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JONATHAN ROSENBERG

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C^*-ALGEBRAS, POSITIVE SCALAR CURVATURE, AND THE NOVIKOV CONJECTURE

by JONATHAN ROSENBERG (1)

This note, an appendix to the foregoing paper [10], is based on the philosophy of [9] that topological obstructions to existence of metrics of positive scalar curvature on non-simply connected manifolds should be closely related to the Novikov conjecture on " higher signatures ". I am very grateful to Mikhael Gromov and to Blaine Lawson for patiently explaining their work to me, for suggesting that I examine its relation to the work of Kasparov and of Miščenko *et al.*, and for helping with many details of this project. I would also like to thank Alain Connes, Paul Baum, Ron Douglas, Jerry Kaminker, John Miller and Larry Taylor for helpful discussions concerning index theorems and/or the Novikov conjecture, and Alain Connes and Mikhael Gromov for useful suggestions about improvements in the exposition of this paper.

We shall show below that certain manifolds do not admit Riemannian metrics of positive scalar curvature. In most situations involving closed manifolds of practical geometric interest, our results of this type will be the same as those of [9] and of Corollary A and Theorem J of [10]. Nevertheless, it is not clear that their results always imply ours or *vice versa*. More significantly, we hope to illustrate another way in which operator algebras can be applied to problems of geometry and topology, and to demonstrate a closer relationship between the positive scalar curvature problem and the Novikov conjecture than may be apparent from the Gromov-Lawson method of attack.

Our main results are Theorems 3.3 and 3.5 below. However, their proofs involve technical complications that tend to obscure the main idea, so we have first given the proof of an easier special case, which we have formulated as Theorem 2.11. The auxiliary results 2.4-2.10 on the Novikov conjecture (especially Theorem 2.6) may be of independent interest.

1. The main construction

As in [9] and [10], our tool will be an analysis of the index of the Dirac operator with coefficients in a bundle, together with the idea of the vanishing theorems of [19]

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and [11]. But whereas Gromov and Lawson use for coefficients ordinary vector bundles of *small* curvature, we shall use flat coefficient bundles (*i.e.* bundles of *zero* curvature), but only at the expense of making the fibres of the bundles infinite-dimensional. This can be explained by the fact that (by Chern-Weil theory) ordinary flat bundles have trivial rational characteristic classes, whereas infinite-dimensional flat bundles sometimes do not. Our main technical tool will therefore be not the Atiyah-Singer index theorem but rather the index theorem of Miščenko and Fomenko [23]. Thus we begin with a review of the key definitions and concepts in the Miščenko-Fomenko theory.

Let A be a (complex) C^*-algebra with unit. If X is a compact space, by an A-*vector bundle* over X we shall mean a locally trivial (Banach) vector bundle E over X, in which the fibres have the structure of finitely generated projective (left) A-modules. Morphisms of such bundles will be required to preserve the A-module structure on the fibres. The Grothendieck group of formal differences of equivalence classes of such bundles (with addition coming from the Whitney sum operation) will be denoted $K^0(X, A)$; it coincides with the algebraic K-group $K_0(C(X) \otimes A)$, where $\otimes$ means the (spatial) C^*-tensor product. (Of course, $C(X) \otimes A \cong C(X, A)$. For all this see not only [23] but also [13], Exercise II.6.14, and [21], Ch. 1.) By the Künneth theorem of [27], or rather by a relatively easy special case thereof, for compact metrizable X there is a natural isomorphism

$$K^0(X, A) \otimes \mathbf{Q} \cong (K^0(X) \otimes K_0(A) \otimes \mathbf{Q}) \oplus (K^1(X) \otimes K_1(A) \otimes \mathbf{Q})$$

(where the tensor products are taken over $\mathbf{Z}$). Thus using the ordinary Chern character $K^*(X) \to H^*(X, \mathbf{Q})$, we obtain the Chern character of Miščenko-Solov'ev [24]:

$$\mathrm{ch}: K^0(X, A) \to H^{\mathrm{even}}(X, \mathbf{Q}) \otimes K_0(A) \oplus H^{\mathrm{odd}}(X, \mathbf{Q}) \otimes K_1(A),$$

which is an isomorphism modulo torsion.

Now suppose X is a closed manifold (for us this will always mean a compact connected C^∞ manifold without boundary), E and F are smooth A-vector bundles over X, and D is an elliptic pseudodifferential A-operator of order n in the sense of [23], § 3, taking smooth sections of E to smooth sections of F. One can define Sobolev spaces $H^s(X, E)$ and $H^s(X, F)$, which are Hilbert C^*-modules (see [23], § 1, or [15]) over A. The fundamental theorem of [23] asserts that D defines a bounded operator $H^s(X, E) \to H^{s-n}(X, F)$, that this operator is an A-module map, with adjoint, which is A-Fredholm (*i.e.* invertible modulo A-*compact* operators), and that the "A-index" of D is independent of the choice of s. One must be careful with the definition of index here; the kernel and cokernel of D are not necessarily finitely generated and A-projective, but this is true of $D + K$ for some A-compact operator K. The A-*index* of D is then the formal difference (in $K_0(A)$):

$$[\ker(D + K)] - [\operatorname{coker}(D + K)],$$

which turns out to be independent of the choice of K ([23], § 1, and [21], Ch. 1). Finally, the A-index of D depends only on the class of the principal symbol of D in $K^0(T^*X, A)$

(where T^*X is the cotangent bundle of X, and where for locally compact spaces, K^* denotes K-theory with compact supports). The A-index is, modulo torsion, given by a formula formally identical to that of Atiyah-Singer [4], the only difference being that the Chern character must be interpreted as just explained above.

We now apply this machinery to the case of spin manifolds of positive scalar curvature. If π is a discrete group, we denote by $B\pi$ a classifying space for π, *i.e.* a $K(\pi, 1)$-space. For any (connected) manifold M with fundamental group π, the universal covering of M is associated to a classifying map $f: M \to B\pi$, well-defined up to homotopy and inducing the identity on π_1.

THEOREM 1.1. — *Let* M *be a compact smooth even-dimensional spin manifold without boundary, and suppose* M *admits a Riemannian metric whose scalar curvature is everywhere non-negative and strictly positive at at least one point. Let* A *be any* C*-*algebra with unit and* $\mathscr{V}$ *a flat (*i.e. *pulled back from* $B\pi$, *where* $\pi = \pi_1(M)$*)* A-*vector bundle over* M. *Then*

$$\langle \hat{\mathbf{A}}(M) \cup \mathrm{ch}[\mathscr{V}], [M] \rangle = 0 \text{ in } K_0(A) \otimes \mathbf{Q}.$$

PROOF. — By Proposition 3.8 of [18], we may assume that M has been given a Riemannian metric whose scalar curvature κ is everywhere strictly positive, hence bounded below by a constant $\kappa_0 > 0$. Let $\mathscr{S}^+$ and $\mathscr{S}^-$ be the two half-spinor bundles of M. We may give $\mathscr{V}$ a flat connection and thus define Dirac operators

$$\begin{cases} D^+ : \Gamma^\infty(\mathscr{S}^+ \otimes \mathscr{V}) \to \Gamma^\infty(\mathscr{S}^- \otimes \mathscr{V}) \\ D^- : \Gamma^\infty(\mathscr{S}^- \otimes \mathscr{V}) \to \Gamma^\infty(\mathscr{S}^+ \otimes \mathscr{V}) \end{cases}$$

which are formal adjoints of one another with respect to the A-valued inner products. These are elliptic A-operators, and the usual calculation gives

$$\mathrm{Th}^{-1}(\mathrm{Td}(M) \cup \mathrm{ch}[\sigma(D^+)]) = \hat{\mathbf{A}}(M) \cup \mathrm{ch}[\mathscr{V}],$$

where $\hat{\mathbf{A}}(M)$ is the total Â-class of M, a certain polynomial in the rational Pontrjagin classes, and where $\mathrm{Th}: H^*(M, \mathbf{Q}) \to H_c^*(T^*M, \mathbf{Q})$ is the Thom isomorphism. Thus by the Miščenko-Fomenko Theorem, we get

$$\mathrm{ind}_A D^+ = \langle \hat{\mathbf{A}}(M) \cup \mathrm{ch}[\mathscr{V}], [M] \rangle \in K_0(A) \otimes \mathbf{Q}.$$

We claim, however, that D^+ has A-index zero. Indeed, with respect to the A-valued L^2-inner products on $\Gamma(\mathscr{S}^\pm \otimes \mathscr{V})$, we have

$$D^\mp D^\pm = \nabla^* \nabla + \frac{\kappa}{4},$$

by exactly the same calculation as in [9], § 1, and in [10], § 1. (There is no $\mathscr{R}_0$ term since $\mathscr{V}$ is flat.) We can conclude from this (together with the fact that $\kappa \geq \kappa_0 > 0$) that D^+D^- and D^-D^+ are one-to-one with dense ranges and bounded inverses. Here we must use spectral theory and the fact that if a and b are positive self-adjoint elements of a C*-algebra, then $a + b \geq b$ and hence $\|a + b\| \geq \|b\|$.

To justify this step precisely, one can for instance first embed $\mathscr{V}$ as a direct summand in a trivial bundle of the form $M \times A^n$ (using [21], Proposition 4.3). The operator $\nabla^*\nabla + \frac{\kappa}{4}$ extends in an obvious way to $\Gamma^\infty(\mathscr{S}^\pm \otimes A^n)$, and since the coefficient bundle is trivial, we can use ordinary spectral theory on the first factor. Since $\nabla^*\nabla + \frac{\kappa}{4} \geq \kappa_0/4$, we conclude that D^+D^- and D^-D^+ have bounded inverses with respect to the L^2-norms. A compact perturbation of D^+ will then be invertible as an operator from $H^{s+1}(\mathscr{S}^+)$ to $H^s(\mathscr{S}^-)$, and tensoring with the identity operator on A^n, we see that D^+ has A-index zero. Hence $\langle \hat{\mathbf{A}}(M) \cup \mathrm{ch}[\mathscr{V}], [M]\rangle = 0$.

2. C*-algebras and the Novikov Conjecture

The usefulness of Theorem 1.1 depends on the possibilities for $\mathrm{ch}[\mathscr{V}]$, *i.e.* for the characteristic classes of flat A-vector bundles over M. The best one could hope for is that these might generate all the "characteristic cohomology" of M, *i.e.* that if $f: M \to B\pi$ is the classifying map for the universal covering of M, one has

CONJECTURE **2.1** (Vanishing of higher Â-genera). — *If* M *is a closed spin manifold admitting a metric of positive scalar curvature, and if* $\pi \cong \pi_1(M)$, *then for all* $a \in H^*(B\pi, \mathbf{Q})$,

$$\langle \hat{\mathbf{A}}(M) \cup f^*(a), [M]\rangle = 0.$$

Note that Conjecture 2.1 can be rephrased in the form $f_*(\hat{\mathbf{A}}(M) \cap [M]) = 0$. Modulo torsion, this is the same as the conjecture in [10] that the image of the class of (M, f) must vanish under the "$\hat{\mathscr{A}}$-homomorphism" $\Omega_*^{\mathrm{Spin}}(B\pi) \to KO_*(B\pi)$.

We shall see now that Conjecture 2.1 follows from the same intermediate step needed by Kasparov and by Miščenko in their proofs of the Novikov Conjecture for certain fundamental groups π. Thus Conjecture 2.1 is valid for many (and conceivably all) finitely presented groups π. One can deduce from this that spin manifolds of certain homotopy types do not admit metrics of positive scalar curvature, and that within other homotopy types of manifolds, such metrics may exist but only subject to certain constraints on the Pontrjagin classes. We shall also see in § 3B below that the somewhat annoying spin condition in 2.1 can be relaxed to give

CONJECTURE **2.2**. — *If* M *is a closed oriented manifold admitting a metric of positive scalar curvature, if the universal covering of* M *is spin, and if* $\pi = \pi_1(M)$, *then for all* $a \in H^*(B\pi, \mathbf{Q})$, $\langle \hat{\mathbf{A}}(M) \cup f^*(a), [M]\rangle = 0$.

COROLLARY (TO THE CONJECTURE) **2.3**. — *No closed* $K(\pi, 1)$*-manifold can admit a metric of positive scalar curvature.*

PROOF OF 2.3 FROM 2.2. — Let M be a closed $K(\pi, 1)$-manifold of positive scalar curvature. If M is not orientable, pass to a double covering and replace π by a subgroup of index 2. We can take $M = B\pi$ and $f = id$, and of course the universal covering of M is spin since it is contractible. Taking $a \neq 0$ in the top-degree cohomology of M, we get $\langle a, [M] \rangle = 0$, a contradiction.

It seems that Conjecture 2.1 (for a fixed finitely presented group π) is closely related to the (generalized) Novikov Conjecture for the same group, which we state here for reference:

NOVIKOV CONJECTURE. — *Among the class of closed oriented manifolds* M *with fundamental group* $\pi_1(M) \cong \pi$, *if* $f: M \to B\pi$ *denotes the classifying map, then the " higher signatures "* $\langle \mathbf{L}(M) \cup f^*(a), [M] \rangle$, $a \in H^*(B\pi, \mathbf{Q})$, *are oriented homotopy invariants.*

Despite the obvious similarity between higher signatures and higher $\hat{A}$-genera, we know of no direct way of deducing Conjecture 2.1 from the Novikov Conjecture. However, it seems that for most groups for which one can prove one of these, one can prove the other as well. It is therefore worth briefly summarizing some of the ideas of [17], § 9.

If π is any countable group, one can choose the space $B\pi$ to be a countable CW-complex (but usually not to be a finite complex). Following Kasparov, we denote by RK^* representable topological K-theory, the representable cohomology theory coinciding with K^* on compact spaces. For a countable CW-complex X with finite skeletons X_n, one can also define $LK^*(X) = \varprojlim K^*(X_n)$, and there is a surjection $RK(X) \to LK^*(X)$ (with kernel given by $\varprojlim^{(1)} K^{*-1}(X_n)$ [20]). We denote by K_* the homology theory on compact metrizable spaces dual to K^*, and let $RK_*(X) = \varinjlim K_*(X_n)$ for X as above. For a separable C*-algebra A, we denote by $K^*(A)$ the group $KK^*(A, \mathbf{C})$ " dual " to $K_*(A) = KK^*(\mathbf{C}, A)$. (Caution: our convention about raised and lowered indices is reversed from that of [16] and [17], in order to agree with the usual convention that K^* should denote a contravariant functor. Thus $K^*(C(X)) \cong K_*(X)$.) As in [16], $\otimes_A$ denotes the " intersection product " $K_*(A) \otimes K^*(A) \to K_*(\mathbf{C}) = \mathbf{Z}$, generalizing the Kronecker pairing between K-homology and K-cohomology. If $C^*(\pi)$ denotes the group C*-algebra of π (the completion of $L^1(\pi)$ in the greatest C*-norm), Kasparov defines maps

$$\alpha : K^*(C^*(\pi)) \to RK^*(B\pi)$$

and

$$\beta : RK_*(B\pi) \to K_*(C^*(\pi))$$

which are dual to one another in the sense that if $\tilde{\alpha}$ denotes the projection of α into $LK^*(B\pi)$, then $\tilde{\alpha}(x) \otimes_{B\pi} y = x \otimes_{C^*(\pi)} \beta(y)$ for $x \in K^*(C^*(\pi))$, $y \in RK_*(B\pi)$. Consider now the following possible properties of π:

PROPERTY SNC 1. — The image of $\tilde{\alpha}$ (after tensoring with $\mathbf{Q}$) is dense in the projective limit topology on $LK^*(B\pi) \otimes \mathbf{Q}$.

PROPERTY SNC 2. — The map β is injective after tensoring with $\mathbf{Q}$.

PROPERTY SNC 3. — The map β is injective (even without tensoring with $\mathbf{Q}$).

PROPERTY SNC 4. — The map β is an isomorphism from $RK_*(B\pi) \to K_*(C^*(\pi))$.

Here the abbreviation " SNC " stands for " Strong Novikov Conjecture ", terminology which is defensible because of the string of implications

$$(\text{SNC } 4) \Rightarrow (\text{SNC } 3) \Rightarrow (\text{SNC } 2) \Leftarrow (\text{SNC } 1),$$

and the theorem of Kasparov ([17], § 9) that if π is finitely presented (so that the Novikov Conjecture for π makes sense), then SNC 1 or SNC 2 implies the Novikov Conjecture for π (in the form stated above). This can be explained as follows. Suppose M is a closed oriented manifold with fundamental group π. Then we can form the " universal flat $C^*(\pi)$-vector bundle " $\mathscr{V} = C^*(\pi) \times_\pi \widetilde{M}$ over M, where $\widetilde{M}$ is the universal covering of M and π acts on $C^*(\pi)$ by right translation and on $\widetilde{M}$ by covering transformations. The *generalized signature* of M is defined by $\langle \mathbf{L}(M) \cup \mathrm{ch}[\mathscr{V}], [M] \rangle$; this may be thought of as the $C^*(\pi)$-index of the signature operator with coefficients in $\mathscr{V}$ (in the case where $\dim M \equiv 0 \pmod 4$) (¹). By [22] or by [17], § 9, Theorem 2, this generalized signature is an oriented homotopy invariant. Applying $\otimes_{C^*(\pi)} y$ for classes $y \in K^*(C^*(\pi))$, it follows that $\langle \mathbf{L}(M) \cup f^*(a), [M] \rangle$ is an oriented homotopy invariant if $a \in H^*(B\pi, \mathbf{Q})$ lies in the image (under the Chern character) of the image of α, and hence SNC implies the Novikov Conjecture.

In the remainder of this section we list a number of situations where various forms of SNC are known, and then broaden the list a bit by discussing various stability properties of the class of groups with SNC. (One may view this as a C^*-algebraic counterpart to [6].) Then we describe consequences for the positive-scalar-curvature problem. The fundamental (and deep) results are the following:

THEOREM (Miščenko, see [12]). — *Suppose there exists a closed orientable* $K(\pi, 1)$-*manifold admitting a Riemannian metric with non-positive sectional curvatures. Then* SNC 1 *holds for* π.

THEOREM (Kasparov, see [17], § 9, Theorem 1). — *Suppose* π *can be embedded as a discrete subgroup in a connected Lie group. Then* SNC 1 *holds for* π *(even without tensoring with* $\mathbf{Q}$*).*

One should note that Miščenko's theorem is stated in terms of existence of certain Fredholm representations, but the above formulation is equivalent. Also, these results appear to lend support to Conjecture 2.3, since there do not seem to be any published examples of closed oriented $K(\pi, 1)$-manifolds for which the hypotheses of both theorems

(¹) We are cheating slightly. One should really replace the Hirzebruch class $\mathbf{L}(M)$ here by the Atiyah-Singer modification ([4], § 6) of it, $\mathscr{L}(M)$, and then multiply in front by a suitable power of 2. However, for purposes of formulating the Novikov Conjecture, it does not matter whether one uses $\mathbf{L}$ or $\mathscr{L}$.

fail. However, M. Gromov has suggested that there are probably many $K(\pi, 1)$-manifolds not satisfying these conditions, so that we are still far from a proof of 2.3.

For purposes of proving stability properties of SNC, there is a considerable technical advantage to working with SNC 2-SNC 4 rather than with SNC 1. The reason is that the functors RK_* and (for C*-algebras) K_* commute with direct limits, whereas RK^* and K^* do not. This makes it possible to avoid dealing with the Milnor $\varprojlim^{(1)}$ sequence and with problems arising from I-adic completion.

PROPOSITION 2.4. — *Suppose π is a countable direct limit $\pi = \varinjlim \pi_n$ and the groups π_n all satisfy* SNC 2, SNC 3, *or* SNC 4. *Then the same holds for π.*

PROOF. — We have $B\pi = \varinjlim B\pi_n$ and $C^*(\pi) = \varinjlim C^*(\pi_n)$ (C*-algebra inductive limit). So use naturality of β and the fact that RK_* and K_* commute with direct limits.

PROPOSITION 2.5. — *Suppose π is an extension*

$$1 \to \pi_1 \to \pi \to \mathbf{Z} \to 1$$

and SNC 4 *holds for the group π_1. Then* SNC 4 *holds for π.*

PROOF (cf. [26], Theorem 3.3). — Since $\mathbf{Z}$ is free, the group extension must split, and we can write $\pi = \pi_1 \rtimes \mathbf{Z}$, where the semidirect product structure is determined by an automorphism θ of π_1. This gives a corresponding decomposition of $C^*(\pi)$ as a crossed product $C^*(\pi_1) \rtimes_\theta \mathbf{Z}$ and of $B\pi$ as a fibre bundle over $S^1 = B\mathbf{Z}$ with fibres $\cong B\pi_1$. Then we may compute $K_*(C^*(\pi))$ from $K_*(C^*(\pi_1))$ via the Pimsner-Voiculescu exact sequence [25] and $RK_*(B\pi)$ as the homology of the mapping torus of the self-map θ_* of $B\pi_1$. As in the proof of [26], Theorem 3.3, we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
K_i(C^*(\pi_1)) & \xrightarrow{\theta_*-1} & K_i(C^*(\pi_1)) & \longrightarrow & K_i(C^*(\pi)) & \xrightarrow{\partial} & K_{i-1}(C^*(\pi_1)) & \xrightarrow{\theta_*-1} & K_{i-1}(C^*(\pi_1)) \\
\uparrow{\scriptstyle\beta_1} & & \uparrow{\scriptstyle\beta_1} & & \uparrow{\scriptstyle\beta} & & \uparrow{\scriptstyle\beta_1} & & \uparrow{\scriptstyle\beta_1} \\
RK_i(B\pi_1) & \xrightarrow{\theta_*-1} & RK_i(B\pi_1) & \longrightarrow & RK_i(B\pi) & \xrightarrow{\partial} & RK_{i-1}(B\pi_1) & \xrightarrow{\theta_*-1} & RK_{i-1}(B\pi_1)
\end{array}$$

Thus if the maps β_1 are isomorphisms, so is β, by the 5-Lemma.

REMARK. — Because of the need to use the 5-Lemma, it is clear that the proof of 2.5 breaks down if we replace SNC 4 by SNC 3 or SNC 2.

THEOREM 2.6. — *If π is a countable solvable group having a composition series in which the composition factors are torsion-free abelian, then* SNC 4 *holds for π.*

PROOF. — We argue by induction on the length of such a composition series. Suppose $\pi_1 \lhd \pi$ and π/π_1 is torsion-free abelian. We may assume that the theorem already holds for π_1. (To start the induction, take $\pi_1 = \{1\}$ if π is abelian.) Now π/π_1 is the direct limit of its finitely generated subgroups, hence is a countable direct limit of free abelian groups. Thus $\pi = \varinjlim \pi_n$, where $\pi_1 \lhd \pi_n$ for each n and where π_n/π_1 is free abelian. By 2.4, it is enough to prove the theorem for π_n. For this it is enough to use the inductive hypothesis and 2.5 (repeated a finite number of times).

REMARK. — Theorem 2.6 applies to some cases not covered by Kasparov's theorem, since there are many groups (even finitely presented) satisfying the hypothesis of 2.6 but not polycyclic.

PROPOSITION **2.7**. — *Suppose π contains a subgroup π_1 of finite index for which* SNC 2 *holds. Then* SNC 2 *holds for π.*

PROOF. — The group π_1 acts freely on $E\pi$ so we obtain a finite covering $E\pi/\pi_1 \sim B\pi_1 \xrightarrow{p} E\pi/\pi = B\pi$. (If π_1 is normal then the covering is Galois, but this is irrelevant for our purposes.) We also have an inclusion $i : C^*(\pi_1) \hookrightarrow C^*(\pi)$ making $C^*(\pi)$ into a finitely generated free $C^*(\pi_1)$-module. Thus we have maps $i_* : K_*(C^*(\pi_1)) \to K_*(C^*(\pi))$ as well as a transfer map $t : K_*(C^*(\pi)) \to K_*(C^*(\pi_1))$ (coming from the forgetful functor from finitely generated projective $C^*(\pi)$-modules to finitely generated projective $C^*(\pi_1)$-modules). Note that $t \circ i_*$ is multiplication by $[\pi : \pi_1]$, hence i_* is split mono after tensoring with $\mathbf{Q}$ (to invert $[\pi : \pi_1]$). Arguing similarly with the transfer in K-homology (see, *e.g.*, [1], Ch. 4), we see $p_* : RK_*(B\pi_1) \to RK_*(B\pi)$ is split epi after tensoring with $\mathbf{Q}$. The conclusion follows from consideration of the commutative diagram

$$\begin{tikzcd} RK_*(B\pi_1) \otimes \mathbf{Q} \arrow[r, "\beta_1"] \arrow[d, "p_*"] & K_*(C^*(\pi_1)) \otimes \mathbf{Q} \arrow[d, "i_*"] \\ RK_*(B\pi) \otimes \mathbf{Q} \arrow[r, "\beta"] & K_*(C^*(\pi)) \otimes \mathbf{Q}. \end{tikzcd}$$

PROPOSITION **2.8**. — *Suppose π contains a finite normal subgroup π_1 and* SNC 2 *holds for π/π_1. Then* SNC 2 *holds for π.*

PROOF. — Consider the commutative diagram

$$\begin{tikzcd} RK_*(B\pi) \otimes \mathbf{Q} \arrow[r, "\beta"] \arrow[d] & K_*(C^*(\pi)) \otimes \mathbf{Q} \arrow[d] \\ RK_*(B(\pi/\pi_1)) \otimes \mathbf{Q} \arrow[r, "\beta'"] & K_*(C^*(\pi/\pi_1)) \otimes \mathbf{Q} \end{tikzcd}$$

If β' is mono, then to show β is mono, it is enough to show that

$$\mathrm{RK}_*(\mathrm{B}\pi) \otimes \mathbf{Q} \to \mathrm{RK}_*(\mathrm{B}(\pi/\pi_1)) \otimes \mathbf{Q}$$

is mono, or that $\mathrm{H}_*(\mathrm{B}\pi, \mathbf{Q}) \to \mathrm{H}_*(\mathrm{B}(\pi/\pi_1), \mathbf{Q})$ is mono. But this follows from collapsing of the Lyndon-Hochschild-Serre spectral sequence $\mathrm{H}_*(\pi/\pi_1, \mathrm{H}_*(\pi_1, \mathbf{Q})) \Rightarrow \mathrm{H}_*(\pi, \mathbf{Q})$.

PROPOSITION **2.9**. — *Suppose* π_1 *and* π_2 *are groups for which* SNC 2 *(resp.,* SNC 4*) holds, and also suppose* $\mathrm{C}^*(\pi_1)$ *is in the category of* [27] *for which the Künneth theorem holds (this is the case, for instance, if* π_1 *satisfies the hypothesis of* 2.6, *but is never the case if* π_1 *is not amenable). Then* SNC 2 *(resp.,* SNC 4*) holds for the direct product* $\pi_1 \times \pi_2$.

PROOF. — Using the Künneth theorems for K_* of C*-algebras [27] and for RK_* of spaces (which follows by Spanier-Whitehead duality and passage to limits from the Künneth theorem [2] for K^* of finite complexes), we obtain a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{RK}_*(\mathrm{B}\pi_1) \otimes \mathrm{RK}_*(\mathrm{B}\pi_2) \arrow[r] \arrow[d, "\beta_1 \otimes \beta_2"] & \mathrm{RK}_*(\mathrm{B}(\pi_1 \times \pi_2)) \arrow[r] \arrow[d, "\beta"] & \mathrm{Tor}(\mathrm{RK}_*(\mathrm{B}\pi_1), \mathrm{RK}_*(\mathrm{B}\pi_2)) \arrow[r] \arrow[d, "\beta_1 * \beta_2"] & 0 \\
0 \arrow[r] & \mathrm{K}_*(\mathrm{C}^*(\pi_1)) \otimes \mathrm{K}_*(\mathrm{C}^*(\pi_2)) \arrow[r] & \mathrm{K}_*(\mathrm{C}^*(\pi_1 \times \pi_2)) \arrow[r] & \mathrm{Tor}(\mathrm{K}_*(\mathrm{C}^*(\pi_1)), \mathrm{K}_*(\mathrm{C}^*(\pi_2))) \arrow[r] & 0
\end{tikzcd}$$

The result for SNC 4 follows by diagram chasing. The case of SNC 2 is even easier since we can disregard the Tor terms. Note, however, that the argument breaks down for SNC 3, since β_1 and β_2 mono does not imply $\beta_1 \otimes \beta_2$ is mono.

PROPOSITION **2.10**. — *Suppose* π_1 *and* π_2 *are groups for which* SNC 2 *(resp.,* SNC 3, SNC 4*) holds. Then* SNC 2 *(resp.,* SNC 3, SNC 4*) holds for the free product* $\pi_1 * \pi_2$.

PROOF. — We have, of course,

$$\mathrm{B}(\pi_1 * \pi_2) = \mathrm{B}\pi_1 \vee \mathrm{B}\pi_2 \quad \text{and} \quad \mathrm{C}^*(\pi_1 * \pi_2) = \mathrm{C}^*(\pi_1) \underset{\mathbf{C}}{*} \mathrm{C}^*(\pi_2)$$

(the free product of the C*-algebras amalgamated over the common identity element). The result now follows trivially from [8].

REMARK. — Undoubtedly there are results like 2.9 and 2.10 for direct products of more general groups and for certain amalgamated free products. However, one encounters serious technical difficulties (such as non-uniqueness of C*-cross-norms on tensor products of non-nuclear algebras); and in the case of amalgamated products, [6] suggests the problems may be more than technical.

It should be evident from the Miščenko and Kasparov theorems, as well as from 2.4-2.10, that the Strong Novikov Conjecture holds for quite a large class of groups.

We proceed to the applications to manifolds of positive scalar curvature. With appropriate conditions on π, we obtain slightly more than Conjecture 2.1.

THEOREM 2.11 (cf. [10], Theorem 13.8). — *Assume the Strong Novikov Conjecture* (SNC 2) *holds for* π, *and suppose* M *is a closed spin manifold and* $g : M \to B\pi$ *a continuous map. Then if* M *admits a metric of positive scalar curvature,* $g_*(\hat{\mathbf{A}}(M) \cap [M]) = 0$ *in* $H_*(B\pi, \mathbf{Q})$, *or equivalently, for all* $a \in H^*(B\pi, \mathbf{Q})$,

$$\langle \hat{\mathbf{A}}(M) \cup g^*(a), [M] \rangle = 0.$$

In particular, if $g_*([M]) \neq 0$ *in* $H_*(B\pi, \mathbf{Q})$, *then* M *(or even any closed manifold homotopy equivalent to* M*) does not admit a metric of positive scalar curvature.*

NOTE. — In Theorem 3.5 below, we will show that the condition that $w_2(M) = 0$ can be weakened to the assumption that M has a spin covering.

PROOF. — Suppose M admits a metric of positive scalar curvature and g is as indicated. Then g is determined up to homotopy by a homomorphism $\pi_1(M) \to \pi$, and hence the pull-back under g of the universal $C^*(\pi)$-bundle $\mathscr{V} = C^*(\pi) \underset{\pi}{\times} E\pi$ may be viewed as a flat bundle over M, which can be taken to be smooth. First assume M is even-dimensional. Applying Theorem 1.1 to $g^*(\mathscr{V})$ with $A = C^*(\pi)$, we get

$$\langle \hat{\mathbf{A}}(M) \cup g^*(\mathrm{ch}[\mathscr{V}]), [M] \rangle = 0,$$

or
$$0 = \langle g^*(\mathrm{ch}[\mathscr{V}]), \hat{\mathbf{A}}(M) \cap [M] \rangle = \langle \mathrm{ch}[\mathscr{V}], g_*(\hat{\mathbf{A}}(M) \cap [M]) \rangle,$$

i.e. $\beta(g_*(\mathrm{ch}^{-1}(\hat{\mathbf{A}}(M) \cap [M])))$. But ch is an isomorphism modulo torsion so this together with SNC 2 says $g_*(\hat{\mathbf{A}}(M) \cap [M]) = 0$ in $H_*(B\pi, \mathbf{Q})$. If $g_*([M]) \neq 0$, this is impossible, since

$$\hat{\mathbf{A}}(M) = 1 + (\text{terms of degree} \geq 4),$$

and in fact only the homotopy type of M is relevant in this case (since we've used the orientation of M but not the Pontrjagin classes).

If M is odd-dimensional, consider instead $N = M \times S^1$ with the product metric obtained from a metric of positive scalar curvature on M and the flat metric on S^1. We replace g by $g \times id : M \times S^1 \to B\pi \times S^1 = B(\pi \times \mathbf{Z})$. By 2.9, SNC 2 holds for $\pi \times \mathbf{Z}$, so by the case already considered, $(g \times id)_*(\hat{\mathbf{A}}(N) \cap [N]) = 0$. However, $\hat{\mathbf{A}}(N)$ is the external product of $\hat{\mathbf{A}}(M)$ and of $1 \in H^0(S^1)$, and $[N]$ is the external product of $[M]$ and $[S^1]$. Since $id_* : H_*(S^1) \to H_*(S^1)$ is an isomorphism, we obtain $g_*(\hat{\mathbf{A}}(M) \cap [M]) = 0$.

3. Refinements

A) *Taking the torsion into account*

As we mentioned previously, there is good evidence in [10] that if M is a closed spin manifold with fundamental group π and classifying map (for the universal covering) $f: M \to B\pi$, then the condition for existence of a metric of positive scalar curvature on M should be the vanishing of the image of the class of (M, f) under the $\hat{\mathscr{A}}$-homomorphism $\Omega_*^{\mathrm{Spin}}(B\pi) \to \mathrm{RKO}_*(B\pi)$. Our methods so far have verified one direction of this conjecture (positive scalar curvature $\Rightarrow$ vanishing of the " $\hat{\mathscr{A}}$-genus"), *modulo torsion*, in the case where the Strong Novikov Conjecture holds for π. Our objective now is to improve this result by taking the torsion in $\mathrm{RKO}_*(B\pi)$ into account.

It is clear (from [11], §§ 4.1-4.4) that to obtain the best possible results about 2-primary torsion, it will be necessary to deal with the KR- or KO-index of *real* Dirac operators. This already causes serious difficulties, since to carry out the same program we have discussed above, it would be necessary to formulate an analogue of the Miščenko-Fomenko index theorem using real or "Real" C*-algebras. However, [16] suggests that this is feasible. A more serious obstacle is that it appears that the obvious analogues of SNC 3 and SNC 4 in real K-theory are not likely to be true in very many cases, so that some essentially new idea will be required. We therefore leave this as a difficult but promising area for future work.

On the other hand, since $\mathrm{KO} \otimes \mathbf{Z}[1/2]$ is a direct summand in $\mathrm{KU} \otimes \mathbf{Z}[1/2]$, it is natural to try to deal with p-primary torsion $(p \neq 2)$ by using complex K-theory. For this, the methods we have used so far "almost work"—the only difficulty is that the Miščenko-Fomenko theorem is not quite optimal; it only computes the index of an A-elliptic operator in $K_0(A) \otimes \mathbf{Q}$, instead of computing "on the nose" in $K_0(A)$. For purposes of analogy, one might compare the case of the index theorem for families—the K-theoretic version in [5] takes torsion into account, whereas the homological formulation in [28] does not.

We begin, therefore, with a K-theoretic version of the Miščenko-Fomenko index theorem. Since this is probably well-known to Kasparov and Connes, who may publish more complete accounts elsewhere, we shall omit some details. It is necessary first to recall the version of the index theorem formulated in [7] (first four pages). If X is a closed manifold, not necessarily orientable, then the cotangent bundle T^*X always carries a canonical Spin^c-structure. This defines a class, which we shall call $\mathscr{I}$ for "index", in $K_0(T^*X) = KK(C_0(T^*X), \mathbf{C})$. This is essentially the class of the Dirac operator defined by the Spin^c-structure, or in the language of [7], $f!$, where $f: T^*X \to \mathrm{pt}$. Now if D is an elliptic operator over X (more accurately, taking sections of one vector bundle over X to sections of another), then D defines a class $[D] \in K_0(X) = KK(C(X), \mathbf{C})$. On the other hand, the symbol σ of D defines a class $[\sigma] \in K^0(T^*X)$. The index theorem of [7] (or of [14], § 7) says that $\operatorname{ind} D = p_*([D]) \in K_0(\mathrm{pt}) = \mathbf{Z}$ (where $p: X \to \mathrm{pt}$)

can also be computed as $[\sigma] \otimes_{C_0(T^*X)} \mathscr{I}$. If X is itself an even-dimensional Spin^c-manifold and $D = D_E^+$ is the Dirac operator with coefficients in a vector bundle E over X, then $p! \in K_0(X)$ is also defined, and the index formula simplifies to $\operatorname{ind} D_E^+ = [E] \otimes_{C(X)} p!$. We are ready for

Theorem 3.1 (K-theoretic Miščenko-Fomenko theorem). — *Let* X *be a closed manifold, let* A *be a* C*-*algebra with unit, and let* D *be a pseudodifferential elliptic* A-*operator over* X *with symbol class* $[\sigma] \in K^0(T^*X, A)$. *Then* D *defines a class* [D] *in* KK(C(X), A) *and*

$$\operatorname{ind}_A D = p_*([D]) = [\sigma] \otimes_{C_0(T^*X)} \mathscr{I} \in KK(\mathbf{C}, A) = K_0(A).$$

Here $p : X \to \mathrm{pt}$ *and* $\mathscr{I}$ *is the index class in* $K_0(T^*X)$ *as above. If* X *is an even-dimensional Spin^c-manifold and* $D = D_E^+$ *is the Dirac operator with coefficients in an* A-*vector bundle* E, *then also*

$$\operatorname{ind}_A D = [E] \otimes_{C(X)} p!$$

(where $p!$ *is defined as in* [7] *using the Spin^c-structure on* X*).*

Proof (Sketch). — Suppose D is an n-th order operator from $\Gamma^\infty(E)$ to $\Gamma^\infty(F)$, where E and F are A-vector bundles over X. Then $\mathscr{H}_A = H^s(X, E) \oplus H^{s-n}(X, F)$ is a graded Hilbert A-module. Without loss of generality, we may assume $n = 0$ and $s = 0$, in which case the Sobolev spaces become just $L^2(X, E)$ and $L^2(X, F)$ (interpreted in terms of the A-valued inner product). Then we get a representation $\varphi : C(X) \to \mathscr{L}(\mathscr{H}_A)$ by pointwise multiplication, and $T = \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix}$ is a self-adjoint operator in $\mathscr{L}(\mathscr{H}_A)$ (of degree 1 with respect to the grading), which because of the " A-Fredholmness theorem " of Miščenko-Fomenko ([23], Theorem 3.4) is invertible modulo $\mathscr{K}(\mathscr{H}_A)$. The commutator of D with multiplication by any $a \in C^\infty(X)$ is pseudo-differential of order -1; thus $[T, \varphi(C(X))] \subseteq \mathscr{K}(\mathscr{H}_A)$ by [23], Lemma 3.3. Thus according to the prescription of [16], § 4, the triple $(\mathscr{H}_A, \varphi, T)$ defines an element of KK(C(X), A). (To put it in standard form, we need $T^2 - 1 \in \mathscr{K}(\mathscr{H}_A)$, but by [21], Ch. 1, this can be arranged by replacing D by a partial isometry with the same A-index.) From the definition of p_* in [16], § 4, one can easily check that $p_*([D])$ coincides with the A-index of D as defined in [23].

As in [7], one can now check that $[D] = [\sigma] \otimes_{C_0(T^*X)} \mathscr{P}_X$ with $\mathscr{P}_X \in K_0(T^*X \times X)$ as defined by Connes-Skandalis. Then the index formula follows from [7], Corollaire 4, since

$$\operatorname{ind}_A D = p_*([\sigma] \otimes_{C_0(T^*X)} \mathscr{P}_X) = [\sigma] \otimes_{C_0(T^*X)} p_*(\mathscr{P}_X) = [\sigma] \otimes_{C_0(T^*X)} \mathscr{I}.$$

In the case where X is an even-dimensional Spin^c-manifold, the same argument of [7], p. 872, that works when E is an ordinary vector bundle basically works here.

Theorem 3.2 (revised version of Theorem 1.1). — *Let* M *be a closed even-dimensional spin manifold, and suppose* M *admits a Riemannian metric whose scalar curvature is everywhere* *non-negative and strictly positive at at least one point. Let* A *be any* C^**-algebra with unit and* $\mathscr{V}$ *a flat* A*-vector bundle over* M. *Then* $[\mathscr{V}] \otimes_{C(X)} p! = 0$ *in* $K_0(A)$, *where* $p : M \to \mathrm{pt}$ *and* $p!$ *is defined using the spin structure on* M.

PROOF. — The proof of Theorem 1.1 applies word-for-word, except that one must replace the Miščenko-Fomenko index calculation by the second part of Theorem 3.1.

THEOREM **3.3** (revised version of Theorem 2.11). — *Let* M *be a closed spin manifold admitting a metric of positive scalar curvature. Let* $\pi = \pi_1(M)$ *and let* $f : M \to B\pi$ *be the classifying map for the universal covering of* M. *If the Strong Novikov Conjecture* (SNC 3) *holds for* π, *then* $f_*(p!) = 0$ *in* $RK_*(B\pi)$. *(Here* $p!$ *is the class of the Dirac operator in* $K_0(M)$ *if* M *is even-dimensional or in* $K_1(M)$ *if* M *is odd-dimensional.)*

PROOF. — First suppose M is even-dimensional, and apply Theorem 3.2 with $\mathscr{V} = C^*(\pi) \times_\pi \widetilde{M}$ the universal $C^*(\pi)$-bundle over M. We conclude that $[\mathscr{V}] \otimes_{C(X)} p! = 0$. However, $\mathscr{V}$ is the pull-back under f of the universal $C^*(\pi)$-bundle $\mathscr{V}_{B\pi}$ over $B\pi$ used to define the Kasparov map β. So by the naturality property of the intersection product, $f^*([\mathscr{V}_{B\pi}]) \otimes_{C(X)} p! = [\mathscr{V}_{B\pi}] \otimes_{B\pi} f_*(p!)$, *i.e.* $\beta(f_*(p!)) = 0$ ([16], § 4, Theorem 4). But β is injective by hypothesis. If M is odd-dimensional, repeat the argument with $M \times S^1$ as in the proof of 2.11, then " de-suspend ".

Combining 3.3 with Theorem 2.6, we get one direction of the Gromov-Lawson conjecture modulo 2-primary torsion for a large class of solvable fundamental groups. (The map $[(M, f)] \mapsto f_*(p_M!)$ gives the analogue of the Milnor $\widehat{\mathscr{A}}$-homomorphism in complex K-theory: $\Omega_*^{\mathrm{Spin}^c}(B\pi) \to RK_*(B\pi)$.)

B) *Weakening the spin condition*

The notion of enlargeability used in [10] (as opposed to that of [9]) suggests that we ought to be able to weaken the spin condition on M in Theorem 2.11, and assume only that the universal covering $\widetilde{M}$ of M should carry a spin structure. (For instance, there might conceivably be non-spin $K(\pi, 1)$-manifolds with π not having any finite quotients, in which case this improvement might be needed in order to prove non-existence of positive-scalar-curvature metrics on such manifolds.) It turns out that techniques using operator algebras are very convenient for handling this extra generality. (It is possible one could even take torsion into account at the same time, as in § 3A above, but we are not sure how to do this at the moment.)

The essential idea is the following. Suppose M is a closed oriented even-dimensional manifold with fundamental group π, whose universal covering $\widetilde{M}$ (usually non-compact, of course) carries a spin structure. Also let E be some A-vector bundle over M. We would like to compute the " index of the Dirac operator D_E^+ with coefficients in E " over M, but of course this does not make any sense. However, we have half-spinor bundles $\mathscr{S}^\pm$ and Dirac operators $D^\pm$ defined over $\widetilde{M}$, and the action of π on $\widetilde{M}$ by covering transformations "almost" commutes with $D^{\pm}$, in the sense that we have a "projective action" of π on spinors and the Dirac operator, or a true action of some double covering $\hat{\pi}$ of π on $\mathscr{S}^{\pm}$, commuting with $D^{\pm}$. If P_{SO} denotes the principal SO-bundle of orthonormal oriented frames on $\widetilde{M}$, and P_{Spin} denotes the principal spin bundle, then π acts on P_{SO} (since M is oriented—of course we choose the orientation of $\widetilde{M}$ compatible with that on M), and the group $\hat{\pi}$ is constructed by "pull-back":

$$\begin{tikzcd} \hat{\pi} \arrow[r] \arrow[d] & \mathrm{Aut}(P_{Spin}) \arrow[d] \\ \pi \arrow[r] & \mathrm{Aut}(P_{SO}). \end{tikzcd}$$

Then $\hat{\pi}$ acts on the spinor bundle $\mathscr{S}$, and preserves the splitting into half-spinor bundles since this is defined by the orientation. To see $\hat{\pi}$ commutes with D^+, just compute in local coordinates.

Now suppose M carries a metric of positive scalar curvature, and for the moment take E to be trivial. If the Dirac operator D^+ existed on M, we would know by [19] that $\hat{A}(M) = \langle \hat{\mathbf{A}}(M), [M] \rangle = 0$. By the L^2-index theorem of Atiyah [3], we also know that

$$\mathrm{ind}_M D^+ = 0 \Leftrightarrow L^2 - \mathrm{ind}_{\widetilde{M}} D^+ = 0.$$

However, even if D^+ does not make sense on M, one can still show that there are no L^2-harmonic spinors on $\widetilde{M}$ (because the scalar curvature is strictly bounded below). Mikhael Gromov and Blaine Lawson were kind enough to inform me that they had used this fact together with a slight modification of Atiyah's proof to conclude that $\hat{A}(M) = 0$ even when M does not have a spin structure. (Here the L^2-index theorem is being used the reverse of the way it is usually applied—data about the non-compact manifold $\widetilde{M}$ yields information about M!)

However, this argument does not generalize to the case where E is a flat C^*-vector bundle, so we prefer to construct a genuine elliptic operator on M and apply the Miščenko-Fomenko theorem. For this, note that we have a split exact sequence of C^*-algebras

$$0 \to C^*(\hat{\pi})_{odd} \to C^*(\hat{\pi}) \to C^*(\pi) \to 0,$$

where $C^*(\hat{\pi})_{odd}$ is universal for unitary representations of $\hat{\pi}$ that are non-trivial on the kernel of the covering map $\hat{\pi} \to \pi$. The trace coming from the regular representation of $\hat{\pi}$ defines a non-zero finite trace α on $C^*(\hat{\pi})_{odd}$, hence a non-zero map $K_0(C^*(\hat{\pi})_{odd}) \to \mathbf{R}$. Observe now that since the cocycle of the group extension

$$\{1\} \to \{\pm 1\} \to \hat{\pi} \to \pi \to \{1\}$$

exactly cancels that for the "action" of π on $\mathscr{S}^{\pm}$, the $C^*(\hat{\pi})_{odd} \otimes A$-vector bundles $(C^*(\hat{\pi})_{odd} \times_{\pi} \mathscr{S}^{\pm}) \otimes E$ actually make sense on M, and we obtain a genuine operator D^+. Now we obtain

THEOREM 3.4 (revised version of Theorem 1.1). — *Let* M *be a closed oriented even-dimensional manifold whose universal covering is spin, and suppose* M *admits a metric of positive scalar curvature. Let* A *be any* C*-*algebra with unit and* E *a flat* A-*vector bundle over* M. *Then*

$$\langle \hat{\mathbf{A}}(\mathrm{M}) \cup \mathrm{ch}[\mathrm{E}], [\mathrm{M}] \rangle = 0 \text{ in } \mathrm{K}_0(\mathrm{A}) \otimes \mathbf{Q}.$$

PROOF. — Since we are still dealing with flat bundles, the index of the Dirac operator D^+ constructed above is zero in $\mathrm{K}_0(\mathrm{C}^*(\hat{\pi})_{\mathrm{odd}} \otimes \mathrm{A})$. But from the trace α on $\mathrm{C}^*(\hat{\pi})_{\mathrm{odd}}$ we obtain a "partial trace"

$$\alpha \otimes \mathrm{id} : \mathrm{C}^*(\hat{\pi})_{\mathrm{odd}} \otimes \mathrm{A} \to \mathrm{A}$$

and thus a map $\mathrm{K}_0(\mathrm{C}^*(\hat{\pi})_{\mathrm{odd}} \otimes \mathrm{A}) \to \mathrm{K}_0(\mathrm{A}) \otimes \mathbf{R}$ (1). Applying this to the index formula now gives

$$c \langle \hat{\mathbf{A}}(\mathrm{M}) \cup \mathrm{ch}[\mathrm{E}], [\mathrm{M}] \rangle = 0 \text{ in } \mathrm{K}_0(\mathrm{A}) \otimes \mathbf{R},$$

where $c > 0$ is some number which we do not need to compute, but which comes out to 1 if we normalize α correctly.

The same arguments as in § 2 now give a revised version of Theorem 2.11, and proofs of Conjectures 2.2 and 2.3, assuming π satisfies SNC 2.

THEOREM 3.5. — *Assume the Strong Novikov Conjecture* (SNC 2) *holds for* π, *and suppose* M *is a closed oriented manifold whose universal covering has a spin structure. Let* $g : \mathrm{M} \to \mathrm{B}\pi$ *be any continuous map. Then if* M *admits a metric of positive scalar curvature,* $g_*(\hat{\mathbf{A}}(\mathrm{M}) \cap [\mathrm{M}]) = 0$ *in* $\mathrm{H}_*(\mathrm{B}\pi, \mathbf{Q})$, *or equivalently, for all* $a \in \mathrm{H}^*(\mathrm{B}\pi, \mathbf{Q})$,

$$\langle \hat{\mathbf{A}}(\mathrm{M}) \cup g^*(a), [\mathrm{M}] \rangle = 0.$$

In particular, if $g_*([\mathrm{M}]) \neq 0$ *in* $\mathrm{H}_*(\mathrm{B}\pi, \mathbf{Q})$, *then no manifold homotopy-equivalent to* M *admits a metric of positive scalar curvature.*

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(1) At least when the Künneth theorem of [27] holds. However, we do not need to worry about this here, since the Chern character of the symbol of D^+ clearly lies in $\mathrm{H}_c^*(\mathrm{T}^*\mathrm{M}, \mathbf{Q}) \otimes \mathrm{K}_*(\mathrm{C}^*(\hat{\pi})_{\mathrm{odd}}) \otimes \mathrm{K}_*(\mathrm{A})$.

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Department of Mathematics,
University of Maryland,
College Park, Maryland 20742

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