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HOMOTOPY REPRESENTATIONS OF FINITE GROUPS

by TAMMO TOM DIECK *and* TED PETRIE

0. Introduction

Our aim is to develop a theory of actions of finite groups on homotopy spheres in analogy with the theory of representations of finite groups. The starting point is the notion of a homotopy representation (§ 1). This is a finite-dimensional G-CW-complex X such that for each subgroup H of G the fixed set X^H of H is homotopy-equivalent to a sphere. The Grothendieck group of equivalence classes of such actions with addition defined by join is the homotopy representation group V(G) (§ 2). It is the homotopy analogue of the representation ring of G.

Homotopy representations, as we shall show, are distinguished by two integral valued functions whose domain is the set of conjugacy classes of subgroups of G. These are: the *dimension function* which assigns to each homotopy representation X of G the function Dim X whose value at the subgroup H is the dimension of X^H plus 1 and the *degree function* which assigns to each pair of homotopy representations X and Y having the same dimension function and G-map $f: X \to Y$ the function $d(f)$ whose value $d(f)(H)$ at H is the degree of f^H. Given X and Y with $\text{Dim}\, X = \text{Dim}\, Y$ there is always an f such that $d(f)(H)$ is prime to the order $|G|$ of G for all H. (For such an f, $d(f)$ is said to be an *invertible degree function*.) In a suitable sense to be made precise in section 3 $d(f)$ depends only on $X - Y$ in the representation group of G and vanishes exactly when $X - Y$ is zero.

Since the dimension function and degree function distinguish homotopy representations, the structure of V(G) is determined by the relations among the values of these functions on the subgroups of G. Put another way the determination of V(G) as an abelian group is equivalent to characterizing those integral valued functions on the set of conjugacy classes of subgroups which occur as Dim X and $d(f)$ for some homotopy representation X resp. some $f: X \to Y$ with $\text{Dim}\, X = \text{Dim}\, Y$. The characterization required involves among other things the group cohomology of subquotients of G and the projective class groups of the integral group rings of these subquotients.

As an example of the interplay between geometry and algebra we note that the existence of a homotopy representation X of G with $\text{Dim}\, X(1) \neq 0$ and $\text{Dim}\, X(H) = 0$ for $H \neq 1$ is equivalent to G having periodic cohomology (§ 12). The values Dim X(1) for such X depend on the projective class group of **Z**G (provided X is a finite CW-complex and not just a finite-dimensional CW-complex).

Our set-up allows us to study homotopy representations with certain side condition (abbreviated by λ in (2.2)) in the same framework. The corresponding group is denoted $V(G, \lambda)$. From the point of view of obtaining invariants of smooth actions on homotopy spheres we are naturally led to study homotopy representations X of finite type *i.e.* X is a finite CW-complex. This is the case $\lambda = h$ ((2.1)). The relation between the geometry and algebra of representations is complicated by imposing this finiteness condition. It turns out to be more efficient to deal with the case $\lambda = h^\infty$ where representations are not required to have finite type. Then $V(G, h)$ is the kernel of the homomorphism $\sigma : V(G, h^\infty) \to \mathscr{K}(G)$ where $\mathscr{K}(G)$ is a group fashioned from the reduced projective class groups of the integral group ring of subquotients of G. In particular $\operatorname{rank} V(G, h) = \operatorname{rank} V(G, h^\infty)$.

The dimension function defines a homomorphism Dim from $V(G, \lambda)$ to the set $C(G)$ of all integral valued functions on conjugacy classes of subgroups of G. Its kernel is denoted $v(G, \lambda)$. The set of invertible functions in $C(G)$ modulo an equivalence relation defines a multiplicative group $\operatorname{Pic}(G)$ ((3.6)) and the degree function defines a homomorphism $d : v(G, \lambda) \to \operatorname{Pic}(G)$. The group $\operatorname{Pic}(G)$ is a finite group and $d : v(G, \lambda) \to \operatorname{Pic}(G)$ is injective ((3.8) and (3.9)). In particular $\operatorname{rank} V(G, \lambda) = \operatorname{rank} \operatorname{Dim} V(G, \lambda)$. In section 10 we compute $\operatorname{rank} V(G, h^\infty)$ in terms of the subgroup structure of G. This uses actions on Brieskorn varieties and a theorem of Borel about p-torus actions on spheres. Since $\operatorname{rank} V(G, \lambda) = \operatorname{rank} \operatorname{Dim} V(G, \lambda)$, Theorem (10.3) counts the number of linearly independent rational linear relations among the values $\{\operatorname{Dim} X(H) \mid H \subset G\}$ as X ranges over homotopy representations of G. In particular (10.2) shows that in general $\operatorname{rank} V(G, h^\infty)$ exceeds the rank of the subgroup $JO(G)$ generated by the unit spheres of real representations of G. In section 6 we show that d maps $v(G, \lambda)$ isomorphically onto $\operatorname{Pic}(G)$ when $\lambda = h^\infty$. In words this means: Every invertible function is the degree function of some $f : X \to Y$ with $\operatorname{Dim} X = \operatorname{Dim} Y$. This is not the case if we insist that X and Y be of finite type since $v(G, h)$ is the kernel of σ restricted to $v(G, h^\infty)$. When G is abelian this point can be made quite explicit in terms of the Swan homomorphism $s_L : (\mathbf{Z}/|L|)^* \to \widetilde{K}_0(L)$ (§ 11). In this case there is an isomorphism

$$\mu : \operatorname{Pic}(G) \to \Pi_{(H)}(\mathbf{Z}/|G/H|^*)/B = A$$

such that $x \in \operatorname{Pic}(G)$ is $d(f)$ for some $f : X \to Y$ with X and Y of finite type if and only if $s\mu(x) = 0$ where $s : A \to K(G)$ is the product of the Swan homomorphisms $s_{G/H}$ for $H \subset G$ and B is a suitable subgroup (see (11.5)). Note that the condition $s\mu(df) = 0$ expresses linear relations among the values $df(K)$, $K \subset G$. Sections 11 and 12 are devoted to illustrate these results for various groups G.

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1. Homotopy representations of finite groups

Let G be a finite group.

Definition (1.1). — A *homotopy representation of* G is a G-CW-complex X such that for each subgroup H of G the fixed point set X^H is an $n(H)$-dimensional CW-complex which is homotopy-equivalent to the sphere $S^{n(H)}$. If X^H is empty we put $n(H) = -1$. The homotopy representation is called *finite* if X is a finite G-CW-complex. A finite-dimensional G-CW-complex is called a *generalized homotopy representation* if each fixed set X^H is homotopy-equivalent to some sphere $S^{n(H)}$ (not necessarily of the dimension of X^H).

We make some remarks concerning these definitions. Since we are mainly interested in homotopy types, the actual CW-structure is not considered as part of the structure. In some parts of the following we could also work with spaces of the G-homotopy type of a G-complex. (Henceforth " complex " shall mean " CW-complex ".)

Example (1.2). — Let V be a finite-dimensional representation of G over the real numbers and let S(V) be the unit sphere of V. Then S(V) is a finite homotopy-representation (use the triangulation theorem of Illman [15]).

Definition (1.3). — A homotopy representation X is called *linear* if it is G-homotopy-equivalent to S(V) for some G-representation V.

Example (1.4). — Let $G = \mathbf{Z}/p$. There exist finite generalized homotopy representations X with the following property: X and X^G are homotopy-equivalent to the same sphere S^n. The inclusion $i : X^G \to X$ has a degree j which can be any integer prime to p (Bredon [2], p. 391). We shall see later that such an X is not G-homotopy-equivalent to a homotopy representation.

Since our main interest lies in finite homotopy representations, because only these can be realized as manifolds, it seems that we could avoid generalized homotopy representations. Nevertheless it turns out that examples of the type (1.4) have value in the development of the general theory.

Homotopy representations have two pieces of structure associated to them, the dimension function and the orientation behavior. We are going to explain this.

The set $\mathcal{S}(G)$ of subgroups of G is partially ordered by inclusion written $\subset$ and $<$ for strict inclusion. This induces a partial order on $\varphi(G)$ the set of conjugacy classes of subgroups of G. The conjugacy class of H is written (H).

A subset S of $\mathcal{S}(G)$ is *closed* by definition if $K \in S$ and $H \in \mathcal{S}(G)$ with $H > K$ implies $H \in S$. Let C(G) be the ring of integral valued functions on $\varphi(G)$. If X is a generalized homotopy representation, then X^H is homotopy equivalent to a sphere $S^{n(H)}$ (where $\varnothing = S^{-1}$) and if H is conjugate to K, X^H is homeomorphic to X^K; so $n(H) = n(K)$. Thus we can give the

Definition (**1.5**). — The *dimension function*

$$\mathrm{Dim}\, X : \varphi(G) \to \mathbf{Z}$$

of the generalized homotopy representation X is defined by

$$(\mathrm{Dim}\, X)(H) = n(H) + 1.$$

We have to use two different notions of dimension in this paper. By dim X we mean the geometric dimension of X as a complex; whereas, h-dim $X = n$ means that X is homotopy-equivalent to S^n.

Let CY denote the cone over Y (which is a point if Y is empty!). If X is a generalized homotopy representation then

$$\text{(1.6)} \qquad H^{n(H)+1}(CX^H, X^H; \mathbf{Z}) \cong \mathbf{Z}$$

(even for $n(H) = 0, -1$). The group $WH = NH/H$ acts on X^H and on the cohomology group (1.6). We put

$$e_H(g) = e_H^X(g) = 1 \ \ (\text{resp.} \ = -1)$$

if $g \in WH$ preserves a generator of (1.6) (resp. changes a generator). We obtain a homomorphism

$$\text{(1.7)} \qquad e_H^X : \ WH \to \mathbf{Z}^* = \{+1, -1\}.$$

Definition (**1.8**). — The *orientation behavior* of X is the collection of the *orientation homomorphisms* e_H^X. We call X *orientable* if all e_H^X are trivial.

Definition (**1.9**). — An *orientation* for an orientable generalized homotopy representation is a choice for each (H) of a generator for the group $H^{n(H)+1}(CX^H, X^H; \mathbf{Z})$.

This notion of orientation is well-defined in the following sense: If K is another representative of (H), say $gHg^{-1} = K$, then left translation $\ell_g : X^H \to X^K : x \mapsto gx$ induces an isomorphism

$$\ell_g^* : \ H^{n(H)+1}(CX^K, X^K) \to H^{n(H)+1}(CX^H, X^H)$$

which is independent of the choice of $g \in G$ with $gHg^{-1} = K$, because e_H^X is assumed to be trivial.

The unit sphere in the direct sum of two linear representations is G-homeomorphic to the join of the individual unit spheres, in symbols

$$S(V \oplus W) \cong S(V) * S(W).$$

Therefore we study in general the join operation on homotopy representations. If X and Y are (generalized, resp. finite) homotopy representations then $X * Y$ is a (generalized, resp. finite) homotopy representation. Note that

$$\text{(1.10)} \qquad (\mathrm{Dim}\, X * Y)(H) = (\mathrm{Dim}\, X)(H) + (\mathrm{Dim}\, Y)(H)$$

which is the reason for taking $n(H) + 1$ instead of $n(H)$ in definition (1.5).

If X and Y are oriented there is a canonical induced orientation on $X * Y$ which is associative. Note also that

(1.11) $$e^X_H . e^Y_H = e^{X*Y}_H$$

(pointwise multiplication of functions $WH \to \mathbf{Z}^*$).

Definition (1.12). — Two (oriented) homotopy representations X and Y are called *equivalent (oriented equivalent)* if there exists a G-homotopy-equivalence $f: X \to Y$ (such that f^H has degree one with respect to the given orientations, for all $H \subset G$).

Actually, if for all $H \subset G$ the map f^H has degree ± 1 then f is a G-homotopy-equivalence (Hauschild [13], James-Segal [16] and Illman [33]).

Finally, we can try to imitate complex representations in our context.

Definition (1.13). — A (generalized) homotopy representation X is called *even* if Dim X takes only even values and if all homomorphisms e^X_H are trivial.

There are many variants and generalizations of the above notions. In particular we mention simple-homotopy-type, sphere bundles, rational homotopy spheres.

Probably the notion of homotopy representation should be more restrictive, at least if one thinks of actions on manifolds as being the most important models. In that case, if H and K are different isotropy groups and $H < K$, then $\dim X^H > \dim X^K$. One might conjecture that under this condition there exists a function $b(n)$ such that a group which acts effectively on a homotopy representation of dimension n is a subgroup of $O(b(n))$.

We now give a simple example (generalizing (1.4)) which shows that such finiteness results do not hold if we drop the condition $\dim X^H \neq \dim X^K$ for different isotropy groups H, K. Let G be any finite group. Let r be an integer prime to $|G|$. There exist free $\mathbf{Z}$G-modules F_1 and F_2 and an isomorphism

$$\varphi: \mathbf{Z} \oplus F_1 \to \mathbf{Z} \oplus F_2$$

such that

$$\mathbf{Z} \xrightarrow[C]{} \mathbf{Z} \oplus F_1 \xrightarrow[\varphi]{} \mathbf{Z} \oplus F_2 \xrightarrow[pr]{} \mathbf{Z}$$

is multiplication by r; we say in this case φ has degree r. (This is due to Swan [23]. Compare section 6 of this paper.) Now consider the exact sequence

$$0 \to F_1 \to \mathbf{Z} \oplus F_2 \to \mathbf{Z} \to 0$$

and realize $F_1 \to \mathbf{Z} \oplus F_2$ geometrically as the cellular chain complex of a space X as follows: Start with S^n and trivial G action. Attach cells of type $G \times D^n$ to S^n by trivial attaching maps ($n > 2$), one for each element of a $\mathbf{Z}$G-basis of F_2. Let Y be the resulting G-complex. Then $\pi_n(Y) \cong H_n(Y) \cong \mathbf{Z} \oplus F_2$. For each basis element e of F_1 attach a cell of type $G \times D^{n+1}$ with attaching map $\{1\} \times S^n \to Y$ representing $\varphi(0, e) \in \mathbf{Z} \oplus F_2 \cong \pi_n(Y)$. The resulting space X is homotopy-equivalent to S^n and $X^G \subset X$ has as degree the degree of φ^{-1}.

2. Homotopy representation groups

The homotopy representation groups now to be defined are the analogues of the representation ring. We consider equivalence classes of the various types of homotopy representations introduced in section 1. We use the join as composition law. This yields commutative semi-groups. The unified notation $V^+(G, \lambda)$ will be used for these semi-groups, where λ refers to the category under question. We mention in particular the following possibilities for λ:

(2.1)
h^∞: homotopy representations
h: finite homotopy representations
$\widetilde{h}^\infty$: generalized homotopy representations
h: finite generalized homotopy representations
ℓ: linear homotopy representations.

The Grothendieck group associated to $V^+(G, \lambda)$ is denoted

(2.2) $\quad V(G, \lambda)$

and is called the *homotopy representation group of* G.

Because of (1.10) taking dimension functions yields a homomorphism

(2.3) $\quad \mathrm{Dim}: V(G, \lambda) \to C(G).$

The kernel of this homomorphism is denoted $v(G, \lambda)$.

The computation of $V(G, \lambda)$ and description of its structure is the main objective of this paper. There are essentially two different steps in the calculation: first—the determination of the image of Dim (which is a free abelian group), second—the computation of $v(G, \lambda)$ (which turns out to be a finite abelian group).

Inclusion of categories gives canonical homomorphisms

(2.4)
$$\begin{array}{ccccc} V(G, \ell) & \longrightarrow & V(G, h) & \longrightarrow & V(G, h^\infty) \\ & & \Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle \beta} \\ & & V(G, \widetilde{h}) & \longrightarrow & V(G, \widetilde{h}^\infty) \end{array}$$

The next Proposition collects a few of the results which we prove in later sections.

Proposition **(2.5)**. — *The horizontal maps in* (2.4) *are injective, the vertical maps are bijective.*

Proof. — It follows immediately from (6.6) that α and β are surjective. We show in section 8 that given any generalized homotopy representation Y there exists a homotopy representation Z such that $Y * Z$ has the G-homotopy-type of a linear homotopy representation. Using the definition of the groups $V(G, \lambda)$ this yields immediately the injectivity of all the maps in (2.4).

Because of (1.11) we obtain for each subgroup H of G a homomorphism

(2.6) $$e_H : V(G, \lambda) \to \operatorname{Hom}(WH, \mathbf{Z}^*)$$

which describes the orientation behavior at H. If $gHg^{-1} = K$ then $x \mapsto gxg^{-1}$ induces a homomorphism $\alpha_g : WH \to WK$ and the diagram

$$\begin{array}{ccc} & V(G, \lambda) & \\ {}^{e_H}\swarrow & & \searrow^{e_K} \\ \operatorname{Hom}(WH, \mathbf{Z}^*) & \xleftarrow[\operatorname{Hom}(\alpha_g, \mathbf{Z}^*)]{} & \operatorname{Hom}(WK, \mathbf{Z}^*) \end{array}$$

is commutative. Moreover $\operatorname{Hom}(\alpha_g, \mathbf{Z}^*)$ is independent of the choice of g with $gHg^{-1} = K$ because $\mathbf{Z}^*$ is abelian. Hence e_H essentially only depends on the conjugacy class (H).

The group $v(G, \ell)$ was called $jO(G)$ in tom Dieck [6] and was computed (using representation theory) for p-groups G.

We also point out that the isomorphisms α and β in diagram (2.4) are stable phenomena. Unstably there exist many generalized homotopy representations which are not homotopy representations. Similarly a homotopy representation X may be in the image of $V(G, \ell) \to V(G, h)$ without being a linear homotopy representation (so is only virtually linear). A general question asks for the properties of the canonical map $V^+(G, \lambda) \to V(G, \lambda)$: When is this map injective? Can one describe the image?

3. Homotopy representations and Burnside modules

This section introduces another basic invariant for homotopy representations: the degree function. For the convenience of the reader we collect various known results.

We begin with the equivariant Hopf theorem. Let X be a finite-dimensional G-complex. Let $\dim X^H = n(H) \geq 1$ for $H \in \operatorname{Iso}(X)$. Here $\operatorname{Iso}(X)$ is the set of isotropy groups of the G-action on X. If $H, K \in \operatorname{Iso}(X)$, $H < K$, $H \neq K$ we assume $n(H) \geq n(K) + 2$. We assume that $H^{n(H)}(X^H; \mathbf{Z}) \cong \mathbf{Z}$. The action of WH on X^H then induces an orientation homomorphism $e_H^X : WH \to \mathbf{Z}^* = \operatorname{Aut} \mathbf{Z}$. Let Y be another G-space. For $H \in \operatorname{Iso}(Y)$ we assume that Y^H is $(n(H) - 1)$-connected and $\pi_{n(H)} Y^H \cong \mathbf{Z}$. Then $H^{n(H)}(Y^H; \mathbf{Z}) \cong \mathbf{Z}$ and we obtain an orientation homomorphism e_H^Y. We assume that $e_H^X = e_H^Y$ for all $H \in \operatorname{Iso}(X)$. This is the case *e.g.* if $X - Y \in v(G, \lambda)$. We orient X by choosing a generator of $H^{n(H)}(X^H; \mathbf{Z})$ for every H and similarly for Y. We assume that X and Y have been oriented. Then, given a G-map $f : X \to Y$, the fixed point mapping f^H has a well-defined degree $d(f)(H) \in \mathbf{Z}$ and $d(f) \in C(G)$.

If $K = gHg^{-1}$ then left translation by g maps a generator of $H^{n(K)}(X^K)$ to the chosen generator of $H^{n(H)}(X^H)$. Using these generators gives a degree $d(f)(K)$ which is independent of the choice of g with $K = gHg^{-1}$ because $e_H^X = e_H^Y$.

Proposition (3.1). — *Under the assumption above the equivariant homotopy set* $[X, Y]_G$ *is not empty. Elements* $[f] \in [X, Y]_G$ *are determined by the set of* $d(f)(H)$, $H \in \mathrm{Iso}(X)$. *The value* $d(f)(H)$ *is modulo* $|WH|$ *determined by the* $d(f)(K)$, $K > H$, $K \neq H$ *and fixing these* $d(f)(K)$ *the possible* $d(f)(H)$ *fill the whole residue class* mod $|WH|$.

Proof. — Tom Dieck [8], (8.4.1) and Petrie unpublished Chicago lectures 1978.

We still assume that X and Y are as above. We define the stable equivariant homotopy group

$$(3.2) \qquad \omega(X, Y)$$

to be the direct limit over linear homotopy representations Z of $[X * Z, Y * Z]$. (By stability of suspension it is not necessary to pass to the limit. A sufficiently large Z will do. See Hauschild [12], Satz (2.4).)

If $x \in \omega(X, Y)$ is represented by $f: X * Z \to Y * Z$, then $d(f)(H)$ is the same for all representatives of x. We denote it by $d_H(x)$.

Definition (3.3). — The *degree function* $d(x) \in C(G)$ of $x \in \omega(X, Y)$ is given by $(H) \mapsto d_H(x)$.

As a corollary of (3.1) we obtain

Proposition (3.4). — *For* X, Y *as above the assignment* $x \mapsto d(x)$ *defines an injective homomorphism* $d: \omega(X, Y) \to C(G)$.

It is quite straightforward to show that $v(G, \lambda)$ is a finite group using the degree function. We note that $\omega(X, X)$ is a ring for any homotopy representation X of G. It is independent of the homotopy representation X. This ring is historically denoted by ω_G^0 (Segal [32]) and we abbreviate it here by ω. The degree function d identifies ω with a subring of $C(G) = C$. This provides an isomorphism of ω with the Burnside ring A(G) of G. By definition this is the Grothendieck group of the category of finite G-sets with addition defined by disjoint union and multiplication by product of finite sets. The Burnside ring is identified as a subring of C by regarding a finite G-set X as the function on $\varphi(G)$ which sends H to the cardinality of X^H. The ring obtained this way is $d\omega$. This shows $A(G) \cong \omega$. See tom Dieck-Petrie [9].

Proposition (3.5). — $|G| . C \subset \omega$.

Proof. — In tom Dieck-Petrie [9, Theorem 3], we have shown that $\omega \subset C$ is described by a set of congruence relations; *i.e.* $d \in C$ is contained in ω if and only if it satisfies a certain set of congruences

$$\sum_{(K)} n_{H,K}\, d(K) \equiv 0 \bmod |WH|$$

for $(H) \in \varphi(G)$. Here $n_{H,K}$ is an integer with $n_{H,H} = 1$ and the sum if taken over conjugacy classes (K) of subgroups such that H is normal in K and K/H is cyclic. Obviously any multiple of $|G|$ in C satisfies these congruences.

The multiplicative group of units of a ring S is denoted by S^*. Note that C^* is the group of functions whose values are ± 1 at every conjugacy class of subgroups of G.

(3.6) Define $\mathrm{Pic}(G) = \overline{C}^*/C^*.\overline{\omega}^*$ where $\overline{C} = C/|G|.C$ and $\overline{\omega} = \omega/|G|.C$.

It follows from tom Dieck-Petrie [9], (3.32) that $\mathrm{Pic}(G) = \mathrm{Pic}(A(G))$ is the Picard group of the Burnside ring.

We use the degree function to define a homomorphism

(3.7) $$D : v(G, \lambda) \to \mathrm{Pic}(G).$$

Theorem **(3.8)**. — *There is an injective homomorphism* $D : v(G, \lambda) \to \mathrm{Pic}(G)$.

Proof. — The proof depends on these two points: Let X and Y be homotopy representations with $X - Y = x \in v(G, \lambda)$.

i) There is an $f \in \omega(X, Y)$ such that $d(f)(H)$ is prime to $|G|$ for all $H \subset G$.

ii) There is an $f' \in \omega(Y, X)$ such that $d(f)(H).d(f')(H) \equiv 1(|G|)$ for all $H \subset G$.

Both i) and ii) are proved in the same way using (3.1). First i). If $X_H = \{x \in X \mid (G_x) > (H)\}$ and $f_H : X_H \to Y$ has been defined such that degree f_H^K is prime to $|G|$ for $(K) > (H)$, then f_H^H can be extended to a WH-map h of X^H to Y^H because $\pi_i(Y^H) = 0$ for $i < \dim X^H$. Note $d(h)(1)$ is determined mod $|WH|$ by (3.1). In fact $d(h)(1)$ is prime to $|WH|$ because $d(h)(1) = \text{degree } h \not\equiv 0 \Leftrightarrow \text{degree } f^{\mathbf{Z}/p} \not\equiv 0 \bmod p$ whenever $\mathbf{Z}/p \subset WH$ is cyclic of prime order p. But $h^{\mathbf{Z}/p} = f_H^K$ for some $K > H$. The degree of this map is prime to p. Now use (3.1) again to modify h without changing h on X_H^H so that $d(h)(1) = \text{degree } h$ is in fact prime to $|G|$. Then there is a unique G-map $f : X_H \cup GX^H \to Y$ which extends $f_H \cup h$. Thus we may assume $f : X \to Y$ and $d(f)(H)$ is prime to $|G|$ for all $H \subset G$.

To establish ii) reverse the roles of X and Y to inductively construct $f' : Y \to X$ satisfying ii). Suppose $f'_H : Y_H \to X$ has been defined such that $\text{degree} f'^K_H \, \text{degree} f^K \equiv 1(|G|)$ for all $(K) > (H)$. Let $h' : Y^H \to X^H$ extend f'^H_H. Then $\text{degree} f^H.\text{degree } h' = \text{degree}(f^H \circ h') \equiv 1(|WH|)$. To see this note $u = f^H \circ h'$ and 1_{Y^H} are both in $\omega(Y^H, Y^H)$ and $d(u)(L) \equiv d(1_{Y^H})(L) \bmod |WH|$ for $1 \neq L \subset WH$. By (3.1) then $d(u)(1) \equiv d(1_{Y^H}) \equiv 1(|WH|)$. Now use (3.1) again to modify h' so that $\text{degree } h' \equiv (\text{degree} f^H)^{-1} \bmod |G|$. Then there is a unique G-map $f' : Y_H \cup GY^H \to X$ which extends $f'_H \cup h$; so f' is constructed inductively.

Now define $D(x)$ to be the class of $d(f)$ (f in i) above) in $\mathrm{Pic}(G)$. To verify D is well defined, suppose $f'' \in \omega(X, Y)$ also satisfies i). Let $f' \in \omega(Y, X)$ satisfy $d(f'')(H).d(f')(H) \equiv 1(|G|)$ for all H. Then in $\mathrm{Pic}(G)$ we have

$$d(f)d(f'')^{-1} = d(f)d(f') = d(ff') \in \overline{\omega}^*$$

(because $\omega = \omega(Y, Y)$ for any homotopy representation Y).

To see that D is injective suppose $D(x) = 0$. Then $d(f) \in \overline{\omega}^*.C^*$ so there is an $h: X \to X$ with $d(h)(H) \equiv \pm d(f)(H)^{-1} \bmod |G|$ for all $(H) \in \varphi(G)$ because the values of functions in C^* are ± 1. Then $fg: X \to Y$ and $d(fg)(H) \equiv \pm 1 \bmod |G|$ for all $(H) \in \varphi(G)$. By (3.1) there is an $f': X \to Y$ such that $d(f')(H) = \pm 1$ for all H. Then f' is a G-homotopy-equivalence; so $x = 0$.

Corollary **(3.9)**. — *$v(G, \lambda)$ is a finite group.*

Proof. — Clearly Pic(G) is finite.

In (6.5) we show D is an isomorphism for $\lambda = h^\infty$.

4. Modifications and finite approximations

In this section we modify a G-map $h: A \to Y$ extending it to a G-map $f: X \to Y$ such that f^H is highly-connected for all $H \subset G$. This is done in such a way that X/A is a finite complex and the dimension function of X is controlled.

Let M_f denote the mapping cone of f and Z_f the mapping cylinder. Note that M_f is a pointed G-space with a natural base point in M^G. The integral group ring of G is denoted by **Z**G. Note that **Z**G acts on $H_*(M_f)$.

We often have to use the following well-known

Lemma **(4.1)**. — *Given a commutative diagram*

$$\begin{array}{ccc} A & \xrightarrow{i} & X \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle f} & \\ & Y & \end{array}$$

of G-maps. Then there exist G-maps f' and f'' such that

$$M_i \xrightarrow{f'} M_h \xrightarrow{f''} M_f$$

is up to G-homotopy a cofibration sequence.

In the following lemma let $h: A \to Y$ be a G-map. We assume that A is 1-connected and $h: A \to Y$ is 1-connected, in order to apply the Hurewicz theorem.

Lemma **(4.2)**. — *Suppose $\widetilde{H}_j(M_h) = 0$ for $j < n \geq 2$. Let F be a free **Z**G-module. Given $\psi \in \mathrm{Hom}_{\mathbf{Z}G}(F, \widetilde{H}_n(M_h))$, there exists a G-space X obtained from A by attaching cells of type $G \times D^n$ and an extension $f: X \to Y$ of h such that:*

(i) $H_n(X, A) = \widetilde{H}_n(M_i) \cong F$;

(ii) $f'_*: F \cong \widetilde{H}_n(M_i) \to \widetilde{H}_n(M_h)$ is ψ.

(We have used the notation of (4.1) and integral homology.)

Proof. — Let $(e_j \mid j \in J)$ be a **Z**G-basis of F. Choose any base point in A and the resulting base point in Z_h and Y to make h and $A \subset Z_h$ pointed. We have the Hurewicz isomorphism $\rho : \pi_n(h) \cong \pi_n(Z_h, A) \to H_n(Z_h, A) \cong \widetilde{H}_n(M_h)$. Let

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & D^n \\ \downarrow{\scriptstyle \varphi_j} & & \downarrow{\scriptstyle \varphi_j} \\ A & \xrightarrow[h]{} & Y \end{array}$$

represent $\rho^{-1}\psi(e_j)$. We use the φ_j to attach $\coprod_{j \in J} G \times D^n \times \{j\}$ equivariantly to A thus forming X. There is a unique G-map $f : X \to Y$ extending h such that $f(g, x, j) = g\varphi_j(x)$ for $(g, x) \in G \times D^n$. Moreover $H_n(X, A) \cong F$ by the isomorphism which sends e_j to the image of $1 \in H_n(D^n, S^{n-1})$ under the characteristic map $(D^n, S^{n-1}) \to (X, A)$. Then (ii) is obvious, by the choice of φ_j.

Remark (4.3). — If F is finitely generated, then (X, A) is a relatively finite complex.

Still assume that we are in the situation of (4.2). We look at the exact homology sequence

$$\to \widetilde{H}_j(M_i) \to \widetilde{H}_j(M_h) \to \widetilde{H}_j(M_f) \to$$

and obtain, because of $\widetilde{H}_j(M_i) = 0$ for $i \neq n$, the exact sequences

$$(4.4) \qquad \begin{array}{ccccccc} \widetilde{H}_n(M_i) & \longrightarrow & \widetilde{H}_n(M_h) & \longrightarrow & \widetilde{H}_n(M_f) & \longrightarrow & 0 \\ \| \wr & & & & \| \wr & & \\ F & & & & \text{Cokernel } \psi & & \end{array}$$

$$(4.5) \qquad \widetilde{H}_k(M_h) \cong \widetilde{H}_k(M_f) \qquad k \neq n, n+1$$

$$(4.6) \qquad 0 \to \widetilde{H}_{n+1}(M_h) \to \widetilde{H}_{n+1}(M_f) \to \text{kernel } \psi \to 0.$$

Proposition (4.7). — *Let* $h : A \to Y$ *be a* G*-map. Suppose* Y *is* 1*-connected. Let* $n \geq 1$ *be an integer. There exists a* G*-space* X_n *obtained from* A *by attaching cells of type* $G \times D^i$, $i \leq n$ *and an extension* $f_n : X_n \to Y$ *of* h *such that* f_n *is* n*-connected. If* $\pi_0 A$ *is finite and* $\pi_1(A, a)$ *and* $H_*(M_h)$ *are finitely generated then* (X_n, A) *can be chosen relatively finite.*

Proof. — By attaching cells of type $G \times D^1$, we build from A a connected space X_1 and extend h to f_1. Because of $\pi_0 Y = 0$, $\pi_1 Y = 0$ this means that f is 1-connected. Then we kill the fundamental group of X_1 and extend f_1 to $f' : X_1' \to Y$; so for $n \geq 2$ we can assume that A and h are 1-connected.

Assume that h is $(n-1)$-connected. By the Hurewicz theorem then $\widetilde{H}_j(M_h) = 0$

for $j \leq n-1$. Let $\psi: F \to \widetilde{H}_n(M_h)$ be a surjection of a free **Z**G-module F, finitely generated if $\widetilde{H}_n(M_h)$ is finitely generated. Apply lemma (4.2) to this situation to obtain $f: X \to Y$. By (4.4) and (4.5) $\widetilde{H}_i(M_f) = 0$ for $i \leq n$ and by the Hurewicz theorem f is n-connected.

Proposition **(4.8)**. — *Assume the hypothesis of* (4.7) *and moreover* A *and* Y *are* G-*complexes such that* $n \geq \dim Y$, $\dim A < n$ *and* $\widetilde{H}_*(M_h^H; \mathbf{Z}/r) = 0$ *for* $H \neq \{1\}$ *and an integer* $r \equiv 0 \bmod |G|$. *Let* $f = f_n$ *be provided by* (4.7). *Then* $P = \widetilde{H}_n(M_f; \mathbf{Z})$ *is a projective* **Z**G-*module*, $\widetilde{H}_i(M_f; \mathbf{Z}) = 0$ *for* $i \neq n$, *and*

$$(4.9) \qquad 0 \to H_n(Y) \to H_n(M_f) \xrightarrow{\partial} H_{n-1}(X) \to H_{n-1}(Y) \to 0$$

is exact.

Proof. — The homology sequence of $f: X \to Y$ together with the hypothesis implies $\widetilde{H}_i(M_f) = 0$ for $i \neq n$ and the exactness of the sequence (4.9). Since X is obtained from A by adding cells of type $G \times D^i$ we have $f^H = h^H$ for $H \neq \{1\}$ and therefore $\widetilde{H}_*(M_f^H; \mathbf{Z}/r) = 0$ for $H \neq \{1\}$. These hypotheses imply that P is projective (Petrie [19]).

Now assume the following: $h: A \to Y$ is a G-map between G-complexes; Y is 1-connected; $2 \leq n \geq \dim Y$, $\dim A < n$; $H_*(M_h)$ is finitely generated; $\pi_0 A$ is finite and $\pi_1(A, a)$ is finitely generated; $\widetilde{H}_*(M_h^H; \mathbf{Z}/r) = 0$ for $H \neq 1$ for an integer $r \equiv 0 \bmod |G|$. Then we have

Proposition **(4.10)**. — *There exists a* G-*complex* X *obtained from* A *by attaching a finite number of cells of type* $G \times D^i$, $i \leq n$, *and an extension* $f: X \to Y$ *of* h *such that:*

(i) f *is* $(n-1)$-*connected*;

(ii) $\widetilde{H}_i(M_f) = 0$ *for* $i \neq n$;

(iii) $\widetilde{H}_n(M_f)$ *is a torsion group of order prime to* r.

Proof. — Let $f_1: X_1 \to Y$ be the extension provided by (4.7). Since $P = H_n(M_{f_1})$ is projective by (4.8), there is a projective module Q such that $P \oplus Q$ is a free module. There exists a free module F and a monomorphism $\mu: Q \to F$ with cokernel T a torsion group of order prime to r (Swan [22]). Attach cells of type $G \times D^{n-1}$ to X_1 by null-homotopic attaching maps forming a G-complex X_2 and extending f_1 to $f_2: X_2 \to Y$ such that $H_{n-1}(X_2, X_1) = F$ and the sequence (4.9) with (X, f) replaced by (X_2, f_2) is altered in the middle two terms by adding F to both and ∂ is replaced by $\partial' = \partial \oplus \mathrm{id}_F$. Let ψ be the monomorphism $\mathrm{id}_P \oplus \mu: P \oplus Q \to P \oplus F = H_n(M_{f_2})$. Apply lemma (4.2) to ψ and (X_2, f_2) to produce $f: X \to Y$ extending f_2. From the exact homology sequence for $M_i \to M_{f_2} \to M_f$, $i: X_2 \to X$, and the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_n(Y) & \longrightarrow & H_n(M_{f_*}) & \xrightarrow{\partial'} & H_{n-1}(X_2) & \longrightarrow & H_{n-1}(Y) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle f} & & \uparrow{\scriptstyle \psi} & & \uparrow{\scriptstyle =} & & \uparrow & & \\
0 & \longrightarrow & H_n(X) & \longrightarrow & H_n(M_i) & \longrightarrow & H_{n-1}(X_2) & \longrightarrow & H_{n-1}(X) & \longrightarrow & 0
\end{array}$$

we find $T = \text{cokernel}\ \psi \cong H_n(M_f)$ and $\widetilde{H}_i(M_f) = 0$ for $i \neq n$.

We now want to apply the preceding results essentially to each orbit bundle of a G-complex. The next lemma supplies a technical detail for this procedure.

Lemma (**4.11**). — *Let* $h : A \to Y$ *be a* G*-map between* G*-complexes. For a subgroup* K *of* G *let* W *be a* WK*-complex containing* A^K *as a subcomplex. Let* $k : W \to Y^K$ *be a* WK*-map extending* h^K. *If* WK *acts freely on* $W \setminus A^K$, *there is a unique* G*-complex* X *containing* $A \cup W$ *and an extension* f *of* h *and* k *such that* $X/A = G \times_{NK} W/G \times_{NK} A^K$.

Proof. — Let $A^K = W_{-1} \subset W_0 \subset \ldots \subset W_r = W$ where W_i is obtained from W_{i-1} by adding cells of type $WK \times D^i$. Let $A = X_{-1} \subset \ldots X_r = X$ where X_i is obtained from X_{i-1} by adding cells of type $G/K \times D^i$ whose attaching maps $G/K \times S^{i-1} \to X_{i-1}$ are the unique G-extensions of the attaching maps $WK \times S^{i-1} \to W_{i-1} \subset X_{i-1}$ for W_i. Define f by $f(x) = h(x)$ for $x \in A$ and $f(gx) = gk(x)$ for $g \in G$ and $x \in W$.

Note that in the situation of (4.11) $X^H = A^H$, $f^H = h^H$ for $H > K$. Also $X^K = W^K$, $f^K = k$.

We now introduce one of the main notions in order to handle geometrically the finiteness obstruction for G-complexes.

Definition (**4.12**). — Let Y be a G-complex. A *finite approximation* to Y consists of a finite G-complex X and a G-map $f : X \to Y$ such that $\widetilde{H}_*(M_f^H; \mathbf{Z}/|G|) = 0$ for all $H \subset G$.

We are going to show the existence of finite approximations under the following assumptions:

(**4.13**) Y^H is 1-connected whenever $H \notin S_0 : = \{H \subset G \mid \dim Y^H \leq 1\}$.

(**4.14**) $H_*(Y^H)$ is finitely generated for all $H \subset G$.

(**4.15**) $\dim Y^H < \infty$ for all $H \subset G$.

Theorem (**4.16**). — *Suppose* (4.13)-(4.15) *holds for the* G*-complex* Y. *Let* A_0 *be a finite* G*-complex such that* $A_0 = \bigcup_{H \in S_0} A_0^H$ *and* $\dim A_0^K < \dim Y^K$ *whenever* $K \notin S_0$. *Let* $h_0 : A_0 \to Y$ *be a* G*-map with* h_0^H *a homotopy-equivalence for* $H \in S_0$. *Let* m *be an integer larger than* $\dim Y$. *Then there is a finite* G*-complex* X *containing* A_0 *with* $X^H = A_0^H$ *for* $H \in S_0$ *and an extension* $f : X \to Y$ *of* h *such that for all* H *we have* $\widetilde{H}_*(M_f^H; \mathbf{Z}/|G|) = 0$ *and* f^H *is* m*-connected.*

Proof. — Let $S_0 \subset S \subset S(G)$ be closed (§ 1). Let $h: A \to Y$ be a G-map from a finite complex A such that $A = \bigcup_{H \in S} A^H$, $\widetilde{H}_*(M_h^H; \mathbf{Z}/|G|) = 0$, and h^H is m-connected for $H \in S$. Let $K \in S(G) \backslash S$ be a maximal element. Let $n > \max(m, \dim Y^K, \dim A^K)$. Since $K \notin S_0$, Y^K is 1-connected by (4.13). Use (4.10) to find a WK-complex $W \supset A^K$ and an extension $k: W \to Y^K$ of h^K such that k is n-connected and $\widetilde{H}_*(M_k; \mathbf{Z}/|G|) = 0$. Let $X \supset A \cup W$ and $f: X \to Y$ be given by (4.11). Then f^H is m-connected and $\widetilde{H}_*(M_f^H; \mathbf{Z}/|G|) = 0$ for $H \in S \cup (K)$. The result follows by induction starting with S_0.

5. Modifications and homotopy representations

Out task in this section is to convert a G-space X into a homotopy representation by attaching cells. Of course some basic structure of X like the dimension function should be preserved.

Suppose that X is a G-complex. Set $n(H) = \dim X^H$. We suppose X has the following properties:

(5.1) X is a G-complex of finite dimension.
For all $H \subset G$:

(5.2) If $n(H) \leq 2$, then X^H is homotopy-equivalent to $S^{n(H)}$.

(5.3) If $n(H) \geq 3$, then $H_{n(H)}(X^H) = \mathbf{Z}$ and $H_{n(H)-1}(X^H)$ is $\mathbf{Z}$-free.

(5.4) For each p-Sylow subgroup W_pH of WH there exists a (generalized) homotopy representation $S(H, p)$ for the group W_pH with $\operatorname{Dim} S(H, p) = \operatorname{Dim} X^H$ and a W_pH-map

$$f(H, p): X^H \to S(H, p).$$

(5.5) For $L < W_pH$ the degree of $f(H, p)^L$ is prime to p.

Remark **(5.6)**. — If X^H and $S(H, p)$ are oriented WH-manifolds then (5.5) follows if we only assume that the degree of $f(H, p)$ is prime to p. See Bredon [3].

Lemma **(5.7)**. — *Suppose* (5.1)-(5.5) *holds for* X. *Let* X^H *be a homology sphere for* $H \neq \{1\}$. *If* $\widetilde{H}_i(X) = 0$ *for* $i < n - 1 = \dim X - 1$ *then* $\widetilde{H}_{n-1}(X)$ *is a projective* **Z**G-*module.*

Proof. — Put $A = H_{n-1}(X)$. By Rim [21] it suffices to show that $A \otimes \mathbf{Z}/p$ is a projective $\mathbf{Z}/p(G_p)$-module for each p-Sylow subgroup G_p of G.

Denote the mapping cone of $f(1, p): X \to S(1, p)$ by M. Then

$$\widetilde{H}_n(M; \mathbf{Z}/p) \cong \widetilde{H}_{n-1}(X; \mathbf{Z}/p) = A \otimes \mathbf{Z}/p,$$

$$\widetilde{H}_i(M; \mathbf{Z}/p) = 0 \quad \text{for } i \neq n,$$

$$\widetilde{H}_*(M^L; \mathbf{Z}/p) = 0 \quad \text{for } \{1\} \neq L \subset G_p.$$

The last condition implies that $\widetilde{H}_*(M; \mathbf{Z}/p) \cong \widetilde{H}_*(M, M_s; \mathbf{Z}/p)$, where $M_s = \bigcup_{L \neq \{1\}} M^L$. The relative cellular chain complex $C_*(M, M_s; \mathbf{Z}/p)$ is a complex of free $\mathbf{Z}/p(G_p)$-modules having homology in only one dimension n. Therefore this homology group has finite homological dimension as a $\mathbf{Z}/p(G_p)$-module and is therefore a projective $\mathbf{Z}/p(G_p)$-module.

Proposition **(5.8)**. — *Suppose* (5.1)-(5.5) *holds for* X. *Then there exists a homotopy representation* Z *containing* X *as a subcomplex such that* $\operatorname{Dim} Z = \operatorname{Dim} X$.

Proof. — Let $S_1 := \{H \mid \dim X^H \leq 2\}$. Suppose $S_1 \subset S \subset S(G)$ and S is closed. Let $K \in S(G) \backslash S$ be maximal. Suppose that X^H is homotopy-equivalent to a sphere for $H \in S$. Put $n = n(K)$. Add cells of type $G/K \times D^i$ to X for $i \leq n-1$ to make X^K $(n-2)$-connected. Let W be the resulting space. Then $H_{n-1}(W^K)$ is a free **Z**-module. This uses the assumption that $H_{n-1}(X^K)$ is a free **Z**-module.

Extend $f(H, p)$ to a WH-map $h(H, p) : W^H \to S(H, p)$. This extension exists by equivariant obstruction theory. The key fact is that for all $L \subset W_p H$

$$\dim (W^H)^L = \dim S(H, p)^L \quad (5.4)$$

and $$\pi_i(S(H, p)^L) = 0 \quad \text{for } i < \dim S(H, p)^L.$$

Apply Lemma (5.7) with G replaced by WK. This shows that $\widetilde{H}_{n-1}(W^H)$ is a projective WK-module. Let F′ be a free WK-module such that $\widetilde{H}_{n-1}(W^K) \oplus F' = F$ is a free module. This exists by the Eilenberg Swindle. Use this fact to add cells of type $G/K \times D^{n-1}$ to W with null-homotopic attaching maps $S^{n-2} \to W^K$ converting $H_{n-1}(W^K)$ to F. So we suppose $H_{n-1}(W^K)$ is a free WK-module. Add cells of type $G/K \times D^n$ to W to form Y such that $H_n(Y^K, W^K) = F$ and $H_n(Y^K, W^K) \to H_{n-1}(W^K)$ is an isomorphism. Then Y^K is n-dimensional, simply-connected, and has the homology of S^n, hence $Y^K \simeq S^n$. As above we can extend $h(H, p)$ to Y^H. By induction over S the proposition follows.

Finally we describe another construction of homotopy representations which has some similarity to the modification procedure of (5.8).

Proposition **(5.9)**. — *Let* Y *be a generalized homotopy representation of dimension at least* 3. *Set* $1 + n(H) = \operatorname{Dim} Y(H)$. *Let* A *be a* G-*complex and let* $f: A \to Y$ *be a* G-*map such that the following holds*:

(5.10) $A^H \simeq Y^H \simeq S^{n(H)}$ *and* $\dim A^H = n(H)$ *for* $H \neq \{1\}$.

(5.11) *The degree of* f^H *is prime to* $|G|$ *for* $H \neq \{1\}$.

(5.12) $\dim A < n(1)$. *Set* $n = n(1)$.

Then there exists a homotopy representation X *obtained from* A *by attaching cells of type* $G \times D^i$, $i \leq n$, *and a* G-*map* $F : X \to Y$ *extending* f. *The degree of* F *is necessarily prime to* $|G|$.

Proof. — By attaching cells of type $G \times D^i$, $i \leq n-1$, we can obtain an $(n-2)$-connected space $B \supset A$ and a G-map $g : B \to Y$ such that (5.10)-(5.12) is satisfied for (B, g) instead of (A, f). The homology sequence for g shows that $\widetilde{H}_i(M_g)$ is zero for $i \neq n$. Since B is an $(n-2)$-connected $(n-1)$-dimensional complex $H_n(M_g)$ is **Z**-free. By (5.11) $\widetilde{H}_*(M_g^H; \mathbf{Z}/|G|) = 0$ for $H \neq 1$. By Petrie [19] $M = H_n(M_g)$ is a projective **Z**G-module. Attaching cells of type $G \times D^{n-1}$ to B by null-homotopic attaching maps and extending g trivially amounts to attaching cells of type $G \times D^n$ to M_g by trivial maps. This changes M by adding free **Z**G-modules. Hence by further enlarging B as in the proof of (5.8), we can actually arrange that M is a free **Z**G-module. The exact homology sequence of $g : B \to Y$ (compare (4.9)) now yields the exact sequence

$$0 \to \mathbf{Z} \to M \xrightarrow{d} H_{n-1}(B) \to 0.$$

We choose a **Z**G-basis $(e_j \mid j \in J)$ of M and attach cells of type $G \times D^n$ to B by using $d(e_j) \in H_{n-1}(B) \cong \pi_{n-1}(B)$ as homotopy classes of attaching maps. The resulting space X is n-dimensional, 1-connected, has the homology of S^n, and is therefore homotopy-equivalent to S^n. Because of $X^H = A^H$ for $H \neq \{1\}$ and (5.10) X is a homotopy representation. Let $F : X \to Y$ be an extension of f (which obviously exists). Let P be a p-Sylow subgroup of G. By Smith theory $\deg F^P \not\equiv 0 \bmod p$ implies $\deg F \not\equiv 0 \bmod p$. Since $F^P = f^P$, (5.11) implies $\deg F^P$ and hence $\deg F$ is prime to p.

6. Swan-Modifications

We show in this section that the homomorphism (3.7)

$$D : v(G, h^\infty) \to \operatorname{Pic}(G)$$

is an isomorphism. We have already seen in (3.8) that D is injective. Using the structure of Pic(G) as described in (3.6) we see that D is surjective if we can find a map $f : X \to Y$ between homotopy representations such that its degree function (3.3) $(H) \mapsto \operatorname{degree} f^H = d(f)(H)$ has given values prime to $|G|$.

We achieve this aim by modifying a given homotopy representation Y so that the modified sphere X admits a map $f : X \to Y$ with suitable degree function. A basic ingredient in this modification procedure will be taken from Swan's paper [23]. Therefore we call X a Swan-modification of Y.

Here are our assumptions.

(6.1) Y is a generalized homotopy representation with the following properties.

i) Iso(Y) is closed under intersections.
ii) h-dim $Y^H = \dim Y^H$ whenever

$$H \in S_0(Y) := \{K \in \operatorname{Iso}(Y) \mid h\text{-}\dim Y^K \leqslant 2\}.$$

iii) If $H \in \mathrm{Iso}(Y)$, $H \notin S_0(Y)$ then for all $H < K$, $H \neq K$

$$h\text{-dim } Y^H \geqslant h\text{-dim } Y^K + 2.$$

It follows from (6.1) i) that to each $H \subset G$ with $Y^H \neq \varnothing$ there exists a unique minimal $m(H) \in \mathrm{Iso}(Y)$ such that $H \subset m(H)$. This is used in the sequel.

(6.2) Let $z \in C(G)$ be a function with the following properties (depending on Y):

i) $z(H) = z(m(H))$ for each $H \subset G$ with $Y^H \neq \varnothing$.

ii) $z(H) = 1$ if $H \in S_0(Y)$.

iii) $z(H)$ is prime to $|G|$ for all $H \subset G$.

Theorem **(6.3)**. — *Let* Y *be a generalized homotopy representation satisfying* (6.1) *and* $z \in C(G)$ *a function satisfying* (6.2). *Then there exists a homotopy representation* X *with* $\mathrm{Dim}\, X = \mathrm{Dim}\, Y$ *and a* G-*map* $f: X \to Y$ *with degree function* z.

The proof of 6.3 will be given by induction over orbit types. The next proposition is used in the induction step.

Proposition **(6.4)**. — *Let* Z *be an n-dimensional* G-*complex which is homotopy-equivalent to* S^n $(n \geqslant 3)$. *Suppose* Z *is obtained from its* $(n-1)$-*skeleton* Z_{n-1} *by attaching cells of type* $G \times D^n$. *Let* $k \in \mathbf{Z}$ *be prime to* $|G|$. *Then there exists an n-dimensional* G-*complex* B *obtained from* Z_{n-1} *by attaching cells of type* $G \times D^{n-1}$, $G \times D^n$ *and a* G-*map* $\varphi: B \to Z$ *such that*:

i) B *is homotopy-equivalent to* S^n.

ii) degree $\varphi = k$.

iii) $\varphi \mid B_{n-2} = \mathrm{id}$ (*note*: $B_{n-2} = Z_{n-2}$).

Proof. — Let $\mathbf{Z}_\varepsilon = H_n(Z; \mathbf{Z})$ be the $\mathbf{Z}G$-module where $\varepsilon: G \to \mathrm{Aut}(\mathbf{Z})$ indicates the G-action. Then there exist free $\mathbf{Z}G$-modules F_1 and F_2 and a $\mathbf{Z}G$-isomorphism

$$\alpha: \mathbf{Z}_\varepsilon \oplus F_2 \to \mathbf{Z}_\varepsilon \oplus F_1$$

of degree k, *i.e.* the composition of φ with the injection $\mathbf{Z} \to \mathbf{Z} \oplus F_2$ and the projection $\mathbf{Z} \oplus F_1 \to \mathbf{Z}$ is multiplication by k. This is proved as Lemma (6.1) in Swan [23], using the left ideals $(r, N_\varepsilon) \subset \mathbf{Z}G$ with $N_\varepsilon = \sum_{g \in G} \varepsilon(g)\, g \in \mathbf{Z}G$ and $rk \equiv 1 \bmod |G|$. The cellular chain complex

$$0 \to \mathbf{Z}_\varepsilon \to C_n \to C_{n-1} \to \dots$$

of Z can then be modified by Swan [23], Lemma (2.1), so as to yield an exact sequence

$$0 \to \mathbf{Z}_\varepsilon \to C_n \oplus F_1 \underset{d}{\to} C_n \oplus F_2 \to C_{n-1} \to \dots$$

so that the obvious projection onto the complex C_* is a chain map of degree k, *i.e.* induces multiplication by k on $\mathbf{Z}_\varepsilon$. We now realize this chain complex and this chain map geometrically (compare the example at the end of § 1). Attach cells of type $G \times D^{n-1}$ to Z_{n-1} by trivial attaching maps, one cell for each element of a $\mathbf{Z}G$-basis of F_2. Let

B_{n-1} be the resulting space. Let $(a_j \mid j \in J)$ be a $\mathbf{Z}G$-basis of $C_n \oplus F_1$. The image of d is contained in $H_{n-1}(B_{n-1})$. Since $n \geq 3$ we have a Hurewicz isomorphism $h : \pi_{n-1}(B_{n-1}) \cong H_{n-1}(B_{n-1})$. For each $j \in J$ we attach a cell of type $G \times D^n$ to B_{n-1} using $h^{-1}d(a_j) \in \pi_{n-1}(B_{n-1})$ as class of an attaching map $\varphi'_j : \{1\} \times S^{n-1} \to B_{n-1}$. The resulting space B has the correct cellular chain complex and is therefore homotopy-equivalent to S^n. The identity obviously has an extension $\varphi' : B_{n-1} \to Z_{n-1}$. An extension of φ' over the n-cell corresponding to a_j is given by a map

$$\varphi_j : (D^n, S^{n-1}) \to (Z_n, Z_{n-1})$$

such that $\varphi_j \mid S^{n-1} = \varphi' \varphi'_j$ and $\varphi_{j*}(1) = \mathrm{pr}(a_j) \in C_n = H_n(Z_n, Z_{n-1})$. Using the Hurewicz isomorphism $\pi_n(Z_n, Z_{n-1}) \cong H_n(Z_n, Z_{n-1})$ we can find φ_j having these properties. The resulting map $\varphi : B \to Z$ induces the correct chain map and therefore has degree k.

Proof of Theorem (6.3). — For each closed family F of subgroups we construct a G-complex X(F) and a G-map $f_F : X(F) \to Y$ with the following properties:

a) $\mathrm{Iso}(X(F)) = F \cap \mathrm{Iso}(Y)$.

b) For $K \in F$ the space $X(F)^K$ is homotopy-equivalent to Y^K and

$$\dim X(F)^K = h\text{-}\dim Y^K.$$

c) For $K \in F$ the map f^K has degree $z(K)$.

The set $S_0(Y)$ is a closed family. We put $X(S_0(Y)) = \bigcup_{H \in S_0(Y)} Y^H$ and $f_{S_0(Y)}$ shall be the inclusion. Now let $F \supset S_0(Y)$ be closed. Take a maximal H not in F and put $F' = F \cup (H)$. We want to show the existence of $f_{F'} : X(F') \to Y$ satisfying *a)-c)*. If $H \notin \mathrm{Iso}(Y)$ we simply take $f_{F'} = f_F$. If $H \in \mathrm{Iso}(Y)$ we apply (5.9) to the WH-map $f^H : X(F)^H \to Y^H$. We obtain a WH-space $X'(F')$ and a WH-map $f' : X'(F') \to Y^H$. But f' may have the wrong degree. Let ℓ be its degree. By Smith theory ℓ is prime to $|G|$. Choose k such that $k\ell \equiv z(H) \bmod |G|$. We apply (6.4) in case $X'(F')$ for Z and WH for G and obtain a WH-map $\varphi : B \to X'(F')$ of degree k. The construction of (6.4) shows $B \supset X(F)^H$. We use (3.1) in order to alter $f'\varphi$ so as to obtain a map $f'' : B \to Y^H$ of degree k which coincides with f_F^H on $X(F)^H$. Now we apply (4.11) to f'' and obtain the desired map $f_{F'}$.

Theorem **(6.5)**. — *The homomorphism*

$$D : v(G, h^\alpha) \to \mathrm{Pic}(G)$$

is an isomorphism.

Proof. — We have already mentioned that this follows from (3.8) and (6.3).

Theorem **(6.6)**. — *Let* Y *be a generalized homotopy representation satisfying* (6.1). *Then* Y *is* G*-homotopy-equivalent to a homotopy representation.*

Proof. — Apply (6.3) to the function z with constant value one.

7. Finiteness obstructions and finite approximations

Our aim in this section is to compare $V(G, h)$ and $V(G, h^\infty)$. Needed are conditions under which a G-complex is homotopy-equivalent to a finite G-complex. The obvious tool for expressing these conditions is an equivariant generalization of the Swan [23], Wall [26] finiteness obstruction. The straightforward generalized definition for the equivariant finiteness obstruction does not relate well to the geometric aspects of our homotopy representation groups and moreover does not give an additive function on these groups. We obtain more insight using the definition (7.23).

We consider G-complexes Y having the properties (4.13)-(4.15) which we recall for completeness

(7.1) Y^H is 1-connected whenever
$$H \notin S_0 := \{H \subset G \mid \dim Y^H \leq 1\}, \qquad H \in \operatorname{Iso} Y.$$

(7.2) $H_*(Y^H)$ is finitely generated for all $H \subset G$.

(7.3) $\dim Y^H < \infty$ for all $H \subset G$.

(7.4) $Y(S_0) := \bigcup_{H \in S_0} Y^H$ is a finite G-complex.

Under these assumptions there exists a *finite approximation* $f: X \to Y$ which is a G-map from a finite complex X such that for all $H \subset G$

(7.5) f^H is m-connected ($m > \dim Y$ given).

(7.6) $\tilde{H}_*(M_f^H; \mathbf{Z}/r) = 0$, ($r \equiv 0 \bmod |G|$ given).

See Theorem (4.16).

Let r be a given multiple of $|G|$.

Let $K_0(G, r)$ be the Grothendieck group of finitely generated $\mathbf{Z}G$-modules M with $M \otimes \mathbf{Z}/r = 0$. (If $0 \to A \to B \to C \to 0$ is exact, then $B = A + C$ in $K_0(G, r)$.) Since a module M with $M \otimes \mathbf{Z}/r = 0$ has projective dimension less than or equal to 1 (see [24]) there is a natural homomorphism

(7.7) $\tau: K_0(G, r) \to \tilde{K}_0(G)$.

Here $K_0(G)$ is the Grothendieck group of finitely generated $\mathbf{Z}G$-modules of finite projective dimension and $\tilde{K}_0(G)$ the quotient of this group by the subgroup generated by free $\mathbf{Z}G$-modules.

When X is a finite-dimensional pointed G-complex with base-point $x_0 \in X^G$ and

(7.8) $\tilde{H}_j(X) = H_j(X, x_0)$ has finite projective dimension over $\mathbf{Z}G$ for all j (and is finitely generated for all j)

resp.

(7.9) $\tilde{H}_j(X; \mathbf{Z}/r) = 0$ for all j, we define

(7.10) $$\chi(X) = \Sigma(-1)^i \widetilde{H}_i(X) \in \widetilde{K}_0(G)$$

resp.

(7.11) $$\chi'(X) = \Sigma(-1)^i \widetilde{H}_i(X) \in K_0(G, r).$$

Note that (7.9) implies $\widetilde{H}_j(X) \otimes \mathbf{Z}/r = 0$ by the Universal Coefficient Theorem. Note also

Lemma **(7.12)**. — *$\chi(X)$ is defined whenever $\chi'(X)$ is defined and $\chi(X) = \tau\chi'(X)$.*

The essential elementary fact about χ is this: If C_* is a chain complex of finitely generated projective **Z**G-modules such that each homology group $H_j(C_*)$ has finite projective dimension, then

(7.13) $$\Sigma(-1)^i H_i(C_*) = \Sigma(-1)^i C_i$$

in $\widetilde{K}_0(G)$. The left hand side is zero if the C_i are free. In particular, if X is a finite G-complex such that G acts freely on $X \backslash \{x_0\}$ and (7.8) holds then $\chi(X) = 0$.

Lemma **(7.14)**. — *If $A \subset X$ is a G-subcomplex and $\widetilde{H}_*(-; \mathbf{Z}/r) = 0$ on two of the three spaces A, X or X/A then χ' is defined on all three and*

$$\chi'(X) = \chi'(A) + \chi'(X/A).$$

Proof. — Use the long exact homology sequence for $A \to X \to X/A$.

Notation **(7.15)**. — $\kappa(G) = \prod_{(H) \in \varphi(G)} \widetilde{K}_0(WH)$.

For a G-space X and a subgroup H of G we put

(7.16) $$X^H_s := \{x \in X \mid H \subset G_x \subset NH, H \neq G_x\}.$$

Lemma **(7.17)**. — *If $\widetilde{H}_*(M^H; \mathbf{Z}/r) = 0$ for all H, then $\widetilde{H}_*(M^H_s; \mathbf{Z}/r) = 0$ for all H.*

Proof. — Use $M^K \cap M^L = M^{K \cdot L}$ and the Mayer-Vietoris sequence.

Corollary **(7.18)**. — *Whenever $\widetilde{H}_*(M^H; \mathbf{Z}/r) = 0$ for all H and an integer $r \equiv 0 \bmod |G|$ then $\chi'(M^H)$, $\chi'(M^H_s)$ and $\chi'(M^H/M^H_s)$ are defined as elements of $K_0(WH, r)$ for all H and*

$$\chi'(M^H) = \chi'(M^H_s) + \chi'(M^H/M^H_s).$$

Corollary **(7.19)**. — *If $\widetilde{H}_*(M^H; \mathbf{Z}/r) = 0$ for all H, $r \equiv 0 \bmod |G|$, and M is a finite G-complex, then*

$$\chi(M^H/M^H_s) = 0 = \chi(M^H) - \chi(M^H_s).$$

Proof. — WH acts freely on $M^H \backslash M^H_s$. Use the remarks following (7.13).

Definition (**7.20**). — Let Y be a G-complex satisfying (7.1)-(7.4). Let $f: X \to Y$ be a finite approximation. We define $\sigma(Y,f) \in \kappa(G)$ by

$$\sigma(Y,f)(H) = \chi(M_f^H) - \chi(M_{f_s}^H) \in \widetilde{K}_0(WH).$$

We are going to show that $\sigma(Y,f)$ is independent of f and is the obstruction for Y being G-equivalent to a finite complex. The first observation is the following consequence of (7.19).

(**7.21**) If Y is a finite complex then $\sigma(Y,f) = 0$ for any finite approximation $f: X \to Y$.

Proposition (**7.22**). — *Let* $f: X \to Y$ *and* $f_1: X_1 \to Y$ *be finite approximations to* Y. *Then* $\sigma(Y,f) = \sigma(Y,f_1)$.

Proof. — There exists a finite approximation $f': X' \to Y$ such that $(f')^H$ is $(2 + \dim X)$-connected for all H (see Theorem (4.16)). Then there exists a map $h: X \to X'$ such that $f'h$ is G-homotopic to f. Now apply (4.1), (7.18), and (7.19) to show that $\sigma(Y,f) = \sigma(Y,f')$.

Definition (**7.23**). — Define the *finiteness obstruction* $\sigma(Y) \in \kappa(G)$ for the complex Y satisfying (7.1)-(7.4) to be $\sigma(Y,f)$ for any finite approximation $f: X \to Y$.

Theorem (**7.24**). — *Let* Y *be a* G-*complex such that*

i) $\dim Y^H$ *is* 1-*connected whenever* $\dim Y^H > 2$.
ii) Y^H *is finite whenever* $\dim Y^H \leq 2$.
iii) $\dim Y^H$ *is finite for all* H.
iv) $H_*(Y^H)$ *is finitely generated.*
v) $\sigma(Y) = 0$.

Then there exists a finite G-*complex* X *which is* G-*homotopy-equivalent to* Y *and such that* $\dim X^H = \dim Y^H$ *for all* H.

Proof. — By induction it suffices to prove the following proposition.

Proposition (**7.25**). — *Let* Y *be a* 1-*connected* G-*complex of dimension at least* 3. *Suppose* $\pi_0(Y_s)$, $\pi_1(Y_s)$ *and* $H_*(Y, Y_s)$ *are finitely generated. Suppose moreover that* $\sigma(Y)(1) = 0$. *Let* $f: A = A_s \to Y_s$ *be a* G-*homotopy-equivalence. Then there exists a* G-*space* X *obtained from* A *by attaching a finite number of cells of type* $G \times D^k$, $k \leq \dim Y$ *and a* G-*homotopy-equivalence* $F: X \to Y$ *with* $F|A = f$.

Proof. — Let $n = \dim Y$. By (4.7) we can find $h: B \to Y$ such that h is $(n-1)$-connected and (B, A) is a relative G-free complex of relative dimension $n-1$. This implies that $\widetilde{H}_i(M_h)$ is zero for $i \neq n$ and $\widetilde{H}_*(M_h^H) = 0$ for $H \neq 1$. Then $M = H_n(M_h)$ is a projective **Z**G-module. (See proof of (5.3).) In fact $0 = \sigma(Y)(1) = \pm M \in \widetilde{K}_0(G)$. (Grant this for the moment.) Thus M is stably free. Add cells of type $G \times D^{n-1}$ to B with trivial attaching map to make $H_n(M_h)$ a free module F. Apply (4.2) with A replaced by B and ψ the identity map of M. This lemma produces a complex X from B and an extension $f: X \to Y$ of h which satisfies (4.2) (i)-(ii). From (4.4)-(4.6) and $H_{n+1}(M_h) = 0$, we see $\widetilde{H}_*(M_f) = 0$. Since $\pi_i(f) = 0$, $i \leq n-1$, f is a homotopy-equivalence.

To see that $\pm M = \sigma(Y)(1)$, let $f': X' \to Y$ be a finite approximation (4.12) with $\pi_i(M_{f'}^H) = 0$ for $i \leq m > \operatorname{Dim} Y(1)$. This exists by (4.16). Then there is a G-map $k: X \to X'$ such that $f = f'k$. Use the cofibration $M_k \to M_f \to M_{f'}$ and the corresponding one for k_s, f_s and f'_s to show $\sigma(Y, f) = \sigma(X', k) + \sigma(Y, f')$. This requires (7.12) and (7.14). By (7.19) $\sigma(X', k) = 0$ because M_k and M_{k_s} are finite complexes. Since $f_s = h_s$ is a homotopy-equivalence (because h^H is for $H \neq 1$), $\chi(M_{f_s}) = 0$. Thus $\sigma(Y)(1) = \sigma(Y, f') = \sigma(Y, f) = \chi(M_f) = \pm M$.

Here is a brief comparison of $\sigma(Y)$ with the Wall-Swan type definition for an equivariant finiteness obstruction. Consider the relative chain complex $C_*(Y, Y_s)$ and let $f: P_* \to C_*(Y, Y_s)$ be a chain-homotopy-equivalence where P_* consists of finitely generated projective **Z**G-modules. (The existence uses (7.2).) Then

$$\sigma'(Y)(1) = \Sigma(-1)^i P_i \in \widetilde{K}_0(G)$$

is independent of the choice of f; moreover, (Y, Y_s) is relative Y_s G-homotopy-equivalent to a finite complex (Y', Y_s) if and only if $\sigma'(Y)(1)$ is zero. We put

(7.26) $$\sigma'(Y)(H) = \sigma'(Y^H)(1) \in \widetilde{K}_0(WH).$$

It is not difficult to see that $\sigma'(Y) = \sigma(Y)$.

8. The product theorem for finiteness obstructions

The aim of this section is to convert $\sigma(Y)$ into a function $\rho(Y)$ which is additive for homotopy representations and vanishes when Y is G-homotopy-equivalent to a finite homotopy representation. Since σ and ρ are defined in terms of the χ' from the preceding section, we first develop some additional properties of χ'. Let $p_i: X_i \to Y_i$ be a finite approximation (7.5) (7.6) considered as an inclusion and let $\chi'(Y_i, X_i)$ denote $\chi'(M_{f_i})$. Additivity of χ' (compare (7.14)) gives

(8.1) $$\chi'(Y_1 \times Y_2, X_1 \times X_2) = \chi'(Y_1 \times Y_2, X_1 \times Y_2 \cup Y_1 \times X_2) + \chi'(X_1 \times Y_2, X_1 \times X_2) + \chi'(Y_1 \times X_2, X_1 \times X_2).$$

Lemma **(8.2)**. — $\chi'(Y_1 \times Y_2, X_1 \times Y_2 \cup Y_1 \times X_2) = 0$.

Proof. — Using the Künneth-formula for $H_k(Y_1 \times Y_2, X_1 \times Y_2 \cup Y_1 \times X_2)$ we see that it suffices to show the following:

$$H_i(Y_1, X_1) \otimes H_j(Y_2, X_2) \quad \text{and} \quad \operatorname{Tor}(H_i(Y_1, X_1), H_j(Y_2, X_2))$$

define the same element in $K_0(G, r)$. This is proved in the next lemma.

Lemma **(8.3)**. — *Let* M *and* N *be finitely generated* **Z**G*-modules which are torsion groups with torsion prime to r. Then* $M \otimes_{\mathbf{Z}} N = \mathrm{Tor}_{\mathbf{Z}}(M, N)$ *in* $K_0(G, r)$.

Proof. — Let $0 \to F \to P \to M \to 0$ be a resolution with P finitely generated and projective and F free. Then we obtain the exact sequence

$$0 \to \mathrm{Tor}(M, N) \to F \otimes N \to P \otimes N \to M \otimes N \to 0.$$

Moreover there exists an embedding $F \to P$ with cokernel $B = P/F$ of order prime to the order of N (see Swan [22]). This gives $\mathrm{Tor}(B, N) = 0$, $B \otimes N = 0$ and therefore $F \otimes N \cong P \otimes N$. The exact sequence above now yields the desired result.

Note

$$\textbf{(8.4)} \qquad \chi'(Z \times (Y, X)) = \sum_k (-1)^k H_k(Z \times (Y, X)) = \sum_{i,j} (-1)^{i+j} H_i(Z; H_j(Y, K)).$$

Let $C_*(Z)$ denote the cellular chain complex of Z. Then $H_*(Z; H_j(Y, X))$ is the homology of $C_*(Z) \otimes H_j(Y, X)$; so that we obtain

$$\textbf{(8.5)} \qquad \sum_i (-1)^i H_i(Z; H_j(Y, X)) = \sum_i (-1)^i C_i(Z) \otimes H_j(Y, Z)$$

and from (8.4) and (8.5)

$$\textbf{(8.6)} \qquad \chi'(Z \times (Y, X)) = \sum_j (-1)^j \Big(\sum_i (-1)^i C_i(Z)\Big) \otimes H_j(Y, Z).$$

In order to deduce from (8.6) a more conceptual result we need an action of the Burnside ring A(G) on $K_0(G, r)$. This is defined as follows. Let S be a finite G-set and F(S) the free abelian group on S considered as **Z**G-module. Let M be a **Z**G-torsion module of torsion prime to *r*. We put

$$\textbf{(8.7)} \qquad [S][M] := [F(S) \otimes_{\mathbf{Z}} M] \in K_0(G, r).$$

By exactness of $F(S) \otimes_{\mathbf{Z}}$ this is additive in S and M and yields a well-defined module structure

$$\textbf{(8.8)} \qquad A(G) \otimes K_0(G, r) \to K_0(G, r),$$

written $x \otimes y \mapsto xy$.

Coming back to (8.6) we note that in A(G) the relation $\Sigma(-1)^i[S_i] = [Z]$ holds, where S_i is the G-set of *i*-cells of Z. Therefore we obtain from (8.1), (8.2) and (8.6)

Proposition **(8.9)**

$$\chi'(Z \times (Y, X)) = [Z]\chi'(Y, X)$$

and

$$\chi'(Y_1 \times Y_2, X_1 \times X_2) = [Y_1]\chi'(Y_2, X_2) + [Y_2]\chi'(Y_1, X_1).$$

As to the second equality we remark that Y_i defines an element in A(G) because Y_i^H and X_i^H have the same Euler-characteristic and X_i is a finite G-complex. (Compare tom Dieck [4], [8].)

Suppose Y_1, Y_2 are homotopy representations with finite approximations $f_i: X_i \to Y_i$. In order to study finiteness obstructions we can assume that f_i is an inclusion. If we write the join $A * B$ as $A \times CB \cup CA \times B$ with $A \times CB \cap CA \times B = A \times B$ (CA is the cone on A), the additivity of χ' applied to this decomposition gives:

(8.10) $$\chi'(Y_1 * Y_2, X_1 * X_2) + \chi'(Y_1 \times Y_2, X_1 \times X_2) = \chi'(Y_1, X_1) + \chi'(Y_2, X_2)$$

so that $\chi'(Y_1 \times Y_2, X_1 \times X_2)$ is responsible for the deviation from additivity. Using (8.9) we can rewrite (8.10) as follows

(8.11) $$\chi'(Y_1 * Y_2, X_1 * X_2) = (1 - [Y_2])\chi'(Y_1, X_1) + (1 - [Y_1])\chi'(Y_2, X_2).$$

If we identify elements in A(G) (via Euler characteristics of fixed point sets) with functions in C(G) (see tom Dieck [4], [8]) then $1 - [Y]$ for a homotopy representation [Y] is the function $(H) \mapsto (-1)^{\mathrm{Dim}\, Y(H)}$. This is a unit in A(G) which we denote by $e(Y)$. Note that

(8.12) $$e(Y_1 * Y_2) = e(Y_1)e(Y_2).$$

From (8.11) and (8.12) we see that

(8.13) $$e(Y)\chi'(Y, X)$$

is additive for pairs (Y, X) under the join operation.

Remark **(8.14)**. — The reader should keep in mind that $\chi'(Y, X)$ and $\chi(Y, X)$ depend on the choice of X. In the sequel we have to use the fact that the finiteness obstruction $\sigma(Y)$ can be computed from χ'-invariants of fixed point sets.

If $H < G$ we have a restriction homomorphism.

(8.15) $$\mathrm{res}^G_H : K_0(G, r) \to K_0(H, r)$$

and an induction homomorphism

(8.16) $$\mathrm{ind}^G_H : K_0(H, r) \to K_0(G, r)$$

the latter being induced by $M \mapsto \mathbf{Z}G \otimes_{\mathbf{Z}H} M$.

Proposition **(8.17)**. — *There exist integers* $a_{H,L}$, $H \subset G$, $L \subset NH$, *depending only on the structure of* G *such that for any finite-dimensional* G*-complex* Z *such that* $H_*(Z^H)$ *is finitely generated for all* $H \subset G$ *and* $\tilde{H}_*(Z^H; \mathbf{Z}/r) = 0$ *for all* $H \subset G$ *the relation*

$$\chi'(Z_s) = \sum_{1 \neq H, L \subset NH} a_{H,L}\, \mathrm{ind}^G_L\, \mathrm{res}^{NH}_L\, \chi'(Z^H)$$

holds. (Here Z^H is considered as NH-space.)

Proof. — Let X be a G-complex which is covered by finitely many subcomplexes X_a, $a \in A$, in such a way that for each $g \in G$ there exists $b \in A$ with $gX_a = X_b$. We put $b = ga$ in this case and obtain a G-action on A. For $B \subset A$ we put $X_B = \bigcap_{b \in B} X_b$.

Suppose for all $B \subset A$ we have $\widetilde{H}_*(X_B; \mathbf{Z}/r) = 0$, so that $\chi'(X_B)$ is defined. Put $G_B = \{g \in G \mid gB = B\}$. Let $P(A)$ be the set of subsets of A with its induced G-action. Then one has, using additivity of χ',

$$\chi'(X) = \sum_{[B] \in P(A)/G} (-1)^{|B|} \chi'(G \times_{G_B} X_B).$$

We apply this to X_s being the union of the X^H, $H \neq 1$.

Now we use $\chi'(G \times_{NH} Z^H) = \mathrm{ind}_{NH}^G \chi'(Z^H)$ and observe that $Z^H \cap Z^K = Z^{HK}$, where HK is the subgroup generated by H and K.

Using (8.17) and the additivity of (8.13) we now define a new invariant $\theta(Y, X) \in K_0(G, r)$ for a homotopy representation Y with finite approximation $X \subset Y$. We put

$$\theta'(Y, X) = \chi'(Y, X) - \chi'(Y_s, X_s)$$

and

(8.18) $$\theta(Y, X) = e(Y)\chi'(Y, X) - \Sigma a_{H,L} \mathrm{ind}_L^G \mathrm{res}_L^{NH} e(Y^H)\chi'(Y^H, X^H).$$

Lemma **(8.19)**. — $\tau\theta(Y, X)$ *is independent of the choice of the finite approximation* X.

Proof. — Using (8.9), $e(Y) = 1 - [Y]$ and $[X] = [Y]$, we obtain

$$\theta(Y, X) = \theta'(Y, X) - \theta'(X \times (Y, X)).$$

Since $\tau\theta'(Y, X) = \sigma(Y)(1)$, (7.22) and (7.23) show $\tau\theta'(Y, X)$ is independent of X. Let X' be a second finite approximation to Y. Then $[X'] = [X]$ in A(G); so by (8.9) and (8.17), $\theta'(X \times (Y, X)) = \theta'(X' \times (Y, X))$. Altogether this shows that $\tau\theta(Y, X)$ is independent of X.

In view of Lemma (8.19) we define

(8.20) $$\rho(Y) \in \kappa(G)$$

by $\rho(Y)(H) = \tau\theta(Y^H, X^H) \in \widetilde{K}_0(WH)$. From the additivity of (8.13) we then obtain

Proposition **(8.21)**. — *Let* Y_1, Y_2 *be homotopy representations. Then we have* $\rho(Y_1 * Y_2) = \rho(Y_1) + \rho(Y_2)$.

Proposition **(8.22)**. — *A homotopy representation* Y *is finite if and only if* $\rho(Y) = 0$.

Proof. — Suppose $\rho(Y) = 0$. Let $S \subset S(G)$ be a closed family. We show by induction that $Y(S) = \bigcup_{H \in S} Y^H$ can be assumed finite. Let $K \in S(G) \backslash S$ be maximal and let Y(S) be finite. Then also Y_s^K considered as WK-space is finite. It is sufficient to show that Y^K is finite as WK-space. Therefore we need only consider the situation $K = 1$, $WK = G$, Y_s finite. Then we can find a finite approximation $X \supset Y_s$ to Y such that $X^H = Y^H$ for $H \neq 1$. In this case therefore $\sigma(Y)(H) = 0$ for $H \neq 1$ and $\sigma(Y)(1) = \tau\chi'(Y, X)$. This follows from (7.20) and (7.23) using (7.12) and

$\chi(M_f) = \chi(Y, X)$ (after f is made an inclusion). Similiarly $\rho(Y)(1) = \tau(e(Y)\chi'(Y, X))$ by (8.18). Since $e(Y)$ is a unit in the Burnside ring, $\chi'(Y, X) = 0$; so $\sigma(Y)(1) = 0$. As $\sigma(Y)$ is then zero, Y is G-homotopy-equivalent to a finite homotopy representation by (7.24). The converse is obvious.

As a corollary to (8.21) and (8.22) we obtain

Theorem **(8.23)**. — *The assignment* $Y \mapsto \rho(Y)$ *induces a homomorphism*

$$\rho : V(G, h^\infty) \to \kappa(G)$$

and the sequence

$$0 \to V(G, h) \to V(G, h^\infty) \underset{\rho}{\to} \kappa(G)$$

is exact.

As a first application of (8.23) we show

Theorem **(8.24)**. — *Let* Y *be a generalized homotopy representation. There exists a homotopy representation* Z *such that* $Y * Z$ *has the* G*-homotopy type of* S(V) *for a representation space* V *of* G.

Proof. — Since $\kappa(G)$ is a finite group there exists an integer n such that the n-fold join $X = Y * \ldots * Y$ is finite. If X is finite we have an equivariant Spanier-Whitehead dual DX and a duality map $X * DX \to S(V)$ for a suitable V (see Wirthmüller [27]). Then DX must be a finite homotopy representation and the duality map must be a G-homotopy-equivalence.

A more abstract approach to the concepts in this section has been presented in tom Dieck [29].

9. Functorial properties

If H is a subgroup of G, then restricting the group action from G to H induces a homomorphism

$$\mathrm{res}^G_H : V(G, \lambda) \to V(H, \lambda).$$

(See section 2 for the possible λ.) There is an induction homomorphism in the other direction. The relation between induction and restriction is an important part of the structure of these groups. This will be evident in section 10.

If X is a homotopy representation of H, $\mathrm{ind}^G_H X$ is the homotopy representation of G defined by the obvious action of G on

$$\mathrm{ind}^G_H X = \mathop{*}_{gH \in G/H} gH \times_H X. \qquad (9.1)$$

An immediate consequence of (9.1) is

Proposition **(9.2)**. — *Let* H *and* K *be subgroups of* G *and let* $G/H = \coprod_i K/K_i$ *be the decomposition of* G/H *into its* K*-orbits. Then*

$$\mathrm{Dim}(\mathrm{ind}_H^G X)(K) = \sum_i \mathrm{Dim}\, X(K_i) = \Sigma\, \mathrm{Dim}\, X(gKg^{-1} \cap H).$$

This last sum is over the double cosets $KgH \in K\backslash G/H$.

The construction (9.1) induces a homomorphism

$$\textbf{(9.3)} \qquad \mathrm{ind}_H^G : V(H, \lambda) \to V(G, \lambda)$$

and makes $V(G, \lambda)$ a module over the Burnside ring A(G) via the following formula:

$$\textbf{(9.4)} \qquad [G/H].x = \mathrm{ind}_H^G\, \mathrm{res}_H^G\, x$$

for $x \in V(G, \lambda)$ and $[G/H] \in A(G)$. Actually $V(G, \lambda)$ has the structure of a Mackey functor in the sense of Dress. Since this is not needed here, the proof is omitted.

10. Dimension functions

In this section we compute the dimension of $V(G) \otimes \mathbf{Q}$ which is the same as the rank of $\mathrm{Dim}\, V(G) \subset C(G)$. Here $V(G) = V(G, h^\infty)$. From our previous results it is clear that the $V(G, \lambda)$ for $\lambda = h$ and h^∞ all have the same rank. By linear algebra, determination of $\mathrm{Dim}\, V(G) \otimes \mathbf{Q} \subset C(G) \otimes \mathbf{Q}$ amounts to finding all linear relations that hold between fixed point dimensions of homotopy representations. Universal relations of this type are provided by the following theorem of Borel [1]. Reformulated it gives an upper bound for the rank of V(G).

Theorem **(10.1)** (Borel). — *Let* $G = \mathbf{Z}/p \times \mathbf{Z}/p$, *p a prime. Let* H_i, $0 \leq i \leq p$ *denote the subgroups of order p in* G. *Then*

$$p\, \mathrm{Dim}\, X(G) = \sum_i (\mathrm{Dim}\, X(H_i) - \mathrm{Dim}\, X(1)).$$

Let H′ denote the commutator subgroup of H. Note that H/H′ is not cyclic if and only if there exists a normal subgroup K of H (written $K \lhd H$) such that $H/K \cong \mathbf{Z}/p \times \mathbf{Z}/p$ for some prime *p*. Let $e_H : C(G) \otimes \mathbf{Q} \to \mathbf{Q}$ be evaluation at H. If H/H′ is not cyclic, we obtain from (10.1) linear forms ℓ_H of the type

$$\ell_H = e_H + \sum_{K<H} a_K e_K, \qquad a_K \in \mathbf{Q}$$

such that $\ell_H(\mathrm{Dim}\, X) = 0$ for all homotopy representations X. This shows that the corank of Dim V(G) in C(G) is at least the cardinality of the set

$$B = \{(H) \mid H/H' \text{ is not cyclic}\}.$$

We show that equality holds. This then shows that there are no more relations between fixed point dimensions than those already obtained from (10.1).

The main result of this section is

Theorem **(10.3).** — *The rank of* $V(G)$ *is equal to the cardinality of* $\{(H) \mid H/H' \text{ is cyclic}\}$.

Using formal properties of $V(G)$, we reduce the proof of (10.3) to the following geometrical result. Compare Dovermann-Petrie [30].

Theorem **(10.4).** — *Let* G *be a finite group such that* G/G' *is cyclic. Then there exist two homotopy representations* X *and* Y *such that* $\mathrm{Dim}\, X(H) = \mathrm{Dim}\, Y(H)$ *if and only if* $H \neq G$.

The structure of the proof is as follows: We construct a G-manifold X such that Dim X coincides with a linear dimension function except for its value at G. Then we show that X satisfies the hypothesis of our general Modification Theorem (5.8), *i.e.* we must show the existence of W_pH-maps $t(H, p) : X^H \to S(V, p)$ of degree prime to p for each prime p. When this is done, (5.8) asserts that X can be converted to a homotopy representation having the same dimension function as X. A version of this procedure was used in Petrie [31] to construct free metacyclic group actions on homotopy spheres.

The manifold X will be given as a Brieskorn variety. We use the following notation. Let V be a complex representation of G with invariant scalar product $\langle\ ,\ \rangle$. Let $f : V \to \mathbf{C}$ be a G invariant polynomial. Then

$$B(V, f) = \{z \in V \mid f(z) = 0,\ \langle z, z\rangle = 1\}.$$

We recall that if f is weighted homogeneous and has an isolated singularity at 0, the intersection of the hypersurface $f^{-1}(0)$ with the unit sphere $S(V)$ of V is transverse and therefore $B(V, f)$ is a closed G-manifold of real codimension 3 in V. See Milnor [18]. Since a weighted homogeneous polynomial is a polynomial which is invariant under a $\mathbf{C}^*$ action on V and $\mathbf{C}$, it is natural for us to treat polynomials invariant by the group $\overline{G} = \mathbf{C}^* \times G$.

First we deal with a suitable representation. Let r_G denote the regular representation of G. View $r_{G/G'}$ as a representation of G. It is a subrepresentation of r_G whose complement we denote by $r_G - r_{G/G'}$.

Proposition **(10.5).** — $|G'|.(r_G - r_{G/G'})$ *is a direct sum of representations which are induced from one-dimensional representations of cyclic subgroups of* G'.

Proof. — For a cyclic group A of order a we define a class function T_A by $T_A(s) = a$ if s generates A and $T_A(s) = 0$ otherwise. Set $L_A = \varphi(a) r_A - T_A$ where φ is the Euler function. For $A = 1$, L_A is 0. For class functions χ and ψ on G set

$$\langle \psi, \chi\rangle_G = |G|^{-1} \sum_{g \in G} \psi(g)\bar{\chi}(g)$$

and let 1_C denote the trivial representation of dimension one. Since $r_G = \mathrm{ind}_1^G 1_C$ Frobenius reciprocity gives $\langle \psi, r_G \rangle_G = \psi(1)$. Since $r_{G/G'}(g)$ is $|G/G'|$ if $g \in G'$ and 0 otherwise, $|G'| \langle \psi, r_{G/G'} \rangle_G = \sum_{g \in G'} \psi(g)$. Thus

$$a) \qquad \langle \psi, |G'|(r_G - r_{G/G'}) \rangle = |G'| \psi(1) - \sum_{g \in G'} \psi(g).$$

We show this is the same as

$$b) \qquad \sum_{A \subset G'} \langle \psi, \mathrm{ind}_A^G L_A \rangle_G$$

where the sum is over all cyclic subgroups of G'.

Again using reciprocity, we find

$$\langle \psi, \mathrm{ind}_A^G L_A \rangle_G = \varphi(a)\psi(1) - \langle \mathrm{res}_A \psi, T_A \rangle_A.$$

Since $\varphi(a)$ is the number of generators of A

$$\sum_{A \subset G'} \varphi(a) = |G'|.$$

By definition of $\langle\ ,\ \rangle$ and T_A

$$\langle \mathrm{res}_A \psi, T_A \rangle_A = \sum_{g \in A^*} \psi(g)$$

where A^* is the set of generators of A; so

$$\Sigma \langle \mathrm{res}_A \psi; T_A \rangle_A = \sum_{g \in G'} \psi(g).$$

These facts imply $a) = b)$; so $\Sigma\, \mathrm{ind}_A^G L_A$ is $|G'|(r_G - r_{G/G'})$. Since L_A is a direct sum of one dimensional characters of G by Lang [17], p. 477, the proof is complete.

We now assume that G/G' is cyclic. Then it is the product of cyclic groups $Z_{q(i)}$, $i = 1, 2, \ldots r$ where $q(i)$ is a power of a prime p_i. Let W_i be a one-dimensional representation of G/G' with kernel of index p_i in G/G'. By (10.5)

$$|G'|(r_G - r_{G/G'}) = \bigoplus_j \mathrm{ind}_{C_j}^G U_j$$

where $C_j \subset G'$ is cyclic and U_j is a one-dimensional representation of C_j. Let

$$\lambda = \prod_{i=1}^{r} p_i . c$$

where $c_j = |C_j|$ and c is the least common multiple of the c_j. Let $\overline{U}_j$ be the one-dimensional representation of $\overline{C}_j$ whose restriction to C_j is U_j and whose restriction to $\mathbf{C}^*$ is defined by having $t \in \mathbf{C}^*$ act by multiplication by t^{λ/c_j}. Similiarly define a $\overline{G}$ representation $\overline{W}_i$ whose restriction to G is W_i and whose restriction to $\mathbf{C}^*$ is defined by having t act by t^{λ/p_i}. Then

$$\textbf{(10.6)} \qquad V = \bigoplus_j \mathrm{ind}_{\overline{C}_j}^{\overline{G}} U_j \oplus \left(\bigoplus_i \overline{W}_i\right)$$

is a representation of $\overline{G}$. We make $\mathbf{C}$ a representation of $\overline{G}$ by having G act trivially and $t \in \mathbf{C}^*$ act by multiplication by t^λ. Note

$$V^{\mathbf{C}^*} = 0 \quad \text{and} \quad \mathbf{C}^{\mathbf{C}^*} = 0. \tag{10.7}$$

Lemma (**10.8**). — $\dim_{\mathbf{C}} V^H \leq 1$ *if and only if* $G' \subset H$ *and* H *has prime power index in* G; *and* $V^H = 0$ *if and only if* $H = G$.

Proof. — For $H \subset G$ we have $\dim(r_G - r_{G/G'})^H = |G/H| - |G/HG'|$. This is zero if and only if $G' \subset H$. Hence $\dim V^H \leq 1$ implies $G' \subset H$ and H fixes exactly one W_i and conversely. The latter happens if and only if $H \subset \operatorname{Ker} W_i$ for exactly one i and this requires $|G/H|$ to be a prime power. By inspection $V^H = 0$ if and only if $H = G$.

We now find a suitable $\overline{G}$-invariant polynomial $f: V \to \mathbf{C}$. If $n = |G|\,|C_j|^{-1}$ and $g_1, \dots g_n$ is a system of coset representatives of C_j in G, then a point $\vec{u}_j$ in $\operatorname{ind}_{\overline{C}_j}^{\overline{G}} U_j$ has coordinates $\{u_{g_i} \mid u_i \in \mathbf{C}\}$. The polynomial $f(\vec{u}_j) = \sum_{i=1}^{n} u_{g_i}^{c_j}$ is $\overline{G}$-invariant. Let w_i be the complex coordinate in $\overline{W}_i$. Then

$$f(\{\vec{u}_j\}, \{w_i\}) = \sum_j f_j(\vec{u}_j) + \sum_i w_i^{p_i} \tag{10.9}$$

is a $\overline{G}$-invariant polynomial from V to $\mathbf{C}$, which has an isolated singularity at 0 and the hypersurface $f^{-1}(0)$ intersects $S(V)$ transversely. Note

$$\operatorname{Dim} B(V,f)(H) = \begin{cases} \operatorname{Dim} SV(H) - 2 & \text{if } V^H \neq 0 \\ \operatorname{Dim} SV(H) & \text{if } V^H = 0. \end{cases} \tag{10.10}$$

Here are a few words to justify (10.10). The restriction f' of f to $S(V)$ is transverse to $0 \in \mathbf{C}$. This trivially implies that $f'^H: S(V)^H \to \mathbf{C}$ is transverse to 0. This means the differential of f'^H at p maps $T_pB(V,f)^H$ surjectively onto $\mathbf{C}$ with kernel $T_pB(V,f)^H$ for each $p \in B(V,f)^H$. Thus $\dim B(V,f)^H = \dim S(V)^H - 2$ whenever $B(V,f)^H$ is not empty or $B(V,f)^H$ is empty and $\dim S(V)^H - 2 < 0$. These two conditions hold exactly when $V^H \neq 0$. To see this note that f^H is a complex polynomial so $\dim(f^H)^{-1}(0) \geq 2\dim_{\mathbf{C}} V^H - 2$. See Milnor [18], § 2. If $\dim_{\mathbf{C}} V^H > 1$, there is a nonzero $z \in (f^H)^{-1}(0)$. Then $tz \in (f^H)^{-1}(0)$ for any $t \in \mathbf{R}^*$ because f is $\overline{G}$-invariant. For a suitable t, $tz \in B(V,f)^H$ since the norm of tz is an increasing function of t. Thus $B(f, V)^H$ is not empty if $\dim_{\mathbf{C}} V^H > 1$. If $\dim_{\mathbf{C}} V^H = 1$, $f^H = w_i^{p_i}$ for some i by the proof of (10.8). Then $(f^H)^{-1}(0) = 0$ and $B(V,f)^H = \varnothing$; so

$$\operatorname{Dim} B(V,f)(H) = \operatorname{Dim} SV(H) - 2.$$

Proposition (**10.11**). — *Let* G *be a compact Lie group and* U *and* $\mathbf{C}$ *be complex representations of* $\overline{G}$ *with* G *acting trivially on* $\mathbf{C}$. *Suppose* $f: U \to \mathbf{C}$ *is an* $\overline{G}$*-invariant polynomial whose hypersurface* $f^{-1}(0)$ *intersects* $S(U)$ *transversely. Then* $f^H: U^H \to \mathbf{C}$ *is an* $\overline{NH}$*-invariant polynomial and* $B(U,f)^H = B(U^H, f^H)$ *is a smooth manifold.*

Proof. — Invariance of f^{H} is clear. The action of G on $B(U,f)$ is smooth; so its H-fixed set $B(U,f)^{H}$ is a smooth manifold.

Lemma **(10.12)**. — *Let* L *be a compact Lie group and* U, W *and* **C** *be complex representations of* $\bar{L}$. *Suppose* 0 *is the only point in each fixed by* $\bar{L}$ *and* $\dim W = 1$. *Let* $f: U \oplus W \to \mathbf{C}$ *be an* $\bar{L}$*-invariant polynomial having the form* $f(u, w) = h(u) + w^q$ *for* $u \in U$ *and* $w \in W$. *Suppose* $B(V \oplus W, f) = B$ *is a smooth manifold. Then there is an* L*-map* $f: B \to S(U)$ *whose degree divides* q.

Proof. — This is essentially in Bredon [2], V. 9. Let $u \in U$, $w \in W$ and $(u, w) \in B$. Since $U^{\mathbf{C}^*} = 0$, the norm of tu is an increasing function of $t \in \mathbf{R}^+ \subset \mathbf{C}^*$; so there is a unique t with $\|t \circ u\| \in S(U)$. Set $\varphi(u, w) = tu$ for this t. The arguments of [2] V. 9 show φ is the orbit map of the $\mathbf{Z}/q$-action on B defined by $g(u, w) = (u, \omega w)$ ω a q^{th}-root of 1 and g a generator of $\mathbf{Z}/q$. This uses the fact that f is $\bar{L}$-invariant. Since the composition of the homology transfer homomorphism and φ_* is multiplication by q, degree φ divides q.

Proposition **(10.13)**. — *For each* $H \subset G$ *and prime* p *there exists an* N_pH*-representation* $U(H, p)$ *and an* N_pH*-map* $t(H, p) : B(V,f)^{H} \to SU(H, p)$ *of degree prime to* p.

Proof. — Let K be N_pH and $B = B(V,f)$. If B^K is not empty, let W be the representation of K on the tangent space to a point x in B^K. The map which collapses the complement of a K-invariant disk in B^H centered at x to a point gives a K-map of B^H to $S(V^H)$ of degree 1.

We now treat the case where $B^K = \varnothing$ but $B^H \neq \varnothing$. Since $B^K = \varnothing$ implies $\dim V^K \leq 1$ by (10.10), G′ is contained in K and G/K is cyclic of order q^r for some prime q by (10.8). There are two cases $q = p$ and $q \neq p$. We rule out this former as follows: We have $H \lhd K \lhd G$. Let K^p be the smallest normal subgroup of K of p-power index. Then K^p is a characteristic subgroup of K hence normal in G. The p-group $G/K^p = L$ must be cyclic otherwise L/L' is not cyclic by [14] III Hilfssatz (7.1). Then G/G′ would have $\mathbf{Z}/p \times \mathbf{Z}/p$ as a quotient group. Since G/G′ is cyclic, this can't happen; so G/K^p is cyclic. Since $K^p \subset H$, G/H is cyclic of p-power order. But then B^H is empty by (10.8). Since B^H is non-empty by assumption, this case doesn't occur.

Thus we have $p \neq q$. Note that W_i with $p_i = q$ is contained in V^H because pq divides the index of H in G which implies $H \subset \mathrm{Ker}(W_i)$. Let $U = V^H - W_i$ so $V^H = U \oplus W_i$. Observe that $f^H: V^H \to \mathbf{C}$ has the form $h(u) + w_i^q$ by (10.9). Apply (10.7), (10.11) and (10.12) to $f^H: V^H \to \mathbf{C}$ to produce a K-map $t: B(V^H, f^H) \to S(U)$ whose degree divides q and is so prime to p.

Proof of (10.4). — Let $B = B(V,f)$ and $S(V) - S(1_{\mathbf{C}}) = S$ where $1_{\mathbf{C}}$ is the one-dimensional trivial representation of G. By (10.10) and (10.8) $\mathrm{Dim}\, B(H) = \mathrm{Dim}\, S(H)$ if and only if $H = G$. By (5.8), there is a homotopy representation X with

$\operatorname{Dim} X = \operatorname{Dim} B$. Then $X * S(1_C)$ and $S(V)$ are homotopy representations of G whose dimension functions differ only at G.

Proof of Theorem (10.3). — From section 9 we recall that $V(G)$ is a module over $A(G)$ which satisfies

(10.14) $$\operatorname{res}_H(x \cdot X) = \operatorname{res}_H(x) . \operatorname{res}_H(X)$$

(10.15) $$\operatorname{Dim}([G/H] \cdot X)(G) = \operatorname{Dim} X(H)$$

for $x \in A(G)$ and $X \in V(G)$. See (9.2) and (9.4). The rank of this module is determined by its localizations at the prime ideals $q(H) = \ker(e_H : A(G) \to \mathbf{Z})$ for $H \subset G$ through this formula:

(10.16) $$\dim_{\mathbf{Q}}(V(G) \otimes \mathbf{Q}) = \sum_{(H)} \dim_{\mathbf{Q}} V(G)_{q(H)}.$$

In order to prove (10.3) we show

Proposition **(10.17)**. — $\dim_{\mathbf{Q}} V(G)_{q(H)}$ *is* 1 *if* H/H' *is cyclic and zero otherwise.*

The proof of (10.17) is a consequence of the following lemmas.

Lemma **(10.18)**. — $\dim_{\mathbf{Q}} V(G)_{q(G)} \leq 1$.

Proof. — $\dim_{\mathbf{Q}} V(G)_{q(G)} = \dim_{\mathbf{Q}} \operatorname{Dim} V(G)_{q(G)} \leq \dim_{\mathbf{Q}} A(G)_{q(G)}$ because localization is an exact functor. Since $q(G) = e_G^{-1}(0)$, $0 \in \mathbf{Z}$, $A(G)_{q(G)}$ is $\mathbf{Z}$ localized at 0 *i.e.* $\mathbf{Q}$.

Lemma **(10.19)**. — $V(G)_{q(G)} = 0$ *if and only if there are integers* a_H, $H \subset G$ *with* $a_G \neq 0$ *and* $\Sigma a_H \operatorname{Dim} X(H) = 0$ *for all* $X \in V(G)$.

Proof. — Suppose integers exist as claimed. Then $x = \Sigma a_H [G/H] \notin q(G)$ but $e_G(x . \operatorname{Dim} X) = 0$ for all $X \in V(G)$ by (10.15). Choose $z \in A(G)$ such that $e_H(z) = 0$ for $H \neq G$ and $e_G(z) = 0$. This is possible because $A(G)$ is a subgroup of finite index in $C(G)$. Since neither x nor z is in $q(G)$, both become units in $A(G)_{q(G)}$. Since $e_H(x . z . \operatorname{Dim} X) = 0$ for all H and all $X \in V(G)$ we see that $x . z . \operatorname{Dim} V(G) = 0$. Since $x . z$ is a unit in $A(G)_{q(G)}$, $\operatorname{Dim} V(G)_{q(G)} = V(G)_q = 0$. Conversely if $V(G)_{q(G)} = 0$, there is an $x \notin q(G)$ with $x . \operatorname{Dim} X = 0$ for all $X \in V(G)$. Then $e_G(x . \operatorname{Dim} X) = 0$ gives the desired linear relation among the integers $\operatorname{Dim} X(H)$, $H \subset G$ by (10.15).

Lemma **(10.20)**. — $\dim_{\mathbf{Q}} V(G)_{q(H)} \leq \dim_{\mathbf{Q}} V(H)_{q(H)}$.

Proof. — $V(G) \xrightarrow{\operatorname{res}_H} V(H) \xrightarrow{\operatorname{ind}_H} V(G)$ is multiplication by $[G/H] \notin q(H)$, hence becomes an isomorphism on localizing at $q(H)$.

Lemma **(10.21)**. — H/H' *cyclic implies* $\dim_{\mathbf{Q}} V(G)_{q(H)} = 1$.

Proof. — By (10.4) we can find a non zero $X \in V(H)$ with $\operatorname{Dim} X(K) = 0$ if and only if $H \neq K$; so by (10.19) $V(H)_{q(H)} \neq 0$. Consider $Y = \operatorname{res}_H \operatorname{ind}_H^G X$.

By (9.2) $\mathrm{Dim}\,Y(H) = |WH|.\mathrm{Dim}\,X(H) \neq 0$ and $\mathrm{Dim}\,Y(K) = 0$ for $K \neq H$. Thus Y is not zero in $V(H)_{q(H)}$ by (10.19); so $\mathrm{ind}_H^G X \neq 0$ in $V(G)_{q(H)}$. Thus $\dim_{\mathbf{Q}} V(G)_{q(H)} = 1$ by (10.18).

Lemma **(10.22).** — H/H' *not cyclic implies* $V(G)_{q(H)} = 0$.

Proof. — By (10.19) and the remarks after (10.1), $V(H)_{q(H)} = 0$. Now use (10.20). Finally we note that (10.3) implies

Proposition **(10.23).** — *All dimension functions for* G *are linear if and only if* G *is nilpotent.*

Proof. — It is shown in tom Dieck [7] and [34] that for a nilpotent group G all dimension functions are linear.

Now suppose that all dimension functions are linear. Then the subgroups H such that H/H' are cyclic are precisely the cyclic subgroups (by (10.3)). We have to show that this implies: G is nilpotent. By induction over G we can assume that all proper subgroups of G are nilpotent. If G/G' were trivial then G would be cyclic, a contradiction. Hence there exists $H \lhd G$ such that $G/H \cong \mathbf{Z}/q$ for a prime q. We know that H is nilpotent hence the product $H = P_0 \times \ldots \times P_r$ of its Sylow subgroups P_i. Suppose P_0 is the Sylow q-group. Then $P_1 \times \ldots \times P_r$ is normal in G and we have a semidirect product

$$1 \to P_1 \times \ldots \times P_r \to G \to Q \to 1.$$

Each P_i is Q-invariant. It suffices to show that a subgroup of the type

$$1 \to P_i \to H_i \to Q \to 1$$

is nilpotent, *i.e.* Q acts trivially on P_i. By induction we can assume that $Q = \mathbf{Z}_q r$. If P_i were cyclic then H_i/H_i' were cyclic hence H_i cyclic. So assume that P_i is not cyclic. Then there exists a unique minimal normal subgroup K of P_i such that P_i/K is an elementary abelian p-group $(\mathbf{Z}/p)^t$ for some p and $t \geq 2$. Moreover K is characteristic in P, hence normal in H_i. Let $(\mathbf{Z}/p)^t \cong A_1 \times \ldots \times A_s$ be a decomposition into isotypical Q-modules, and let $B_j \subset P_i$ be the preimage of A_j. The A_j generate P_i. If A_j is a non-trivial Q-module then a subgroup of the type $1 \to B_j \to K_j \to Q \to 1$ has $K_j/K_j' \cong Q$, hence K_j would be cyclic: a contradiction. Hence $s = 1$ and A_1 is a trivial Q-module. Then all maximal proper subgroups of P_i are Q-invariant and by induction have a trivial Q-action. Hence Q acts trivially on P_i so that H_i is nilpotent.

11. Abelian groups — Examples

In this section we discuss some examples. These should convince the reader that apart from the general theory developed so far the internal algebra and geometry of homotopy representations deserves further study.

To begin with we relate the homomorphism ρ of (8.23) to the Swan homomorphism

(11.1) $\quad s_G : (\mathbf{Z}/|G|)^* \to \widetilde{K}_0(G).$

This homomorphism is zero on ± 1; so s_G factors through the quotient by this group. The resulting homomorphism is denoted also s_G. By definition s_G maps the integer r mod $|G|$ to the class of $\mathbf{Z}/r$ viewed as a $\mathbf{Z}$G-module with trivial G-action. (This module has projective dimension one.)

We first remark that for even homotopy representations (see (1.13)) the finiteness obstructions ρ and σ coincide. This is clear from (7.21), (7.23), (8.17), (8.18) and (8.20). Moreover, since homotopy representations with the same dimension function have the same orientation behavior we can, by stability, work with even homotopy representations if we deal with $v(G, h^\infty)$. Using these remarks and the canonical isomorphism $v(G, h^\infty) \cong \mathrm{Pic}(G)$ we obtain from the finiteness obstruction a homomorphism

(11.2) $$\sigma : \mathrm{Pic}(G) \to \kappa(G).$$

This being a homomorphism between algebraic objects we ask for its computation in algebraic terms. In principle this is achieved using Proposition (8.17).

Suppose $x \in \mathrm{Pic}(G)$ is represented by the invertible degree function $d \in C(G)$; this means $d(H)$ is prime to $|G|$ for all $H \subset G$ and $x \in \mathrm{Pic}(G) = v(G, h^\infty)$ is represented by $Y - X$ such that there exists a map $f : X \to Y$ with degree $f^H = d(H)$. Using the notation of (8.17) we claim

Proposition **(11.3)**. — *The finiteness obstruction* $\sigma(x)(1)$ *equals*

$$s_G(\mathbf{Z}/d(1)) - \sum_{1 \neq H,\, L \subset NH} a_{H,L} \operatorname{ind}_L^G \operatorname{res}_L^{NH} s_{NH}(\mathbf{Z}/d(H)).$$

Proof. — Let $h : A \to X$ be a finite approximation. Then $fh : A \to Y$ is a finite approximation too. Therefore by (4.1) and (7.14)

$$\sigma(Y)(1) - \sigma(X)(1) = \chi(M_f) - \chi(M_{fg}).$$

Now use (8.17).

For abelian groups the homomorphisms ind and res do not appear because all fixed point sets are G-spaces. Moreover the formula of (11.3) can be made more explicit. We recall the computation of Pic(G) from tom Dieck [6], Theorem 5 and tom Dieck-Petrie [9] (3.33). Let $C = C(G)$, $n = |G|$. Let $e_H : (C/nC)^* \to (\mathbf{Z}/n)^*$ be evaluation at H. Define u_H inductively by

(11.4) $$e_H = \prod_{K \supseteq H} u_K.$$

The composition of u_H with $(\mathbf{Z}/n)^* \to (\mathbf{Z}/|G/H|)^*$ is also called u_H. Then we have: The product $u : (C/nC)^* \to \prod_{(H)} (\mathbf{Z}/|G/H|)^*$ of the u_H factors through the canonical map $(C/nC)^* \to \mathrm{Pic}(G)$ and induces an isomorphism

$$u : \mathrm{Pic}(G) \cong \prod_{H \subset G} (\mathbf{Z}/|G/H|)^*/\{\pm 1\}.$$

Proposition (**11.5**). — *The following diagram is commutative*

$$\begin{array}{ccc} & \mathrm{Pic}(G) & \\ {}^{u}\swarrow & & \searrow{}^{\sigma} \\ \prod_{(H)} (\mathbf{Z}/|G/H|)^*/\{\pm 1\} & \xrightarrow[\Pi s_{G/H}]{} & \Pi \widetilde{K}_0(G/H). \end{array}$$

Proof. — Let $x = Y - X \in \mathrm{Pic}(G)$ be represented by the invertible degree function $d \in (C/nC)^*$. We may suppose that X is a finite homotopy representation; so $\sigma(X) = 0$. Then

$$\sigma(x)(H) = \sigma(Y)(H) = \chi(M_f^H) - \chi(M_{fs}^H) = \chi(M_f^H/M_{fs}^H).$$

For an abelian group we have

$$\textbf{(11.6)} \qquad \chi'(M^H) = \sum_{K \geq H} \chi'(M^K/M_s^K)$$

in $K_0(G/H, |G|)$. Solving (11.4) resp. (11.6) for u_H resp. $\chi'(M^H/M_s^H)$ we obtain

$$\textbf{(11.7)} \qquad u_H = \prod_{K \supseteq H} d_K^{a_K}$$

$$\textbf{(11.8)} \qquad \chi'(M^H/M_s^H) = \sum_{K \supseteq H} a_K \chi'(M^K)$$

with the *same* integers a_K in (11.7) and (11.8). Hence

$$s_{G/H} u_H(x) = s_{G/H} \prod_{K \supseteq H} d_K^{a_K} = \sum_{K \supseteq H} a_K s_{G/H} d_K$$
$$= \sum_{K \supseteq H} a_K \chi(M^K) = \chi(M^H/M_s^H) = \sigma(x)(H)$$

as has to be shown. ■

Corollary (**11.5**). — $x \in \mathrm{Pic}(G)$ *is represented as the degree function* $d(f)$ *for some* $f: X \to Y$ *with* X *and* Y *finite homotopy representations if and only if* $s\mu(x) = 0$.

This corollary expresses the relations which exist among the values $d(f)(H)$, $H \subset G$ when f is a mapping between finite homotopy representations. Note that μ is entirely determined by the subgroup structure of G through (11.4) and s is determined by the Swan homomorphism on quotient groups of G.

We consider the group $G = \mathbf{Z}/p \times \mathbf{Z}/p$ in detail. Let $H_0, \dots, H_p$ be the subgroups of order p. Then from (11.4)

$$u_1 = d_1 \prod_{i=0}^{p} d_{H_i}^{-1} d_G^p, \qquad u_{H_i} = d_{H_i} d_G^{-1}, \qquad u_G = d_G.$$

The Swan homomorphism for cyclic groups is zero. The kernel of the Swan homomorphism for p-groups has been determined by Taylor [25]. For $G = \mathbf{Z}/p \times \mathbf{Z}/p$ this kernel is precisely the $(p-1)$-torsion of $(\mathbf{Z}/|G|)^*$. Let X be an even homotopy representation for G. Since by tom Dieck [7], dimension functions for G are linear one can see that there exists a complex representation V and a G-map $f: X \to S(V)$ such that degree $f^{H} = \pm 1$ for $H \neq 1$. The G-homotopy-type of S(V) is uniquely determined by these conditions. Then degree $f = d(f)(1) = u_1 d(f) \in (\mathbf{Z}/|G|)^*/\{\pm 1\}$ measures the deviation of X from linearity; *i.e.* X is linear if and only if degree f is one mod $|G|$. There exist non-linear finite homotopy representations; namely X as above is finite if and only if degree f is a p-th power mod p^2 (because the $(p-1)$-torsion of $(\mathbf{Z}/|G|)^*$ is the subgroup of p-th powers).

In terms of generalized homotopy representations, the non-linearity is easy to explain. Let X be such that only $\{1\}$ and G are isotropy groups and $i : X^G \to X$ has degree d prime to $|G|$. Then X represents a non-linear element in $V(G, h^\infty)$, finite if $d = a^p \bmod p^2$ for some a. If we want to realize homotopy representations by smooth manifolds then non-linear one's cannot have fixed points. Suppose the manifold X has the dimension function of $S(V_0 \oplus V_p)$, where $H_i = \text{kernel } V_i$, $\dim_{\mathbf{C}} V_i \geq 2$ $i = 0, p$. Then we have

Proposition (**11.9**). — $X \backslash X^H$ *is a generalized homotopy representation and* X *is* G-*homotopy-equivalent to* $X^H * (X \backslash X^H)$, *where* H *is* H_0 *or* H_p.

Proof. — Suppose $H = H_0$, $H_p = K$. Using duality $X \backslash X^H$ is seen to have the homology of a sphere of dimension of $S(V_p)$; moreover $X \backslash X^H$ is simply-connected; and the only non-trivial fixed point set is X^K. Therefore $X \backslash X^H$ is a generalized homotopy representation. One has (as in the Spanier-Whitehead theory) a duality map

$$d : X \to X^H * (X \backslash X^H)$$

which is a G-map and a homotopy-equivalence on all fixed point sets, hence a G-equivalence.

Using previous notation we study the inclusion $i : X^K \to X \backslash X^H$. The degree of i measures deviation from linearity and has, moreover, the following geometric interpretation.

Proposition (**11.10**). — *The degree of i is equal to the linking number of* X^H *and* X^K *in* X.

Proof. — The linking number may be defined through the following composition

$$H^0(X^K) \cong H^n(X, X - X^K) \stackrel{\cong}{\leftarrow} H^{n-1}(X - X^K) \to H^{n-1}(X^H)$$

where $n - 1 = \dim X^H$.

Using (11.9) it is easy to see how the degree of i is related to the degree of $f : X \to S(V_0 \oplus V_p)$ with $\deg(f^H) = \deg(f^K) = 1$, namely one has

$$\deg f \deg i \equiv 1 \bmod p^2.$$

Remark (**11.11**). — It does not seem easy to show that there exist manifolds X which realize non-trivial linking numbers. The naive surgery methods do not work because for $S(V_0 \oplus V_p)$ the so called gap hypothesis (Petrie [20]) is not satisfied. But there are natural candidates with which to start the surgery: Brieskorn varieties. We mention the following examples.

Let $V^d(A)$ be the Brieskorn variety consisting of points $(z_0, \ldots, z_n) \in \mathbf{C}^{n+1}$ such that

$$z_0^d + z_1^2 + \ldots + z_n^2 = 0$$
$$|z_0|^2 + \ldots + |z_n|^2 = 1$$

and with G-action induced by the representation $A : G \to O(n) \subset U(n)$ acting on the coordinates $z_1, \ldots, z_n$. The map

$$\varphi : V^d(A) \to S(A)$$
$$(z_0, \ldots, z_n) \mapsto (\Sigma |z_i|^2)^{-1/2}(z_1, \ldots, z_n)$$

is equivariant and has degree d. It induces an analogous map between H-fixed point sets. If d is prime to $|G|$ then by (5.8) we can modify $V^d(A)$ by attaching cells so as to obtain a homotopy representation $X^d(A)$ and a G-map

$$f(A, d) = f : X^d(A) \to S(A)$$

with

$$\deg f^H = d \quad \text{if } A^H \neq \{0\}$$
$$\deg f^H = 1 \quad \text{if } A^H = \{0\}.$$

Proposition **(11.12)**. — *If* G *is abelian* $v(G, h^\infty)$ *is generated by the* $S(A) - X^\ell(A)$ *as* A *ranges over representations of* G *and* ℓ *over integers prime to* $|G|$.

Proof. — We use the isomorphism (6.5) of $v(G, h^\infty)$ with Pic(G) which sends $Y - X$ to the class of $d(f)$ in Pic(G). Here $f : X \to Y$ is any G-map with invertible degree function. (See also section 3.) It suffices to show then that Pic(G) is generated by the function $d(f(A, \ell))$.

Let $m \subset \varphi(G)$ be any subset with characteristic function $c(m) : \varphi(G) \to \{0, 1\}$. Let $M(m, d) \in \text{Pic}(G)$ be represented by the function $(H) \mapsto d^{\tau(m)(H)}$. Note that $M(m, d)^{-1} = M(m, e)$ where $de \equiv 1 \bmod |G|$. One has

(11.13)
$$M(m_2)M(m_1)^{-1} = M(m_2 \backslash m_n) \quad \text{for } m_2 \supset m_1$$
$$M(m_1)M(m_2) = M(m_1 \cap m_2)M(m_1 \cup m_2).$$

More generally suppose that $m = m_1 \cap \ldots \cap m_n$. Then using the combinatorial identity

$$\sum_{\varnothing \not\subset A \subset \{1,\ldots,n\}} (-1)^{|A|} c(\bigcup_{j \in A} m_j) = -c(\bigcap_{j=n}^{n} m_j)$$

and abbreviating $\bigcup_{j \in A} m_j = m_A$, $e(A) = (-1)^{|A|-1}$ we obtain

(11.14)
$$M(m) = \prod_{\varnothing \neq A \subset \{1,\ldots,n\}} M(m_A)^{e(A)}.$$

If V is a representation of G we put

$$m(\mathrm{V}) = \{(\mathrm{H}) \mid \mathrm{V}^{\mathrm{H}} \neq \{0\}\}$$

and our claim is that the $\mathrm{M}(m(\mathrm{V}))$ generate Pic(G). This is equivalent to the statement that all $\mathrm{M}(m)$, $m \subset \varphi(\mathrm{G})$, can be generated. If V is irreducible, then $m(\mathrm{V}) = \{\mathrm{H} \mid \mathrm{H} \subseteq \mathrm{G_V} := \text{kernel } \mathrm{V}\}$. If we put, for $\mathrm{H} \subset \mathrm{G}$, $k(\mathrm{H}) := \{\mathrm{K} \mid \mathrm{K} \subseteq \mathrm{H}\}$ then $m(\mathrm{V}) = k(\mathrm{G_V})$ and $k(\mathrm{H}) = \bigcap_{\mathrm{G_V} \supseteq \mathrm{H}} m(\mathrm{V})$.

Hence using (11.14) we see that $\mathrm{M}(k(\mathrm{H}))$ are obtainable from $\mathrm{M}(m(\mathrm{V}))$. Using $c(m_1 \cup \ldots \cup m_r) = c(m_1 \cup \ldots \cup m_{r-1}) + c(m_r) - c((m_1 \cup \ldots \cup m_{r-1})m_r)$ we see by induction over r that $\mathrm{M}(k(\mathrm{H}_1) \cup \ldots \cup k(\mathrm{H}_r))$ are obtainable. Finally using (11.13) we see that $\mathrm{M}(m_{\mathrm{H}})$ is obtainable where $m_{\mathrm{H}} = \mathrm{H}$. This finishes the proof.

12. Metacyclic groups — Examples and computations

Periodic groups play a special role in our theory. This is indicated by the following proposition.

Proposition (**12.1**). — *There exist homotopy representations* X *and* Y *such that* $\mathrm{Dim}\,\mathrm{X}(\mathrm{H}) = \mathrm{Dim}\,\mathrm{Y}(\mathrm{H})$ *for* $\mathrm{H} \neq 1$, $\mathrm{Dim}\,\mathrm{X}(1) \neq \mathrm{Dim}\,\mathrm{Y}(1)$ *if and only if* G *has periodic cohomology. If such* X *and* Y *exist then the period of* G *divides* $\mathrm{Dim}\,\mathrm{X}(1) - \mathrm{Dim}\,\mathrm{Y}(1)$.

Proof. — If G has cohomology with period q then there exists by the work of Swan [23] a homotopy representation X with $\mathrm{Dim}\,\mathrm{X}(1) = q$ and with free G-action. The result follows with Y the empty set.

Conversely assume that X and Y are homotopy representations with dimension functions differing only at $\{1\}$, say

$$n + k = \dim \mathrm{Y}(1) + k = \dim \mathrm{X}(1), \qquad k > 0.$$

By (5.9) and its proof, we can attach cells of type $\mathrm{G} \times \mathrm{D}^i$, $n \leq i \leq n + k - 1$ to Y to obtain a homotopy representation W which has the same dimension function as X. The relative cellular chain complex of (W, Y) yields an exact sequence

$$0 \to \mathbf{Z} \to \mathrm{C}_{n+k-1}(\mathrm{W}, \mathrm{Y}) \to \ldots\ \mathrm{C}_{n-1}(\mathrm{W}, \mathrm{Y}) \to \mathbf{Z} \to 0$$

with free **Z**G-modules $\mathrm{C}_i(\mathrm{W}, \mathrm{Y})$. This gives a periodic resolution of period k.

The special types of metacyclic groups that are singled out by our theory are those with cyclic Sylow subgroups. Namely we have

Proposition (**12.2**). — $\mathrm{Dim}\,\mathrm{V}(\mathrm{G}, h^\infty) \subset \mathrm{C}(\mathrm{G})$ *has maximal rank if and only if all Sylow subgroups of* G *are cyclic.*

Proof. — By Theorem (10.3) $\operatorname{rank} \mathrm{V}(\mathrm{G}, h^\infty) = \operatorname{rank} \mathrm{C}(\mathrm{G})$ if and only if for each $\mathrm{H} < \mathrm{G}$ the quotient $\mathrm{H}/[\mathrm{H}, \mathrm{H}] = \mathrm{L}$ is cyclic. If this condition holds for the Sylow subgroups they have to be cyclic. If the Sylow subgroups are cyclic then obviously L must be cyclic.

The groups G in question have the following structure (Wolf [28], (5.4.1)): Generators A, B with relations

$$A^m = B^n = 1, \quad BAB^{-1} = A^r,$$
$$((r-1)n, m) = 1, \quad r^n = 1 \bmod m.$$

The commutator subgroup is generated by A, and G has order mn. Having determined the rank of Dim V(G, h) it is necessary to ask for other relations that dimension functions have to satisfy. Obvious relations come from this observation: If X is a homotopy representation then $\mathrm{res}_H X$ must have an NH-invariant homotopy-type. For metacyclic groups as above all additional relations are of this type (see [34]). We consider an example: m and n odd primes and $m = 1 \bmod n$. Let H be generated by A and K be generated by B. We have $\varphi(G) = \{1, H, K, G\}$. The group G has n one-dimensional irreducible representations, lifted from K; and $(m-1)/n$ n-dimensional irreducible representations induced from H. The Galois group of mn-th roots of unity acts on these irreducible representations with three orbits. Representatives 1, V, and W of these orbits have the dimension functions

	1	H	K	G
Dim S(1)	1	1	1	1
Dim SV	1	1	0	0
Dim SW	n	0	1	0

Here we consider complex representations (even homotopy representations) and divide dimensions by 2. There exists a homotopy representation of dimension $2n-1$ with free G-action. Hence

	1	H	K	G
Dim X	n	0	0	0

The functions Dim S(1), Dim SV, Dim SW, Dim X generate a subgroup of C(G) of index n. There can be no more dimension functions because for any homotopy representation Y the relation

$$\dim Y = \dim Y^H \bmod n$$

holds. This follows from the fact that $\mathrm{res}_H Y$ must be $G/H \cong K$-invariant using the known classification of H-homotopy representations. The Burnside ring $A(G) \subset C(G)$ consists of functions z such that

$$z(G) \equiv z(H) \bmod n$$
$$z(H) \equiv z(1) \bmod m$$
$$z(K) \equiv z(1) \bmod n.$$

Using this one shows that the map

$$(C/mnC)^* \xrightarrow{\mu} (\mathbf{Z}/n)^* \times (\mathbf{Z}/m)^* \times (\mathbf{Z}/n)^* = A$$
$$z \mapsto (z(H)z(G)^{-1}, z(1)z(H)^{-1}, z(1)z(K)^{-1})$$

induces an isomorphism

(12.3) $\text{Pic}(G) \cong A/\bar{\omega}^*$ (See (11.4).)

In general terms the first factor of A is $(\mathbf{Z}/|WH|)^*$ and the last two factors give $(\mathbf{Z}/|W1|)^*$. As in (11.5), we find $\sigma = s\mu$, $s = s_{WH} \times s_{W1}$; moreover $s_{WH} = 0$ because WH is cyclic. If $|G|$ is odd $\bar{\omega}^*$ is cyclic of order 2 generated by $-1 = (-1, -1, -1)$ and the kernel of s_G consists of n-th powers mod m. Thus $V(G, h)$ is the quotient of $\{(a, b) \in (\mathbf{Z}/n)^* \times (\mathbf{Z}/mn)^* \mid b \equiv \mu^n \bmod m\}$ by the subgroup generated by -1. Taken altogether we have a complete description of V(G).

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