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AN ANALYTIC PROOF OF NOVIKOV'S THEOREM ON RATIONAL PONTRJAGIN CLASSES

by D. SULLIVAN *and* N. TELEMAN (1)

We give here an *analytic* proof for the following:

Theorem 1 (S. P. Novikov [3]). — *The rational Pontrjagin classes of any compact oriented smooth manifold are topological invariants.*

This problem was previously posed by I. M. Singer [4] and D. Sullivan [5]. Theorem 1 is a direct consequence of the following Theorems 2 and 3.

Theorem 2 (D. Sullivan [5]). — *Any topological manifold of dimension* $\neq 4$ *has a Lipschitz atlas of coordinates, and for any two such Lipschitz structures* $\mathscr{L}_i$, $i = 1, 2$, *there exists a Lipschitz homeomorphism* $h : \mathscr{L}_1 \to \mathscr{L}_2$ *close to the identity.*

Remark 1. — The proof of theorem 2 in general uses Kirby's annulus theorem to know that topological manifolds are stable (2). The proof of Theorem 2 for stable manifolds is more elementary. Simply connected manifolds are stable and these (3) are sufficient for proving Novikov's theorem.

Theorem 3 (N. Teleman [6]). — *For any compact oriented boundary free Riemannian Lipschitz manifold* $M^{2\mu}$, *and for any Lipschitz complex vector bundle* ξ *over* $M^{2\mu}$, *there exists a signature operator* D_ξ^+, *which is Fredholm, and its index is a Lipschitz invariant.*

Theorem 2 allows a strengthening of the statement of Theorem 3.

Theorem 4. — *For any simply-connected compact, oriented, boundary free topological manifold* $M^{2\mu}$ *of dimension* $2\mu \neq 4$, *and for any complex continuous vector bundle* ξ *over* M, *there exists a class* $\mathscr{C}(M, \xi)$ *of signature operators* D_ξ^+ *which are Fredholm operators. The index*

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(2) See also P. Tukia and J. Väisälä [7] and [8].
(3) See remark in [3].

of any of these operators is the same and is a topological invariant of the pair (M, ξ). *When* M *and* ξ *are smooth, the smooth signature operators* D^+_ξ (cf. [1]) *belong to this class* $\mathscr{C}(M, \xi)$.

Proof. — Pick a Lipschitz structure $\mathscr{L}_1$ on M by Theorem 2, and regularize the bundle ξ up to a Lipschitz vector bundle ξ_1. Theorem 3 says that the class $\mathscr{C}(M, \xi)$ is not void, and because the Lipschitz signature operators generalize the smooth signature operators, the last part of the theorem follows.

Suppose now that $\mathscr{L}_i$, $i = 1, 2$, are two Lipschitz structures on M and that ξ_i are corresponding Lipschitz regularizations of ξ.

The Theorem 2 implies that there exists a Lipschitz homeomorphism $h : \mathscr{L}_1 \to \mathscr{L}_2$ close to the identity (isotopic to the identity). As h is isotopic to the identity, the bundle $h^*\xi_2$ is Lipschitz isomorphic to ξ_1; let $\bar{h} : \xi_1 \to \xi_2$ be such an isomorphism. Take any Lipschitz Riemannian metric [6] Γ_i on M, $i = 1, 2$, and any connection Δ_i in ξ_i; the signature operators $D^+_{\xi_i}$ are defined. From Theorem 3 we know that the index of $D^+_{\xi_i}$, i fixed, is independent of the Riemannian metric Γ_i and the connection Δ_i chosen. In order to compare Index $D^+_{\xi_1}$ and Index $D^+_{\xi_2}$ themselves, we chose Γ_2 and Δ_2 arbitrarily, but we take

$$\Gamma_1 = h^*\Gamma_2, \quad \text{and} \quad \Delta_1 = \bar{h}^*\Delta_2.$$

From the very definition of the signature operators, we get that the homeomorphisms $h, \bar{h}$ allow us to identify the corresponding domains and codomains of the operators $D^+_{\xi_1}, D^+_{\xi_2}$; with these natural identifications, $D^+_{\xi_1}$ and $D^+_{\xi_2}$ coincide, and therefore, they have the same index.

Proof of theorem 1. — Suppose that $M^{2\mu}$ is a smooth manifold, and ξ is a smooth complex vector bundle over M. The signature theorem due to F. Hirzebruch, and subsequently generalized by M. F. Atiyah and I. M. Singer [1], asserts that

$$\text{Index } D^+_\xi = \text{ch } \xi . L(p_1, p_2, \ldots, p_{\mu/2})[M]$$

where L is the Hirzebruch polynomial and $p_1, p_2, \ldots, p_{\mu/2}$ are the Pontrjagin classes of M. Theorem 4 implies that the right hand side of this identity is a topological invariant of the pair (M, ξ). By letting ξ to vary, ch ξ generates over the rationals the whole even-cohomology subring of $H^*(M, \mathbf{Q})$. From the Poincaré duality we deduce further that the cohomology class $L(p_1, \ldots, p_{\mu/2})$ is a topological invariant. It is known that the homogeneous cohomology part L_i of degree $4i$ of $L(p_1, \ldots, p_{\mu/2})$ is of the form (see *e.g.* [2])

$$L_i = a_i . p_i + \text{polynomial in } p_1, p_2, \ldots, p_{i-1}, \quad a_i \in \mathbf{Q}, \quad a_i \neq 0.$$

Therefore $p_1, p_2, \ldots, p_{\mu/2}$ are polynomial combinations with rational coefficients of $L_1, L_2, \ldots, L_{\mu/2}$, which, as seen, are topological invariants.

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