}}$. Note that $\mathrm{ind}_1^{\mathrm{G/H}s}$ is the same as $\mathrm{ind}_{\mathrm{H}}^{\mathrm{G}}\mathrm{S}$ as a G manifold. (If $k < \frac{1}{2}\dim \mathrm{X}^{\mathrm{H}}$, the immersion may be taken to be a G embedding. Note 4.9 (i).) Thus we may suppose that $\mathrm{ind}_{\mathrm{H}}^{\mathrm{G}} i$ is an immersion. The G Tubular Neighborhood Theorem [B₁] provides a G immersion $\iota$ of $\mathbf{D}_0 = \mathrm{ind}_{\mathrm{H}}^{\mathrm{G}}\mathrm{S} \times \mathrm{D}(\Omega)$ into X extending $\mathrm{ind}_{\mathrm{H}}^{\mathrm{G}} i$ such that the normal differential $n(\iota)_S$ of $\iota : (\mathbf{D}_0, \mathrm{S}) \to (\mathrm{X}, \mathrm{X}^{\mathrm{H}})$ is $\ell'(\beta)_{\mathrm{H}}$. Since $(\mathbf{D}, \mathbf{D}_0)$ retracts equivariantly into $\mathrm{ind}_{\mathrm{H}}^{\mathrm{G}}(\mathrm{D}, \mathrm{S})$, there is an extension of $h : \mathrm{D} \to \mathrm{Y}$ to $\kappa : \mathbf{D} \to \mathrm{Y}$ giving the diagram $\mathcal{U}$ as in 4.8.

*Proof of* 4.11. — The existence of the extension $\mathcal{U}$ of $\mu$ is a consequence of the preceding lemma. The embedding $\iota$ given there satisfies the hypothesis of 4.15; so $\Gamma(\mathrm{TW}, \mu, \beta) = \mathrm{TW}'\big|_O$. By 3.12 there is a Smith framing $\beta'' : \Gamma \to \widetilde{\mathrm{R}}$. Let $r : \mathbf{W}' \to \mathbf{O}$ be a G retraction, i.e. $r = \{r_\infty, r_p \mid p \in \mathrm{A}\}$ and $r_\infty$ and $r_p$ are G retractions for $p \in \mathrm{A}$. Since Smith framings are functorial with respect to such maps, there is a Smith framing $r_\infty^* \beta'' : r_\infty^* \Gamma \to \widetilde{\mathrm{R}}$. Since $r_\infty$ is a deformation retraction, $\mathrm{TW}' = r_\infty^*(\mathrm{TW}'\big|_O) = r_\infty^* \Gamma$. This identification gives the required Smith framing $\beta' : \mathrm{TW}' \to \widetilde{\mathrm{R}}$. Moreover, $\mathrm{F} = \mathrm{F}'' r_\infty : \mathrm{W}' \to \mathrm{Z}$ extends F (see 4.6) and $\mathcal{W}' = (\mathrm{W}', \mathrm{F}', \beta')$ is the effect of surgery on $\mu$.

A *normal cobordism* between two normal maps $\mathscr{W}^i = (W_i, F_i, \beta_i)$, $i = 0, 1$, with $F_i : W_i \to Y$ consists of a normal map of triads $\mathscr{W} = (W, F, \beta)$, $F : W \to Z$ with these properties: Z is the triad $(Y \times I, \ Y \times 0, \ Y \times 1)$ with $I = [0, 1]$; $\mathscr{W}_i = \mathscr{W}^i$. There is a G embedding $j : (N_0 \times I, N_0 \times 0, N_0 \times 1) \to (W, W_0, W_1)$ whose image is $(N, N_0, N_1)$ where $N = N(\mathscr{S})$, $N_i = N(\mathscr{S}_i)$, $\mathscr{S} = \mathscr{S}(W)$ and $\mathscr{S}_i = \mathscr{S}(W_i)$. Finally require $\mathscr{S} = \{j(A \times I) \mid A \in \mathscr{S}_0\}$ and note that the condition $\mathscr{W}_i = \mathscr{W}^i$ then implies that $\mathscr{S}_0$ and $\mathscr{S}_1$ are canonically isomorphic because $\mathscr{S}_i = \mathscr{S}\big|_{W_i}$ emphasize that surgeries of the kind in 4.11 produce normal cobordisms.

*Remark* **4.17.** — If $\mathscr{W}^i = (W_i, F_i, \beta_i)$, $i = 0, 1$, are G normally cobordant and $G \notin \mathscr{P}$, then $W_0^G$ and $W_1^G$ are canonically isomorphic; so we write $W_0^G = W_1^G$. In addition $T_x W_0 = T_x W_1$ as representations of G for $x \in W_0^G = W_1^G$. For if $\mathscr{W} = (W, F, \beta)$ is a normal cobordism between $\mathscr{W}^0$ and $\mathscr{W}^1$ then $W^K \cong W_0^K \times I$ for any $K \notin \mathscr{P}$, in particular for $K = G$. For by 4.2 (ii) $W^K \in \mathscr{S}(W)$ and $W_0^K \in \mathscr{S}(W_0)$; so $W^K = j(W_0^K \times I)$ where $j$ is the G embedding given in the definition of normal cobordism above.

Let W be a G manifold with boundary X, R a representation of G and $\beta : TW \to \widetilde{R} \oplus \widetilde{\mathbf{R}}$ be a Smith framing rel **W**. If $\mathscr{W} = (W, F, \beta)$ is a normal map with $F : W \to Z$, then $\partial\mathscr{W} = (X, F\big|_X, \beta\big|_X)$ is a new normal map with Smith set $\mathscr{S}(X) = \mathscr{S}(W)\big|_X$, $F\big|_X : X \to Y = \partial Z$ and Smith framing $\beta\big|_X : TX \oplus \widetilde{\mathbf{R}} \to \widetilde{R} \oplus \widetilde{\mathbf{R}}$. If $\mathscr{W}$ is a G normal map and H is a subgroup of G, then $\mathrm{res}_H \mathscr{W}$ is the H normal map $(\mathrm{res}_H W, \ \mathrm{res}_H F, \ \mathrm{res}_H \beta)$.

Before treating the Induction Theorem we need to discuss some related points from [$DP_1$]. There we used a variant of the notion of a normal map called a prenormal map. This is a triple $\mathscr{W} = (W, F, \widetilde{\beta})$ like the normal map in 4.9. The essential difference appears in the vector bundle isomorphisms occurring in these definitions. This is 4.9 (ii) here and 2.4 (iv) there (where $\xi = \widetilde{R}$ for some representation R of G). Briefly 2.4 (iv) requires a stable G vector bundle isomorphism $b : s TW \to s\widetilde{R}$ together with a collection $c = \{c(H) : TW_H \to \widetilde{R}_H \mid H \subset G\}$ of G vector bundle isomorphisms. Let us denote $(b, c)$ by $\widetilde{\beta}$, write $\widetilde{\beta} : TW \to \widetilde{R}$ and call this a bundle isomorphism. This is stronger than a Smith framing of TW; so the condition that TW be a stable G vector bundle (3.34) is not required for a prenormal map.

Let $\mathscr{W} = (W, F, \beta)$, $F : W \to Z$ be a normal map (3.9) which satisfies these conditions:

**(4.18)** (i) $[Z] - [W] \in \Delta(G) + 2\Omega(G, \mathscr{P})$,
(ii) $\partial Z = \partial W = \emptyset$,
(iii) $\dim Z$ is even, $Z^H$ is simply connected, $\dim Z^H = \dim W^H$ and exceeds 5 for $H \in \mathscr{P}$.

*Induction Theorem* **4.19.** — *Let* $\mathscr{W}$ *be a normal map which satisfies* 4.18. *If for each* $H \in \widetilde{\mathscr{P}}$ *there is an* H *normal map* $\mathscr{W}(H)$ *such that* $\partial\mathscr{W}(H) = \text{res}_H \mathscr{W}$, *then* $\mathscr{W}$ *is normally cobordant to* $\mathscr{W}' = (W', F', \beta')$ *where* $F' : W' \to Z$ *is a homotopy equivalence.*

*Proof.* — Since $Z^H$ is connected for $H \in \mathscr{P}$ and since each component of $W^H$ contains a point whose isotropy group is H (4.1), we may alter $\mathscr{W}$ by a normal cobordism without destroying 4.18 and achieve also that $W^H$ is connected. This is a simple standard construction. See e.g. [$DP_2$, 9.1]. We therefore add to the hypothesis 4.18 (iii) the condition that $W^H$ is connected for all $H \in \mathscr{P}$. With this change the conditions of this theorem imply those of [$DP_1$, 2.6] with K there taken to be $\{K \subset G \mid K \notin \mathscr{P}\}$. There is one essential difference namely the induction theorem there involved a pre-normal map and here deals with a normal map. (In the presence of the assumptions 4.18, the only difference is 4.9 (ii) vs. [$DP_1$, 2.4 (iv)].) The proof of the Induction Theorem 2.6 of [$DP_1$] is entirely algebraic but appeals to two geometric results [$DP_1$, 3.12 and 3.13] about surgery on a prenormal map. We have established their analogs here (4.11 and 4.12) for surgery on a normal map. There is one minor difference between [$DP_1$, 3.12 and 3.13] and 4.11 and 4.12. The former deals with arbitrary subgroups H while the latter with subgroups $H \in \mathscr{P}$. The only use of [$DP_1$, 3.12 and 3.13] for $H \notin \mathscr{P}$ occurs in [$DP_1$, 3.15] to achieve $[Z] - [W] \in \Delta(G)$. The assumption that $[Z] - [W] \in \Delta(G) + 2\Omega(G, \mathscr{P})$ (4.18 (i)) means this can be achieved via [$DP_1$, 3.15] using 4.11 and 4.12. This means the proof of the Induction Theorem of [$DP_1$] for prenormal maps applies to prove the Induction Theorem here for normal maps.

*Remark* **4.20.** — We should emphasize one aspect of the equality $\partial\mathscr{W}(H) = \text{res}_H \mathscr{W}$ in 4.19 which might otherwise go unnoticed. Let $\mathscr{W}(H) = (U, F_U, \beta(U))$ be an H normal map and let $\mathscr{W} = (W, F_W, \beta(W))$ be a G normal map. Then from the definitions involved, $\text{res}_H \mathscr{W} = \partial\mathscr{W}(H)$ means in particular that $\text{res}_H \mathscr{S}(W) = \mathscr{S}(U)\big|_W$. For later application we give examples of Smith sets $\mathscr{S}(W) = \mathscr{S}(W, G)$ and $\mathscr{S}(U) = \mathscr{S}(U, H, G)$ defined when $\text{res}_H W = \partial U$ for which $\text{res}_H \mathscr{S}(W) = \mathscr{S}(U)\big|_W$. By definition

$$\mathscr{S}(W, G) = \{W^L \mid L \subset G, L \notin \mathscr{P}\}$$

and

$$\mathscr{S}(U, H, G) = \mathscr{S}(W, G) \cup \{U^L \mid L \subset H, L \notin \mathscr{P}\}.$$

To verify 4.2 (i)-(iii), note that 4.1 implies 4.2 (i) and 4.2 (ii) is clear. Here are some remarks about 4.2 (iii). The G equivariant triangulation theorem of Illman [$I_1$] gives a triangulation of W for which 4.2 (iii) is satisfied for $\mathscr{S}(W, G)$. Take an Illman H triangulation of the closure U′ of the complement of an H collar neighborhood C of W in U. Take an Illman H triangulation of C which agrees on $C \cap W$ and $C \cap U'$ with the given triangulations there. One uses [$I_2$, Theorem 4.3] for the last triangulation. Together, these triangulate U and (iii) is satisfied for $\mathscr{S}(U, H, G)$.

## 5. Proof of theorems A, B and B′

The proof of Theorem A combines 3.13, the Induction Theorem 4.19 and the one fixed point manifolds constructed in $[P_4]$.

Suppose $\eta$ is a G vector bundle over W and R and R′ are complex representations of G whose difference lies in I(G). Then by 3.13 there is a Smith framing

**(5.1)** $$\beta_W(R', R) : \widetilde{R}' \to \widetilde{R} \text{ rel } \mathbf{W}$$

for any Smith decomposition $\mathbf{W}$ of W. If $b : s(\eta) \to s(\widetilde{R}')$ is any stable G vector bundle isomorphism (R′ framing) then there is an obvious Smith framing $\beta = \beta_W(R', R) \circ b : \eta \to \widetilde{R}$ obtained by composing $b$ and the Smith framing $\beta_W(R', R)$. This is used in the proof of Theorem A.

We begin with preparation for the proof of Theorem A by recalling material from $[P_4]$. Remember $\mathbf{R}$ is the one dimensional trivial real representation of G. Henceforth G is as in Theorem A′ (section 1).

*Results from* $[P_4]$: For each representation $R \in \mathscr{R}$, there is a G manifold $X = X(R)$ with these properties:

**(5.2)**
(i) $X^G$ is one point $u$.
(ii) There is a stable G vector bundle isomorphism
$$b(X) : s(TX \oplus \mathbf{R}) \to s(\widetilde{R} \oplus \widetilde{\mathbf{R}}).$$
(iii) $2 - [X]$ lies in the subgroup $\Gamma = \Delta(G) + 2\Omega(G, \mathscr{H})$ of $\Omega(G)$.
(iv) For each $H \in \widetilde{\mathscr{P}}$, there is an H manifold U(H, R) whose boundary is $\mathrm{res}_H\, X(R)$ and there is a stable H vector bundle isomorphism $b(U(H, R)) : s(TU(H, R)) \to \mathrm{res}_H\, s(\widetilde{R} \oplus \widetilde{\mathbf{R}})$ such that
$$\mathrm{res}_H\, b(X(R)) = b(U(H, R))\big|_{X(R)}.$$
(v) dim X is even; $\dim X^H = R^H$ and exceeds 5 (resp. 3) for $H \in \mathscr{P}$ (resp. $H \in \mathscr{H}$); X satisfies 4.1 and $H \in \mathrm{Iso}(X)$.
(vi) TX is a stable G vector bundle and TU(H, R) is a stable H vector bundle for $H \in \mathscr{H}$.

*Remarks.* — Condition (vi) is not explicit in $[P_4]$, but is verified at the end of the section. Actually the main result of $[P_4]$ is that $X = X(R)$ may be taken to be a homotopy sphere with the properties in 5.2. We do not require the additional depth of that condition but note that it implies 5.2 (iii) because then $2 - [X] \in \Delta(G)$. To see this, note from section 2 that that condition is equivalent to the condition that the Euler characteristic $\chi(X^H)$ is 2 whenever $H \in \widetilde{\mathscr{P}}$. To verify this use the fact that H, being in $\widetilde{\mathscr{P}}$, has a subgroup P of prime power order for some prime $p$ such that H/P is cyclic. Note that $X^P$ is an even dimensional mod $p$ homology sphere because $|P|$ is odd and $\dim \Sigma$ is even. The odd order cyclic group H/P acts on $X^P$ with $X^H$ as fixed set. Apply the Lefschetz Fixed Point Theorem to this situation to see that $\chi(X^H)$ is 2. Finally we remark that the existence of $b(X)$ in 5.2 (ii) implies $T_u X(R)$ is isomorphic to R.

We emphasize that the main conceptual conditions involved in the proof of Theorem A below are 5.2 (i)-(iv). The others in 5.2 are necessary but technical.

The next lemma and its corollary are needed to construct a G manifold W which among other properties satisfies the condition that $2 - [W] \in \Delta(G)$. See section 2 for definition of $\Delta(G)$.

*Lemma* **5.3.** — *For* $\varepsilon = 0, 1$, *any* $R \in \mathscr{R}$ *and any* $H \in \mathscr{H}$, *there is an* $R \oplus \mathbf{R}$ *framed* G *manifold* $V_H^\varepsilon$ *whose boundary* $Z_H^\varepsilon$ *has no fixed points and* $[Z_H^\varepsilon] = (2 - 6\varepsilon)[G/H]$ *in* $\Omega(G, \mathscr{H})$.

*Proof.* — For each subgroup H in $\mathscr{H}$, we construct H manifolds $Z^\varepsilon(H)$ and $V^\varepsilon(H)$, $\varepsilon = 0, 1$, with these properties: (i) $V^\varepsilon(H)$ has a $\mathrm{res}_H(R \oplus \mathbf{R})$ framing; (ii) $Z^\varepsilon(H)$ is the boundary of $V^\varepsilon(H)$; (iii) the class $[Z^\varepsilon(H)]$ in $\Omega(H)$ is 2 (resp. $-4$) for $\varepsilon = 0$ (resp. 1). We shall then define $V_H^\varepsilon = G \times_H V^\varepsilon(H)$, $Z_H^\varepsilon = G \times_H Z^\varepsilon(H)$. Begin with an oriented surface S with trivial H action and $\chi(S) = -2$. Since $\dim R^H \geq 2$ by (2.5 (i)), $\mathrm{res}_H R = A \oplus \mathbf{R}^2$ for some representation A of H. Set $Z^0(H) = S(R \oplus \mathbf{R})$, $Z^1(H) = S \times S(A \oplus \mathbf{R})$, $V^0(H) = D(R \oplus \mathbf{R})$ and $V^1(H) = S \times D(A \oplus \mathbf{R})$, where $D(\cdot)$ denotes unit disk. Condition (iii) is a consequence of the fact that $\chi(Z^\varepsilon(H)^K) = 2$ (resp. $-4$) for all $K \subset H$ for $\varepsilon = 0$ (resp. 1). Conditions (i) and (ii) are nearly obvious. Their verification is left to the reader. From properties (i)-(iii), one may verify: (i') $V_H^\varepsilon = G \times_H V^\varepsilon(H)$ has an $R \oplus \mathbf{R}$ framing; (ii') $Z_H^\varepsilon = G \times_H Z^\varepsilon(H)$ is the G boundary of $V_H^\varepsilon$; (iii') the class $[Z_H^\varepsilon]$ in $\Omega(G, \mathscr{H})$ is $(2 - 6\varepsilon)[G/H]$; (iv') the G fixed set of $Z_H^\varepsilon$ is empty (since $G \notin \mathscr{H}$).

*Remark.* — $2 . \Omega(G, \mathscr{H})$ is freely generated by $\{2 . [G/H] \mid H \in \mathscr{H}\}$.

*Corollary* **5.4.** — *Let* K *be a finite set and* $\{R, R_u \mid u \in K\}$ *be a set of representations in* $\mathscr{R}$. *Then there is a* G *manifold* V *with these properties*:

(i) *There is a stable* G *vector bundle isomorphism* $b(V) : s(TV) \to s(R \oplus \mathbf{R})$.

(ii) *The boundary of* V *has no fixed points and* $2 - [X(K)] - [\partial V] \in \Delta(G)$ *for* $X(K) = \coprod_{u \in K} X(R_u)$.

*Proof.* — Assertion: $2 - [X(K)] \in \Gamma$. For this, note from 2.2 that

$$1 \in \Delta(G) + \Omega(G, \mathscr{H}).$$

This implies $2 \in \Gamma$. In addition $2 - [X(R_u)] \in \Gamma$ for each $u \in K$ by (5.2 (iii)); so $[X(R_u)]$ $(u \in K)$ and 2 are in $\Gamma$. This implies the assertion. It means that $2 - [X(K)] = \lambda + 2\omega$ for some $\lambda \in \Delta(G)$ and $2\omega \in 2\Omega(G, \mathscr{H})$. By Lemma 5.3, and the subsequent remark, there is some choice of nonnegative integers $a^{\varepsilon}_{H}$, $\varepsilon = 0, 1$, $H \in \mathscr{H}$; so that $[V_0] = 2\omega$ for $V_0 = \coprod a^{\varepsilon}_{H} . Z^{\varepsilon}_{H}$. Let $V = \coprod a^{\varepsilon}_{H} . V^{\varepsilon}_{H}$. Then $V_0 = \partial V$ and conditions (i) and (ii) of this corollary hold.

*Proof of Theorem A* (§ 1). — Let K be the finite set given in the statement of Theorem A. For each $u \in K$, we are given a representation $R_u \in \mathscr{R}$. Fix a point $z$ in K and set $T = R_z$. Add an additional point $o$ to K to give a new set K′ and set $R_0 = T$; so $R_u$ is defined for all $u \in K'$. Let $Z = S(T \oplus \mathbf{R})$ and $D = D(T \oplus \mathbf{R})$ be the unit sphere resp. unit disk in $T \oplus \mathbf{R}$. These are G manifolds. We shall construct a G normal map $\mathscr{W} = (W, F, \beta(W))$, $F : W \to Z$ and for each $H \in \widetilde{\mathscr{P}}$ an H normal map $\mathscr{W}(H) = (U(H), F(H), \beta(U(H)))$, $F(H) : U(H) \to D$ such that $\mathrm{res}_H \mathscr{W} = \partial \mathscr{W}(H)$.

First we construct W and U(H) for $H \in \widetilde{\mathscr{P}}$. For $u \in K$, let $W_u = X(R_u)$ (5.2) and for each $H \in \widetilde{\mathscr{P}}$, let $U_u(H) = U(H, R_u)$ (5.2 (iv)). Corollary 5.4 provides a G manifold V such that $(\partial V)^G = \emptyset$, $2 - [X(K)] - [\partial V] \in \Delta(G)$ and V has a $T \oplus \mathbf{R}$ framing $b(V)$ of its tangent bundle. Set $W_0 = \partial V$, $U_0(H) = V$ and $W = \coprod_{u \in K'} W_u$; so $W^G = K$ and $T_u W = R_u$ for $u \in K$. Set $U(H) = \coprod_{u \in K'} U_u(H)$ whenever $H \in \widetilde{\mathscr{P}}$.

Then by 5.2 and the construction of V and Z we find

(i) $\mathrm{res}_H W = \partial U(H)$ for each $H \in \widetilde{\mathscr{P}}$,
(ii) $2 - [W] \in \Delta(G)$.

The degree one G map $F : W \to Z$ is obtained by collapsing the complement of an invariant open disk centered at $z \in W$ (and identified with T) to $(0, 1) \in Z^G$ and maps the disk homeomorphically onto $Z - (0, 1)$. Let $F(H) : U(H) \to D$ be any H map which extends $\mathrm{res}_H F$; so it too has degree 1.

Next we construct Smith framings

$$\beta(W) : TW \oplus \widetilde{\mathbf{R}} \to \widetilde{T} \oplus \widetilde{\mathbf{R}} \quad \text{and} \quad \beta(U(H)) : TU(H) \to \widetilde{T} \oplus \widetilde{\mathbf{R}};$$

so that $\mathrm{res}_H \beta(W) = \beta(U(H))\big|_W$. Recall that the definition of a normal map (4.9) requires that these Smith framings are defined rel $\mathbf{W}$ and $\mathbf{U(H)}$ where $\mathbf{W} = \mathbf{W}(\mathscr{S})$ and $\mathbf{U(H)} = \mathbf{U(H)}(\mathscr{S}')$ for Smith sets $\mathscr{S}$ and $\mathscr{S}'$ for W and U(H). The equation $\mathrm{res}_H \beta(W) = \beta(U(H))\big|_W$ requires $\mathrm{res}_H \mathscr{S} = \mathscr{S}'\big|_W$ by 4.20. We take $\mathscr{S} = \mathscr{S}(W, G)$ and $\mathscr{S}' = \mathscr{S}(U(H), H, G)$; so by 4.20 the preceding equality is satisfied. Having said this, set

$$\beta(W) = \coprod_{u \in K'} \beta_{W_u}(R_u \oplus \mathbf{R}, R \oplus \mathbf{R}) \circ b(W_u)$$

$$\beta(U(H)) = \coprod_{u \in K'} \beta_{U_u(H)}(\mathrm{res}_H R_u \oplus \mathbf{R}, \mathrm{res}_H R \oplus \mathbf{R}) \circ b(U_u(H)).$$

See 5.1, 5.2 and 5.4 for notation here. Then

(iii) $\operatorname{res}_H \beta(W) = \beta(U(H))\big|_W$ for each $H \in \mathscr{H}$.

This follows from the naturality of $\beta(\cdot, \cdot)$ with respect to restriction to subgroups and to subspaces (3.13). By construction

(iv) $\operatorname{res}_H F = F(H)\big|_W$.

We assert that $\mathscr{W}$ and each $\mathscr{W}(H)$ for $H \in \widetilde{\mathscr{P}}$ are normal maps (4.9). The conceptual elements of a normal map—an equivariant degree 1 map and Smith framing of tangent bundle—have been defined for $\mathscr{W}$ and $\mathscr{W}(H)$. The added technical requirements of 4.9 which must be verified for $\mathscr{W}$ and $\mathscr{W}(H)$ follow from conditions 5.2 and are left to the reader. In order to apply 4.19, $\mathscr{W}$ must satisfy 4.18 and $\operatorname{res}_H \mathscr{W} = \partial\mathscr{W}(H)$ for each $H \in \widetilde{\mathscr{P}}$. Concerning 4.18, only 4.18 (i) is not obvious and that has been achieved in (ii) above. The equations $\operatorname{res}_H \mathscr{W} = \partial\mathscr{W}(H)$ for $H \in \widetilde{\mathscr{P}}$ have been verified in (i), (iii) and (iv) above. As a consequence of 4.19, $\mathscr{W}$ is normally cobordant to $\mathscr{W}' = (W', F', \beta')$ where $F' : W' \to Z$ is a homotopy equivalence. Then $W'$ is a homotopy sphere with $W'^G = W^G = K$ and $T_u W' = T_u W = R_u$ for $u \in K$ by Remark 4.17. This completes the proof of Theorem A.

As promised earlier we verify 5.2 (vi) which requires that TW is a stable G vector bundle and TU(H) is a stable H vector bundle for $H \in \widetilde{\mathscr{P}}$. These conditions are required as part (4.9 (i)) of the definition of that $\mathscr{W}$ and $\mathscr{W}(H)$ be normal maps. It suffices to show that $TW_u$ and $TU_u(H)$ for $u \in K'$ and $H \in \widetilde{\mathscr{P}}$ are stable G resp. H vector bundles. By 5.2 (ii) and 5.2 (iv), $TW_u$ and $\widetilde{R}_u$ are stably isomorphic G vector bundles and $TU_u(H)$ and $\widetilde{R}_u \oplus \widetilde{\mathbf{R}}$ are stably isomorphic H vector bundles for $H \in \widetilde{\mathscr{P}}$. In particular the $G_x$ representations $T_x W_u$ and $R_u$ are isomorphic for all $x \in W_u$ and similarly for $T_x U_u(H)$ with $x \in U_u(H)$. From this stability (3.34) both for $TW_u$ and $TU_u(H)$ follows from:

(i) $\dim R_u^H < d_\chi m_\chi(R_u)$ whenever H is a subgroup of G and $\chi$ is an irreducible representation of H such that $m_\chi(R_u) \neq 0$ (provided $\chi \neq \mathbf{R}$).

(ii) $m_\chi(R_u \oplus \mathbf{R}) = m_\chi(R_u)$ if $\chi \neq \mathbf{R}$.

We must justify (i). By definition

$$(5.5) \qquad d_\chi = \dim_{\mathbf{R}} D_\chi$$

where $D_\chi$ is the division algebra of real linear H equivariant endomorphisms of $\chi$. In our case G and hence H is odd order abelian; so $d_\chi = 2$ whenever $\chi$ is a non-trivial irreducible representation. By hypothesis $R_u \in \mathscr{R}$; so if $\chi$ is a non-trivial irreducible complex representation of H it remains irreducible as a real representation and 2.5 (ii) gives

$$\dim R_u^H = 2 \dim_{\mathbf{C}} R_u^H < 2n_\chi(R_u) \leq 2m_\chi(R_u)$$

which is (i) because $d_\chi$ is 2.

*Proof of Theorem B* (§ 1). — Let $\{R_u \mid u \in K\}$ be the representations in $\mathcal{S}$ given in the statement of Theorem B. Since each $R_u$ lies in $\mathcal{S}$, $R_u^H = 0$ whenever $G/H \in \mathcal{P}$. This implies that $\mathrm{Iso}(R_u - O) \subset \mathcal{H}$; so by [PR, 9.16, p. 139], there is a representation S of G such that each $R_u \oplus S$ is the realification of a complex representation $\overline{R}_u \in \mathcal{R}$ (2.5).

Now modify the proof of Theorem A. Use the representations $\{\overline{R}_u \mid u \in K\}$ to replace the representations $\{R_u \mid u \in K\}$ there. We do not know (and in general it is false) that $\overline{R}_u - \overline{R}_{u'} \in I'(G)$ for each $u, u' \in K$; however, $r : I(G) \to IO(G)$ is surjective and $r(\overline{R}_u - \overline{R}_{u'}) = R_u - R_{u'} \in IO'(G) \subset IO(G)$. At the point in the proof of Theorem A where Theorem 3.13 is applied, use the fact that $r(\overline{R}_u - \overline{R}_{u'}) = r(z)$ for some $z \in I(G)$. With this modification the proof of Theorem A yields the proof of Theorem B.

*Proof of Theorem B'.* — This is an immediate consequence of Theorem B and Lemma 2.7.

We end this paper with the referee's comments showing that the homotopy sphere $\Sigma$ in Theorem A can be taken to be the standard sphere. The key point to note is the manifold $W = \coprod W_u$ constructed in the proof of Theorem A is a framed boundary and the modification of W to $W' = \Sigma$ using 4.19 is by framed surgeries; so $\Sigma$ is a framed boundary. Since dim $\Sigma$ is even, the famous work of Kervaire-Milnor implies $\Sigma$ is the standard sphere. To see that W is a framed boundary note $W_0$ is by construction. The manifolds $W_u$ for $u \in K$ come from [$P_3$] (see especially 1.7 and 2.10 there) and referring to this one notes the $W_u$ are framed boundaries for $u \in K$. This means W is a framed boundary too.

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