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VOLUME PRESERVING ACTIONS OF LATTICES IN SEMISIMPLE GROUPS ON COMPACT MANIFOLDS[1]

by ROBERT J. ZIMMER [2]

1. Introduction

In this paper we begin a study of volume preserving actions of lattices in higher rank semisimple groups on compact manifolds. More precisely, let G be a connected semisimple Lie group with finite center, such that the **R**-rank of every simple factor of G is at least 2. Let $\Gamma \subset G$ be a lattice subgroup. Assume M is a compact manifold and ω is a smooth volume density on M. The general question we wish to address is to determine how Γ can act on M so as to preserve ω. All presently known actions of this type are essentially of an algebraic nature, and a fundamental general problem is to determine whether or not the known examples are an essentially complete list or whether there are genuinely " geometric " actions of such groups.

The standard arithmetic construction of cocompact lattices shows that in the cocompact case one may have homomorphisms $\Gamma \to K$ where K is a compact Lie group and the image of Γ is dense in K. Thus, Γ acts isometrically (and ergodically) on the homogeneous spaces of K. One way to attempt to exhibit new actions of Γ would be to start with a given action and try to perturb this action. One of our main results shows that if we start with an isometric action and make a sufficiently smooth perturbation, then at least topologically we stay within the class of isometric actions. More precisely:

Theorem **6.1**. — *Let* G *and* Γ *be as above. Let* M *be a compact Riemannian manifold,* $\dim M = n$. *Set* $r = n^2 + n + 1$. *Assume* Γ *acts by isometries of* M. *Let* $\Gamma_0 \subset \Gamma$ *be a finite generating set. Then any volume preserving action of* Γ *on* M *which*

i) *for elements of* Γ_0 *is a sufficiently small* C^r *perturbation of the original action*;

and

ii) *is ergodic*;

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actually leaves a C^0 *Riemannian metric invariant. In particular, there is a* Γ*-invariant topological distance function and the action is topologically conjugate to an action of* Γ *on a homogeneous space of a compact Lie group* K *defined via a dense range homomorphism of* Γ *into* K.

In addition to showing that we obtain no new ergodic actions of Γ by perturbing an isometric action, Theorem 6.1 can be profitably viewed from other vantage points. We recall that if we have an isometric diffeomorphism (i.e. **Z**-action) a volume preserving perturbation of this diffeomorphism is not likely to be isometric. In fact, hyperbolicity rather than isometry is typical of properties of a diffeomorphism that are preserved under a perturbation. A similar remark obviously applies to actions of free groups as well. Thus Theorem 6.1 shows a sharp contrast between the behavior under perturbation of actions of lattices in higher rank semisimple groups and that of free groups. From another point of view, we recall from the work of Weil [19], Mostow [11], Margulis [9], and others that homomorphisms of lattices in higher rank groups into (finite dimensional) Lie groups have strong rigidity properties. Theorem 6.1 can be viewed as a type of (local) rigidity theorem for a class of homomorphisms of Γ into the infinite dimensional group of diffeomorphisms of M.

As all " algebraic " examples of volume preserving actions of Γ on manifolds have dimension restrictions, we put forth the following conjecture.

Conjecture. — Let G, Γ be as above. Let $d(G)$ be the minimal dimension of a non-trivial real representation of the Lie algebra of G, and $n(G)$ the minimal dimension of a simple factor of G. Let M be a compact manifold, $\dim M = n\ (> 0)$.

Assume

i) $n < d(G)$; and

ii) $n(n+1) < 2n(G)$.

Then every volume preserving action of Γ on M is an action by a finite quotient of Γ. In particular, there are no volume preserving ergodic actions of Γ on M.

With one additional hypothesis, we can verify the final assertion of this conjecture. We recall that on any compact manifold, the space of Riemannian metrics has a natural (metrizable) C^r-topology for each integer $r \geqslant 0$. If ξ is a metric, $\mathcal{O}$ a C^r-neighborhood of ξ in the space of metrics with the same volume density as ξ, and S is a finite set of diffeomorphisms, we say that ξ is $(\mathcal{O}, S)$-invariant if $f^*\xi \in \mathcal{O}$ for each $f \in S$. In other words ξ is " nearly C^r-invariant " under S. In general, this is not a very strong condition. For example, let φ_t be a smooth flow on M, preserving the volume density of a metric ξ. Then for any $\mathcal{O}$, we have that for t sufficiently small, ξ is $(\mathcal{O}, \varphi_t)$-invariant. Since highly mixing diffeomorphisms can arise this way, e.g. all φ_t, $t \neq 0$, may be Bernoulli, " near isometry " has no implications in general for the existence of an invariant metric. With the additional hypothesis that the generators of Γ act nearly isometrically we can prove the final assertion of the conjecture.

Theorem **6.2**. — *Let* G, Γ *be as above and* M *a compact manifold,* $\dim M = n$ $(n > 0)$. *Assume* $n(n+1) < 2n(G)$, *where* $n(G)$ *is as in the conjecture. Set* $r = n^2 + n + 1$. *Let* $\Gamma_0 \subset \Gamma$ *be a finite generating set. Then for any smooth Riemannian metric* ξ *on* M, *there is a* C^r *neighborhood* $\mathcal{O}$ *of* ξ *such that there are no volume preserving ergodic actions of* Γ *on* M *for which* ξ *is* $(\mathcal{O}, \Gamma_0)$*-invariant.*

Both Theorem 6.1 and 6.2 are deduced from the following.

Theorem **5.1**. — *Let* G, Γ *be as above, and* $\Gamma_0 \subset \Gamma$ *a finite generating set. Let* M *be a compact n-manifold, and let* $r = n^2 + n + 1$. *Let* ξ *be a smooth Riemannian metric on* M *with volume density* ω. *Then there is a* C^r *neighborhood* $\mathcal{O}$ *of* ξ *such that any* ω*-preserving smooth ergodic action of* Γ *on* M, *with* ξ $(\mathcal{O}, \Gamma_0)$*-invariant, leaves a* C^0*-Riemannian metric invariant.*

Once again, Theorem 5.1 shows a sharp contrast between actions of Γ and actions of **Z** or more general free groups as shown above by the example coming from a flow.

The framework of both the results and conjecture above can be generalized in a number of directions only the most direct of which we shall consider in this paper. (We hope to return to some of these other directions elsewhere.) For example, let k be a totally disconnected local field of characteristic 0, and suppose that G is a connected semisimple algebraic k-group, such that every k-simple factor of G has k-rank $\geqslant 2$. Let $\Gamma \subset G_k$ be a lattice. From results of Margulis [9] (see also the work of Raghunathan [16]), it follows that for any homomorphism $\Gamma \to GL(n, \mathbf{R})$ the image of Γ is precompact. The following version of the previous conjecture would be a generalization of this result where $GL(n, \mathbf{R})$ is replaced by the infinite dimensional group of volume preserving diffeomorphisms of a compact manifold of arbitrary dimension.

Conjecture. — Let $\Gamma \subset G_k$ be as in the preceding paragraph. Then any smooth volume preserving action of Γ on a compact manifold is an action by isometries.

As in the case of real groups, we can establish the final assertion with one additional hypothesis.

Theorem **6.3**. — *Let* $\Gamma \subset G_k$ *as above. Let* M *be a compact n-manifold* $(n > 0)$. *Let* $r = n^2 + n + 1$. *Let* $\Gamma_0 \subset \Gamma$ *be a finite generating set. Then for any smooth Riemannian metric* ξ *on* M *there is a* C^r *neighborhood* $\mathcal{O}$ *of* ξ *such that any volume preserving ergodic action of* Γ *on* M *for which* ξ *is* $(\mathcal{O}, \Gamma_0)$*-invariant leaves a* C^0*-Riemannian metric invariant.*

We now outline the contents of the remainder of this paper, and in doing so indicate some of the tools needed for the proofs of the above theorems. The first major step in the proofs, and the result which leads one to put forward the above conjectures, is the superrigidity theorem for cocycles proved by the author in [26], [27], [28], [29]. This is a generalization of Margulis' superrigidity theorem [9] and its proof draws in part on Margulis' techniques. (See [28], [29] for an exposition of all of this material.)

While in [26] we used the superrigidity theorem to construct homomorphisms between Lie groups, here we use it to construct measurable invariant metrics on vector bundles. In particular, the superrigidity theorem when applied to the derivative cocycle on the tangent bundle, will in our situation give us a Γ-invariant assignment of an inner product to each tangent space on the manifold, but this assignment will only be measurable. To deduce the existence of such a measurable invariant metric from superrigidity requires the use of Kazhdan's property, the Furstenberg-Kesten theorem on the products of random matrices [3], [7], and some general structural properties of algebraic groups. This measure theoretic information that the superrigidity theorem gives us, as well as some other results of a similar type, is developed in section 2. Sections 3-5 are devoted to trying to make the measurable invariant metric continuous.

In section 3 we discuss certain constructions on the frame bundle of a manifold and on some associated jet bundles, and apply some of the results of section 2 to this situation.

Section 4 is devoted to a simple but basic estimate which compares a measurable invariant metric to a smooth metric on a vector bundle over a manifold. Here we make use of the " near isometry " condition, ergodicity of the action, and of Kazhdan's property. We deduce that the measurable invariant metric must satisfy L^2 conditions with respect to a smooth metric.

In section 5 we complete the main argument (the proof of Theorem 5.1). We use the results of sections 2 and 4 applied to a cocycle defined on a jet bundle over the frame bundle of the original manifold. " Near C^r-isometry " and Kazhdan's property are used to construct a fixed point in an associated Sobolev space, and using our L^2 estimates, the Sobolev embedding theorem and some supplementary ergodic theoretic arguments we construct an invariant C^0 metric.

Section 6 contains the deductions of Theorems 6.1, 6.2, 6.3 from the arguments of section 5. In section 7 we present the known examples of volume preserving actions on compact manifolds for the arithmetic groups we have been considering and some questions concerning these actions. Some of the results of this paper were announced in [30].

2. Measurable cocycles and superrigidity

In this section we present some fundamental results we will be using concerning measurable cocycles. We will be applying these in subsequent sections to the actions on various natural vector bundles related to a smooth action on a manifold. As a general background source on measurable cocycles and related matters, see [29].

Let Γ be a locally compact second countable group and (S, μ) a standard Borel Γ-space with μ a Γ-quasi-invariant probability measure. We recall that if H is a standard Borel group, a Borel map $\alpha : S \times \Gamma \to H$ is called a cocycle if for all $\gamma_1, \gamma_2 \in \Gamma$, $\alpha(s, \gamma_1\gamma_2) = \alpha(s, \gamma_1)\alpha(s\gamma_1, \gamma_2)$ for almost all $s \in S$, and that two cocycles $\alpha, \beta : S \times \Gamma \to H$

are called equivalent if there is a Borel function $\varphi : S \to H$ such that for each $\gamma \in \Gamma$, $\alpha(s, \gamma) = \varphi(s)\beta(s, \gamma)\varphi(s\gamma)^{-1}$ almost everywhere. We shall be most often concerned with the case in which H is a linear group (usually algebraic). In this case an important invariant of the cocycle is the algebraic hull, introduced in [24] (see also [28]). We summarize some relevant information.

Proposition **2.1** [24], [28]. — *Suppose* $H \subset GL(n, \mathbf{C})$ *is an algebraic* **R**-*group and* $\alpha : S \times \Gamma \to H_{\mathbf{R}}$ *is a cocycle. Suppose the* Γ-*action on* S *is ergodic. Then there exists an* **R**-*subgroup* $L \subset H$ *such that* $\alpha \sim \alpha_1$ *where* α_1 *takes values in* $L_{\mathbf{R}}$, *and there is no proper* **R**-*subgroup* $M \subset L$ *such that* $\alpha \sim \alpha_2$ *where* α_2 *takes values in* $M_{\mathbf{R}}$. *The group* $L_{\mathbf{R}}$ *is unique up to conjugacy by an element of* $H_{\mathbf{R}}$ *and is called the algebraic hull of* α.

If G is a connected semisimple Lie group with finite center Z(G), then $G/Z(G) \cong (G^*_{\mathbf{R}})^0$ where G^* is a connected semisimple algebraic **R**-group with trivial center. If H is an **R**-group and $\pi : G \to H_{\mathbf{R}}$ is a homomorphism with $\pi \mid Z(G)$ trivial, we shall call π **R**-rational if the induced homomorphism on G/Z(G) is the restriction of an **R**-rational homomorphism $G^* \to H$.

The following is the version of the superrigidity theorem for cocycles that we will need. It is adapted from the version in [27]. (See also [29], Theorem 9.3.14.)

Theorem **2.2** [26], [27], [29]. — *Suppose* G *is a connected semisimple Lie group with finite center such that every simple factor has* **R**-*rank* ≥ 2. *Suppose* $\Gamma \subset G_{\mathbf{R}}$ *is a lattice and that* Γ *acts ergodically on* (S, μ) *where* μ *is an invariant probability measure. Suppose that* H *is a connected, semisimple, adjoint algebraic* **R**-*group, that* $H_{\mathbf{R}}$ *has no compact factors, and that* $\alpha : S \times \Gamma \to H_{\mathbf{R}}$ *is a cocycle whose algebraic hull is* $H_{\mathbf{R}}$ *itself. Then there exists an* **R**-*rational homomorphism* $\pi : G \to H_{\mathbf{R}}$ (*trivial on* Z(G)), *such that* $\alpha \sim \alpha_\pi$ *where* $\alpha_\pi : S \times \Gamma \to H_{\mathbf{R}}$ *is the cocycle* $\alpha(s, \gamma) = \pi(\gamma)$.

We shall need information concerning the situation in which the algebraic hull of α is not necessarily $H_{\mathbf{R}}$. To this end we recall the following fact about cocycles in the case in which Γ has Kazhdan's property. (See [1], [6], [18], [29] for Kazhdan's property.)

Theorem **2.3** [17], [25]. — *Suppose* Γ *has Kazhdan's property and that* A *is an amenable group. Suppose* Γ *acts ergodically with finite invariant measure on* S. *If* $\alpha : S \times \Gamma \to A$ *is a cocycle, then* α *is equivalent to a cocycle taking values in a compact subgroup of* A.

We observe that equivalence of a cocycle into a linear group to a cocycle into a compact subgroup is germane to the problem of finding invariant metrics. If $\alpha : S \times \Gamma \to H$ where H is a group acting continuously on the left on a separable metrizable space X, by an α-invariant function $\varphi : S \to X$ we mean a measurable function φ such that for each $g \in \Gamma$, $\alpha(s, g)\varphi(sg) = \varphi(s)$ for almost all $s \in S$. (See [29] for a general discussion.) If $H \subset GL(n, \mathbf{R})$ is a subgroup, then H acts on the space of inner products on $\mathbf{R}^n$, which we denote by $\mathrm{Inn}(\mathbf{R}^n)$. Fixing the standard basis in $\mathbf{R}^n$ we can identify $\mathrm{Inn}(\mathbf{R}^n)$ with the positive definite symmetric matrices, and hence we may speak of the determinant of any element in $\mathrm{Inn}(\mathbf{R}^n)$. If the determinant is 1 we say the inner product is unimodular.

Definition **2.4.** — *If* $\alpha : S \times \Gamma \to H \subset GL(n, \mathbf{R})$ *is a cocycle, an* α*-invariant function* $B : S \to \mathrm{Inn}(\mathbf{R}^n)$ *is called a measurable* α*-invariant metric.*

We let $SL'(n, \mathbf{R})$ be the subgroup of $GL(n, \mathbf{R})$ consisting of matrices A with $\det(A)^2 = 1$. We then have:

Proposition **2.5.** — *Suppose* $H \subset GL(n, \mathbf{R})$ *and* $\alpha : S \times \Gamma \to H$ *is a cocycle. If* $\alpha \sim \beta$ *where* $\beta(S \times \Gamma) \subset K$, $K \subset H$ *a compact subgroup, then there is a measurable* α*-invariant metric* B. *If* $H \subset SL'(n, \mathbf{R})$, *we can assume* $\det B(s) = 1$ *for all* s.

Conversely, if α *is a cocycle into* $SL'(n, \mathbf{R})$ *and there is a measurable* α*-invariant metric, then* α *is equivalent to a cocycle taking values in* $O(n, \mathbf{R})$.

Proof. — Let $B_0 \in \mathrm{Inn}(\mathbf{R}^n)$ be a K-invariant inner product and let $B(s) = \varphi(s)B_0$ where $\varphi : S \to H$ satisfies $\alpha(s, \gamma) = \varphi(s)\beta(s, \gamma)\varphi(s\gamma)^{-1}$. Then B is the required α-invariant metric. To see the converse, let B_0 be the standard $O(n, \mathbf{R})$-invariant inner product on $\mathbf{R}^n$, and $\mathrm{Inn}_1(\mathbf{R}^n)$ the inner products of determinant 1. We can choose a measurable function $\varphi : \mathrm{Inn}_1(\mathbf{R}^n) \to SL(n, \mathbf{R}^n)$ such that $P = \varphi(P) \cdot B_0$ for all $P \in \mathrm{Inn}_1(\mathbf{R}^n)$. If $s \to B(s)$ is the measurable α-invariant metric, let $\psi : S \to SL(n, \mathbf{R})$ be $\psi(s) = \varphi(\det(B(s))^{-1/n}B(s))$. Then $\beta(s, \gamma) = \psi(s)^{-1}\alpha(s, \gamma)\psi(sg) \in O(n, \mathbf{R})$.

We can now give an application of Theorems 2.2, 2.3 to the existence of measurable α-invariant metrics. We first record an elementary lemma.

Lemma **2.6.** — i) *Suppose* Γ *acts ergodically on* S, $\alpha : S \times \Gamma \to H$ *is a cocycle, and* $H_0 \subset H$ *is a subgroup of finite index. Then there is a finite ergodic extension* $p : T \to S$ *(i.e. all fibers are of fixed finite cardinality) such that the cocycle* $\tilde{\alpha} : T \times \Gamma \to H$, $\tilde{\alpha}(t, \gamma) = \alpha(p(t), \gamma)$ *is equivalent to a cocycle into* H_0.

ii) *Suppose* $\alpha : S \times \Gamma \to H$ *is a cocycle,* $p : H \to H_1$ *is a surjective homomorphism, and* $p \circ \alpha$ *is equivalent to the trivial cocycle. Then* α *is equivalent to a cocycle into* $\ker p$.

iii) *Suppose* $T \to S$ *is a finite extension of ergodic* Γ*-spaces and* $\alpha : S \times \Gamma \to SL'(n, \mathbf{R})$ *is a cocycle. Let* $\tilde{\alpha}$ *be defined as in* (i). *Then there is a measurable* α*-invariant metric if and only if there is a measurable* $\tilde{\alpha}$*-invariant metric.*

Proof. — For (i), (ii) see [29]. For (iii), one direction is clear. Conversely, if there is an $\tilde{\alpha}$-invariant metric, summing the metrics over the fibers produces an α-invariant metric.

We recall the following notation from the introduction.

Definition **2.7**. — *If* G *is a semisimple Lie group with no compact factors, let* $d(G)$ *be the minimal dimension of a non-trivial representation of the Lie algebra of* G *on a real vector space.*

Theorem **2.8**. — *Let* G, Γ *be as in Theorem* 2.2. *Suppose that* (S, μ) *is a* Γ*-space such that (almost) every ergodic component of the* Γ *action on* S *has a finite* Γ*-invariant measure. Let* $\alpha : S \times \Gamma \to SL'(n, \mathbf{R})$ *be a cocycle. If* $n < d(G_{\mathbf{R}})$, *then there is a measurable* α*-invariant metric* B *with* $\det B(s) = 1$ *for all* s.

Proof. — Using a standard ergodic decomposition argument it suffices to consider the case in which Γ acts ergodically with an invariant probability measure. Let $H_{\mathbf{R}} \subset SL'(n, \mathbf{R})$ be the algebraic hull of α (Proposition 2.1). If H is not connected, by Lemma 2.6 (i) we can pass to a finite ergodic extension of S and assume that $\tilde{\alpha}$ (defined as in 2.6) takes values in $(H^0)_{\mathbf{R}}$. If the algebraic hull of $\tilde{\alpha}$ is not $(H^0)_{\mathbf{R}}$, take the algebraic hull. Once again, if this is not connected repeat the process, passing to a finite ergodic extension. In this way, replacing S by a finite ergodic extension if necessary, we can assume (using lemma 2.6 (iii)) that the algebraic hull of α, say $H_{\mathbf{R}}$, is such that H is connected. We can write $H_{\mathbf{R}} = L_{\mathbf{R}} \ltimes U_{\mathbf{R}}$ where L is reductive and U is the unipotent radical of H. Let $q : H_{\mathbf{R}} \to L_{\mathbf{R}}/[L_{\mathbf{R}}, L_{\mathbf{R}}]$ be the natural projection. By Theorem 2.3, $q \circ \alpha$ is equivalent to a cocycle into a compact subgroup of $L_{\mathbf{R}}/[L_{\mathbf{R}}, L_{\mathbf{R}}]$. Since $L_{\mathbf{R}}/[L_{\mathbf{R}}, L_{\mathbf{R}}]$ is the algebraic hull of $q \circ \alpha$, it follows that this group is compact. We have $L = [L, L]Z(L)$ where $Z(L)$ is the center of L and $[L, L] \cap Z(L)$ is finite. It follows that $Z(L)_{\mathbf{R}}$ is also compact.

We can write the $\mathbf{R}$-group $L/Z(L)$ as a product of connected semisimple $\mathbf{R}$-groups $L/Z(L) = L_1 \times L_2$ where $(L_2)_{\mathbf{R}}$ is compact and $(L_1)_{\mathbf{R}}$ is centerfree with no compact factors. Let $q_1 : H_{\mathbf{R}} \to (L_1)_{\mathbf{R}}$ be the projection. Then $q_1 \circ \alpha$ is a cocycle with algebraic hull $(L_1)_{\mathbf{R}}$ and by the superrigidity theorem for cocycles (Theorem 2.2), there is a $\mathbf{R}$-rational homomorphism $\pi : G \to (L_1)_{\mathbf{R}}$ such that $q_1 \circ \alpha$ is equivalent to α_π. We can consider π as a rational homomorphism $\pi : \tilde{G} \to L_1$ where $\tilde{G}$ is the algebraic universal covering group of G^*. (Here G^* is as in the discussion preceding Theorem 2.2.) Then π lifts to a homomorphism $\tilde{\pi} : \tilde{G} \to [L, L] \subset GL(n, \mathbf{C})$. By the definition of $d(G)$, our hypotheses imply $\tilde{\pi}$ is trivial. Since the algebraic hull of $\tilde{\pi}(\tilde{G}_{\mathbf{R}})$ is $(L_1)_{\mathbf{R}}$, L_1 is trivial. It follows that $L_{\mathbf{R}}$ is compact and hence that $H_{\mathbf{R}}$ is amenable. Applying Theorem 2.3 once again, we deduce that α is equivalent to a cocycle taking values in a compact subgroup and hence by Proposition 2.5, there is a measurable α-invariant metric.

If we drop the restriction that $n < d(G)$ the conclusion of Theorem 2.8 is obviously no longer true in general. There are however other hypotheses which will lead to the same conclusion with close to the same proof. If $\alpha : S \times \Gamma \to GL(n, \mathbf{R})$ is a cocycle, $\Gamma_0 \subset \Gamma$ is a finite set, and $\varepsilon > 0$ we say that α is (ε, Γ_0)-admissable if $\|\alpha(s, \gamma)\| \leqslant 1 + \varepsilon$, $\|\alpha(s, \gamma)^{-1}\| \leqslant 1 + \varepsilon$ for all $s \in S$ and all $\gamma \in \Gamma_0$.

Theorem **2.9**. — *Let* Γ, S *be as in Theorem* 2.8, *and suppose* $\Gamma_0 \subset \Gamma$ *is a finite generating set. Fix* $n > 0$. *Then there exists* $\varepsilon > 0$ *such that for any* (ε, Γ_0)-*admissible cocycle* $\alpha : S \times \Gamma \to SL'(n, \mathbf{R})$ *there is a measurable* α-*invariant metric.*

To describe the proof, we first recall the theorem of Furstenberg and Kesten concerning products of random matrices. If Λ is a group, S a Λ-space, and $\alpha : S \times \Lambda \to GL(n, \mathbf{R})$ is a cocycle, we call α a tempered cocycle if for each $h \in \Lambda$, $\|\alpha(\cdot, h)\| \in L^\infty(S)$. Now suppose $\Lambda = \mathbf{Z}$, the group of integers. Define

$$e(\alpha)(s) = \lim_{n \to \infty} \frac{1}{n} \log^+ \|\alpha(s, n)\|$$

if it exists. (We remark that if $\alpha(S \times \mathbf{Z}) \subset SL'(n, \mathbf{R})$, then $\|\alpha(s, n)\| \geqslant 1$, so $\log^+ \|\alpha(s, n)\| = \log \|\alpha(s, n)\|$.)

Proposition **2.10** (Furstenberg-Kesten). — *Let* (S, μ) *be a* $\mathbf{Z}$-*space with invariant probability measure. Suppose* $\alpha : S \times \mathbf{Z} \to GL(n, \mathbf{R})$ *is a tempered cocycle. Then:*

i) $e(\alpha)$ *exists and is a* $\mathbf{Z}$-*invariant function on* S. *Hence, if* $\mathbf{Z}$ *acts ergodically,* $e(\alpha)$ *is a constant.*

ii) *If* α *and* β *are tempered and* $\alpha \sim \beta$, *then* $e(\alpha) = e(\beta)$ *a.e.*

For a proof, see [2], [3], [7]. The following proposition summarizes some other useful properties of $e(\alpha)$.

Proposition **2.11**

1) *If* $\alpha(s, n) = A^n$ *for some matrix* $A \in SL'(n, \mathbf{R})$, *then*

$$e(\alpha) = \max\{\log|\lambda| \mid \lambda \text{ is an eigenvalue of } A\}.$$

2) *More generally, suppose* $A \in SL'(n, \mathbf{R})$ *and* $\alpha(s, n) \equiv A^n \bmod K$ *where* $K \subset SL'(n, \mathbf{R})$ *is a compact subgroup normalized by* A. *Then*

$$e(\alpha) = \max\{\log|\lambda| \mid \lambda \text{ is an eigenvalue of } A\}.$$

3) *Let* $H \subset GL(n, \mathbf{C})$ *be a connected* $\mathbf{R}$-*group,* $H = L \ltimes U$ *where* L *and* U *are respectively reductive and unipotent* $\mathbf{R}$-*subgroups. Let* $q : H_\mathbf{R} \to L_\mathbf{R}$ *be projection. If* $\alpha : S \times \mathbf{Z} \to H_\mathbf{R}$ *is tempered, then* $e(\alpha) = e(q \circ \alpha)$.

4) *More generally, with* H, L, q *as in* (3), *suppose* $\sigma : S \times \mathbf{Z} \to GL(n, \mathbf{R})$ *is tempered, that* $\sigma \sim \alpha$ *where* $\alpha(S \times \mathbf{Z}) \subset H_\mathbf{R}$, *and that* $q \circ \alpha$ *is tempered. Then* $e(q \circ \alpha) \leqslant e(\sigma)$.

5) *If* $p : T \to S$ *is a finite extension, and* α *is a tempered cocycle on* S, *then* $e(\alpha) = e(\tilde{\alpha})$ *where* $\tilde{\alpha}(t, n) = \alpha(p(t), n)$.

We now recall one fact about Lie algebras and one fact about lattices in semisimple algebraic groups.

Proposition **2.12.** — a) *Let* $\mathfrak{I}$ *be a complex semisimple Lie algebra. Then for any n the number of inequivalent representations of* $\mathfrak{I}$ *on a space of dimension n is finite.*

b) *Let* G *be a connected semisimple algebraic* **R**-*group such that* $G_{\mathbf{R}}$ *has no compact factors. Let* $\Gamma \subset G_{\mathbf{R}}$ *be a lattice. Then for any non-trivial* **R**-*rational representation* $\pi : G \to GL(n, \mathbf{C})$, *there exists* $\gamma \in \Gamma$ *such that* $\pi(\gamma)$ *has an eigenvalue* λ *with* $|\lambda| > 1$.

Proof. — *a)* is standard and follows for example from the Weyl character formula [20, p. 363].

b) follows from [12], [15].

With these preliminaries, we can now prove Theorem 2.9.

Proof of Theorem 2.9. — As in the proof of 2.8 it suffices to consider the case in which Γ acts ergodically with an invariant probability measure. Let $f : G \to G^*_{\mathbf{R}}$ be the projection, where G^* is as in the discussion preceding Theorem 2.2. Let $\psi : \widetilde{G} \to G^*$ be the algebraic universal covering of G^*, $\widetilde{\Gamma} = \psi^{-1}(f(\Gamma))$. By 2.12 we can choose a finite set $\widetilde{F} \subset \widetilde{\Gamma}$ and $r > 1$ such that if σ is any non-trivial **R**-rational representation $\sigma : \widetilde{G} \to GL(n, \mathbf{C})$, there exists $\widetilde{\gamma} \in \widetilde{F}$ such that $\sigma(\widetilde{\gamma})$ has an eigenvalue λ with $\log|\lambda| > \log r$. Let $F = f^{-1}(\psi(\widetilde{F}))$. Since Γ_0 generates Γ, there is an integer N with $(\Gamma_0 \cup \Gamma_0^{-1})^N \supset F$. Let $\varepsilon > 0$ be such that $(1 + \varepsilon)^N < r$. For each $\gamma \in \Gamma$ and each tempered cocycle α, let $e_{\alpha,\gamma} = e(\alpha \mid S \times \{\gamma^n\})$. Then if α is (ε, Γ_0)-admissible, we have for $\gamma \in F$ that

$$(*) \qquad |e_{\alpha,\gamma}(s)| \leqslant \log r \quad \text{for all } s \in S.$$

Now consider the proof of Theorem 2.8. We can construct the **R**-rational homomorphism π and the lifted homomorphism $\widetilde{\pi} : \widetilde{G} \to GL(n, \mathbf{C})$ as in that proof. Let $p : H_{\mathbf{R}} \to L_{\mathbf{R}}$ and $p_1 : L_{\mathbf{R}} \to (L_1)_{\mathbf{R}}$ be the projection maps where the groups are defined as in the proof of 2.8. Since $p_1 \circ p \circ \alpha \sim \alpha_\pi$, it follows that $p \circ \alpha \sim \beta$ where $\beta : S \times \Gamma \to L_{\mathbf{R}}$ is a cocycle with $p_1(\beta(s, \gamma)) = \pi(\gamma)$. Thus, if $\widetilde{\gamma} \in \widetilde{G}$ projects to $f(\gamma) \in f(\Gamma)$, we have $p_1(\beta(s, \gamma)) = p_1(\widetilde{\pi}(\widetilde{\gamma}))$. In other words, $\beta(s, \gamma) = \widetilde{\pi}(\widetilde{\gamma}) . b(s, \widetilde{\gamma})$ where $b(s, \widetilde{\gamma}) \in \ker(p_1)$. However, $\ker(p_1)$ is a compact normal subgroup of $L_{\mathbf{R}}$. It follows from Proposition 2.11 (2) that $e_{\beta,\gamma} = \max\{\log|\lambda| \mid \gamma \text{ is an eigenvalue of } \widetilde{\pi}(\widetilde{\gamma})\}$. However, $e_{\beta,\gamma} \leqslant e_{\alpha,\gamma}$ by Proposition 2.10 and 2.11, and hence it follows from (*) that for all $\gamma \in F$, $e_{\beta,\gamma} \leqslant \log r$. But this contradicts the choice of r unless π is trivial. We can then complete the argument as in the proof of Theorem 2.8.

A similar argument shows the following.

Theorem **2.13.** — *Let* Γ *be as in* 2.8, S *a* Γ-*space with an invariant probability measure. Suppose* $\alpha : S \times \Gamma \to SL'(n, \mathbf{R})$ *is a tempered cocycle. If* $e_{\alpha,\gamma} = 0$ *for all* $\gamma \in \Gamma$, *then there is a measurable* α-*invariant metric.*

An argument similar to that in the proof of Theorem 2.8 yields another result of the same type.

Theorem **2.14.** — *Let* G, Γ, S *as in* 2.8. *Suppose* $H \subset GL(n, \mathbf{C})$ *is an* **R**-*group and* $\alpha : S \times \Gamma \to H_{\mathbf{R}}$ *is a cocycle. Suppose* $\mathbf{R}\text{-rank}(H) < \mathbf{R}\text{-rank}(G_i)$ *for every non-trivial almost* **R**-*simple factor* G_i *of* G. *Then there is a measurable* α-*invariant metric.*

3. On principal bundles and jet bundles

In this section we recall some basic information concerning principal bundles and jet bundles incorporating some of the measure theoretic information of the previous section.

Let H be a Lie group, M a manifold, and $\pi : P \to M$ a principal H-bundle. We shall take H to be acting on the right of P, so that M is the quotient space P/H. The trivial principal H-bundle is just the product bundle $M \times H$ with H acting on the second coordinate by right translation. We recall that any section $\varphi : M \to P$ of π defines a trivialization of P, i.e. an equivalence of P with the trivial H-bundle. If φ is smooth (resp. continuous, measurable), then the equivalence of P with $M \times H$ will be smooth (resp. continuous, measurable) as well. If X is any left H-space, then H acts on the right of $P \times X$ via $(z, x).h = (zh, h^{-1}x)$ and the quotient space $E = (P \times X)/H$ will be a bundle over M with fiber X, the " associated bundle " to P with fiber X. Suppose now that Γ acts (on the left) by automorphisms of the principal H-bundle P. Letting Γ act trivially on X, the Γ action and the H action on $P \times X$ commute, so that Γ acts on the bundle E in such a way that the projection map $E \to M$ is a Γ-map.

For any principal H-bundle $\pi : P \to M$ one can always find a Borel section of π. This defines an associated measurable trivialization $P \cong M \times H$. If Γ acts by automorphisms of P, then under this trivialization we have the Γ-action given by $\gamma.(m, h) = (\gamma.m, \alpha(m, \gamma)^{-1}h)$ where $\alpha(m, \gamma) \in H$. Since the trivialization is Borel, $\alpha : M \times \Gamma \to H$ will be a Borel function, and if we write the action of Γ on M on the right (as in section 2) instead of the left, it follows directly that α is a cocycle. It is easy to check that different trivializations yield equivalent cocycles. In other words, whenever Γ acts by automorphisms of the principal H-bundle $P \to M$, we have a naturally defined equivalence class of measurable cocycles $M \times \Gamma \to H$. Furthermore, if M is compact we can choose the image of the Borel section to lie in a compact subset of P. It follows that for M compact and H linear we can choose the cocycle α to be tempered. (See the definition preceding Proposition 2.10.)

If $E \to M$ is the bundle associated to P by the action of H on X, the measurable trivialization of P defines a measurable trivialization of $E \cong M \times X$. If Γ acts by automorphisms of P then the associated action on $M \times X$ is simply given by $\gamma.(m, x) = (\gamma m, \alpha(m, \gamma)^{-1}x)$ where α is the above cocycle. We record the following trivial observation.

Lemma **3.1.** — *There is a measurable* Γ*-invariant section of the associated bundle* $E \to M$ *if and only if there is an* α*-invariant measurable function* $M \to X$ *(in the sense of the discussion following Theorem 2.3).*

If $X = \mathbf{R}^n$ and H acts by a differentiable linear representation on $\mathbf{R}^n$, then the associated bundle $\pi : E \to M$ will be a vector bundle and Γ will act on E by vector bundle automorphisms. As usual a Γ-invariant (Riemannian) metric on E will mean a Γ-invariant assignment $m \to B_m$, where B_m is an inner product on $\pi^{-1}(m)$. This assignment may be smooth (resp., continuous, measurable, etc.), and for a given bundle we may enquire as to the existence of a smooth (resp., continuous, measurable, etc.) Γ-invariant metric. If $E = TM$, the tangent bundle of M, we speak of a smooth (resp. continuous, measurable) metric on M. Via Lemma 3.1, we may translate some of the results in section 2 into the present situation. It will be convenient to introduce some notation.

If V is a real vector space and η is an inner product on V, let $\| \ \|_\eta$ denote the corresponding norm on V.

Definition **3.2.** — *If* η, ξ *are two inner products, let*

$$M(\eta/\xi) = \max\{\|v\|_\eta \mid v \in V, \|v\|_\xi = 1\}$$
$$= \max_{v \neq 0}\{\|v\|_\eta/\|v\|_\xi\}.$$

If $E \to M$ is a vector bundle over M and η, ξ are two measurable Riemannian metrics on E, then $M(\eta/\xi)(s) = M(\eta(s)/\xi(s))$ is a measurable function of $s \in M$.

If $\varepsilon > 0$ and F is a finite set of vector bundle automorphisms of E, we call a metric η on E (ε, F)-invariant if for each $f \in F$,

$$M(f^*\eta/\eta),\ M(\eta/f^*\eta) < \varepsilon.$$

For Theorems 3.3-3.5 we make the following hypotheses. Let G be a connected semisimple Lie group with finite center such that every simple factor of G has $\mathbf{R}$-rank ≥ 2. Let $\Gamma \subset G$ be a lattice and $\Gamma_0 \subset \Gamma$ a fixed finite generating set. Suppose (N, ω) is a manifold with a smooth volume density ω, and $E \to N$ is a vector bundle.

Theorem **3.3.** — *Suppose* $\mathrm{degree}(E) < d(G)$. *Then any action of* Γ *by vector bundle automorphisms of* E *which covers an* ω*-preserving smooth action of* Γ *on* N, *for which the ergodic components (of the action on* N*) have a finite invariant measure, leaves a measurable Riemannian metric on* E *invariant. If* $E = TM$, *the associated volume density of this metric can be taken to be* ω.

Theorem **3.4**. — *Let* η *be a smooth metric on* E. *Then there exists* $\varepsilon > 0$ *such that any action of* Γ *by vector bundle automorphisms of* E *which covers a smooth* ω*-preserving action of* Γ *on* N *such that*:

i) *the ergodic components of* Γ *on* N *have finite invariant measure*; *and*

ii) η *is* (ε, Γ_0)*-invariant*;

leaves a measurable Riemannian metric on E *invariant. If* $E = TM$, *the associated volume density can be chosen to be* ω.

We recall that if $H \subset GL(n, \mathbf{R})$ is a closed subgroup, $n = \text{degree}(E)$, then an H-structure on E is a reduction of the structure group of E to H. Equivalently, it can be defined as a smooth section of the bundle with fiber $GL(n, \mathbf{R})/H$ associated to the principal $GL(n, \mathbf{R})$-bundle of frames of E. If Γ acts by vector bundle automorphisms of E, one can clearly speak of a Γ-invariant H-structure on E.

Theorem **3.5**. — *Suppose* Γ *acts by vector bundle automorphisms of* E *covering an* ω*-preserving smooth action on* N *for which the ergodic components have finite invariant measure. Let* $n = \text{degree}(E)$ *and* $H \subset GL(n, \mathbf{C})$ *an* $\mathbf{R}$*-group with* $\mathbf{R}\text{-rank}(H) < \mathbf{R}\text{-rank}(G_i)$ *for every non-trivial simple factor of* G_i *of* G. *If* Γ *leaves an* $H_{\mathbf{R}}$*-structure invariant, then there is a measurable* Γ*-invariant Riemannian metric on* E.

Theorems 3.3-3.5 follow from Lemma 3.1 and the results of section 2. When applied to the tangent bundle of a manifold on which Γ acts, Theorems 3.3-3.5 show that the superrigidity theorem for cocycles yields, under very natural and general hypotheses, the existence of a measurable Γ-invariant Riemannian metric. The next main problem, to which most of this paper is devoted, is to obtain the existence of a C^0 Γ-invariant Riemannian metric, which we achieve via the existence of the measurable invariant metric. Before proceeding, however, we make some general remarks about diffeomorphisms with a measurable invariant metric.

Let $\varphi : M \to M$ be any volume preserving diffeomorphism of a compact manifold. Consider the following properties:

i) there is a C^0 φ-invariant metric on M;

ii) there is a measurable φ-invariant metric on M;

iii) $h(\varphi) = 0$ where $h(\varphi)$ is the entropy of φ.

For a general diffeomorphism, we have (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). (The first assertion is trivial. As for the second, from Pesin's formula [14] for the entropy of a volume preserving diffeomorphism, to see that $h(\varphi) = 0$ it suffices to see that $e(\alpha) = 0$ where α is the cocycle corresponding to the derivative under a bounded measurable trivialization of the tangent bundle. However, if there is measurable α-invariant metric, then $\alpha \sim \beta$ where β takes values in $O(n)$, $n = \dim M$. Since α and β are tempered and equivalent, $e(\alpha) = e(\beta)$, and clearly $e(\beta) = 0$.) On the other hand, the converses are not true in general. For example, the time 1 diffeomorphism of a classical horocycle flow satisfies (iii) but not (ii). Thus, for a single diffeomorphism the existence of a measurable invariant metric lies properly between the two classical notions of isometry and 0 entropy. For the groups we have been considering we have:

Theorem **3.6.** — *Let* Γ *be as in* 3.3 *and suppose* Γ *acts on a compact manifold by smooth diffeomorphisms preserving a smooth volume density. Then the following are equivalent:*

a) *There is a measurable* Γ*-invariant metric.*

b) $h(\gamma) = 0$ *for all* $\gamma \in \Gamma$.

Proof. — From Pesin's formula for the entropy of a volume preserving diffeomorphism [14], we have that under a bounded Borel trivialization of the tangent bundle $h(\gamma) = 0$ implies $e_{\alpha,\gamma} = 0$ where α is the cocycle corresponding to the trivialization. The result then follows from Theorem 2.13.

Conjecture. — For Γ as in 3.6, conditions (a) (or (b)) imply the existence of a C^0-invariant Riemannian metric.

It also follows from this discussion that the above consequences of the superrigidity theorem for cocycles, Theorems 3.3-3.5, without approaching the question of the existence of a C^0 invariant metric, already have strong implications for smooth Γ-actions. For example:

Corollary **3.7.** — *Suppose* M *is a compact manifold,* $\dim M < d(G)$. *Then for any smooth volume preserving* Γ*-action on* M *we have* $h(\gamma) = 0$ *for all* $\gamma \in \Gamma$.

This of course follows from 3.3. Similar results may be deduced from 3.4, 3.5. For other results of this type, see [4], [29].

In our proof of the existence of Γ-invariant C^0 Riemannian metrics on TM where M is compact (under suitable hypotheses of course), we shall be applying the above consequences of superrigidity not only to TM but to certain jet bundles on the frame bundle of M which is of course a non-compact manifold. This accounts for our formulation of Theorems 3.3-3.5 for certain non-compact manifolds.

We now recall some basic facts and establish some notation concerning jet bundles. Our basic reference here is [13, Chapter 4].

Let N be a manifold and $E \to N$ a (smooth) vector bundle. (Vector bundles will always be assumed to be finite dimensional unless explicitly indicated otherwise.) Then for each integer k we have another vector bundle $J^k(N; E)$ over N, the k-jet bundle of E, where the fiber at each point $x \in N$ consists of smooth sections of E, two sections being identified if they agree up to order k at x. For any vector bundle E we let $C^\infty(N; E)$ be the space of smooth sections of E. There is a natural map

$$j^k : C^\infty(N; E) \to C^\infty(N; J^k(N; E)),$$

$j^k(f)$ being called the k-jet extension of $f \in C^\infty(N; E)$. We can naturally identify the bundles E and $J^0(N; E)$.

If V, W are finite dimensional real vector spaces, we let $S^r(V, W)$ be the space of symmetric r-linear maps $V^r \to W$. Similarly, if E, F are vector bundles over N, we let $S^r(E, F)$ denote the vector bundle whose fiber at x is $S^r(E_x, F_x)$. If $N \subset \mathbf{R}^n$ is open and E is a product bundle over N, then we can naturally identify $J^k(N; E) \cong \sum_{r=0}^{k}{}^{\oplus} S^r(TN, E)$, where TN is the tangent bundle of N. However this decomposition cannot be carried over in a natural way to an arbitrary N (even for trivial bundles) via coordinate charts. In general there are natural maps

$$J^k(N; E) \to J^{k-1}(N; E) \quad \text{and} \quad S^k(TN, E) \to J^k(N; E)$$

such that

$$0 \to S^k(TN, E) \to J^k(N; E) \to J^{k-1}(N; E) \to 0$$

is exact. (This is the jet bundle exact sequence.) There is no natural splitting of this sequence. (One exception of course is that the composition map $J^k(N; \mathbf{C}) \to J^0(N; \mathbf{C}) \to 0$ where $\mathbf{C}$ denotes the trivial 1-dimensional bundle, is naturally split by the " constant map ".) However, given a connection ∇_E on E and a connection ∇_{T^*N} on T^*N, then there is a natural splitting of the jet bundle exact sequence and hence a natural (given the connections) identification $J^k(N; E) \cong \sum_{r=0}^{k} S^r(TN, E)$. (See [13, p. 90].) If $f: (N, E) \to (N', E')$ is a vector bundle isomorphism covering a diffeomorphism $N \to N'$ (which we still denote by f), then f induces natural maps $S^r(f) : S^r(TN, E) \to S^r(TN', E')$ and $J^k(f) : J^k(N, E) \to J^k(N', E')$. If f is connection preserving on E and T^*N, then under the above identification, $J^k(f) \sim \sum_{r=0}^{k}{}^{\oplus} S^r(f)$. This of course is no longer true if f fails to preserve the connections.

Suppose now that $E = \mathbf{C}$, the 1-dimensional trivial bundle. Any smooth Riemannian metric ξ on N defines in a canonical way a connection on TN, hence on T^*N, and hence an isomorphism $J^k(N; \mathbf{C}) \cong \sum_{r=0}^{k} S^r(TN, \mathbf{C})$. This isomorphism depends upon the k-th iteration of the covariant derivative defined by ξ [13, p. 90] and hence, in local coordinates upon derivatives of the coordinates of the metric up to order k. The metric ξ also defines in a natural way a metric on each $S^r(TN, \mathbf{C})$ and hence via the above isomorphism a metric ξ_k on the bundle $J^k(N; \mathbf{C})$. Our remarks imply the following.

Proposition **3.8.** — i) *The map* $\xi \to \xi_k$ *is natural*; *i.e. if* $f: N \to N'$ *is a diffeomorphism with* $f^*(\xi') = \xi$, *then the induced map* $J^k(f) : J^k(N; \mathbf{C}) \to J^k(N', \mathbf{C})$ *satisfies* $J^k(f)^*(\xi'_k) = \xi_k$.

ii) *The map* $\xi \to \xi_k$ *is continuous where metrics on* N *have the* C^k*-topology (defined by* C^k*-convergence on compact subsets of* N*) and metrics on* $J^k(N; \mathbf{C})$ *have the* C^0*-topologv (defined by* C^0*-convergence on compact subsets of* N*).*

4. Integrability of a Measurable Invariant Metric

In this section we make an estimate giving conditions under which a measurable invariant metric will satisfy L^p conditions with respect to a smooth metric. This will be a consequence of the following general fact about actions of groups having Kazhdan's property.

Theorem **4.1.** — *Let Γ be a (countable) discrete group having Kazhdan's property, and $\Gamma_0 \subset \Gamma$ a finite generating set. Then there exists $K > 1$ (depending on Γ_0) with the following property: If (S, μ) is any ergodic Γ-space with invariant probability measure, and $f : S \to \mathbf{R}$ is a measurable function satisfying $|f(s\gamma)| \leqslant K\,|f(s)|$ for all $s \in S$, $\gamma \in \Gamma_0$, then $f \in L^1(S)$.*

We recall the definition of Kazhdan's property. If π is a unitary representation of a locally compact group Γ on a Hilbert space $\mathscr{H}$, $F \subset \Gamma$ is compact, and $\varepsilon > 0$, a unit vector $x \in \mathscr{H}$ is called (ε, F)-invariant if $\|\pi(\gamma)x - x\| < \varepsilon$ for all $\gamma \in F$. We say that Γ has Kazhdan's property if there is some (ε, F) such that any π with (ε, F)-invariant unit vectors actually has invariant unit vectors [1], [6], [29]. Any discrete group with Kazhdan's property is finitely generated and F can clearly be taken to be a generating set. Furthermore, if Γ_0, $F \subset \Gamma$ are two finite sets with $\Gamma_0^n \supset F$ for some integer n, then any $(\varepsilon/n, \Gamma_0)$-invariant unit vector will be (ε, F)-invariant. (To see this just observe that if x is $(\varepsilon/n, \Gamma_0)$-invariant and $\gamma_i \in \Gamma_0$, then

$$\begin{aligned} \|\pi(\gamma_1 \dots \gamma_n)x - x\| &\leqslant \|\pi(\gamma_1)\pi(\gamma_2 \dots \gamma_n)x - \pi(\gamma_1)x\| + \|\pi(\gamma_1)x - x\| \\ &\leqslant \|\pi(\gamma_2 \dots \gamma_n)x - x\| + \varepsilon/n. \end{aligned}$$

Repeating the argument we see that x is $(\varepsilon, \Gamma_0^n)$-invariant.) Thus, in the defining condition of Kazhdan's property we may assume F is any predetermined finite generating set.

Proof of Theorem 4.1. — We apply Kazhdan's property to the unitary representation π of Γ on $L^2(S) \ominus \mathbf{C}$ defined by translation. By ergodicity of Γ on S (with respect to an invariant probability measure), there are no invariant unit vectors for π. Hence, given Γ_0 there exists $\varepsilon > 0$ (independent of the ergodic action) such that there are no (ε, Γ_0)-invariant unit vectors. In other words, for every unit vector x, there exists $\gamma_0 \in \Gamma$ such that $\|\pi(\gamma_0)x - x\| \geqslant \varepsilon$. For any measurable set $A \subset S$, let χ_A be the characteristic function, p_A the projection of χ_A onto $L^2(S) \ominus \mathbf{C}$, and $f_A = p_A/\|p_A\|$ when A is neither null nor conull. If $A, B \subset S$ with $\mu(A) = \mu(B)$, and A neither null nor conull, then a straightforward calculation shows that

$$\|f_A - f_B\|^2 = \mu(A \,\Delta\, B)/\mu(A)(1 - \mu(A)).$$

Choose a measurable set $A_0 \subset S$ such that $|f|$ is bounded on A_0, say $|f(x)| \leqslant B$ for $x \in A_0$ and with $\mu(A_0) > 1/2$. If $\mu(A_0) = 1$, we are done. If not, there exists $\gamma_0 \in \Gamma_0$ such that

$$\|\pi(\gamma_0)f_{A_0} - f_{A_0}\| \geqslant \varepsilon, \quad \text{i.e. } \|f_{A_0\gamma_0} - f_{A_0}\| \geqslant \varepsilon.$$

Thus, $\mu(A_0\gamma_0 \,\Delta\, A_0) \geqslant \frac{\varepsilon^2}{2}(1 - \mu(A_0))$, so that

$$\mu(A_0\gamma_0 \cap (S - A_0)) \geqslant \frac{\varepsilon^2}{4}(1 - \mu(A_0)).$$

Thus we can choose $A_1 \subset (A_0\gamma_0 \cap (S - A_0))$ such that $\mu(A_1) = \frac{\varepsilon^2}{4}(1 - \mu(A_0))$. (We remark that we can assume S is non-atomic for otherwise ergodicity implies S is finite.) Suppose $|f(x\gamma)| \leqslant K|f(x)|$ for $x \in S$, $K > 1$. Then we have $|f(x)| \leqslant KB$ for $x \in A_1$.

We now repeat the argument applied to $A_0 \cup A_1$ instead of A_0. We deduce the existence of $A_2 \subset (S - (A_0 \cup A_1))$ with $\mu(A_2) = \frac{\varepsilon^2}{4}(1 - \mu(A_0 \cup A_1))$ and $|f(x)| \leqslant K^2B$ for $x \in A_2$. Continuing inductively, we find a disjoint collection of measurable sets $A_i \subset S$ such that $|f(x)| \leqslant K^nB$ for $x \in A_n$, and letting $a_i = \mu(A_i)$ we have, for $n \geq 0$, $a_{n+1} = \frac{\varepsilon^2}{4}(1 - \sum_{i=0}^{n} a_i)$. Thus if $n \geq 1$,

$$\begin{aligned} a_{n+1} &= \frac{\varepsilon^2}{4}\Big(1 - \sum_{i=0}^{n-1} a_i - a_n\Big) \\ &= a_n - \frac{\varepsilon^2}{4} a_n \\ &= a_n\left(1 - \frac{\varepsilon^2}{4}\right). \end{aligned}$$

$\left(\text{For } n = 0, \text{ we have } a_1 = \frac{\varepsilon^2}{4}(1 - a_0).\right)$ We clearly have $\sum_{i=0}^{\infty} a_i = 1$, and hence $\int |f| \leqslant \sum_{n=0}^{\infty} a_n K^n B$. This will be finite if $\lim \frac{a_{n+1}K^{n+1}}{a_n K^n} < 1$, i.e. if $K < \frac{1}{1 - \varepsilon^2/4}$.

This completes the proof.

We now apply Theorem 4.1 to a measurable invariant metric. For two measurable metrics η, ξ on a vector bundle, recall from Definition 3.2 the measurable function $M(\eta/\xi)$. We remark that if η, ξ, τ are metrics, then $M(\eta/\tau) \leqslant M(\eta/\xi)M(\xi/\tau)$. We also note that if $T: W \to V$ is a vector space isomorphism and η, ξ are inner products on V, then $M(T^*\eta/T^*\xi) = M(\eta/\xi)$.

Corollary 4.2. — *Let* Γ *be a discrete group with Kazhdan's property,* $\Gamma_0 \subset \Gamma$ *a finite generating set. Let* $E \to N$ *be a continuous vector bundle over a separable metrizable space* N *and suppose that* Γ *acts by vector bundle automorphisms of* E *so that the action on* N *preserves a probability measure* μ. *Suppose* η *and* ξ *are measurable Riemannian metrics on* E *such that:*

i) η *is* Γ*-invariant; and*

ii) *for* $\gamma \in \Gamma_0$, $M(\gamma^*\xi/\xi)$ *and* $M(\xi/\gamma^*\xi)$ *are uniformly bounded by* $K^{1/p}$ $(p \geq 1)$ *where* K *is as in Theorem* 4.1.

Then $M(\eta/\xi)$, $M(\xi/\eta) \in L^p(N, \mu)$.

Proof. — For $s \in N$ and $\gamma \in \Gamma_0$, we have

$$\begin{aligned} M(\eta/\xi)(s\gamma) &= M(\gamma^*\eta/\gamma^*\xi)(s) \\ &\leqslant M(\eta/\xi)(s) \,.\, M(\xi/\gamma^*\xi)(s) \\ &\leqslant K^{1/p} M(\eta/\xi)(s). \end{aligned}$$

Thus $M(\eta/\xi) \in L^p(N)$ by Theorem 4.1. That $M(\xi/\eta) \in L^p$ is proved similarly.

5. The Main Argument

The point of this section is to prove the following theorem stated in the introduction.

Theorem 5.1. — *Let* G *be a connected semisimple Lie group with finite center such that the* **R**-*rank of every simple factor of* G *is at least* 2. *Let* $\Gamma \subset G$ *be a lattice and* $\Gamma_0 \subset \Gamma$ *a finite generating set. Let* M *be a compact n-manifold, and let* $r = n^2 + n + 1$. *Let* ξ *be a smooth Riemannian metric on* M *with volume density* ω. *Then there is a* C^r*-neighborhood* $\mathcal{O}$ *of* ξ *such that any* ω*-preserving smooth ergodic action of* Γ *on* M *with* ξ $(\mathcal{O}, \Gamma_0)$*-invariant leaves a* C^0*-Riemannian metric invariant.*

We begin the proof with some observations on the frame bundle. The volume density ω defines a $SL'(n, \mathbf{R})$-structure on M. We let $P \to M$ be the corresponding principal $SL'(n, \mathbf{R})$-bundle, where the fiber $F_m \subset P$ over $m \in M$ is the space of frames in TM_m spanning a parallelepiped of volume 1 with respect to $\omega(m)$. Thus, $SL'(n, \mathbf{R})$ acts on the right of P and we have an action of Γ on the left of P which commutes with the $SL'(n, \mathbf{R})$-action. Fix an inner product on the Lie algebra $\mathfrak{sl}(n, \mathbf{R})$ which is $Ad(O(n, \mathbf{R}))$-invariant. In the standard manner this defines a smooth metric on each fiber $F_m \subset P$ which is right invariant under $O(n, \mathbf{R})$ and in any admissable chart for P, under which we identify $F_m \cong SL'(n, \mathbf{R})$, we have that the metric is left invariant under $SL'(n, \mathbf{R})$. In other words, we obtain a smooth metric τ on Vert(P), the vertical subbundle of the tangent bundle to P, which is invariant under the right action of $O(n, \mathbf{R})$ on P. Now let ξ be a smooth metric on M whose volume density is ω. This defines a reduction of P to the subgroup $O(n, \mathbf{R}) \subset SL'(n, \mathbf{R})$, or equivalently a smooth section $\varphi : M \to P/O(n, \mathbf{R})$ of the map $P/O(n, \mathbf{R}) \to M$. Let $q : P \to P/O(n, \mathbf{R})$ be the natural map. The space $q^{-1}(\varphi(\mathrm{M})) \subset \mathrm{P}$ is $\mathrm{O}(n, \mathbf{R})$-invariant, and in fact consists of exactly one $\mathrm{O}(n, \mathbf{R})$-orbit in each fiber over M. The metric τ on Vert(P) is $\mathrm{O}(n, \mathbf{R})$ invariant on $q^{-1}(\varphi(\mathrm{M}))$ and since $\mathrm{SL}'(n, \mathbf{R})$ acts freely on P, $\tau \mid q^{-1}(\varphi(\mathrm{M}))$ extends in a unique manner to a metric on Vert(P) which is invariant under the right action of $\mathrm{SL}'(n, \mathbf{R})$ on P. (We note that this metric on Vert(P) is not equal to τ on all P.) Furthermore, the metric ξ on M defines in a canonical way a connection on P, i.e. a $\mathrm{SL}'(n, \mathbf{R})$-invariant subbundle $\mathrm{Hor(P)} \subset \mathrm{T(P)}$ complementary to Vert(P). The metric ξ lifts in a canonical way to a $\mathrm{SL}'(n, \mathbf{R})$ invariant metric on Hor(P). Putting together these metrics on Vert(P) and Hor(P) we have, given a smooth metric ξ on M, a canonically defined smooth metric ξ^* on P. The map $\xi \to \xi^*$ has the following readily verified properties.

Proposition **5.2.** — i) *For any* $r \geqslant 1$, *the map* $\xi \to \xi^*$ *is continuous where metrics on* M *have the* C^r*-topology and metrics on* P *have the* C^{r-1}*-topology. (On a non-compact manifold the* C^r*-topology is given by* C^r*-convergence on compact sets. The loss of one degree of differentiability stems from the fact that the connection form on* P *defined by* ξ *is expressed locally in terms of the Cristoffel symbols which contain first derivatives of the components of* ξ.)

ii) *The map* $\xi \to \xi^*$ *is natural. I.e., if* $f: \mathrm{M}_1 \to \mathrm{M}_2$ *is a diffeomorphism such that* $f^*(\xi_2) = \xi_1$, *then the induced map* $\widetilde{f}: \mathrm{P}_1 \to \mathrm{P}_2$ *on the (special) frame bundles satisfies* $\widetilde{f}^*(\xi_2^*) = \xi_1^*$.

iii) ξ^* *is* $\mathrm{SL}'(n, \mathbf{R})$*-invariant.*

We also remark that the volume density on P induced by ξ^* is independent of ξ as long as the volume density of ξ is ω. When we speak of the measure on P, we shall henceforth mean this measure.

Combining Propositions 5.2 and 3.8 we obtain:

Proposition **5.3.** — *Let* M *be compact. Then the map* $\xi \to \xi_k^*$ *from metrics on* TM *to metrics on* $\mathrm{J}^k(\mathrm{P}; \mathbf{C})$ *satisfies*:

i) *If* $f: \mathrm{M} \to \mathrm{M}'$ *is a diffeomorphism such that* $f^*(\xi') = \xi$, *then*

$$\mathrm{J}^k(\widetilde{f})^*((\xi')_k^*) = \xi_k^*.$$

ii) *If* $\xi^j \to \xi$ *is a convergent sequence of smooth metrics on* M *with the* C^{k+1}*-topology, then* $(\xi^j)_k^* \to \xi_k^*$ *uniformly on* P *in the sense that* $\mathrm{M}((\xi^j)_k^*/\xi_k^*)$ *and* $\mathrm{M}(\xi_k^*/(\xi^j)_k^*)$ *converge to* 1 *uniformly on* P.

Proof. — (i) and the fact that $\mathrm{M}((\xi^j)_k^*/\xi_k^*)$ and $\mathrm{M}(\xi_k^*/(\xi^j)_k^*)$ converge to 1 uniformly on compact subsets of P follow directly from corresponding assertions in Propositions 5.2 and 3.8. It remains only to see that this convergence is uniform on P. By construction, the metrics $(\xi^j)^*$ and ξ^* on the bundle $\mathrm{TP} \to \mathrm{P}$ are invariant under the action of $\mathrm{SL}'(n, \mathbf{R})$ on P. Hence, by Proposition 3.8 (i), the metrics $(\xi^j)_k^*$ and ξ_k^* are also invariant under $\mathrm{SL}'(n, \mathbf{R})$, and therefore the functions $\mathrm{M}((\xi^j)_k^*/\xi_k^*)$ and $\mathrm{M}(\xi_k^*/(\xi^j)_k^*)$ are $\mathrm{SL}'(n, \mathbf{R})$ invariant as well. Since M is compact, there is a compact subset of P whose saturation under $SL'(n, \mathbf{R})$ is all of P. (For example, just take the compact set $q^{-1}(\varphi(M))$ constructed preceding Proposition 5.2.) Thus, uniform convergence on P follows from uniform convergence on compact subsets.

We wish to use Proposition 5.3 in conjunction with Theorem 3.4 to deduce the existence of measurable Γ-invariant metrics on $J^k(P; \mathbf{C})$. To this end (as well as others) we now examine the ergodic decomposition of the Γ action on the non-compact infinite volume manifold P.

Let us choose a Borel section $\varphi : M \to P$ of the natural projection $P \to M$. As in the beginning of section 3, this defines a Borel trivialization of P via the map $\Phi_0 : M \times SL'(n, \mathbf{R}) \to P$, $\Phi_0(m, g) = \varphi(m)g$. Under this trivialization the action of Γ on $M \times SL'(n, \mathbf{R})$ is given by $\gamma(m, g) = (\gamma m, \alpha(m, \gamma)^{-1}g)$ where $\alpha : M \times \Gamma \to SL'(n, \mathbf{R})$ is a cocycle if the Γ-action is written on the right. Applying Theorem 3.4 to the Γ-action on the bundle $TM \to M$, we deduce that we can find a C^0-neighborhood $\mathcal{O}$ of ξ (and hence a C^k neighborhood, $k \geq 1$) such that for any ω preserving Γ-action with ξ $(\mathcal{O}, \Gamma_0)$-invariant there exists a measurable Γ-invariant metric. By 3.1 and 2.5, this implies that $\alpha \sim \beta$ where β is a cocycle taking values in $O(n, \mathbf{R})$. Let $K \subset O(n, \mathbf{R})$ be the compact subgroup which is the algebraic hull of β (Proposition 2.1) and $\delta \sim \beta$ with $\delta(M \times \Gamma) \subset K$. (Recall that compact real matrix groups are real points of algebraic $\mathbf{R}$-groups.) From the theory of cocycles into compact groups developed in [21], it follows that if the Γ-action on M is ergodic, then the action of Γ on $M \times K$ given by $\gamma.(m, k) = (\gamma m, \delta(m, \gamma)^{-1}k)$ is ergodic. Furthermore the ergodic components of the action of Γ on $M \times SL'(n, \mathbf{R})$ defined by the cocycle δ will be exactly the sets of the form $M \times Ka$, where $a \in SL'(n, \mathbf{R})$. (In other words, the ergodic components are explicitly in bijective correspondence with the coset space $K \backslash SL'(n, \mathbf{R})$. Since $\alpha \sim \delta$, there is a measurable map $\lambda : M \to SL'(n, \mathbf{R})$ implementing the equivalence, and hence the ergodic components of the Γ action defined by the cocycle α will be sets of the form $\{(m, \lambda(m)Ka) \in M \times SL'(n, \mathbf{R})\}$ for some $a \in SL'(n, \mathbf{R})$. Transferring this back to P via Φ_0, and letting $\psi(m) = \varphi(m).\lambda(m)$, we deduce the following.

Lemma **5.4**. — *There is a measurable section* ψ *of the map* $P \to M$ *and a compact subgroup* $K \subset O(n, \mathbf{R})$ *such that the ergodic components of the action of* Γ *on* P *are exactly the sets of the form* $E = \bigcup_{m \in M} \psi(m)Ka$ *where* $a \in SL'(n, \mathbf{R})$. *Furthermore, the corresponding measure on* E *(deoomposing the measure on* P*) is* $\int_M^{\oplus} \mathrm{Haar}(\psi(m)Ka)\,dm$ *where* $\mathrm{Haar}(\psi(m)Ka)$ *is the measure defined on the submanifold* $\psi(m)Ka \subset P$ *as the image of the Haar measure of* K *under the map* $k \to \psi(m)ka$. *In particular, each ergodic component has finite measure.*

Combining 5.4, 3.4, and 5.3 we deduce:

Corollary **5.5** *(With the hypotheses of Theorem* 5.1*).* — *Let* ξ *be a smooth metric on* M *with volume density* ω. *Then for any* $k > 1$ *there is a* C^{k+1}*-neighborhood* $\mathcal{O}$ *of* ξ *such that any* *smooth ω-preserving Γ-action on* M *with* ξ $(\mathcal{O}, \Gamma_0)$*-invariant leaves invariant a measurable Riemannian metric on* $J^k(P; \mathbf{C})$.

We shall now modify the measurable metric on $J^k(P; \mathbf{C})$ given by this corollary so that it satisfies certain integrability conditions.

Lemma **5.6**. — *Let* M, Γ *as in* 5.1, ξ *a smooth metric on* M *with volume density* ω. *Then for any* $k > 1$, *there is a* C^{k+1} *neighborhood* $\mathcal{O}$ *of* ξ *such that any smooth ω-preserving ergodic action of* Γ *on* M *with* ξ $(\mathcal{O}, \Gamma_0)$*-invariant leaves invariant a measurable Riemannian metric* η *on the jet bundle* $J^k(P; \mathbf{C})$ *such that*:

1) $M(\eta/\xi_k^*)$, $M(\xi_k^*/\eta) \in L^2_{loc}(P)$ *(i.e. they are square integrable on any compact subset of* P*)*;
2) *for any compact* $A \subset P$, $\int_A M(\eta/\xi_k^*)^2 \leqslant c(A)$, *where* $c(A)$ *is a constant depending only on* A *(and on the choice of Haar measure on* $SL'(n, \mathbf{R})$*)*;
3) *on the naturally split trivial subbundle* $J^0(P; \mathbf{C}) \subset J^k(P; \mathbf{C})$, *the metric* η *on each fiber* $\mathbf{C}$ *agrees with the usual metric (i.e.* $z, w) = z\overline{w}$*). (We note that this is also true of the metric* ξ_k^* *on* $J^0(P; \mathbf{C})$ *by construction.)*

Proof. — From Proposition 5.3 and Corollary 5.5 we deduce that for any $\varepsilon > 0$, there is a C^{k+1} neighborhood $\mathcal{O}$ of ξ such that for any ω-preserving smooth ergodic Γ-action on M for which ξ $(\mathcal{O}, \Gamma_0)$-invariant we have:

i) the metric ξ_k^* on $J^k(P; \mathbf{C})$ is (ε, Γ_0)-invariant; and
ii) there exists a measurable Γ-invariant metric η_0 on $J^k(P; \mathbf{C})$.

Since $J^0(P; \mathbf{C}) \subset J^k(P; \mathbf{C})$ is a Γ-invariant subbundle with a natural Γ-invariant complement, by taking the standard metric on $J^0(P; \mathbf{C})$ and η_0 on the complement one can assume that conclusion (3) holds for η_0. Since η_0 is Γ-invariant, $\eta_0 \mid E$ will be Γ-invariant for almost every ergodic component E of Γ acting on P. Fix one such ergodic component E. By Lemma 5.4 we can apply Corollary 4.2 to E, and hence if ε is sufficiently small we deduce that $M((\eta_0 \mid E)/(\xi_k^* \mid E))$, $M((\xi_k^* \mid E)/(\eta_0 \mid E)) \in L^2(E)$. Let us write $\xi_k^* = i \oplus \widetilde{\xi}_k^*$ where i is the standard metric on $J^0(P; \mathbf{C})$, $\widetilde{\xi}_k^*$ is the metric on the complement, and similarly write $\eta_0 = i \oplus \widetilde{\eta}_0$. We note that if V, W are finite dimensional vector spaces, ξ an inner product on V and η_1, η_2 inner products on W, then on $V \oplus W$ we have

$$M(\xi \oplus \eta_1/\xi \oplus \eta_2) \leqslant \max(1, M(\eta_1/\eta_2)) \leqslant 1 + M(\eta_1/\eta_2).$$

Hence, for any $t > 0$ we have

$$\begin{aligned} M(i \oplus t\widetilde{\eta}_0/i \oplus \widetilde{\xi}_k^*) &\leqslant 1 + tM(\widetilde{\eta}_0/\widetilde{\xi}_k^*) \\ &\leqslant 1 + tM(\eta_0/\xi_k^*), \end{aligned}$$

and we have a similar expression for $M(i \oplus \widetilde{\xi}_k^*/i \oplus t\widetilde{\eta}_0)$. Therefore, by choosing t sufficiently small and replacing η_0 by $i \oplus t\widetilde{\eta}_0$, we can assume that η_0 satisfies:

a) $\int_E M(\eta_0/\xi_k^*)^2 \leqslant 2$;

b) $M((\xi_k^* \mid E)/(\eta_0 \mid E)) \in L^2(E)$.

Let K be as in lemma 5.4 and choose a measurable map $\lambda : K\backslash SL'(n, \mathbf{R}) \to SL'(n, \mathbf{R})$ such that $\lambda(Kg) \in Kg$. As in 5.4, we write $E = \bigcup_m \psi(m)Ka$ for some $a \in SL'(n, \mathbf{R})$. If E' is any other ergodic component of the Γ action on P, then we can write $E' = \bigcup_m \psi(m)Kb$ for some $b \in SL'(n, \mathbf{R})$. Thus $E = E'b^{-1}a$. We define the metric η on $J^k(P; \mathbf{C}) \mid E'$ to be $(\lambda(Kb)^{-1}a)^*\eta_0$. Since λ is measurable, the resulting Riemannian metric η on the bundle $J^k(P; \mathbf{C}) \to P$ will be measurable. Furthermore, since the Γ action on P (and hence on $J^k(P; \mathbf{C})$) commutes with the $SL'(n, \mathbf{R})$ action and $\eta_0 \mid E$ is Γ-invariant, the metric η will be Γ-invariant.

We now claim that $M(\eta/\xi_k^*) \in L^2_{loc}(P)$, and in fact that the more exact assertion (2) in the statement of the lemma is valid. Let $A \subset P$ be compact and consider $\Phi^{-1}(A)$ where $\Phi : M \times SL'(n, \mathbf{R}) \to P$ is the measurable trivialization defined by ψ, i.e. $\Phi(m, g) = \psi(m)g$. Since Φ is measure preserving, it suffices to see

$$\int_{\Phi^{-1}(A)} (M(\eta/\xi_k^*) \circ \Phi)^2 \leqslant c(A).$$

Via the measurable section λ we can write $SL'(n, \mathbf{R})$ as a measurable product, $K \times K\backslash SL'(n, \mathbf{R}) \cong SL'(n, \mathbf{R})$, namely by the map $(k, Kg) \to k\lambda(Kg)$. We may view Φ as a map defined on $M \times K \times K\backslash SL'(n, \mathbf{R})$. If we let ν be Haar measure on $SL'(n, \mathbf{R})$ we can write $\nu = \nu_1 \times \nu_2$ under above product decomposition where ν_1 is Haar measure on K and ν_2 is an invariant measure on $K\backslash SL'(n, \mathbf{R})$.

Since ξ_k^* is right invariant under $SL'(n, \mathbf{R})$, from the definition of η it follows that $\Psi = M(\eta/\xi_k^*) \circ \Phi$ satisfies the following condition: $\Psi(m, k, x)$ is independent of $x \in K\backslash SL'(n, \mathbf{R})$. When convenient, we shall write this as $\Psi(m, k)$. The assertion *(a)* above that $\int_E M(\eta_0/\xi_k^*)^2 \leqslant 2$ implies that for each $x \in K\backslash SL'(n, \mathbf{R})$, $\int_{M \times K} \Psi(m, k, x)^2 dm d\nu_1(k) \leqslant 2$. Since A is compact, the sets $A_m \in SL'(n, \mathbf{R})$ given by $A_m = \{g \in SL'(n, \mathbf{R}) \mid \Phi(m, g) \in A\}$ for $m \in M$ have a uniformly bounded Haar measure, say $\nu(A_m) \leqslant c(A)$ for all $m \in M$. (We remark that by the bi-invariance of Haar measure, $c(A)$ depends only on A and the choice of Haar measure, not on the choice of the section ψ.) For $m \in M$, $x \in K\backslash SL'(n, \mathbf{R})$, let $A_{m,x} = \{k \in K \mid \Phi(m, k, x) \in A\}$. Then

$$\int_{\Phi^{-1}(A)} \Psi^2 = \int_{M \times K\backslash SL'} \left(\int_{A_{m,x}} \Psi(m, k, x)^2 d\nu_1(k)\right) dm d\nu_2(x)$$
$$\leqslant \int_{M \times K\backslash SL'} \nu_1(A_{m,x}) \left(\int_K \Psi(m, k, x)^2 dk\right) dm dx.$$

Integrating over x and recalling that $\Psi(m, k, x)$ is independent of x, we obtain

$$\int_{\Phi^{-1}(A)} \Psi^2 \leqslant \int_M \nu(A_m) \int_K \Psi(m, k)^2 dk dm \leqslant c(A) \int_{M \times K} \Psi^2 \leqslant 2c(A).$$

This verifies assertion (2). The proof that $M(\xi_k^*/\eta) \in L^2_{loc}(P)$ is similar, and this completes the proof of Lemma 5.6.

We now consider Sobolev spaces defined using the metric η constructed in Lemma 5.6.

Quite generally, suppose $E \to N$ is a smooth vector bundle over a manifold N and ω is a smooth volume density on N. Suppose σ is a measurable Riemannian metric on E. Then for any p, $1 \leqslant p \leqslant \infty$, we define $L^p(E)_\sigma$ to be the set of measurable sections f of E (with the usual identifications modulo null sets) such that $\|f\|_{p,\sigma} = \left(\int_N \|f(s)\|^p_{\sigma(s)} ds\right)^{1/p} < \infty$. $L^\infty(E)_\sigma$ and $\|f\|_{\infty,\sigma}$ are defined similarly. Then $L^p(E)_\sigma$ is a Banach space, and for $p = 2$ is a Hilbert space. We also recall that we denote by $C^\infty(N; E)$, $C^\infty_c(N; E)$ the space of smooth (respectively smooth with compact support) sections of E. In general, of course, $C^\infty(N; E) \cap L^p(E)_\sigma$ may be trivial if the measurable σ is very badly behaved locally. However, if ξ is a smooth metric on E and $M(\sigma/\xi) \in L^p_{loc}(N)$, then for $f \in C^\infty_c(N; E)$,

$$\|f(s)\|^p_{\sigma(s)} \leqslant M(\sigma/\xi)(s)^p \|f(s)\|^p_\xi \leqslant M(\sigma/\xi)(s)^p \|f\|^p_{\infty,\xi},$$

and since $\operatorname{supp}(f)$ is compact $f \in L^p(E)_\sigma$.

Suppose now that η is a measurable Riemannian metric on $J^k(N; E)$, σ a smooth metric on $J^k(N; E)$ with $M(\eta/\sigma) \in L^p_{loc}(N)$. If $f \in C^\infty(N; E)$, we have the k-jet extension $j^k(f) \in C^\infty(N; J^k(N; E))$, and of course if $f \in C^\infty_c(N; E)$, then $j^k(f)$ also has compact support. Let us set

$$C^\infty(N; E)_{p,k,\eta} = \{f \in C^\infty(N; E) \mid j^k(f) \in L^p(J^k(N; E))_\eta\}.$$

By the remarks of the preceding paragraph, we have $C^\infty_c(N; E) \subset C^\infty(N; E)_{p,k,\eta}$. We have an injective linear map

$$C^\infty(N; E)_{p,k,\eta} \to L^p(J^k(N; E))_\eta, \quad f \to j^k(f),$$

and the completion of $C^\infty(N; E)_{p,k,\eta}$ with the induced norm will be denoted by $L^{p,k}_\eta(N; E)$, the (p, k) Sobolev space with respect to η. The linear embedding j^k extends to a map $L^{p,k}_\eta(N; E) \to L^p(J^k(N; E))_\eta$ which we still denote by j^k.

Returning now to the situation of Lemma 5.6, we observe that because the measurable Riemannian metric η on $J^k(P; \mathbf{C})$ is Γ-invariant the Γ-action on sections of $J^k(P; \mathbf{C})$ defines a unitary representation of Γ on $L^2(J^k(P; \mathbf{C}))_\eta$ and hence on $L^{2,k}_\eta(P; \mathbf{C})$. (This of course is not necessarily the case for $L^{2,k}_{\xi_k^*}(P; \mathbf{C})$.) We also remark that condition (2) of Lemma 5.6 implies that for $f \in C^\infty_c(P)$ with $\operatorname{supp}(f) \subset A$, we have

$$(*) \qquad \|j^k(f)\|_{2,\eta} \leqslant c(A) \|j^k(f)\|_{\infty,\xi_k^*}$$

where $c(A)$ depends only on A and not on η.

We construct an explicit element of $L^{2,k}_\eta(P; \mathbf{C})$. Recall that a smooth metric ξ on M defines in a natural manner a smooth section φ of the natural projection $P/O(n, \mathbf{R}) \to M$, as in the discussion preceding Proposition 5.2. As in that discussion we have the compact submanifold $q^{-1}(\varphi(M)) \subset P$ whose saturation under the right

$SL'(n, \mathbf{R})$ action is all of P. Fix a non-zero function $F \in C_c^\infty(O(n, \mathbf{R})\backslash SL'(n, \mathbf{R}))$ such that $\int |F|^2 = 1$. Then for $p \in P$ define $F_\xi(p) = F(g)$ where $g \in SL'(n, \mathbf{R})$ satisfies $pg^{-1} \in q^{-1}(\varphi(M))$. If $pg_1^{-1}, pg_2^{-1} \in q^{-1}(\varphi(M))$, then there exists $a \in O(n, \mathbf{R})$ such that $pg_1^{-1} = pg_2^{-1}a$, so that by freeness of $SL'(n, \mathbf{R})$ on P, $g_1 \in O(n, \mathbf{R})g_2$. Thus F_ξ is independent of the choice of g. It is easy to see that F_ξ is smooth and in fact $F_\xi \in C_c^\infty(P)$. The following properties of the assignment $\xi \to F_\xi$ are readily verified.

Proposition 5.7

1) *The map* $\xi \to F_\xi$ *is continuous where metrics on* M *have the* C^k *topology and* $C_c^\infty(P)$ *has the topology of uniform* C^k*-convergence on compact sets.*

2) $\int_P |F_\xi|^2 = 1.$

3) *Fix* ξ *and a compact set* $A \subset P$ *which contains an open neighborhood of* $\operatorname{supp}(F_\xi)$. *Then there is a* C^0 *neighborhood* $\mathcal{O}$ *of* ξ *such that if* $\sigma \in \mathcal{O}$, *then* $\operatorname{supp}(F_\sigma) \subset A$.

4) *The assignment is natural. I.e. if* $h : M \to M'$ *is a diffeomorphism and* ξ *is a metric on* M, *then* $F_\xi \circ \tilde{h} = F_{h^*\xi}$ *where* $\tilde{h} : P' \to P$ *is the induced map on the principal bundles.*

Now fix $\xi, \varepsilon > 0$, and a compact set A as in condition (3) of Proposition 5.7. By 5.6 and 5.7, we can choose a C^{k+1}-neighborhood $\mathcal{O}$ of ξ such that the conclusions of 5.6 hold, (3) of 5.7 holds, and using (1) of 5.7, such that $\sigma \in \mathcal{O}$ implies

$$||j^k(F_\sigma) - j^k(F_\xi)||_{\infty, \xi_k^*} < \frac{\varepsilon}{c(A)},$$

where $c(A)$ is as in 5.6. By equation (*) above, this last assertion implies

$$(**) \qquad ||j^k(F_\sigma) - j^k(F_\xi)||_{2,\eta} < \varepsilon.$$

We also remark that $||j^k(F_\xi)||_{2,\eta} \geqslant 1$ by (2) of Proposition 5.7 and (3) of Lemma 5.6. Hence, if Γ acts with ξ $(\mathcal{O}, \Gamma_0)$ invariant, it follows from (**) and (4) of Proposition 5.7 (by taking $\sigma = \gamma^*\xi$ in 5.7 (4)) that we have

$$||\gamma.j^k(F_\xi) - j^k(F_\xi)||_{2,\eta} < \varepsilon \quad \text{for all } \gamma \in \Gamma_0.$$

In other words, the vector $j^k(F_\xi) \in L_\eta^{2,k}(P)$ is (ε, Γ_0)-invariant under the unitary representation of Γ on $L_\eta^{2,k}(P)$ and has norm at least one. Thus, if ε is sufficiently small, Kazhdan's property implies there is a non-zero Γ-invariant function $f \in L_\eta^{2,k}(P)$.

Let $B \subset P$ be a relatively compact open set. By Lemma 5.6 (1), $\int_B M(\xi_k^*/\eta)^2 < \infty$. Since $f \in L_\eta^{2,k}(P)$ we can find a sequence $f_j \in C^\infty(P; \mathbf{C})_{2,k,\eta}$ such that $f_j \to f$ in $L^2(P)$ and $\{j^k(f_i)\}$ is a Cauchy sequence in $L^2(J^k(P; \mathbf{C}))_\eta$. It follows that by restricting to B, we have $f_i \to f$ in $L^2(B)$ and $\{j^k(f_i)\}$ is Cauchy in $L^2(J^k(B; \mathbf{C}))_\eta$. Therefore

$$\int_B ||j^k(f_i)(p) - j^k(f_r)(p)||_{\xi_k^*} dp \leqslant \int_B M(\xi_k^*/\eta)\, ||j^k(f_i)(p) - j^k(f_r)(p)||_\eta dp$$
$$\leqslant \left(\int_B M(\xi_k^*/\eta)^2\right)^{1/2} ||j^k(f_i) - j^k(f_r)||_{2,\eta}.$$

From this we deduce that $f_i \to f$ in $L^1(B)$ and $\{j^k(f_i)\}$ is Cauchy in $L^1(J^k(B; \mathbf{C}))_{\xi_k^*}$. Since ξ_k^* is a smooth metric, it follows that in local coordinate neighborhoods f has weak derivatives up to order k and that these derivatives are in L^1. Hence, the classical Sobolev embedding theorem implies that if $k > \dim P$, $f \in C^0(P)$. Thus, we may summarize our results so far in this section by:

Lemma 5.8. — *Let* $n = \dim M$ *and* $k = n^2 + n$. *Then for any smooth metric* ξ *on* M *there is a* C^{k+1} *neighborhood* $\mathcal{O}$ *of* ξ *such that for any volume preserving smooth ergodic action of* Γ *on* M *with* ξ $(\mathcal{O}, \Gamma_0)$*-invariant there is a non-zero* Γ*-invariant function* $f \in L^2(P) \cap C^0(P)$.

We now proceed to show how the existence of such a function f implies the existence of a Γ-invariant C^0-Riemannian metric on M. For convenience of notation we shall denote $SL'(n, \mathbf{R})$ by H for the remainder of this section.

Let $\Phi : M \times H \to P$ be a measurable trivialization of P. With this identification we identify f as a function $f \in L^2(M \times H)$ and, as usual, the Γ action on P with the Γ-action on $M \times H$ given by $\gamma(m, h) = (\gamma m, \alpha(m, \gamma)^{-1}h)$ where $\alpha : M \times \Gamma \to H$ is a cocycle. For each $m \in M$, let $f_m \in L^2(H)$ be $f_m(h) = f(m, h)$. Let π be the left regular representation of H on $L^2(H)$. Then Γ invariance of f is the assertion that $f_{\gamma m} = \pi(\alpha(m, \gamma))f_m$. The orbits of H in $L^2(H)$ are locally closed in the weak topology [22] and hence the quotient space $L^2(H)/H$ is a countably separated Borel space (see [29], e.g.). Letting $\tilde{f}_m$ be the H-orbit of f_m, we deduce that $\tilde{f}_{\gamma m} = \tilde{f}_m$ in $L^2(H)/H$. Since $L^2(H)/H$ is countably separated and Γ is ergodic on M, $\tilde{f}_m$ is essentially constant. I.e. there is $\lambda \in L^2(H)$ such that $f_m \in \pi(H)\lambda$ for almost all $m \in M$. (Cf. [27], [28, Theorem 2.10].) In particular, $\lambda \in L^2(H) \cap C^0(H)$ and $\lambda \neq 0$.

We can rephrase this in terms of the original bundle $P \to M$ as follows. Let $\mathcal{H} \to M$ be the associated bundle to the left H-action on $L^2(H)$, so that the fiber of $\mathcal{H}$ is $L^2(H)$. We have $\pi(H)\lambda \subset L^2(H)$ is a G-invariant subset and hence we have an associated bundle $\mathcal{H}_\lambda \to M$ with fiber $\pi(H)\lambda$ and a natural inclusion $\mathcal{H}_\lambda \hookrightarrow \mathcal{H}$ (both over M). We remark that the stabilizer of λ in H, say H_λ, is compact (this is true for the regular representation of any locally compact group) and that $\pi(H)\lambda$ is homeomorphic as an H-space to H/H_λ [29]. From the preceding paragraph we see that f defines a measurable section $m \to f_m$ of the associated bundle $\mathcal{H} \to M$ which satisfies the condition that $f_m \in \mathcal{H}_\lambda$ for almost all $m \in M$.

Lemma 5.9. — *Let* $W = \{m \in M \mid f_m \in \mathcal{H}_\lambda\}$. *Then*

a) *if* $m \notin W$, *then* $f_m = 0$;

b) *the map* $m \to f_m$ *is continuous on* W;

c) W *is open and* Γ*-invariant.*

Proof. — Fix $m \in M$ and an open neighborhood U of m over which P is trivial. Then over U we can consider f as a function in $L^2(U \times H) \cap C^0(U \times H)$. To verify (a), suppose $m \notin W$. Since W is conull we can choose $u_n \in W$, $u_n \to m$. Then choose $h_n \in H$ such that $f_{u_n} = \pi(h_n)\lambda$. We consider two cases: (1) $h_n \to \infty$; and (2) h_n has a convergent subsequence. In case (1), given any compact subset $A \subset H$, and $\varepsilon, \delta > 0$, the fact that $\lambda \in L^2(H)$ implies that for n sufficient large, $\nu(\{a \in A \mid (\pi(h_n)\lambda)(a) > \varepsilon\}) < \delta$, where $\nu =$ Haar measure. Since $\pi(h_n)\lambda = f_{u_n}$ and $f_{u_n} \to f_m$ uniformly on compact sets, it follows that $f_m = 0$ on A, so $f_m = 0$. In case (2), we can assume $h_n \to h \in H$. Then $f_{u_n} = \pi(h_n)\lambda \to \pi(h)\lambda$ uniformly on compact subsets of H, and since $f_{u_n} \to f_m$ uniformly on compact sets, we have $f_m = \pi(h)\lambda$. Thus $m \in W$. This verifies (a).

The proof of (b) is similar. To see continuity at $m \in W$, it suffices to assume $u_n \in W$, $u_n \to m$, and to show the existence of a subsequence u_{n_j} such that $f_{u_{n_j}} \to f_m$ in $L^2(H)$. Choose $h_n \in H$ such that $f_{u_n} = \pi(h_n)\lambda$. If $h_n \to \infty$, then as above we deduce that $f_m = 0$, contradicting the assumption that $m \in W$. If $h_{n_j} \to h \in H$, we deduce as above that $f_m = \pi(h)\lambda$. But by continuity of the left regular representation, $\pi(h_{n_j})\lambda \to \pi(h)\lambda$ in $L^2(H)$, so $f_{u_{n_j}} \to f_m$ in $L^2(H)$.

Finally, to see (c), we have that $W = \{m \mid f_m \text{ is not identically } 0\}$ and from this it is clear that W is open. Since f is Γ invariant, W is as well.

Lemma **5.10**. — *With* W *as in* 5.9, *there is a* Γ*-invariant* C^0*-Riemannian metric on* W *(with volume density* ω*).*

Proof. — By Lemma 5.9 and the discussion preceding it, there is a Γ-invariant C^0-section of the bundle $\mathscr{H}_\lambda \to M$ defined on the open set W. However, by our remarks preceding Lemma 5.9, $\mathscr{H}_\lambda$ is C^0-isomorphic as a bundle over M to the associated bundle of the left action of H on H/H_λ. Since H_λ is compact, a conjugate of H_λ is contained in $O(n, \mathbf{R})$. Thus we have a Γ-invariant C^0 reduction of the bundle of frames (of volume 1 with respect to ω) to the orthogonal group, and this implies the existence of a C^0 Γ-invariant Riemannian metric on W.

We now complete the proof of Theorem 5.1. Choose a connected component $W_0 \subset W$. The Γ-invariant C^0 metric on W_0, say η, defines a topological distance function on W_0 in the usual way: $d(x,y) = \inf\left\{\int \|\varphi'(t)\|_\eta \mid \varphi \text{ is a continuous piecewise differentiable path from } x \text{ to } y\right\}$ and the topology defined by this distance function agrees with the original topology on W_0. Γ permutes the set of connected components of W and since the measure on W is finite and invariant, ΓW_0 contains only finitely many connected components. Thus, the subgroup $\Gamma_0 \subset \Gamma$ leaving W_0 invariant is of finite index. We have $\Gamma_0 \subset \mathrm{Iso}(W_0, d)$, the latter being the group of isometries of W_0 with respect to d. Since W_0 is a connected manifold, $\mathrm{Iso}(W_0, d)$ is locally compact [8] and the stabilizers of points in W_0 are compact. Let $\overline{\Gamma}_0$ be the closure of Γ_0 in $\mathrm{Iso}(W_0, d)$. Then $\overline{\Gamma}_0$ is a locally compact isometry group with compact stabilizers, and since Γ_0 leaves the finite measure $\omega \mid W_0$ invariant, so does $\overline{\Gamma}_0$. Since Γ is ergodic on W, Γ_0 is ergodic on W_0, and hence we can find $x \in W_0$ such that $\Gamma_0 x$ is dense in W_0. It follows from [5, proof of IV, 2.2] that $\overline{\Gamma}_0$ is transitive on W_0. The finiteness of the measure on W_0 and compactness of the stabilizers imply that the Haar measure of $\overline{\Gamma}_0$ is finite and hence that $\overline{\Gamma}_0$ is compact. This implies that W_0 is compact. Ergodicity of Γ on W implies $\Gamma W_0 = W$, so that W is also compact. Since W is conull in M, it is dense in M. Thus, $W = M$ and the proof of Theorem 5.1 is complete.

6. Perturbations and Near Isometries

Theorem 5.1 immediately implies the following perturbation theorem stated in the introduction.

Theorem **6.1.** — *Let* G *be connected semisimple Lie group with finite center such that every simple factor of* G *has* **R**-rank $\geqslant 2$. *Let* $\Gamma \subset G$ *be a lattice. Let* M *be a compact Riemannian manifold,* $\dim M = n$. *Set* $r = n^2 + n + 1$. *Assume* Γ *acts by isometries of* M. *Let* $\Gamma_0 \subset \Gamma$ *be a finite generating set. Then any volume preserving action of* Γ *on* M *which*

i) *for elements of* Γ_0 *is a sufficiently small* C^r *perturbation of the original action*; *and*
ii) *is ergodic*;

actually leaves a C^0*-Riemannian metric invariant. In particular there is a* Γ*-invariant topological distance function and the action is topologically conjugate to an action of* Γ *on a homogeneous space of a compact Lie group* K *defined via a dense range homomorphism of* Γ *into* K.

The only part of this result we have not proven is that the compact group K is actually Lie. However, this follows from [10, p. 244].

Theorem 5.1 also implies the following result stated in the introduction.

Theorem **6.2.** — *Let* G, Γ *be as in* 6.1. *Let* M *be a compact manifold,* $\dim M = n$ $(n > 0)$. *Assume* $n(n+1) < 2n(G)$ *where* $n(G)$ *is the minimal dimension of a simple factor of* **G**. *Set* $r = n^2 + n + 1$. *Let* $\Gamma_0 \subset \Gamma$ *be a finite generating set. Then for any smooth Riemannian metric* ξ *on* M *there is a* C^r *neighborhood* $\mathcal{O}$ *of* ξ *such that there are no volume preserving ergodic actions of* Γ *on* M *for which* ξ *is* $(\mathcal{O}, \Gamma_0)$*-invariant.*

Proof. — By Theorem 5.1 we can choose $\mathcal{O}$ such that for any ergodic volume preserving action with ξ $(\mathcal{O}, \Gamma_0)$-invariant there is a dense range homomorphism of Γ into a compact Lie (by [10, p. 244]) group of isometries K of M. By results of Margulis [9] concerning homomorphisms of Γ into Lie groups (see also the work of Raghunathan [16]), it follows that $\dim K \geq n(G)$. On the other hand, it is well known that $\dim K \leqslant \frac{n(n+1)}{2}$ [10, p. 246]. Thus $2n(G) \leq n(n+1)$, which contradicts our hypotheses.

We now turn to the p-adic case.

Theorem **6.3**. — *Let k be a totally disconnected local field of characteristic* 0. *Let* G *be a connected semisimple algebraic k-group such that the k-rank of every k-simple factor of* G *is* $\geqslant 2$. *Suppose* $\Gamma \subset G_k$ *is a lattice. Let* M *be a compact n-manifold* $(n > 0)$, *and let* $r = n^2 + n + 1$. *Let* $\Gamma_0 \subset \Gamma$ *be a finite generating set. Then for any smooth Riemannian metric* ξ *on* M *there is a* C^r *neighborhood* $\mathcal{O}$ *of* ξ *such that any volume preserving ergodic action of* Γ *on* M *for which* ξ *is* $(\mathcal{O}, \Gamma_0)$*-invariant leaves a* C^0*-Riemannian metric invariant.*

Proof. — The proof is basically the same as that of Theorem 6.2 (and Theorem 5.1) once we have two properties of Γ:

(i) superrigidity for measurable cocycles defined on ergodic Γ-spaces with finite invariant measure;
(ii) Kazhdan's property.

For (i), see [28]. In this case the superrigidity theorem asserts that for any cocycle $\alpha : S \times \Gamma \to H_{\mathbf{R}}$ where H is a connected semisimple adjoint $\mathbf{R}$-group such that the algebraic hull of α is $H_{\mathbf{R}}$, we have $H_{\mathbf{R}}$ is compact. Property (ii) is well-known [1], [6].

7. Concluding Remarks

Let G be a connected semisimple Lie group with finite center such that every simple factor of G has $\mathbf{R}$-rank $\geqslant 2$. Let $\Gamma \subset G$ be a lattice. We have the following examples of smooth volume preserving actions of Γ on compact manifolds.

Example **7.1**

a) Suppose H is a Lie group, $\Lambda \subset H$ is a cocompact subgroup such that H/Λ has a finite H-invariant volume density, and there is a continuous homomorphism $G \to H$. Then Γ acts on H/Λ.

b) Suppose N is a Lie group and that G acts by unimodular automorphisms of N. Suppose $\Lambda \subset N$ is a cocompact lattice such that $\gamma(\Lambda) = \Lambda$ for all $\gamma \in \Gamma$. Then Γ acts on N/Λ.

c) If K is a compact Lie group acting smoothly on a compact manifold M and there is a dense range homomorphisms $\Gamma \to K$, then Γ acts on M. This includes of course the actions of Γ that factor through a finite quotient of Γ.

We can then formulate the following question.

Problem. — Can every volume preserving Γ-action (or at least every such ergodic action) on a compact manifold be obtained from these examples by elementary constructions (e.g. products, finite covers and quotients)?

In particular, in case the action preserves a stronger structure than a volume density, we put forward the following conjecture.

Conjecture. — Let $H \subset SL'(n, \mathbf{C})$ be an $\mathbf{R}$-group and suppose Γ acts on a compact manifold M so as to preserve an $H_{\mathbf{R}}$-structure on M. If $\mathbf{R}\text{-rank}(H) < \mathbf{R}\text{-rank}(G_i)$ for every simple factor G_i of G, then there is a C^0 Γ-invariant Riemannian metric on M.

In this regard, see Theorem 2.14.

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