PUBLICATIONS MATHÉMATIQUES DE L'I.H.É.S.

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Publications mathématiques de l'I.H.É.S., tome 53 (1981), p. 79-162
<http://www.numdam.org/item?id=PMIHES_1981__53__79_0>

REAL ALGEBRAIC STRUCTURES ON TOPOLOGICAL SPACES

by SELMAN AKBULUT *and* HENRY C. KING

A real algebraic set is a set of the form $p^{-1}(0)$ for some real polynomial $p: \mathbf{R}^n \to \mathbf{R}^k$. This paper is part of an attempt to understand what these real algebraic sets look like. In particular, which topological spaces are homeomorphic to real algebraic sets?

It is known that real algebraic sets are triangulable and by [7] we know that each simplex of this triangulation is contained in an odd number of closed simplices. We also know by [5] that the singularities of a real algebraic set can be resolved, but it is not clear to us what this means topologically.

The purpose of this paper is to prove that certain topological spaces (which we call A-spaces) are homeomorphic to real algebraic sets (Theorem (8.1)). These A-spaces are smooth stratified sets which admit a certain topological resolution of singularities. One can show that any P.L. manifold is an A-space [3].

Roughly speaking, A-spaces are topological spaces built up from smooth manifolds by the operations of coning over boundaries, crossing with smooth manifolds and taking unions along the boundary.

Although not every real algebraic set is homeomorphic to an A-space (for instance the Whitney umbrella is not) it seems likely that the techniques of this paper combined with a few more from [2] will allow a topological characterization of real algebraic sets (i.e. a topologically defined class of spaces which up to homeomorphism is exactly the class of real algebraic sets).

In this paper we use very little algebraic geometry, a great deal of elementary differential topology and not much else. Thus very little background is needed to read the paper although at times it would be helpful to have read our previous work [1].

We start with Section 0 in which we give a rough sketch of the proof, making a few simplifications so as not to obscure the main ideas. In Section 1 through 7 we develop the ideas which we need to prove the main theorem in Section 8. Unfortunately we are in the position of developing the subject from scratch so the paper is rather long.

We recall a few definitions from [1]. A polynomial $p: \mathbf{R}^n \to \mathbf{R}$ is called *overt* if $p_d^{-1}(0)$ is empty or 0. Here p_d is the homogeneous part of p of maximal degree. An algebraic set $V \subset \mathbf{R}^n$ is called *projectively closed* if $V = p^{-1}(0)$ for some overt polynomial $p : \mathbf{R}^n \to \mathbf{R}$.

Let M be a smooth manifold and let U and V be smooth submanifolds of M. We say U and V intersect *cleanly* if $U \cap V$ is a smooth submanifold of M and the tangent space of $U \cap V$ is the intersection of the tangent spaces of U and of V. This is equivalent to having a coordinate system at each point of $U \cap V$ so that U and V are both linear subspaces in this coordinate system. Some examples of clean intersections are when U and V are transverse or when $U \subset V$.

Let us set up a bit of notation. For a topological space X, $cl(X)$ denotes the closure of X, $c(X)$ denotes the cone on X, i.e. $X \times [0, 1]/X \times 0$, and $\mathring{c}(X)$ denotes the open cone on X, i.e. $X \times [0, 1)/X \times 0$. In either case we denote the vertex $X \times 0/X \times 0$ by $*$. By convention:

$$c(\text{empty set}) = \mathring{c}(\text{empty set}) = \text{a point } *.$$

We set $\varepsilon c(X) = X \times [0, \varepsilon]/X \times 0$, $\varepsilon\mathring{c}(X) = X \times [0, \varepsilon)/X \times 0$ and $I = [0, 1]$.

We set $\varepsilon B^n = \{x \in \mathbf{R}^n \mid |x| \leq \varepsilon\}$. Also if T is a linear subspace of some $\mathbf{R}^n$ we set $\varepsilon T = \{x \in T \mid |x| < \varepsilon\}$. If T is a linear subspace of $\mathbf{R}^n$ and $y \in \mathbf{R}^n$ we set:

$$y + T = \{y + x \in \mathbf{R}^n \mid x \in T\}.$$

Also $\varepsilon\mathring{B}^n = \{x \in \mathbf{R}^n \mid |x| < \varepsilon\}$ and $\varepsilon S^{n-1} = \{x \in \mathbf{R}^n \mid |x| = \varepsilon\}$.

If M is a smooth manifold, then ∂M denotes its boundary and $\mathring{M}$ or int M denotes its interior.

A smooth submanifold N of a smooth boundaryless manifold M is called *proper* if it is a closed subset and ∂N is empty.

The reader may take a *spine* of a smooth manifold M to mean the complement of an open collar on ∂M. Thus if K is a spine of M then M/K is homeomorphic to $c\,\partial M$. Since A-spaces have boundaries and collars on boundaries we may similarly have a spine of an A-space. We only use this term spine loosely.

CONTENTS

0. — A SKETCH OF THE PROOF

The purpose of this section is to summarize this paper, by giving an overview of some of the constructions and the proof of the main theorem.

Firstly we construct certain stratified spaces which we call A-spaces. An A_0-space is a smooth manifold. An A_1-space is given by $Y = Y_0 \bigcup_{\varphi} \coprod (N_i \times c\Sigma_i)$ where Y_0 and N_i's are smooth manifolds and each Σ_i is a closed smooth manifold which bounds a compact smooth manifold, $\varphi = \{\varphi_i\}$, and each $\varphi_i : N_i \times \Sigma_i \to \partial Y_0$ is a smooth imbedding where we identify $N_i \times \Sigma_i$ with $N_i \times (\Sigma_i \times 1) \subset N_i \times c\Sigma_i$. Define:

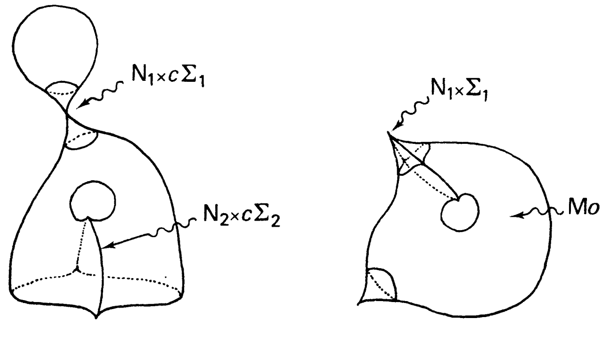
$$\partial Y = (\partial Y_0 - \bigcup \varphi_i(\overset{\circ}{N}_i \times \Sigma_i)) \bigcup_{\varphi} \coprod \partial N_i \times c\Sigma_i.$$

Inductively we assume that we have defined A_{k-1}-spaces and the notion of boundary for A_{k-1}-spaces is well defined. Then an A_k-space is given by: $Y = Y_0 \bigcup_{\varphi} \coprod_{i=1}^{\ell} (N_i \times c\Sigma_i)$ where Y_0 is an A_{k-1}-space with boundary, N_i's are smooth manifolds and each Σ_i is an A_{k-1}-space which is the boundary of a compact A_{k-1}-space, $\varphi = \{\varphi_i\}$, and each $\varphi_i : N_i \times \Sigma_i \to \partial Y_0$ is an A_{k-1}-imbedding (this means a piecewise differentiable imbedding preserving and respecting all the strata and the links of the strata). Also define:

$$\partial Y = (\partial Y_0 - \bigcup \varphi_i(\overset{\circ}{N}_i \times \Sigma_i)) \bigcup_{\varphi} \coprod \partial N_i \times c\Sigma_i$$

(see Figure 0.1). We say Y is *closed* if it is compact and $\partial Y = \varnothing$.

We call an entity an A-*space* if it is an A_k-space for some k. Also the notion of A-subspaces of an A-space and transversality between them are defined in the obvious ways. Define an A-isomorphism between two A-spaces to be a stratum preserving homeomorphism which restricts to a diffeomorphism on each stratum and preserves



an A_1-space with boundary an A_2-space

FIG. 0.1

the links (Σ_i's). Let $\dim(Y) =$ dimension of the highest dimensional stratum; as usual the superscript of Y^m denotes $\dim(Y)$.

In Section 1 a slightly different (but equivalent) definition of A-spaces is given.

A-spaces are constructed so that they can be topologically blown up to smooth manifolds: Let Y be an A_k-space given by the usual decomposition $M = Y_0 \underset{\varphi}{\cup} \amalg (N_i \times c\Sigma_i)$ and let W_i's be compact A_{k-1}-spaces with $\partial W_i = \Sigma_i$. Then the A_{k-1}-space

$$Y_1 = Y_0 \underset{\varphi}{\cup} \amalg (N_i \times W_i)$$

can be considered a one stage *blow up* of Y. There is the obvious map $\pi_1 : Y_1 \to Y$ which is obtained by collapsing spines of the W_i's to points. The disjoint union $\amalg N_i$ is called the *center* of π_1. Continuing in this way we get the resolutions

$$Y_k \xrightarrow{\pi_k} Y_{k-1} \xrightarrow{\pi_{k-1}} \dots \longrightarrow Y_1 \xrightarrow{\pi_1} Y$$

where Y_i's are A_{k-i}-spaces, in particular Y_k is a smooth manifold. The composition map $\pi = \pi_1 \circ \pi_2 \circ \dots \circ \pi_k : Y_k \to Y$ can be considered as a topological analogue of the resolution map for algebraic sets.

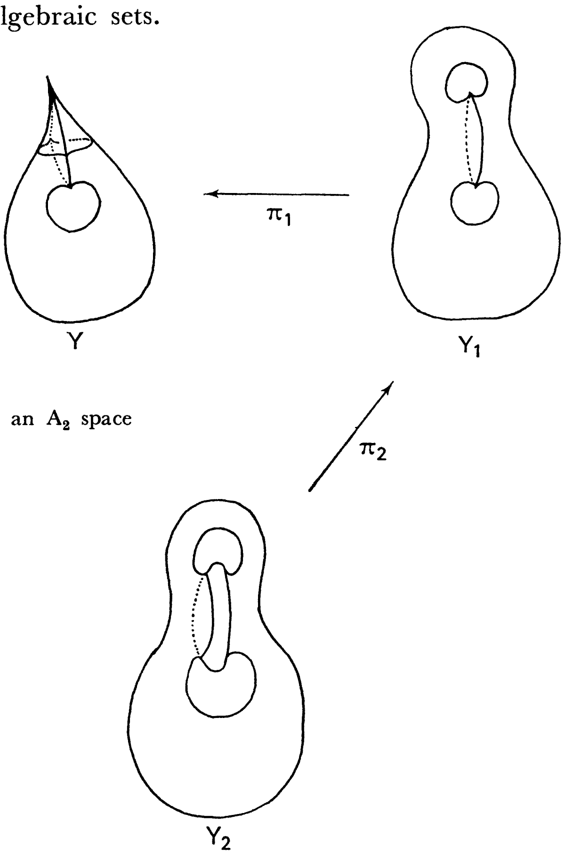


FIG. 0.2

We now state the main theorem which we will prove in Section 8.

Theorem **(8.1)**. — *The interior of any compact* A*-space is homeomorphic to a real algebraic set.*

In fact this homeomorphism is a stratified set isomorphism between the singular stratification of the variety and the A*-space.*

The resolution theorem of [3] shows that the class of A-spaces is large enough to contain all P.L. manifolds; hence we have the following corollary.

Theorem **(8.2)**. — *The interior of any compact P.L. manifold is P.L. homeomorphic to a real algebraic set.*

The idea of the proof of Theorem (8.1) is to make the smooth manifold Y_k a nonsingular algebraic set, where $\pi: Y_k \to Y$ is a resolution of an A_k space Y. Then blow down this set " algebraically " to a singular algebraic set which is homeomorphic to Y.

Roughly the proof goes in four steps:

1. Finding nice spines for A-spaces
2. Approximating submanifolds with subvarieties
3. Algebraic tower construction
4. Algebraic blowing down.

1. Finding nice Spines for A-spaces

Given an A_k-space Y^m which bounds a compact A_k-space, we prove that Y bounds a compact A_k-space W such that a spine of W consists of transversally intersecting codimension one closed A_k-subspaces with certain nice properties. For example if $Y = S^1$ then $W = T^2 - \mathring{D}^2$, and $\text{Spine}(W) = S^1 \times b \cup a \times S^1 \approx S^1 \vee S^1$ where $(a, b) \in T^2$.

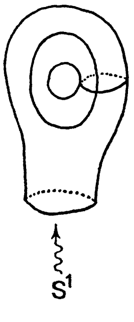


FIG. 0.3

This interesting topological property shows that one can resolve an A_k-space Y

$$Y_k \xrightarrow{\pi_k} Y_{k-1} \xrightarrow{\pi_{k-1}} \dots \longrightarrow Y_1 \xrightarrow{\pi_1} Y$$

in such a way that each π_j is obtained by collapsing A_{k-j}-subspaces $N_{ji}\times L_{ji}$ of Y_j to N_{ji} where L_{ji} are unions of codimension one A_{k-j}-spaces without boundaries and the N_{ji} are smooth. This makes the resolution $Y_k \to Y$ very analogous to a resolution of singularities of an algebraic set (every stage corresponds to resolving along the centers N_{ji}, and L_{ji} correspond to algebraic sets lying over N_{ji}). We call this resolution of an A_k-space a *good resolution.*

We discuss the method of obtaining these spines in Section 7. Proposition (0.1) gives the main idea (which derives from the smooth version in [1], Fact (3.2)).

Proposition **(0.1)**. — *If* Y^m *is an* A_k*-space which bounds a compact* A_k*-space* W_1^{m+1}, *then* Y^m *bounds a compact* A_k*-space* W^{m+1} *such that there are a finite number of codimension one closed* A_k*-subspaces* $\{S_i\}$ *in the interior of* W *with the properties*:

(i) $\bigcup S_i$ *is a spine of* W.

(ii) $\partial S_i = \varnothing$.

(iii) *Each* S_i *has a trivial normal bundle in* W.

Proof. — For the sake of clarity we first discuss the proof when Y is smooth (i.e. $k=0$) (cf. [1], Fact (3.2)). We prove this by induction on $m=\text{dimension}(Y)$. Let W_1^{m+1} be a compact smooth manifold with $\partial W_1 = Y$. Pick a collection of balls $\{D_i^{m+1}\}$ in W_1 so that:

(1) $\bigcup D_i$ is a spine of W_1.

(2) $\bigcup D_i - \bigcup \partial D_i = \bigcup\limits_j \text{interior}(B_j')$, where B_j''s are $(m+1)$-balls with disjoint interiors.

See Figure 0.4.

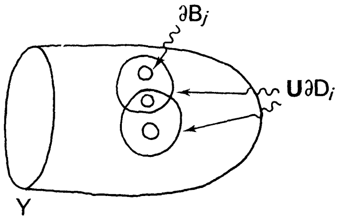


FIG. 0.4

Then remove an $(m+1)$-ball B_j from each B_j'. Then the smooth manifold

$$W_2 = W_1 - \bigcup_j \text{interior}(B_j)$$

has $\bigcup \partial D_i$ as a spine.

Then by attaching 1-handles onto ∂W_2 as in Figure 0.5 we get a manifold W_3 with $\partial W_3 = M \# S^m \# \dots \# S^m \approx M$ with spine $\bigcup \partial D_i \cup \bigcup C_j$ (Figure 0.6). But each circle C_j has a neighborhood N_j diffeomorphic to $S^1 \times B^m$. By induction we can find an m-manifold U^m with $\partial U^m \approx S^{m-1}$ and satisfying the conclusions of the propo-

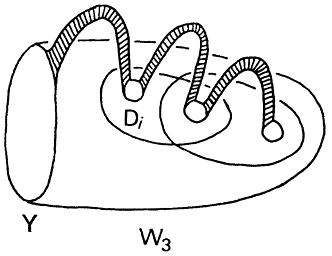


FIG. 0.5

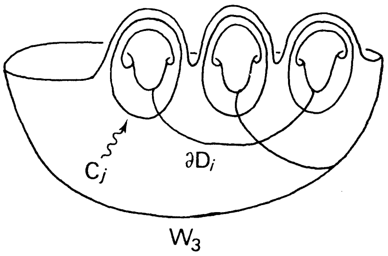


FIG. 0.6

sition. By replacing each N_j by $S \times U^m$ in W_3 we get a manifold W^{m+1} with the required properties.

For instance the example in Figure 0.3 can be obtained by applying this process to D^2 which S^1 bounds.

Proof in the general case. — Let Y^m be an A_k-space as above. We prove this again by induction on m, clearly the theorem is true for $m = 0$. Assume that the theorem holds for $m - 1$. Without loss of generality we can assume that all the strata of W_1 meet the boundary; because if $W_1 = W_1' \cup N \times c\Sigma$ where N is a statum of W_1 which is a closed manifold contained in the interior of W_1, we simply replace W_1 by $W_1' \cup N \times W_1''$ where W_1'' is an A_k-space with $\partial W_1'' = \Sigma$ (cf. Lemma (1.4)).

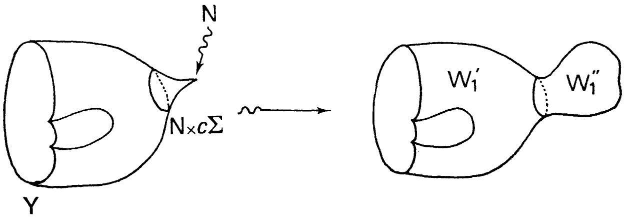


FIG. 0.7

Pick a fine triangulation of W_1 compatible with the stratification coming from the A_k-structure. Let K be a subcomplex of W_1 which is a spine of W_1. Cover K with $\bigcup_j \text{star}(v_j)$ where $\{v_i\}$ are the barycenters of the simplicies in K, and star (v_i) is the closure of the union of all simplicies meeting v_i. The star (v_i)'s can naturally be isotoped to codimension zero A_k-subspaces of Y, namely $(D^r)' \times c\Sigma$ if v_i lies on an r-dimensional smooth stratum N^r and Σ is the link of this stratum and $(D^r)'$ is a smooth ball in N^r (in the smooth case star (v_i) are just smooth balls).

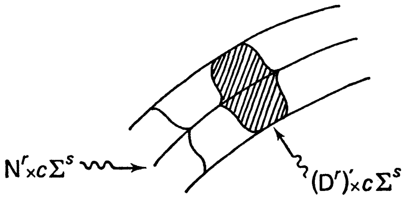


FIG. 0.8

By shrinking the star (v_i)'s slightly, call them D_i, we can assume that:

(1) $\bigcup D_i$ is a spine of W_1.

(2) $D_i = D'_i \times c\Sigma_i$, D'_i, Σ_i are as above.

(3) $\bigcup D_i - \bigcup_i \partial D_i = \bigcup_j \text{interior}(B'_j)$, where B'_j's are disjoint A_k-subspaces of W_1 with disjoint interiors and with the A_k-structures $B'_j = B''_j \times c\Sigma_j$, where B''_j are discs lying in some strata and Σ_j are the links of these strata in W_1 (compare [1], Fact (3.2)).

(4) $\{\partial D_i\}$ have trivial normal bundles in W_1 and they are in general position with respect to each other.

Let B_j be a slightly shrunken copy of B'_j inside of B'_j, i.e. $B_j = \frac{1}{2} B''_j \times \frac{1}{2}(c\Sigma_j)$ where $\frac{1}{2} B''_j \subset B''_j$ and $\frac{1}{2}(c\Sigma_j)$ is a subcone of $c\Sigma_j$. Then clearly $\bigcup_i \partial D_i$ is a spine of the A_k-space $W_2 = W_1 - \bigcup_{j=1}^{s} \text{interior}(B_j)$.

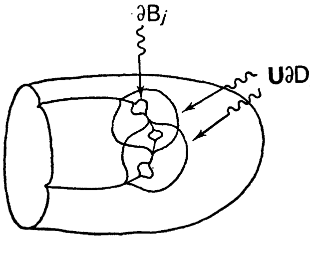


FIG. 0.9

The spine $\bigcup_i \partial D_i$ of W_2 satisfies the conclusions (i), (ii) and (iii) of the proposition but unfortunately ∂W_2 is not equal to Y, $\partial W_2 = Y \amalg \bigcup_{j=1}^{s} \partial B_j$. Inductively we reduce s by changing W_2 to another A_k-space whose spine satisfies the conditions (i), (ii) and (iii) until we get such an A_k-space W with $\partial W = Y$. Hence it remains to show that s can be made to be $s-1$. Pick an $\alpha\in\{1, 2, \dots, s\}$ such that there is a smooth arc on the stratum N, on which B''_α lies, connecting $\partial\left(\frac{1}{2}B''_\alpha\right)\times * \subset \partial B_\alpha$ to ∂W_1 (where $*$ is the vertex in $c\Sigma_\alpha$), meeting only one of the ∂D_i, and meeting this ∂D_i at a single point. Let $N' = N\cap\partial W_1 = \partial N$ (recall that $N' \neq \varnothing$ by the hypothesis). Then the link of N′ in ∂W_1 is Σ_α since the link of N in W_1 is Σ_α. Let $U'_\alpha = D'_\alpha \times c\Sigma_\alpha$ where D'_α is a small codimension zero ball in N'_α; hence U'_α is a small closed neighborhood of D'_α in ∂W_1. Let

$$U''_\alpha = D''_\alpha \times c\Sigma_\alpha$$

where D''_α is a small codimension zero ball in $\partial\left(\frac{1}{2}B''_\alpha\right)\times *$ and hence U''_α is a small closed neighborhood of D''_α in ∂W_2. Let $\overline{W} = W_2 \cup I\times(D\times c\Sigma_\alpha)$ where $I\times(D\times c\Sigma_\alpha)$ is glued onto ∂W_2 along $\partial I\times(D\times c\Sigma_\alpha) \approx U'_\alpha \cup U''_\alpha$ and D is a smooth ball of the same dimension as D'_α and D''_α.

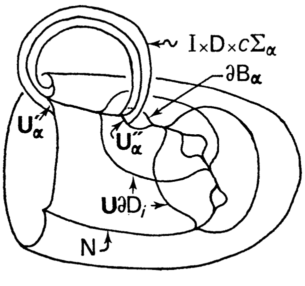


Fig. 0.10

Note that $\overline{W}$ is a compact A_k-space with $\partial\overline{W} = Y \amalg \bigcup_{j\neq\alpha} \partial B_j$ (because connecting Y to ∂B_α in this manner just gives Y back) and $\bigcup_i \partial D_i \cup C_\alpha$ is a spine of $\overline{W}$ where C_α is the smooth circle $C'_\alpha \cup C''_\alpha$ with $C'_\alpha = I\times o\times * \subset I\times D\times c\Sigma_\alpha$, and C''_α is a smooth arc in N connecting the end points of C'_α and intersecting $\bigcup \partial D_i$ only at a single point.

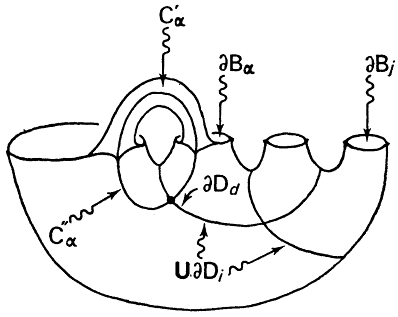


Fig. 0.11

The circle C_α has a neighborhood $S^1\times D\times c\Sigma_\alpha$ in $\overline{W}$. Since $\partial(D\times c\Sigma_\alpha)$ is an A_k-space of dimension less then m bounding $D\times c\Sigma_\alpha$, by the induction hypothesis there is a compact A_k-space W_α with $\partial W_\alpha = \partial(D \times c\Sigma_\alpha)$ and codimension one A_k-subspaces $\{Z_r\}$ of W_α satisfying (i), (ii) and (iii) of the proposition. Let

$$W = [\overline{W} - \text{interior}(S^1 \times D \times c\Sigma_\alpha)] \cup S^1 \times W_\alpha .$$

(This should be considered as blowing up $\overline{W}$ along the circle C_α.)

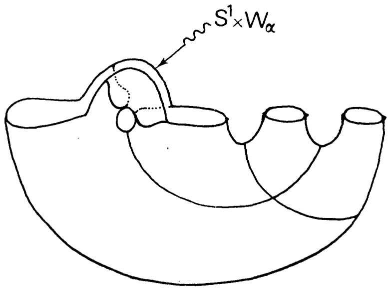


FIG. 0.12

This process alters ∂D_d, where d is the unique index with $\partial D_d \cap C_\alpha$ consisting of a single point, to $\widetilde{D}_d = (\partial D_d - \text{interior}(D \times c\Sigma_\alpha)) \cup W_\alpha$. Then $\widetilde{D}_d \cup \bigcup_{j \neq d} \partial D_j \cup \bigcup_r S^1 \times Z_r$ is a spine of W, and the codimension one A_k-subspaces $\{\widetilde{D}_d, D_j, S^1 \times Z_r\}$ satisfy (i), (ii) and (iii) of the proposition by the construction and the inductive hypothesis on $\{Z_r\}$'s. Also $\partial W = \partial W_3 = Y \amalg \bigcup_{\substack{j=1 \\ j \neq \alpha}}^{s} \partial B_j$, i.e. after reindexing $\partial W = Y \amalg \bigcup_{j=1}^{s-1} \partial B_j$. We are done. ■

Remark. — It should be emphasized that the W which satisfies the conclusion of Proposition (0.1) is obtained by first finding an A_k-space $\overline{W}$ with $\partial\overline{W} = Y$ and with Spine($\overline{W}$) consisting of codimension one closed separating A_k-subspaces $\{Z_r\}$ and circles $\{C_j\}$ such that each C_j intersects only one Z_r at a single point; then blowing up $\overline{W}$ along these circles. More precisely each circle C_j has a neighborhood $N(C_j) \approx S^1 \times (D_j \times c\Sigma_j')$ in $\overline{W}$ where D_j is a smooth ball and Σ_j' is a closed A_k-space which is a boundary. Then inductively $\partial(D_j \times c\Sigma_j') = \partial W_j''$ where W_j'' are compact A_k-spaces satisfying the conditions of the proposition, and:

$$W = (\overline{W} - \bigcup_j N(C_j)) \cup \bigcup S^1 \times W_j''$$

where the union is taken along $\partial N(C_j) \approx S^1 \times \partial W_j''$ for each j.

2. Approximating Submanifolds with Algebraic Subsets

This is discussed in Section 2. The Nash-Tognoli theorem states that any closed smooth manifold is diffeomorphic to a nonsingular algebraic subset of $\mathbf{R}^n$ for some n. More generally we can start with a closed smooth submanifold M^m of a nonsingular algebraic set V and try to isotop it to a nonsingular algebraic subset of $V \times \mathbf{R}^n$. Such a result can be achieved under a bordism restriction: M gives rise to an element $[M^m]$ of the unoriented bordism group $\mathcal{N}_m(V)$. We say that [M] is algebraic in V if [M] contains a representative which is in the form: $P \times N \xrightarrow{\text{projection}} P \hookrightarrow V \times \mathbf{R}^q$ where P is a nonsingular, algebraic subset of $V \times \mathbf{R}^q$ for some q. (For example in case $V = \mathbf{R}^n$ then every such [M] is algebraic.) Then we have the following result which is proved in Proposition (2.3).

Proposition (**0.2**). — *If* [M] *is algebraic in* V *then* M *is isotopic to a nonsingular algebraic set in* $V \times \mathbf{R}^q$ *for some* q *by a small isotopy.*

3. Algebraic Tower Construction

Let Y be a closed A_k-space and $Y_k \xrightarrow{\pi_k} Y_{k-1} \xrightarrow{\pi_{k-1}} \dots \longrightarrow Y_1 \xrightarrow{\pi_1} Y$ be a good resolution. Let N_i be the center of π_i. Here we construct a tower of nonsingular algebraic sets

$$U_k \xrightarrow{p_k} U_{k-1} \xrightarrow{p_{k-1}} \dots \longrightarrow U_1 \xrightarrow{p_1} U_0 = U$$

with $N_{i+1} \subset U_i$ as a nonsingular algebraic set (and with U_{i+1} some kind of algebraic resolution of U_i along N_{i+1} discussed in Section 4), and imbeddings $Y \subset U_i$ which commute with the projections and which are in some sense stable over the projections. (For the proof of Theorem (8.3) the imbeddings in (8.3) don't actually need to commute but for the purposes of this summary the reader should assume they commute with the projections.) This means that if Y_i' is a nearby copy of Y_i then $p_i(Y_i')$ is a nearby copy of $p_i(Y_i) = Y_{i-1}$. Thus if Q is diffeomorphic to Y_k and is close to Y_k in U_k then $p(Q) \approx p(Y_k) \approx Y$ where $p = p_1 \circ p_2 \circ \dots \circ p_k$. To do this construction we need to make the following definition which is made more precise in Section 4. For a given nonsingular algebraic set U and a nonsingular algebraic subset $N \subset U$ we say that $p : \hat{U} \to U$ is a *super multiblowup* with center N if p is the composition of the maps

$$\hat{U} \to \dots \to (\widetilde{U \times \mathbf{R}^{q_1}}) \times \mathbf{R}^{q_2} \xrightarrow{\pi_2} (\widetilde{U \times \mathbf{R}^{q_1}}) \xrightarrow{p_1} U \times \mathbf{R}^{q_1} \xrightarrow{\pi_1} U$$

where the π_i are the obvious projections and p_i are multiblowups whose centers are some nonsingular algebraic subsets ($\approx N \times L$ for some L) lying over N. A multiblowup of Q with center P means you first blow Q up along P, then find a copy of P upstairs lying over P, then blow up again, etc., a certain number of times. (There is a technical difficulty of blowing up a nonsingular variety Q along a subvariety P more than once because after the first blow up $\tilde{Q}$ we might no longer be able to find an algebraic copy of P in $\tilde{Q}$ to blow up $\tilde{Q}$ along P again. But this can be done if we cross the space with **R** before every blowup.)

Let $Y = Y_0^1 \cup \coprod_{i=1}^{r} (N_i^1 \times c\Sigma_i^1)$ be a usual decomposition of the A_k-space Y. Furthermore without loss of generality assume $r=1$ (otherwise repeat the following process), i.e. $Y=Y_0^1 \cup N_1 \times c\Sigma_1$. Since N_1 is a smooth closed manifold it can be made an algebraic subset of $U_0=\mathbf{R}^{q_1}$ for some large q_1 (Proposition (0.2)). Extend this imbedding to an imbedding of Y such that N_1 is identified by $N_1 \times * \hookrightarrow N_1 \times c\Sigma_1 \hookrightarrow Y$. Let W_1 be a compact A_{k-1}-space with $\partial W_1=\Sigma_1$ with a spine of transversally intersecting codimension one closed A_{k-1}-subspaces (Proposition (0.1)). We claim that we can find a super multiblowup with center N_1, $\bar{p}_1 : \bar{U}_1 \to \mathbf{R}^{q_1}$ and an imbedding of $N_1 \times W_1 \hookrightarrow \bar{U}_1$ so that:

(i) $N_1 \times W_1$ is transverse to $\bar{p}_1^{-1}(N_1)$,
(ii) $\bar{p}_1^{-1}(N_1) \cap (N_1 \times W_1) = N_1 \times \text{Spine}(W_1)$,
(iii) $\bar{p}_1(N_1 \times W_1) \approx N_1 \times c\Sigma_1$.

We prove this by induction on the dimension of W_1.

By Proposition (0.1) and the Remark following it, we can assume that there is an A_{k-1}-space $\overline{W}_1$ such that a spine of $\overline{W}_1$ consists of transversally intersecting codimension one closed A_{k-1}-subspaces Z_r's and circles C_j's such that each C_j intersects a unique Z_r at a single point. Furthermore W_1 is obtained by blowing $\overline{W}_1$ along each circle C_j, i.e.:

$$W_1 = (\overline{W}_1 - \bigcup_j N(C_j)) \cup \bigcup S^1 \times W_j''$$

where $N(C_j)$ is an open neighborhood of the circle C_j in $\overline{W}_1$, in fact the closure of $N(C_j) \approx S^1 \times W_j'$ where W_j' is an A_{k-1}-space (it is the cone over the link of C_j in $\overline{W}_1$) and W_j'' is a compact A_{k-1}-space (obtained inductively) satisfying the properties of the proposition (0.1) with $\partial W_j' = \partial W_j''$. Also the A_{k-1}-spaces Z_r separate each other.

First identify $\mathbf{R}^{q_1}$ by $\mathbf{R}^{q_1} \times 0 \subset \mathbf{R}^{q_1+2}$, and then multiblowup $\mathbf{R}^{q_1+2}$ (algebraically) along the algebraic subset N_1 several times. Let π_1 be the composition of the multiblowup map and the obvious projection

$$\widetilde{\mathbf{R}}^{q_1+2} \xrightarrow{\pi_{11}} \mathbf{R}^{q_1+2} = \mathbf{R}^{q_1} \times \mathbf{R}^2 \xrightarrow{\pi_{12}} \mathbf{R}^{q_1} \supset N_1.$$

For any $p \in N_1 \subset \mathbf{R}^{q_1}$, $\pi_1^{-1}(p)$ is transversally intersecting codimension one algebraic subsets $\pi_{11}^{-1}(p, 0)$ and a codimension q_1 nonsingular algebraic subset R_p (= the strict preimage of $p \times \mathbf{R}^2$) of $\pi_1^{-1}(\mathcal{O}_p)$, where $\mathcal{O}_p$ is the normal plane to N_1 in $\mathbf{R}^{q_1}$.

Hence in $\pi_1^{-1}(\mathcal{O}_p)$, $\pi_1^{-1}(p)$ locally looks like transversally intersecting hyperplanes and a codimension q_1-dim N_1 plane in a euclidean space. Because the Z_r's have trivial normal bundles and separate each other in $\overline{W}_1$, we can imbed a neighborhood Q of $\bigcup Z_r$ (in $\overline{W}_1$) into $\pi_1^{-1}(\mathcal{O}_p) \subset \widetilde{\mathbf{R}}^{q_1+2}$ as in Figure (0.13) such that:

(1) Q is transverse to $\pi_{11}^{-1}(p, 0)$.

(2) $\pi_{11}^{-1}(p, 0) \cap Q \approx \bigcup Z_r$.

(3) $R_p \cap Q \approx \bigcup I_j$ where $\bigcup I_j$ is a disjoint union of arcs corresponding to the intersection $Q \cap \bigcup_j C_j$ in $\overline{W}_1$.

(4) $\pi_{11}(Q) \approx c(\partial Q)$.

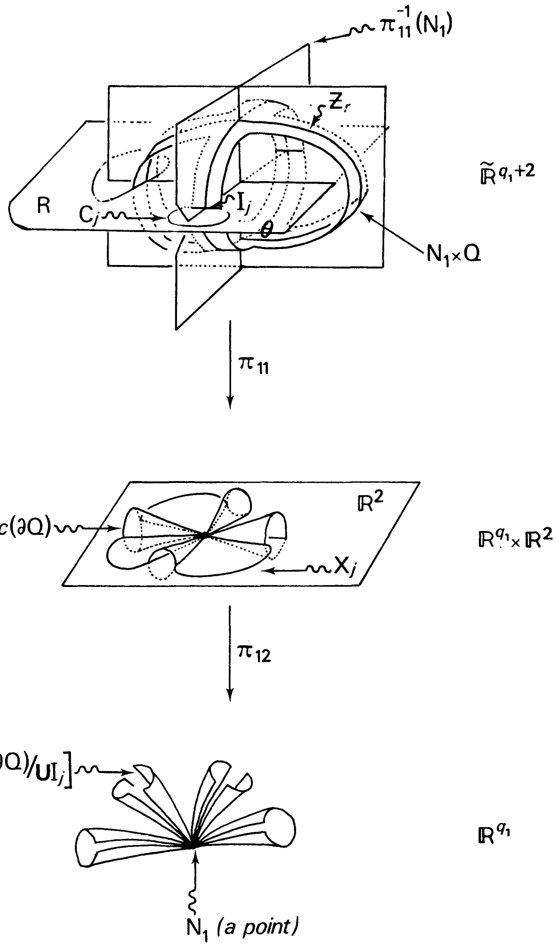


FIG. 0.13

Now as in Figure 0.14 we extend the imbedding $Q \hookrightarrow \pi_1^{-1}(\mathcal{O}_p)$ to an imbedding $\overline{W}_1 \hookrightarrow \pi_1^{-1}(\mathcal{O}_p)$ by attaching 1-handles to Q (in $\pi_1^{-1}(\mathcal{O}_p)$) along the end points of I_j's, so that:

(1) $\overline{W}_1$ is transverse to $\pi_{11}^{-1}(p, 0)$
(2) $\pi_{11}^{-1}(p, 0) \cap \overline{W}_1 \approx \bigcup Z_r$
(3) $R_p \cap \overline{W}_1 \approx \bigcup C_j$ (extending the imbedding of $\bigcup I_j$) with each C_j intersecting $\pi_{11}^{-1}(p, 0)$ only at a single point (in I_j).

In particular $\pi_1^{-1}(p) \cap \overline{W}_1 \approx (\bigcup Z_r) \cup (\bigcup C_j) = \text{Spine } \overline{W}_1$. In fact first π_{11} collapses $\bigcup Z_r$ to $(p, 0)$ then π_{12} collapses $\bigcup C_j$ to p so that $\pi_1(\overline{W}_1) \approx c(\Sigma_1)$.

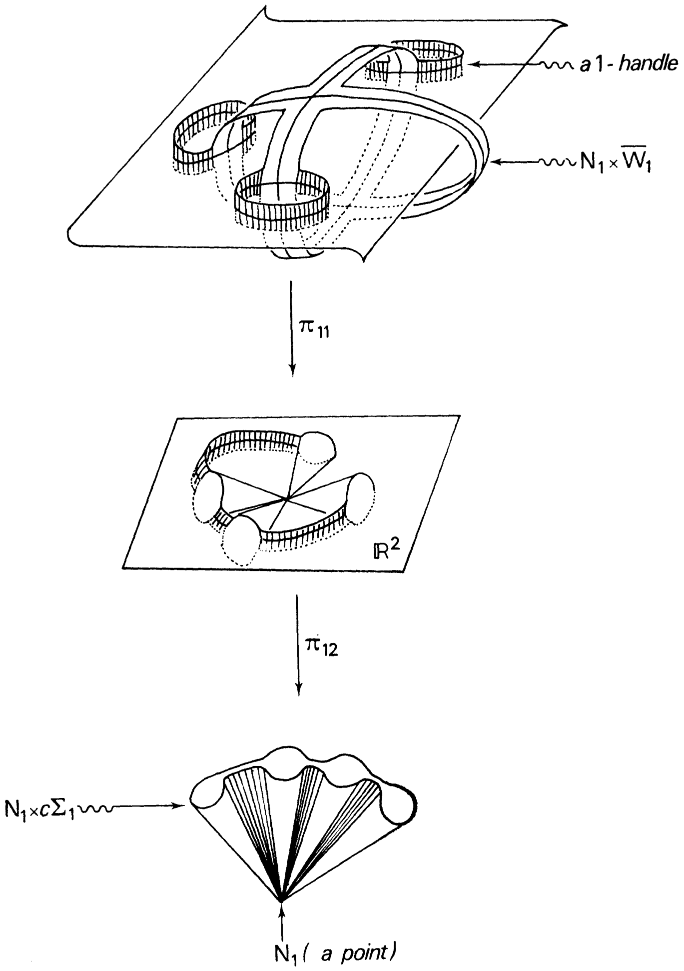


FIG. 0.14

By doing this process continuously for each $p \in N_1$ we end up with an imbedding $N_1 \times \overline{W}_1 \hookrightarrow \widetilde{\mathbf{R}}^{q_1+2}$ such that:

(1) $N_1 \times \overline{W}_1$ is transverse to $\pi_{11}^{-1}(N_1)$

(2) $\pi_1^{-1}(N_1) \cap (N_1 \times \overline{W}_1) \approx N_1 \times \text{Spine } \overline{W}_1$

(3) $\pi_1(N_1 \times \overline{W}_1) \approx N_1 \times c\Sigma_1$.

We can assume that the each naturally imbedded copy of $N_1 \times C_j$ in $\widetilde{\mathbf{R}}^{q_1+1}$ is a nonsingular algebraic set (and they are disjoint for different j's). This is not obvious to see, but here is a way of visualizing it: we can imbed a bouquet of circles $\bigcup X_j$ (Figure 0.13) into $\mathbf{R}^2$ each as a nonsingular algebraic set with various degree's of tangencies with respect to each other so that the map π_{11} will lift $N_1 \times \bigcup X_j$ to the right places in

$\widetilde{\mathbf{R}}^{q_1+2}$, this will in turn imply that $\pi_{11}^{-1}(N_1\times X_j)$ contains an algebraic copy of $N_1\times C_j$ as above (=the strict preimage of $N_1\times X_j$).

We are now in a position to apply the induction. We have the disjoint nonsingular algebraic sets $N_1\times C_j$ in $\widetilde{\mathbf{R}}^{q_1+2}$, so by induction there is a super multiblowup with center $\bigcup N_1\times C_j$, $\overline{U}_1\xrightarrow{\pi_0}\widetilde{\mathbf{R}}^{q_1+2}$ and an imbedding $\bigcup(N_1\times C_j)\times W_j''\hookrightarrow\overline{U}_1$ so that:

(1) $\bigcup(N_1\times C_j)\times W_j''$ is transversal to $\pi_0^{-1}(\bigcup N_1\times C_j)$

(2) $\pi_0^{-1}(N_1\times C_j)\cap(N_1\times C_j\times W_j'')\approx(N_1\times C_j)\times\mathrm{Spine}(W_j'')$

(3) $\pi_0(N_1\times C_j\times W_j'')\approx N_1\times C_j\times c(\partial W_j'')$.

We then extend this imbedding to an imbedding of

$$N_1\times W_1=N_1\times(\overline{W}_1-\bigcup N(C_j))\cup\bigcup N_1\times C_j\times W_j''\hookrightarrow\overline{U}_1$$

by simply lifting the imbedding $N_1\times(\overline{W}_1-\bigcup N(C_j))\hookrightarrow\widetilde{\mathbf{R}}^{q_1+2}$ by π_0 and piecing with the above imbedding. This the imbedding $N_1\times W_1\hookrightarrow\overline{U}_1$ and the composite multiresolution $\bar{p}_1:\overline{U}_1\xrightarrow{\pi_0}\widetilde{\mathbf{R}}^{q_1+2}\xrightarrow{\pi_1}\mathbf{R}^{q_1}$, $\bar{p}_1=\pi_1\circ\pi_0$ satisfies the requirements of the claim. Of course the above process requires some care (in fact it should be done slightly differently) and we don't indulge in the details in this section.

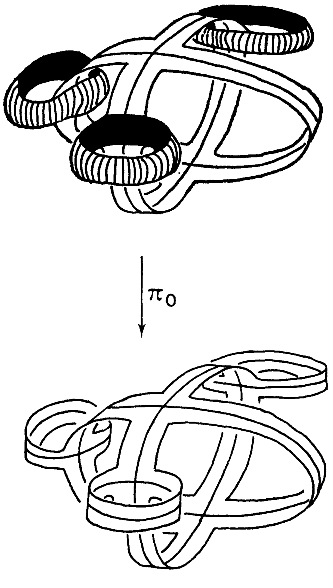


FIG. 0.15

Having constructed $\bar{p}_1$ we extend the imbedding of $N_1\times W_1\subset\overline{U}_1$ to an imbedding of $Y_1=Y_0^1\cup N_1\times W_1$. This is done by lifting the imbedding $Y_0^1\hookrightarrow\mathbf{R}^{q_1}$ via $\bar{p}_1$ and piecing together with the imbedding of $N_1\times W_1$. This imbedding easily can be arranged so that Y_1 is transverse to $\bar{p}_1^{-1}(N_1)$, $\bar{p}_1(Y_1)=Y$ and $\bar{p}_1$ is stable. Since Y_1 is an A_{k-1}-space let $Y_1=Y_0^2\cup N_2\times c\Sigma_2$ be the usual A_{k-1}-decomposition. Now suppose $[N_2]$ is algebraic in $\overline{U}_1$

then by (0.2), N_2 can be made a nonsingular algebraic subset in $U_1 = \overline{U}_1 \times \mathbf{R}^{s_1}$ for some s_1. Let W_2 be an A_{k-2}-space satisfying requirements of Proposition (0.1) so $\partial W_2 = \Sigma_2$. Then just as above we find a multiresolution $\bar{p}_2 : \overline{U}_2 \to U_1$ with center N_2 and an imbedding of $Y_2 = Y_0^2 \cup N_2 \times W_2 \hookrightarrow \overline{U}_2$ such that Y_2 is transverse to $\bar{p}_2^{-1}(N_2)$, $\bar{p}_2(Y_2) = Y_1$ and $\bar{p}_2$ is stable. Continuing in this fashion we get the following tower:

$$\begin{tikzcd}
 & \overline{U}_k \times \mathbf{R}^{s_k} \arrow[d] & = U_k \arrow[dd, "p_k"] \\
Y_k \arrow[r, hook] \arrow[ur, hook] \arrow[dd] & \overline{U}_k \arrow[d, "\bar{p}_k"] & \\
 & \overline{U}_{k-1} \times \mathbf{R}^{s_{k-1}} \arrow[d] & = U_{k-1} \supset N_k \arrow[dd] \\
Y_{k-1} \arrow[r, hook] \arrow[ur, hook] \arrow[d] & \overline{U}_{k-1} \arrow[d] & \\
\vdots \arrow[dd] & \vdots \arrow[d, "\bar{p}_3"] & \vdots \arrow[d] \\
 & \overline{U}_2 \times \mathbf{R}^{s_2} \arrow[d] & = U_2 \supset N_3 \arrow[dd, "p_2"] \\
Y_2 \arrow[r, hook] \arrow[ur, hook] \arrow[dd] & \overline{U}_2 \arrow[d, "\bar{p}_2"] & \\
 & \overline{U}_1 \times \mathbf{R}^{s_1} \arrow[d] & = U_1 \supset N_2 \arrow[dd, "p_1"] \\
Y_1 \arrow[r, hook] \arrow[ur, hook] \arrow[d] & \overline{U}_1 \arrow[d, "\bar{p}_1"] & \\
Y \arrow[r, hook] & \mathbf{R}^{q_1} & = U \supset N_1
\end{tikzcd}$$

where Y_k is a smooth manifold. Supposing $[Y_k]$ is algebraic in $\overline{U}_k$, Y_k can be isotoped (by a small isotopy) to a nonsingular algebraic set Q in $\overline{U}_k \times \mathbf{R}^{s_k}$ for some s_k. Call $U_k = \overline{U}_k \times \mathbf{R}^{s_k}$. The tower

$$\begin{tikzcd}
U_k \arrow[r, "p_k"] & U_{k-1} \arrow[r, "p_{k-1}"] & \cdots \arrow[r] & U_1 \arrow[r, "p_1"] & U = \mathbf{R}^{q_1} \\
\cup & \cup & & \cup & \cup \\
Y_k \arrow[r, hook] & Y_{k-1} \arrow[r] & \cdots \arrow[r] & Y_1 \arrow[r] & Y
\end{tikzcd}$$

has the required properties where p_i is the composition $U_i = \overline{U}_i \times \mathbf{R}^{s_i} \to \overline{U}_i \xrightarrow{\bar{p}_i} U_{i-1}$. In particular this tower is stable in the sense described earlier in this section. This is visualized by the Figure 0.16. Isotop Y_k to an algebraic set Q which is close to Y_k then $p(Q) \approx Y$ where $p = p_1 \circ p_2 \circ \ldots \circ p_k$ ($k = 1$ in the picture). This is true because of the properties of p_i's.

One has to prove that $[Y_k]$ and $[N_i]$ for $i = 1, 2, \ldots$ are algebraic. This is not obvious, one has to complicate the whole construction to see this, we refrain and refer the reader to the main proof. (Note: Set $\hat{p}_i = p_i \circ p_{i+1} \circ \ldots \circ p_k$. Then in the proof of (8.3), the projections $\hat{p}_i : \hat{p}_i^{-1}(N_i) \rightarrow N_i$ are fins in U_k.)

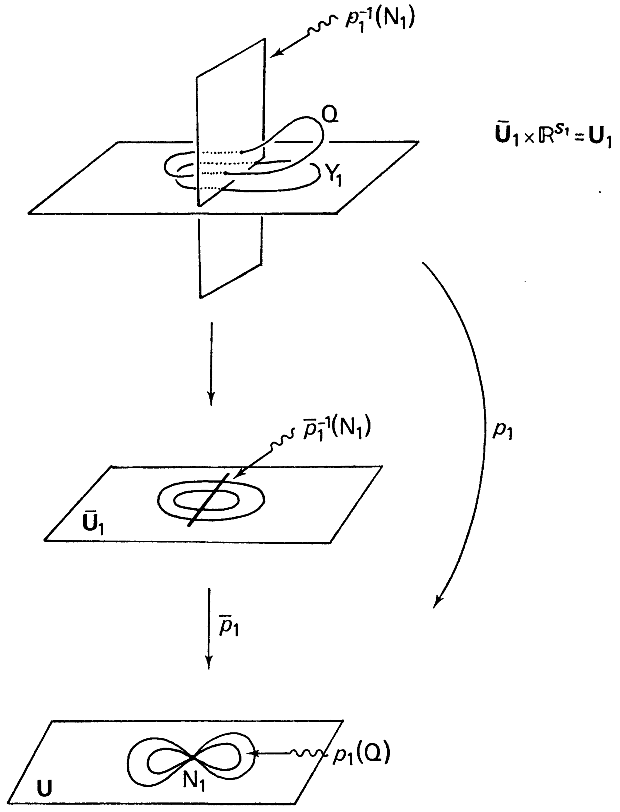


FIG. 0.16

4. Blowing Down

Having found an algebraic set $Q \subset U_k$ with $p(Q) \approx Y$ where $p : U_k \rightarrow U$ is as above, we need to prove that $p(Q)$ is still an algebraic set. A priori there is no reason to assume that p takes algebraic sets to algebraic sets (in fact a linear projection $\mathbf{R}^2 \rightarrow \mathbf{R}$ takes the unit circle to an interval). So we have to do a more complicated algebraic blowdown than mere projection p. This is discussed in section 3.

I. — STRATIFIED SETS AND A-SPACES

The natural topological object to use when dealing with algebraic sets is a smooth stratified set. Any real algebraic set has a smooth stratification. In fact it is not hard to show, using the Whitney conditions, that every real algebraic set is homeomorphic to the interior of a compact TCSS space (these TCSS spaces are smooth stratified sets with some extra structure defined below). The converse is not true, although we know of no compact TCSS space which satisfies Sullivan's even local Euler characteristic condition and is not homeomorphic to any real algebraic set. In fact in dimensions ≤ 2 we can show that no such example exists, thus we topologically characterize real algebraic sets of dimension ≤ 2 [2].

Resolution of singularities tells us that links of strata of algebraic sets must bound in some sense, but what this bounding means is not clear to us. What we can show is that if the links of the strata of a compact TCSS space bound in a certain naive sense then this TCSS space is homeomorphic to a real algebraic set. We call TCSS spaces with this naive bounding condition on the links A-spaces. Not all real algebraic sets are homeomorphic to A-spaces, for instance the Whitney umbrella is not. However the class of A-spaces is big enough to include all PL manifolds [3].

In this section we give definitions of stratified sets, TCSS spaces, A-spaces and various auxiliary notions. Also we show how we may " resolve the singularities " of A-spaces by blowing up along closed strata.

The following definition of a smooth stratified set with boundary is the same as the usual definition of smooth stratified set except that we allow a stratum to have a boundary.

Roughly speaking a TCSS space is a smooth stratified set together with a trivialization of a " tubular neighborhood " of each stratum. The trivializations of different strata are required to fit together nicely.

An A-space is a TCSS space so that the link of each stratum bounds. Thus we can resolve the singularities of an A-space by replacing a (closed stratum)×(cone on its link) by (closed stratum)×(A-space which its link bounds).

Definition. — A *smooth stratified set with boundary* is a topological space X with a locally finite collection of disjoint subsets $\{X_\alpha\}$ $\alpha\in\mathscr{A}$ (which we call strata of X) so that:

1) $X = \bigcup_{\alpha \in \mathscr{A}} X_\alpha$;
2) each stratum X_α is given the structure of a smooth manifold with (possibly empty) boundary;
3) the closure of each stratum is a union of strata;
4) the closure of the boundary of each stratum is a union of boundaries of strata.

A stratified set is called *finite* if it has only a finite number of strata.

The *dimension* of a smooth stratified set is the maximum dimension of its strata (assuming a maximum exists).

The *boundary* of a smooth stratified set with boundary $(X, \{X_\alpha\}\ \alpha \in \mathscr{A})$ is the smooth stratified set $(\bigcup_{\alpha \in \mathscr{A}} \partial X_\alpha, \{\partial X_\alpha\}\ \alpha \in \mathscr{A})$. We denote this by $\partial(X, \{X_\alpha\})$ or, loosely, ∂X.

If $(X, \{X_\alpha\}\ \alpha \in \mathscr{A})$ and $(Y, \{Y_\beta\}\ \beta \in \mathscr{B})$ are smooth stratified sets with boundary then we may define a cartesian product $(X \times Y, \{X_\alpha \times Y_\beta\}\ (\alpha, \beta) \in \mathscr{A} \times \mathscr{B})$ by standard corner rounding on the $X_\alpha \times Y_\beta$.

If $(X, \{X_\alpha\}\ \alpha \in \mathscr{A})$ is a finite smooth stratified set without boundary we may define the *cone* and *open cone* on X by:

$$(cX, \{*\} \cup \{X_\alpha \times (0, 1]\}\ \alpha \in \mathscr{A}) = c(X)$$

and

$$(\overset{\circ}{c}X, \{*\} \cup \{X_\alpha \times (0, 1)\}\ \alpha \in \mathscr{A}) = \overset{\circ}{c}(X).$$

If X and Y are finite stratified sets with empty boundary we may define the *join* $X * Y$ of X and Y to be the union $X \times \overset{\circ}{c}Y \cup \overset{\circ}{c}X \times Y$ with $(x, (y, t)) \in X \times \overset{\circ}{c}Y$ identified with $((x, 1-t), y) \in \overset{\circ}{c}X \times Y$.

Definition. — A *smooth stratified morphism* between stratified sets $(X, \{X_\alpha\})$ and $(Y, \{Y_\beta\})$ is a continuous map $\varphi : X \to Y$ so that the image of each stratum of X is contained in a stratum of Y and φ is a smooth map on each stratum.

Definition. — A *trivially conelike smooth stratified space* (or TCSS *space* for short) is a 4-tuple $(X, \{X_\alpha\}_{\alpha \in \mathscr{A}}, \{\gamma_\alpha\}_{\alpha \in \mathscr{A}}, \{\Sigma_\alpha\}_{\alpha \in \mathscr{A}})$ where:

1) $(X, \{X_\alpha\}_{\alpha \in \mathscr{A}})$ is a finite smooth stratified set with boundary.

2) Each Σ_α is either the empty set or (inductively) a compact TCSS space $(\Sigma_\alpha, \{\Sigma_{\alpha\beta}\}_{\beta \in \mathscr{B}_\alpha}, \{\eta_{\alpha\beta}\}, \{\Lambda_{\alpha\beta}\})$ with empty boundary.

3) Each γ_α is the germ at $X_\alpha \times *$ of a smooth stratified morphism $c_\alpha : X_\alpha \times \overset{\circ}{c}\Sigma_\alpha \to X$ so that:

a) $c_\alpha(x, *) = x$ for all $x \in X_\alpha$.

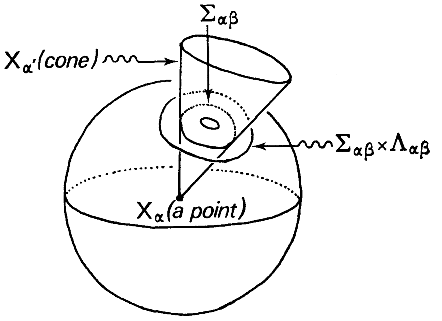
b) c_α is a smooth stratified isomorphism onto a neighborhood of X_α in X.

c) $\partial X_\alpha \times \overset{\circ}{c}\Sigma_\alpha = c_\alpha^{-1}(\partial X)$.

d) For any $\beta \in \mathscr{B}_\alpha$ let $\alpha' \in \mathscr{A}$ be such that $c_\alpha(X_\alpha \times (\Sigma_{\alpha\beta} \times (0, 1))) \subset X_{\alpha'}$. Then $\Sigma_{\alpha'} = \Lambda_{\alpha\beta}$ and the following diagram commutes.

$$\begin{tikzcd}
(X_\alpha \times (\Sigma_{\alpha\beta} \times (0, 1))) \times \mathring{c}\Sigma_{\alpha'} \arrow[d, "\gamma_\alpha \times \mathrm{id}"] & {=} & X_\alpha \times (\Sigma_{\alpha\beta} \times \mathring{c}\Lambda_{\alpha\beta}) \times (0, 1) \arrow[d, "\mathrm{id} \times \eta_{\alpha\beta} \times \mathrm{id}"] \\
X_{\alpha'} \times \mathring{c}\Sigma_{\alpha'} \arrow[dr, "\gamma_{\alpha'}"'] & & X_\alpha \times (\Sigma_\alpha \times (0, 1)) \arrow[dl, "\gamma_\alpha"] \\
& X &
\end{tikzcd}$$

That is, $c_\alpha(x, (c_{\alpha\beta}(y, z), t)) = c_{\alpha'}(c_\alpha(x, y, t)), z)$ for all $x \in X_\alpha$, $y \in \Sigma_{\alpha\beta}$, $z \in \mathring{c}\Sigma_{\alpha'}$, $t \in (0, 1)$ with z near $*$ and t near 0. Here $c_{\alpha'}$ and $c_{\alpha\beta}$ represent $\gamma_{\alpha'}$ and $\eta_{\alpha\beta}$. (See Fig 1.1.)



($\Sigma_{\alpha'}$ is the link of $X_{\alpha'}$ in $\mathbf{R}^3$ which is $\Lambda_{\alpha\beta}$)

FIG. 1.1

Notice that 3) *c)* above implies that the boundary of a TCSS space inherits the structure of a TCSS space.

Definition. — Let $(X, \{X_\alpha\}, \{\gamma_\alpha\}, \{\Sigma_\alpha\})$ be a TCSS space and let X_α be a stratum of X. Then a *neighborhood trivialization* of X_α is a map $c_\alpha : X_\alpha \times \mathring{c}\Sigma_\alpha \to X$ satisfying 3 *a)*, *b)*, *c)*, *d)* in the above definition so that γ_α is the germ at $X_\alpha \times *$ of c_α. The TCSS space Σ_α is called the *link* of X_α.

Notice that the cone and open cone on a boundaryless TCSS space set have an induced structure of a TCSS space and the join and cartesian product of two TCSS spaces have induced TCSS structures.

For instance, if X and Y are TCSS spaces and U and V are strata of X and Y and Σ and Λ are the links of U and of V, then the link of the stratum $U \times V$ in $X \times Y$ is the join $\Sigma * \Lambda$. (Note that $\mathring{c}(\Sigma * \Lambda) \approx \mathring{c}\Sigma \times \mathring{c}\Lambda$).

Lemma (1.1). — *For any* TCSS *space* X *there is a smooth stratified imbedding* $f: X \to \mathbf{R}^n$ *for some n.*

The proof is clear.

Definition. — A TCSS *isomorphism* between two TCSS spaces $(X, \{X_\alpha\}, \{\gamma_\alpha\}, \{\Sigma_\alpha\})$ and $(Y, \{Y_\beta\}, \{\delta_\beta\}, \{\Lambda_\beta\})$ is a smooth stratified isomorphism $h: X \to Y$ such that for each α if Y_β is the stratum of Y so that $h(X_\alpha) = Y_\beta$ then there is a smooth stratified isomorphism $h_\alpha : \Sigma_\alpha \to \Lambda_\beta$ so that the following diagram commutes; where $h'_\alpha : \overset{\circ}{c}\Sigma_\alpha \to \overset{\circ}{c}\Lambda_\beta$ is the map $h'_\alpha(x, t) = (h_\alpha(x), t)$.

$$\begin{tikzcd}
Y_\beta \times \overset{\circ}{c}\Lambda_\beta \arrow[r, "\delta_\beta"] & Y \\
X_\alpha \times \overset{\circ}{c}\Sigma_\alpha \arrow[r, "\gamma_\alpha"] \arrow[u, "h \times h'_\alpha"] & X \arrow[u, "h"']
\end{tikzcd}$$

In short a TCSS isomorphism is a smooth stratified isomorphism which preserves links and neighborhood trivializations.

Notice that the h_α above is automatically a TCSS isomorphism also.

Definition. — Let $(X, \{X_\alpha\}, \{\gamma_\alpha\}, \{\Sigma_\alpha\})$ be a TCSS space. Then $Y \subset X$ is a *TCSS subspace* if $\left(Y, \{Y \cap X_\alpha\}, \left\{\gamma_\alpha\Big|_{(Y\cap X_\alpha)\times c\Sigma_\alpha}\right\}, \{\Sigma_\alpha\}\right)$ is a TCSS space. In other words, $Y \cap X_\alpha$ must be a smooth submanifold of each X_α and for each α there must be a neighborhood trivialization $c_\alpha : X_\alpha \times \overset{\circ}{c}\Sigma_\alpha \to X$ so that $c_\alpha((Y \cap X_\alpha) \times \overset{\circ}{c}\Sigma_\alpha)$ is an open subset of Y.

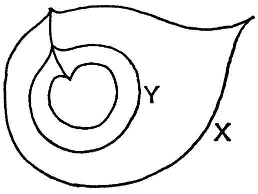


Fig. 1.2

Notice open subsets always are TCSS subspaces.

Definition. — A *TCSS imbedding* $\alpha : X \to Y$ is a TCSS isomorphism onto a TCSS subset of Y. We say α is *open* if its image is open in Y.

Lemma (1.2) *(Collaring).* — *Suppose* $(X, \{X_\alpha\}, \{\gamma_\alpha\}, \{\Sigma_\alpha\})$ *is a TCSS space with compact boundary and* $U \subset \partial X$ *is an open subset of* ∂X *and* $K \subset U$ *is compact and* $\theta : U \times [0, 1) \to X$ *is an open TCSS imbedding with* $\theta(u, 0) = u$ *for all* $u \in U$. *Then there* *is an open TCSS imbedding* $\varphi : \partial X \times [0, 1) \to X$ *so that* $\varphi\big|_{K\times[0,1)} = \theta\big|_{K\times[0,1)}$ *and* $\varphi(x, 0) = x$ *for* $x \in X$.

Such a φ is called a *collaring* of ∂X. The U, K and θ only appear in order to make the proof easier. Normally they would all be empty.

Proof. — The proof is standard. We prove by induction on the number of strata of ∂X not contained in K. If all strata are contained in K we are done since $K = \partial X$. Otherwise, pick a stratum N of least dimension among those strata not contained in K. Let L be the stratum of X with $N = \partial L$ and let $c : L \times \mathring{c}\Sigma \to X$ be a neighborhood trivialization for L. Let $K' \subset U$ be a compact neighborhood of K in ∂X. By the relative collaring theorem for smooth manifolds there is an open imbedding $\alpha : N \times [0, 1) \to L$ so that $\alpha(x, 0) = x$ for all $x \in N$ and $\alpha\big|_{(N \cap K') \times [0,1)} = \theta\big|_{(N \cap K') \times [0,1)}$. We may define U' to be the interior of K' in ∂X union a neighborhood of N in ∂X. We let $\beta : U' \times [0, 1) \to X$ be defined so that $\beta(x, t) = \theta(x, t)$ for $x \in K'$, $t \in [0, 1)$ and the following diagram commutes

$$\begin{tikzcd}
(N\times\mathring{c}\Sigma)\times[0,1) \arrow[d, "\gamma\times \mathrm{id}"] & = & (N\times[0,1))\times\mathring{c}\Sigma \arrow[d, "\alpha\times \mathrm{id}"] \\
U'\times[0,1) \arrow[dr, "\beta"] & & L\times\mathring{c}\Sigma \arrow[dl, "\gamma"] \\
& X &
\end{tikzcd}$$

i.e. $\beta(c(x, (y, s)), t) = c(\alpha(x, t), (y, s))$ for $x \in N$, $y \in \Sigma$ and s small. Well definedness of β follows since θ is a TCSS imbedding. Then $\beta\big|_{(K \cup N)\times[0,1)} = \theta\big|_{(K \cup N)\times[0,1)}$ so by induction we are done. ■

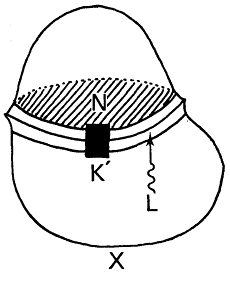


Fig. 1.3

Definition. — A TCSS subspace $X \subset Y$ is *full* if $X - \partial X$ is an open subset of Y. The basic idea is that X is " codimension 0 " in Y.

Lemma **(1.3)** *(Bicollaring). — Suppose* Y *is a TCSS space and* $X \subset Y$ *is a full TCSS subspace with* $X \cap \partial Y$ *empty and* ∂X *compact. Suppose* $U \subset \partial X$ *is open in* ∂X *and* $K \subset U$ *is compact and* $\theta : U \times (-1, 1) \to Y$ *is an open TCSS imbedding with* $\theta(u, 0) = u$ *for all* $u \in U$. *Then there is an open TCSS imbedding* $\varphi : \partial X \times (-1, 1) \to Y$ *so that* $\varphi\Big|_{K \times (-1,1)} = \theta\Big|_{K \times (-1,1)}$ *and* $\varphi(x, 0) = x$ *for all* $x \in \partial X$.

Proof. — The proof is similar to the proof of Lemma (1.2). ■

Definition. — A TCSS space X *bounds* if there is a compact TCSS space Y so that $X = \partial Y$ and X and Y have the same number of strata. (In particular X must have empty boundary and be compact.)

Definition. — An A-*space* is a TCSS space $(X, \{X_\alpha\}, \{\gamma_\alpha\}, \{\Sigma_\alpha\})$ so that each Σ_α bounds.

Definition. — An A-*subspace* Y of an A-space X is a TCSS subspace Y of X. (Note Y is automatically an A-space itself.) (See Figure 1.2.)

We define an A-*map* to be a smooth stratified morphism between A-spaces and an A-*isomorphism* to be a TCSS isomorphism between A-spaces.

Suppose X is an A-space, N is a closed stratum of X and $M \subset N$ is a union of connected components of N. Let $c : N \times c\Sigma \to X$ be a neighborhood trivialization for N and suppose W is a compact A-space with the same number of strata as Σ so that $\partial W = \Sigma$.

We define an A-space B(X, M, W) as follows. As a point set B(X, M, W) is $X - M \cup M \times W$. Let $\eta : \Sigma \times [0, 1) \to W$ be a collaring of $\partial W = \Sigma$. Define:

$$\lambda : \ M \times \Sigma \times (-1, 1) \to B(X, M, W)$$

by $\lambda(x, y, t) = (x, \eta(y, t)) \in M \times W$ if $t \geq 0$ and $\lambda(x, y, t) = c(x, (y, -t)) \in X - M$ if $t < 0$. We put the unique A-structure on B(X, M, W) so that the three maps

$$X - M \to B(X, M, W), \qquad M \times (W - \partial W) \to B(X, M, W)$$

and

$$\lambda : \ M \times \Sigma \times (-1, 1) \to B(X, M, W)$$

are all A-imbeddings. The basic idea is that we replace M by $M \times W$, a process analogous to blowing up in algebraic geometry.

We can define a collapsing map $\pi(X, M, W) : B(X, M, W) \to X$ by letting $\pi(X, M, W)\Big|_{X-M}$ be inclusion and letting $\pi(X, M, W)\Big|_{M \times W}$ be projection onto M. Note that although $\pi(X, M, W)$ is continuous it is not an A-map since the image of a stratum is not contained in a single stratum.

We call B(X, M, W) an A-*blowup* of X. Likewise an A-*blowup map* is $\pi(X, M, W)$ for some X, M, and W.

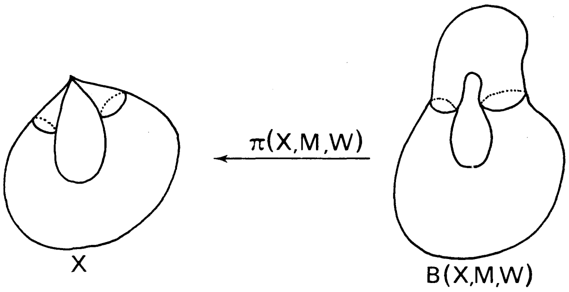


FIG. 1.4

Lemma **(1.4).** — *Let* X *be an* A-*space which bounds. Then* $X=\partial Y$ *where* Y *is a compact* A-*space so that each connected component of each stratum of* Y *has nonempty boundary. (In particular* Y *has the same number of strata as* X.*)*

Proof. — The proof will be by induction on the dimension of X. If $\dim X<0$ then X is empty so the lemma is true (with Y empty) in this case.

In the general case, suppose the lemma is not true for some X. Let Y be a compact A-space with the least number of components of strata with empty boundary such that $X=\partial Y$ and Y has the same number of strata as X. Pick a component M of a stratum N of Y so that ∂M is empty and M has least possible dimension. Then notice that M must be closed. Let $c: N\times \mathring{c}\Sigma \rightarrow Y$ be a neighborhood trivialization for N. Note $\dim N>0$, otherwise Y would have more strata than X. Hence $\dim\Sigma<\dim X$ so by induction there is a compact A-space W with $\partial W=\Sigma$ so that each connected component of each stratum of W has nonempty boundary.

Now let $Y'=B(Y, M, W)$. Then $\partial Y'=X$, Y′ has as many strata as X and Y′ has fewer connected components of strata with empty boundary than Y. Hence we have a contradiction and the lemma is proven. (See Fig. 0.7.) ■

Definition. — An A-*disc* is an A-space of the form $B^k\times cX$ where X is a compact A-space which bounds and $B^k=\{x\in\mathbf{R}^k \mid |x|\leq 1\}$. An *open* A-*disc* is the interior of an A-disc, i.e. an A-space of the form $\mathbf{R}^k\times\mathring{c}X$ where X is a compact A-space which bounds. An A-*sphere* is the boundary of an A-disc.

Definition. — Let X be an A-space and let $Y_i\subset X$ be A-subspaces $i=1, \ldots, k$. Then $Y_1, Y_2, \ldots, Y_k$ are in *general position* if for each stratum X_α of X, $Y_1\cap X_\alpha$, $Y_2\cap X_\alpha, \ldots, Y_k\cap X_\alpha$ are in general position in X_α.

II. — A-BORDISM, ALGEBRAIC BORDISM AND ALGEBRAIC SETS

Given an algebraic set X, a closed, smooth manifold Y and a map $f: \mathrm{Y} \rightarrow \mathrm{X}$ it is useful to know when for some k we may approximate $f \times 0: \mathrm{Y} \rightarrow \mathrm{X} \times \mathbf{R}^k$ by an imbedding onto a nonsingular projectively closed algebraic set. This turns out to depend only on the unoriented bordism class of f (this is an immediate corollary of Proposition (2.3)). For this reason, it is useful to study A-bordism which we define below. Also, the question of representing Z/2Z homology classes by algebraic sets turns out to be crucial.

Definition. — Let X be a topological space and let $\alpha_i: \mathrm{X}_i \rightarrow \mathrm{X}$ be continuous maps where X_i are compact A-spaces $i=0, 1$. We say α_0 and α_1 are *bordant* if there is a compact A-space Y and a continuous map $\beta: \mathrm{Y} \rightarrow \mathrm{X}$ so that $\partial \mathrm{Y}$ is the disjoint union $\mathrm{X}_0 \cup \mathrm{X}_1$ and $\beta\big|_{\mathrm{X}_i} = \alpha_i$.

This bordism relation is an equivalence relation as usual so it gives rise to a bordism theory $\mathscr{N}^{\mathrm{A}}$ where $\mathscr{N}^{\mathrm{A}}(\mathrm{X})$ is the group of bordism classes of maps from compact A-spaces into X. The group operation is disjoint union. Every element has order two.

Suppose $\mathrm{K} \subset \mathbf{R}^n$. We may define a subgroup $\hat{\mathscr{N}}^{\mathrm{A}}(\mathrm{K}) \subset \mathscr{N}^{\mathrm{A}}(\mathrm{K})$ to be the subgroup generated by maps $\mathrm{Z} \times \mathrm{Y} \rightarrow \mathrm{Y} \rightarrow \mathrm{K}$ where Y is a nonsingular projectively closed algebraic subset of $\mathrm{K} \times \mathbf{R}^k$ for some k and where Z is any compact A-space and the map $\mathrm{Z} \times \mathrm{Y} \rightarrow \mathrm{Y}$ is the projection and the map $\mathrm{Y} \rightarrow \mathrm{K}$ is induced by the projection $\mathrm{K} \times \mathbf{R}^k \rightarrow \mathrm{K}$.

Now if $\mathrm{Y} \subset \mathrm{X} \subset \mathbf{R}^n$ we may define $\mathscr{N}^{\mathrm{A}}(\mathrm{X} : \mathrm{Y})$ to be the quotient group $\mathscr{N}^{\mathrm{A}}(\mathrm{X}) / i_* \hat{\mathscr{N}}^{\mathrm{A}}(\mathrm{Y})$ where $i_*: \mathscr{N}^{\mathrm{A}}(\mathrm{Y}) \rightarrow \mathscr{N}^{\mathrm{A}}(\mathrm{X})$ is induced by the inclusion.

We may define $\mathscr{N}_i^{\mathrm{A}}(\mathrm{X})$ and $\mathscr{N}_i^{\mathrm{A}}(\mathrm{X} : \mathrm{Y})$ to be the subgroups of $\mathscr{N}^{\mathrm{A}}(\mathrm{X})$ and $\mathscr{N}^{\mathrm{A}}(\mathrm{X} : \mathrm{Y})$ generated by maps $\alpha: \mathrm{Z} \rightarrow \mathrm{X}$ where $\dim \mathrm{Z} = i$.

Definition. — Let $\mathrm{U} \subset \mathbf{R}^n$. Then $\mathrm{H}_i^{\mathrm{A}}(\mathrm{U})$ will be the subgroup of the singular homology group $\mathrm{H}_i(\mathrm{U}, \mathrm{Z}/2\mathrm{Z})$ generated by homology classes of the form $\pi_*([\mathrm{V}])$ where $\mathrm{V} \subset \mathrm{U} \times \mathbf{R}^k$ is an i dimensional nonsingular projectively closed algebraic subset of $\mathbf{R}^n \times \mathbf{R}^k$ for some k; $\pi: \mathrm{V} \rightarrow \mathrm{U}$ is induced by projection $\mathrm{U} \times \mathbf{R}^k \rightarrow \mathrm{U}$ and $[\mathrm{V}] \subset \mathrm{H}_i(\mathrm{V}, \mathrm{Z}/2\mathrm{Z})$ is the fundamental class of V. Equivalently, $\mathrm{H}_i^{\mathrm{A}}(\mathrm{U})$ is generated by $f_*([\mathrm{V}])$ where V is a compact algebraic set and $f: \mathrm{V} \rightarrow \mathrm{U}$ is an entire rational function (see [1]).

Definition. — Suppose U is an algebraic set and $\mathrm{H}_i^{\mathrm{A}}(\mathrm{U}) = \mathrm{H}_i(\mathrm{U}, \mathrm{Z}/2\mathrm{Z})$ for all $i \leq n$. Then we say U has *algebraic homology up to n*. If U has algebraic homology up to n for all n, then we say U has *totally algebraic homology*.

Lemma (**2.1**). — *Let* $V\subset \mathbf{R}^n$ *be a set and* $U\subset V$ *a subset and suppose that:*

$$j_*: H_i^A(U) \to H_i(V, Z/2Z)$$

is onto for all $i\leq n$ *where* $j: U\to V$ *is inclusion. Then* $\mathcal{N}_n^A(V:U)$ *is the trivial group.*

Proof. — For any $\gamma\in\mathcal{N}_n^A(V:U)$ let $\alpha: X\to V$ be a map representing γ so that $\dim X = n$. We will prove this Lemma by induction on the number of strata of X.

Pick a closed stratum N of X. Let $c: N\times \mathring{c}\Sigma\to X$ be a neighborhood trivialization and let Z be a compact A-space so that $\partial Z = \Sigma$. We know by [4] that generators of $\mathcal{N}_*(V)$ = smooth unoriented bordism of V are of the form $P_i\times M_j\to P_i\to V$ where the M_j are generators for smooth unoriented bordism of a point and the P_i generate $H_*(V, Z/2Z)$. Hence there is a smooth manifold W and a map $\beta: W\to V$ so that ∂W is the disjoint union $N\cup\bigcup_{i=1}^{k} P_i\times U_i$ and $\beta|_N = \alpha|_N$ and $\beta|_{P_i\times U_i}: P_i\times U_i\to V$ is projection. Here each U_i is a smooth closed manifold and each P_i is a projectively closed nonsingular algebraic set contained in $U\times\mathbf{R}^{k_i}$ given by the hypothesis.

Let $Z' = c\Sigma\cup Z$ with $\Sigma\times 1\subset c\Sigma$ identified with $\partial Z = \Sigma$. Let:

$$X' = X\times[0, 1]\cup W\times c\Sigma\cup\bigcup_{i=1}^{k} P_i\times U_i\times Z'\times[0, 1]$$

with $(c(x, (y, t)), 1)\in X\times[0, 1]$ identified with $(x, (y, 2t))\in W\times c\Sigma$ for $x\in N$, $y\in\Sigma$, $t\in\left[0, \frac{1}{2}\right]$ and with $P_i\times U_i\times c\Sigma\subset W\times c\Sigma$ identified with:

$$P_i\times U_i\times c\Sigma\times 1\subset P_i\times U_i\times Z'\times[0, 1]. \quad \text{(See Figure 2.1.)}$$

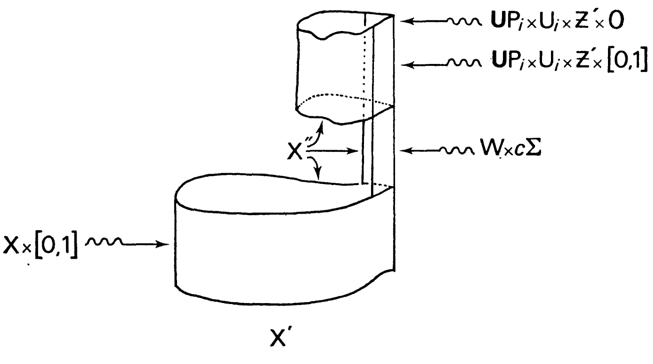


FIG. 2.1

Now define $\alpha': X'\to V$ as follows: On $X\times[0, 1]$:

$$\alpha'(x, t) = \alpha(x) \quad\text{for}\quad (x, t)\in(X - c(N\times\mathring{c}\Sigma))\times[0, 1],$$

$$\alpha'(c(x, (y, s)), t) = \alpha c(x, (y, s+t(s-1))) \quad\text{if}\quad t(1-s)\leq s$$

$$\alpha'(c(x, (y, s)), t) = \alpha(x) \quad\text{if}\quad t(1-s)\geq s.$$

On $W \times c\Sigma$ $\alpha'(w, (y, t)) = \beta(w)$. On $P_i \times U_i \times Z' \times [0, 1]$, $\alpha'(p, u, y, t) = \pi_i(p)$ where $\pi_i : P_i \to V$ is induced by projection. Notice that $\alpha' | P_i \times U_i \times Z' \times 0$ represents an element of $\hat{\mathcal{N}}^A(U)$ and $\alpha' \Big|_{X \times 0}$ represents γ, hence $\alpha' \Big|_{X''}$ represents γ where:

$$X'' = (X \times 1 - N \times c\Sigma) \cup W \times (\Sigma \times 1) \cup \bigcup_{i=1}^{k} P_i \times U_i \times Z \times 1.$$

But X'' has less strata than X or it is empty. So by induction, $\mathcal{N}_n^A(V; U) = 0$. ■

Lemma (**2.2**). — *Let* X *be a compact* A-*space, let* $M \subset \mathbf{R}^n$ *be a smooth manifold, let* $K \subset M$ *and suppose* $\alpha : X \to M$ *represents* 0 *in* $\mathcal{N}^A(M : K)$. *Then*

a) *If* $X' \subset X$ *is a closed union of strata then* $\alpha \Big|_{X'} : X' \to M$ *represents* 0 *in* $\mathcal{N}^A(M : K)$.

b) *If* $\pi : X' \to X$ *is an* A-*blow up map then* $\alpha \circ \pi : X' \to M$ *represents* 0 *in* $\mathcal{N}^A(M : K)$.

c) *If* $Y \subset \mathbf{R}^m$ *is a projectively closed nonsingular algebraic set then*:

$$\text{identity} \times \alpha : \; Y \times X \to Y \times M$$

represents 0 *in* $\mathcal{N}^A(Y \times M : Y \times K)$.

d) *If* $L \subset M - (K \cup \alpha(X))$ *is a sub-manifold and* $\dim X + \dim L + 1 < \dim M$, *then* $\alpha : X \to M - L$ *represents* 0 *in* $\mathcal{N}^A(M - L : K)$.

e) *If* M' *is a smooth manifold and* $K' \subset M'$, $M \subset M'$, $K \subset K'$ *then* $\alpha : X \to M'$ *represents* 0 *in* $\mathcal{N}^A(M' : K')$.

f) *If* $\gamma : X \to \mathbf{R}^m$ *is any map and* $p \in \mathbf{R}^m$ *then* $\alpha \times \gamma : X \to M \times \mathbf{R}^m$ *represents* 0 *in* $\mathcal{N}^A(M \times \mathbf{R}^m : K \times p)$.

g) *There is an integer* u, *a compact* A-*space* Z *with the same number of strata as* X *and a map* $\beta : Z \to M$ *so that* ∂Z *is the disjoint union* $X \cup \bigcup_{i=1}^{k} P_i \times U_i$ *where* $P_i \subset K \times \mathbf{R}^u$ *are projectively closed nonsingular real algebraic sets,* U_i *are compact* A-*spaces and* $\beta \Big|_X = \alpha$ *and* $\beta : P_i \times U_i \to K$ *is the composition of projections and an inclusion* $P_i \times U_i \to P_i \to K \times \mathbf{R}^u \to K$.

Proof. — We first prove g). Let $\beta : Z \to M$ be a map as in *g* above except that Z might have more strata than X, but pick $\beta : Z \to M$ so that Z has the least number of strata possible. If Z has more strata than X then we may pick a closed stratum N of Z which is disjoint from X. Let $\pi : Z' \to Z$ be an A-blowup of Z along N. Note $\partial N = \bigcup_{i=1}^{k} P_i \times N_i$ for some closed stratum N_i of each U_i. Then consider $\beta \circ \pi : Z' \to M$. Z' has less strata than Z and $\partial Z'$ is the disjoint union $X \cup \bigcup_{i=1}^{k} P_i \times U_i'$ where U_i' is an A-blowup of U_i along N_i. Also $\beta \circ \pi \Big|_X = \alpha$ and $\beta \circ \pi \Big|_{P_i \times U_i'}$ is the composition

$$P_i \times U_i' \to P_i \to K \times \mathbf{R}^u \to K$$

so we have a contradiction.

So Z and X have the same number of stratum and g) is proven. (See Figure (2.2.)

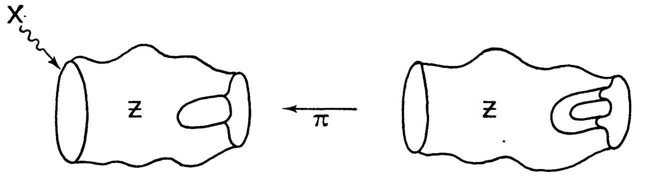


Fig. 2.2

For the remainder of this proof we take a $\beta : Z \to M$ as in g).

To prove a), let Z′ be the union of the strata of Z which intersect X′. Then $\beta\big|_{Z'} : Z' \to M$ is a bordism from X′ to maps representing elements of $\hat{\mathcal{N}}^A(K)$.

To prove b), suppose $X' = B(X, N, W)$ (see Section 1) then let L be the stratum of Z containing N, note that L is closed, otherwise Z would have more strata than X. Then $\beta \circ \pi(Z, L, W) : B(Z, L, W) \to M$ is a bordism from $\alpha \circ \pi(X, N, W) = \alpha \circ \pi : X' \to M$ to some maps representing elements of $\hat{\mathcal{N}}^A(K)$.

To prove c), note that $\mathrm{id} \times \beta : Y \times Z \to Y \times M$ is a bordism from $\mathrm{id} \times \alpha$ to some maps representing elements of $\hat{\mathcal{N}}^A(Y \times K)$.

To prove d), by general position we may assume that $\beta(Z) \cap L$ is empty so the result is proven.

To prove f), let $\delta : Z \to [0, 1]$ be a smooth function which is 1 on X and 0 on $\partial Z - X$. Define $\beta' : Z \to M \times \mathbf{R}^m$ by $\beta'(z) = (\beta(z), p + \delta(z)(\gamma(z) - p))$. Then $\beta'\big|_X = \alpha \times \gamma$ and $\beta'\big|_{\partial Z - X}$ represents an element of $\hat{\mathcal{N}}^A(K \times p)$ so we are done.

The proof of e) is a triviality, notice the obvious homomorphism

$$\mathcal{N}^A(M : K) \to \mathcal{N}^A(M' : K'). \quad ■$$

Proposition **(2.3)**. — *Suppose* W *is a nonsingular real algebraic set and* $M \subset W$ *is a smooth compact boundaryless submanifold so that the inclusion* $M \hookrightarrow W$ *represents* 0 *in* $\mathcal{N}^A(W : W)$. *Then for some* k, M *is isotopic in* $W \times \mathbf{R}^k$ *to a nonsingular projectively closed algebraic set* $V \subset W \times \mathbf{R}^k$. *We can make this isotopy as* C^∞ *small as we wish.*

Proof. — We know by Lemma (2.2) g) that there is a smooth manifold N and a map $\alpha : N \to W$ so that ∂N is the disjoint union $M \cup \bigcup_{i=1}^{b} P_i \times U_i$ and $\alpha\big|_M =$ inclusion and $\alpha\big|_{P_i \times U_i} : P_i \times U_i \to P_i \hookrightarrow W \times \mathbf{R}^u \to W$ is the projection where P_i are nonsingular projectively closed algebraic subsets of $W \times \mathbf{R}^u$ and U_i are closed smooth manifolds. By [1], Proposition (2.8), we may assume the U_i are nonsingular projectively closed algebraic sets $U_i \subset \mathbf{R}^n$ for some n. By translating we may assume the P_i's are pairwise disjoint. So if n is large enough we have a smooth imbedding $\beta : N \times [-1, 1] \to (W \times \mathbf{R}^u) \times \mathbf{R}^n \times \mathbf{R}$

so that $\beta(x, t)=((x, 0), 0, t)$ for $x \in M$, t near 0 and $\beta((p, u), t)=(p, u, t)$ for $p \in P_i$, $u \in U_i$ and t near 0. The proof now proceeds as in the proof of Proposition (2.8) of [1]. More specifically, by Proposition (2.8) of [1] we may isotop $\beta \times 0 \big|_{\partial(N \times [-1, 1])}$ fixing $\bigcup P_i \times U_i \times 0$ to an imbedding $\beta' : \partial(N \times [-1, 1]) \to W \times \mathbf{R}^u \times \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^m$ onto a projectively closed nonsingular algebraic set X. By Lemmas (2.2) and (1.6) of [1] there is a nonsingular projectively closed algebraic subset V of X isotopic to $\beta'(M \times 0)$. ■

III. — BLOWING DOWN

In this section we describe a procedure for " algebraically blowing down " an algebraic set. The map $\mathscr{D}_q$ is a quotient map which collapses certain algebraic subsets.

Suppose $W \subset \mathbf{R}^m$ is a real algebraic set and $q : \mathbf{R}^m \to \mathbf{R}$ is a polynomial of degree d. We define for any n a function $\mathscr{D}_q : W \times \mathbf{R}^n \to W \times \mathbf{R}^n$ by

$$\mathscr{D}_q(x,y) = (x, y \,.\, |y|^{-2d/(2d+1)}(q(x))^{2/(2d+1)}).$$

Notice that $\mathscr{D}_q$ is a homeomorphism on $(W - q^{-1}(0)) \times \mathbf{R}^n$ and a diffeomorphism on $(W - q^{-1}(0)) \times (\mathbf{R}^n - 0)$. Also, if $X \subset W \times \mathbf{R}^n$ then $\mathscr{D}_q(X)$ is homeomorphic to the quotient space of X by the equivalence relation $(x,y) \sim (x^1, y^1)$ if $x = x^1 \in q^{-1}(0)$. The usefulness of this $\mathscr{D}_q$ is indicated in the following Proposition.

Proposition **(3.1)**. — *If* $V \subset W \times \mathbf{R}^n$ *is a projectively closed algebraic set and* $q : W \to \mathbf{R}$ *is an overt polynomial, then* $\mathscr{D}_q(V) \cup (q^{-1}(0) \times 0)$ *is a projectively closed algebraic set.*

Proof. — Let $W \subset \mathbf{R}^m$ and let $p : \mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}$ be an overt polynomial of degree e with $V = p^{-1}(0)$. Let d be the degree of q. Define a polynomial $r : \mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}$ by

$$r(x,y) = (q(x))^{2e} p(x, y|y|^{2d}/q^2(x))$$

for $x \in \mathbf{R}^m$ and $y \in \mathbf{R}^n$. (This r is a polynomial after clearing denominators.) Note that:

$$(x,y) \to (x, y|y|^{2d}/q^2(x))$$

is the inverse of $\mathscr{D}_q$ on $(W - q^{-1}(0)) \times \mathbf{R}^n$. Clearly $r(\mathscr{D}_q(V)) = 0$ and $r(q^{-1}(0) \times 0) = 0$. In fact:

$$r^{-1}(0) = \mathscr{D}_q(V) \cup q^{-1}(0) \times 0$$

for suppose $r(x,y) = 0$. If $q(x) \neq 0$, then $(x, y|y|^{2d}/q^2(x)) \in V$, so $(x,y) \in \mathscr{D}_q(V)$. But if $q(x) = 0$ then $r(x,y) = p_e(0, y|y|^{2d})$ where p_e is the homogeneous part of p of degree e. Hence $y = 0$, so $r^{-1}(0) = \mathscr{D}_q(V) \cup q^{-1}(0) \times 0$.

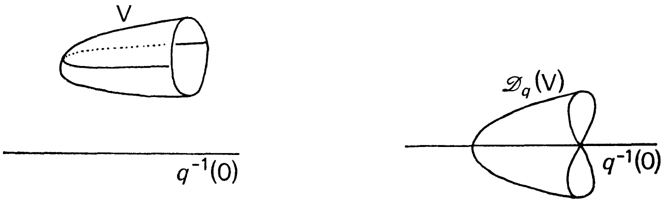


FIG. 3.1

To see that $\mathscr{D}_q(\mathrm{V})$ is projectively closed, we show that r is overt. Notice that the highest degree terms of r are $r'(x,y)=q_d^{2e}(x)p_e(x,y|y|^{2d}/q_d^2(x))$ where q_d is the highest degree terms of q. If $r'(x,y)=0$ and $q_d(x)\neq 0$ then $p_e(x,y|y|^{2d}/q_d^2(x))=0$ so $x=0$ and $y=0$ by overtness of p_e. But if $r'(x,y)=0$ and $q_d(x)=0$ then $r'(x,y)=p_e(0,y|y|^{2d})$ so $y=0$ by overtness of p and $x=0$ by overtness of q. So $r'(x,y)=0$ if and only if $x=0$ and $y=0$ so r is overt. ■

For example, if $\mathrm{W}=\mathbf{R}^2$, $q(x,y)=x^2+y^2$, $n=1$, then:

$$\mathscr{D}_q(x,y,z)=(x,y,z^{\frac{1}{5}}.(x^2+y^2)^{\frac{2}{5}}).$$

If we let $\mathrm{X}=\{(x,y,z)\in\mathbf{R}^3 \mid x^2+y^2+z^2=4\}$ then:

$$\mathscr{D}_q(\mathrm{X})=\{(x,y,z)\in\mathbf{R}^3 \mid (x^2+y^2)^5+z^{10}=4(x^2+y^2)^4\}.$$

For another cxample, suppose $q^1(x,y)=x^2+y^2-1$. Then:

$$\mathscr{D}_{q^1}(x,y,z)=(x,y,z^{\frac{1}{5}}(x^2+y^2-1)^{\frac{2}{5}})$$

and $$\mathscr{D}_{q^1}(\mathrm{X})=\{(x,y,z)\in\mathbf{R}^3 \mid (x^2+y^2)(x^2+y^2-1)^4+z^{10}=4(x^2+y^2-1)^4\}.$$

IV. — BLOWING UP

In this section we describe algebraic blowing up. We start out with the usual blowing up of a nonsingular algebraic set along a nonsingular algebraic subset. It will be useful to combine this process with the process of crossing with Euclidean space which leads to the multiblowup $B_k(V, N)$. Even this multiblowup is not good enough so we must do a sequence of multiblowups and crossing with Euclidean spaces. Such a super-multiblowup is determined by resolution data defined below.

We need these processes to set the stage for the algebraic blowing down of Section 3.

The whole point of this process is that it is an algebraic process which has the local description given by Proposition (4.6) and this local description is analogous to a description which can be given to A-blowups. (The links of an A-space can bound a compact A-space with a spine of codimension one transversely intersecting A-subspaces.) This connection will be made less tenuous in Sections 5, 6 and 7.

Definition. — Let V_i be nonsingular algebraic sets and let $U_i \subset V_i$ be algebraic subsets $i=0, 1$. A map $f: V_0 - U_0 \rightarrow V_1 - U_1$ is called a *birational diffeomorphism* if f is a diffeomorphism and both f and f^{-1} are rational functions. That is, if $V_i \subset \mathbf{R}^{n_i}$ then there are polynomials $p_0: V_0 \rightarrow \mathbf{R}^{n_1}$, $p_1: V_1 \rightarrow \mathbf{R}^{n_0}$ and $q_i: V_i \rightarrow \mathbf{R}$ so that $q_i^{-1}(0) \subset U_i$, $f(x) = p_0(x)/q_0(x)$ for all $x \in V_0 - U_0$ and $f^{-1}(x) = p_1(x)/q_1(x)$ for all $x \in V_1 - U_1$.

It is easy to see that if $f: V_0 - U_0 \rightarrow V_1 - U_1$ is a birational diffeomorphism and $W_i \subset V_i$ are algebraic sets then both $f(W_0) \cup U_1$ and $f^{-1}(W_1) \cup U_0$ are algebraic sets (see Lemma (1.3) of [1]).

Consider the map $\nu: \mathbf{RP}^{k-1} \rightarrow \mathbf{R}^{k^2} = k \times k$ real matrices defined by:

$$\nu_{ij}([x_1 : \ldots : x_k]) = x_i x_j / (\sum_{n=1}^{k} x_i^2) \qquad i=1, \ldots, k, \ j=1, \ldots, k$$

where ν_{ij} is the i, j-th coordinate of ν.

The map ν imbeds $\mathbf{RP}^{k-1}$ onto the nonsingular projectively closed algebraic set

$$\{L \in \mathbf{R}^{k^2} \mid L \text{ is symmetric, } L^2 = L \text{ and } \operatorname{Trace} L = 1\}$$

where L is thought of as a $k \times k$ matrix. In fact $X \subset \mathbf{RP}^{k-1}$ is a projective algebraic set if and only if $\nu(X)$ is an algebraic set.

Let U and V be nonsingular algebraic sets with $U \subset V$. Then we may blow up V along U to get a nonsingular algebraic set $\mathscr{B}(V, U)$. This is a standard procedure in algebraic geometry.

We indicate how it is done. One takes a polynomial map $p:(V, U) \to (\mathbf{R}^k, 0)$ for some k so that $U = p^{-1}(0)$ and in fact the coordinates of p generate the ideal of polynomials vanishing on U. Then $\mathscr{B}(V, U)$ is the closure in $V \times \mathbf{R}^{k^2}$ of

$$\{(x, \nu\theta p(x)) \in V \times \mathbf{R}^{k^2} \mid x \in V - U\}$$

where $\theta : \mathbf{R}^k - 0 \to \mathbf{RP}^{k-1}$ is the quotient map $\theta(x_1, \dots, x_k) = [x_1 : x_2 : \dots : x_k]$ and ν is as above. The projection from $V \times \mathbf{R}^{k^2}$ to V gives us a map

$$\pi(V, U) : \mathscr{B}(V, U) \to V.$$

We denote $\mathscr{E}(V, U) = \pi(V, U)^{-1}(U)$. This blowup has certain nice properties:

$$\pi(V, U)\Big|_{\pi(V, U)^{-1}(V-U)}$$

is a birational diffeomorphism onto $V - U$, $\mathscr{E}(V, U)$ has codimension 1 in $\mathscr{B}(V, U)$ and $\pi(V, U)\Big|_{\mathscr{E}(V, U)} : \mathscr{E}(V, U) \to U$ is the projectivized normal bundle of U in V. The actual set $\mathscr{B}(V, U)$ is not canonically defined, it depends on a choice of p, but any two choices will give blowups which are birationally diffeomorphic via a diffeomorphism which commutes with projections to V. This diffeomorphism and its inverse are entire rational functions so they take algebraic sets to algebraic sets.

For instance, let us find $\mathscr{B}(\mathbf{R}^n \times \mathbf{R}^k, \mathbf{R}^n \times 0)$. Let $y_1, \dots, y_k$ be coordinates for $\mathbf{R}^k$. Then:

$$\mathbf{R}^n \times 0 = \bigcap_{i=1}^{k} y_i^{-1}(0).$$

Hence $\mathscr{B}(\mathbf{R}^n \times \mathbf{R}^k, \mathbf{R}^n \times 0)$ is the closure in $\mathbf{R}^n \times \mathbf{R}^k \times \mathbf{R}^{k^2}$ of:

$$\{(x, y, \nu([y_1 : \dots : y_k]) \mid x \in \mathbf{R}^n, y \in \mathbf{R}^k - 0\}$$

which is:

$$\{(x, y, z) \in \mathbf{R}^n \times \mathbf{R}^k \times \mathbf{R}^{k^2} \mid |y|^2 z_{ij} = y_i y_j, \ \text{trace}\ z = 1, \ z^2 = z, \ z \text{ symmetric}\}$$

where z is thought of as a $k \times k$ matrix. We have a diffeomorphism

$$\gamma : \mathbf{R}^n \times (\mathbf{RP}^k - \text{point}) \to \mathscr{B}(\mathbf{R}^n \times \mathbf{R}^k, \mathbf{R}^n \times 0)$$

given by:

$$\gamma(x, [t : y_1 : y_2 : \dots : y_k]) = (x, ty/|y|^2, (y_i y_j/|y|^2))$$

for all $x \in \mathbf{R}^n$ and $[t : y_1 : \dots : y_k] \in \mathbf{RP}^k - [1 : 0 : 0 : \dots : 0]$ where $y = (y_1, \dots, y_k)$.

We may describe $\mathscr{B}(V, U)$ (up to diffeomorphism) as follows. Let $p : P \to U$ be the projectivization of the normal bundle of U in V (so for each $x \in U$, $p^{-1}(x)$ is the projective space of lines in $q^{-1}(x)$ where $q : Q \to U$ is the normal bundle of U in V). We have the canonical line bundle $r : L \to P$ (where $r^{-1}(y) =$ the set of points in the line y if $y \in P$ is a line in Q). We include P in L and include U in Q as the zero section. Notice there is a natural diffeomorphism $\lambda : L - P \to Q - U$. Then $\mathscr{B}(V, U)$ as a point set is $(V-U)\cup P$ and we put the smooth structure on $\mathscr{B}(V, U)$ so that inclusion $V-U\to\mathscr{B}(V, U)$ is an open smooth imbedding and if $\eta: Q\to V$ is a tubular neighborhood of U then $\theta: L\to\mathscr{B}(V, U)$ is an open smooth imbedding where $\theta|_P$ is the identity and $\theta(x)=\eta\lambda(x)$ for $x\in L-P$.

For many purposes we are only interested in $\mathscr{B}(V, U)$ up to diffeomorphism. The above description then gives a definition of $\mathscr{B}(V, U)$ where V is a smooth boundaryless manifold and U is a proper submanifold of V. It should be understood, however, that in case V and U are nonsingular algebraic sets, $\mathscr{B}(V, U)$ will denote an algebraic subset of some $V\times\mathbf{R}^{k^2}$ as described above.

Suppose now that U, V and W are smooth boundaryless manifolds, $W\subset V$, U is a proper submanifold of V and U and W intersect cleanly. Then we have a natural inclusion

$$\mathscr{B}(W, W\cap U)\hookrightarrow\mathscr{B}(V, U)$$

so that
$$\pi(W, W\cap U)=\pi(V, U)\Big|_{\mathscr{B}(W, W\cap U)}.$$

In addition, if U, V and W are nonsingular algebraic sets then $\mathscr{B}(W, W\cap U)$ is a nonsingular algebraic subset of $\mathscr{B}(V, U)$.

If X is a smooth boundaryless manifold we have a natural diffeomorphism

$$X\times\mathscr{B}(V, U)\approx\mathscr{B}(X\times V, X\times U),$$

so that $\mathrm{id}\times\pi(V, U)=\pi(X\times V, X\times U)$. If U, V and X are nonsingular algebraic sets then this natural diffeomorphism and its inverse are entire rational functions.

The following Lemma gives the well known local description of blowing up.

Lemma (4.1). — *Suppose* V *is a smooth manifold,* $U\subset V$ *is a proper smooth submanifold and* $\alpha:\mathbf{R}^n\times\mathbf{R}\times\mathbf{R}^k\to V$ *is an open imbedding so that* $\alpha^{-1}(U)=\mathbf{R}^n\times o\times o$. *Then:*

$$\pi^{-1}(V, U)(\alpha(o))\cap\mathscr{B}(\alpha(\mathbf{R}^n\times\mathbf{R}\times o), \alpha(\mathbf{R}^n\times o\times o))$$

is a single point q *and there is an open imbedding*

$$\beta:(\mathbf{R}^n\times\mathbf{R}\times\mathbf{R}^k, o)\to(\mathscr{B}(V, U), q)$$

so that
$$\pi(V, U)\circ\beta(x, t, y)=\alpha(x, t, ty)$$

and
$$\beta^{-1}(\mathscr{B}(\alpha(\mathbf{R}^n\times\mathbf{R}\times o), \alpha(\mathbf{R}^n\times o\times o)))=\mathbf{R}^n\times\mathbf{R}\times o.$$

Proof. — Let:

$$\pi=\pi(V, U)\qquad \pi'=\pi(\mathbf{R}^n\times\mathbf{R}\times\mathbf{R}^k, \mathbf{R}^n\times o\times o)$$

$$B'=\mathscr{B}(\mathbf{R}^n\times\mathbf{R}\times\mathbf{R}^k, \mathbf{R}^n\times o\times o)$$

and
$$B''=\mathscr{B}(\mathbf{R}^n\times\mathbf{R}\times o, \mathbf{R}^n\times o\times o)\subset B'.$$

Then α induces a diffeomorphism $\psi : \mathrm{B}' \to \pi^{-1}$ (Image α) so that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{B}' \arrow[r, "\psi"] \arrow[d, "\pi'"] & \pi^{-1}(\mathrm{Im}\,\alpha) \arrow[d, "\pi"] \\
\mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^k \arrow[r, "\alpha"] & \mathrm{Im}\,\alpha
\end{tikzcd}$$

and so that:

$$\mathrm{B}'' = \psi^{-1}(\mathscr{B}(\alpha(\mathbf{R}^n \times \mathbf{R} \times 0),\ \alpha(\mathbf{R}^n \times 0 \times 0))).$$

We have a diffeomorphism

$$\gamma : \ \mathbf{R}^n \times (\mathbf{RP}^{1+k} - [1:0:0]) \to \mathrm{B}'$$

so that $\quad \pi'\gamma(x, [s:t:y]) = (x, st/(t^2+|y|^2), sy/(t^2+|y|^2))$

for all $x \in \mathbf{R}^n$, $s \in \mathbf{R}$, $t \in \mathbf{R}$ and $y \in \mathbf{R}^k$ and so that:

$$\gamma^{-1}(\mathrm{B}'') = \{(x, [s:t:0]) \subset \mathbf{R}^n \times (\mathbf{RP}^{1+k} - [1:0:0])\}.$$

Thus $\quad \gamma^{-1}\psi^{-1}(\pi^{-1}\alpha(0) \cap \mathscr{B}(\alpha(\mathbf{R}^n \times \mathbf{R} \times 0), \alpha(\mathbf{R}^n \times 0 \times 0)))$

$$= \gamma^{-1}\pi'^{-1}(0) \cap \gamma^{-1}\mathrm{B}'' = (0, [0:1:0]),$$

a point.

Define $\theta : \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^k \to \mathbf{R}^n \times (\mathbf{RP}^{1+k} - [1:0:0])$ to be the open imbedding $\theta(x, t, y) = (x, [t(1+|y|^2) : 1 : y])$. Then $\pi'\gamma\theta(x, t, y) = (x, t, ty)$ so we may let $\beta = \psi\gamma\theta$ and we are done. ■

Suppose V and N are smooth manifolds, $\mathrm{N} \subset \mathrm{V}$, N is a proper submanifold of V and $k \geq 0$ is an integer. We inductively define manifolds $\mathrm{B}_k(\mathrm{V}, \mathrm{N})$ and $\mathrm{N}_k(\mathrm{V}, \mathrm{N})$ by setting:

$$\mathrm{B}_0(\mathrm{V}, \mathrm{N}) = \mathrm{V}, \qquad \mathrm{N}_0(\mathrm{V}, \mathrm{N}) = \mathrm{N},$$

$$\mathrm{B}_{k+1}(\mathrm{V}, \mathrm{N}) = \mathscr{B}(\mathrm{B}_k(\mathrm{V}, \mathrm{N}) \times \mathbf{R},\ \mathrm{N}_k(\mathrm{V}, \mathrm{N}) \times 0)$$

$$\mathrm{N}_{k+1}(\mathrm{V}, \mathrm{N}) = \mathscr{E}(\mathrm{N}_k(\mathrm{V}, \mathrm{N}) \times \mathbf{R},\ \mathrm{N}_k(\mathrm{V}, \mathrm{N}) \times 0)$$

$$\subset \mathscr{E}(\mathrm{B}_k \times \mathbf{R},\ \mathrm{N}_k \times 0) \subset \mathscr{B}(\mathrm{B}_k \times \mathbf{R},\ \mathrm{N}_k \times 0) = \mathrm{B}_{k+1}(\mathrm{V}, \mathrm{N}).$$

For instance, if $\mathrm{V} = \mathbf{R}$, $\mathrm{N} = 0$, $k = 1$ then:

$$\mathrm{B}_1(\mathbf{R}, 0) = \mathscr{B}(\mathbf{R}^2, 0) = \{((x, y), \nu([s:t])) \in \mathbf{R}^2 \times \nu(\mathbf{RP}^1) \mid sy = tx\}$$

$$= \{((x, y), (z_{11}, z_{12}, z_{21}, z_{22})) \in \mathbf{R}^2 \times \mathbf{R}^4 \mid z_{12} = z_{21},\ z_{11} + z_{22} = 1,$$

$$z_{11} = z_{11}^2 + z_{12}^2,\ \ z_{22} = z_{22}^2 + z_{12}^2 \text{ and } z_{11}y^2 + z_{22}x^2 = 2z_{12}xy\}$$

$$\mathrm{N}_1(\mathbf{R}, 0) = \mathscr{E}(0 \times \mathbf{R}, 0) = ((0, 0), \nu([0:1])) = ((0, 0), (0, 0, 0, 1)) \in \mathbf{R}^2 \times \mathbf{R}^4.$$

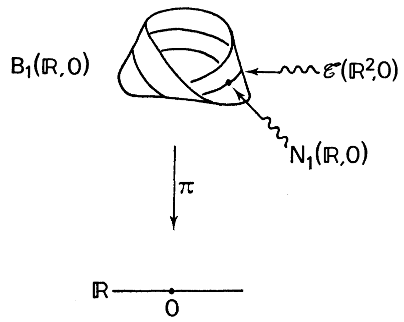


Fig. 4.1

There is also a projection

$$\pi_k(V, N) : B_k(V, N) \to V$$

defined by the composition

$$\mathcal{B}(B_{k-1}(V, N) \times \mathbf{R}, N_{k-1} \times o) \to B_{k-1}(V, N) \times \mathbf{R} \to B_{k-1}(V, N) \to V$$

where the first map is the usual projection, the second is projection and the third is π_{k-1}, (π_0 is the identity). Notice that if V and N are algebraic sets then $B_k(V, N)$ is an algebraic subset of $V \times (\mathbf{R}^n - o)$ for some n and $\pi_k(V, N) : B_k(V, N) \to V$ is induced by projection $V \times \mathbf{R}^n \to V$. Also notice that π_k restricted to N_k is a diffeomorphism onto N. This is because we have a diffeomorphism

$$N_{k-1}(V, N) \times \mathcal{E}(\mathbf{R}, o) = \mathcal{E}(N_{k-1}(V, N) \times \mathbf{R}, N_{k-1}(V, N) \times o) = N_k(V, N)$$

and $\mathcal{E}(\mathbf{R}, o)$ is a point.

We may define a diffeomorphism

$$\lambda_k(V, N) : (V-N) \times \mathbf{R}^k \to B_k(V, N) - \pi_k(V, N)^{-1}(N)$$

as follows. Let $\lambda_0(V, N)$ be inclusion

$$(V-N) \times \mathbf{R}^0 = V-N \hookrightarrow V = B_0(V, N).$$

Let $\lambda_1(V, N)$ be the inclusion

$$(V-N) \times \mathbf{R} \hookrightarrow V \times \mathbf{R} - N \times o = \mathcal{B}(V \times \mathbf{R}, N \times o) - \mathcal{E}(V \times \mathbf{R}, N \times o)$$
$$\to \mathcal{B}(V \times \mathbf{R}, N \times o) = B_1(V, N).$$

We may then inductively define:

$$\lambda_k(V, N)(x, (y_1, \ldots, y_k))$$
$$= \lambda_{k-1}(B_1(V, N), N_1(V, N))(\lambda_1(V, N)(x, y_1), (y_2, y_3, \ldots, y_k))$$

for $x \in V-N$ and $(y_1, \ldots, y_k) \in \mathbf{R}^k$.

Notice that if V and N are algebraic sets then λ is a birational diffeomorphism.

If K is a smooth boundaryless manifold, we have a natural diffeomorphism from

$K \times B_k(V, N)$ to $B_k(K \times V, K \times N)$ which takes $K \times N_k(V, N)$ to $N_k(K \times V, K \times N)$. This natural diffeomorphism is defined inductively to be the composition

$$\begin{aligned} K \times B_k(V, N) &= K \times B_1(B_{k-1}(V, N), N_{k-1}(V, N)) \\ &= K \times \mathcal{B}(B_{k-1}(V, N) \times \mathbf{R}, N_{k-1}(V, N) \times o) \\ &\approx \mathcal{B}(K \times B_{k-1}(V, N) \times \mathbf{R}, K \times N_{k-1}(V, N) \times o) \\ &\approx \mathcal{B}(B_{k-1}(K \times V, K \times N) \times \mathbf{R}, N_{k-1}(K \times V, K \times N) \times o) \\ &\approx B_k(K \times V, K \times N). \end{aligned}$$

The projection $\pi_k(K \times V, K \times N)$ becomes (identity on $K) \times \pi_k(V, N)$. If V, N and K are algebraic sets then this diffeomorphism is birational.

Lemma **(4.2)**. — *Suppose* U, V *and* W *are smooth manifolds,* U *is a proper submanifold of* V, $W \subset V$ *and suppose* U *intersects* W *cleanly. Then* $\mathcal{E}(U \times \mathbf{R}, U \times o)$ *intersects* $\mathcal{B}(W \times \mathbf{R}, (U \cap W) \times o)$ *cleanly in* $\mathcal{B}(V \times \mathbf{R}, U \times o)$ *and*

$$\mathcal{E}(U \times \mathbf{R}, U \times o) \cap \mathcal{B}(W \times \mathbf{R}, (U \cap W) \times o) = \mathcal{E}((U \cap W) \times \mathbf{R}, (U \cap W) \times o).$$

(Here $\mathcal{B}(W \times \mathbf{R}, (U \cap W) \times o) \subset \mathcal{B}(V \times \mathbf{R}, U \times o)$ *is the natural inclusion.)*

Proof. — Let $B = \mathcal{B}(V \times \mathbf{R}, U \times o)$, $B' = \mathcal{B}(W \times \mathbf{R}, (U \cap W) \times o)$, $T = \mathcal{E}(U \times \mathbf{R}, U \times o)$ and $\pi = \pi(V \times \mathbf{R}, U \times o)$. For any $p \in U \cap W$ we have an open imbedding

$$\varphi : (\mathbf{R}^a \times \mathbf{R}^b \times \mathbf{R}^c \times \mathbf{R}^d, o) \to (V, p)$$

so that $\varphi^{-1}(U) = \mathbf{R}^a \times \mathbf{R}^b \times o \times o$ and $\varphi^{-1}(W) = \mathbf{R}^a \times o \times \mathbf{R}^c \times o$. (This is equivalent to cleanness of the intersection of U and W.) Let:

$$N = (\text{Image } \varphi) \times \mathbf{R} \subset V \times \mathbf{R}.$$

Then φ induces a diffeomorphism

$$\varphi_* : \mathbf{R}^a \times \mathbf{R}^b \times \mathcal{B}(\mathbf{R}^c \times \mathbf{R}^d \times \mathbf{R}, o) \to \pi^{-1}(N)$$

so that $\varphi_*(\mathbf{R}^a \times \mathbf{R}^b \times \mathcal{E}(o \times o \times \mathbf{R}, o)) = \pi^{-1}(N) \cap T$

and $\varphi_*(\mathbf{R}^a \times o \times \mathcal{B}(\mathbf{R}^c \times o \times \mathbf{R}, o)) = \pi^{-1}(N) \cap B'$.

Thus $\pi^{-1}(N) \cap T \cap B' = \varphi_*(\mathbf{R}^a \times o \times \mathcal{E}(o \times o \times \mathbf{R}, o))$

since $\mathcal{E}(o \times o \times \mathbf{R}, o) \subset \mathcal{B}(\mathbf{R}^c \times o \times \mathbf{R}, o)$.

Also T intersects B′ cleanly in $\pi^{-1}(N)$. But:

$$\begin{aligned} \varphi_*(\mathbf{R}^a \times o \times \mathcal{E}(o \times o \times \mathbf{R}, o)) &= \varphi_*(\mathcal{E}(\mathbf{R}^a \times o \times o \times o \times \mathbf{R}, \mathbf{R}^a \times o \times o \times o \times o)) \\ &= \pi^{-1}(N) \cap \mathcal{E}((U \cap W) \times \mathbf{R}, (U \cap W) \times o). \end{aligned}$$

So we have shown that T intersects B′ cleanly and

$$T \cap B' = \mathcal{E}((U \cap W) \times \mathbf{R}, (U \cap W) \times o). \quad \blacksquare$$

As a consequence of Lemma (4.2) we see that if U, V and W are as in Lemma (4.2), we have a natural inclusion

$$B_k(W, U \cap W) \subset B_k(V, U)$$

so that
$$\pi_k(W, U \cap W) = \pi_k(V, U) \Big|_{B_k(W, U \cap W)}$$

and so that $B_k(W, U \cap W)$ intersects $N_k(V, U)$ cleanly and:

$$B_k(W, U \cap W) \cap N_k(V, U) = N_k(W, U \cap W).$$

For $k=0$ this inclusion is $B_0(W, U \cap W) = W \subset V = B_0(V, U)$. Suppose by induction that we have such an inclusion $B_k(W, U \cap W) \subset B_k(V, U)$. Then $B_k(W, U \cap W) \times \mathbf{R}$ intersects $N_k(V, U) \times 0$ cleanly and their intersection is $N_k(W, U \cap W) \times 0$. We thus have a natural inclusion

$$\mathscr{B}(B_k(W, U \cap W) \times \mathbf{R}, N_k(W, U \cap W) \times 0) \subset \mathscr{B}(B_k(V, U) \times \mathbf{R}, N_k(V, U) \times 0),$$

i.e. an inclusion $B_{k+1}(W, U \cap W) \subset B_{k+1}(V, U)$. Lemma (4.2) shows this inclusion has the required properties.

In addition, if U, V and W are algebraic sets then this inclusion is a birational diffeomorphism onto its image.

Note also that:

$$\lambda_k(W, U \cup W) = \lambda_k(V, U) \Big|_{(W-U) \times \mathbf{R}^k}.$$

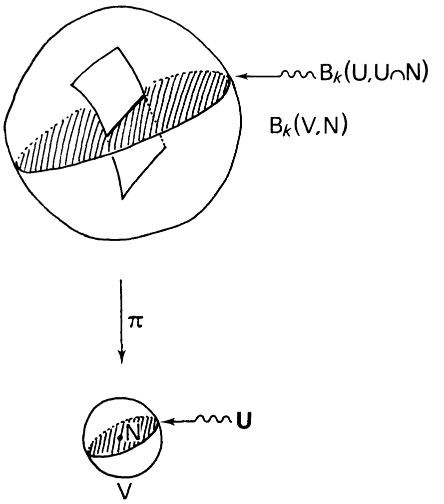


Fig. 4.2

Now we investigate $\pi_k(V, U)^{-1}(U)$. Define submanifolds $S_{ki}(V, U) \subset B_k(V, U)$ $i = 0, 1, \ldots, k$ as follows. Let:

$$S_{k0}(V, U) = B_k(U, U) \subset B_k(V, U)$$

(the inclusion exists because U intersects itself cleanly)

and
$$S_{kk}(V, U) = \mathscr{C}(B_{k-1}(V, U)\times\mathbf{R}, N_{k-1}(V, U)\times 0)$$
$$\subset \mathscr{B}(B_{k-1}(V, U)\times\mathbf{R}, N_{k-1}(V, U)\times 0) = B_k(V, U).$$

Then
$$S_{ki}(V, U) = B_{k-i}(S_{ii}(V, U), N_i(V, U))$$
$$\subset B_{k-i}(B_i(V, U), N_i(V, U)) = B_k(V, U)$$

for $1 \leq i < k$ (the inclusion exists because $N_i \subset S_{ii}$ hence they intersect cleanly). Notice that S_{ki} has codimension 1 for $i \geq 1$ and S_{k0} has codimension equal to the codimension of U in V by Lemma (4.2).

Lemma (**4.3**). — $\pi_k(V, U)^{-1}(U) = \bigcup_{i=0}^{k} S_{ki}(V, U).$

Proof. — Suppose by induction that:
$$\pi_{k-1}(V, U)^{-1}(U) = \bigcup_{i=0}^{k-1} S_{k-1,i}(V, U).$$

Then the lemma follows from the observations that:
$$\pi_k(V, U) = \pi_{k-1}(V, U) \circ \pi_1(B_{k-1}(V, U), N_{k-1}(V, U))$$

and that if $U \subset W \subset V$ then:
$$\pi_1(V, U)^{-1}(W) = B_1(W, U) \cup \mathscr{C}(V\times\mathbf{R}, U\times 0).$$

(Note $N_k(V, U) \subset S_{ki}(V, U)$ for all i.) ■

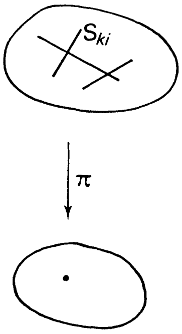


FIG. 4.3

Lemma (**4.4**). — *For all non-negative integers k and n there is an open imbedding*
$$\alpha : (\mathbf{R}^n\times\mathbf{R}^k, 0) \to (B_k(\mathbf{R}^n, 0), N_k(\mathbf{R}^n, 0))$$

so that:

a) $\alpha^{-1}S_{k0}(\mathbf{R}^n, 0) = 0\times\mathbf{R}^k$;

b) $\alpha^{-1}S_{ki}(\mathbf{R}^n, 0) = \mathbf{R}^n\times R_i$ *where*:
$$R_i = \{(y_1, \ldots, y_k) \in \mathbf{R}^k \mid y_i = 0\} \qquad i = 1, 2, \ldots, k;$$

c) *for any linear subspace* $T \subset \mathbf{R}^n$:

$$\alpha^{-1}B_k(T, 0) = T \times \mathbf{R}^k.$$

Proof. — By induction on k we have an open imbedding

$$\beta : (\mathbf{R}^n \times \mathbf{R}^{k-1}, 0) \to (B_{k-1}(\mathbf{R}^n, 0), N_{k-1}(\mathbf{R}^n, 0))$$

so that $\beta^{-1}S_{k-1,i}(\mathbf{R}^n, 0) = \mathbf{R}^n \times R_i'$

where $R_i' = \{(y_1, \ldots, y_{k-1}) \in \mathbf{R}^{k-1} \mid y_i = 0\}$

and so that:

$$\beta^{-1}B_{k-1}(T, 0) = T \times \mathbf{R}^{k-1}$$

for any linear subspace $T \subset \mathbf{R}^n$.

Let $\pi = \pi(B_{k-1}(\mathbf{R}^n, 0) \times \mathbf{R}, N_{k-1}(\mathbf{R}^n, 0) \times 0) : B_k(\mathbf{R}^n, 0) \to B_{k-1}(\mathbf{R}^n, 0) \times \mathbf{R}.$

By Lemma (4.1) there is an open imbedding

$$\alpha : (\mathbf{R}^n \times \mathbf{R}^{k-1} \times \mathbf{R}, 0) \to (B_k(\mathbf{R}^n, 0), N_k(\mathbf{R}^n, 0))$$

so that $\pi\alpha(x, y, t) = (\beta(tx, ty), t).$

Now for $T \subset \mathbf{R}^n$ a linear subspace

$$\alpha^{-1}B_k(T, 0) = \alpha^{-1}(C\ell(\pi^{-1}(B_{k-1}(T, 0) \times (\mathbf{R} - 0))))$$
$$= C\ell\alpha^{-1}\pi^{-1}(\beta(T \times \mathbf{R}^{k-1}) \times (\mathbf{R} - 0)) = T \times \mathbf{R}^{k-1} \times \mathbf{R}.$$

In particular:

$$\alpha^{-1}S_{k0}(\mathbf{R}^n, 0) = \alpha^{-1}B_k(0, 0) = 0 \times \mathbf{R}^{k-1} \times \mathbf{R}.$$

Likewise for $i < k$:

$$\alpha^{-1}S_{ki}(\mathbf{R}^n, 0) = \alpha^{-1}(C\ell(\pi^{-1}(S_{k-1,i}(\mathbf{R}^n, 0) \times (\mathbf{R} - 0))))$$
$$= C\ell\alpha^{-1}\pi^{-1}(\beta(\mathbf{R}^n \times R_i') \times (\mathbf{R} - 0)) = \mathbf{R}^n \times R_i' \times \mathbf{R}.$$

Also $\alpha^{-1}S_{kk}(\mathbf{R}^n, 0) = \alpha^{-1}\pi^{-1}(N_{k-1}(\mathbf{R}^n, 0) \times 0)$

$$= \alpha^{-1}\pi^{-1}(\beta(0, 0), 0) = \mathbf{R}^n \times \mathbf{R}^{k-1} \times 0. \quad \blacksquare$$

Definition. — We call $\mathscr{A} = (M, (N_1, N_2, \ldots, N_m), (k_1, \ldots, k_m), (s_1, \ldots, s_m))$ *resolution data* if N_i and M are smooth boundaryless manifolds

$N_1 \subset M;$

$N_2 \subset B_{k_1}(M \times \mathbf{R}^2, N_1 \times 0) \times \mathbf{R}^{s_1},$

$N_3 \subset B_{k_2}(B_{k_1}(M \times \mathbf{R}^2, N_1 \times 0) \times \mathbf{R}^{s_1} \times \mathbf{R}^2, N_2 \times 0) \times \mathbf{R}^{s_2}$

and so on. (Of course the above inclusions must all be proper.)

We define a manifold $B(\mathscr{A})$ as follows:

We set $B((M, (N), (k), (s))) = B_k(M\times\mathbf{R}^2, N\times o)\times\mathbf{R}^s$,

$$B((M, (N_1, N_2), (k_1, k_2), (s_1, s_2)))$$
$$= B_{k_2}(B_{k_1}(M\times\mathbf{R}^2, N_1\times o)\times\mathbf{R}^{s_1}\times\mathbf{R}^2, N_2\times o)\times\mathbf{R}^{s_2}$$

and so on. We define:

$$\mathscr{A}(i) = (M, (N_1, N_2, \ldots, N_i), (k_1, \ldots, k_i), (s_1, \ldots, s_i))$$

for any $o\leq i\leq m$ and we define:

$$\mathscr{A}-\mathscr{A}(i) = (B(\mathscr{A}(i)), (N_{i+1}, \ldots, N_m), (k_{i+1}, \ldots, k_m), (s_{i+1}, \ldots, s_m)).$$

Notice $N_i\subset B(\mathscr{A}(i-1))$ for any $i = 1, \ldots, m$. Also notice that $B(\mathscr{A}) = B(\mathscr{A}-\mathscr{A}(i))$ for any $i = o, \ldots, m$. We define $\pi(\mathscr{A}) : B(\mathscr{A})\to M$ by the rules

$$\pi(\mathscr{A}) = \pi(\mathscr{A}(i))\circ\pi(\mathscr{A}-\mathscr{A}(i))$$

and $\pi((M, (N), (k), (s)))$ is the composition of the projection

$$B_k(M\times\mathbf{R}^2, N\times o)\times\mathbf{R}^s\to B_k(M\times\mathbf{R}^2, N\times o)$$

and $\pi_k(M\times\mathbf{R}^2, N\times o) : B_k(M\times\mathbf{R}^2, N\times o)\to M\times\mathbf{R}^2$

and the projection $M\times\mathbf{R}^2\to M$. We define $P(\mathscr{A})\subset M$ and $T(\mathscr{A})\subset B(\mathscr{A})$ by:

$$P(\mathscr{A}) = \bigcup_{i=0}^{m-1}\pi(\mathscr{A}(i))(N_{i+1})$$

and
$$T(\mathscr{A}) = \bigcup_{i=0}^{m-1}\pi(\mathscr{A}-\mathscr{A}(i))^{-1}(N_{i+1}).$$

Then we have a diffeomorphism

$$\lambda(\mathscr{A}) : (M-P(\mathscr{A}))\times\mathbf{R}^{2m}\times\mathbf{R}^k\times\mathbf{R}^s\to B(\mathscr{A})-T(\mathscr{A}).$$

Where $k = \sum_{i=1}^{m} k_i$ and $s = \sum_{i=1}^{m} s_i$ defined by the rules

1) $\lambda((M, (N), (k), (s))) : (M-N)\times\mathbf{R}^2\times\mathbf{R}^k\times\mathbf{R}^s$
$$\to B_k(M\times\mathbf{R}^2, N\times o)\times\mathbf{R}^s - \pi_k(M\times\mathbf{R}^2, N\times o)^{-1}(N\times\mathbf{R}^2)\times\mathbf{R}^s$$

is defined by:

$$\lambda((M, (N), (k), (s)))(x, u, v, w) = (\lambda_k(M\times\mathbf{R}^2, N\times o)((x, u), v), w).$$

2) $\lambda(\mathscr{A})(x, (u_0, u_1), (v_0, v_1), (w_0, w_1))$
$$= \lambda(\mathscr{A}-\mathscr{A}(i))(\lambda(\mathscr{A}(i))(x, u_0, v_0, w_0), u_1, v_1, w_1)$$

for all $i = o, 1, \ldots, m$ and $x\in M-P(\mathscr{A})$

$$u_0\in\mathbf{R}^{2i},\quad u_1\in\mathbf{R}^{2m-2i},\quad v_0\in\mathbf{R}^{k^i},\quad v_1\in\mathbf{R}^{k-k^i},\quad w_0\in\mathbf{R}^{s^i},\quad w_1\in\mathbf{R}^{s-s^i}$$

where $k^i = \sum_{j=1}^{i} k_j$ and $s^i = \sum_{j=1}^{i} s_j$.

If V is a smooth boundaryless manifold we may define $V \times \mathscr{A}$ to be the resolution data

$$V \times \mathscr{A} = (V \times M, (V \times N_1, V \times N_2, \ldots, V \times N_m), (k_1, \ldots, k_m), (s_1, \ldots, s_m)).$$

Notice that:

$$B(V \times \mathscr{A}) = V \times B(\mathscr{A})$$

in such a way that:

$$\pi(V \times \mathscr{A}) = \mathrm{id}_V \times \pi(\mathscr{A}),$$

$$P(V \times \mathscr{A}) = V \times P(\mathscr{A}), \quad T(V \times \mathscr{A}) = V \times T(\mathscr{A})$$

and $\lambda(V \times \mathscr{A}) : V \times (M - P(\mathscr{A})) \times \mathbf{R}^{2m} \times \mathbf{R}^k \times \mathbf{R}^s \to B(V \times \mathscr{A}) = V \times B(\mathscr{A})$

is the map

$$\lambda(V \times \mathscr{A})((y, x), u, v, w) = (y, \lambda(\mathscr{A})(x, u, v, w)).$$

If $U \subset M$ intersects N cleanly then we have a natural inclusion

$$B((U, (U \cap N), (k), (s)) \subset B((M, N, (k), (s))$$

since we have an inclusion

$$B_k(U \times \mathbf{R}^2, (U \cap N) \times 0) \subset B_k(M \times \mathbf{R}^2, N \times 0).$$

Thus it makes sense to say that:

$$\mathscr{A}' = (U, (K_1, \ldots, K_m), (k_1, \ldots, k_m), (s_1, \ldots, s_m)) \subset \mathscr{A} = (M, (N_1, \ldots, N_m), (k_1, \ldots, k_m), (s_1, \ldots, s_m)).$$

This will mean that $U \subset M$ and U intersects N cleanly and $K_1 = N_1 \cap U$ hence:

$$B(\mathscr{A}'(1)) \subset B(\mathscr{A}(1)).$$

Then we require that $B(\mathscr{A}'(1))$ intersect N_2 cleanly and $K_2 = N_2 \cap B(\mathscr{A}'(1))$ and so on, so that we have a natural inclusion

$$B(\mathscr{A}'(i)) \subset B(\mathscr{A}(i))$$

for all i and $B(\mathscr{A}'(i))$ intersects N_{i+1} cleanly and

$$B(\mathscr{A}'(i)) \cap N_{i+1} = K_{i+1} \quad i = 0, 1, \ldots, m-1.$$

In particular we have a natural inclusion

$$B(\mathscr{A}') \subset B(\mathscr{A}).$$

We also have the properties

$$\pi(\mathscr{A})\Big|_{B(\mathscr{A}')} = \pi(\mathscr{A}'),$$

$$P(\mathscr{A}') = U \cap P(\mathscr{A}), \quad T(\mathscr{A}') = B(\mathscr{A}') \cap T(\mathscr{A})$$

and $\lambda(\mathscr{A}') : (U - P(\mathscr{A}')) \times \mathbf{R}^{2m} \times \mathbf{R}^k \times \mathbf{R}^s \to B(\mathscr{A}') - T(\mathscr{A}')$

is the restriction of $\lambda(\mathscr{A})$.

In case $\mathscr{A}' \subset \mathscr{A}$ as above, we will denote $\mathscr{A}'$ by $\mathscr{A} \cap \mathrm{U}$. We will say $\mathscr{A} \cap \mathrm{U}$ is defined if $\mathscr{A}'$ exists satisfying the above requirements.

Let:

$$\mathscr{A} = (\mathrm{M}, (\mathrm{N}_1, \ldots, \mathrm{N}_m), (k_1, \ldots, k_m), (s_1, \ldots, s_m))$$

be resolution data and let:

$$\mathscr{A}' = (\mathrm{M}', (\mathrm{N}'_1, \ldots, \mathrm{N}'_n), (k'_1, \ldots, k'_n), (s'_1, \ldots, s'_n))$$

be resolution data also with $\mathrm{M}' \subset \mathrm{B}(\mathscr{A})$. Then we may define resolution data $\mathscr{A}' * \mathscr{A}$ by

$$\mathscr{A}' * \mathscr{A} = (\mathrm{M}, (\mathrm{N}_1, \ldots, \mathrm{N}_m, \mathrm{N}'_1, \mathrm{N}'_2, \ldots, \mathrm{N}'_n), \\ (k_1, \ldots, k_m, k'_1, \ldots, k'_n), (s_1, \ldots, s_m, s'_1, \ldots, s'_n)).$$

Let:

$$\mathscr{A} = (\mathrm{M}, (\mathrm{N}_1, \ldots, \mathrm{N}_m), (k_1, \ldots, k_m), (s_1, \ldots, s_m))$$

be resolution data and let $\mu : \mathrm{M} \to \mathrm{M}'$ be an imbedding. Then sometimes we can define resolution data $\mu_\#(\mathscr{A})$ and an imbedding

$$\mu_*(\mathscr{A}) : \mathrm{B}(\mathscr{A}) \to \mathrm{B}(\mu_\#(\mathscr{A}))$$

by

$$\mu_\#(\mathscr{A}) = (\mathrm{M}', (\mu_0(\mathrm{N}_1), \mu_1(\mathrm{N}_2), \ldots, \mu_{m-1}(\mathrm{N}_m)), (k_1, \ldots, k_m), (s_1, \ldots, s_m))$$

where $\mu_0 = \mu$ and $\mu_i : \mathrm{B}(\mathscr{A}(i)) \to \mathrm{B}(\mu_\#(\mathscr{A}(i)))$ is $\mu_i = \mu_*(\mathscr{A}(i))$. We define $\mu_*(\mathscr{A})$ to be $(\mu_i)_*(\mathscr{A} - \mathscr{A}(i))$ for any $i = 1, 2, \ldots, m-1$ with:

$$\mu_*(\mathrm{M}, (\mathrm{N}), (k), (s)) : \mathrm{B}_k(\mathrm{M}, \mathrm{N}) \times \mathbf{R}^s \to \mathrm{B}_k(\mathrm{M}', \mu(\mathrm{N})) \times \mathbf{R}^s$$

defined to be the composition of the isomorphism

$$\mathrm{B}_k(\mathrm{M}, \mathrm{N}) \times \mathbf{R}^s = \mathrm{B}_k(\mu(\mathrm{M}), \mu(\mathrm{N})) \times \mathbf{R}^s$$

and the natural inclusion

$$\mathrm{B}_k(\mu(\mathrm{M}), \mu(\mathrm{N})) \times \mathbf{R}^s \to \mathrm{B}_k(\mathrm{M}', \mu(\mathrm{N})) \times \mathbf{R}^s.$$

(Of course for this to work we must have each $\mu_i(\mathrm{N}_{i+1})$ a proper submanifold. This will be true if for instance μ is a proper imbedding or if each N_i is compact.)

Definition. — We say that the resolution data

$$\mathscr{A} = (\mathrm{M}, (\mathrm{Q}_1, \mathrm{Q}_2, \ldots, \mathrm{Q}_m), (k_1, \ldots, k_m), (s_1, \ldots, s_m))$$

is *controlled* if:

a) $\pi(\mathscr{A}(i-1))(\mathrm{Q}_i) \subset \mathrm{Q}_1$ for all $i = 1, \ldots, m$ (i.e. $\mathrm{P}(\mathscr{A}) = \mathrm{Q}_1$).

b) $\mathscr{A} \cap \mathrm{Q}_1$ is defined.

c) If $\mathscr{A}' = (\mathrm{Q}_1, (\mathrm{Q}_1, \ldots, \mathrm{Q}_m), (k_1, \ldots, k_m), (s_1, \ldots, s_m))$ then $\mathscr{A}' = \mathscr{A} \cap \mathrm{Q}_1$. That is:

$$\mathrm{Q}_i \subset \mathrm{B}(\mathscr{A}'(i-1)) \subset \mathrm{B}(\mathscr{A}(i-1))$$

for all $i = 1, \ldots, m$.

d) Q_i is in a general position with:

$$\mathrm{C}\ell(\pi(\mathscr{A}(i-1))^{-1}(x)-\mathrm{B}(\mathscr{A}'(i-1)))$$

for each $x \in Q_1$ and $i=1, \ldots, m$.

e) $k_1>0$ and $M-Q_1$ is dense in M. (This is just to assure that *d)* is not vacuous.)

Notice for controlled resolution data that:

$$T(\mathscr{A})=\pi(\mathscr{A})^{-1}(Q_1).$$

We define $\quad S(\mathscr{A})=B(\mathscr{A} \cap Q_1) \subset B(\mathscr{A})$.

For instance:

$$S(\mathscr{A}(1))=S_{k_1 0}(M \times \mathbf{R}^2, Q_1 \times 0) \times \mathbf{R}^{s_1}.$$

Note that if $\mathscr{A}$ is controlled resolution data and K is a smooth manifold then $K \times \mathscr{A}$ is controlled resolution data and $S(K \times \mathscr{A})=K \times S(\mathscr{A})$.

Lemma (4.5). — *Suppose* $S_i \subset \mathbf{R}^n$ $i=1, \ldots, k$ *are linear subspaces and suppose* $M \subset \mathbf{R}^m \times \mathbf{R}^n$ *is a smooth manifold with* $0 \in M$ *so that* M *is transverse to*

$$0 \times \bigcap_{i=1}^{k} S_i$$

at 0. *Then there is an* $\varepsilon>0$ *a linear subspace* $T \subset \mathbf{R}^n$ *and a smooth function*

$$h: (\varepsilon \overset{\circ}{B}{}^m \times \mathbf{R}^n, 0) \rightarrow (\mathbf{R}^n, 0)$$

so that if $h': \varepsilon \overset{\circ}{B}{}^m \times \mathbf{R}^n \rightarrow \varepsilon \overset{\circ}{B}{}^m \times \mathbf{R}^n$ *is the function* $h'(x,y)=(x, h(x,y))$ *then* h' *is an open imbedding*

$$h'^{-1}(M)=\varepsilon \overset{\circ}{B}{}^m \times T$$

and

$$h'^{-1}(\varepsilon \overset{\circ}{B}{}^m \times S_i)=\varepsilon \overset{\circ}{B}{}^m \times S_i.$$

Proof. — Let $T \subset \mathbf{R}^n$ be the tangent space to $M \cap 0 \times \mathbf{R}^n$ at 0. Pick:

$$S \subset \bigcap_{i=1}^{k} S_i$$

so that S is a complementary subspace to T in $\mathbf{R}^n$. Let $\pi: \mathbf{R}^m \times \mathbf{R}^n \rightarrow \mathbf{R}^m \times T$ be projection along S. Then by the inverse function theorem, there is a neighborhood V of 0 in M and an $\varepsilon>0$ so that:

$$\pi\Big|_V : V \rightarrow \varepsilon B^m \times (T \cap \varepsilon B^n)$$

is a diffeomorphism.

Define:

$$g: \varepsilon B^m \times (T \cap \varepsilon B^n) \rightarrow S$$

by $\quad g=\left(\pi\Big|_V\right)^{-1}-\mathrm{id},$

i.e. g is the unique function so that:

$$(x, y+g(x,y)) \in \mathrm{V}$$

for any $(x,y) \in \varepsilon \mathrm{B}^m \times (\mathrm{T} \cap \varepsilon \mathrm{B}^n)$.

Pick $\delta > 0$ so that if:

$$(x,y) \in (\mathrm{M}-\mathrm{V}) \cap \varepsilon \mathrm{B}^m \times ((\mathrm{T} \cap \varepsilon \mathrm{B}^n)+\mathrm{S})$$

then $|\pi'(x,y) - g\pi(x,y)| > \delta$

where $\pi' : \mathbf{R}^m \times \mathbf{R}^n \to \mathrm{S}$

is the projection $\mathrm{id}-\pi$.

Let $\varphi : \mathbf{R}^n \to (\mathrm{T} \cap \varepsilon \mathring{\mathrm{B}}^n) + (\mathrm{S} \cap \delta \mathring{\mathrm{B}}^n)$

be any radial diffeomorphism (i.e. $|\varphi(x)| . x = |x| . \varphi(x)$ for all $x \in \mathbf{R}^n$). Define:

$$h(x,y) = g\pi(x, \varphi(y)) + \varphi(y)$$

and we are done. ■

The following proposition gives a very useful local description of a supermultiblowup.

Proposition **(4.6).** — *Suppose*:

$$\mathscr{A} = (\mathrm{M}, (\mathrm{Q}_1, \dots, \mathrm{Q}_m), (k_1, \dots, k_m), (s_1, \dots, s_m))$$

is controlled resolution data and

$$\varphi : \mathbf{R}^a \times \mathbf{R}^b \times \mathbf{R}^c \to \mathrm{M}$$

is an open imbedding with:

$$\varphi^{-1}(\mathrm{Q}_1) = \mathbf{R}^a \times 0 \times 0.$$

Pick any $q \in \mathrm{B}(\mathscr{A} \cap \varphi(\mathbf{R}^a \times \mathbf{R}^b \times 0))$

so that $\pi(\mathscr{A})(q) = \varphi(0)$.

Then there is an $\varepsilon > 0$, *and integer* d, *an open imbedding*

$$\theta : (\varepsilon \mathring{\mathrm{B}}^a \times \mathbf{R}^b \times \mathbf{R}^d \times \mathbf{R}^c, 0) \to (\mathrm{B}(\mathscr{A}), q)$$

and smooth functions

$$f : \varepsilon \mathring{\mathrm{B}}^a \times \mathbf{R}^b \times \mathbf{R}^d \to \mathbf{R}$$

and $g : \varepsilon \mathring{\mathrm{B}}^a \times \mathbf{R}^b \times \mathbf{R}^d \to \mathbf{R}^b$

and $\mathrm{L} \subset \mathbf{R}^d$ *so that*:

a) $\theta^{-1}(\mathrm{B}(\mathscr{A} \cap \varphi(\mathbf{R}^a \times \mathbf{R}^b \times 0))) = \varepsilon \mathring{\mathrm{B}}^a \times \mathbf{R}^b \times \mathbf{R}^d \times 0$

b) $\pi(\mathscr{A}) \circ \theta(x, y, z, w) = \varphi(x, g(x,y,z), f(x,y,z) . w)$

c) L *is a union of codimension one linear subspaces in general position*

d) $f^{-1}(0) = \varepsilon \mathring{\mathrm{B}}^a \times \mathbf{R}^b \times \mathrm{L}$

e) $g^{-1}(0) = f^{-1}(0)$ *unless* $q \in \mathrm{S}(\mathscr{A})$ *in which case* $g^{-1}(0) = f^{-1}(0) \cup \varepsilon \mathring{\mathrm{B}}^a \times 0 \times \mathbf{R}^d$.

Proof. — The proof will be by induction on m. For $m=0$ we may let $d=0$, $\theta=\varphi$, $f=1$, $g=y$, $L=\varnothing$.

Let $\mathcal{A}'=\mathcal{A}(m-1)$ and let $q'=\pi(\mathcal{A}-\mathcal{A}')(q)$. Suppose first that $q'\notin Q_m$. By induction we have an open imbedding

$$\theta' : (\varepsilon'\mathring{B}^a\times\mathbf{R}^b\times\mathbf{R}^e\times\mathbf{R}^c, 0) \to (B(\mathcal{A}'), q'),$$

smooth functions

$$f' : \varepsilon'\mathring{B}^a\times\mathbf{R}^b\times\mathbf{R}^e\to\mathbf{R}$$

and

$$g' : \varepsilon'\mathring{B}^a\times\mathbf{R}^b\times\mathbf{R}^e\to\mathbf{R}^b$$

and $L'\subset\mathbf{R}^e$ satisfying a), b), c), d) and e) with everything primed.

We may construct θ as follows. Pick:

$$w_0\in\mathbf{R}^{2+k_m+s_m}$$

so that $$q=\lambda(\mathcal{A}-\mathcal{A}')(q', w_0).$$

Define $$\theta : \varepsilon'\mathring{B}^a\times R^b\times(\mathbf{R}^e\times\mathbf{R}^{2+k_m+s_m})\times\mathbf{R}^c\to B(\mathcal{A})$$

by $$\theta(x,y,(u,w),z)=\lambda(\mathcal{A}-\mathcal{A}')(\theta'(x,y,u,z), w+w_0).$$

Let $L=L'\times\mathbf{R}^{2+k_m+s_m}$ and define:

$$f: \varepsilon'\mathring{B}^a\times\mathbf{R}^b\times\mathbf{R}^e\times\mathbf{R}^{2+k_m+s_m}\to\mathbf{R}$$

and $$g : \varepsilon'\mathring{B}^a\times\mathbf{R}^b\times\mathbf{R}^e\times\mathbf{R}^{2+k_m+s_m}\to\mathbf{R}^b$$

by $$f(x,y,u,w)=f'(x,y,u)$$

and $$g(x,y,u,w)=g'(x,y,u).$$

So we have done this case.

Now suppose $q'\in Q_m$. Then $q'\in S(\mathcal{A}')$ so:

$$q'\in B(\mathcal{A}'\cap\varphi(\mathbf{R}^a\times 0\times 0)).$$

Hence by induction (with $\mathbf{R}^c$ replaced by $\mathbf{R}^b\times\mathbf{R}^c$ and $\mathbf{R}^b$ replaced by 0) we have an open imbedding

$$\theta'' : (\varepsilon''\mathring{B}^a\times\mathbf{R}^e\times\mathbf{R}^b\times\mathbf{R}^c, 0) \to (B(\mathcal{A}'), q'),$$

a smooth function $f'' : \varepsilon''\mathring{B}^a\times\mathbf{R}^e\to\mathbf{R}$ and a subset $L''\subset\mathbf{R}^e$ so that:

$$\theta''^{-1}(S(\mathcal{A}'))=\varepsilon''\mathring{B}^a\times\mathbf{R}^e\times 0\times 0,$$

$$\pi(\mathcal{A}')\circ\theta''(x,u,y,z)=\varphi(x, f''(x,u)y, f''(x,u)z),$$

L'' is a union of codimension one linear subspaces of $\mathbf{R}^e$ in general position and $f''^{-1}(0)=\varepsilon''\mathring{B}^a\times L''$.

If $b\neq 0$:

$$\theta''^{-1}(B(\mathcal{A}'\cap\varphi(\mathbf{R}^a\times\mathbf{R}^b\times 0)))$$
$$=\theta''^{-1}(C\ell(\pi(\mathcal{A}')^{-1}(\varphi(\mathbf{R}^a\times(\mathbf{R}^b-0)\times 0))))=\varepsilon''\mathring{B}^a\times\mathbf{R}^e\times\mathbf{R}^b\times 0.$$

If $b=0$:

$$\theta''^{-1}(B(\mathscr{A}'\cap\varphi(\mathbf{R}^a\times\mathbf{R}^b\times 0)))=\theta''^{-1}(S(\mathscr{A}'))$$
$$=\varepsilon''\mathring{B}^a\times\mathbf{R}^e\times 0\times 0=\varepsilon''\mathring{B}^a\times\mathbf{R}^e\times\mathbf{R}^b\times 0.$$

Notice that:

$$\theta''^{-1}\pi(\mathscr{A}')^{-1}(\varphi(0))=0\times L''\times\mathbf{R}^b\times\mathbf{R}^c\cup 0\times\mathbf{R}^e\times 0\times 0.$$

Hence the fact that A is controlled implies that:

$$\theta''^{-1}(Q_m)\subset\varepsilon''\mathring{B}^a\times\mathbf{R}^e\times 0\times 0$$

and $\theta''^{-1}(Q_m)$ is in general position with the subspaces $0\times L''\times\mathbf{R}^b\times\mathbf{R}^c$.

By Lemma (4.5) there is an $\varepsilon>0$ and a smooth function

$$h:\ \varepsilon\mathring{B}^a\times\mathbf{R}^e\to\mathbf{R}^e$$

and a linear subspace $T\subset\mathbf{R}^e$ so that if:

$$h':\ \varepsilon\mathring{B}^a\times\mathbf{R}^e\to\varepsilon\mathring{B}^a\times\mathbf{R}^e$$

is the map $\quad h'(x,y)=(x,\ h(x,y))$

then h' is a smooth imbedding:

$$(h'\times\mathrm{id})^{-1}\theta''^{-1}(Q_m)=\varepsilon\mathring{B}^a\times T\times 0$$

and $\quad h'^{-1}(\varepsilon\mathring{B}^a\times L'')=\varepsilon\mathring{B}^a\times L''.$

Pick a linear subspace T′ complementary to T in $\mathbf{R}^e$ so that T′ is contained in all codimension one subspaces of $\mathbf{R}^e$ which are in L″.

Define an open imbedding

$$\psi:\ \varepsilon\mathring{B}^a\times T\times T'\times\mathbf{R}^b\times\mathbf{R}^c\to B(\mathscr{A}')$$

by $\quad \psi(x,\ u,\ v,y,\ z)=\theta''(x,\ h(x,\ u+v),y,\ z).$

Note $\quad \psi^{-1}(Q_m)=\varepsilon\mathring{B}^a\times T\times 0\times 0\times 0.$

Suppose the proposition were true with $m=1$ and with the additional conclusion that if $K\subset R^b$ is any particular linear subspace and $q\notin B(\mathscr{A}\cap\varphi(\mathbf{R}^a\times K\times 0))$ then we could pick θ, f, g and L so that $g^{-1}(K)=f^{-1}(0)$.

Since $m=1$ for $\mathscr{A}-\mathscr{A}'$ we would then (after perhaps making ε smaller) have an open imbedding

$$\theta:\ (\varepsilon\mathring{B}^a\times\mathbf{R}^b\times\mathbf{R}^c\times T\times T'\times\mathbf{R}^j,\ 0)\to(B(\mathscr{A}),\ q)$$

and smooth functions

$$f^*:\ \varepsilon\mathring{B}^a\times\mathbf{R}^b\times T\times T'\times\mathbf{R}^j\to\mathbf{R}$$

and $\quad (g_1^*,\ g_2^*):\ \varepsilon\mathring{B}^a\times\mathbf{R}^b\times T\times T'\times\mathbf{R}^j\to\mathbf{R}^b\times T'$

and $\quad L^*\subset\mathbf{R}^j$

so that $\quad \pi(\mathscr{A}-\mathscr{A}')\circ\theta(x,y,\ z,\ u,\ v,\ w)=\psi(x,\ u,\ g_2^*,\ g_1^*,\ z\cdot f^*),$

$$f^{*-1}(0)=\varepsilon\mathring{B}^a\times\mathbf{R}^b\times T\times T'\times L^*$$

and L^* is a union of codimension one linear subspaces in general position. In addition, if $q\notin S(\mathcal{A})$ then $q\notin B((\mathcal{A}-\mathcal{A}')\cap\psi(\varepsilon\mathring{B}^a\times T\times T'\times 0\times 0))$ so we may assume (by our additional conclusion) that:

$$(g_1^*, g_2^*)^{-1}(0\times T')=(g_1^*, g_2^*)^{-1}(0\times 0)=f^{*-1}(0).$$

On the other hand, if $q\in S(\mathcal{A})-S(\mathcal{A}-\mathcal{A}')$ then:

$$q\in B((\mathcal{A}-\mathcal{A}')\cap\psi(\varepsilon\mathring{B}^a\times T\times T'\times 0\times 0))-S(\mathcal{A}-\mathcal{A}')$$

so we may assume that $g_1^*(x,y,u,v,w)=f^*(x,y,u,v,w)\cdot y$, $g_2^{*-1}(0)=f^{*-1}(0)$ and f^* is independent of y. Finally, if $q\in S(\mathcal{A}-\mathcal{A}')$ then we may assume that $(g_1^*, g_2^*)=(y,v)\cdot f^*$ and f^* is independent of y and v.

We may now define $\mathbf{R}^d=T\times T'\times\mathbf{R}^j$, $L=T\times T'\times L^*\cup(T\cap L'')\times T'\times\mathbf{R}^j$ and let f and g be defined by:

$$f(x,y,u,v,w)=f''h'(x,u+g_2^*(x,y,u,v,w))\cdot f^*(x,y,u,v,w)$$

and

$$g(x,y,u,v,w)=f''h'(x,u+g_2^*(x,y,u,v,w))\cdot g_1^*(x,y,u,v,w).$$

Then these are the θ, f, g and L we want, as the reader may verify.

So it only remains to prove the proposition and the above extra conclusion with $m=1$. We may as well prove it with $s_1=0$. But in this case, if:

$$\mathcal{A}''=(M, (Q_1), (k_1-1), (1)) \quad\text{and if}\quad N\subset B(\mathcal{A}'')$$

is $N_{k_1-1}(M\times R^2, Q_1\times 0)\times 0$, then $B(\mathcal{A})=\mathcal{B}(B(\mathcal{A}''), N)$. Thus the result follows readily by induction on k_1 and by Lemma (4.1). ∎

Lemma (4.7). — *Let* $\mathcal{A}=(M, (Q_1, \ldots, Q_m), (k_1, \ldots, k_m), (s_1, \ldots, s_m))$ *be controlled resolution data and suppose* $\mu: M\to M'$ *is a smooth imbedding. Then* $\mu_\#(\mathcal{A})$ *is controlled resolution data (assuming* $\mu_\#(\mathcal{A})$ *is defined).*

Proof. — Pick any $p\in Q_m$. Let $\varphi: (\mathbf{R}^a\times\mathbf{R}^b\times\mathbf{R}^c, 0)\to(M', \mu\circ\pi(\mathcal{A})(p))$ be an open imbedding so that:

$$\varphi^{-1}(\mu(M))=\mathbf{R}^a\times\mathbf{R}^b\times 0$$

and

$$\varphi^{-1}(\mu(Q_1))=\mathbf{R}^a\times 0\times 0.$$

Notice:

$$\mu_*(\mathcal{A}(m-1))(p)\subset\mu_*(\mathcal{A}(m-1))(S(\mathcal{A}(m-1)))\subset S(\mu_\#(\mathcal{A}(m-1))).$$

By induction we know that $\mu_\#(\mathcal{A}(m-1))$ is controlled resolution data so by Proposition (4.6) there is an open imbedding

$$\theta:\ (\varepsilon\mathring{B}^a\times\mathbf{R}^b\times\mathbf{R}^c\times\mathbf{R}^d, 0)\to(B(\mu_\#(\mathcal{A}(m-1))), \mu_*(\mathcal{A}(m-1))(p)),$$

a smooth function $f:\varepsilon\mathring{B}^a\times\mathbf{R}^d\to\mathbf{R}$ and an $L\subset\mathbf{R}^d$ so that:

$$\theta^{-1}(S(\mu_\#(\mathcal{A}(m-1))))=\varepsilon\mathring{B}^a\times 0\times 0\times\mathbf{R}^d,$$

$$\pi(\mu_\#(\mathcal{A}(m-1)))\circ\theta(x,y,z,w)=\varphi(x,f(x,w)y,f(x,w)z),$$

L is a union of codimension one linear subspaces in general position and $f^{-1}(0)=\varepsilon\mathring{B}^a\times L$. Notice:

$$\theta^{-1}(\mu_*(\mathscr{A}(m-1))(B(\mathscr{A}(m-1))))$$
$$=\theta^{-1}(C\ell(\pi(\mu_\#(\mathscr{A}(m-1)))^{-1}(\mu(M-Q_1))))=\varepsilon\mathring{B}^a\times\mathbf{R}^b\times 0\times\mathbf{R}^d.$$

Also
$$\theta^{-1}\pi(\mu_\#(\mathscr{A}(m-1)))^{-1}(\mu\circ\pi(\mathscr{A}(m-1))(p))=0\times 0\times 0\times\mathbf{R}^d\cup 0\times\mathbf{R}^b\times\mathbf{R}^c\times L.$$

Since $\mathscr{A}$ is controlled resolution data, we know that $\theta^{-1}\mu_*(\mathscr{A}(m-1))(Q_m)$ is in general position with $0\times\mathbf{R}^b\times 0\times L$ as a submanifold of $\varepsilon\mathring{B}^a\times\mathbf{R}^b\times 0\times\mathbf{R}^d$. Hence $\theta^{-1}\mu_*(\mathscr{A}(m-1))(Q_m)$ is in general position with $0\times\mathbf{R}^b\times\mathbf{R}^c\times L$ as a submanifold of $\varepsilon\mathring{B}^a\times\mathbf{R}^b\times\mathbf{R}^c\times\mathbf{R}^d$. Thus $\mu_\#(\mathscr{A})$ is controlled resolution data. ■

Lemma **(4.8)**. — *Let*:

$$\mathscr{A}=(M, (Q_1, \ldots, Q_m), (k_1, \ldots, k_m), (s_1, \ldots, s_m))$$

and
$$\mathscr{A}'=(B(\mathscr{A}), (Q'_1, \ldots, Q'_u), (k'_1, \ldots, k'_u), (s'_1, \ldots, s'_u))$$

be two controlled resolution data and suppose $\mathscr{A}'(1)*\mathscr{A}$ is controlled resolution data. Then $\mathscr{A}'*\mathscr{A}$ is controlled resolution data.

Proof. — Conditions a), b), c) and e) in the definition of controlled resolution data are clearly satisfied so it remains to prove d).

Pick $i=1, \ldots, u-1$ and pick $p\in Q'_{i+1}$. Pick an open imbedding

$$\varphi:\ (\mathbf{R}^a\times\mathbf{R}^b, 0)\to(M, \pi(\mathscr{A})\circ\pi(\mathscr{A}'(i))(p)),$$

so that $\varphi^{-1}(Q_1)=\mathbf{R}^a\times 0$. By Proposition (4.6), we have an open imbedding

$$\theta:\ (\varepsilon\mathring{B}^a\times\mathbf{R}^b\times\mathbf{R}^c, 0)\to(B(\mathscr{A}), \pi(\mathscr{A}'(i))(p)),$$

an $L\subset\mathbf{R}^c$ and an $f:\varepsilon\mathring{B}^a\times\mathbf{R}^c\to\mathbf{R}$ so that:

$$\pi(\mathscr{A})\circ\theta(x,y,z)=\varphi(x,f(x,z)y),\qquad f^{-1}(0)=\varepsilon\mathring{B}^a\times L,$$

and L is a union of codimension one linear subspaces in general position.

By Lemma (4.5) we may also assume there is a linear subspace $T\subset\mathbf{R}^c$ and a smooth $h:(\varepsilon\mathring{B}^a\times\mathbf{R}^c, 0)\to(\mathbf{R}^c, 0)$ so that if $h':\varepsilon\mathring{B}^a\times\mathbf{R}^c\to\varepsilon\mathring{B}^a\times\mathbf{R}^c$ is $h'(x,y)=(x, h(x,y))$ then h' is a smooth imbedding,

$$(h'\times\mathrm{id})^{-1}\theta^{-1}(Q'_1)=\varepsilon\mathring{B}^a\times T\times 0$$

and
$$h'^{-1}(\varepsilon\mathring{B}^a\times L)=\varepsilon\mathring{B}^a\times L.$$

Pick a complementary subspace T' to T in $\mathbf{R}^c$ so that T' is contained in every codimension one subspace of $\mathbf{R}^c$ which is contained in L. Thus $L=(T\cap L)+T'$.

Again by Proposition (4.6) there is an open imbedding

$$\theta':\ (\varepsilon'\mathring{B}^a\times\mathbf{R}^b\times T\times T'\times\mathbf{R}^d, 0)\to(B(\mathscr{A}'(i)*\mathscr{A}), p),$$

an $L' \subset \mathbf{R}^d$ and an $f' : \varepsilon'\mathring{B}^a \times T \times \mathbf{R}^d \to \mathbf{R}$ so that:

$$\pi(\mathscr{A}'(i)) \circ \theta'(x, y, u, v, w) = \theta(x, f'(x, u, w)y, h(x, u + f'(x, u, w)v)),$$
$$f'^{-1}(0) = \varepsilon'\mathring{B}^a \times T \times L',$$

and L' is a union of codimension one linear subspaces in general position.

Notice that:

$$\theta'^{-1}\pi(\mathscr{A}'(i))^{-1}(\pi(\mathscr{A}'(i))(p)) = 0 \times 0 \times 0 \times 0 \times \mathbf{R}^d \cup 0 \times \mathbf{R}^b \times 0 \times T' \times L'$$

and $$\theta'^{-1}S(\mathscr{A}'(i)) = 0 \times 0 \times 0 \times 0 \times \mathbf{R}^d$$

so since $\mathscr{A}'$ is controlled, $\theta'^{-1}(Q'_{i+1})$ must be in general position with $0 \times \mathbf{R}^b \times 0 \times T' \times L'$.

Also:

$$\begin{aligned}&\theta'^{-1}\pi(\mathscr{A}'(i))^{-1}\pi(\mathscr{A})^{-1}(\pi(\mathscr{A}) \circ \pi(\mathscr{A}'(i))(p)) \\ &\quad = 0 \times 0 \times T \times T' \times \mathbf{R}^d \cup 0 \times \mathbf{R}^b \times T \times T' \times L' \cup 0 \times \mathbf{R}^b \times (T \cap L) \times T' \times \mathbf{R}^d.\end{aligned}$$

But $\theta'^{-1}(Q'_{i+1})$ is in general position with:

$$0 \times \mathbf{R}^b \times T \times T' \times L' \cup 0 \times \mathbf{R}^b \times (T \cap L) \times T' \times \mathbf{R}^d$$

at 0. Hence Q'_{i+1} is in general position with:

$$\mathrm{Cl}(\pi(\mathscr{A}'(i) * \mathscr{A})^{-1}(\pi(\mathscr{A}'(i) * \mathscr{A})(p)) - S(\mathscr{A}'(i) * \mathscr{A}))$$

at p, so $\mathscr{A}' * \mathscr{A}$ is controlled resolution data. ■

V. — FINS

The local description of $B(\mathscr{A})$ given by Proposition (4.6) will be very useful and it will be convenient to put its important features in a formal setting. Thus we define a fin.

Let us recall the promised method of proof, for now we have defined enough concepts to describe it more accurately. We start with a compact A-space Y without boundary and pick a closed stratum $N_1 \subset Y$. Then let Y_1 be an A-blowup of Y with center N_1. We then pick a closed stratum $N_2 \subset Y_1$ and let Y_2 be an A-blowup of Y_1 with center N_2. Keep on doing this until finally we get Y_m which is a smooth manifold.

Now we pick an imbedding $Y \subset \mathbf{R}^n$ so that N_1 is a nonsingular algebraic set. We then find some controlled resolution data $\mathscr{A}_1 = (\mathbf{R}^n, (N_1, \ldots), \ldots, \ldots)$ and an imbedding $Y_1 \subset B(\mathscr{A}_1)$ so that N_2 is a nonsingular algebraic set and so if $N_1 = q_1^{-1}(0)$ then $\mathscr{D}_{q_1}(Y_1)$ is isotopic to Y. We keep on doing this, i.e. we find controlled resolution data $\mathscr{A}_i = (B(\mathscr{A}_{i-1}), (N_i, \ldots), \ldots, \ldots)$ and imbeddings $Y_i \subset B(\mathscr{A}_i)$ so that N_{i+1} is a nonsingular algebraic set and so $\mathscr{D}_{q_1} \circ \mathscr{D}_{q_2} \circ \ldots \circ \mathscr{D}_{q_i}(Y_i)$ is isotopic to Y.

In the end we make sure Y_m is a nonsingular projectively closed algebraic set, hence $\mathscr{D}_{q_1} \circ \ldots \circ \mathscr{D}_{q_m}(Y_m)$ is an algebraic set homeomorphic to Y.

The value of fins is that they keep track of the collapsing that the maps $\mathscr{D}_{q_i}$ do. Each of the controlled resolution data $\mathscr{A}_i$ gives rise to a fin, hence we also wish to define what it means for a collection of fins to be compatible enough so that we can work with them.

Notice that in the tower construction above we wish to make sure the N_i's and Y_m are nonsingular algebraic sets. We do this by isotoping a smooth manifold to an algebraic set. We must then make sure that after doing this isotopy the $\mathscr{D}_{q_i}$ still collapse the same subsets. This requires the notion of stability and its consequence, Prop. (5.5).

Let M and N be smooth manifolds, let $V \subset M$ be a union of proper immersed submanifolds in general position and let $\pi : V \to N$ be a map. Then π is called a *fin* in M if π is a " submersion ". In other words, for each $p \in V$ there is an open neighborhood U of 0 in $\mathbf{R}^m$, a smooth open imbedding $\varphi : (U, 0) \to (M, p)$, linear subspaces $R_j \subset \mathbf{R}^m$ $j = 1, 2$ and a subset $L \subset R_2$ so that:

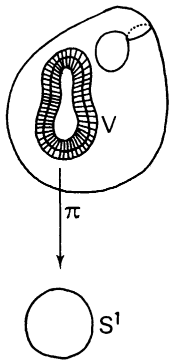
a) R_1 and R_2 are complementary subspaces of $\mathbf{R}^m$.

b) L is a union of linear subspaces of R_2 in general position.

c) $\pi\varphi|_{R_1 \cap U} : R_1 \cap U \to N$ is an open imbedding.

d) $\varphi^{-1}\pi^{-1}(x) = (y + L) \cap U$ for each $x \in N$ and $y \in R_1$ such that $\pi\varphi(y) = x$.

e) $\varphi^{-1}(V) = (R_1 + L) \cap U$.



A fin in a torus

FIG. 5.1

Given U, φ, R_1, R_2 and L as above it will be convenient to define an $R_3 \subset L$. This R_3 is a subspace of R_2 and is defined to be the intersection of all maximal linear subspaces contained in L. Equivalently R_3 is the subspace of vectors v such that translation by v leaves L invariant. The importance of R_3 is that a linear subspace is transverse to R_3 if and only if it is in general position with L.

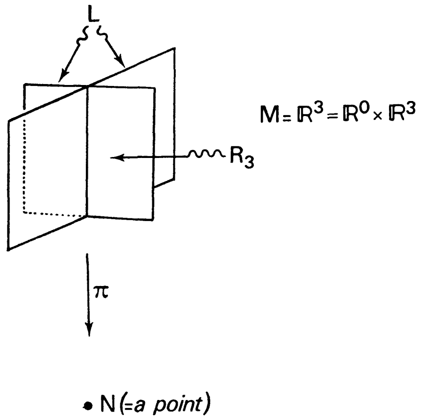


FIG. 5.2

We call $(U, \varphi, \{R_1, R_2, R_3\}, L)$ as above *local data* at p for the fin π. It will be convenient to define local data at points $q \in M - V$ also. This will be $(U, \varphi, \{R_1, R_2, R_3\}, L)$ with $\varphi : (U, 0) \to (M - V, q)$ an open imbedding, L empty, $R_1 = 0$ and $R_2 = R_3 = \mathbf{R}^m$.

Lemma (**5.1**). — *Let* $(U, \varphi, \{R_j\}, L)$ *and* $(U', \varphi', \{R'_j\}, L')$ *be two local data at* p *for a fin* $\pi : V \to N$ *in* N. *Then* $\varphi^{-1}\varphi'(R'_3 \cap U') \subset R_3$, $\varphi^{-1}\varphi'((R'_1 + R'_3) \cap U') \subset R_1 + R_3$ *and* $\varphi^{-1}\varphi'(L' \cap U') \subset L$.

Proof. — The set $\pi^{-1}(\pi(p))\cap\varphi(U)\cap\varphi'(U')=\varphi(L\cap U)\cap\varphi'(U')=\varphi'(L'\cap U')\cap\varphi(U)$ is a union of cleanly intersecting manifolds whose intersection is:

$$\varphi(R_3\cap U)\cap\varphi'(U')=\varphi'(R_3'\cap U')\cap\varphi(U).$$

Hence $\varphi^{-1}\varphi'(R_3'\cap U')\subset R_3$ and $\varphi^{-1}\varphi'(L'\cap U')\subset L$. Likewise:

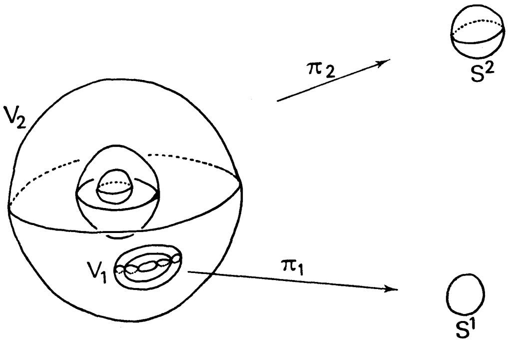
$$V\cap\varphi(U)\cap\varphi'(U')=\varphi((R_1+L)\cap U)\cap\varphi'(U')=\varphi'((R_1'+L')\cap U')\cap\varphi(U)$$

is a union of transversely intersecting manifolds whose intersection is:

$$\varphi((R_1+R_3)\cap U)\cap\varphi'(U')=\varphi'((R_1'+R_3')\cap U')\cap\varphi(U).$$

So $\varphi^{-1}\varphi'((R_1'+R_3')\cap U')\subset R_1+R_3$. ■

Definition. — Suppose $\pi_i: V_i\rightarrow N_i$ $i=1,\ldots,k$ is a collection of fins in M. We call the fins π_i compatible if for each $p\in M$ there is a U and φ and linear subspaces $R_{ij}\subset\mathbf{R}^m$ $i=1,\ldots,k$ $j=1,2,3$ and subsets $L_i\subset R_{i2}$ $i=1,\ldots,k$ so that for each $i=1,\ldots,k$, $(U,\varphi,\{R_{i1},R_{i2},R_{i3}\},L_i)$ is local data at p for the fin π_i. We call such $(U,\varphi,\{R_{ij}\},\{L_i\})$ *local data* at p for the fins π_i $i=1,\ldots,k$.



$V_2=S^2\times\mathbf{R}$
$\cup$
$V_1=S^1\times L$

L = figure eight

Compatible fins in $\mathbf{R}^3$

FIG. 5.3

Definition. — Let $\pi_i: V_i\rightarrow N_i$ $i=1,\ldots,k$ be compatible fins in M and suppose $N\subset M$ is a submanifold and $\pi: X\rightarrow M$ is a smooth map from a manifold X. Then we say that π and N *augment* the fins if $\pi_i': V_i'\rightarrow N_i'$ $i=1,\ldots,k+1$ are compatible fins in X where $V_i'=\pi^{-1}(V_i)$ $i=1,\ldots,k$, $V_{k+1}'=\pi^{-1}(N)$, $N_i'=N_i$ $i=1,\ldots,k$, $N_{k+1}=N$, $\pi_i'=\pi_i\pi\big|_{V_i'}$ $i=1,\ldots,k$ and $\pi_{k+1}'=\pi\big|_{V_{k+1}'}$. We say that the fins $\pi_i': V_i'\rightarrow N_i'$ $i=1,\ldots,k+1$ are the *augmented fins*.

Likewise we say that π *extends* the fins if $\pi_i': V_i'\rightarrow N_i$ $i=1,\ldots,k$ are compatible fins. We call $\pi_i': V_i'\rightarrow N_i$ $i=1,\ldots,k$ the *extended fins*.

Definition. — Suppose $\pi_i: V_i\rightarrow N_i$ $i=1,\ldots,k$ are compatible fins in M and suppose $\alpha: Y\rightarrow M$ is a smooth stratified morphism from an A space Y. We say that

α is *stable* over the fins π_i if for each $x\in Y$ there is local data $(U, \varphi, \{R_{ij}\}, \{L_i\})$ at $\alpha(x)$ for the fins π_i $i=1, \ldots, k$ so that if X is the stratum of Y containing x and $c: X\times\mathring{c}\Sigma\to Y$ is a neighborhood trivialization then:

a) $\alpha\big|_X$ is transverse to $\varphi(\bigcap_{i=1}^{k} R_{i3}\cap U)$.

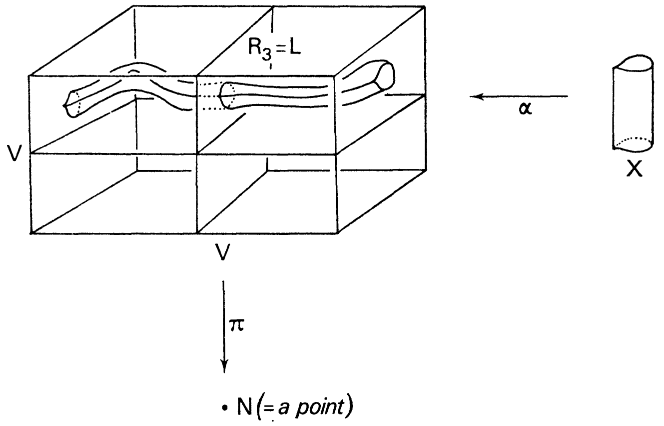
b) The following diagram commutes (if we take the germ at $(x, *)\in X\times\mathring{c}\Sigma$):

$$\begin{tikzcd}
& X \arrow[r, "\alpha"] & \varphi(U) \arrow[r, "\varphi^{-1}"] & \mathbf{R}^m \arrow[dr, "\pi"] & \\
X\times\mathring{c}\Sigma \arrow[ur] \arrow[dr, "c"'] & & & & (\bigcap_{i=1}^{k} R_{i3})^{\perp} \\
& Y \arrow[r, "\alpha"] & \varphi(U) \arrow[r, "\varphi^{-1}"] & \mathbf{R}^m \arrow[ur, "\pi"'] &
\end{tikzcd}$$

where $\pi: \mathbf{R}^m\to(\bigcap_{i=1}^{k} R_{i3})^{\perp}$ is orthogonal projection. In other words:

$$\varphi^{-1}\alpha c(y, z)\in\varphi^{-1}\alpha(y)+\bigcap_{i=1}^{k} R_{i3}$$

for all $v\in X$ near x and $z\in\mathring{c}\Sigma$ near $*$.



α is stable over the fin $V\xrightarrow{\pi}N$

FIG. 5.4

We call such local data $(U, \varphi, \{R_{ij}\}, \{L_i\})$ compatible with α. Notice that if X is a smooth manifold and $\alpha: X\to M$ is stable over the fins π_i then every local data is compatible with α by Lemma (5.1).

By the following Lemma we can straighten out any stable submanifold.

Lemma **(5.2).** — *Suppose* $\pi_i: V_i\to N_i$ $i=1, \ldots, k$ *are compatible fins in* M, Q *is a smooth manifold and* $\alpha: Q\to M$ *is a smooth imbedding which is stable over the fins* π_i. *Then* *for each* $q \in Q$ *there is a linear space* T *and local data* $(U, \varphi, \{R_{ij}\}, \{L_{ij}\})$ *at* $\alpha(q)$ *so that* $T \cap U = \varphi^{-1}\alpha(Q)$ *and* $R_{i1} \subset T$ *for each* $i = 1, \ldots, k$.

Proof. — This follows from the inverse function theorem. Pick any local data $(U', \varphi', \{R'_{ij}\}, \{L_i\})$ at $\alpha(q)$. Then $\varphi'^{-1}(\alpha(Q))$ is transverse to $\bigcap_{i=1}^{k} R'_{i3}$.

Let T be the tangent space to $\varphi'^{-1}(\alpha(Q))$ at 0. We may pick a linear subspace $S \subset \bigcap_{i=1}^{k} R'_{i3}$, so that S is a complimentary subspace to T. Let $\pi : \mathbf{R}^m \rightarrow T$ be projection along S (so $\pi(x+y) = y$ for $x \in S$, $y \in T$). We may pick a smooth submersion $f : \mathbf{R}^m \rightarrow S$ and a neighborhood V of 0 in $\mathbf{R}^m$ so that $V \cap f^{-1}(0) = V \cap \varphi'^{-1}\alpha(Q)$. Define $\psi : V \rightarrow \mathbf{R}^m$ by $\psi(x) = \pi(x) + f(x)$. Then $d\psi$ has rank m at 0 so by the inverse function theorem there is neighborhood U of 0 in $\mathbf{R}^m$ and an open imbedding $\theta : (U, 0) \rightarrow (V, 0)$ so that $\theta^{-1} = \psi\big|_{\theta(U)}$. Notice that $\theta^{-1}(y + L_i) \subset y + L_i$ for each i and $y \in R'_{i1}$ since

$$x - \psi(x) \in S \subset \bigcap_{i=1}^{k} R'_{i3} \quad \text{for all } x.$$

Define $R_{i1} = \pi(R'_{i1})$ and $R_{ij} = R'_{ij}$ for $i = 1, 2, \ldots, k$ and $j = 2, 3$. Then $(U, \varphi'\theta, \{R_{ij}\}, \{L_i\})$ is local data at $\alpha(q)$ and:

$$(\varphi'\theta)^{-1}(\alpha(Q)) = \psi(\varphi'^{-1}\alpha(Q) \cap \theta(U)) = T \cap U. \blacksquare$$

Lemma (5.3). — *Let* $\pi_i : V_i \rightarrow N_i$ $i = 1, \ldots, k$ *be compatible fins in* M. *If* N *is a smooth manifold then the projection* $\pi : M \times N \rightarrow M$ *extends the fins.*

Proof. — Take any (p, q) in $M \times N$, pick any open smooth imbedding

$$\psi : (\mathbf{R}^n, 0) \rightarrow (N, q)$$

and pick local data $(U, \varphi, \{R_{ij}\}, \{L_i\})$ at p for the fins π_i. Let $R'_{i1} \subset \mathbf{R}^m \times \mathbf{R}^n$ be $R_{i1} \times 0$ and let $R_{ij} = R_{ij} \times \mathbf{R}^n$ $j = 2, 3$. Then $(U \times \mathbf{R}^m, \varphi \times \psi, \{R'_{ij}\}, \{L_i \times \mathbf{R}^n\})$ is local data at (p, q) for the fins $\pi_i \circ \pi : V_i \times N \rightarrow N_i$ $i = 1, \ldots, k$. So π extends the fins. $\blacksquare$

Lemma (5.4). — *Let* $\pi_i : V_i \rightarrow N_i$, $i = 1, \ldots, k$ *be compatible fins in* M *and let* $\alpha : Y \rightarrow M$ *be a mapping from an* A-*space* Y *which is stable over these fins.*

a) *If* N *is a smooth manifold and* $\beta : Y \rightarrow N$ *is a smooth stratified morphism then* $\alpha \times \beta : Y \rightarrow M \times N$ *is stable over the extended fins* $V_i \times N \rightarrow V_i \rightarrow N_i$ $i = 1, \ldots, k$ *where* $V_i \times N \rightarrow V_i$ *is projection.*

b) *If* N *is a smooth manifold then* $\alpha \times \mathrm{id}_N : Y \times N \rightarrow M \times N$ *is stable over the extended fins.*

c) *If* M *is a smooth submanifold of* M′ *and* $\pi'_i : V'_i \rightarrow N_i$ $i = 1, \ldots, k$ *are compatible fins in* M′ *so that* $V_i = V'_i \cap M$ *and* $\pi_i = \pi'_i\big|_{V_i}$ *and the inclusion* $M \rightarrow M'$ *is stable over the fins* π' *then* $\alpha : Y \rightarrow M'$ *is stable over the fins* π'_i.

d) *If* $k = 1$, $N_1 =$ a point *and* M′ *is a smooth manifold and* $V' \rightarrow$ point *is a fin in* M′, *then* $\alpha \times \mathrm{id} : Y \times M' \rightarrow M \times M'$ *is stable over the fin* $V_1 \times M' \cup M \times V' \rightarrow$ point.

Proof. — To prove a) pick any $p \in Y$ and let $(U, \varphi, \{R_{ij}\}, \{L_i\})$ be local data at $\alpha(p)$ compatible with α. Then if $\psi : (\mathbf{R}^n, 0) \to (N, \beta(p))$ is an open smooth imbedding then $(U \times \mathbf{R}^n, \varphi \times \psi, \{R'_{ij}\}, \{L'_i\})$ is local data at $(\alpha(p), \beta(p)) \in M \times N$ where $R'_{i1} = R_{i1}$, $R'_{i2} = R_{i2} \times \mathbf{R}^n$, $R'_{i3} = R_{i3} \times \mathbf{R}^n$ and $L'_i = L_i \times \mathbf{R}^n$. Then notice that this local data is compatible with $\alpha \times \beta$ so a) is proven.

To prove b) pick $(p, q) \in Y \times N$ and let $(U, \varphi, \{R_{ij}\}, \{L_i\})$ be local data at $\alpha(p)$ compatible with α and let $\psi : (\mathbf{R}^n, 0) \to (N, q)$ be an open smooth imbedding, then $(U \times \mathbf{R}^n, \varphi \times \psi, \{R'_{ij}\}, \{L'_i\})$ is local data at $(\alpha(p), q) \in M \times N$ compatible with $\alpha \times \mathrm{id}$ where R'_{ij} and L'_i are defined as above.

To prove c), pick $p \in Y$ and let X be the stratum of Y which contains p. Let $(U, \varphi, \{R_{ij}\}, \{L_i\})$ be local data at $\alpha(p)$ for the fins $\pi'_i : V'_i \to N_i$ so that $\varphi^{-1}M = U \cap T$ for some linear subspace T of $\mathbf{R}^m$ and $R_{i1} \subset T$ $i = 1, \ldots, k$ (we may do this by Lemma (5.2)).

Let $T' \subset \bigcap_{i=1}^{k} R_{i3}$ be a linear subspace complementary to T. Then let $(U', \varphi', \{R'_{ij}\}, \{L'_i\})$ be local data at $\alpha(p)$ for the fins $\pi_i : V_i \to N_i$ $i = 1, \ldots, k$ compatible with α. Suppose also that $\varphi'(U') \subset \varphi(U)$. Define local data $(U'', \varphi'', \{R''_{ij}\}, \{L''_i\})$ at $\alpha(p)$ for the fins $\pi'_i : V'_i \to N_i$ by letting $U'' = \{(u, t) \in U' \times T' \mid \varphi^{-1}\varphi'(u) + t \in U\}$, and $\varphi''(u, t) = \varphi(\varphi^{-1}\varphi'(u) + t)$. Let $R''_{i1} = R'_{i1} \times 0$ and let $R''_{ij} = R'_{ij} \times T'$ $j = 2, 3$ and $L''_i = L'_i \times T'$. Then $(U'', \varphi'', \{R''_{ij}\}, \{L''_i\})$ is compatible with α.

To prove d) pick any $(p, q) \in Y \times M'$. Let $(U, \varphi, \{R_j\}, L)$ be local data for the fin $\pi_1 : V_1 \to$ point at $\alpha(p)$ compatible with α and let $(U', \varphi', \{R'_j\}, L')$ be local data at q for the fin $V' \to$ point. Then $(U \times U', \varphi \times \varphi', \{R_j \times R'_j\}, \{L \times \mathbf{R}^{m'} \cup \mathbf{R}^m \times L'\})$ is local data for the fin $V_1 \times M' \cup M \times V' \to$ point which is compatible with $\alpha \times \mathrm{id}$. ∎

Definition. — Let $\pi_i : V_i \to N_i$ $i = 1, \ldots, k$ be compatible fins in M and let $h_t : X \to M \times \mathbf{R}^n$ $t \in [0, 1]$ be a homotopy. Then we say that h_t is a *homotopy over the fins* $\pi_i : V_i \to N_i$ if for each $i = 1, \ldots, k$, $y \in N_i$ and $t \in [0, 1]$:

$$h_t^{-1}(\pi_i^{-1}(y) \times \mathbf{R}^n) = h_0^{-1}(\pi_i^{-1}(y) \times \mathbf{R}^n).$$

This definition also applies to isotopies.

Proposition (5.5). — *Let* $\pi_i : V_i \to N_i$ $i = 1, \ldots, k$ *be compatible fins in* M. *Suppose* $Q \subset M$ *is a compact boundaryless smooth submanifold so that* $Q \hookrightarrow M$ *is stable over the fins* π_i. *Suppose also that* Q *is isotopic to a smooth submanifold* Q' *by a* C^1 *small isotopy. Then there is an isotopy* $H_t : M \to M$, $t \in [0, 1]$ *over the fins* $\pi_i : V_i \to N_i$ $i = 1, \ldots, k$ *so that*:

$$H_0 = \text{identity} \quad \textit{and} \quad H_1(Q) = Q'.$$

Proof. — The proof will be by the usual Thom isotopy device. By the covering isotopy theorem we may pick a C^1 small isotopy $h_t : M \to M$ with $h_0 = \text{identity}$ and $h_1(Q) = Q'$. We wish to construct a vector field $v = \left(v', \frac{\partial}{\partial t}\right)$ on $M \times [0, 1]$ which is tangent to

$$Y = \{(h_t(x), t) x \in Q, \ t \in [0, 1]\}$$

and is "tangent" to each fiber $\pi_i^{-1}(x)\times[0, 1] x\in N_i$, $i=1, \ldots, k$ (the fibers $\pi_i^{-1}(x)\times[0, 1]$ are immersed submanifolds so at multiple intersections we would want the vector field to lie in several tangent spaces at once). Since we may piece together with a partition of unity and retain these properties, we only need to construct v locally. In addition we wish v to have compact support. Once we have this vector field, we integrate it to obtain a flow μ on $M\times[0, 1]$ and then define $H_t(x)=\rho\mu_t(x, 0)$ where $\rho: M\times[0, 1]\to M$ is projection.

So take any $(p, u)\in M\times[0, 1]$. If $p\notin h_u(Q)$ we may take $v=\left(0, \frac{\partial}{\partial t}\right)$ locally. If $p\in h_u(Q)$ we pick local data at p $(U, \varphi, \{R_{ij}\}, \{L_i\})$. Then $h_u(Q)$ is transverse to $\varphi(\bigcap_{i=1}^{k} R_{i3}\cap U)$ since Q is transverse to $\varphi(\bigcap_{i=1}^{k} R_{i3}\cap U)$ and the isotopy h_t is C^1 small. Hence we may by Lemma (5.2) assume that there is a linear subspace $T\subset \mathbf{R}^m$ so that $R_{i1}\subset T$ $i=1, \ldots, k$ and $U\cap T=\varphi^{-1}h_u(Q)$. Let $T'\subset \bigcap_{i=1}^{k} R_{i3}$ be a complementary subspace to T. Let $\pi: \mathbf{R}^m\to T$ and $\pi': \mathbf{R}^m\to T'$ be the projections so that:

$$\pi+\pi'=\text{identity}.$$

We may find a neighborhood W of (p, u) in $\varphi(U)\times[0, 1]$ so that the map

$$\psi: W\to \mathbf{R}^m\times[0, 1]$$

is defined and is an open imbedding where $\psi(x, t)=(\pi\varphi^{-1}(x)+\pi'\varphi^{-1}h_u h_t^{-1}(x), t)$. Let our vector field v be locally defined on W by $v=d\psi^{-1}\left(\frac{\partial}{\partial t}\right)$ where $d\psi$ is the map on tangent spaces induced by ψ. This v works since:

$$W\cap Y=\psi^{-1}(T\times[0, 1])$$

and $$W\cap(\pi_i^{-1}(x)\times[0, 1])=\psi^{-1}((y+L_i)\times[0, 1])$$

for any $x\in N_i$ and $y\in R_{i1}$ so that $\pi_i\varphi(y)=x$.

Hence v is tangent to Y and to each $\pi_i^{-1}(x)\times[0, 1]$.

If $v=\left(v', \frac{\partial}{\partial t}\right)$ is the global v we get after piecing together the local v's with a partition of unity, then v' has compact support since the local v's equalled $\left(0, \frac{\partial}{\partial t}\right)$ except in a small neighborhood of Y which we could take to be compact. ■

Lemma **(5.6)**. — *Suppose* $\alpha: Y\to M$ *is a map from an* A-*space* Y *which is stable over some compatible fins* $\pi_i: V_i\to N_i$ $i=1, \ldots, k$. *Suppose also that* $h: M\to M$ *is a diffeomorphism so that* $h(\pi_i^{-1}(x))=\pi_i^{-1}(x)$ *for each* $i=1, \ldots, k$ *and* $x\in N_i$. *Then* $h\alpha$ *is stable over the fins* π_i.

Proof. — Suppose $p\in Y$ and $(U, \varphi, \{R_{ij}\}, \{L_i\})$ is local data at $\alpha(p)$ compatible with α. Then $(U, h\varphi, \{R_{ij}\}, \{L_i\})$ is local data at $h\alpha(p)$ compatible with $h\alpha$. Hence $h\alpha$ is stable. ■

VI. — THE RELATIONS BETWEEN BLOWING UP FINS AND A-SPACES

In this section we tie together some of the notions we have been discussing. Proposition (6.1) indicates what happens to the representability of $Z/2Z$ homology classes when we blow up. Proposition (6.2) and Lemma (6.3) relate the notions of controlled resolution data and fins which you may recall was the reason for defining fins in the first place.

Proposition **(6.1)**. — *Suppose* V *and* U *are nonsingular algebraic sets with* $U \subset V$ *and* U *compact and suppose* V *and* U *both have all algebraic homology up to* n. *Then* $\mathscr{B}(V, U)$ *has all algebraic homology up to* n.

Proof. — Denote $\pi = \pi(V, U)$, $B = \mathscr{B}(V, U)$, $T = \mathscr{E}(V, U)$ $v = \dim V$ and $u = \dim U$. Let $r : L \to T$ be the canonical line bundle over $T =$ the projectivized normal bundle of U in V. We may identify L with a tubular neighborhood of T in B. Let $q : Q \to U$ be the normal bundle of U in V, we may identify Q with a tubular neighborhood of U in V, and in fact assume that $\pi|_{L-T}$ is a diffeomorphism onto $Q - U$.

Pick $\alpha \in H_i(V)$ with $i \leq n$ (the coefficients of all homology groups will be $Z/2Z$). Pick a nonsingular projectively closed algebraic set $X \subset V \times \mathbf{R}^m$ for some m so that $[X]$ represents α. Then by Proposition (2.3) we may isotop X in some $V \times \mathbf{R}^m \times \mathbf{R}^k$ to a nonsingular projectively closed algebraic set $Y \subset V \times \mathbf{R}^m \times \mathbf{R}^k$ so that Y is transverse to $U \times \mathbf{R}^m \times \mathbf{R}^k$. In particular, Y intersects $U \times \mathbf{R}^m \times \mathbf{R}^k$ cleanly so:

$$\mathscr{B}(Y, Y \cap U \times \mathbf{R}^m \times \mathbf{R}^k) \hookrightarrow \mathscr{B}(V \times \mathbf{R}^m \times \mathbf{R}^k, U \times \mathbf{R}^m \times \mathbf{R}^k) = B \times \mathbf{R}^m \times \mathbf{R}^k.$$

The nonsingular projectively closed algebraic set $\mathscr{B}(Y, Y \cap U \times \mathbf{R}^m \times \mathbf{R}^k)$ represents some $\alpha' \in H_i(B)$ and clearly $\pi_*(\alpha') = \alpha$. Hence the maps $\pi_* : H_i(B) \to H_i(V)$ and $H_i^A(B) \to H_i(V)$ are onto. So if there is a $\beta \in H_i(B) - H_i^A(B)$ then we may assume that β is in the kernel of π_*. From the exact sequences

$$\begin{tikzcd}
{} \arrow[r] & H_i(L) \arrow[r, "j_*"] \arrow[d] & H_i(B) \arrow[r] \arrow[d, "\pi_*"] & H_i(B, L) \arrow[r] \arrow[d, "\approx"] & {} \\
{} \arrow[r] & H_i(Q) \arrow[r] & H_i(V) \arrow[r] & H_i(V, Q) \arrow[r] & {}
\end{tikzcd}$$

we see that there is a $\gamma \in \text{kernel}(H_i(L) \to H_i(Q))$ so that $j_*(\gamma) = \beta$. (Note that

$$H_i(B, L) \to H_i(V, Q)$$

is an isomorphism since $H_i(B, L) \approx H_i(B-T, L-T) \overset{\pi_*}{\approx} H_i(V-U, Q-U) \approx H_i(V, Q)$.) So if we can show $j_*(H_i(L)) \subset H_i^A(B)$ we will be done since this would imply $H_i(B) = H_i^A(B)$.

Since $T \hookrightarrow L$ is a homotopy equivalence it is sufficient to represent $H_i(T)$. The cohomology $H^*(T)$ is a free $H^*(U)$ module with generators $1, \omega, \omega^2, \dots, \omega^{v-u-1}$ where $\omega \in H'(T)$ is the first Stiefel-Whitney class of the line bundle $r : L \to T$. (For instance this is implied by Theorem (5.7.9) of [6] after observing that the map

$$\theta : H^*(\mathbf{RP}^{v-u-1}) \to H^*(T)$$

is a cohomology extension of the fiber for the bundle $\pi : T \to U$ where θ(generator of $H^j(\mathbf{RP}^{v-u-1})) = \omega^j$.)

Let $\psi : H_*(T) \to H^*(T)$ and $\varphi : H_*(U) \to H^*(U)$ be the Poincare duality isomorphisms. We claim that $\psi^{-1}\pi^*\varphi H_*^A(U) \subset H_*^A(T)$ and $\psi^{-1}(\omega \cup \psi(H_*^A(T))) \subset H_*^A(T)$. This implies the $H_i^A(T) = H_i(T)$ for all $i \leq n$ since if $\alpha \in H_i(T)$ then $\psi(\alpha) = \sum_{j=0}^{v-u-1} \beta_j \cup \omega^j$ where $\beta_j \in \pi^* H^{v-1-i-j}(U) = \pi^*\varphi H_{i+j+1+u-v}(U) \subset \pi^*\varphi H_*^A(U)$ (since $i+j+1+u-v \leq i$) so $\beta_j \in \psi H_*^A(T)$. But if $H_i^A(T) = H_i(T)$ for all $i \leq n$ then $j_* H_i(L) \subset H_i^A(B)$ so we are done once the claim is proven.

So take $\alpha \in H_i^A(U)$ and a nonsingular projectively closed algebraic set $Z \subset U \times \mathbf{R}^m$ so $[Z]$ represents α. Consider $Z' = \{(x, y) \in T \times \mathbf{R}^m \mid (\pi(x), y) \in Z\}$, so Z' is the projective space bundle over Z induced from the bundle $T \to U$. Then $[Z']$ represents:

$$\psi^{-1}\pi^*\varphi(\alpha) \in H_{i+v-u-1}(T).$$

Hence $\psi^{-1}\pi^*\varphi H_*^A(U) \subset H_*^A(T)$.

Now take $\alpha \in H_*^A(T)$ and a nonsingular projectively closed algebraic set $W \subset T \times \mathbf{R}^m$ so that $[W]$ represents α. By Proposition (2.3) there is a small isotopy of W to a nonsingular projectively closed algebraic set $W' \subset L \times \mathbf{R}^m \times \mathbf{R}^k$ so that W' is transverse to $T \times \mathbf{R}^m \times \mathbf{R}^k$. Then $[W' \cap T \times \mathbf{R}^m \times \mathbf{R}^k]$ represents $\psi^{-1}(\omega \cup \psi(\alpha))$. Hence:

$$\psi^{-1}(\omega \cup \psi H_*^A(T) \subset H_*^A(T)).$$

(Recall that $\psi^{-1}(\omega) = [T \cap T']$ where $T' \subset L$ is a copy of T isotoped a little until it is transverse to T.) ■

Proposition **(6.2)**. — *Suppose* $\pi_i : V_i \to N_i$ $i = 1, \dots, k$ *are compatible fins in a smooth manifold* M *and* $\mathscr{A} = (M, (Q_1, Q_2, \dots, Q_u), (k_1, \dots, k_u), (s_1, \dots, s_u))$ *is controlled resolution data with* $Q_1 \hookrightarrow M$ *stable over the fins* π_i. *Then*:

a) $\pi(\mathscr{A}) : B(\mathscr{A}) \to M$ *and* $Q_1 \subset M$ *augment the fins* $\pi_i : V_i \to N_i$ $i = 1, \dots, k$.

b) *If* $P \subset M$ *is any smooth submanifold so that* $P \hookrightarrow M$ *is stable over the fins* π_i *and* $Q_1 \subset P$, *then* $B(\mathscr{A} \cap P) \hookrightarrow B(\mathscr{A})$ *is stable over the augmented fins.*

Proof. — Let $b = 2u + \sum_{i=1}^{u} (k_i + s_i)$. We have a diffeomorphism

$$\lambda(\mathscr{A}) : (M - Q_1) \times \mathbf{R}^b \to B(\mathscr{A}) - T(\mathscr{A})$$

so that $\pi(\mathscr{A})\circ\lambda(\mathscr{A})$ is projection to $M-Q_1$. Hence to prove a), it suffices by Lemma (5.3) to find local data for the augmented fins at any $p\in T(\mathscr{A})$. To prove b), note that we have a commutative diagram

$$\begin{tikzcd}
B(\mathscr{A}\cap P)-T(\mathscr{A}\cap P) \arrow[r, hook] & B(\mathscr{A})-T(\mathscr{A}) \\
(P-Q_1)\times\mathbf{R}^b \arrow[r, hook] \arrow[u, "\approx", "\lambda(\mathscr{A}\cap P)"'] & (M-Q_1)\times\mathbf{R}^b \arrow[u, "\approx", "\lambda(\mathscr{A})"']
\end{tikzcd}$$

so by Lemma (5.4) b) it suffices to find local data compatible with $B(\mathscr{A}\cap P)\hookrightarrow B(\mathscr{A})$ at all $p\in T(\mathscr{A}\cap P)$.

Let us now prove b). Pick any $p\in T(\mathscr{A}\cap P)$. By Lemma (5.2) we may pick local data $(U, \varphi, \{R_{ij}\}, \{L_i\})$ at $\pi(\mathscr{A})(p)$ for the fins $\pi_i: V_i\to N_i$ and a linear subspace T so that $\varphi^{-1}(Q_1)=T\cap U$, $R_{i1}\subset T$ for each $i=1, \dots, k$ and T is transverse to $\bigcap_{i=1}^{k} R_{i3}$. Let $T'\subset\bigcap_{i=1}^{k} R_{i3}$ be a complementary subspace to T.

By a relative version of Lemma (5.2) we may in fact assume that there is a linear subspace $S\subset T'$ so that $\varphi^{-1}(P)=(T+S)\cap U$.

Let $S'\subset T'$ be a complementary subspace to S in T'. By Proposition (4.6) we have an open imbedding $\theta: (\varepsilon T\times S\times\mathbf{R}^b\times S', 0)\to(B(\mathscr{A}), p)$, smooth functions

$$f:\ \varepsilon T\times S\times\mathbf{R}^b\to\mathbf{R} \quad\text{and}\quad g:\ \varepsilon T\times S\times\mathbf{R}^b\to S$$

and a $K\subset S\times\mathbf{R}^b$ so that:

$$\varepsilon T\subset U,\quad \theta^{-1}(B(\mathscr{A}\cap P))=\varepsilon T\times S\times\mathbf{R}^b\times 0,$$
$$\pi(\mathscr{A})\circ\theta(x, y, w, z)=\varphi(x+g(x, y, w)+f(x, y, w).z),$$

K is a union of linear subspaces in general position, and $f^{-1}(0)\subset g^{-1}(0)=\varepsilon T\times K$.

Now for any $i=1, \dots, k$, $x\in N_i$ and $y\in R_{i1}\cap\varepsilon T$ with $\pi_i\varphi(y)=x$ we have $\theta^{-1}\pi(\mathscr{A})^{-1}\pi_i^{-1}(x)=((y+L_i)\cap\varepsilon T)\times S\times\mathbf{R}^b\times S'$. Also for any $x\in\varphi(\varepsilon T)$ we have:

$$\theta^{-1}\pi(\mathscr{A})^{-1}(x)=\varphi^{-1}(x)\times K\times S'.$$

Define $L_i'=(L_i\cap T)\times S\times\mathbf{R}^b\times S'$, $R_{i1}'=R_{i1}\times 0\times 0\times 0$, $R_{ij}'=(R_{ij}\cap T)\times S\times\mathbf{R}^b\times S'$ $j=2, 3$ for $i=1, \dots, k$. Let:

$$L_{k+1}'=0\times K\times S',\quad R_{k+1,1}'=T\times 0\times 0\times 0,$$
$$R_{k+1,2}'=0\times S\times R^b\times S',\quad R_{k+1,3}'=0\times N\times S'$$

where N is the intersection of the maximal subspaces of K. Then $(\varepsilon T\times S\times\mathbf{R}^b\times S', \theta, \{R_{ij}'\}, \{L_i'\})$ is local data at p for the fins π_i augmented by $\pi(\mathscr{A})$ and $Q_1\subset M$. Since $\bigcap_{i=1}^{k+1} R_{i3}'=(\bigcap_{i=1}^{k} R_{i3}\cap T)\times N\times S'$ and $\theta^{-1}(B(\mathscr{A}\cap P))=T\times S\times\mathbf{R}^b\times 0$, this local data is compatible with $B(\mathscr{A}\cap P)\hookrightarrow B(\mathscr{A})$.

Since we could have let $P=M$ we would have obtained local data at all points of $B(\mathscr{A})$, so both a) and b) are proven. ■

Lemma **(6.3)**. — *Suppose* $\pi_i : V_i \to N_i$ $i=1, \ldots, k$ *are compatible fins in a smooth manifold* M, $\mathscr{A}=(M, (Q_1, \ldots, Q_m), (k_1, \ldots, k_m), (s_1, \ldots, s_m))$ *is controlled resolution data and* $Q_1 \hookrightarrow M$ *is stable over the fins* π_i. *Suppose* P *is a smooth submanifold of some* $B(\mathscr{A}(i))$ *so that* P *intersects* $S(\mathscr{A}(i))$ *cleanly*, $P \cap S(\mathscr{A}(i)) = Q_{i+1}$, $(\mathscr{A}-\mathscr{A}(i)) \cap P$ *is defined*, $(\mathscr{A}-\mathscr{A}(i)) \cap P$ *is controlled resolution data and* $P-Q_{i+1} \hookrightarrow B(\mathscr{A}(i))$ *is stable over the fins* π_i *augmented by* $\pi(\mathscr{A}(i))$ *and* $Q_1 \subset M$.

Then $B((\mathscr{A}-\mathscr{A}(i)) \cap P) - S((\mathscr{A}-\mathscr{A}(i)) \cap P) \hookrightarrow B(\mathscr{A})$ *is stable over the fins* π_i *augmented by* $\pi(\mathscr{A})$ *and* $Q_1 \subset M$.

Proof. — Let $\mathscr{A}'=(\mathscr{A}-\mathscr{A}(i)) \cap P$. Notice that by Lemma (5.4) b) the stability of $P-Q_{i+1}$ implies the stability of $B(\mathscr{A}')-T(\mathscr{A}') \hookrightarrow B(\mathscr{A})$ since:

$$B(\mathscr{A}')-T(\mathscr{A}') \subset B(\mathscr{A})-T(\mathscr{A}-\mathscr{A}(i)) \quad \text{and if} \quad c=\sum_{j=i+1}^{m}(2+k_i+s_i)$$

then
$$\lambda(\mathscr{A}-\mathscr{A}(i))((P-Q_{i+1}) \times \mathbf{R}^c) = B(\mathscr{A}')-T(\mathscr{A}').$$

So it suffices to find compatible local data at all $p \in T(\mathscr{A}')-S(\mathscr{A}')$. So pick any $p \in T(\mathscr{A}')-S(\mathscr{A}')$. Let $(U, \varphi, \{R_{ij}\}, \{L_i\})$ be local data for the fins π_i at $\pi(\mathscr{A})(p)$ so that there is a linear subspace T so that $R_{i1} \subset T$ for all $i=1, \ldots, k$ and:

$$U \cap T = \varphi^{-1}(Q_1).$$

Pick $T' \subset \bigcap_{i=1}^{k} R_{i3}$ so that T' and T are complementary subspaces.

By Proposition (4.6) there is an open imbedding

$$\theta : (\varepsilon T \times \mathbf{R}^d \times T', 0) \to (B(\mathscr{A}(i)), \pi(\mathscr{A}')(p))$$

and a smooth $f : \varepsilon T \times \mathbf{R}^d \to R$ and a subset $L \subset \mathbf{R}^d$ so that $\theta^{-1}(S(\mathscr{A}(i))) = \varepsilon T \times \mathbf{R}^d \times 0$, $\pi(\mathscr{A}(i)) \circ \theta(x, y, z) = \varphi(x+f(x,y)z)$, L is a union of codimension one linear subspaces in general position and $f^{-1}(0) = \varepsilon T \times L$.

By Lemma (4.5) we may make ε smaller and find a linear subspace $J \subset \mathbf{R}^d$ and an $h : (\varepsilon T \times \mathbf{R}^d, 0) \to (R^d, 0)$ so that if $h'(x, y) = (x, h(x, y))$ then h' is an open imbedding, $h'^{-1}\theta\ {}^{1}(Q_{i+1}) = \varepsilon T \times J$ and $h'^{-1}(\varepsilon T \times L) = \varepsilon T \times L$. By a similar arguement there is a linear subspace $K \subset T'$ and an $(h_1, h_2) : \varepsilon T \times \mathbf{R}^d \times T' \to \mathbf{R}^d \times T'$ so that if:

$$h''(x, y, z) = (x, h(x, y) + h_1(x, y, z), h_2(x, y, z))$$

then h'' is an open imbedding, $h''^{-1}\theta^{-1}(P) = \varepsilon T \times J \times K$, $h_2^{-1}(0) = \varepsilon T \times \mathbf{R}^d \times 0$ and $h_1^{-1}(0) \supset h_2^{-1}(0)$, and $h''^{-1}(\varepsilon T \times L \times T') = \varepsilon T \times L \times T'$. (Hence $h''^{-1}\theta^{-1}(Q_{i+1}) = \varepsilon T \times J \times 0$.)

Let J' and K' be complementary subspaces to J and K in $\mathbf{R}^d$ and T' respectively. We also require J' to be contained in each maximal subspace of L.

By Proposition (4.6) there is a smooth open imbedding

$$\theta' : (\varepsilon T \times J \times K \times \mathbf{R}^e \times J' \times K', 0) \to (B(\mathscr{A}), p)$$

and an $L' \subset \mathbf{R}^e$ and smooth functions $f' : \varepsilon T \times J \times K \times \mathbf{R}^e \to \mathbf{R}$ and $g' : \varepsilon T \times J \times K \times \mathbf{R}^e \to K$ so that $\pi(\mathscr{A}') \circ \theta'(x, y, z, u, v, w) = \theta h''(x, y + v . f'(x, y, z, u), g'(x, y, z, u) + w . f'(x, y, z, u))$, L' is a union of codimension one linear subspaces in general position and:

$$f'^{-1}(0) = g'^{-1}(0) = \varepsilon T \times J \times K \times L'.$$

For $i = 1, \ldots, k$ define:

$$R'_{i1} = R_{i1} \times 0 \times 0 \times 0 \times 0 \times 0 \quad \text{and} \quad R_{ij} = (R_{ij} \cap T) \times J \times K \times \mathbf{R}^e \times J' \times K'$$

for $j = 2, 3$ and $L'_i = (L_i \cap T) \times J \times K \times \mathbf{R}^e \times J' \times K'$. Also let:

$$R'_{k+1,1} = T \times 0 \times 0 \times 0 \times 0 \times 0,$$
$$R'_{k+1,2} = 0 \times J \times K \times \mathbf{R}^e \times J' \times K',$$
$$L'_{k+1} = 0 \times (L \cap J) \times K \times \mathbf{R}^e \times J' \times K' \cup 0 \times J \times K \times L' \times J' \times K',$$
$$R'_{k+1,3} = 0 \times J'' \times K \times L'' \times J' \times K'$$

J'' and L'' are the intersections of the maximal subspaces of $L \cap J$ and L' respectively.

Then $(\varepsilon T \times J \times K \times \mathbf{R}^e \times J' \times K', \theta', \{R'_{ij}\}, \{L'_i\})$ is local data at p for the augmented fins which is compatible with $B(\mathscr{A}') - S(\mathscr{A}')$. ■

VII. — GETTING NICE SPINES

This section gives the final ingredient in the proof of our main Theorem (8.3). Recall that when doing an A-blowup we need an A-space W which a link Σ bounds. We want to be careful picking this W. We want to make sure that W has a spine of codimension one transversely intersecting sub A-spaces with empty boundaries. Furthermore, we wish to imbed W in some supermultiblowup so that W intersected with the associated fin is this spine and so W is in fact stable over the associated fin (the W we refer to above is the W′ of Proposition (7.2)).

The following Proposition (7.1) is an A-version of Fact (3.2) of [1]. It is extremely technical, the reader should first read the proof of Proposition (0.1) which contains a more intuitive and less precise form of Proposition (7.1). Proposition (0.1) also contains some of Proposition (7.2).

Proposition **(7.1)**. — *Let* Y *be a compact* A-*space so that each connected component of each stratum of* Y *has nonempty boundary. Then there are* A-*spheres* $A_i \subset Y$, *smooth stratified morphisms* $f_i : Y \rightarrow [-1, 1]$ $i=1, \ldots, k$, *pairwise disjoint* A-*discs* $B_{ij} \subset Y - \bigcup_{s=1}^{k} A_s$, A-*discs* Y_{ij}, A-*imbeddings* $\alpha_{ij} : Y_{ij} \times [-1, 1] \rightarrow Y - \bigcup_{s=1}^{k} \bigcup_{t=1}^{a_s} \operatorname{int} B_{st} = V$ $i=1, \ldots, k$ $j=1, \ldots, a_i$, *a collaring* $\kappa : \partial V \times [0, 1) \rightarrow V$ *and integers* $\sigma_{sij} = \pm 1$ $s=1, \ldots, k$, $i=1, \ldots, k$ $j=1, \ldots, a_i$ *so that:*

a) *if* $R_i \subset \mathbf{R}^k$ *is defined by* $R_i = \{(x_1, x_2, \ldots, x_k) \in \mathbf{R}^k \mid x_i = 0\}$ *then:*

$$(f_1, f_2, \ldots, f_k) : \operatorname{int} Y \rightarrow \mathbf{R}^k$$

is stable over the fin $\bigcup_{i=1}^{k} R_i \rightarrow \text{point}$;

b) $f_i^{-1}(0) = A_i$ $i=1, \ldots, k$. (*Hence* $(f_1, \ldots, f_k)^{-1}(\bigcup_{i=1}^{k} R_i) = \bigcup_{i=1}^{k} A_i)$;

c) $\kappa(\partial V \times [0, 1)) = V - \bigcup_{i=1}^{k} A_i$ (*so* $\bigcup A_i$ *is a* “*spine*” *of* V);

d) $\partial(V - \bigcup_{i=1}^{k} \bigcup_{j=1}^{a_i} \alpha_{ij}(\operatorname{int} Y_{ij} \times [-1, 1]))$ *is* A-*isomorphic with* ∂Y;

e) B_{ij} *is a full* A-*subspace of* Y *and is* A-*isomorphic with* $Y_{ij} \times [-1, 1]$ $i=1, \ldots, k$ $j=1, \ldots, a_i$.

f) $\partial Y \subset f_i^{-1}(1)$ $i=1, \ldots, k$;

g) $f_s \alpha_{ij}(y, t) = \sigma_{sij}$ *for all* $(y, t) \in Y_{ij} \times [-1, 1]$ *unless* $s=i$ *in which case* $f_i \alpha_{ij}(y, t) = \sigma_{iij} t$;

h) $\alpha_{ij}^{-1}(\partial V)=Y_{ij}\times\{-1, 1\}$ *and* $\alpha_{ij}^{-1}(\partial B_{ij})=Y_{ij}\times 1$;
i) $\kappa(\alpha_{ij}(y, \pm 1), t)=\alpha_{ij}(y, \pm(1-t))$ *for all* $y\in Y_{ij}$, $t\in[0, 1)$.

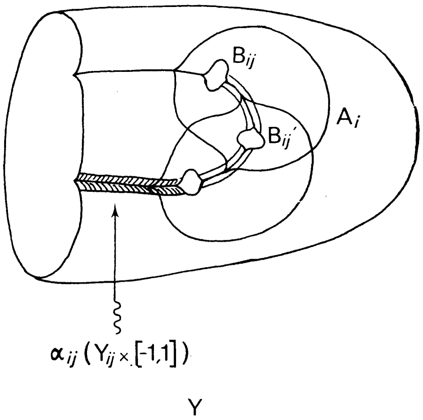


FIG. 7.0

Proof. — The first step is to get full A-discs $D_i\subset \operatorname{int} Y$ so that the $A_i=\partial D_i$ are in general position, there is a collaring $\kappa' : \partial Y\times[0, 1)\to Y$ with $\operatorname{Image}(\kappa')=Y-\bigcup_{i=1}^{k} D_i$ and so that each component of $\bigcup_{i=1}^{k} D_i-\bigcup_{i=1}^{k} A_i$ is an open A-disc. This is done as in Proposition (0.1) so we will not repeat the construction. The general idea of this proof will be as follows. We can construct f_i satisfying a) and b) because the A_i are in general position. In each component of $\bigcup D_i-\bigcup A_i$ we take a full A-disc B_{ij}, then we may find a κ satisfying c). The α_{ij} are thickened paths connecting various boundary components of V. These paths are the γ of Proposition (0.1), hence we have d). The conclusions f), g), h) and i) are technical conditions which we need later.

Let us now proceed with the proof. Let $\mathscr{B}$ be the set of connected components of $\bigcup_{i=1}^{k} D_i-\bigcup_{i=1}^{k} A_i$. Then for each $\beta\in\mathscr{B}$ pick an A-disc B_β in the component β of $\bigcup D_i-\bigcup A_i$ so that $\beta-\operatorname{int} B_\beta$ is A-isomorphic with $\partial B_\beta\times[0, 1)$. Let $V=Y-\bigcup_{\beta\in\mathscr{B}} \operatorname{int} B_\beta$. We may then extend κ' to a collaring $\kappa'' : \partial V\times[0, 1)\to V$ with $\operatorname{Image}(\kappa'')=V-\bigcup_{i=1}^{k} A_i$.

Let $\mathscr{B}'\subset\mathscr{B}$ be a maximal subset so that for each $\beta\in\mathscr{B}'$ there exists a $\gamma_\beta : [-1, 1]\to V$ so that γ_β is a smooth imbedding into a stratum of V, $\gamma_\beta^{-1}(\partial V)=\{-1, 1\}$ and $\gamma_\beta(1)$ is in the lowest dimensional stratum of ∂B_β, $\gamma^{-1}(\bigcup_{i=1}^{k} A_i)=\{0\}$ and each $\gamma_\beta(0)$ is in exactly one A_i, the $\gamma_\beta([-1, 1])$ are pairwise disjoint and the complex $\mathscr{C}$ is contractible where $\mathscr{C}=\partial Y\cup\bigcup_{\beta\in\mathscr{B}'}(\gamma_\beta([-1, 1])\cup\partial B_\beta)$ with ∂Y identified to a point and each ∂B_β identified to a point.

Suppose that $\mathscr{B}'\neq\mathscr{B}$. Take the stratum Y_α of least dimension so that $Y_\alpha\cap\bigcup_{\beta\notin\mathscr{B}'} B_\beta$

is nonempty. Since each component of Y_α has nonempty boundary we may choose a path in Y_α from some component in $\mathscr{B}-\mathscr{B}'$ to ∂Y_α. We may assume this path is in general position with the $Y_\alpha \cap A_i$. Then at some point this path must leave a component β' in $\mathscr{B}-\mathscr{B}'$ and enter either a component in $\mathscr{B}'$ or $Y-\bigcup D_i$. In either case we may then choose an imbedding $\gamma_{\beta'}:[-1, 1]\to V\cap Y_\beta$ so that $\gamma_{\beta'}^{-1}(\partial V)=\{-1, 1\}$, $\gamma_{\beta'}(1)$ is in the lowest dimensional stratum of $B_{\beta'}$, $\gamma_{\beta'}([-1, 1])$ is disjoint from all $\gamma_\beta([-1, 1])$ $\beta\in\mathscr{B}'$, $\gamma_{\beta'}^{-1}(\bigcup A_i)=\{0\}$ and $\gamma_{\beta'}$ intersects exactly one A_i and:

$$\gamma_{\beta'}(-1)\in \bigcup_{\beta\in\mathscr{B}'} \partial B\cup\partial Y.$$

But then the complex $\mathscr{C}'=\mathscr{C}\cup\gamma_{\beta'}([-1, 1])\cup\partial B_{\beta'}$ with $\partial B_{\beta'}$ collapsed to a point is contractible so $\mathscr{B}'$ was not maximal, hence $\mathscr{B}'=\mathscr{B}$.

So for each $\beta\in\mathscr{B}$ pick γ_β so that the above properties are true. Note also that for each β, if V_r is the stratum of V containing Image(γ_β) then we may assume that γ_β is in general position with the $V_r\cap A_i$ $i=1, \ldots, k$.

We rename the B_β and γ_β as B_{ij} and γ_{ij} $i=1, \ldots, k$ $j=1, \ldots, a_i$ so that $\gamma_\beta(0)\in A_i$ means that γ_β and B_β are renamed as γ_{ij} and B_{ij} for some j. Now the normal bundle of each $\gamma_{ij}([-1, 1])$ in its stratum $V_{r_{ij}}$ is trivial so we have pairwise disjoint open imbeddings $\gamma'_{ij}:[-1, 1]\times\mathbf{R}^{n_{ij}}\to V_{r_{ij}}$ so that $\gamma_{ij}'^{-1}(\partial V)=\{-1, 1\}\times\mathbf{R}^{n_{ij}}$ and $\gamma_{ij}'^{-1}(A_i)=0\times\mathbf{R}^{n_{ij}}$ and $\gamma_{ij}'^{-1}(A_s)$ is empty for $s\neq i$. If $c_{ij}:V_{r_{ij}}\times\mathring{c}\Sigma_{r_{ij}}\to V$ is a neighborhood trivialization of the stratum $V_{r_{ij}}$ then we let:

$$Y_{ij}=\left\{(y, (z, t))\in\mathbf{R}^{n_{ij}}\times\mathring{c}\Sigma_{r_{ij}} \mid |y|\leq 1 \text{ and } t\leq\frac{1}{2}\right\}\approx n_{ij}\text{-disc}\times c\Sigma_{r_{ij}}$$

and define α_{ij} by $\alpha_{ij}((y, (z, t)), s)=c_{ij}(\gamma'_{ij}(s, y), (z, t))$ for $(y, (z, t))\in Y_{ij}\subset\mathbf{R}^{n_{ij}}\times\mathring{c}\Sigma_{r_{ij}}$ and $s\in[-1, 1]$. Notice that h) is satisfied.

We now adjust the collaring κ'' to a collaring $\kappa:\partial V\times[0, 1)\to V$ so that c) and i) are satisfied. Notice that e) is satisfied by our choice of Y_{ij} and d) follows because the complex $\mathscr{C}$ above is contractible (cf. [1], Fact (3.2)). So it remains to find σ_{sij} and $f_i:Y\to[-1, 1]$ so that a), b), f) and g) are satisfied.

Let $\sigma_{sij}=-1$ if $B_{ij}\subset D_s$ and $\sigma_{sij}=1$ otherwise. Then if $\sigma_{iij}=-1$, $Y_{ij}\times[0, 1]=\alpha_{ij}^{-1}(D_i)$ and if $\sigma_{iij}=1$, $Y_{ij}\times[-1, 0]=\alpha_{ij}^{-1}(D_i)$.

By Lemma (1.3) we may pick bicollarings, open A-imbeddings

$$g_i:\ A_i\times(-1, 1)\to Y-\partial Y$$

so that $g_i(A_i\times 0)=A_i$, Image$(g_s)\cap$ Image(α_{ij}) is empty if $i\neq s$ and so:

$$\alpha_{ij}(x, t)=g_i(\alpha_{ij}(x, 0), \sigma_{iij}t) \quad \text{for all} \quad x\in Y_{ij} \quad \text{and} \quad t\in[-1, 1].$$

Then we may define f_i by $f_i g_i(x, t)=t$ for $(x, t)\in A_i\times(-1, 1)$ and $f_i(y)=-1$ $y\in D_i-$ Image g_i and $f_i(y)=1$ for $y\in Y-(D_i\cup$ Image $g_i)$. We also smooth out the f a little near the frontier of Image g_i. Then these f_i will satisfy a), b), f) and g) so our proposition is proven. ■

The reader should be aware that the important conclusions of Proposition (7.2) are that $X = \partial W'$, γ is stable, W' has the same number of strata as X, the collaring ρ exists and $\gamma(\Sigma) \subset B(\mathscr{A}) - T(\mathscr{A})$. Conclusions 5 and 6 are there only to allow an algebraic bordism condition to be satisfied in Theorem (8.3). If the algebraic homology conjecture were true, they would be unnecessary [8].

Proposition **(7.2)**. — *Suppose* Σ *is a compact* A-*space which bounds and suppose* $n > \dim \Sigma + 2$. *Then there are compact* A-*spaces* W, W′ *and* W″ *and controlled resolution data* $\mathscr{A} = (\mathbf{R}^n, (0, Q_2, \ldots, Q_u)\ (k_1, \ldots, k_u), (s_1, \ldots, s_u))$ *and a smooth stratified imbedding* $\gamma : W \rightarrow B(\mathscr{A})$ *so that*:

1) $W = W' \cup W''$ *and* $W' \cap W'' = \partial W' = \partial W'' = \Sigma$;
2) W *has the same number of strata as* Σ;
3) γ *is stable over the fin* $\pi(\mathscr{A}) : T(\mathscr{A}) \rightarrow 0$;
4) *there is a collaring* $\rho : \Sigma \times [0, 1) \rightarrow W'$ *so that*:

$$\text{Image}(\rho) = W' - \gamma^{-1}T(\mathscr{A})$$

(*i.e.* $\gamma(W') \cap T(\mathscr{A})$ *is a spine of* $\gamma(W')$);
5) γ *represents* 0 *in* $\mathscr{N}^A(B(\mathscr{A}) : S(\mathscr{A}))$;
6) $\gamma(W'') \subset B(\mathscr{A}) - T(\mathscr{A})$.

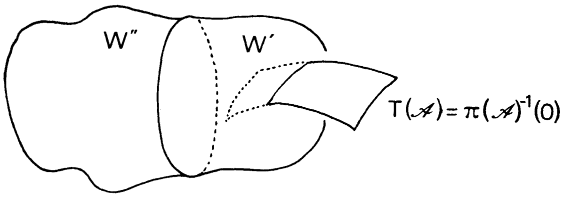


Fig. 7.1

Proof. — The proof will be by induction on the dimension of Σ. If $\dim \Sigma < 0$ then Σ is empty so we may let W, W′ and W″ be empty and $\mathscr{A} = (\mathbf{R}^n, (0), (0), (0))$. So we may assume the proposition is true for dimensions less than the dimension of Σ.

By Lemma (1.4) we may pick a compact A-space Y so that Σ is A-isomorphic to ∂Y and each component of each stratum of Y has nonempty boundary. Now by Proposition (7.1) there are A_i, f_i, σ_i, B_{ij}, Y_{ij}, α_{ij}, V and κ satisfying a)-i) of Proposition (7.1).

Notice that $\dim \partial Y_{ij} < \dim \Sigma$ so for each $i = 1, \ldots, k$ $j = 1, \ldots, a_i$ we may by induction find compact A-spaces W_{ij}, W'_{ij} and W''_{ij} and controlled resolution data $\mathscr{A}_{ij} = (\mathbf{R}^{n-1}, (0, Q_2^{ij}, \ldots, Q_m^{ij}), (k_1^{ij}, \ldots, k_m^{ij}), (s_1^{ij}, \ldots, s_m^{ij}))$ and a smooth stratified morphism $\gamma^{ij} : W_{ij} \rightarrow B(\mathscr{A}_{ij})$ so that γ^{ij} represents 0 in $\mathscr{N}^A(B(\mathscr{A}_{ij}) : S(\mathscr{A}_{ij}))$ an γ^{ij} is stable over the fin $T(\mathscr{A}_{ij}) \rightarrow$ point. Also:

$$W_{ij} = W'_{ij} \cup W''_{ij}, \qquad W'_{ij} \cap W''_{ij} = \partial W'_{ij} = \partial W''_{ij} = \partial Y_{ij}$$

and there are collarings $\rho_{ij}: \partial Y_{ij}\times[0, 1)\to W'_{ij}$ with:

$$\rho_{ij}(\partial Y_{ij}\times[0, 1)) = W'_{ij} - (\gamma^{ij})^{-1}(T(\mathscr{A}_{ij})).$$

In addition, W_{ij} has the same number of strata as ∂Y_{ij} and $\gamma^{ij}(W''_{ij})\subset B(\mathscr{A}_{ij})-T(\mathscr{A}_{ij})$. Note also that $\gamma^{ij}(W_{ij})\cap S(\mathscr{A}_{ij})$ is empty since γ^{ij} is stable and the dimension of W_{ij} is less than the codimension of $S(\mathscr{A}_{ij})$ which is $n-1$.

We define now an A-space Z. Loosely speaking, Z is a boundary connected sum of $Y\times[0, 1]$ with the $(Y_{ij}\cup_{\Sigma} W''_{ij})\times$annulus and $W_{ij}\times$annulus. (See Figures (7.2) and (7.3).) More precisely, let $\psi: [-1, 1]\to S^1$ be a smooth imbedding onto a subset K of the circle S^1. Define Z to be:

$$Y\times[0, 1]\cup\bigcup_{i=1}^{k}\bigcup_{j=1}^{a_i}(Y_{ij}\times[-1, 1]\times[1, 2]$$
$$\cup(Y_{ij}\cup_{\Sigma} W''_{ij})\times S^1\times[2, 3]$$
$$\cup W''_{ij}\times[-1, 1]\times[3, 4]\cup W_{ij}\times S^1\times[4, 5])$$

with the identifications $(\alpha_{ij}(y, t), 1)\in Y\times[0, 1]$ equals $(y, t, 1)\in Y_{ij}\times[-1, 1]\times[1, 2]$ for all $y\in Y_{ij}$, $t\in[-1, 1]$, also $(y, t, 2)\in Y_{ij}\times[-1, 1]\times[1, 2]$ equals:

$$(y, \psi(t), 2)\in(Y_{ij}\cup W''_{ij})\times S^1\times[2, 3] \quad\text{for all } y\in Y_{ij},\quad t\in[-1, 1],$$

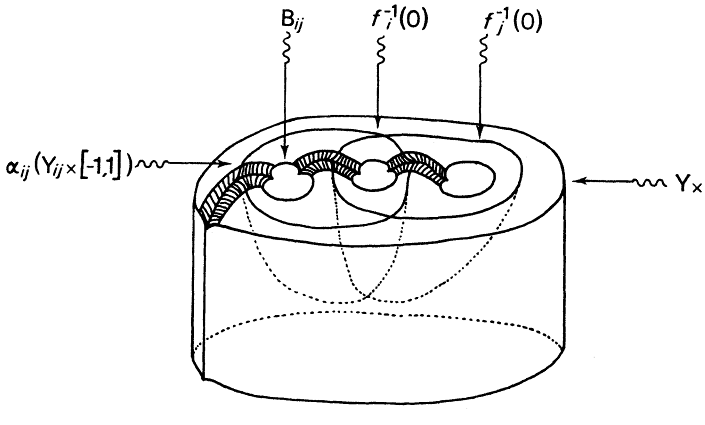
in addition $\partial Y_{ij}\times S^1\times[2, 3] = \partial W''_{ij}\times S^1\times[2, 3]$ and $(y, \psi(t), 2)\in(Y_{ij}\cup W''_{ij})\times S^1\times[2, 3]$ equals $(y, t, 3)\in W''_{ij}\times[-1, 1]\times[3, 4]$ for $y\in W''_{ij}$ and $t\in[-1, 1]$ and also:

$$(y, t, 4)\in W''_{ij}\times[-1, 1]\times[3, 4]$$

equals $(y, \psi(t), 4)\in W_{ij}\times S^1\times[4, 5]$ for all $y\in W''_{ij}$, $t\in[-1, 1]$.

In other words the $Y_{ij}\times[-1, 1]\times[1, 2]$ are tubes reaching from $Y\times[0, 1]$ to $(Y_i\cup W''_{ij})\times S^1\times[2, 3]$ and the $W''_{ij}\times[-1, 1]\times[3, 4]$ are tubes reaching from

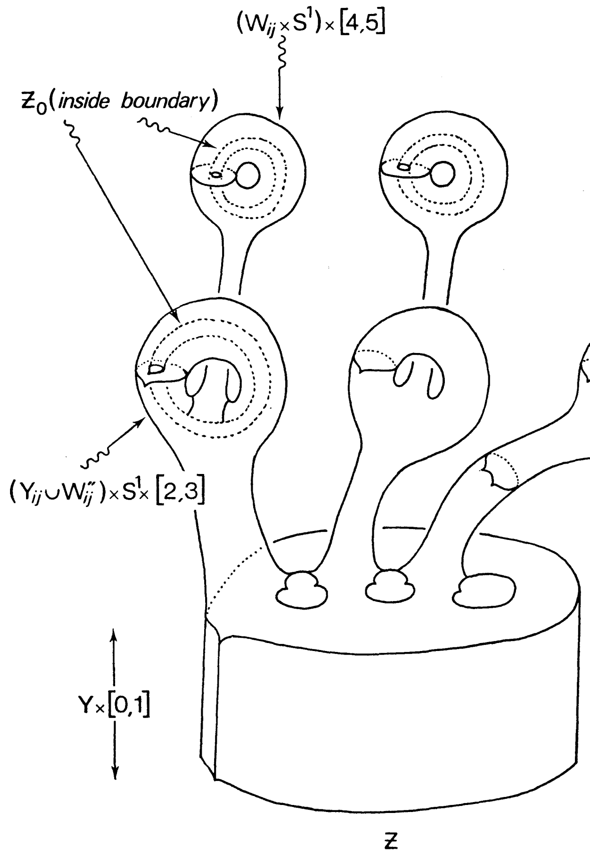
$$(Y_{ij}\cup W''_{ij})\times S^1\times[2, 3] \quad\text{to}\quad W_{ij}\times S^1\times[4, 5].$$



$Y\times[0,1]$

(flower pot)

FIG. 7.2



(second generation growth)

FIG. 7·3

We define $Z_i \subset \partial Z$ $i=0, 1, 2, 3$ as follows:

$$Z_0=\bigcup_{i,j}((Y_{ij}\cup W''_{ij})\times S^1\times 3\cup W_{ij}\times S^1\times 5)$$

$$\begin{aligned}Z_1=(V-\bigcup_{i,j}\alpha_{ij}(\operatorname{int} Y_{ij}\times[-1,1]))\times 1\cup\bigcup_{i,j}(\partial Y_{ij}\times[-1,1]\times[1,2]\\ \cup\,\partial W''_{ij}\times[-1,1]\times[3,4]\cup W'_{ij}\times S^1\times 4\end{aligned}$$

$$\begin{aligned}Z_2=(\partial(Y\times[0,1])-V\times 1)\cup\bigcup_{i,j}(Y_{ij}\times\{-1,1\}\times[1,2]\\ \cup(Y_{ij}\cup W''_{ij})\times(S^1-K)\times 2\cup W''_{ij}\times\{-1,1\}\times[3,4]\cup W''_{ij}\times(S^1-K)\times 4)\end{aligned}$$

$$\begin{aligned}Z_3=(\partial V-\bigcup_{i,j}\alpha_{ij}(Y_{ij}\times\{-1,1\}))\times 1\cup\bigcup_{i,j}(\partial Y_{ij}\times\{-1,1\}\times[1,2]\\ \cup\,\partial W''_{i_j}\times\{-1,1\}\times[3,4]\cup\partial W''_{ij}\times(S^1-K)\times 4).\end{aligned}$$

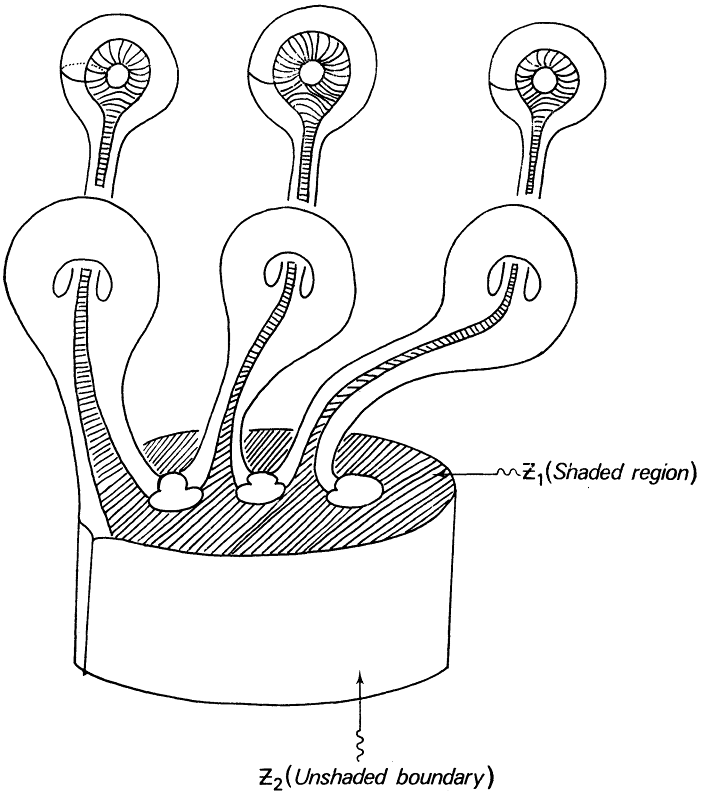


FIG. 7.4

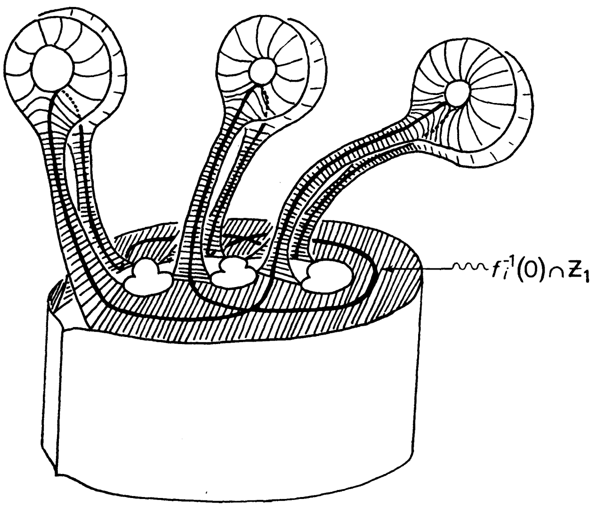
Notice that Z is a bordism from Z_0 to $Z_1 \cup Z_2$ and also $Z_1 \cap Z_2 = Z_3 = \partial Z_1 = \partial Z_2$. Also, Z_3 is A-isomorphic to Σ (see Figure (7.4)) since Z_3 is A-isomorphic with:

$$\partial(V - \bigcup_{i=1}^{k} \bigcup_{j=1}^{a_i} \alpha_i(\operatorname{int} Y_{ij} \times [-1, 1])$$

which by Proposition (7.1) d) is A-isomorphic with ∂Y.

Our plan is to let $W = Z_1 \cup Z_2$, $W' = Z_1$ and $W'' = Z_2$. We will find controlled resolution data $\mathscr{A}$ and a map $\gamma : W \to B(\mathscr{A})$ with the desired properties. This Z will be the bordism we use to show that γ represents 0 in $\mathscr{N}^{A}(B(\mathscr{A}) : S(\mathscr{A}))$.

Let $S = B_k(0 \times \mathbf{R}^2, 0) \subset B_k(\mathbf{R}^n \times \mathbf{R}^2, 0) = B$, let $\pi : B \to \mathbf{R}^n \times \mathbf{R}^2$ be $\pi_k(\mathbf{R}^n \times \mathbf{R}^2, 0)$ let $\lambda = \lambda_k(\mathbf{R}^n \times \mathbf{R}^2, 0)$ and $S_i = S_{ki}(\mathbf{R}^n \times \mathbf{R}^2, 0)$, $S_i' = \bigcup_{\substack{j=0 \\ j \neq i}}^{k} S_j$ for $i = 0, 1, 2, \ldots, k$. Note that $\pi^{-1}(0 \times \mathbf{R}^2) = S \cup \pi^{-1}(0) = S \cup \bigcup_{i=1}^{k} S_i$ by Lemma (4.3). By Lemma (4.4) there is an imbedding $\alpha : (\mathbf{R}^n \times \mathbf{R}^2 \times \mathbf{R}^k, 0) \to (B, N_k(\mathbf{R}^n \times \mathbf{R}^2, 0))$ so that $\alpha^{-1}(S) = 0 \times \mathbf{R}^2 \times \mathbf{R}^k$,



Z_1 (*Shaded region*)

FIG. 7.5

$\alpha^{-1}(S_0) = 0 \times 0 \times \mathbf{R}^k$ and $\alpha^{-1}(S_i) = \mathbf{R}^n \times \mathbf{R}^2 \times R_i$ where $R_i = \{(x_1, \ldots, x_k) \in \mathbf{R}^k \mid x_i = 0\}$ $i = 1, 2, \ldots, k$. Define $\beta : Y \times [0, 1] \to \mathbf{R}^k$ by:

$$\beta(y, t) = (f_1(y), \ldots, f_k(y)) + (1-t)(2, 2, 2, \ldots, 2).$$

Define $\gamma_1 : Y \times [0, 1] \to B$ by $\gamma_1(y, t) = \alpha((1, 0, \ldots, 0), (0, 0), \beta(y, t))$. Notice:

$$\gamma_1(Y \times [0, 1] \cap Z_2) \subset \pi^{-1}((\mathbf{R}^n - 0) \times \mathbf{R}^2)$$

and

$$\gamma_1^{-1}\pi^{-1}(0 \times \mathbf{R}^2) \cap Z_1 = \gamma_1^{-1}(S \cup \bigcup_{i=1}^{k} S_i) \cap Z_1$$
$$= \beta^{-1}(\bigcup_{i=1}^{k} R_i) \cap Z_1 = (\bigcup_{i=1}^{k} A_i \times 1) \cap Z_1.$$

Also, by Proposition (7.1) a) and Lemma (5.4) a), $\gamma_1\big|_{Y \times 1 \cap Z_1}$ is stable over the fin $\pi^{-1}(0 \times \mathbf{R}^2) \to 0$.

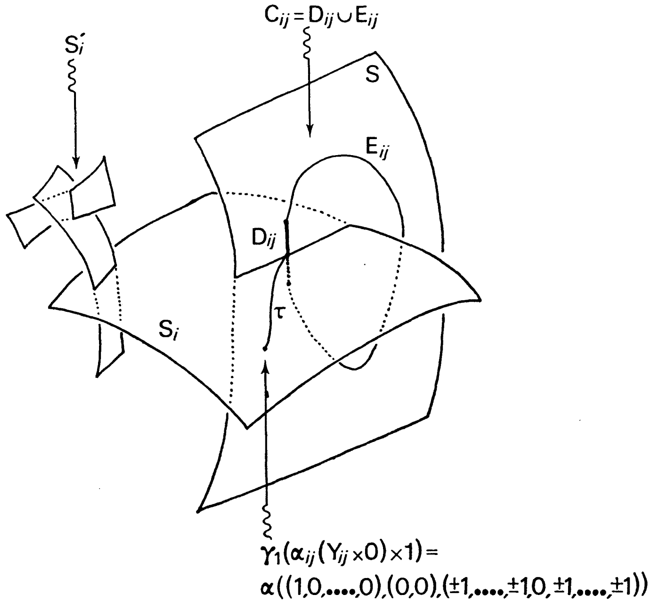
Let $q_{ij} \in S_i - S_i'$ be the point

$$\alpha((1, 0, 0, \ldots, 0), (0, 0), (\sigma_{1ij}, \sigma_{2ij}, \ldots, \sigma_{i-1, ij}, 0, \sigma_{i+1i, j}, \ldots, \sigma_{kij}))$$

$i = 1, \ldots, k$ $j = 1, \ldots, a_i$. Notice that $\alpha_{ij}(Y_{ij} \times 0) \times 1 \subset \gamma_1^{-1}(q_{ij})$. There is a smooth imbedded circle $C_{ij} \subset S - S_i'$ so that C_{ij} is transverse to S_i, $C_{ij} \cap S_i$ is a point and $C_{ij} \cap S_i$ is in the same path component of $S_i - S_i'$ as q_{ij}. To see this, take a smooth arc D_{ij} in $S - S_i'$ transverse to S_i so that $D_{ij} \cap S_i$ is a point and $D_{ij} \cap S_i$ is in the same path component of $S_i - S_i'$ as q_{ij}. For instance we may let the arc D_{ij} be:

$$\alpha(0, (1, 0), (\sigma_{1ij}, \sigma_{2ij}, \ldots, [-1, 1], \ldots, \sigma_{kij})).$$

Let a_{ij} and b_{ij} be the endpoints of D_{ij}. Then we may pick a smooth arc E_{ij} in $(0\times R^2-0)\times R^k$ from $\lambda^{-1}(a_{ij})$ to $\lambda^{-1}(b_{ij})$. We may then construct C_{ij} by smoothing out the two corners of the circle $D_{ij}\cup\lambda(E_{ij})$.



The arc's τ and D_{ij} are given by:

$D_{ij}: t \rightsquigarrow \alpha((0, ., 0), (1, 0), (\pm 1, \pm 1, ., t, \pm 1, ., \pm 1))$

$\tau \quad : t \rightsquigarrow \alpha((t, 0, ., 0), (1-t, 0), (\pm 1, ., \pm 1, 0, \pm 1, ., \pm 1))$

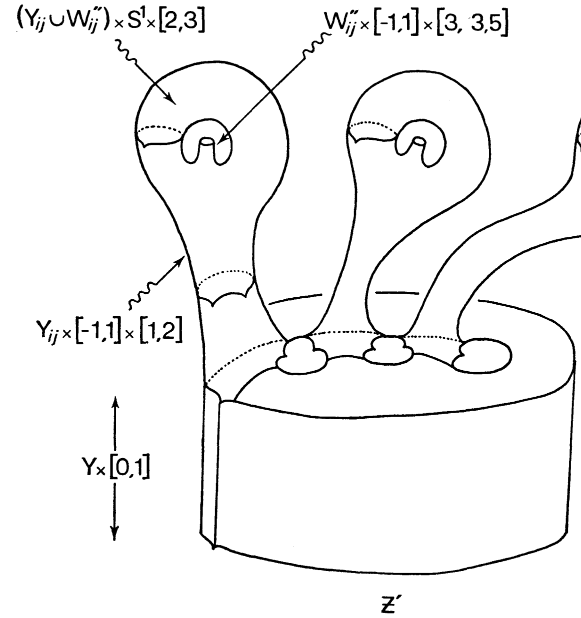
FIG. 7.6

Now by Proposition (6.1), $H_i^A(S)=H_i(S, Z/2Z)$ for all i so by Lemma (2.1) and Proposition (2.3) we may find an s and pairwise disjoint nonsingular projectively closed algebraic subsets $L_{ijr}\subset S\times\mathbf{R}^s$ which are two parallel copies of C_{ij}, i.e. for $i=1, \ldots, k$ $j=1, \ldots, a_i$ $r=0, 1$, each L_{ijr} is isotopic in $S\times\mathbf{R}^s$ to $C_{ij}\times 0$ by a small isotopy. In particular, $L_{ijr}\subset(S-S_i')\times\mathbf{R}^s$, $S_i\times\mathbf{R}^s\cap L_{ijr}$ is a point in the same component of $(S_i-S_i')\times\mathbf{R}^s$ as $(q_{ij}, 0)$, and L_{ijr} is transverse to $S_i\times\mathbf{R}^s$.

We now define $\mathscr{A}'=(\mathbf{R}^n, (0), (k), (s))$ (so $B(\mathscr{A}')=B\times\mathbf{R}^s$). Define a full A-subspace $Z'\subset Z$ by:

$$Z'=Y\times[0, 1]\cup\bigcup_{i,j}(Y_{ij}\times[-1, 1]\times[1, 2]$$
$$\cup(Y_{ij}\cup W_{ij}'')\times S^1\times[2, 3]\cup W_{ij}''\times[-1, 1]\times[3, 7/2]$$

(see Figure 7.7).



(first generation of growth)

FIG. 7.7

We also define:

$$Z_{ij} = Z' \cup \bigcup_{v,w \in A_{ij}} (W''_{v,w} \times [-1, 1] \times [7/2, 4] \cup W_{v,w} \times S^1 \times [4, 5])$$

where $A_{ij} = \{(v, w) \mid v = 1, 2, \ldots, i;\ w = 1, \ldots, a_v \text{ and } w \leq j \text{ if } v = i\}$

$= \{(v, w) \mid (v, w) \leq (i, j) \text{ where} < \text{is the lexicographical order}\}.$

Notice that $Z_{u,v} \subset Z_{ij}$ if $(u, v) \leq (i, j)$.

Our plan will be to extend $\gamma_1 \times 0$ to $\gamma_2 : Z' \to B(\mathscr{A}') = B \times \mathbf{R}^s$. We will then blow up $B(\mathscr{A}')$ some and extend γ_2 to $\gamma_{11} : Z_{11} \to$ blown up $B(\mathscr{A}')$, then blow up some more and extend to $\gamma_{12} : Z_{12} \to$ more blown up $B(\mathscr{A}')$, etc. until we have:

$$\gamma_{ka_k} : Z_{ka_k} = Z \to \text{very blown up } B(\mathscr{A}').$$

For a final step we will change $\gamma_{ka_k}\big|_W$ a little bit so that it becomes an imbedding.

Notice $S \times \mathbf{R}^s = S(\mathscr{A}')$ has codimension n in $B(\mathscr{A}') = B \times \mathbf{R}^s$ so the normal bundle of $S \times \mathbf{R}^s$ restricted to the circle L_{ijr} contains a trivial $n-1$ bundle. Hence there are imbeddings $\mu_{ijr} : S^1 \times \mathbf{R}^{n-1} \to B(\mathscr{A}') - S'_i \times \mathbf{R}^s$ so that $\mu_{ijr}(S^1 \times 0) = L_{ijr}$, $\mu_{ijr}^{-1}(S(\mathscr{A}')) = S^1 \times 0$, $\mu_{ijr}(S^1 \times \mathbf{R}^{n-1})$ intersects $S(\mathscr{A}')$ cleanly and $\mu_{ijr}^{-1}(S_i \times \mathbf{R}^s) = \psi(0) \times \mathbf{R}^{n-1}$. There are also imbeddings $\varphi_{ij} : [-1, 1] \times [0, 2] \to B(\mathscr{A}') - (S'_i \cup S) \times \mathbf{R}^s$ so that:

$$\varphi_{ij}^{-1}(S_i \times \mathbf{R}^s) = 0 \times [0, 2],$$

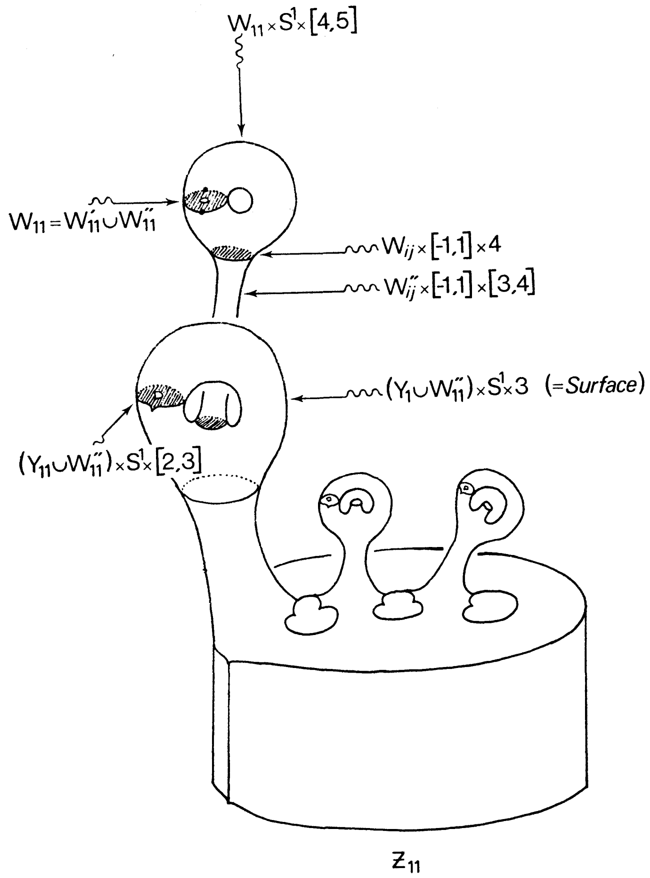


FIG. 7.8

φ_{ij} is transverse to $S_i\times\mathbf{R}^s$, and for some $p\in\mathbf{R}^{n-1}-0$:

$$\varphi_{ij}(t,1)=\mu_{ij0}(\psi(t),p), \qquad \varphi_{ij}(t,2)=\mu_{ij1}(\psi(t),p)$$

and
$$\varphi_{ij}(t,0)=\alpha((1,0,\ldots,0),(0,0),(\sigma_{1ij},\ldots,t\sigma_{iij},\ldots,\sigma_{kij}))$$

for all $t\in[-1,1]$. (See Fig. 7.9.)

We define $\gamma_2:Z'\rightarrow B(\mathscr{A}')$ by:

$$\begin{aligned}
&\gamma_2(z)=\gamma_1(z)\times 0 && \text{for } z\in Y\times[0,1],\\
&\gamma_2(y,t,u)=\varphi_{ij}(t,u-1) && \text{for } (y,t,u)\in Y_{ij}\times[-1,1]\times[1,2],\\
&\gamma_2(y,t,u)=\mu_{ij0}(t,(3-u)p) && \text{for } (y,t,u)\in(Y_{ij}\cup W''_{ij})\times S^1\times[2,3]\\
\text{and}\quad &\gamma_2(y,t,u)=\varphi_{ij}(t,2u-5) && \text{for } (y,t,u)\in W''_{ij}\times[-1,1]\times[3,7/2].
\end{aligned}$$

In particular γ_2 first crushes:

$$Y_{ij}\times[-1,1]\times[1,2]\cup(Y_{ij}\cup W''_{ij})\times S^1\times[2,3]\cup W''_{ij}\times[-1,1]\times[3,7/2]$$

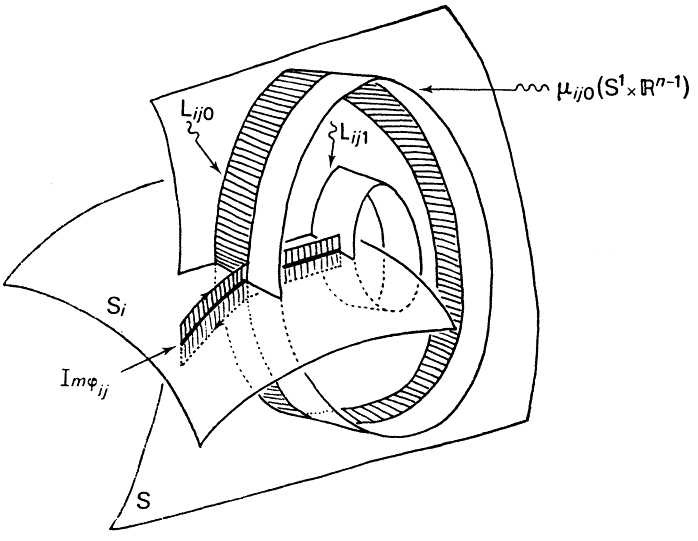


FIG. 7.9

to a disc union an annulus in an obvious way (going from Figure 7.7 to Figure 7.11) then maps them into $B(\mathscr{A}')$ as in Figure 7.9 (i.e. the shaded region in Figure 7.10 gets mapped to the shaded region in Figure 7.9).

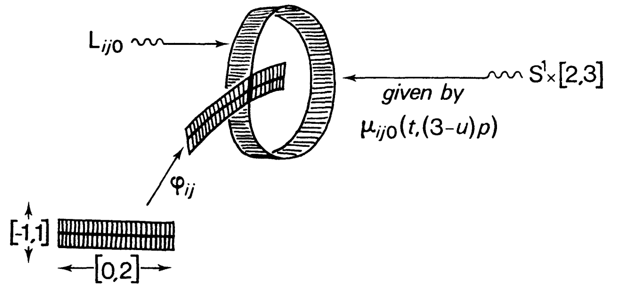


FIG. 7.10

Notice that

$$\gamma_2(Z_2\cap Z')\subset B(\mathscr{A}')-T(\mathscr{A}'),$$

$$\gamma_2^{-1}(T(\mathscr{A}'))\cap Z_1$$
$$=(\bigcup_{i=1}^{k} A_i\times 1\cap Z_1)\cup\bigcup_{i,j}(\partial Y_{ij}\times 0\times[1,2]\cup\partial W_{ij}''\times 0\times[3,7/2]),$$

$\gamma_2\Big|_{Z_1\cap Z'}$ is stable over the fin $T(\mathscr{A}')\to$ point and $\gamma_2\Big|_{Z_0\cap Z'}$ represents 0 in:

$$\mathscr{N}^A(B(\mathscr{A}'):S(\mathscr{A}'))\quad(\text{note } S(\mathscr{A}')=B_k(0\times\mathbf{R}^2,0\times 0)\times\mathbf{R}^s=S\times\mathbf{R}^s).$$

The stability of $\gamma_2\Big|_{Z_1\cap Z'}$ follows from stability of γ_1, Lemma (5.4) a) and the fact that the φ_{ij} are transverse to $S_i\times\mathbf{R}^s$.

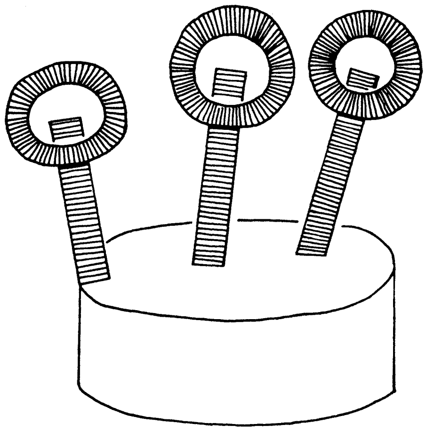


FIG. 7.11

We now consider the resolution data

$$\mathscr{A}'_{11}=\mu_{111\#}(\mathrm{S}^1\times\mathscr{A}^{11})$$
$$=(\mathrm{B}(\mathscr{A}'),\ (\mathrm{L}_{111},\ \mathrm{L}_{111}\times\mathrm{Q}_2^{11},\ \ldots,\ \mathrm{L}_{111}\times\mathrm{Q}_m^{11}),\ (k_1^{11},\ \ldots),\ (s_1^{11},\ \ldots))$$

and $$\mathscr{A}''_{11}=\mathscr{A}'_{11}*\mathscr{A}'=(\mathbf{R}^n,\ (0,\ \mathrm{L}_{111},\ \mathrm{L}_{111}\times\mathrm{Q}_2^{11},\ \ldots),\ (k,\ k_1^{11},\ \ldots),\ (s,\ s_1^{11},\ \ldots)).$$

Note $\mathrm{B}(\mathscr{A}'_{11})=\mathrm{B}(\mathscr{A}''_{11})$. The imbedding $\mu_{111}:\mathrm{S}^1\times\mathbf{R}^{n-1}\to\mathrm{B}(\mathscr{A}')$ gives us the imbedding $\mu_{111*}:\mathrm{S}^1\times\mathrm{B}(\mathscr{A}_{11})\to\mathrm{B}(\mathscr{A}'_{11})=\mathrm{B}(\mathscr{A}''_{11})$. Notice that $\mathscr{A}'_{11}$ is controlled resolution data by Lemma (4.7) and $\mathscr{A}''_{11}$ is controlled resolution data by Lemma (4.8). We may now define $\gamma_{11}:\mathrm{Z}_{11}\to\mathrm{B}(\mathscr{A}''_{11})$ by:

$$\gamma_{11}(x)=\lambda(\mathscr{A}'_{11})(\gamma_2(x),\ 0)\qquad\text{for}\quad x\in\mathrm{Z}'$$

and $$\gamma_{11}(z,\ t,\ u)=\mu_{111*}(t,\ \gamma^{11}(z))\qquad\text{for}\quad(z,\ t,\ u)\in\mathrm{W}_{11}\times\mathrm{S}^1\times[4,\ 5].$$

Notice in particular if we let $\lambda(\mathscr{A}_{11})^{-1}\gamma^{11}(z)=(\varphi''(z),\ \varphi'(z))\in(\mathbf{R}^{n-1}-0)\times\mathbf{R}^q$ for some q for $z\in\mathrm{W}''_{11}$ then

$$\gamma_{11}(z,\ t,\ u)=\mu_{111*}(t,\ \lambda(\mathscr{A}_{11})(\varphi''(z),\ \varphi'(z)))$$
$$=\lambda(\mathscr{A}'_{11})(\mu_{111}(t,\ \varphi''(z)),\ \varphi'(z))$$

(by the following commutative diagram):

$$\begin{tikzcd}
\mathrm{B}(\mathscr{A}')\times\mathbf{R}^q\supset(\mathrm{B}(\mathscr{A}')-\mathrm{L}_{111})\times\mathbf{R}^q \arrow[drr, "\lambda(\mathscr{A}'_{11})"] & & \\
\mathrm{S}^1\times(\mathbf{R}^{n-1}-0)\times\mathbf{R}^q \arrow[u, "\mu_{111}\times\mathrm{id}"] \arrow[r, hook, "\mathrm{id}\times\lambda(\mathscr{A}_{11})"] & \mathrm{S}^1\times(\mathrm{B}(\mathscr{A}_{11})-\mathrm{T}(\mathscr{A}_{11})) \arrow[r, "\mu_{111*}"] & \mathrm{B}(\mathscr{A}'_{11}) \\
\mathrm{S}^1\times\mathrm{W}''_{11}\times\{4\} \arrow[u, "\mathrm{id}\times\varphi''\times\varphi'"] \arrow[urr, dashed] & &
\end{tikzcd}$$

we then extend γ_{11} on the tube $(=W''_{11}\times[-1, 1]\times[7/2, 4])$ connecting Z' to $W_{11}\times S^1\times[4, 5]$ (see Figure 7.8). Since $\dim W''_{11}<n-2$ there is a homotopy

$$\varphi''_u : W''_{11}\to \mathbf{R}^{n-1}-0 \quad 7/2\leq u\leq 4 \qquad \text{with} \qquad \varphi''_{7/2}(z)=p \quad \varphi''_4(z)=\varphi''(z).$$

Let $\gamma_{11}(z, t, u)=\lambda(\mathscr{A}'_{11})(\mu_{111}(\psi(t), \varphi''_u(z)), (2u-7)\varphi'(z))$; then it is easy to check that γ_{11} is well defined.

We claim that:

a) $\gamma_{11}(Z_2\cap Z_{11})\subset B(\mathscr{A}''_{11})-T(\mathscr{A}''_{11})$

b) $$\gamma_{11}^{-1}(T(\mathscr{A}''_{11}))\cap Z_1=(\bigcup_{i=1}^{k} A_i\times 1\cap Z_1)\cup\bigcup_{i,j}(\partial Y_{ij}\times 0\times[1, 2]$$
$$\cup\,\partial W''_{ij}\times 0\times[3, 7/2])\cup\partial W''_{ij}\times 0\times[7/2, 4]\cup W'_{11}\times\psi(0)\times 4$$
$$\cup(W_{11}\cap(\gamma^{11})^{-1}(T(\mathscr{A}_{11})))\times S^1\times 4$$

c) $\gamma_{11}\big|_{Z_1\cap Z_{11}}$ is stable over the fin $T(\mathscr{A}''_{11})\to$point

d) $\gamma_{11}\big|_{Z_0\cap Z_{11}}$ represents 0 in $\mathscr{N}^{A}(B(\mathscr{A}''_{11}) : S(\mathscr{A}''_{11}))$.

Claims *a)* and *b)* follow immediately from the definition of γ_{11}. To see claim *d)* note that $\gamma^{11} : W_{11}\to B(\mathscr{A}_{11})$ is bordant to a map $\eta : X\to S(\mathscr{A}_{11})$ which represents 0 in $\mathscr{N}^{A}(S(\mathscr{A}_{11}) : S(\mathscr{A}_{11}))$. Hence $\gamma_{11}\big|_{W_{11}\times S^1\times 5}$ is bordant to

$$\mathrm{id}\times\eta : L_{111}\times X\to L_{111}\times S(\mathscr{A}_{11})=S(L_{111}\times\mathscr{A}_{11})\subset S(\mathscr{A}'_{11})\subset S(\mathscr{A}''_{11}).$$

But $\mathrm{id}\times\eta$ represents 0 in $\mathscr{N}^{A}(L_{111}\times S(\mathscr{A}_{11}) : L_{111}\times S(\mathscr{A}_{11}))$ by Lemma (2.2) c), hence it represents 0 in $\mathscr{N}^{A}(S(\mathscr{A}''_{11}) : S(\mathscr{A}''_{11}))$ by Lemma (2.2) e). To see that $\gamma_{11}\big|_{Z_0\cap Z'}$ represents 0 in $\mathscr{N}^{A}(B(\mathscr{A}''_{11}) : S(\mathscr{A}''_{11}))$ use Lemma (2.2) f).

We must now show claim *c)*, that $\gamma_{11}\big|_{Z_1\cap Z_{11}}$ is stable over the fin $T(\mathscr{A}''_{11})\to$point. Lemma (5.4) a) shows us that $\gamma_{11}\big|_{Z_1\cap Z_{11}-W'_{11}\times S^1\times 4}$ is stable. Lemma (6.3) shows us that $\mu_{111*}\big|_{S^1\times(B(\mathscr{A}_{11})-S(\mathscr{A}_{11}))}$ is stable so by Lemmas (5.4) c) and (5.4) d), $\gamma_{11}\big|_{W'_{11}\times S^1\times 4}$ is stable, so *c)* is demonstrated.

In the same manner we may find $\gamma_{12} : Z_{12}\to B(\mathscr{A}''_{12})$ where:

$$\mathscr{A}''_{12}=\mu'_{121\#}(S^1\times\mathscr{A}^{12}) * \mathscr{A}''_{11} \qquad \text{and} \qquad \mu'_{121} : S^1\times\mathbf{R}^{n-1}\to B(\mathscr{A}''_{11})$$

is defined by $\mu'_{121}(x,y)=\lambda(\mathscr{A}'_{11})(\mu_{121}(x,y), 0)$.

By repeating this process we eventually find $\gamma_{ka_k} : Z_{ka_k}\to B(\mathscr{A}''_{ka_k})$. But $Z=Z_{ka_k}$ so we may let $\gamma'=\gamma_{ka_k}$ and $\mathscr{A}''=\mathscr{A}''_{ka_k}$. This γ' was constructed so that:

a) $\gamma'(Z_2)\subset B(\mathscr{A}'')-T(\mathscr{A}'')$

b) $$\gamma'^{-1}(T(\mathscr{A}''))\cap Z_1=(\bigcup_{i=1}^{k} A_i\times 1\cap Z_1)\cup\bigcup_{i,j}(\partial Y_{ij}\times 0\times[1, 2]$$
$$\cup\,\partial W''_{ij}\times 0\times[3, 4]\cup W'_{ij}\times\psi(0)\times 4$$
$$\cup(W_{ij}\cap(\gamma^{ij})^{-1}(T(\mathscr{A}_{ij})))\times S^1\times 4)=\text{a “spine” of } Z_1.$$

c) $\gamma'\big|_{Z_1}$ is stable over the fin $T(\mathscr{A}'') \to$ point

d) $\gamma'\big|_{Z_0}$ represents 0 in $\mathscr{N}^{\Lambda}(B(\mathscr{A}'') : S(\mathscr{A}''))$.

Figure 7.12 geometrically describes the map γ', note that it factors through the complex C.

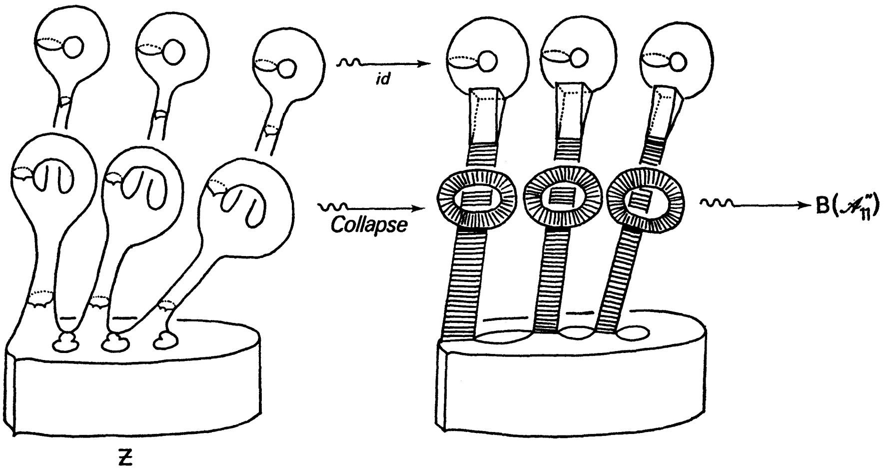


Fig. 7.12

The map γ' is almost everything we want except that $\gamma'\big|_{W}$ might not be an imbedding. This is easily fixed up. By Lemma (1.1) there is a smooth stratified imbedding $\beta : W \to \mathbf{R}^m$ for some integer m. Let $\mathscr{A} = (B(\mathscr{A}''), (\emptyset), (0), (m)) * \mathscr{A}''$, so $B(\mathscr{A}) = B(\mathscr{A}'') \times \mathbf{R}^2 \times \mathbf{R}^m$. Then we let $\gamma : W \to B(\mathscr{A})$ be $\gamma = \gamma' \times 0 \times \beta$. Then γ satisfies all our required properties.

We may define $\rho : Z_3 \times [0, 1) \to Z_1 - \gamma^{-1}T(\mathscr{A})$ by $\rho((y, 1), t) = (\kappa(y, t), 1)$ for $t \in [0, 1)$, $y \in \partial V - \bigcup_{i,j} \alpha_{ij}(Y_{ij} \times \{-1, 1\})$ where κ comes from Proposition (7.1) and

$$\rho((y, \pm 1, u), t) = (y, \pm t, u), \text{ for } t \in [0, 1),$$

$$(y, \pm 1, u) \in \partial Y_{ij} \times \{-1, 1\} \times [1, 2] \cup \partial W''_{ij} \times \{-1, 1\} \times [3, 4]$$

and $$\rho((y, u, 4), t) = (\rho_{ij}(y, \delta(u, t)), \varepsilon(u, t), 4) \text{ for } y \in \partial W''_{ij},\ u \in S^1 - K,\ t \in [0, 1)$$

where $(\delta, \varepsilon) : (S^1 - K) \times [0, 1) \to (S^1 - \psi(0)) \times [0, 1)$ is a homeomorphism with:

$$\delta(\psi(\pm 1), t) = \psi(\pm t) \quad \text{and} \quad (\delta, \varepsilon)(u, 0) = (u, 0) \quad \text{for} \quad u \in S^1 - K.$$

Figures 7.5 and 7.13 describe the inverse images $\gamma'^{-1}(\bigcup S_i)$ on Z and on Z_1. Figure 7.5 factors through 7.14 under the map γ'. ■

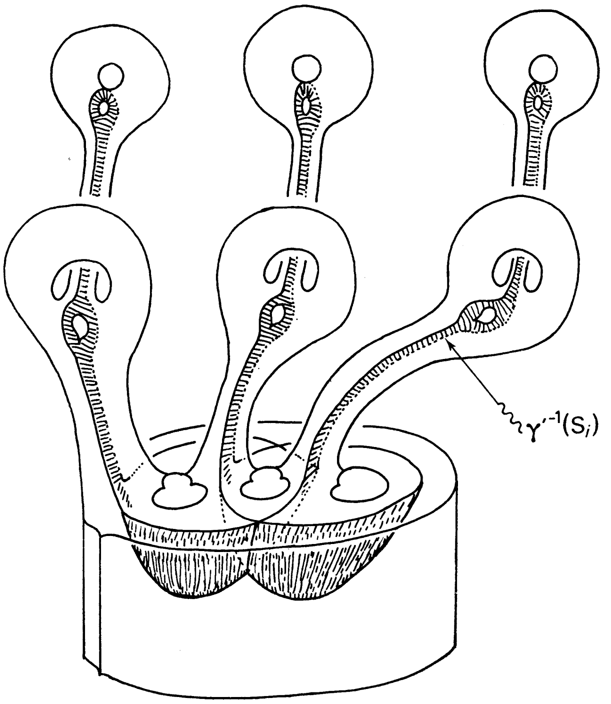


FIG. 7.13

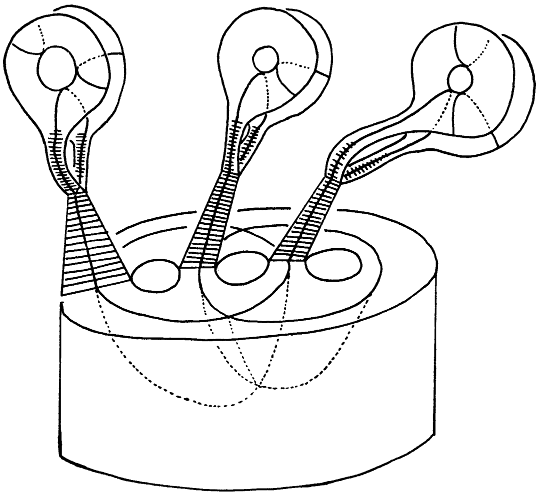


FIG. 7.14

VIII. — THE MAIN THEOREM

We have finally done enough preparation to be able to prove the main theorems we are after. Theorems (8.1) and (8.2) are essentially corollaries of Theorem (8.3) the theorem which gives the actual proof. We do not explicitly do the tower construction of Section 0, but this construction is implicit in the proof. Instead we prove (8.3) by an inductive argument so that we only have to do one story of the tower construction. The fins are there to keep track of the collapsing we want to do.

More explicitly, suppose we wish to show an A-space Y is homeomorphic to an algebraic set. The idea is to find controlled resolution data $\mathscr{A}_i$, overt polynomials $q_i : \mathrm{B}(\mathscr{A}_{i-1}) \rightarrow \mathrm{R}$, and a tower of imbeddings

$$\begin{tikzcd}
\mathrm{Y}_m \subset \mathrm{B}(\mathscr{A}_m) \arrow[d, "\pi(\mathscr{A}_m)"'] \\
\mathrm{N}_m \subset \mathrm{Y}_{m-1} \subset \mathrm{B}(\mathscr{A}_{m-1}) \arrow[d] \\
\vdots \arrow[d] \\
\mathrm{N}_2 \subset \mathrm{Y}_1 \subset \mathrm{B}(\mathscr{A}_1) \arrow[d, "\pi(\mathscr{A}_1)"'] \\
\mathrm{N}_1 \subset \mathrm{Y}_0 \subset \mathrm{B}(\mathscr{A}_0)
\end{tikzcd}$$

so that:

1) $\mathrm{N}_i = q_i^{-1}(0) \quad i=1, \dots, m$;
2) $\mathscr{A}_i = (\mathrm{B}(\mathscr{A}_{i-1}), (\mathrm{N}_i, \mathrm{Q}_{i1}, \dots), \dots, \dots) \quad i=1, \dots, m$;
3) N_i is a closed stratum of Y_{i-1} $i=1, \dots, m$;
4) Y_i is an A blowup of Y_{i-1} along N_i $i=1, \dots, m$;
5) for each $i=1, \dots, m$, Y_i is stable over the compatible fins $\pi_{ij} : \mathrm{V}_{ij} \rightarrow \mathrm{N}_j$ $j=1, \dots, i$ where $\pi_{ij} = \pi(\mathscr{A}_j) \circ \pi(\mathscr{A}_{j+1}) \circ \dots \circ \pi(\mathscr{A}_i)$ and $\mathrm{V}_{ij} = \pi_{ij}^{-1}(\mathrm{N}_j)$;
6) for each $i=1, 2, \dots, m$, $\mathscr{D}_{q_i}(\mathrm{Y}_i)$ is isotopic to Y_{i-1} over the fins $\pi_{i-1,j} : \mathrm{V}_{i-1,j} \rightarrow \mathrm{N}_j$ $j=1, \dots, i-1$;
7) Y_m, Q_{ij} and $\mathrm{B}(\mathscr{A}_0)$ are nonsingular algebraic sets;
8) $\mathrm{Y}=\mathrm{Y}_0$.

Suppose we can construct k stories of this tower so we have $\mathrm{Y}_k \subset \mathrm{B}(\mathscr{A}_k)$ and $\mathrm{N}_{k+1} \subset \mathrm{Y}_k$ so N_{k+1} is a nonsingular projectively closed algebraic set. Then in the proof of Theorem (8.3) we are essentially showing that we can construct the next story of the tower $\mathrm{Y}_{k+1} \subset \mathrm{B}(\mathscr{A}_{k+1})$.

Theorem **(8.1).** — *Let* X *be a compact* A-*space. Then* int X *is homeomorphic to a real algebraic set* W.

Proof. — Let $Y = X \cup c\partial X$ where $\partial X \times 1 \subset c\partial X$ is identified with $\partial X \subset X$. Then Y is a compact A-space without boundary. By Lemma (1.1) there exists a smooth stratified imbedding $\alpha : Y \to \mathbf{R}^n$. Note that α is stable over the empty fin in $\mathbf{R}^n$. Hence by Theorem (8.3) below there is an imbedding H_1 of Y onto a real algebraic set $V \subset \mathbf{R}^n \times \mathbf{R}^b$. By Proposition (4.2) of [1] there is a real algebraic set W homeomorphic to $V - H_1(*)$ where $*$ is the vertex of $c\partial X$. But $V - H_1(*)$ is homeomorphic to $Y - *$ which is homeomorphic to int X. ■

Remark. — In fact it follows from the proof of (8.3) that the natural singular stratification of the real algebraic set W is isomorphic with Y as a stratified set.

Theorem **(8.2).** — *The interior of any compact P.L. manifold is homeomorphic to a real algebraic set.*

Proof. — By [3] every P.L. manifold is homeomorphic to an A-space in such a way that boundaries go to boundaries. The result now follows from Theorem (8.1). ■

Theorem **(8.3).** — *Let* M *be a nonsingular algebraic set and let* $\pi_i : V_i \to N_i$ $i = 1, \ldots, k$ *be compatible fins in* M. *Let* Y *be a compact* A-*space without boundary and suppose* $\alpha : Y \to M$ *is a smooth stratified imbedding which represents* 0 *in* $\mathcal{N}^A(M : M)$ *and suppose* α *is stable over the fins* π_i.

Then for some b *there is a projectively closed real algebraic set* $V \subset M \times \mathbf{R}^b$ *and smooth stratified isotopy* $H_t : Y \to M \times \mathbf{R}^b$, $t \in [0, 1]$ *over the fins* π_i *so that* $H_0 = \alpha \times 0$ *and* $H_1(Y) = V$.

Proof. — Take any closed stratum N of Y. Then by Lemma (2.2) a), $\alpha\big|_N : N \to M$ represents 0 in $\mathcal{N}^A(M : M)$ so by Proposition (2.3) there is a p and a C^1 small isotopy of $\alpha(N) \times 0$ to a projectively closed nonsingular algebraic set $Q \subset M \times \mathbf{R}^p$. By Proposition (5.5) and Lemma (5.3) there is a small smooth isotopy $h_t : M \times \mathbf{R}^p \to M \times \mathbf{R}^p$ $t \in [0, 1]$ over the extended fins $V_i \times \mathbf{R}^p \to N_i$ so that h_0 is the identity and:

$$h_1(\alpha(N) \times 0) = Q.$$

(Here we use stability of $\alpha\big|_N$.) Note that by Lemma (5.6), $h_1 \circ (\alpha \times 0) : Y \to M \times \mathbf{R}^p$ is stable over the extended fins. Also $h_1 \circ (\alpha \times 0)$ represents 0 in $\mathcal{N}^A(M \times \mathbf{R}^p : M \times \mathbf{R}^p)$ since it is homotopic to $\alpha \times 0$ which represents 0 by Lemma (2.2) f).

Hence we have reduced to the case where $\alpha(N)$ is a nonsingular projectively closed algebraic subset $Q \subset M$. Notice in particular that if Y had just one stratum then we would be done since then $\alpha(Y) = \alpha(N) = Q$.

Now let $c : N \times \overset{\circ}{c}\Sigma \to Y$ be a neighborhood trivialization of N in Y and pick any $n > \dim \Sigma + 2$. Then by Proposition (7.2) there are compact A-spaces W, W′ and W″, controlled resolution data $\mathscr{A} = (\mathbf{R}^n, (0, Q_2, \ldots, Q_u), (k_1, \ldots, k_u), (s_1, \ldots, s_u))$, a smooth stratified imbedding $\gamma : W \to B(\mathscr{A})$ and a collaring $\rho : \Sigma \times [0, 1) \to W'$ satisfying the conclusions of Proposition (7.2). Let:

$$\mathscr{A}' = (M \times \mathbf{R}^n, (Q \times 0, Q \times Q_2, \ldots, Q \times Q_u), (k_1, \ldots, k_u), (s_1, \ldots, s_u)).$$

Lemma (5.4) a) implies that $Q \times 0 \hookrightarrow M \times \mathbf{R}^n$ is stable and Lemma (4.7) implies that $\mathscr{A}'$ is controlled. Hence $\pi(\mathscr{A}') : B(\mathscr{A}') \to M \times \mathbf{R}^n$ and $Q \times 0 \subset M \times \mathbf{R}^n$ augment the fins $V_i \times \mathbf{R}^n \to V_i \to N_i$ $i = 1, \ldots, k$ by Proposition (6.2) a).

Let $\pi'_i : V'_i \to N'_i$ $i = 1, \ldots, k+1$ be the compatible fins in $B(\mathscr{A}')$ obtained by extending π_i by $M \times \mathbf{R}^n \to M$ and then augmenting by $\pi(\mathscr{A}')$ and $Q \times 0 \subset M \times \mathbf{R}^n$. (That is, $N'_i = N_i$ $i = 1, \ldots, k$, $N'_{k+1} = Q \times 0$, $V'_i = \pi(\mathscr{A}')^{-1}(V_i \times \mathbf{R}^n)$ $i = 1, \ldots, k$, $V'_{k+1} = \pi(\mathscr{A}')^{-1}(Q \times 0)$, $\pi'_i = \pi_i \circ \pi \circ \pi(\mathscr{A}')$ $i = 1, \ldots, k$ $\pi_{k+1} = \pi(\mathscr{A}')$ where: $\pi : M \times \mathbf{R}^n \to M$ is projection.)

Let $q : M \times \mathbf{R}^n \to \mathbf{R}$ be any overt polynomial with $q^{-1}(0) = Q \times 0$ and let d be such that $B(\mathscr{A}') \subset M \times \mathbf{R}^n \times (\mathbf{R}^d - 0)$ and $\pi(\mathscr{A}')$ is induced by projection onto $M \times \mathbf{R}^n$.

Let $Y' = B(Y, N, W')$. We claim that there exists a smooth stratified imbedding $\alpha' : Y' \to B(\mathscr{A}')$ so that:

1) $\alpha'^{-1}\pi(\mathscr{A}')^{-1}(x, 0) = \alpha^{-1}(x) \times \gamma^{-1}(T(\mathscr{A}))(= \alpha^{-1}(x) \times (\text{spine of } W'))$ for each $x \in Q$. (So $\mathscr{D}_q\alpha'$ crushes $N \times$(spine of W') in Y' to $N \times$point, see Section 3.)
2) α' is stable over the fins π'_i $i = 1, \ldots, k+1$.
3) α' represents 0 in $\mathscr{N}^{\Lambda}(B(\mathscr{A}') : B(\mathscr{A}'))$.
4) There is a smooth stratified isotopy of imbeddings $g_t : Y \to M \times \mathbf{R}^n \times \mathbf{R}^d$ over the fins π_i $i = 1, \ldots, k$ so that $g_0 = \alpha \times 0 \times 0$ and $g_1(Y) = \mathscr{D}_q\alpha'(Y')$.

Given this claim we may finish the proof as follows. By induction and 2) and 3) there is an e, a projectively closed algebraic set $V' \subset B(\mathscr{A}') \times \mathbf{R}^e$ and a smooth stratified isotopy of imbeddings $H'_t : Y' \to B(\mathscr{A}') \times \mathbf{R}^e$, $t \in [0, 1]$ over the fins π'_i $i = 1, \ldots, k+1$ so that $H'_0 = \alpha' \times 0$ and $H'_1(Y') = V'$. Since H'_t is an isotopy over the fin

$$\pi(\mathscr{A}') : T(\mathscr{A}') \to Q \times 0,$$

then for each $x \in Q$ and $t \in [0, 1]$:

$$\begin{aligned}(H'_t)^{-1}(\pi(\mathscr{A}')^{-1}(x, 0) \times \mathbf{R}^e) &= (H'_0)^{-1}(\pi(\mathscr{A}')^{-1}(x, 0) \times \mathbf{R}^e)\\ &= \alpha^{-1}(x) \times \gamma^{-1}(T(\mathscr{A}))\end{aligned}$$

by condition 1). Also for each $x \in Q$:

$$\pi(\mathscr{A}')^{-1}(x, 0) \times \mathbf{R}^e = B(\mathscr{A}') \times \mathbf{R}^e \cap x \times 0 \times \mathbf{R}^d \times \mathbf{R}^e$$

since $\pi(\mathscr{A}')$ is induced by projection $M \times \mathbf{R}^n \times \mathbf{R}^d \to M \times \mathbf{R}^n$. Thus we have an isotopy

$$F_t : \mathscr{D}_q(\alpha'(Y') \times 0) \to M \times \mathbf{R}^n \times \mathbf{R}^d \times \mathbf{R}^e$$

over the fins $\pi_i = V_i \to N_i$ $i = 1, \ldots, k$ defined by letting $F_t(\mathscr{D}_q(\alpha'(y), 0)) = \mathscr{D}_q H'_t(y)$ for all $y \in Y'$. This is well defined because H'_t is an isotopy over the fin $\pi(\mathscr{A}')$. Also for each stratum S of Y', F_t imbeds $\mathscr{D}_q(\alpha'(S - N \times \gamma^{-1}(T(\mathscr{A}))) \times 0)$ since $\mathscr{D}_q$ is a diffeomorphism on $(M \times \mathbf{R}^n - Q \times 0) \times (\mathbf{R}^d \times \mathbf{R}^e - (0, 0))$. In addition, F_0 is inclusion.

Now by 4) of the claim $g_t \times 0$ isotops $\alpha(Y) \times 0 \times 0 \times 0 \subset M \times \mathbf{R}^n \times \mathbf{R}^d \times \mathbf{R}^e$ over the fins π_i to $\mathscr{D}_q(\alpha'(Y')) \times 0$ and then F_t isotops $\mathscr{D}_q(\alpha'(Y')) \times 0 = \mathscr{D}_q(\alpha'(Y') \times 0)$ over the fins π_i to $\mathscr{D}_q \mathrm{H}'_1(\mathrm{Y}') = \mathscr{D}_q(\mathrm{V}')$. But $\mathscr{D}_q(\mathrm{V}')$ is a projectively closed algebraic set by Proposition (3.1) so we are done, modulo proving claim 1), 2), 3) and 4) above.

Let us now prove the claim. Let $a = 2u + \Sigma k_i + \Sigma s_i$. Let $\alpha_1 : \mathrm{B}(\mathrm{Y}, \mathrm{N}, \mathrm{W}'') \to \mathrm{M}$ be $\alpha_1 = \alpha \circ \pi(\mathrm{Y}, \mathrm{N}, \mathrm{W}'')$. Pick any $(\alpha_2, \alpha_3) : \mathrm{B}(\mathrm{Y}, \mathrm{N}, \mathrm{W}'') \to (\mathbf{R}^n - 0) \times \mathbf{R}^a$ so that $(\alpha_2(y, w), \alpha_3(y, w)) = \lambda(\mathscr{A})^{-1}(\gamma(w))$ for all $(y, w) \in \mathrm{N} \times \mathrm{W}''$. (Hence $\alpha_2(y, w) = \pi(\mathscr{A}) \circ \gamma(w)$ for all $(y, w) \in \mathrm{N} \times \mathrm{W}''$.) Let Z be the A-space

$$\mathrm{Z} = \mathrm{B}(\mathrm{Y}, \mathrm{N}, \mathrm{W}'') \times [0, 1] \cup \mathrm{N} \times \mathrm{W} \times [1, 2]$$

where $\mathrm{N} \times \mathrm{W}'' \times 1 \subset \mathrm{B}(\mathrm{Y}, \mathrm{N}, \mathrm{W}'') \times [0, 1]$ is identified with $\mathrm{N} \times \mathrm{W}'' \times 1 \subset \mathrm{N} \times \mathrm{W} \times [1, 2]$, and corners are rounded off. Then $\partial \mathrm{Z}$ has three pieces, $\mathrm{B}(\mathrm{Y}, \mathrm{N}, \mathrm{W}'') \times 0$, $\mathrm{N} \times \mathrm{W} \times 2$ and $(\mathrm{B}(\mathrm{Y}, \mathrm{N}, \mathrm{W}'') - \mathrm{N} \times \mathrm{W}'') \times 1 \cup \mathrm{N} \times \mathrm{W}' \times 1$. The last piece is A-isomorphic with $\mathrm{B}(\mathrm{Y}, \mathrm{N}, \mathrm{W}') = \mathrm{Y}'$. (See Figure 8.1.)

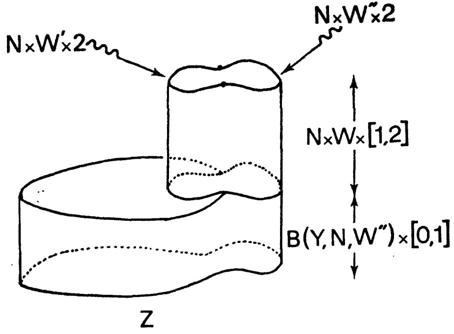


Fig. 8.1

We define $\alpha'' : \mathrm{Z} \to \mathrm{B}(\mathscr{A}')$ as follows:

Let $\mu : \mathrm{Q} \times \mathrm{B}(\mathscr{A}) \to \mathrm{B}(\mathscr{A}')$ be the imbedding induced by $\mathrm{Q} \times \mathbf{R}^n \hookrightarrow \mathrm{M} \times \mathbf{R}^n$. Let $\alpha''(y, w, t) = \mu(\alpha(y), \gamma(w))$ for $(y, w, t) \in \mathrm{N} \times \mathrm{W} \times [1, 2]$. Otherwise let:

$$\alpha''(z, t) = \lambda(\mathscr{A}')(\alpha_1(z), \alpha_2(z), \alpha_3(z)) \quad \text{for} \quad (z, t) \in \mathrm{B}(\mathrm{Y}, \mathrm{N}, \mathrm{W}'') \times [0, 1].$$

To see α'' is well defined on the intersection pick $(y, w, 1) \in \mathrm{N} \times \mathrm{W}'' \times \{1\}$, then:

$$\alpha_1(y, w) = \alpha(y)$$

and

$$\begin{aligned} \mu(\alpha(y), \gamma(w)) &= \mu(\alpha(y), \lambda(\mathscr{A})(\alpha_2(y, w), \alpha_3(y, w))) \\ &= \lambda(\mathscr{A}')(\alpha_1(y, w), \alpha_2(y, w), \alpha_3(y, w)) \end{aligned}$$

because we have the following commutative diagram.

$$\begin{tikzcd}
\mathrm{Q} \times (\mathbf{R}^n - 0) \times \mathbf{R}^a \arrow[rr, "\mathrm{id} \times \lambda(\mathscr{A})"] \arrow[d, equal] & & \mathrm{Q} \times \mathrm{B}(\mathscr{A}) \arrow[ddl, "\mu"] \\
(\mathrm{Q} \times \mathbf{R}^n - \mathrm{Q} \times 0) \times \mathbf{R}^a \arrow[dr, "\lambda(\mathscr{A}')"'] & & \\
& \mathrm{B}(\mathscr{A}') &
\end{tikzcd}$$

We define $\alpha' = \alpha''\big|_{Y'}$.

Notice that $\alpha''\big|_{N\times W\times 2}$ represents 0 in $\mathcal{N}^A(B(\mathcal{A}') : B(\mathcal{A}'))$ by Lemma (2.2) c) and $\alpha''\big|_{B(Y,N,W'')\times 0}$ represents 0 by Lemma (2.2) b), d), f). Hence α' represents 0 in $\mathcal{N}^A(B(\mathcal{A}') : B(\mathcal{A}'))$ also. In addition $\alpha'\big|_{N\times\partial W'\times 1}$ and $\alpha'\big|_{N\times \mathrm{int}\,W'\times 1}$ are stable over the fins π_i' by Lemma (5.4) b) and (5.4) c) since μ is stable over the fins π_i' by Proposition (6.2) b). Also $\alpha'\big|_{(B(Y,N,W'')-N\times W'')\times 1}$ is stable over the fins π_i' by Lemma (5.4) a). Hence α' is stable over these fins.

Let $Y''=Y'-N\times\gamma^{-1}(T(\mathcal{A}))$. We define an isotopy

$$h_t : Y'' \to (M\times\mathbf{R}^n - Q\times 0)\times\mathbf{R}^a$$

as follows. For $z\in Y-c(N\times\mathring{c}\Sigma)\subset Y''$ we define:

$$h_t(z)=(\alpha_1(z),\, t\alpha_2(z),\, t\alpha_3(z))\in M\times\mathbf{R}^n\times\mathbf{R}^a.$$

For $z=c(y,\,(x,\,s))\in c(N\times(\mathring{c}\Sigma-*))\subset Y''$:

$$h_t(z)=(\alpha c(y,\,(x,\,(st+s-t+1)/2)),\, t\alpha_2(z),\, t\alpha_3(z))\in M\times\mathbf{R}^n\times\mathbf{R}^a$$

and for $z=(y,\,\rho(w,\,s))\in N\times(W'-\gamma^{-1}(T(\mathcal{A})))$:

$$h_t(z)=(\alpha c(y,\,(w,\,(1-t)(1-s)/2),\, t\lambda(\mathcal{A})^{-1}\gamma\rho(w,\,s))\in M\times(\mathbf{R}^n\times\mathbf{R}^a).$$

(Recall $\rho : \Sigma\times[0,\,1) \to W'-\gamma^{-1}(T(\mathcal{A}))$ is the collaring given by Proposition (7.2).)

Notice that $\lambda(\mathcal{A}')h_t : Y'' \to B(\mathcal{A}')-T(\mathcal{A}')$ is a continuous isotopy over the fins $\pi_i : V_i\to N_i$ and that $\lambda(\mathcal{A}')h_1=\alpha''\big|_{Y''}$ and $\lambda(\mathcal{A}')h_0$ is an imbedding of Y'' onto $\lambda(\mathcal{A}')(\alpha(Y-N)\times 0\times 0)$.

Let $\varphi : Y-N \to \mathbf{R}^n\times\mathbf{R}^d$ be the function so that:

$$\mathcal{D}_q(\lambda(\mathcal{A}')(\alpha(y),\,0,\,0))=(\alpha(y),\,\varphi(y))\in M\times(\mathbf{R}^n\times\mathbf{R}^d)$$

for all $y\in Y-N$, i.e. φ is the composition

$$Y-N \xrightarrow{\alpha\times 0\times 0} (M\times\mathbf{R}^n-Q\times 0)\times\mathbf{R}^a \xrightarrow{\lambda(\mathcal{A}')} B(\mathcal{A}') \hookrightarrow M\times\mathbf{R}^n\times\mathbf{R}^d$$

$$\xrightarrow{\mathcal{D}_q} M\times\mathbf{R}^n\times\mathbf{R}^d \to \mathbf{R}^n\times\mathbf{R}^d.$$

Let $\psi : Y-N\to Y''$ be the homeomorphism so that $\mathcal{D}_q\lambda(\mathcal{A}')h_0\psi(y)=(\alpha(y),\,\varphi(y))$ for all $y\in Y-N$. Then the isotopy $g_t : Y\to M\times\mathbf{R}^n\times\mathbf{R}^d$ required by condition 4) of the claim is obtained by smoothing out the following isotopy.

Let $g_t(y)=\alpha(y)\times 0\times 0$ for $y\in N$. For $y\in Y-N$ and $0\leq t\leq 1/2$ let $g_t(y)$ be $(\alpha(y),\,2t\varphi(y))$. (This takes $Y-N$ to $\lambda(\mathcal{A}')(\alpha(Y-N)\times 0\times 0)$.) For $y\in Y-N$ and $1/2\leq t\leq 1$ let $g_t(y)$ be $\mathcal{D}_q\lambda(\mathcal{A}')h_{2t-1}\psi(y)$. (This takes $Y-N$ to $\mathcal{D}_q\alpha''(Y'')$.)

This isotopy g_t is in fact continuous. (The hardest part is for $1/2\leq t\leq 1$. Take $z\in Q$ and pick r so that:

$$\alpha'(N\times W')\subset M\times\mathbf{R}^n\times rB^d$$

where $$rB^d=\{x\in\mathbf{R}^d \mid |x|\leq r\}.$$

For any $\varepsilon>0$ and neighbourhood U of z in M pick $\delta>0$ and a neighborhood U^1 of $\alpha^{-1}(z)$ in N so that:

$$\alpha c(U^1\times\delta\mathring{c}\Sigma)\times\pi(\mathscr{A})\gamma\rho(\partial W'\times(1-\delta, 1))$$
$$\subset U\times\varepsilon B^n\cap q^{-1}((-(\varepsilon^{2b+1}/r)^{1/2}, (\varepsilon^{2b+1}/r)^{1/2}))$$

where b is the degree of q and $\delta\mathring{c}\Sigma=\{(x, t)\in\mathring{c}\Sigma \mid t<\delta\}$. Then:

$$c(U^1\times(\delta/2)\mathring{c}\Sigma)=U^1\cup\psi^{-1}(U^1\times\rho(\partial W'\times(1-\delta, 1)))$$

is a neighborhood of $\alpha^{-1}(z)$ in Y which is mapped by each g_t into $U\times\varepsilon B^n\times\varepsilon B^d$, $1/2\leq t\leq 1$. Hence g_t is continuous for $1/2\leq t\leq 1$.)

To make the proof of (8.3) precise, one would have to complicate the various constructions to make the maps all smooth. This can be left to the reader. ■

There are a number of alternate methods of proof of Theorem (8.1). One attractive method is to change Theorem (8.3) by assuming that the stratification on $\alpha(Y)$ satisfies the Whitney conditions, deleting all mention of fins and concluding that the isotopy H_t is very small. Then you would get stability from the Thom isotopy theorems. The proof of Theorem (8.3) would be more or less the same, but a bit messier. The advantage would be that the notion of compatible fins would be unnecessary. The disadvantage would be that a certain amount of messy analysis would be needed. The concepts developed in this paper will be useful in our future work.

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Manuscrit reçu le 29 octobre 1979.