PUBLICATIONS MATHÉMATIQUES DE L'I.H.É.S.

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Publications mathématiques de l'I.H.É.S., tome 58 (1983), p. 5-18
<http://www.numdam.org/item?id=PMIHES_1983__58__5_0>

THE NERON MODEL
FOR FAMILIES OF INTERMEDIATE JACOBIANS
ACQUIRING "ALGEBRAIC" SINGULARITIES

by HERBERT CLEMENS

1. Introduction

Let V be an irreducible complex projective manifold of dimension $2m-1$. Let $\{Z_s\}_{s\in S}$ be an algebraic family of algebraic $(m-1)$-cycles on V whose members Z_s are all homologically equivalent. In Appendix A of his paper "Periods of integrals on algebraic manifolds, III" ([G]; p. 165), P. Griffiths defines an analytic map, called the *Abel-Jacobi homomorphism*

(1.1) $$S \to J(V).$$

Here $J(V) = \dfrac{[F^m H^{2m-1}(V; \mathbf{C})]^*}{H_{2m-1}(V; \mathbf{Z})}$ is the Jacobian variety of V, and (1.1) is defined by picking a basepoint $s_0 \in S$ and sending

(1.2) $$s \mapsto \int_{Z_{s_0}}^{Z_s}.$$

Next let X be a complex projective manifold of dimension $2m$, let Δ be the unit disc, and let

(1.3) $$V_t, \quad t \in \Delta,$$

be an analytic family of divisors on X which are irreducible and non-singular as long as $t \neq 0$. Suppose

(1.4) $$\{Z_s\}_{s\in S_t}$$

is an algebraic family of algebraic $(m-1)$-cycles on V_t for each $t \in \Delta$. Suppose further that, for fixed t, all cycles Z_s, $s \in S_t$, are *homologous* to one another. Finally suppose that

$$\mathscr{S} = \bigcup_{t\in\Delta} (\{t\} \times S_t)$$

is a smooth analytic variety with

$$(1.5) \qquad \mathscr{S} \to \Delta$$

everywhere of maximal rank. We make no assumptions about properness of (1.5) or connectivity of fibres.

Let $\mathscr{J}^* \to \Delta^* = (\Delta - \{0\})$ be the bundle of complex tori over the punctured disc whose fibre over t is $J(V_t)$. Then for every section $\tau : \Delta \to \mathscr{S}$ of (1.5) there is a commutative diagram of proper morphisms

$$(1.6) \qquad \begin{array}{ccc} \mathscr{S}^* & \longrightarrow & \mathscr{J}^* \\ & \searrow \quad \swarrow & \\ & \Delta^* & \end{array}$$

defined fibrewise by (1.2) with $s_0 = \tau(t)$. The point of this paper is to complete (1.6) to a commutative diagram

$$(1.7) \qquad \begin{array}{ccc} \mathscr{S} & \longrightarrow & \mathscr{J} \\ & \searrow \quad \swarrow & \\ & \Delta & \end{array}$$

where $\mathscr{J}$ is obtained from $\mathscr{J}^*$ by filling in over $t = 0$ with a commutative complex Lie group. The complex Lie group in question is the fibre over $t = 0$ of an analytic analogue of the *Neron minimal model.*

We will be able to carry out this program only after putting some severe restrictions on the family (1.3). We devote the rest of the introduction to explaining these restrictions.

Let $\mathscr{V}^* = \bigcup_{t \in \Delta} (\{t\} \times V_t)$ and

$$(1.8) \qquad \mu : \mathscr{V}^* \to \Delta^*.$$

We pull-back the bundle (1.8) via the universal covering map

$$\tilde{\Delta} \to \Delta^*$$

$$u \mapsto t = e^{2\pi i u}$$

to obtain $\qquad \tilde{\mu} : \tilde{\mathscr{V}} \to \tilde{\Delta}.$

Abusing notation we write

$$\tilde{\mu}^{-1}(u) = V_u = V_t = \mu^{-1}(t)$$

whenever $t = e^{2\pi i u}$. The derived bundles $R_{2m-1}\tilde{\mu}_*(\mathbf{Z})$ and $R^{2m-1}\tilde{\mu}_*(\mathbf{Z})$ are trivial and we will denote their modules of global sections, taken modulo torsion, by $H_{\mathbf{Z}}$ and $H^{\mathbf{Z}}$ respectively. Also $H^{\mathbf{C}} = H^{\mathbf{Z}} \otimes \mathbf{C}$.

The natural isomorphism

$$H_{\mathbf{Z}} \cong H_{2m-1}(V_{u+1}) = H_{2m-1}(V_t) = H_{2m-1}(V_u) \cong H_{\mathbf{Z}}$$

is not the identity map on $H_{\mathbf{Z}}$ but rather the *monodromy isomorphism*

$$T_* : H_{\mathbf{Z}} \to H_{\mathbf{Z}}.$$

Let T^* be the adjoint of T_* with respect to the natural unimodular pairing

$$H_{\mathbf{Z}} \times H^{\mathbf{Z}} \to \mathbf{Z}$$
$$(\gamma, \omega) \to \int_\gamma \omega.$$

Our first major assumption is that

(1.9) $$(T_* - I)^2 = (T^* - I)^2 = 0.$$

Let $N_* = \log T_*$, $N^* = \log T^*$. We then have a filtration $\{W_*\}$ on $H^{\mathbf{Z}}$ defined by

$$W_{2m-3}H^{\mathbf{Z}} = 0$$
$$W_{2m-2}H^{\mathbf{Z}} = (\ker N_*)^\perp$$
$$W_{2m-1}H^{\mathbf{Z}} = (\text{image } N_*)^\perp$$
$$W_{2m}H^{\mathbf{Z}} = H^{\mathbf{Z}}.$$

This filtration is called the *asymptotic weight filtration.*

Under the identification

$$H^{\mathbf{C}} = H^{2m-1}(V_u; \mathbf{C})$$

the Hodge filtration on $H^{2m-1}(V_u; \mathbf{C})$ induces a filtration F_u^* on $H^{\mathbf{C}}$. This filtration varies with $u \in \widetilde{\Delta}$, but there is a well-defined filtration

(1.10) $$F_\infty^* = \lim_{t\to 0} \exp(-uN^*)F_u^*$$

on $H^{\mathbf{C}}$ called the *asymptotic Hodge filtration.*

In [S], W. Schmid shows that the array $(H^{\mathbf{Z}}, W_*, F_\infty^*)$ is a *mixed Hodge structure* such that

$$N^* : H^{\mathbf{C}} \to H^{\mathbf{C}}$$

is a morphism of mixed Hodge structures of type $(-1, -1)$. In fact in this situation which "comes from geometry", we have that

$$N^* : H^{\mathbf{Z}}/W_{2m-1} \to W_{2m-2}$$

is an isomorphism over $\mathbf{Q}$. Our second major assumption is:

(1.11) The Hodge structure of weight $2m$ on $H^{\mathbf{C}}/W_{2m-1}$ induced by (1.10) is of pure type (m, m).

Finally we make an assumption which is not essential but will simplify the exposition:

(1.12) All of $H^{2m-1}(V_t; \mathbf{C})$ is primitive cohomology.

2. Growth of normal functions

Let

(2.1) $$\tau : \Delta \to \mathcal{S}$$

be a section of the fibration $\mathcal{S} \to \Delta$ considered in (1.7). Let

$$Z_t \subseteq V_t, \quad t \in \Delta,$$

be the corresponding family of $(m-1)$-cycles. By Kleiman's smoothing theorem ([K]; p. 297), we can assume that, if $t \neq 0$,

(2.2) $$Z_t = Z_t' - Z_t''$$

where Z_t' and Z_t'' are smooth and do not meet. We can resolve the family

(2.3) $$\bigcup_{t \in \Delta} (\{t\} \times V_t) \subseteq \Delta \times X$$

along V_0 so that:

i) the fibre over $t = 0$ is a normal crossing variety in a smooth ambient space of dimension $2m$;
ii) the proper transform $\widetilde{Z}$ of
$$\bigcup_{t \in \Delta} (\{t\} \times Z_t)$$
is smooth and meets the fibre over zero transversely.

In ([C[1]]; p. 245), we explicitly construct an action of the semigroup $[0, 1] \times \mathbf{R}$ on the resolved ambient space which is equivariant with the action

$$(r, \theta) . t = re^{2\pi i\theta}t$$

of $[0, 1] \times \mathbf{R}$ on Δ. It is easy to see that this action can be defined so as to respect $\widetilde{Z}$ since it is constructed first locally and then pieced together via fibrations which can be constructed to be compatible with $\widetilde{Z}$.

As before, let

(2.4) $$u = \frac{1}{2\pi i} \log t$$

and take, for some fixed u_0, a $(2m-1)$-chain Γ_{u_0} such that

$$\partial\Gamma_{u_0} = Z_{t_0}.$$

For $(r, \theta) \in [0, 1] \times \mathbf{R}$, define

$$\Gamma_u = (r, \theta) . \Gamma_{u_0}$$

where $u = u_0 + \left(\theta + \frac{\log r}{2\pi i}\right)$. Then Γ_u has as its boundary the algebraic cycle Z_t with $t = e^{2\pi i u}$.

Lemma **(2.5)**. — *The cycle* $(\Gamma_{u+1} - \Gamma_u) \in H_{2m-1}(V_t; \mathbf{Z})$ *has zero intersection number with any cycle which is invariant under the monodromy transformation*

$$T_*: H_{2m-1}(V_t; \mathbf{Z}) \to H_{2m-1}(V_t; \mathbf{Z})$$
$$\gamma \mapsto (1, 1).\gamma.$$

Proof. — The cycle $(\Gamma_{u+1} - \Gamma_u)$, by construction, bounds in the ambient space of the resolution of the family (2.3). Therefore, by the Local Invariant Cycle Theorem ([C[1]]; p. 230), this cycle integrates to zero against any invariant element of $H^{2m-1}(V_t; \mathbf{C})$. Since the Poincaré duals of the invariant cocycles are the invariant cycles, the lemma is proved.

Next let $\omega(t)$ be a section of the canonical prolongation of

$$F^m R^{2m-1}\mu_*(\mathbf{C}),$$

where $\mu: \mathscr{V}^* \to \Delta^*$

is our family (1.8) and the canonical prolongation is as established in ([D]; pp. 91-92).

Lemma **(2.6)**. — *The integrals*

$$\int_{Z_t''}^{Z_t'} \omega(t) = \int_{\Gamma_u} \omega(t)$$

are all of the form

$$a(t) \log t + f(t)$$

where $a(t)$ *and* $f(t)$ *are holomorphic on* Δ.

Proof. — The idea of the proof is to make the integrals in question into period functions, *i.e.* integrals over *cycles*, for some two-parameter family of varieties. Let X as in (1.3) denote the ambient variety for the family $\{V_t\}$ and let

$$U_t \subseteq X, \quad t \in \Delta,$$

be an analytic family of very ample hypersurfaces such that:

i) if $t \neq 0$, U_t is smooth and meets V_t transversely,
ii) $Z_t \subseteq (U_t \cap V_t)$;
iii) Z_t is homologous to zero in U_t.

That such U_t exist is an application of the results of [K]. (See the Columbia University Thesis of Spencer Bloch, 1971, pp. 9-10.) If we choose the family U_t sufficiently amply, there will be a two-parameter family

$$W_{(t,t')}, \quad (t, t') \in \Delta \times \Delta,$$

of hypersurfaces in X such that:

i) $W_{(t,t')}$ is smooth and irreducible for $(t, t') \in \Delta^* \times \Delta^*$,
ii) $W_{(t,0)} = U_t \cup V_t$.

Next choose chains $\Sigma_u \subseteq U_t$ such that

$$\partial \Sigma_u = Z_t$$

and define $\quad \Gamma_{(u,0)} = \Gamma_u - \Sigma_u.$

Finally because of the simple nature of the degenerations $W_{(t,t')} \to W_{(t,0)}$ we can form a continuously varying family of *cycles*

$$\Gamma_{(u,u')} \subseteq W_{t,t')}$$

such that $\quad \lim_{t' \to 0} \Gamma_{(u,u')} = \Gamma_{(u,0)}.$

Now the assumption (1.12) that the differential $\omega(t)$ on V_t is primitive implies that there is a two-parameter family of differentials

$$\omega(t, t') \in F^m H^{2m-1}(W_{(t,t')}; \mathbf{C})$$

such that:

i) $\omega(t, 0) = \omega(t)\Big|_{V_t} + 0\Big|_{U_t}$;

ii) the family $\omega(t, t')$ extends over $\Delta \times \Delta$ to give a section of the canonical prolongation of the Hodge bundle for the two-variable degeneration ([1]).

In fact, let $\varphi_{(u,u')} = \Gamma_{(u,u'+1)} - \Gamma_{(u,u')}$. Then $\varphi_{(u,u'+1)} = \varphi_{(u,u')}$ and, by i),

$$\lim_{t' \to 0} \int_{\varphi_{(u,u')}} \omega(t, t') = 0.$$

Therefore the integrals

$$\int_{\Gamma_{(u,u')}} \omega(t, t') - u' \int_{\varphi_{(u,u')}} \omega(t, t')$$

are well-defined functions of the variables u and

$$t' = e^{2\pi i u'}.$$

Also since periods with respect to the canonical prolongation have at most logarithmic growth, these functions must be holomorphic along $\Delta^* \times \{0\}$, and have value

$$\int_{\Gamma_u} \omega(t)$$

at $(t, 0)$. Finally we use logarithmic growth of periods with respect to the canonical prolongation once again, this time in the t-direction. This allows us to conclude that

$$a(t) = \frac{1}{2\pi i} \int_{\Gamma_{u+1} - \Gamma_u} \omega(t)$$

is bounded at $t = 0$, since, by Lemma (2.5), it is a well-defined function of t. So

$$\int_{\Gamma_u} \omega(t) - a(t) \log t$$

is well-defined and so also bounded at $t = 0$ and the lemma is proved.

([1]) For a more complete discussion of this point, see the Appendix.

3. The Neron model

Recall that in § 1 we defined

$$N_*: H_{\mathbf{Z}} \to H_{\mathbf{Z}}.$$

Now (image N_*) is not in general a direct summand of $H_{\mathbf{Z}}$ so we enlarge it:

$$(3.1) \qquad E_{van} = \{\varphi \in H_{\mathbf{Z}} : \text{some integral multiple of } \varphi \text{ lies in (image } N_*)\}$$

is called the module of *vanishing cycles*. Also

$$(3.2) \qquad E_{inv} = (\ker N_*)$$

is called the module of *invariant cycles*. E_{van} is totally isotropic for the intersection pairing on $H_{\mathbf{Z}}$, in fact, the intersection bilinear form is identically zero on

$$E_{inv} \times E_{van}.$$

So it easy to see that there is a splitting of $H_{\mathbf{Z}}$

$$(3.3) \qquad H_{\mathbf{Z}} = L \oplus E \oplus E_{van}$$

such that $E \oplus E_{van} = E_{inv}$, and the intersection pairing is unimodular on $L \oplus E_{van}$ and on E, and these two symplectic modules are orthogonal. Furthermore we can adjust the definition of L so that L is totally isotropic.

Let $\{\varphi_j\}$ be a basis for E_{van} satisfying (1.12). Let $\{\delta_\ell, \varepsilon_\ell\}_{\ell=1}^{s}$ be a symplectic basis for E and let $\{\lambda_j\} \subseteq L$ be such that $\{\lambda_j, \varphi_j\}_{j=1}^{r}$ is a symplectic basis for $L \oplus E_{van}$. The Riemann relations imply that if $\{\omega_i(t), \eta_k(t)\}$ is a framing of $F^m R^{2m-1}\mu_*(\mathbf{C})$ for μ as in (1.8), then the matrix

$$(3.4) \qquad \begin{bmatrix} \int_{\varphi_j} \omega_i(t) & \int_{\varepsilon_\ell} \omega_i(t) \\ \int_{\varphi_j} \eta_k(t) & \int_{\varepsilon_\ell} \eta_k(t) \end{bmatrix}$$

is invertible for each $t \neq 0$. So we can normalize the choice of the framing to make (3.4) be the identity matrix for each $t \neq 0$. (Notice that the matrix (3.4) is well-defined as a function of t since the cycles φ_j and ε_ℓ are invariant.)

Now if " * " denotes Poincaré dual, we can write any framing $\{\omega_i(t), \eta_k(t)\}$ of $F^m H^{2m-1}(V_u; \mathbf{C})$ for $t = e^{2\pi i u}$ as

$$(3.5) \quad \text{i)} \quad \sum_j \Big(\int_{\varphi_j} \omega_i(t)\Big) \lambda_j^* + \sum_\ell \Big(\int_{\varepsilon_\ell} \omega_i(t)\Big) \delta_\ell^* + \sum_j \Big(\int_{\lambda_j} \omega_i(t)\Big) \varphi_j^* + \sum_\ell \Big(\int_{\delta_\ell} \omega_i(t)\Big) \varepsilon_\ell^*$$

$$\text{ii)} \quad \sum_j \Big(\int_{\varphi_j} \eta_k(t)\Big) \lambda_j^* + \sum_\ell \Big(\int_{\varepsilon_\ell} \eta_k(t)\Big) \delta_\ell^* + \sum_j \Big(\int_{\lambda_j} \eta_k(t)\Big) \varphi_j^* + \sum_\ell \Big(\int_{\delta_\ell} \eta_k(t)\Big) \varepsilon_\ell^*.$$

The elements (3.5) therefore frame $F_u^m \subseteq H^{\mathbf{C}}$ (see (1.10)).

Now Schmid's theory and our normalization of (3.4) imply a considerable amount about the entries in (3.5). First of all we wish to compute F^m_∞ as in (1.10). To do this, suppose

$$\textbf{(3.6)} \qquad N_*(\lambda_j) = \Sigma m_{jj'}\varphi_{j'}.$$

Then

$$\textbf{(3.7)} \qquad \int_{\lambda_j} - u\sum_{j'} m_{jj'}\int_{\varphi_{j'}}$$

is a well-defined function of $t = e^{2\pi i u}$, and F^m_∞ is obtained by replacing $\int_{\lambda_j}$ in (3.5) by the operator (3.7) and taking limits as $t \to 0$.

Now our assumption in (1.11) is that

$$(F^m_\infty + W_{2m-1}) \supseteq L^* = \Sigma \mathbf{C}\lambda^*_j$$

for L as in (3.3). But we have arranged that the $\left(\int_{\varphi_j}\omega_i(t)\right) \equiv$ Kronecker δ_{ij} and $\left(\int_{\varphi_j}\eta_k(t)\right) \equiv 0$ in (3.5). So replacing $\int_{\lambda_j}$ by (3.7), all entries in (3.5 i)) *stay bounded* as $t \to 0$. Thus in particular

$$\textbf{(3.8)} \qquad \int_{\lambda_j}\omega_i(t) = m_{ji}u + (\text{holo. fn. of } t).$$

Also, the fact that $F^m_\infty \cap W_{2m-1}$ induces a Hodge structure on E^* implies that all entries in (3.5) ii) are bounded and therefore holomorphic functions of t at $t = 0$. So we can rewrite (3.5) as follows:

$$\textbf{(3.9)} \quad \text{i)} \qquad \omega_i(t) = \lambda^*_i + \sum_j (m_{ij}u + \omega_{ij}(t))\varphi^*_j + \sum_\ell (\omega_{i\ell}(t))\varepsilon^*_\ell$$

$$\text{ii)} \qquad \eta_k(t) = \delta^*_k + \sum_j (\eta_{kj}(t))\varphi^* + \sum_\ell (\eta_{k\ell}(t))\varepsilon^*_\ell$$

where all functions of t on the right-hand-side are holomorphic at $t = 0$. We were able to replace m_{ji} with m_{ij} in the above formula because

$$m_{ji} = (\lambda_i . N_*\lambda_j) = (\lambda_i . T_*\lambda_j) = (T_*^{-1}\lambda_i . \lambda_j)$$
$$= (-N_*\lambda_i . \lambda_j) = (\lambda_j . N_*\lambda_i) = m_{ij}.$$

From (3.9) and the characterization of the canonical prolongation in ([Z]; p. 189), one concludes that the framing (3.9) of

$$\textbf{(3.10)} \qquad F^m R^{2m-1}\mu_*(\mathbf{C})$$

is in fact a framing defining the canonical prolongation of Deligne. Therefore Lemma (2.6) applies to the families of differentials $\{\omega_i(t), \eta_k(t)\}$.

Also if we use the dual basis to (3.9) to frame the dual bundle to (3.10), then the "Jacobian bundle"

$$\mathscr{J}^* \to \Delta^*$$

in (1.6) can be described as the bundle whose fibre over t is obtained by dividing the affine space $\mathbf{C}^{r+s}$ by the lattice generated by the columns of the matrix

$$\begin{bmatrix} I & 0 & (m_{ij}u + \omega_{ij}(t)) & (\omega_{i\ell}(t)) \\ 0 & I & (\eta_{kj}(t)) & (\eta_{k\ell}(t)) \end{bmatrix}$$

where, as always, $u = \frac{1}{2\pi i}\log t$. Let

$$\mathscr{J}' \to \Delta \tag{3.11}$$

be the analytic fibration obtained by filling in over $t = 0$ with the quotient of $\mathbf{C}^{r+s}$ by the partial lattice generated by the columns of

$$\begin{bmatrix} I & 0 & (\omega_{i\ell}(0)) \\ 0 & I & (\eta_{k\ell}(0)) \end{bmatrix}. \tag{3.12}$$

(The fact that these columns are indeed independent over $\mathbf{R}$ follows from the fact that $[I(\eta_{k\ell}(0))]$ is the period matrix for the Hodge structure of weight $2m-1$ on W_{2m-1}/W_{2m-2} in the asymptotic mixed Hodge structure.) The fibration (3.11) is as in ([Z]; p. 191).

Our next step is to enlarge the fibre over $t = 0$ in (3.11) as is done in the construction of the Neron model associated to the degeneration of a family of abelian varieties. The purpose is the same, namely, so that the sections of $\mathscr{J}^* \to \Delta^*$ considered in § 2

$$\int_{Z_i'}^{Z_i} \in J(V_t)$$

extend over $t = 0$.

To accomplish this we refer to (3.3) and define

$$\hat{L} = \{\lambda \in L \otimes \mathbf{Q} : \lambda \text{ has } \textit{integral} \text{ intersection number with each element of } N_* L\}. \tag{3.13}$$

Then the group $\hat{L}/L$ is naturally the dual of the group $E_{van}/N_* L$ and so has order equal to $\det(m_{ij})$ where (m_{ij}) is the non-degenerate symmetric matrix in (3.6). Furthermore the natural map

$$N_* : \hat{L}/L \to E_{van}/N_* L \tag{3.14}$$

is an isomorphism.

Since each element $\lambda \in \hat{L}$ is a section of $R_{2m-1}\mu_*(\mathbf{Q})$ which is invariant modulo elements of $R_{2m-1}\mu_*(\mathbf{Z})$, it gives a well-defined section

$$\Delta^* \to \mathscr{J}^*. \qquad t \mapsto \int_\lambda . \tag{3.15}$$

Such a section is zero if and only if $\lambda \in L$. Thus we have an isomorphism of $\hat{L}/L$ with a group of sections (3.15). From now on we will denote this group simply as

$$\mathscr{G}. \tag{3.16}$$

We are now ready to define the Neron model associated to the degeneration

(3.17) $\mathscr{J}^* \to \Delta^*$

of complex tori. We take $|\mathscr{G}|$ copies of $\mathscr{J}'$ in (3.11) and index them by the elements of $\mathscr{G}$. We identify a point x in the fibre of $\mathscr{J}'_{g_1}$ over $t \neq 0$ with a point y in the fibre of $\mathscr{J}'_{g_2}$ over the same t if and only if

(3.18) $x - y = g_1 - g_2$

in $J(V_t)$. The result is a smooth complex manifold

(3.19) $\mathscr{J} \to \Delta$

whose restriction to Δ^* is (3.17), and whose fibre J_0 over $t = 0$ fits into the exact sequence

(3.20) $0 \to J_0' \to J_0 \to \mathscr{G} \to 0$

where J_0' is the fibre of $\mathscr{J}'$ over $t = 0$.

By Lemma (2.6), if $\omega(t)$ is any section of the canonical prolongation of $F^m R^{2m-1}\mu_*(\mathbf{C})$, then

$$\int_\lambda \omega(t) - u\int_{N_*(\lambda)} \omega(t)$$

is a well-defined function of t holomorphic at $t = 0$ for each $\lambda \in \hat{L}$. Thus

(3.21) $u\int_{N_*(\lambda)}$

is a well-defined section of (3.19) which extends over $t = 0$. In fact (3.21) gives a section of (3.19) which passes through the same component of J_0 that $\int_\lambda$ does, that is, the component given by $g = \int_\lambda$.

So now suppose we have an analytic family of algebraic $(m-1)$-cycles

$$Z_t = Z_t' - Z_t''$$

as in (2.2). Let

$$\partial\Gamma_u = Z_t' - Z_t''$$

as before. By Lemma (2.5),

(3.22) $\Gamma_{u+1} - \Gamma_u = \varphi \in E_{\text{van}}$

so that there is $\lambda \in \hat{L}$ such that $N_*(\lambda) = \varphi$. By Lemma (2.6), the integrals

$$\int_{\Gamma_u} \omega(t) - u\int_\varphi \omega(t)$$

are bounded holomorphic functions of t for $\omega(t) \in \{\omega_i(t), \eta_k(t)\}$. Thus:

Theorem **(3.23)**. — *The Abel-Jacobi map*

$$\Delta^* \to \mathcal{J}^*$$
$$t \mapsto \int_{Z_t''}^{Z_t'} = \int_{\Gamma_u}$$

extends over $t = 0$ *to a section of*

$$\mathcal{J} \to \Delta$$

whose value at $t = 0$ *lies in the component of* J_0 *given by*

$$g = \int_\lambda \in \mathcal{G}$$

with $N_*\lambda = \Gamma_{u+1} - \Gamma_u$.

Since all our constructions can be carried out holomorphically with respect to auxiliary parameters we conclude:

Corollary **(3.24)**. — *The diagram* (1.6) *extends to a commutative diagram*

$$\begin{array}{ccc} \mathcal{S} & \longrightarrow & \mathcal{J} \\ & \searrow \quad \swarrow & \\ & \Delta & \end{array}$$

This last corollary is just the analytic analogue of the " universal property " of the Neron model in the algebraic case.

Appendix

The purpose of this appendix is to justify the assertion, made during the proof of Lemma (2.6), that the integrals

(A.1) $$\int_{\Gamma_u} \omega(t)$$

appear as coefficients, along $t' = 0$, of sections of the canonical prolongation of

(A.2) $$\{F^2 H^3(W_{(t,t')})\}_{(t,t') \in \Delta^* \times \Delta^*}$$

with respect to a flat basis of

(A.3) $$\{H^3(W_{(t,t')})_{\mathbf{Z}}\}_{(t,t') \in \Delta^* \times \Delta^*}.$$

It is this fact that allows us to conclude from ([D]; pp. 91-92) that the normal functions (A.1) have at worst logarithmic growth.

Let

$$\pi : \widetilde{\Delta} \times \widetilde{\Delta} \to \Delta^* \times \Delta^*$$
$$(u, u') \mapsto (t, t')$$

be the universal covering map as in § 2. Using π^*, we pull the bundle (A.3) back to a trivial bundle on $\widetilde{\Delta} \times \widetilde{\Delta}$ whose global sections will be denoted by $K^{\mathbf{Z}}$. There are two commuting, nilpotent endomorphisms of $K^{\mathbf{Z}}$, namely

$$N = \text{logarithm of monodromy around } t = 0,$$

$$N' = \text{logarithm of monodromy around } t' = 0.$$

Notice that $W_{(t,0)} = V_t \cup U_t$, the union of two smooth manifolds meeting transversely. So $(N')^2 = 0$, in fact, the topological part of our analysis in § 3 applies to the one-variable degeneration

$$W_{(t,t')} \to W_{(t,0)}.$$

We want to apply the Cattini-Kaplan-Schmid theory of asymptotic mixed Hodge structures to the two-parameter family (A.3). This theory says, first of all, that $K^{\mathbf{Z}}$ has a mixed Hodge structure whose weight filtration is defined by the nilpotent endomorphism $N + N'$, and whose Hodge filtration is given by

(A.4) $$F^* = \lim_{(t,t')\to(0,0)} \exp(-uN - u'N')\pi^*(F^*H^3(W_{(t,t')})).$$

If $H^{\mathbf{Z}}$ is as in (1.8)-(1.11), then there is a subquotient of $K^{\mathbf{Z}}$ which can be identified with $H^{\mathbf{Z}}$. Namely let

(A.5) $$Gr_3^{N'} = \frac{\text{kernel}(N' : K^{\mathbf{Z}} \to K^{\mathbf{Z}})}{((\text{image } N') \otimes \mathbf{Q}) \cap K^{\mathbf{Z}}}.$$

The filtrations on $H^{\mathbf{Z}} \otimes \mathbf{C}$ induced by the weight and Hodge filtrations on $K^{\mathbf{Z}} \otimes \mathbf{C}$ define a mixed Hodge structure on (A.5) which is isomorphic to (the mixed Hodge structure on)

(A.6) $$H^{\mathbf{Z}} \oplus G^{\mathbf{Z}}.$$

Here $H^{\mathbf{Z}}$ has the mixed Hodge structure in (1.8)-(1.11), and $G^{\mathbf{Z}}$ has the asymptotic mixed Hodge structure for the family $\{U_t\}_{t\in\Delta^*}$ constructed in the proof of Lemma (2.6).

Now let φ_j^* and ε_l^* be as in (3.5). These are flat, N-invariant sections of $\{H^3(V_t)_{\mathbf{Z}}\}_{t\in\Delta^*}$ and can be extended to flat sections of (A.3) which are both N and N′ invariant. We compute the limit (A.4) in two steps, first letting t' go to zero and then letting t go to zero. As in § 3, we see that after the first step, the vector space

$$F_t^2 = \lim_{t'\to 0} \exp(-u'N')\pi^*F^2H^3(W_{(t,t')})$$

can be written as the direct sum of two subspaces,

(A.7) $$M_t = (F_t^2 \cap \{\Sigma\mathbf{Z}\varphi_j^* + \Sigma\mathbf{Z}\varepsilon_l^*\}^{\perp})$$

and a second subspace, which we will call

(A.8) $$L_t,$$

which lies in $(\ker N') \otimes \mathbf{C}$ and gives $F^2H^3(V_t)$ in the weight three graded quotient of the asymptotic mixed Hodge structure associated to the one-parameter family

$$\textbf{(A.9)} \qquad W_{(t,t')} \to W_{(t,0)} = V_t \cup U_t.$$

The important point is that

$$L = \lim_{t\to 0} \exp(-uN)L_t$$

and

$$M = \lim_{t\to 0} \exp(-uN)M_t$$

is a direct-sum decomposition of F^2 (see (A.4)). This is because

i) $\dim L_t = \frac{1}{2} \dim H^3(V_t) = \dim\{\Sigma\mathbf{C}\varphi_j^* + \Sigma\mathbf{C}\varepsilon_\ell^*\} = \dim F^2(W_3(H^{\mathbf{C}}))$;

ii) $M \subseteq \{\Sigma\mathbf{C}\varphi_j^* + \Sigma\mathbf{C}\varepsilon_\ell^*\}^\perp$ and so, by § 3, $M \cap (\ker N') \otimes \mathbf{C}$ projects to zero in $F^2(W_3(H^{\mathbf{C}}))$.

Therefore there is a framing $\omega(t)$ of L_t which extends to a partial framing $\omega(t, t')$ of the canonical prolongation of (A.2). These are the differentials which occur in the proof of Lemma (2.6).

The differentials

$$\omega(t)\Big|_{V_t} + 0\Big|_{U_t}$$

framing L_t are well-defined modulo

$$F^2((\text{image } N') \otimes \mathbf{C})$$

by the fact that they are dual to the basis $\{\varphi_j\}$, $\{\varepsilon_\ell\}$ of $F^2(W_3(H^{\mathbf{C}}))^*$. More precisely, these differentials map isomorphically to a framing of L_t under the natural morphism of mixed Hodge structures

$$H^3(V_t \cup U_t) \to (\text{asymptotic mixed Hodge structure for the family (A.9)}).$$

For Γ_u as in (A.1), $\partial\Gamma_u$ is algebraic so that the integral (A.1) will occur as the coefficient of an *algebraic* basis element of

$$(\text{image } N') \otimes \mathbf{Q} \underset{\S 3}{=} \frac{H^2(U_t \cap V_t)_{\mathbf{Q}}}{\{\text{hyperplane section}\}}.$$

So if we change the framing $\omega(t)$ by an element of

$$F^2((\text{image } N') \otimes \mathbf{C}),$$

the value of the coefficient is unchanged.

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Manuscrit reçu le 12 janvier 1981.
Révisé le 5 mai 1982.