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# THE $C^{1+\alpha}$ HYPOTHESIS IN PESIN THEORY

*by* CHARLES C. PUGH*

**1. Introduction.** — The stable manifold theory developed by Pesin and others [1, 4, 7, 8, 9] contains a hypothesis that the given dynamics be of differentiability class $C^{1+\alpha}$ for some $\alpha > 0$. That is, first derivatives must obey $\alpha$-Hölder conditions. Here is an example showing that $C^1$ alone, i.e. $\alpha = 0$, is insufficient. It uses the auxiliary function

$$g(u) = \begin{cases} \dfrac{u}{\log(1/u)} & \text{if } 0 < u < 1 \\ 0 & \text{if } u = 0 \end{cases}$$

which is $C^1$, strictly monotone increasing, has $g'(0) = 0$, and is not $C^{1+\alpha}$ for any $\alpha > 0$. Near 0, $g$ grows faster than $u^{1+\alpha}$ for all $\alpha > 0$. Functions like $g$ have been seen before, for instance in S. Sternberg's example of a non $C^1$-linearizable $C^1$ contraction of $\mathbf{R}$ [10, p. 101].

Suppose $f: M \to M$ is a $C^1$ diffeomorphism of the compact manifold M. Let $p \in M$, $v \in T_p M$, $v \neq 0$, be given. The *Lyapunov exponents* of $v$ are

$$\chi^-(v) = \lim_{-n \to -\infty} \frac{1}{-n} \log | T_p f^{-n}(v) |$$

$$\chi^+(v) = \lim_{n \to \infty} \frac{1}{n} \log | T_p f^n(v) |$$

if the limits exist. For vectors in most tangent spaces, Oseledec [6] proves that these Lyapunov exponents do exist and that there is a kind of uniformity referred to as *regularity* of the orbit $\mathcal{O}(p) = \{f^n p\}_{n \in \mathbf{Z}}$; namely, if $E_p^\lambda$ denotes $\{v : v = 0 \text{ or } \chi^-(v) = \lambda = \chi^+(v)\}$ then $T_p M = \bigoplus_\lambda E_p^\lambda$ and

$$\lim_{|n| \to \infty} \frac{1}{n} \log \left( \frac{\| Tf^n \mid E_p^\lambda \|}{\| (Tf^n \mid E_p^\lambda)^{-1} \|} \right) = 0. \qquad (1)$$

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See also [1, 8]. (A better name for this property might be *subexponential conformality* since $Tf^n \mid E_p^\lambda$ grows like a sequence of dilations, up to a subexponential error.)

*Definition.* — The orbit $\mathcal{O}(p)$ is *asymptotically hyperbolic* if $\mathcal{O}(p)$ is regular and $\chi^{\pm}(v) \neq 0$ for all non-zero $v \in T_p M$.

Assume $\mathcal{O}(p)$ is asymptotically hyperbolic and write

$$E_p^u = \bigoplus_{\lambda > 0} E_p^\lambda, \qquad E_p^s = \bigoplus_{\lambda < 0} E_p^\lambda.$$

Pesin's Stable Manifold Theorem [1, 5, 7, 9] asserts in part that the *asymptotically exponentially stable set of* $p$

$$W^s(p) = \left\{ x \in M : \lim_{n \to \infty} \frac{1}{n} \log d(f^n x, f^n p) < 0 \right\}$$

is an injectively immersed $C^1$ manifold tangent at $p$ to $E_p^s$, provided $f$ is $C^{1+\alpha}$ for some $\alpha > 0$.

*Theorem.* — *There exists a* $C^1$ *diffeomorphism of a* 4*-manifold having an asymptotically hyperbolic orbit* $\mathcal{O}(p)$ *such that* $W^s(p)$ *is not an injectively immersed manifold tangent to* $E_p^s$.

See § 3 for the example cited.

In the proof of Pesin's Theorem, $f$ is lifted from M to TM along $\mathcal{O}(p)$ via the exponential map. The composites

$$f_n = \exp_{f^{n+1}p}^{-1} \circ f \circ \exp_{f^n p} \qquad \begin{array}{ccc} T_{f^n p} M & \xrightarrow{f_n} & T_{f^{n+1}p} M \\ \downarrow \exp & & \downarrow \exp \\ M & \xrightarrow{f} & M \end{array}$$

$$n \in \mathbf{Z}$$

$$\begin{array}{ccccccccccc} \cdots \xrightarrow{f_{-2}} & T_{f^{-1}p} M & \xrightarrow{f_{-1}} & T_p M & \xrightarrow{f_0} & T_{fp} M & \xrightarrow{f_1} & T_{f^2 p} M & \xrightarrow{f_2} \cdots \\ & \downarrow \exp & & \downarrow \exp & & \downarrow \exp & & \downarrow \exp & \\ \cdots \longrightarrow & \bullet & \xrightarrow{f} & \bullet & \xrightarrow{f} & \bullet & \xrightarrow{f} & \bullet & \cdots \\ & f^{-1}p & & p & & fp & & f^2 p & \end{array}$$

locally represent $f$ in exponential coordinates along the orbit $\mathcal{O}(p)$. The crucial consequence of $C^{1+\alpha}$ differentiability of $f$ is that the $f_n$ are $C^{1+\alpha}$-equicontinuous. That is, there are uniform constants K, $\delta > 0$ such that

$$\| (Df_n)_x - (Df_n)_y \| \leq K \, |x - y|^\alpha$$

for all $x, y \in T_{f^n p} M$, with $|x|, |y| \leq \delta$.

In § 2, we give an example of a sequence of maps

$$\mathbf{R}^2 \xrightarrow{f_n} \mathbf{R}^2, \quad f_n(z) = A_n z + G_n(z), \quad n \in \mathbf{Z}$$

such that $A_n$ is a diagonal $2 \times 2$ matrix, $||A_n||$ and $||A_n^{-1}||$ are uniformly bounded,

$$\lim_{n\to\infty} \frac{1}{n} \log |A_n \circ \ldots \circ A_0(v)| = \lambda$$

$$v \in \mathbf{R}^2 - \{0\}$$

$$\lim_{n\to\infty} \frac{1}{-n} \log |A_{-n}^{-1} \circ \ldots \circ A_{-1}^{-1}(v)| = \lambda$$

where $\lambda$ is a negative constant, $G_n$ is $C^1$, $(DG_n)_0 = 0$, and ($C^1$ equicontinuity of $f_n$)

$$||(DG_n)_x - (DG_n)_y|| \to 0 \text{ uniformly as } |x-y| \to 0,$$

but there exist points $z$ arbitrarily near 0 with

$$|f_n \circ \ldots \circ f_0(z)| \to \infty \quad \text{as } n \to \infty.$$

Thus, 0 should be asymptotically stable under $\{f_n\}$ since $\{Tf_n\}$ contracts asymptotically, but, due to lack of smoothness of $\{f_n\}$, it is not. See Theorem 1 in § 2. (Regularity in this context is implied by the fact that $A_n$ is diagonal.)

In § 3 the maps $f_n$ are realized as lifts along an orbit of some $C^1$ diffeomorphism of a compact 4-manifold. See Theorem 2.

*Conjecture.* — If $\mathcal{O}(p)$ is an asymptotically hyperbolic orbit of the $C^1$ diffeomorphism $f: M \to M$, M has dimension two, and $\dim E_p^u = \dim E_p^s = 1$, then Pesin's result holds: $W^s(p)$ is $C^1$ and tangent at $p$ to $E_p^s$. Indeed this might be true whenever $E_p^s$ has dimension one. Regularity is automatic on one-dimensional subspaces.

*Thanks.* — In writing this paper, I benefitted from conversations with M. Herman and A. Fathi at the École Polytechnique in Paris. Comments by the referee were also useful.

**2. Nonlinear shear.** — Let $g : (0, \infty) \to (0, \infty)$ be any smooth function such that

$$(2) \quad \begin{aligned} g(u) &= \frac{u}{\log(1/u)} && \text{if } 0 < u \leq 1/e \\ g'(u) &> 1 && \text{if } u \geq 1/e \\ g'(u) &\text{ is constant} && \text{if } u \geq 1. \end{aligned}$$

Extend $g$ to all of $\mathbf{R}$ by setting $g(-u) = g(u)$ and $g(0) = 0$. Then $g : \mathbf{R} \to (0, \infty)$ is $C^1$ and is $C^\infty$ on $\mathbf{R} - \{0\}$. The graph of $g$ is shown in Figure 1.

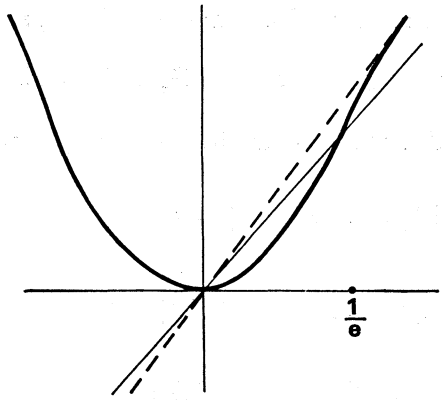


FIG. 1. — The graph of $g$

Choose constants $0 < a < ab < 1 < b$ and call

$$S = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad T = \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix},$$

$$f_S = \begin{pmatrix} a & g \\ 0 & b \end{pmatrix}, \quad f_T = \begin{pmatrix} b & 0 \\ g & a \end{pmatrix}.$$

Thus, $f_S$ and $f_T$ are $C^1$ origin preserving maps $\mathbf{R}^2 \to \mathbf{R}^2$

$$f_S : z \mapsto \begin{pmatrix} ax + g(y) \\ by \end{pmatrix}, \quad f_T : z \mapsto \begin{pmatrix} bx \\ g(x) + ay \end{pmatrix}, \quad z = \begin{pmatrix} x \\ y \end{pmatrix}.$$

The linear parts of $f_S, f_T$ at 0 are S, T and $f_S, f_T$ are easily seen to be invertible on all of $\mathbf{R}^2$.

As $n = 1, 2, 3, \ldots$ choose $A_n$ equal to S or T according to the pattern

$$\begin{matrix} S & T & T & S\,S\,S & T\,T\,T\,T & S\,S\,S\,S\,S & \ldots \\ \| & \| & \| & & & \| & \\ A_1 & A_2 & A_3 & \ldots & \ldots & A_{15} & \ldots \end{matrix}$$

Thus, if $L_k$ denotes $1 + \ldots + k = k(k+1)/2$ then $0 = L_0 < L_1 < L_2 < \ldots$, the sets

$$\mathcal{S} = \{n \in \mathbf{N} : L_{k-1} < n \leq L_k \text{ for some odd } k\}$$
$$\mathcal{E} = \{n \in \mathbf{N} : L_{k-1} < n \leq L_k \text{ for some even } k\}$$

decompose $\mathbf{N}$ into disjoint subsets, and

$$A_n = \begin{cases} S & n \in \mathcal{S} \\ T & n \in \mathcal{E} \end{cases}.$$

Correspondingly define

$$f_n = \begin{cases} f_S & n \in \mathscr{S} \\ f_T & n \in \mathscr{T} \end{cases}.$$

To suggest local iteration along the origin-orbit, write

$$A^n = A_{n-1} \circ \dots \circ A_0, \quad f^n = f_{n-1} \circ \dots \circ f_0, \quad n > 0,$$

where we take $A_0 = T$ and $f_0 = f_T$. To complete the picture, if $-n < 0$, set

$$A_{-n} = A_n, \qquad f_{-n} = f_n,$$
$$A^{-n} = A_{-n}^{-1} \circ \dots \circ A_{-1}^{-1}, \quad f^{-n} = f_{-n}^{-1} \circ \dots \circ f_{-1}^{-1},$$

and call $A^0 = f^0 =$ identity. Then $A^n$ is the linear part of $f^n$ at 0, $n \in \mathbf{Z}$.

For $|n|$ large

$$A^n = \begin{pmatrix} c_n & 0 \\ 0 & d_n \end{pmatrix}$$

where $c_n$ and $d_n$ are products of approximately equal numbers of $a$'s and $b$'s. As $|n| \to \infty$, $c_n^{1/n}$ and $d_n^{1/n} \to (ab)^{1/2}$, so the family $\{A^n\}_{n\in\mathbf{Z}}$ has double Lyapunov exponent equal to

$$\lambda = \frac{1}{2}\log(ab) < 0.$$

Since the $A_n$ are diagonal, regularity (see (1) in § 1) of this Lyapunov splitting is automatic. In sum, $\{A^n\}_{n\in\mathbf{Z}}$ contracts asymptotically. Note, however, that the contraction is *only* asymptotic. Along the orbit, the length of $v = (1, 0)$ expands by $b$, shrinks by $a$, grows by $b$ at the next two points, shrinks by $a$ at the next three points, etc.

*Theorem 1.* — *$\{f^n\}_{n\in\mathbf{Z}}$ does not contract asymptotically although its linear part does. In fact if $z = (x, y)$ is any point in the first quadrant of $\mathbf{R}^2$, $x > 0$ and $y > 0$, then*

$$f^n(z) \to \infty \quad \text{as } n \to \infty$$

*although* $(\mathrm{D}f^n)_0(z) \to 0$ *as* $n \to \infty$.

*Remarks.* — *a)* $f_S$ is a $C^1$ diffeomorphism of $\mathbf{R}^2$ onto itself leaving invariant the foliation by horizontal lines. It *shears* the $y$-axis onto the curve $y \mapsto g(y/b)$ and is an affine contraction of each horizontal line by the constant factor $a$. See Figure 2.

*b)* $f_T$ is conjugate to $f_S$ by a 90° rotation:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \circ f_S \circ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = f_T.$$

*c)* $\{f_n\}_{n\in\mathbf{Z}}$ is uniformly $C^1$-equicontinuous.

*d)* The stable set of o under $f^n$ can be shown to lie wholly in the third quadrant, $x < 0$ and $y < 0$; it seems to be a curve which converges to o in the manner of $\sin(1/x)/\log(1/x)$ as $x \to 0$. All other orbits converge to $\infty$ as $n \to \infty$.

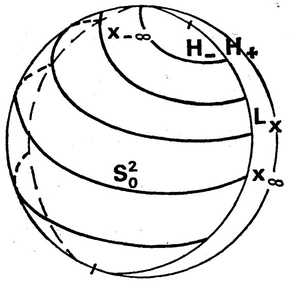


FIG. 2. — The non-linear horizontal shear $f_S$

*Lemma 1.* — *If* $0 < \beta_n \to 1$ *as* $n \to \infty$ *and* $b > 1$, $c > 0$ *are fixed, then*

$$\lim_{n \to \infty} b^{cn^2} \beta_n \beta_{n-1}^2 \ldots \beta_1^n \beta_0^{n+1} = \infty.$$

*Proof.* — Fix $s$, $l$ such that

$$n > s \Rightarrow \beta_n \geq b^{-c}$$
$$\beta_0, \ldots, \beta_s \geq b^{-l}.$$

Then

$$\begin{aligned} b^{cn^2} \beta_n \ldots \beta_0^{n+1} &= b^{cn^2} \beta_n \ldots \beta_{s+1}^{n-s} \beta_s^{n-s+1} \ldots \beta_0^{n+1} \\ &\geq b^{cn^2} (b^{-c})^{1+\ldots+(n-s)} (b^{-\ell})^{(n+1)(s+1)} \\ &> b^{(cn^2/2) - \ell(s+1)(n+1)} \to \infty \end{aligned}$$

as $n \to \infty$. Q.E.D.

For any function $f(u)$ that vanishes at $u = 0$ and is defined for $u \geq 0$, let

$$\sigma_f(u) = \frac{f(u)}{u}.$$

$\sigma_f(u)$ is the *shrinking factor* of $f$ at $u$: under $f$, the distance from $u$ to o is shrunk by the factor $\sigma_f(u)$.

*Lemma 2.* — *Let* $g(u) = u/\log(1/u)$ *as above,* $0 < u < 1/e$. *Call* $\sigma = \sigma_g$. *Then*

$$\frac{\sigma(gu)}{\sigma(u)} \to 1 \quad \textit{as } u \to 0.$$

*Proof.*

$$\frac{\sigma(gu)}{\sigma(u)} = \frac{\log(u)}{\log(gu)} = \frac{\log(u)}{\log u - \log|\log u|} = \frac{1}{1 + \dfrac{\log|\log u|}{|\log u|}} \to 1$$

as $u \to 0$. Q.E.D.

*Remark.* — Since $g'(0) = 0$, $\sigma(gu)$ and $\sigma(u)$ both tend to 0 with $u$. It is somewhat remarkable that they do so at the same order because $gu$ is *far* closer to 0 than is $u$. If $g$ were $C^{1+\alpha}$, $\alpha > 0$, the lemma would fail. For example if $\tilde{g} = u^{1+\alpha}$ and $\tilde{\sigma}$ is the shrinking factor for $\tilde{g}$, then

$$\frac{\tilde{\sigma}(\tilde{g}u)}{\tilde{\sigma}(u)} = u^{\alpha^2} \to 0 \quad \text{as } u \to 0.$$

Likewise, in the next lemma, $g$ must be $u/\log(1/u)$ or some similar function which grows too fast near 0 to be $C^{1+\alpha}$, $\alpha > 0$.

*Lemma 3.* — *Let $g$ be as in Figure 1 and (2). Fix $u_0 > 0$, $b > 1$. Set*

$$u_1 = bg(u_0), \ldots, u_k = b^k g(u_{k-1}).$$

*Then*
$$\lim_{k \to \infty} u_k = \infty.$$

*Proof.* — If $u_0 > 1/e$ then $u_1 = bg(u_0) \geq b/e > 1/e$, and, by induction,

$$u_k = b^k g(u_{k-1}) \geq b^k/e \to \infty \quad \text{as } k \to \infty,$$

so we may suppose $0 < u_0 < 1/e$. On $(0, 1/e)$,

$$\sigma'(u) = \frac{1}{u(\log(1/u))^2} > 0,$$

so $\sigma$ is monotone there. Clearly $g$ and $g^k = g \circ \ldots \circ g$, $k$ times, are monotone also. Thus,

$$\begin{aligned} g^k(u_0) &= g^{k-1}(gu_0) < g^{k-1}(bgu_0) = g^{k-1}(u_1) \\ &= g^{k-2}(gu_1) < g^{k-2}(b^2 gu_1) = g^{k-2}(u_2) \\ &= \ldots < u_k \end{aligned}$$

and

$$\begin{aligned} u_{k+1} &= b^{k+1} g(u_k) = b^{k+1} \sigma(u_k)\, u_k = b^{k+1} \sigma(u_k)\, b^k \sigma(u_{k-1})\, u_{k-1}. \\ &= \ldots = b^{(k+2)(k+1)/2} \sigma(u_k) \ldots \sigma(u_0) \\ &> b^{(k+2)(k+1)/2} \sigma(g^k u_0)\, \sigma(g^{k-1} u_0) \ldots \sigma(u_0) \\ &> b^{k^2/2} \left(\frac{\sigma(g^k u_0)}{\sigma(g^{k-1} u_0)}\right) \left(\frac{\sigma(g^{k-1} u_0)}{\sigma(g^{k-2} u_0)}\right)^2 \ldots \left(\frac{\sigma(gu_0)}{\sigma(u_0)}\right)^k (\sigma(u_0))^{k+1}. \end{aligned}$$

By Lemma 2, all the factors

$$\frac{\sigma(g^n u_0)}{\sigma(g^{n-1} u_0)} \to 1$$

as $n \to \infty$, so by Lemma 1, $u_{k+1} \geq 1/e$ for some first $k+1$. Beyond this $k+1$, $u_m \geq b^{m-(k+1)}/e \to \infty$ as $m \to \infty$, as observed at the outset. Q.E.D.

*Proof of Theorem 1.* — Recall that $0 < a < ab < 1 < b$ and

$$S = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad T = \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}, \quad f_S = \begin{pmatrix} a & g \\ 0 & b \end{pmatrix}, \quad f_T = \begin{pmatrix} b & 0 \\ g & a \end{pmatrix},$$

$$A_n = \begin{cases} S & \text{if } n \in \mathscr{S} \\ T & \text{if } n \in \mathscr{E} \end{cases}, \quad A^n = A_{n-1} \circ \ldots \circ A_0,$$

$$f_n = \begin{cases} f_S & \text{if } n \in \mathscr{S} \\ f_T & \text{if } n \in \mathscr{E} \end{cases}, \quad f^n = f_{n-1} \circ \ldots \circ f_0,$$

where

$$\mathscr{S} = \{n \in \mathbf{N} : 1 + \ldots + (k-1) < n \leq 1 + \ldots + k \text{ for some even } k \in \mathbf{N}\},$$
$$\mathscr{E} = \mathbf{N} - \mathscr{S},$$

and $g$ is the function in Figure 1 or (2). Write

$$f_n(z) = A_n z + G_n(z),$$

where $G_n : \mathbf{R}^2 \to \mathbf{R}^2$ is given by

$$G_n(z) = \begin{cases} \begin{pmatrix} gy \\ 0 \end{pmatrix} & \text{if } n \in \mathscr{S} \\ \begin{pmatrix} 0 \\ gx \end{pmatrix} & \text{if } n \in \mathscr{E} \end{cases}, \quad z = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Let $\pi_1$ and $\pi_2$ be the projections onto the $x$-axis and $y$-axis. Identify each axis with $\mathbf{R}$. Then

$$\pi_1 \circ G_n = \begin{cases} g \circ \pi_2 & \text{if } n \in \mathscr{S}, \\ 0 & \text{if } n \in \mathscr{E}, \end{cases} \qquad \pi_2 \circ G_n = \begin{cases} 0 & \text{if } n \in \mathscr{S}, \\ g \circ \pi_1 & \text{if } n \in \mathscr{E}. \end{cases}$$

Now fix some $z = (x, y)$ with $x > 0$ and $y > 0$. We must prove

(3) $$f^n(z) \to \infty \quad \text{as } n \to \infty.$$

By construction, $f^n$ carries the first quadrant into itself, so all quantities in the following estimates are positive. Set

$$s_0 = \pi_2(f^1 z) = \pi_2(f_0 z),$$
$$s_1 = \pi_1(f^3 z),$$
$$s_2 = \pi_2(f^6 z),$$
$$\vdots$$
$$s_k = \pi_i(f^m z), \quad m = 1 + \ldots + (k+1), \quad i = \begin{cases} 1 & k \text{ is odd} \\ 2 & k \text{ is even} \end{cases},$$
$$\vdots$$

We claim

(4) $$s_k \geq b^k g(s_{k-1}) \quad k \geq 1.$$

Suppose $k$ is odd, for instance $k = 1$. Then

$$\left.\begin{aligned} A_n &= T \\ \pi_1 \circ G_n &= 0 \\ \pi_2 \circ G_n &= g \circ \pi_1 \end{aligned}\right\} \quad \text{for } m - (k+1) + 1 \leq n \leq m,$$

$$\left.\begin{aligned} A_n &= S \\ \pi_1 \circ G_n &= g \circ \pi_2 \\ \pi_2 \circ G_n &= 0 \end{aligned}\right\} \quad \text{if} \quad n = m - (k+1).$$

Thus
$$\begin{aligned} s_k &= \pi_1(f^m z) = \pi_1(f_{m-1}(f^{m-1} z)) = \pi_1(T(f^{m-1} z) + G_{m-1}(f^{m-1} z)) \\ &= b\pi_1(f^{m-1} z) = \ldots = b^{k-1} \pi_1(f^{m-k+1} z) \\ &= b^{k-1} \pi_1(f_{m-k}(f^{m-k} z)) = b^{k-1} \pi_1(T(f^{m-k} z) + G_{m-k}(f^{m-k} z)) \\ &= b^k \pi_1(f^{m-k} z) = b^k \pi_1(f_{m-k-1}(f^{m-k-1} z)) \\ &= b^k \pi_1(S(f^{m-k-1} z) + G_{m-k-1}(f^{m-k-1} z)) \\ &= b^k a\pi_1(f^{m-k-1} z) + b^k g(\pi_2(f^{m-k-1} z)) \\ &> b^k g(\pi_2(f^{m-k-1} z)) = b^k g(s_{k-1}), \end{aligned}$$

since $(1 + \ldots + k) = m - k - 1$. Similarly, if $k$ is even, $k \geq 2$, $m = 1 + \ldots (k+1)$, then

$$\left.\begin{aligned} A_n &= S \\ \pi_1 \circ G_n &= g \circ \pi_2 \\ \pi_2 \circ G_n &= 0 \end{aligned}\right\} \quad \text{for } m - (k+1) + 1 \leq n \leq m,$$

$$\left.\begin{aligned} A_n &= T \\ \pi_1 \circ G_n &= 0 \\ \pi_2 \circ G_n &= g \circ \pi_1 \end{aligned}\right\} \quad \text{if} \quad n = m - (k+1)$$

and
$$\begin{aligned} s_k &= \pi_2(f^m z) = \ldots = b^k \pi_2(f_{m-k-1}(f^{m-k-1} z)) \\ &= b^k \pi_2(T(f^{m-k-1} z) + G_{m-k-1}(f^{m-k-1} z)) \\ &= b^k a\pi_2(f^{m-k-1} z) + b^k g(\pi_1(f^{m-k-1} z)) > b^k g(s_{k-1}), \end{aligned}$$

which proves (4).

Call $u_0 = s_0$ and $u_k = b^k g(u_{k-1})$, $k \geq 1$. By (4), induction, and monotonicity of $g$,

(5) $$s_k \geq u_k \quad \text{if } k \geq 0,$$

for $s_k \geq b^k g(s_{k-1}) \geq b^k(u_{k-1}) = u_k$. By Lemma 3, $u_k \to \infty$ as $k \to \infty$. From (5)

(6) $$s_k \to \infty \quad \text{as } k \to \infty.$$

This proves (3) for a certain subsequences of $n \to \infty$, namely for $n = m$ with $m$ of the form $1 + \ldots + (k+1)$.

To handle general $n$, observe that

(7) $\pi_1(f^{m+1}z)$ and $\pi_2(f^{m+1}z)$ both tend to $\infty$ as $m = 1 + \ldots + (k+1) \to \infty$.

For if $k$ is odd then

$$\begin{aligned}\pi_1(f^{m+1}z) &= \pi_1(\mathrm{T}(f^m z) + \mathrm{G}_m(f^m z)) \\ &= b\pi_1(f^m z) = bs_k \to \infty \quad \text{as } k \to \infty, \\ \pi_2(f^{m+1}z) &= \pi_2(\mathrm{T}(f^m z) + \mathrm{G}_m(f^m z)) \\ &= a\pi_2(f^m z) + g(\pi_1(f^m z)) > g(s_k) \to \infty \quad \text{as } k \to \infty.\end{aligned}$$

Similarly, if $k$ is even. Now the $k+1$ iterates $f_n$, $m < n < m + (k+2)$, increase one of these coordinates, $\pi_1(f^{m+1}z)$ or $\pi_2(f^{m+1}z)$, by powers of $b$, so, for these $n$,

$$\max(\pi_1(f^{n+1}z), \pi_2(f^{n+1}z)) > \min(\pi_1(f^{m+1}z), \pi_2(f^{m+1}z)).$$

But this means, by (7), that

$$|f^{n+1}z| \geq \min(\pi_1(f^{m+1}z), \pi_2(f^{m+1}z)) \to \infty \quad \text{as } k \to \infty,$$

for $1 + \ldots + (k+1) \leq n < 1 + \ldots + (k+2)$, which completes the proof of (3) and Theorem 1.

*Remark.* — The strategy of $\{f_n\}$ is to expand one component of $z$ $k$ times by $b$ and then transfer as much as possible of this expanded component to the opposite component for the next $k+1$ iterates. The non-smoothness of $f_n$ permits just enough transfer.

**3. Realizing the Example.** — Consider the diffeomorphisms $f_n : \mathbf{R}^2 \to \mathbf{R}^2$ as in § 2. We want to lift them to the 2-sphere by central projection, that is, by projection from the center of a unit 2-sphere whose south pole rests at the origin of $\mathbf{R}^2$. For polynomial vector fields this is a standard construction due to Poincaré [2]. Let

$$r = \frac{\mathrm{R}}{1 + \mathrm{R}^2}$$

$$\mathrm{R} = \tan(\varphi)$$

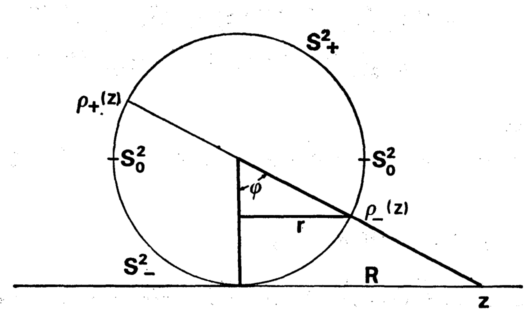


FIG. 3. — Central Projection

$\rho_- : \mathbf{R}^2 \to S^2_-$ be this projection where $S^2_-$ is the southern hemisphere. Let $\alpha$ be the antipodal map of $S^2$ and define $\rho_+ : \mathbf{R}^2 \to S^2_+$, the central projection to the northern hemisphere, $S^2_+$, by $\rho_+ = \alpha \circ \rho_-$. See Figure 3. (Stereographic projection, by the way, is unsuitable for such lifting.)

Any map $f : \mathbf{R}^2 \to \mathbf{R}^2$ lifts to a map $\rho_- f \cup \rho_+ f : S^2 - S^2_0 \to S^2 - S^2_0$ making

$$\begin{array}{ccc} S^2_\pm & \xrightarrow{\rho_\pm f} & S^2_\pm \\ \Big\downarrow{\scriptstyle \rho_\pm} & & \Big\downarrow{\scriptstyle \rho_\pm} \\ \mathbf{R}^2 & \xrightarrow{f} & \mathbf{R}^2 \end{array}$$

commute. ($S^2_0$ is the equator of $S^2$.) The next lemma gives sufficient conditions that $\rho_- f \cup \rho_+ f$ extend to a map $\rho_\# f$ on all of $S^2$. Note first, however, that any linear (or affine) isomorphism $A : \mathbf{R}^2 \to \mathbf{R}^2$ lifts to a diffeomorphism $\rho_\# A : S^2 \to S^2$. See [2] or below.

*Lemma* 4. — *Suppose* $A = \begin{pmatrix} a & c \\ 0 & b \end{pmatrix}$, $ab \neq 0$, *and* $h : \mathbf{R} \to \mathbf{R}$ *is a* $C^1$ *function with compact support. Then the map*

$$f = \begin{pmatrix} a & c + h \\ 0 & b \end{pmatrix} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} ax + cy + h(y) \\ by \end{pmatrix}$$

*lifts to a unique continuous map* $\rho_\# f : S^2 \to S^2$ *which agrees with* $\rho_\pm f$ *on* $S^2_\pm$. *Moreover,* $\rho_\# f$ *is a* $C^1$ *diffeomorphism whose* 1*-jet at the equator* $S^2_0$ *is the same as that of* $\rho_\# A$. *At the points* $x_{\pm\infty}$ *where the* $x$*-axis-longitude* $L_x$ *meets* $S^2_0$, *this* 1*-jet is independent of* $c$; *it is*

$$x_{\pm\infty} \mapsto \left(x_{\pm\infty}, \begin{pmatrix} 1/a & 0 \\ 0 & b/a \end{pmatrix}\right)$$

*respecting the splitting* $T_{x_{\pm\infty}}(S^2) = T_{x_{\pm\infty}}(L_x) \oplus T_{x_{\pm\infty}}(S^2_0)$.

*Proof.* — This is basically a chain rule calculation. Let $(\varphi, \theta)$ be the natural angular coordinates on $S^2_-$ and let $(r, \theta)$ be the polar coordinates in $\mathbf{R}^2$. Then

$$\begin{array}{lll} r = \tan\varphi, & \theta = \theta, & \rho^{-1}(\varphi, \theta) = (\tan\varphi, \theta), \\ x = r\cos\theta, & & f_1 = ax + cy + h(y), \\ y = r\sin\theta, & & f_2 = by. \end{array}$$

Express $\rho_- f : S^2_- \to S^2_-$ in the $(\varphi, \theta)$-coordinates as

$$\begin{array}{l} (\varphi, \theta) \mapsto (\Phi, \Theta) \\ \Phi = \tan^{-1}(R), \quad R = |f| = (f_1^2 + f_2^2)^{1/2}, \\ \Theta = \tan^{-1}(f_2/f_1). \end{array}$$

Since $R = |f|$ we have

$$\frac{R^2}{r^2} = a^2\cos^2\theta + c^2\sin^2\theta + \frac{(h(r\sin\theta))^2}{r^2} + 2ac\cos\theta\sin\theta$$
$$+ 2a\cos\theta\frac{h(r\sin\theta)}{r} + 2c\sin\theta\frac{h(r\sin\theta)}{r} + b^2\sin^2\theta.$$

Let $\varphi \to \pi/2$. Then $r = \tan\varphi \to \infty$ and

(8) $$\frac{R^2}{r^2} \rightrightarrows a^2\cos^2\theta + c^2\sin^2\theta + 2ac\cos\theta\sin\theta + b^2\sin^2\theta > 0.$$

By $\rightrightarrows$ we denote uniform convergence respecting $\theta$. From (8) follows

(9) $$\Phi \rightrightarrows \pi/2 \quad \text{as } \varphi \to \pi/2.$$

Similarly

(10) $$\Theta(\varphi,\theta) = \tan^{-1}(f_2/f_1) = \tan^{-1}\left(\frac{b\sin\theta}{a\cos\theta + c\sin\theta + \frac{1}{r}h(r\sin\theta)}\right)$$
$$\rightrightarrows \tan^{-1}\left(\frac{b\sin\theta}{a\cos\theta + c\sin\theta}\right).$$

Some care is needed here since $a\cos\theta + c\sin\theta$ can equal 0. Fix some small $\theta_0 > 0$ and let $N = \{\theta : 0 \le \theta \le \theta_0, \text{ or } \pi - \theta_0 \le \theta \le \pi + \theta_0, \text{ or } 2\pi - \theta_0 \le \theta < 2\pi\}$. If $\theta_0$ is small and $\theta \in N$ then the argument of $\tan^{-1}$ converges uniformly and (10) is immediate. If $\theta \notin N$ and $z = (r, \theta)$, $r = \tan\varphi \to \infty$, then

$$f(z) = \begin{pmatrix} ax + cy \\ by \end{pmatrix}$$

since $h(y) = h(r\sin\theta)$, $h$ has compact support, and $r\sin\theta \to \infty$. Since $\Theta$ refers to the angle made by $f(z)$, $\Theta(\varphi,\theta)$ converges uniformly for $\theta \notin N$ also, proving (10).

From (9) and (10) we see that $\rho_- f : S^2_- \to S^2_-$ extends to a continuous map on $S^2_0 \cup S^2_-$, sending the equator into itself according to

(11) $$\theta \mapsto \tan^{-1}\left(\frac{b\sin\theta}{a\cos\theta + c\sin\theta}\right).$$

Note that (11) changes by $\pi$ if $\theta$ is replaced by $\theta + \pi$. Thus, $\rho_+ f$ extends to the same map on the equator; *i.e.* $f$ lifts to a (necessarily unique) continuous map $\rho_\# f : S^2 \to S^2$ agreeing with $\rho_\pm f$ on $S^2$. It is easily seen to be a homeomorphism which is a diffeomorphism except perhaps at the equator.

To calculate the derivatives of $\rho_{\#}f$ we compute

$$\frac{\partial\Phi}{\partial\varphi} = \left(\frac{1}{1+R^2}\right)\left(\frac{1}{R}\right)\left\{f_1\left(\frac{\partial f_1}{\partial x}\frac{\partial x}{\partial r}\frac{\partial r}{\partial\varphi} + \frac{\partial f_1}{\partial y}\frac{\partial y}{\partial r}\frac{\partial r}{\partial\varphi}\right) + f_2\left(\frac{\partial f_2}{\partial x}\frac{\partial x}{\partial r}\frac{\partial r}{\partial\varphi} + \frac{\partial f_2}{\partial y}\frac{\partial y}{\partial r}\frac{\partial r}{\partial\varphi}\right)\right\}$$

$$= \left(\frac{1+r^2}{1+R^2}\right)\left(\frac{r}{R}\right)\left\{\left(a\cos\theta + c\sin\theta + \frac{1}{r}h(r\sin\theta)\right)\right.$$
$$\left.(a\cos\theta + (h'(r\sin\theta) + c)\sin\theta) + b^2\sin^2\theta\right\}.$$

By (8), the first two factors converge uniformly. As above, the terms $h(r\sin\theta)/r$ and $h'(r\sin\theta)\sin\theta$ go to 0 when $r\to\infty$. Thus

$$(12)\qquad \frac{\partial\Phi}{\partial\varphi} \rightrightarrows \lim_{\varphi\to\frac{\pi}{2}}\left(\frac{r}{R}\right)^3\{a^2\cos^2\theta + 2ac\cos\theta\sin\theta + c^2\sin^2\theta + b^2\sin^2\theta\}$$
$$= \lim_{\varphi\to\frac{\pi}{2}}\left(\frac{r}{R}\right).$$

Second,

$$\frac{\partial\Phi}{\partial\varphi} = \left(\frac{1}{R^2+1}\right)\left(\frac{1}{R}\right)\left\{f_1\left(\frac{\partial f_1}{\partial x}\frac{\partial x}{\partial\theta} + \frac{\partial f_1}{\partial y}\frac{\partial y}{\partial\theta}\right) + f_2\left(\frac{\partial f_2}{\partial x}\frac{\partial x}{\partial\theta} + \frac{\partial f_2}{\partial y}\frac{\partial y}{\partial\theta}\right)\right\}$$

$$= \left(\frac{r}{R^2+1}\right)\left(\frac{r}{R}\right)\left\{\left(a\cos\theta + c\sin\theta + \frac{1}{r}h(r\sin\theta)\right)\right.$$
$$\left.(-a\sin\theta + (c + h'(r\sin\theta))\cos\theta) + b^2\sin\theta\cos\theta\right\}.$$

As $r\to\infty$, $r/(R^2+1)\to 0$ while the other factors approach finite limits. Hence

$$(13)\qquad \frac{\partial\Phi}{\partial\theta} \rightrightarrows 0 \quad \text{as } \varphi\to\frac{\pi}{2}.$$

Third,

$$\frac{\partial\Theta}{\partial\varphi} = \frac{1}{1+(f_2/f_1)^2}\left\{\frac{\partial(f_2/f_1)}{\partial x}\frac{\partial x}{\partial r}\frac{\partial r}{\partial\varphi} + \frac{\partial(f_2/f_1)}{\partial y}\frac{\partial y}{\partial r}\frac{\partial r}{\partial\varphi}\right\}$$

$$= \frac{f_1^2}{R^2}\left\{\frac{-aby}{f_1^2}\frac{\cos\theta}{\cos^2\varphi} + \frac{abx + bh(y) - byh'(y)}{f_1^2}\frac{\sin\theta}{\cos^2\varphi}\right\}$$

$$= \frac{1+r^2}{R^2}\{-abr\sin\theta\cos\theta + abr\cos\theta\sin\theta + (bh(y) - byh'(y))\sin\theta\}$$

$$= \frac{1+r^2}{R^2}\{bh(r\sin\theta)\sin\theta - br\sin\theta h'(r\sin\theta)\sin\theta\}.$$

Now as $\varphi \to \frac{\pi}{2}$, either $h$ and $h'$ equal 0 or else $r \sin \theta$ stays bounded: $\sin \theta \to 0$ while $r \to \infty$. Thus, the bracketed terms go to 0 while $(1 + r^2)/R^2$ tends to $(\lim r/R)^2 \neq \infty$. Therefore

$$(14) \qquad \frac{\partial \Theta}{\partial \varphi} \rightrightarrows 0 \quad \text{as } \varphi \to \frac{\pi}{2}.$$

Finally,

$$\begin{aligned}\frac{\partial \Theta}{\partial \theta} &= \frac{1}{1 + (f_2/f_1)^2}\left\{\frac{\partial f_2/f_1}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f_2/f_1}{\partial y}\frac{\partial y}{\partial \theta}\right\} \\ &= \left(\frac{r}{R}\right)^2 \left\{ab \sin^2 \theta + ab \cos^2 \theta \right. \\ &\qquad \left. + \left(\frac{1}{r} bh(r \sin \theta) - bh'(r \sin \theta) \sin \theta\right) \cos \theta \right\}.\end{aligned}$$

As above, the terms involving $h$ and $h'$ tend to 0 as $\varphi \to \pi/2$. Thus

$$(15) \qquad \frac{\partial \Theta}{\partial \theta} \rightrightarrows \left(\lim \frac{r}{R}\right)^2 ab \quad \text{as } \varphi \to \pi/2.$$

The limits (12)-(15) commute with the antipodal map, so $\rho_\# f$ is $C^1$; at $(\pi/2, \theta) \in S^2_0$ it has derivative

$$\begin{bmatrix} \gamma & 0 \\ 0 & ab\gamma^2 \end{bmatrix}, \qquad \gamma = ((a \cos \theta + c \sin \theta)^2 + (b \sin \theta)^2)^{-1/2},$$

respecting the $(\varphi, \theta)$-coordinates. This is clearly invertible and independent of $h$. Hence $\rho_\# f$ is a $C^1$ diffeomorphism whose 1-jet agrees with that of $\rho_\# A$ at the equator. The points $x_{-\infty}$, $x_{+\infty}$ correspond to $\theta = \pi$, $\theta = 0$ and give $\gamma = a^{-1}$, verifying the fact that the derivative of $\rho_\# f$ at $x_{\pm\infty}$ is independent of $c$. Q.E.D.

Now return to the proof of Theorem 1. Since $f_S$ does not satisfy the hypotheses of Lemma 4, it is convenient to introduce the odd version of the function $g$ in § 2,

$$g_o(u) = \begin{cases} g(u) & \text{if } u \geq 0, \\ -g(u) & \text{if } u \leq 0. \end{cases}$$

Then $g_o'(u) \equiv c$ for some constant $c > 1$, provided $|u| \geq 1$. Call

$$A_\pm = \begin{pmatrix} a & \pm c \\ 0 & b \end{pmatrix}, \quad f_\pm = \left(A_\pm \pm \begin{pmatrix} 0 & g_o - c \\ 0 & 0 \end{pmatrix}\right) : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} ax \pm g_o(y) \\ by \end{pmatrix}.$$

Clearly

$$f_\pm(x, y) = f_S(x, y) \quad \text{for } \pm y \geq 0.$$

Since $g_0(y) - cy$ has compact support, $f_\pm$ satisfies the hypotheses of Lemma 4 and lifts to $S^2$ as $\rho_\#(f_\pm)$. At the equator, the 1-jet of $\rho_\#(f_\pm)$ agrees with that of $\rho_\#(A_\pm)$.

Divide $S^2$ into two hemispheres $H_\pm$ along the $x$-axis longitude $L_x$, say

> $H_\pm$ is the hemisphere containing the quarter sphere $\rho_-\{(x,y) \in \mathbf{R}^2 : \pm y > 0\}$.

See Figure 4.

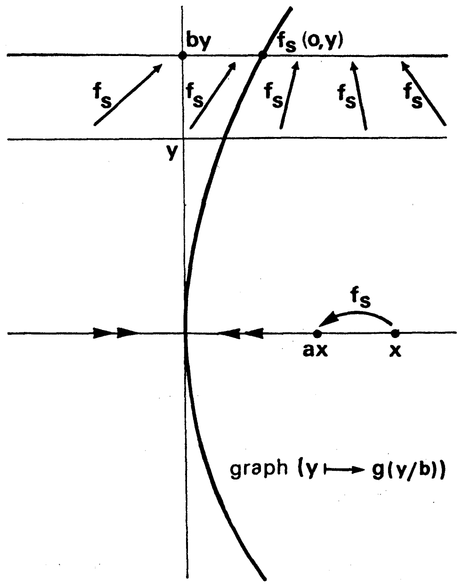


FIG. 4. — The hemispheres $H_\pm$ with half latitudes drawn in $H_-$

Define $F_S : S^2 \to S^2$ by

$$F_S(z) = \begin{cases} \rho_\# f_+(z) & \text{if } z \in H_+, \\ \rho_\# f_-(z) & \text{if } z \in H_-. \end{cases}$$

Then $F_S$ lifts $f_S$ to $S^2$, but *not* as $\rho_- f_S \cup \rho_+ f_S$! In fact this canonical lift $\rho_\# f_S$ fails to be $C^1$ at the equator.

At all finite points of the $x$-axis, $f_+ - f_-$ vanishes to first order (since $g(0) = g'(0) = 0$) so $F_S$ is well defined and continuous on $S^2_-$; in fact

$$T_z F_S = T_z(\rho_\# f_+) = T_z(\rho_\# f_-)$$

for all $z \in L_x \cap S^2_-$. Thus, $F_S \mid S^2_-$ is a $C^1$ diffeomorphism of $S^2_-$.

Although $F_S$ does not commute with the antipodal map $\alpha$, there is enough symmetry that differentiability of $F_S$ on $S^2_-$ implies it on $S^2_+$. If $z \in S^2_+ \cap H_+$ then

$$F_S(z) = \rho_\# f_+(z) = \alpha \circ \rho_- f_+ \circ \alpha(z),$$
$$T_z F_S = (T\alpha)_{\rho_- f(\alpha z)} \circ T_{\alpha z}(\rho_- f_+) \circ T_z \alpha,$$

while if $z' \in S^2_+ \cap H_-$ then

$$F_S(z') = \rho_\# f_-(z') = \alpha \circ \rho_- f_- \circ \alpha(z'),$$
$$T_{z'} F_S = (T\alpha)_{\rho_- f_-(\alpha z')} \circ T_{\alpha z'}(\rho_- f_-) \circ T_{z'} \alpha.$$

Now if $z = z' \in L_x \cap S^2_+$ then

$$\rho_- f_+(\alpha z) = \rho_- f_-(\alpha z'),$$
$$T_{\alpha z}(\rho_- f_+) = T_{\alpha z'}(\rho_- f_-),$$

since $\rho_- f_+$ equals $\rho_- f_+$ to first order along $L_x \cap S^2_-$. Thus, $F_S \mid S^2_+$ is a well defined $C^1$ diffeomorphism of $S^2_+$ also.

Since $F_S$ is the $C^1$ diffeomorphism $\rho_\# f_\pm$ on the interior of $H_\pm$, it remains only to check $F_S$ at $S^2_0 \cap \partial H_\pm = x_{\pm\infty}$. But by Lemma 4, $\rho_\# f_\pm$ has at the equator a 1-jet equal to that of the diffeomorphism $\rho_\# A_\pm$ and at $x_{\pm\infty}$ the latter 1-jet does not depend on $c$. That is,

$$T_{x_{\pm\infty}}(\rho_\# f_+) = T_{x_{\pm\infty}}(\rho_\# A_+),$$
$$T_{x_{\pm\infty}}(\rho_\# f_-) = T_{x_{\pm\infty}}(\rho_\# A_-),$$

and so $T_{x_{\pm\infty}} F_S$ exists and is invertible. Hence

(16) $f_S$ lifts to a (somewhat noncanonical) $C^1$ diffeomorphism $F_S$ of $S^2$; similarly $f_T$ lifts to $F_T$.

*Remarks.* — It is because the dynamics of the sequence $\{f_n\}$ is sensitive to perturbations at infinity that we took pains to lift the global map $f_n$ to $S^2$, not just its germ near 0.

We are now ready to embed the example in § 2 into a diffeomorphism of a compact manifold.

Let $h : M^2 \to M^2$ be any diffeomorphism having a hyperbolic invariant set H on which $h$ is topologically conjugate to the full 2-shift and

(17) $T_H h$ dominates $TF_S$.

By (17) we mean that if $E^{uu} \oplus E^{ss} = T_H M^2$ is the hyperbolic splitting then

$$|Th(v)| > |TF_S(u)| \quad \text{whenever } v \in E^{uu}, \ |v| = 1, \ u \in TS^2, \ |u| = 1,$$
$$|Th(v)| < |TF_S(u)| \quad \text{whenever } v \in E^{ss}, \ |v| = 1, \ u \in TS^2, \ |u| = 1.$$

That is, the spectrum of $T_H h$ lies outside the annular hull of the spectrum of $TF_S$. We could, for instance, take H to be a horse-shoe basic set.

Let $H_0$ be the set of points of H which correspond to symbol sequences with a 0 in the initial position and $H_1$ be those with a 1 in the initial position. Then

$$H = H_0 \sqcup H_1$$

and $H_0$, $H_1$ are compact. Choose a smooth bump function

$$\mu : M^2 \to [0, \pi/2]$$

such that $H_0 = \mu^{-1}(0) \cap H$, $H_1 = \mu^{-1}(\pi/2) \cap H$, and $\mu^{-1}(\{0, \pi/2\})$ is a neighborhood of H.

Let $R_\theta$ be the rotation of $S^2$ by angle $\theta$ which fixes the poles. Form the skew product of $h$ and $F_S$

$$F : M^2 \times S^2 \to M^2 \times S^2$$

$$(w, z) \mapsto (h(w), R_{\mu(w)} \circ F_S \circ R_{-\mu(w)}(z)).$$

F leaves invariant the foliation $\mathscr{S}$ by 2-spheres $w \times S^2$, $w \in M^2$, and by (17), F is normally hyperbolic to $\mathscr{S}$. See [3, p. 116]. Besides

$$(18) \qquad T_{(w,z)} F = \begin{bmatrix} \dfrac{\partial h}{\partial w} & \dfrac{\partial R_\mu \circ F_S \circ R_{-\mu}}{\partial w} \\ 0 & \dfrac{\partial R_\mu \circ F_S \circ R_{-\mu}}{\partial z} \end{bmatrix}.$$

Since $h \mid H$ is the 2-shift, there is a (unique) orbit $\mathcal{O}(p)$ in H such that

$$h^n(p) \in H_0 \quad \text{iff } A_n = S, \qquad h^n(p) \in H_1 \quad \text{iff } A_n = T.$$

That is, we consider the orbit $\mathcal{O}(p)$ whose symbol is

$$000110.10110000 1111 \ldots$$

respecting the division $H = H_0 \sqcup H_1$. Let

$$P = (p, z_0)$$

where $z_0$ is the south pole of $S^2$. The F-orbit of P is $\{(h^n p, z_0)\}$ since $z_0$ is fixed under $F_S$ and $R_\theta$. By (16), (18), and constancy of $\mu$ near H,

$$(19) \qquad T_{(h^n p, z_0)} F = \begin{bmatrix} T_{h^n p} h & 0 \\ 0 & A_n \end{bmatrix}.$$

Indeed by choice of $\mu$ and the fact that

$$f_T = R_{\pi/2} \circ f_S \circ R_{-\pi/2}$$

one sees that

$$(20) \qquad \begin{array}{ccc} h^n p \times S^2 & \xrightarrow{F} & h^{n+1} p \times S^2 \\ \downarrow & & \downarrow \\ S^2 & \xrightarrow{F_n} & S^2 \end{array}$$

commutes, where $F_n = F_S$ or $F_n = F_T$ according as $A_n = S$ or $A_n = T$.

From (19) $\mathcal{O}(P)$ has one positive Lyapunov exponent corresponding to $Th \mid E^{uu}$ and three negative Lyapunov exponents: one corresponding to $Th \mid E^{ss}$ and the other two being $\lambda = \frac{1}{2} \log (ab)$ which correspond to the $A_n$'s as in § 2. Let $E^s$ denote the space of vectors with negative Lyapunov exponents

$$E^s_P \supset T(p \times S^2).$$

The orbit $\mathcal{O}(P)$ is regular because $TF^n \mid E^s_P$ is diagonal repecting $E^{ss} \oplus (x\text{-axis}) \oplus (y\text{-axis})$.

*Theorem 2. — The stable set of* P *is not an immersed manifold tangent to* $E^s_P$.

*Proof.* — Since F is normally hyperbolic to $\mathcal{S}$, a point $(w, z)$ is asymptotic with P under F if and only if $(w, z)$ lies on the strong stable manifold of some point

$$(p, z') \in W^s(P) \cap (p \times S^2).$$

That is,

$$W^s(P) = W^{ss}(W^s(P) \cap (p \times S^2))$$

where $W^{ss}$ denotes the strong stable manifolds. See [3, p. 71]. But, by (20), $W^s(P) \cap (p \times S^2)$ is just the stable set of 0 under the maps $f^n$ as in § 2 and this set is not a neighborhood of 0; it misses the whole first quadrant. Thus, $W^s(P)$ is contained in the three dimensional set $W^{ss}(p \times S^2)$ but does not include a neighborhood of P in it. It is therefore not able to be an immersed manifold tangent to $E^s_P$. Q.E.D.

*Remarks and Questions.* — *a)* More can probably be proved about $W^s(P)$. It seems to consist of the $W^{ss}$ fibers over a curve tending to P in $p \times S^2$ in an oscillatory fashion. In particular, it seems to have dimension two.

*b)* Can the dimension of M be reduced form 4 to 3 in the above example by the introduction of a solenoid?

*c)* Do $C^1$ diffeomorphisms of 2-manifolds have $C^1$ stable manifolds at asymptotically hyperbolic orbits?

*d)* Which orbits in H (*i.e.* which symbol sequences) exhibit this anti-Pesin behavior? Do they form a residual set in H? A set of measure zero for every *h*-invariant probability measure on H?

*e)* Is the set of points where the stable set of *f* is a $C^1$ injectively immersed manifold a set of measure one for every *f*-invariant probability measure on M?

*f)* Does the *generic* $C^1$ diffeomorphism (for example, one near the F above) have $C^1$ stable manifolds at its *generic* asymptotically hyperbolic orbits?

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