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LIFTING SMOOTH HOMOTOPIES OF ORBIT SPACES

by GERALD W. SCHWARZ ([1])

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0. Introduction.

This paper establishes a C^∞ analogue of Palais' covering homotopy theorem. Techniques from differential analysis, invariant theory, and commutative algebra are used in the proof.

Let K be a compact Lie group, and let X be a smooth ($=C^\infty$) K-manifold. We denote by $\pi_{X,K}$ the canonical map from X to the orbit space X/K. We give X/K the quotient topology and differentiable structure, i.e. if $U \subseteq X/K$ is open, then $C^\infty(U)$ is the set of real-valued functions on U whose pull-backs to $\pi_{X,K}^{-1}(U)$ are smooth. Thus $C^\infty(U) \simeq C^\infty(\pi_{X,K}^{-1}(U))^K$. If Y is another smooth K-manifold, we say that $\psi : X/K \to Y/K$ is **smooth** if $\psi^* C^\infty(Y/K) \subseteq C^\infty(X/K)$. The notions of **diffeomorphism**, **isotopy**, etc., of orbit spaces have their usual categorical meaning.

Let $x \in X$. The **slice representation** at x is the representation of the isotropy group K_x on the normal space at x to the orbit Kx. Two K-orbits are said to have the same **normal type** if there are points in each with the same isotropy group and isomorphic slice representations (up to trivial factors). The subsets of $B = X/K$ of given normal type are C^∞ manifolds, and they form a locally finite stratification of B. Above each stratum, $X \to B$ is a smooth fiber bundle, so we may view $X \to B$ as a *collection* of smooth fiber bundles. Many beautiful and deep results concerning regular actions of the classical groups have been proved by classifying these types of bundle collections over a fixed B ([7], [8], [12], [13], [14], [15], [16], [23], [36], [41], [42]). These classification results have all hinged upon proving some form of homotopy lifting theorem. For the category of continuous K-actions one has the celebrated covering homotopy theorem of Palais [61]. In the case of ordinary fiber bundles Palais' theorem reduces to the statement that pull-backs by homotopic maps are isomorphic. There is a natural smooth analogue of Palais' theorem which lies behind the classification results cited above, and this smooth Palais theorem is equivalent to the

Isotopy Lifting Conjecture **(0.1)**. — *Let* $\overline{F} : X/K \times [0, 1] \to X/K$ *be a smooth isotopy starting at the identity. Then there is a smooth* K*-equivariant isotopy* $F : X \times [0, 1] \to X$ *starting at the identity and inducing* $\overline{F}$.

The above conjecture is due to Bredon. In [7] he proved the conjecture for "special G-manifolds." Davis [12] showed that (0.1) holds for a large class of regular actions of the classical groups, and Bierstone [2] showed that (0.1) holds for smooth actions all of whose isotropy groups have the same dimension. In this paper we show that (0.1) holds in general (corollary (2.4) below). The smooth analogue of Palais' covering homotopy theorem is our theorem (2.3).

We now briefly describe the contents of the chapters of this paper. Each chapter begins with a more detailed précis of its contents.

In chapter I we reduce (0.1) to a lifting problem for vector fields: Let X and K be as above, and let $\mathrm{Der}(C^\infty(X/K))$ denote the real-linear derivations of $C^\infty(X/K)$. We refer to elements of $\mathrm{Der}(C^\infty(X/K))$ as **smooth vector fields** on X/K. (See § 3 for justification of this terminology.) An element of $\mathrm{Der}(C^\infty(X/K))$ is **strata preserving** if it preserves the ideals of $C^\infty(X/K)$ vanishing on the various strata of X/K. We denote by $\mathfrak{X}^\infty(X/K)$ the collection of strata preserving smooth vector fields on X/K, and we denote by $\mathfrak{X}^\infty(X)$ the smooth vector fields on X. We prove that the following result implies (0.1):

Smooth Lifting Theorem **(0.2)**. — *The canonical map*

$$(\pi_{X,K})_*: \mathfrak{X}^\infty(X)^K \to \mathrm{Der}(C^\infty(X/K))$$

has image $\mathfrak{X}^\infty(X/K)$.

Using the differentiable slice theorem we reduce to proving

(0.3) $$(\pi_{W,L})_*\mathfrak{X}^\infty(W)^L = \mathfrak{X}^\infty(W/L)$$

for all representation spaces W of closed subgroups L of K. We prove that (0.3) is equivalent to analogous statements involving polynomial, real analytic, or complex analytic vector fields.

In chapters II, III, and IV we concentrate on the algebraic and complex analytic analogues of (0.3). In chapter II we reduce (0.3) to a cohomology problem which we can solve provided the representation of L on W has finite principal isotropy groups and no S^3 strata (conditions on slice representations). In chapter III we show how to classify representations with infinite principal isotropy groups or S^3 strata. Using this classification and some theorems of § 11 we are able to reduce (0.3) to the case of representations of the simple compact Lie groups which have trivial principal isotropy groups and S^3 strata. In chapter IV we develop a method for handling these remaining cases.

We found it necessary to develop several techniques for calculating rings of invariants of representations, and in chapter IV we exhibit many cases where the rings of invariants are regular. Further work along these lines can be found in [67], [68], and references therein. Also see remarks (17.28) and (17.30) below.

During this work I have benefited from conversations with many mathematicians, and I would especially like to thank E. Bierstone, D. Buchsbaum, D. Eisenbud, M. Hochster, D. Lieberman, D. Luna, J. Mather, and Th. Vust.

I. — THE COVERING HOMOTOPY THEOREM

In § 1 we recall the basic results concerning orbit spaces of smooth transformation groups. In § 2 we show how to reduce the isotopy lifting conjecture (0.1) and the smooth Palais theorem (2.3) to the smooth lifting theorem (0.2). Much of § 1 and § 2 overlaps with [14]. In § 3 we reduce (0.2) to a polynomial analogue of (0.3). In § 5 we study representations of reductive complex algebraic groups and complexifications of representations of compact Lie groups, and in § 6 we show that polynomial, real analytic, and complex analytic versions of (0.3) are equivalent. In § 4 we prove that the kernel of the map $(\pi_{X,K})_*$ of (0.2) has a closed complementary subspace in $\mathfrak{X}^\infty(X)^K$ (C^∞ topology). As we explain, this result is a step towards a strengthening of (0.1).

1. Orbit Spaces.

We fix notation and review some of the main theorems concerning orbit spaces. Proofs of unreferenced claims can be found in [7]. We end the section by proving an orbit space version of the inverse function theorem.

Z, **R**, **C**, and **Q** will denote, respectively, the integers, real numbers, complex numbers, and quaternions. The non-negative integers (resp. reals) are denoted $\mathbf{Z}^+$ (resp. $\mathbf{R}^+$), and $\mathbf{Z}_n$ will denote $\mathbf{Z}/n\mathbf{Z}$, $n \in \mathbf{Z}^+$.

All manifolds will be assumed to be second countable and are allowed to have a boundary.

If G, K, ... are Lie groups or linear algebraic groups over **C**, then $\mathfrak{g}$, $\mathfrak{k}$, ... will denote their Lie algebras and G^0, K^0, ... will denote their identity components. If L is a subgroup of a group G, then (L) denotes the conjugacy class of L. If L_1 and L_2 are subgroups of G, we write $(L_1) \leq (L_2)$ if L_1 is conjugate to a subgroup of L_2. If $(L_1) \leq (L_2)$ and $(L_1) \neq (L_2)$, we write $(L_1) < (L_2)$.

Throughout this paper, K *will denote a compact Lie group.* A **representation** of K will mean a finite dimensional real vector space W (the **representation space**) together with a continuous homomorphism ρ from K to the general linear group GL(W) of W. We will denote the representation by ρ or by the pair (W, K). The direct sum of m copies of ρ is denoted by $m\rho$ or (mW, K). If $\rho' = (W', K)$, then $\rho + \rho'$ or (W+W', K) denotes the direct sum of ρ and ρ'. The trivial real m-dimensional representation of K is denoted θ_m, and we will also use θ_m to denote the corresponding representation space $\mathbf{R}^m$. A trivial representation of unspecified dimension is denoted θ.

We use the (standard) notation of [7] when referring to the classical groups. We denote the basic representations of $O(n)$, $U(n)$, $Sp(n)$, etc., by $(\mathbf{R}^n, O(n))$, $(\mathbf{C}^n, U(n))$, $(\mathbf{Q}^n, Sp(n))$, etc. We embed $O(m)$ into $O(n)$ via the natural action of the former group on the first m co-ordinates of $\mathbf{R}^n$, $m \leq n$. We similarly consider $U(m)$, $SU(m)$, etc., as embedded in $U(n)$, $SU(n)$, etc.

Let $\rho = (W, K)$ be a representation of K, and let X be a smooth K-manifold. Following Bredon [8], we say that X is **modelled** on ρ if each $x \in X$ has a K-invariant neighborhood which is K-diffeomorphic to an open subset of W. We say that X is **stably modelled** on ρ if for each component X′ of X there are r, $s \in \mathbf{Z}^+$ such that $X' \times \theta_r$ is modelled on $\rho + \theta_s$. If K is the classical group $O(n)$ (resp. $U(n)$, resp. $Sp(n)$), then X is called a **regular K-manifold** if X is stably modelled on $(m\mathbf{R}^n, O(n))$ (resp. $(m\mathbf{C}^n, U(n))$, resp. $(m\mathbf{Q}^n, Sp(n))$), $m \in \mathbf{Z}^+$.

Let L be a closed subgroup of K, and let P be a smooth L-manifold. The **twisted product** $K \times_L P$ is the orbit space $(K \times P)/L$ where $\ell(k, p) = (k\ell^{-1}, \ell p)$; $\ell \in L$, $k \in K$, $p \in P$. We denote the orbit of (k, p) by $[k, p]$. The twisted product $K \times_L P$ is a smooth K-manifold, where $k'[k, p] = [k'k, p]$; $k, k' \in K$, $p \in P$. Note that $(K \times_L P)/K$ is diffeomorphic to P/L. If X is a smooth K-manifold and $x \in X$, then the normal bundle to Kx is K-diffeomorphic to $K \times_{K_x} N_x$, where $N_x = T_x(X)/T_x(Kx)$ is the normal space to Kx at x.

The following is a variant of a theorem of Koszul:

Differentiable Slice Theorem **(1.1)**. — *Let* X *be a smooth* K*-manifold.*

(1) *If* $x \notin \partial X$, *then a* K*-invariant neighborhood of* x *is* K*-diffeomorphic to* $K \times_{K_x} N_x$.

(2) *If* $x \in \partial X$, *then a* K*-invariant neighborhood of* x *is* K*-diffeomorphic to* $\mathbf{R}^+ \times (K \times_{K_x} \bar{N}_x)$, *where* $\bar{N}_x = T_x(\partial X)/T_x(Kx)$. ■

Statement (1) is the usual differentiable slice theorem, and (2) follows from (1) and the fact that ∂X has a collar equivariantly diffeomorphic to $\mathbf{R}^+ \times \partial X$, where K acts trivially on $\mathbf{R}^+$. We will refer to theorem (1.1) as the **DST**. If X and the K-action are real analytic (e.g. X is a representation space of K), then the diffeomorphisms of the DST can be chosen to be real analytic. Note that the DST implies that X/K is locally diffeomorphic to (perhaps the product of $\mathbf{R}^+$ and) orbit spaces of linear actions.

Let L be a subgroup of K, and let X be a smooth K-manifold. The union of those orbits whose isotropy groups are in (L) is denoted $X^{(L)}$, and $X^{\langle L \rangle}$ denotes $X^{(L)} \cap X^L$. Suppose that $X^{(L)} \neq \emptyset$. We then call (L) an **isotropy class** of the K-action on X. There is a canonical embedding of $X^{(L)}/K$ into X/K, and the image $(X/K)_{(L)}$ is called an **isotropy type stratum** of X/K. We give $(X/K)_{(L)}$ the smooth structure of the orbit space $X^{(L)}/K$.

Proposition **(1.2)**. — *Let* X *be a smooth* K*-manifold.*

(1) *The isotropy type strata* $\{(X/K)_{(L)}\}$ *are smooth manifolds, and the inclusions* $(X/K)_{(L)} \to X/K$ *are smooth.*

(2) *The components of the isotropy type strata are a locally finite collection of subsets of* X/K *whose boundaries (as manifolds) are a locally finite collection of subsets of* $\partial X/K$.

(3) *Let* $\{\sigma_\alpha\}$ *be the set of normal type strata of* X/K. *Then the components of the* σ_α *are the same as the components of the* $(X/K)_{(L)}$, *and* $\{\sigma_\alpha\}=\{(X/K)_{(L)}\}$ *if* X *is stably modelled on a representation of* K.

Proof. — Using the fact that ∂X has a K-collar $\partial X \times \mathbf{R}^+$, we may easily reduce to the case $\partial X = \emptyset$. Let $x \in X$. By the DST, a K-invariant neighborhood U of Kx is K-diffeomorphic to $K \times_{K_x} N_x$. Let W be a K_x-complement to $N_x^{K_x}$ in N_x. Then $X^{(K_x)} \cap U \simeq K \times_{K_x} (N_x^{K_x})$, and $(X^{(K_x)} \cap U)/K \simeq N_x^{K_x}$ is a C^∞ manifold which embeds smoothly in $U/K \simeq N_x^{K_x} \times W/K_x$. Thus (1) is proved. Part (2) is well-known: Since $U/K \simeq N_x/K_x$, one can reduce to the case of representations, and then one proceeds by induction (see [62]).

We now prove (3). Let W be a representation space of K, and let $w \in W$. Then there is an isomorphism of K_w-representations:

$$N_w = T_w(W)/T_w(Kw) \simeq W/(\mathfrak{k}/\mathfrak{k}_w).$$

It follows that isotropy type determines normal type for stably modelled actions, and that isotropy type determines normal type locally. ■

Corollary **(1.3)**. — *Let* X *be a smooth* K*-manifold. Then*

$$(\pi_{X,K})_* \mathfrak{X}^\infty(X)^K \subseteq \mathfrak{X}^\infty(X/K).$$

Proof. — Let $A \in \mathfrak{X}^\infty(X)^K$, let $x \in X$, and let U and W be as in the proof of (1.2). Let $f \in C^\infty(U/K) \simeq C^\infty(N_x = N_x^{K_x} \times W)^{K_x}$, and suppose that f vanishes on $N_x^{K_x} \times \{0\}$. Since the image of $A(x)$ in N_x lies in $N_x^{K_x}$, $A(f)(x) = 0$. It follows that $(\pi_{X,K})_* A$ preserves the ideals in $C^\infty(X/K)$ vanishing on the strata of X/K, i.e. $(\pi_{X,K})_* A \in \mathfrak{X}^\infty(X/K)$. ■

The following result is due to Montgomery, Samelson, and Yang:

Theorem **(1.4)**. — *Let* X *be a connected smooth* K*-manifold. There is a unique isotropy class* (H) *such that*

(1) $(X/K)_{(H)}$ *is connected and open and dense in* X/K.

(2) (H) *is a minimum among all isotropy classes of* X.

(3) *The slice representations at points of* $X^{(H)}$ *are trivial.*

(4) $\dim(X/K)_{(H)} = \dim X - \dim K + \dim H$. ■

We call (H) the **principal isotropy class**, H is called a **principal isotropy group**, and orbits Kx with $x \in X^{(H)}$ are called **principal orbits**. The covering dimension $\dim X/K$ of X/K equals $\dim(X/K)_{(H)}$; see ([7], Ch. III).

We now require small digressions on stratifications and invariant theory. Let S be a (for simplicity closed) semi-analytic subset of $\mathbf{R}^d$ (see [48] or [55] for definitions). A **primary stratification** of S is a locally finite collection $\{E_i\}$ of connected semi-analytic submanifolds of $\mathbf{R}^d$, called strata, such that $S = \bigcup_i E_i$ and such that, for each i, closure$(E_i) - E_i$ is a union of lower dimensional strata. Łojasiewicz [48] gives an algorithm for constructing such a stratification of S, and we call the resulting $\{E_i\}$ the **primary strata** of S. The C^∞ structure sheaf of S is the sheaf of germs of functions on S which have local smooth extensions to $\mathbf{R}^d$. Since S is closed, $C^\infty(S) = C^\infty(\mathbf{R}^d)|_S$. A remark of Mather's ([55], p. 210) shows that the primary stratification of S only depends on the C^∞ structure of S.

Let W be a representation space of K. By a theorem of Hilbert (see [80], p. 274) the algebra of K-invariant polynomials $\mathbf{R}[W]^K$ is noetherian. Let $p_1, \ldots, p_d$ be homogeneous generators, and let $p = (p_1, \ldots, p_d) : W \to \mathbf{R}^d$. Then p is proper and induces a homeomorphism of W/K with the closed subset $S = p(W)$ of $\mathbf{R}^d$ ([66]). Since p is polynomial, S is semi-algebraic ([69]). If W, K, p, S, and d are as above, we call p and the quintuple (W, K, p, S, d) **orbit maps**. If d is minimal, we say that (W, K, p, S, d) and p are **minimal orbit maps**. We will confuse $p : W \to \mathbf{R}^d$ with the associated map from W to S. We denote by $\bar{p}$ the induced map from W/K to S (or $\mathbf{R}^d$).

Theorem **(1.5).** — *Let* (W, K, p, S, d) *be an orbit map. Then*

(1) $\bar{p}$ *maps the components of the normal* (= *isotropy*) *type strata of* W/K *in a one-to-one manner onto the primary strata of* S.

(2) $\bar{p}((W/K)_{(L)})$ *is semi-algebraic for each isotropy class* (L).

(3) *There is a continuous linear map* $\varphi : C^\infty(W)^K \to C^\infty(\mathbf{R}^d)$ (C^∞ *topologies, see* [25]) *such that* $p^* \circ \varphi$ *is the identity. In particular,*

(4) $$C^\infty(W)^K = p^* C^\infty(\mathbf{R}^d)$$

and $\bar{p} : W/K \to S$ *is a diffeomorphism.* ∎

Part (1) is due to Bierstone [2], and most of the results we derive from it were known to him. Part (2) is the following observation: If $W^{(L)} \neq \emptyset$, then $p(W^{(L)}) = p(W^{\langle L \rangle})$ is semi-algebraic since it is the difference $p(W^L) - \bigcup_{i=1}^m p(W^{L_i})$, where $(L_1), \ldots, (L_m)$ are the isotropy classes strictly larger than (L). Part (3) is due to Mather [56], and (4) was first shown by the author [66] (see also [52]).

In [2], Bierstone observes that the primary stratification of S satisfies Whitney's conditions. Moreover, from his proof of (1.5.1) one can see that the primary stratification of S only depends on its C^1 structure. The key point is that if $W^K = \{0\}$, then S contains no non-singular arcs through 0 (see (3.4) below).

Corollary (**1.6**). — *Let* W_1 *and* W_2 *be representation spaces of* K, *and let* $p: W_1 \to \mathbf{R}^d$ *and* $q: W_2 \to \mathbf{R}^e$ *be orbit maps. If* $\psi: W_1/K \to W_2/K$ *is a smooth map, then there is a smooth map* η *making the following diagram commute*:

$$\begin{array}{ccc} W_1/K & \xrightarrow{\psi} & W_2/K \\ \big\downarrow{\scriptstyle \bar{p}} & & \big\downarrow{\scriptstyle \bar{q}} \\ \mathbf{R}^d & \xrightarrow{\eta} & \mathbf{R}^e \end{array}$$

Proof. — Let $\eta = (\eta_1, \ldots, \eta_e)$ where η_i is any smooth function on $\mathbf{R}^d$ such that $\bar{p}^* \eta_i = \psi^* \bar{q}_i$, $i = 1, \ldots, e$. ■

Unless otherwise specified, we give orbit spaces their stratification by normal orbit type, and we give images S of orbit maps the induced stratification. The canonicity of the primary stratification yields

Corollary (**1.7**). — *Let* X *be a smooth* K-*manifold. Then*

(1) *The partition of* X/K *given by the components of the interiors and boundaries of its strata (as manifolds) is determined by the* C^∞ *structure of* X/K.

(2) *Let* ψ_t, $0 \leq t \leq 1$, *be a smooth isotopy of* X/K *starting at the identity. Then each* ψ_t *is strata preserving.* ■

In the remaining part of this section we prove the orbit space analogue of the inverse function theorem.

Let X be a smooth K-manifold. Let $\xi \in X/K$, and let $\mathcal{M}_\xi$ (or $\mathcal{M}_\xi(X/K)$) denote the elements in the ring of germs of smooth functions at ξ which vanish at ξ. As usual, we define the (Zariski) **cotangent space** $T^*_\xi(X/K)$ of X/K at ξ to be $\mathcal{M}_\xi/\mathcal{M}_\xi \cdot \mathcal{M}_\xi$, and the dual space $T_\xi(X/K)$ is the (Zariski) **tangent space**. The DST and (1.5) show that $T_\xi(X/K)$ and $T^*_\xi(X/K)$ are always finite dimensional vector spaces. If Y is a smooth K-manifold and $\psi: X/K \to Y/K$ is smooth, then ψ induces a linear map $(d\psi)_\xi : T_\xi(X/K) \to T_{\psi(\xi)}(Y/K)$.

Lemma (**1.8**). — *Let* (W, K, p, S, d) *be a minimal orbit map, and let* $\bar{0}$ *denote* $\pi_{W/K}(0)$. *Then*

$$(d\bar{p})_{\bar{0}} : T_{\bar{0}}(W/K) \to T_0(\mathbf{R}^d) \simeq \mathbf{R}^d$$

is an isomorphism.

Proof. — Since $\bar{p}^*$ is surjective and $\bar{p}$ is proper, $\bar{p}^* \mathcal{M}_0(\mathbf{R}^d) = \mathcal{M}_{\bar{0}}(W/K)$, hence $(d\bar{p})_{\bar{0}}$ is injective. If $(d\bar{p})_{\bar{0}}$ is not surjective, then there is a non-zero element $(a_1, \ldots, a_d) \in T_0(\mathbf{R}^d)$ which is perpendicular to $\operatorname{Im} d\bar{p}_{\bar{0}}$. In other words, there is a relation

$$(\mathbf{1.9}) \qquad \sum_{i=1}^{d} a_i p_i \in \mathcal{M}_0(W)^K \cdot \mathcal{M}_0(W)^K$$

where the a_i are not all zero. Without loss of generality, suppose $a_1 \neq 0$. Taking Taylor series in (1.9) and restricting to terms homogeneous of degree $\deg p_1$, we see that p_1 is a polynomial in $p_2, \ldots, p_d$, contradicting the minimality of p. ■

Lemma **(1.10)**. — *Let* (W, K, p, S, d) *be an orbit map. Let* U_1 *and* U_2 *be neighborhoods of* 0 *in* $\mathbf{R}^d$, *and suppose that* $\psi : U_1 \to U_2$ *is a diffeomorphism such that* $\psi(S \cap U_1) \subseteq S$, $\psi(0) = 0$. *If* $S \cap U_2$ *is connected, then* $\psi^{-1}(S \cap U_2) \subseteq S$.

Proof. — Let $S' \subseteq S$ denote the image of the principal orbits. Then (1.7) shows that $\psi(S' \cap U_1) \subseteq S'$ and that $\psi((S - S') \cap U_1) \subseteq S - S'$. Thus $\psi : S' \cap U_1 \to S' \cap U_2$ is an open embedding with image a closed subset of $S' \cap U_2$. Now $S \cap U_2$ is connected, and it follows from the DST and (1.4.1) that $S' \cap U_2$ must then also be connected. Hence $\psi(S' \cap U_1) = S' \cap U_2$. Since S' is dense in S and since ψ is proper,

$$\psi(S \cap U_1) = S \cap U_2.$$

Consequently, $\psi^{-1}(S \cap U_2) \subseteq S$. ■

Inverse Function Theorem **(1.11)**. — *Let* X *and* Y *be smooth* K-*manifolds. Suppose* $\psi : X/K \to Y/K$ *is a smooth strata preserving map,* $\psi(\partial X/K) \subseteq \partial Y/K$, *and* $(d\psi)_\xi$ *is an isomorphism at* $\xi \in X/K$. *Then* ψ *is a diffeomorphism near* ξ.

Proof. — Suppose $\xi \notin \partial X/K$. Using the DST we may reduce to the case where $X = Y$ is a representation space W of a closed subgroup L of K, $\xi = \psi(\xi) = \pi_{W,L}(0)$. Let (W, K, p, S, d) be a minimal orbit map. Then by (1.6), (1.8), and (1.10) there is a germ of a C^∞ map $\eta : \mathbf{R}^d \to \mathbf{R}^d$, $\eta(S) \subseteq S$, such that $\bar{p}^{-1} \circ \eta \circ \bar{p}$ is a local smooth inverse for ψ near ξ. If $\xi \in \partial X/K$, we may reduce to a case $X = Y = W \times \mathbf{R}^+$, $\xi = \psi(\xi) = \pi_{W,L}(0)$. By (1.5), we may identify $W/K \times \mathbf{R}^+$ with $(W \times \mathbf{R})/(K \times \{\pm 1\})$, where $\{\pm 1\}$ acts by multiplication on $\mathbf{R}$. Our previous argument then applies. ■

2. Covering Smooth Homotopies.

We begin with some preliminaries on pull-backs (fiber products). Let X and Y be smooth K-manifolds, and let $\psi : X/K \to Y/K$ be smooth. We define the **pull-back** $\psi^* Y$ to be $\{(\xi, y) \in X/K \times Y : \psi(\xi) = \pi_{Y,K}(y)\}$. We give $\psi^* Y$ the C^∞ structure induced from that on $X/K \times Y$. Then K acts smoothly on $\psi^* Y$, where $k(\xi, y) = (\xi, ky)$; $k \in K$, $(\xi, y) \in \psi^* Y$. The pull-back $\psi^* Y$ has the usual universal properties of fiber products ([7], [14]).

Let $\xi \in X/K$, and let σ_ξ denote the stratum of X/K containing ξ. Then $T_\xi(\sigma_\xi)$ is a subspace of $T_\xi(X/K)$, and we let $\mathcal{N}_\xi(X/K)$ denote $T_\xi(X/K)/T_\xi(\sigma_\xi)$—the normal space to σ_ξ at ξ. Suppose that $\psi : X/K \to Y/K$ is smooth and strata preserving. Then $(d\psi)_\xi$ induces a linear map $(\delta\psi)_\xi : \mathcal{N}_\xi(X/K) \to \mathcal{N}_{\psi(\xi)}(Y/K)$, and we say that ψ is **normally transverse** if $(\delta\psi)_\xi$ is an isomorphism for all $\xi \in X/K$.

Let $x \in X$, let N_x denote $T_x(X)/T_x(Kx)$ as before, and let $\mathcal{N}_x(X)$ denote $N_x/N_x^{K_x}$.

If $\psi: X \to Y$ is smooth and equivariant, we say that ψ is **strata preserving** if it preserves the normal type of orbits. In this case, $(d\psi)_x$ induces $(\delta\psi)_x : \mathcal{N}_x(X) \to \mathcal{N}_{\psi(x)}(Y)$, and we say that ψ is **normally transverse** if $(\delta\psi)_x$ is an isomorphism for all $x \in X$.

Proposition **(2.1)**. — *Let* X *and* Y *be smooth* K*-manifolds,* $\partial Y = \emptyset$. *Let* $\bar{f}: X/K \to Y/K$ *be a smooth strata preserving map, and let* $f: X \to Y$ *be smooth, equivariant, and strata preserving.*

(1) *If* $\bar{f}$ *is normally transverse, then* $\bar{f}^* Y$ *is a smooth* K*-manifold whose boundary is* $(\bar{f}|_{\partial X})^* Y$.

(2) *If* f *induces* $\bar{f}$ *and both are normally transverse, then the canonical map from* X *to* $\bar{f}^* Y$ *is a* K*-diffeomorphism.*

(3) *If* f *induces* $\bar{f}$, *then* f *is normally transverse if and only if* $\bar{f}$ *is normally transverse.*

Proof. — By the DST it suffices to consider the case $X = W \times \mathbf{R}^n \times \mathbf{R}^+$, $Y = W \times \mathbf{R}^m$, where W is a representation space of K and $W^K = \{0\}$. To prove (1), write $\bar{f} = (\bar{f}_1, \bar{f}_2)$ where $\operatorname{Im} \bar{f}_1 \subseteq W/K$, $\operatorname{Im} \bar{f}_2 \subseteq \mathbf{R}^m$. Let $\bar{F}(\xi, x, t) = (\bar{f}_1(\xi, x, t), x, t)$; $\xi \in W/K$, $x \in \mathbf{R}^n$, $t \in \mathbf{R}^+$. Then $\bar{F}$ is a map from X/K to X/K, $\bar{F}(\partial X/K) \subseteq \partial X/K$, and $\bar{F}$ has a non-singular differential at each point since $\bar{f}$ is normally transverse. By (1.11), $\bar{F}$ is locally smoothly invertible, so we may further reduce to the case $\bar{f}_1(\xi, x, t) = \xi$. Then

$$\bar{f}^* Y = \{(\xi, x, t, w, y) : \xi = \pi_{W,K}(w), y = \bar{f}_2(\pi_{W,K}(w), x, t)\},$$

so clearly $\bar{f}^* Y$ is a smooth manifold with the indicated boundary. We have proved (1).

To prove (2), express f as (f_1, f_2) where $\operatorname{Im} f_1 \subseteq W$, $\operatorname{Im} f_2 \subseteq \mathbf{R}^m$. As in the proof of (1), an inverse function theorem argument reduces us to the case where $f_1(w, x, t) = w$. Then the canonical map of X to $\bar{f}^* Y$ is

$$X \ni (w, x, t) \mapsto (\pi_{W,K}(w), x, t, w, f_2(w, x, t)) \in \bar{f}^* Y,$$

clearly an isomorphism.

To prove (3), we may reduce to the case where $X = Y = W$, $W^K = \{0\}$, and both f and $\bar{f}$ are origin preserving. If f is normally transverse, then f is an equivariant diffeomorphism and $\bar{f}$ is then clearly normally transverse. Suppose $(df)_0$ is singular. Let W_0 denote the kernel of $(df)_0$, considered as a subspace of W. Clearly W_0 is K-invariant. Let r_0^2 denote the square of the radius function on W_0 relative to some K-invariant inner product. Since $W^K = \{0\}$, no non-zero element of $\mathbf{R}[W]^K$ is homogeneous of degree 1, and using Taylor series one sees that r_0^2 cannot be written as $f^* h$ in any neighborhood of 0, $h \in C^\infty(W)^K$. Hence $\bar{f}$ is not a diffeomorphism near the origin of W/K, i.e. $\bar{f}$ is not normally transverse. This completes the proof of (3). ■

Example **(2.2)**. — Let $X = Y = \mathbf{R}$, $K = \{\pm 1\}$. Let $n \in \mathbf{Z}$, $n > 0$, and let $\bar{f}$ denote the map $\xi \mapsto \xi^n$ from $\mathbf{R}^+ = X/K$ to $\mathbf{R}^+ = Y/K$. One easily sees that $\bar{f}^* Y$ is a smooth submanifold of $X/K \times Y$ if and only if $n = 1$, the only case in which $\bar{f}$ is normally transverse.

We now show that the smooth lifting theorem (0.2) implies the

Smooth Palais Theorem **(2.3)**. — *Let* X *and* Y *be smooth* K-*manifolds without boundary, and let* $f: X \to Y$ *be smooth and equivariant. Let* $\overline{F} : X/K \times [0, 1] \to Y/K$ *be a smooth map such that* $\overline{F}_t = \overline{F}(\cdot, t) : X/K \to Y/K$ *is normally transverse for each* $t \in [0, 1]$, *and suppose that* f *induces* $\overline{F}_0$. *Then there is an equivariant normally transverse homotopy* $F : X \times [0, 1] \to Y$ *inducing* $\overline{F}$ *and starting at* f. *Moreover, any two such liftings of* $\overline{F}$ *differ by composition with an equivariant isotopy of* X *which starts at the identity map of* X *and induces the trivial isotopy on* X/K.

Proof. — We form the pull-back diagram

$$\begin{array}{ccc} \overline{F}^* : \ \overline{F}^*Y & \longrightarrow & Y \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi_{Y,K}} \\ \overline{F} : \ X/K \times [0, 1] & \longrightarrow & Y/K \end{array}$$

where π denotes $\pi_{\overline{F}^*Y, K}$. By (0.2) we may choose a smooth K-invariant vector field A on $\overline{F}^*Y$ such that $\pi_* A$ is the vector field $(0, d/dt)$ on $X/K \times [0, 1]$. Since π is proper, A integrates to a K-diffeomorphism of $(\overline{F}_0)^* Y \times [0, 1]$ with $\overline{F}^*Y$. But $(\overline{F}_0)^* Y \simeq X$ by (2.1.2), so we obtain a commutative diagram

$$\begin{tikzcd} X \times [0, 1] \arrow[r, "\Theta"] \arrow[dr, "\pi_{X,K} \times \mathrm{id}"'] & \overline{F}^*Y \arrow[r, "\overline{F}^*"] \arrow[d, "\pi"] & Y \arrow[d, "\pi_{Y,K}"] \\ & X/K \times [0, 1] \arrow[r, "\overline{F}"] & Y/K. \end{tikzcd}$$

$F = \overline{F}^* \circ \Theta$ is the required lift of $\overline{F}$.

If F' is another lift of $\overline{F}$ starting at f, then by the universal property of fiber products, F' factors through $\overline{F}^*Y \simeq X \times [0, 1]$, i.e. $F' = F \circ \Phi$ where $\Phi = (\Psi, t) : X \times [0, 1] \to X \times [0, 1]$ is a smooth equivariant map which induces the identity on $X/K \times [0, 1]$. By (2.1.3), Ψ is normally transverse, and it follows that Ψ is an equivariant smooth isotopy of X. Since Ψ_0 induces the identity on X/K, Ψ_0 maps each K-orbit into itself. But f is isovariant (since it is strata preserving), and $f = F'_0 = F_0 \circ \Psi_0 = f \circ \Psi_0$. It follows that Ψ_0 is the identity on each K-orbit, hence Ψ_0 is the identity. ■

Corollary **(2.4)**. — *Let* X *be a smooth* K-*manifold,* $\partial X = \emptyset$. *Suppose that* $\overline{F}$ *is a smooth isotopy of* X/K *starting at the identity. Then there is a smooth equivariant isotopy* F *of* X *starting at the identity and inducing* $\overline{F}$. ■

Remarks **(2.5)**.

(1) One can easily formulate and prove versions of (2.3) and (2.4) for K-manifolds with boundary (or corners).

(2) We conjecture that theorems (2.3) and (2.4) remain true if we only require that F and $\bar{F}$ are continuous in t. See § 4 for a related conjecture. Relevant is [79].

Example **(2.6)** *(Bierstone)*. — Let $X = \mathbf{R}$, $K = \{\pm 1\}$. We show that (0.2) holds and that the transversality conditions in (2.3) are necessary.

By (1.5), the map $x \mapsto x^2$ induces an isomorphism of $\mathbf{R}/\{\pm 1\}$ with $\mathbf{R}^+$. Via this isomorphism, $\mathfrak{X}^\infty(\mathbf{R}/\{\pm 1\})$ corresponds to the smooth vector fields on $\mathbf{R}^+$ vanishing at 0. Hence $\mathfrak{X}^\infty(\mathbf{R}/\{\pm 1\}) \simeq C^\infty(\mathbf{R}^+) y\, d/dy$. But any smooth vector field $f(y) y\, d/dy$ is induced by $\frac{1}{2} f(x^2) x\, d/dx \in \mathfrak{X}^\infty(\mathbf{R})^{\{\pm 1\}}$, so (0.2) holds in this case.

Let $\bar{F}_t$ denote the map $y \mapsto (1-t)y + ty^2$ from $\mathbf{R}^+$ to $\mathbf{R}^+$, and consider $\bar{F}_t$ as a strata preserving homotopy of the identity map of $\mathbf{R}/\{\pm 1\}$. Then $\bar{F}_t$ lifts to the equivariant map F_t, where $F_t(x) = x(1 - t + tx^2)^{\frac{1}{2}}$, $x \in \mathbf{R}$. The map F_t is smooth for $0 \leq t < 1$, while $\bar{F}_1 = y^2$ has only $\pm x|x|$ as continuous equivariant lifts. But $\bar{F}_t$ only fails to be normally transverse at $t = 1$, so (2.3) is not contradicted.

In the above example, $\bar{F}_1$ has a smooth non-equivariant lift, namely x^2. We can construct a similar example where $\bar{F}_1$ has no smooth lift at all near 0: By Glaeser [24] there is a smooth even function $f: \mathbf{R} \to \mathbf{R}$ such that $f(x) > 0$ for $x \neq 0$, f and all its derivatives vanish at 0, yet $\sqrt{f}$ is not twice differentiable. Setting $\bar{F}_t(y) = (1-t)y + tf(\sqrt{y})$ gives the desired example.

3. Smooth Vector Fields on Orbit Spaces.

We begin a deeper study of the strata preserving vector fields on an orbit space X/K. We find conditions under which $\mathfrak{X}^\infty(X/K) = \mathrm{Der}(C^\infty(X/K))$. Our final result is a proof of the smooth lifting theorem, modulo an algebraic analogue (theorem (3.7)). Chapters II, III, and IV contain the proof of (3.7).

We begin with an alternate characterization of strata preserving vector fields. Let X be a smooth K-manifold. If $\{U_\alpha\}$ is a cover of X by K-invariant open sets, then clearly there is a partition of unity in $C^\infty(X)^K$ subordinate to $\{U_\alpha\}$. Thus X/K admits smooth partitions of unity. Let $\xi \in X/K$, let $\tilde{f} \in \mathcal{M}_\xi$, and let $A \in \mathrm{Der}(C^\infty(X/K))$. Standard partition of unity arguments show that $\tilde{f}$ has a representative f in $C^\infty(X/K)$, and if f_1 is another representative, then $f - f_1 = h_1 h_2$ where $h_1, h_2 \in C^\infty(X/K)$ and $h_1(\xi) = h_2(\xi) = 0$. It follows that $A(f)(\xi)$ only depends on $\tilde{f}$, and $\tilde{f} \mapsto A(f)(\xi)$ gives rise to a linear functional $A(\xi)$ on $T^*_\xi(X/K)$, i.e. an element of $T_\xi(X/K)$. Let σ be a stratum of X/K. We say that A is **tangent** to σ if $A(\xi) \in T_\xi(\sigma) \subseteq T_\xi(X/K)$ for all $\xi \in \sigma$. Using the DST one easily proves

Proposition (**3.1**). — *Let* X *be a smooth* K-*manifold, and let* A *be a smooth vector field on* X/K.

(1) *Let* σ *be a stratum of* X/K. *Then* A *is tangent to* σ *if and only if* A *preserves the ideal in* $C^\infty(X/K)$ *vanishing on* σ.

(2) *The vector field* A *is in* $\mathfrak{X}^\infty(X/K)$ *if and only if it is tangent to all the strata of* X/K. ■

Let S be a closed semi-algebraic subset of $\mathbf{R}^d$, and suppose that S is given a stratification by semi-algebraic submanifolds of $\mathbf{R}^d$. We denote by $\mathbf{R}[S]$ the polynomial functions on S. If σ is a stratum of S, then $I^\infty(\sigma)$ (resp. $I(\sigma)$) denotes the ideal in $C^\infty(S)$ (resp. $\mathbf{R}[S]$) vanishing on σ. We use $\mathfrak{X}^\infty(S)$ (resp. $\mathfrak{X}(S)$) to denote the real-linear derivations of $C^\infty(S)$ (resp. $\mathbf{R}[S]$) leaving all the ideals $I^\infty(\sigma)$ (resp. $I(\sigma)$) invariant.

Let W be a representation space of K. We define the polynomial functions $\mathbf{R}[W/K]$ on W/K to be $(\pi^*_{W,K})^{-1}\mathbf{R}[W]^K$. If σ is a stratum of W/K, then $I^\infty(\sigma)$ (resp. $I(\sigma)$) denotes the ideal of smooth (resp. polynomial) functions vanishing on σ. We use $\mathfrak{X}(W/K)$ to denote the real-linear derivations of $\mathbf{R}[W/K]$ preserving the ideals $I(\sigma)$, and $\mathfrak{X}(W)$ denotes the polynomial vector fields on W.

Proposition (**3.2**). — *Let* (W, K, p, S, d) *be an orbit map. Then* $\bar{p}$ *induces isomorphisms*

(1) $\bar{p}_* : \mathfrak{X}^\infty(W/K) \xrightarrow{\sim} \mathfrak{X}^\infty(S)$

and

(2) $\bar{p}_* : \mathfrak{X}(W/K) \xrightarrow{\sim} \mathfrak{X}(S)$

and p *induces maps*

(3) $p_* : \mathfrak{X}^\infty(W)^K \to \mathfrak{X}^\infty(S)$

and

(4) $p_* : \mathfrak{X}(W)^K \to \mathfrak{X}(S)$.

Proof. — Part (1) is a consequence of (1.5), and (2) is a tautology. By corollary (1.3), $(\pi_{W,K})_*$ maps $\mathfrak{X}^\infty(W)^K$ into $\mathfrak{X}^\infty(W/K)$, and (3) follows from the fact that $p_* = \bar{p}_* \circ (\pi_{W,K})_*$. Since $p_*\mathfrak{X}(W)^K \subseteq \mathfrak{X}^\infty(S) \cap \mathrm{Der}(\mathbf{R}[S]) \subseteq \mathfrak{X}(S)$, (4) also holds. ■

Remark (**3.3**). — Let S be as above, and let σ be a stratum of S. In § 6 we show that $I^\infty(\sigma) = C^\infty(S) \cdot I(\sigma)$. It follows that $\mathfrak{X}(S) = \mathfrak{X}^\infty(S) \cap \mathrm{Der}(\mathbf{R}[S])$ and that

$$\mathfrak{X}(W/K) = \mathfrak{X}^\infty(W/K) \cap \mathrm{Der}(\mathbf{R}[W/K]).$$

Lemma (**3.4**). — *Let* (W, K, p, S, d) *be an orbit map,* $W^K = \{0\}$. *Let* $c : (-\varepsilon, \varepsilon) \to \mathbf{R}^d$ *be a* C^1 *curve with image in* S, *and suppose that* $c(0) = 0$. *Then* $c'(0) = 0$.

Proof. — Give W a K-invariant inner product. By perhaps increasing d, we may assume that p_1 is the square of the radius function. Let D be a constant dominating $|p_1(x)|, \ldots, |p_d(x)|$ for x in the unit sphere of W, and let $e_i = \deg p_i$, $1 \leq i \leq d$. Then $S \subseteq \{(y_1, \ldots, y_d) : y_1 \geq 0, \quad |y_i| \leq D y_1^{e_i/2} \quad i = 2, \ldots, d\}$. Clearly $(y_1 \circ c)'(0) = 0$. Since $W^K = \{0\}$, $e_i \geq 2$ and $(y_i \circ c)'(0) = 0 \quad i = 2, \ldots, d$. Hence $c'(0) = 0$. ∎

The following result was shown in [2] for finite K:

Proposition **(3.5)**. — *Let* X *be a smooth* K-*manifold, and let* $A \in \mathrm{Der}(C^\infty(X/K))$. *Then* A *is in* $\mathfrak{X}^\infty(X/K)$ *if and only if* A *is tangent to the codimension one strata of* X/K. *In particular,* $\mathfrak{X}^\infty(X/K) = \mathrm{Der}(C^\infty(X/K))$ *if (and only if)* X/K *has no codimension one strata.*

Proof. — It suffices to consider the case where X is a representation space W of K, $W^K = \{0\}$. Let (W, K, p, S, d) be an orbit map, and let $B = \bar{p}_* A \in \mathrm{Der}(C^\infty(S))$. By induction we may assume that the proposition holds for smooth actions of compact Lie groups L such that $\dim L < \dim K$ or $\dim L = \dim K$ and L has fewer components than K. In particular, we may assume the proposition for the slice representations at all points $x \in W$, $x \neq 0$. Using the DST we may then assume that B is tangent to the strata of S, except perhaps for $\{0\}$. We may also assume $\dim S \geq 2$.

Let $y_1, \ldots, y_d$ be co-ordinates on $\mathbf{R}^d$. Choose $C = \Sigma a_i \partial/\partial y_i \in \mathfrak{X}^\infty(\mathbf{R}^d)$ so that $a_i | S = B(y_i | S)$, $i = 1, \ldots, d$. Let φ_t denote the local one-parameter group generated by C. Then φ_t is defined for small t in some neighborhood of 0. If $c(t) = \varphi_t(0)$ lies in S for t near 0, then lemma (3.4) shows that $c'(0) = 0$. It follows that $C(0) = 0$, and $B \in \mathfrak{X}^\infty(S)$.

Suppose $\varphi_t(0) \notin S$ for arbitrarily small t (which we may assume are positive). Since S is closed, we may find a neighborhood U of 0 and $t > 0$ such that $\varphi_t(U) \cap S = \emptyset$. If $u \in U \cap S$, then the curve $c(r) = \varphi_r(u)$ last lies in S at some time $r_0 < t$. Since C is tangent to the strata of $S - \{0\}$, $\varphi_{r_0}(u) = 0$. Hence $U \cap S$ is contained in the curve $\{\varphi_{-r}(0) : 0 \leq r \leq t\}$, yet $\dim U \cap S \geq 2$. Thus $\varphi_t(0) \in S$ for small t, $C(0) = 0$, and $B \in \mathfrak{X}^\infty(S)$. ∎

Let (W, K, p, S, d) be an orbit map. We grade $\mathbf{R}[\mathbf{R}^d] \simeq \mathbf{R}[y_1, \ldots, y_d]$ by setting $\deg y_i = e_i = \deg p_i$, $i = 1, \ldots, d$. Then $p^* : \mathbf{R}[\mathbf{R}^d] \to \mathbf{R}[W]^K$ is degree preserving, hence $\mathrm{Ker}\, p^*$ is a homogeneous ideal in $\mathbf{R}[\mathbf{R}^d]$, and $\mathbf{R}[S] \simeq \mathbf{R}[\mathbf{R}^d]/\mathrm{Ker}\, p^*$ is naturally a graded ring. We say that $A \in \mathfrak{X}(\mathbf{R}^d)$ is of **degree** n if it maps forms of degree m to forms of degree $m+n$ for all m. We grade $\mathfrak{X}(W)$ and $\mathfrak{X}(W)^K$ similarly, and $\mathrm{Der}(\mathbf{R}[S])$ is given the grading induced from $\mathfrak{X}(\mathbf{R}^d)$. If E is a graded ring or module, we denote by E_n the elements of E of degree n.

Let σ be a stratum of S, and let (L) be the corresponding isotropy class. Since $W^{(L)}$ is stable under multiplication by $\mathbf{R}^+$, the ideal in $\mathbf{R}[W]^K$ vanishing on $W^{(L)}$ is homogeneous, hence $I(\sigma)$ is homogeneous. It follows that $\mathfrak{X}(S)$ is a graded $\mathbf{R}[S]$-submodule of $\mathrm{Der}(\mathbf{R}[S])$. Let $\mathbf{R}[[S]]$ denote the product $\prod_n \mathbf{R}[S]_n \simeq \mathbf{R}[[W]]^K$, let $I^\wedge(\sigma)$ denote $\prod_n I(\sigma)_n \subseteq \mathbf{R}[[S]]$, and let $\mathfrak{X}^\wedge(S)$ denote $\prod_n \mathfrak{X}(S)_n \subseteq \prod_n \mathrm{Der}(\mathbf{R}[S])_n = \mathrm{Der}(\mathbf{R}[[S]])$. Note that there is a Taylor series map $T=\prod_n T_n : C^\infty(S)\to \mathbf{R}[[S]]$, and it follows from E. Borel's lemma ([73], p. 78) that T is surjective. If $A\in \mathrm{Der}(C^\infty(S))$, then, as in the proof of (3.5), A lifts to a vector field $\sum_i a_i\partial/\partial y_i \in \mathfrak{X}^\infty(\mathbf{R}^d)$. It follows that A preserves Ker T and that A induces $T_*A=\sum_i T(a_i|_S)\,\partial/\partial y_i\in \mathrm{Der}(\mathrm{Im}\, T=\mathbf{R}[[S]])$.

Lemma **(3.6)**. — *Let* (W, K, p, S, d) *be an orbit map,* σ *a stratum of* S. *Then*

(1) $p_*: \mathfrak{X}(W)^K\to\mathfrak{X}(S)$ *is degree preserving.*

(2) $T(I^\infty(\sigma))\subseteq I^\wedge(\sigma)$.

(3) $T_*(\mathfrak{X}^\infty(S))\subseteq \mathfrak{X}^\wedge(S)$.

Proof. — Part (1) is an immediate consequence of the definitions. Let (L) be the isotropy class of σ. If $f\in C^\infty(W)^K$ vanishes on the cone $W^{(L)}$, then Taylor's theorem and an induction on degree show that any Taylor polynomial of f vanishes on $W^{(L)}$, so (2) holds.

Let $A\in\mathfrak{X}^\infty(S)$. Then $T_*A=\prod_n A_n$ for some $A_n\in \mathrm{Der}(\mathbf{R}[S])_n$. If $f\in I(\sigma)_m$, then $A(f)\in I^\infty(\sigma)$, and $T_{n+m}(A(f))=A_n(f)$ is in $I(\sigma)$ by (2). Hence $A_n(I(\sigma))\subseteq I(\sigma)$, $T_*A\in\mathfrak{X}^\wedge(S)$, and (3) is proved. ■

We can now prove the smooth lifting theorem modulo the following result.

Algebraic Lifting Theorem **(3.7)**. — *Let* W *be a representation space of* K. *Then*

$$(\pi_{W,K})_*\mathfrak{X}(W)^K=\mathfrak{X}(W/K).$$

Proof of (0.2). — Suppose $\{U_\alpha\}$ is a cover of X by K-invariant open sets such that (0.2) holds for each U_α. Then a partition of unity argument (using K-invariant functions) shows that (0.2) holds for X. Using the DST we may reduce to proving (0.2) when $X=K\times_L W$ or $X=(K\times_L W)\times\mathbf{R}^+$, where W is a representation space of a closed subgroup L of K. Let $A\in\mathfrak{X}^\infty(W)^L$. Then the trivial extension of A to a vector field on $K\times W$ is $K\times L$ invariant, where $(k,\ell)\cdot(k',w)=(kk'\ell^{-1},\ell w)$; $k,k'\in K$, $\ell\in L$, $w\in W$. Quotienting by L we obtain a vector field $\widetilde{A}$ on $K\times_L W$ which induces the same derivation of $C^\infty(K\times_L W)^K\simeq C^\infty(W)^L$ as does A. Thus if $(\pi_{W,L})_*$ is surjective, so is $(\pi_{K\times_L W,K})_*$. The proof for the case $(K\times_L W)\times\mathbf{R}^+$ is the same. It thus suffices to prove (0.2) for representations.

Let W and L be as above, and let (W, L, p, S, d) be an orbit map. Let $A\in\mathfrak{X}^\infty(S)$, and let $a_i=p^*A(y_i|_S)$ where $y_1,\dots,y_d$ are the co-ordinate functions on $\mathbf{R}^d$. Let $x_1,\dots,x_n$ be co-ordinates on W. Our problem is to find $\sum_j b_j\partial/\partial x_j\in\mathfrak{X}^\infty(W)^L$ such that

$$\textbf{(3.8)}\qquad \sum_j b_j\frac{\partial p_i}{\partial x_j}=a_i\qquad i=1,\dots,d.$$

By (3.6) and (3.7) we can find power series $\hat{b}_j$ solving (3.8) formally at 0. Using the DST as above, we can find formal power series solutions to (3.8) at the other points of W. By a theorem of Malgrange [53], we can then find $b_j \in C^\infty(W)$ satisfying (3.8). Averaging over L we can further arrange that $\sum_j b_j \partial/\partial x_j$ is L-invariant. ∎

The idea of using [53] in the proof above is due to E. Bierstone.

4. Split Surjectivity.

In this section we prove a strengthened form of the smooth lifting theorem. No results from this section are needed elsewhere in this paper. First some motivation: Let X be a compact smooth K-manifold. For simplicity we assume $\partial X = \emptyset$. The group Diff(X) of diffeomorphisms of X can be given the structure of an infinite dimensional Lie group modelled on a Frechet space, and the corresponding Lie algebra is $\mathfrak{X}^\infty(X)$ (see [47]). One can show that $\mathrm{Diff}(X)^K$ is a Lie subgroup of Diff(X), and the corresponding Lie algebra is $\mathfrak{X}^\infty(X)^K$. Let $\mathrm{Diff}_K(X)^K$ denote the subgroup of $\mathrm{Diff}(X)^K$ acting trivially on $C^\infty(X)^K$, and let $\mathfrak{X}^\infty_K(X)^K$ denote the corresponding subalgebra of $\mathfrak{X}^\infty(X)^K$. Then $\mathfrak{X}^\infty(X)^K$ and $\mathfrak{X}^\infty_K(X)^K$ are closed subalgebras of $\mathfrak{X}^\infty(X)$, and we give them the induced C^∞ topology. By the smooth lifting theorem, $\mathfrak{X}^\infty(X/K) \simeq \mathfrak{X}^\infty(X)^K/\mathfrak{X}^\infty_K(X)^K$, and we give $\mathfrak{X}^\infty(X/K)$ the quotient topology. Alternately, X/K is locally diffeomorphic to stratified closed semi-algebraic sets S, and one can define the same topology on $\mathfrak{X}^\infty(X/K)$ by patching together a more intrinsic topology on the spaces $\mathfrak{X}^\infty(S)$ (see lemma (4.5) below).

Now $\mathrm{Diff}_K(X)^K$ is a normal subgroup of $\mathrm{Diff}(X)^K$. It is not unreasonable to expect that $\mathrm{Diff}_K(X)^K$ is a Lie subgroup of $\mathrm{Diff}(X)^K$ and that Diff(X/K) has a Lie group structure. Then, letting a superscript o denote identity component, one is led to conjecture that there is an exact sequence of Lie groups

$$(4.1) \qquad 1 \to (\mathrm{Diff}_K(X)^K)^0 \to (\mathrm{Diff}(X)^K)^0 \xrightarrow{\pi} \mathrm{Diff}(X/K)^0 \to 1$$

such that the canonical map π has local smooth sections. Then the map $(\pi_{X,K})_*$ in

$$(4.2) \qquad 0 \longrightarrow \mathfrak{X}^\infty_K(X)^K \longrightarrow \mathfrak{X}^\infty(X)^K \xrightarrow{(\pi_{X,K})_*} \mathfrak{X}^\infty(X/K) \longrightarrow 0$$

will be **split surjective**, i.e. it will have a continuous linear section.

Theorem **(4.3)**. — *Let* X *be a smooth* K-*manifold. Then* $(\pi_{X,K})_*$ *is split surjective.*

We know two proofs of (4.3). The first is similar to those of (1.5.3) and (1.5.4) given in [52], [56], and [66]. The non-routine part of the proof involves applying Mather's generalization of E. Borel's lemma ([56]). The second proof is based upon recent progress in functional analysis due to Vogt and Wagner ([74], [75], [76]). We present this (shorter) proof below. We assume the smooth lifting theorem and we use some results from § 6.

Let W be a representation space of K, and let $\mathfrak{X}_K(W)$ denote the elements of $\mathfrak{X}(W)$ annihilating $\mathbf{R}[W]^K$. Then there is an exact sequence of $\mathbf{R}[W]^K$-modules

(4.4) $$0 \to \mathfrak{X}_K(W)^K \to \mathfrak{X}(W)^K \to \mathfrak{X}(W/K) \to 0.$$

It follows from proposition (6.8) that the modules above are noetherian.

Lemma **(4.5)**. — *Let* (W, K, p, S, d) *be an orbit map. Then*

(1) $\mathfrak{X}^\infty_K(W)^K$ *is (topologically) isomorphic to a quotient of a finite free* $C^\infty(W)^K$*-module.*

(2) $\mathfrak{X}^\infty(S)$ *and* $\mathfrak{X}^\infty(W/K)$ *are (topologically) isomorphic to a closed submodule of* $(C^\infty(W)^K)^d$.

Proof. — It follows from (6.11.1), (6.14), and (6.15) below that one can obtain the exact sequence (4.2) (with X replaced by W) by tensoring (4.4) with $C^\infty(W)^K$ over $\mathbf{R}[W]^K$. In particular, $\mathfrak{X}^\infty_K(W)^K$ is generated over $C^\infty(W)^K$ by $\mathfrak{X}_K(W)^K$. Since $\mathfrak{X}_K(W)^K$ is a noetherian $\mathbf{R}[W]^K$-module, there is an $m \in \mathbf{Z}^+$ and a continuous linear surjection from $(C^\infty(W)^K)^m$ to $\mathfrak{X}^\infty_K(W)^K$. Since $C^\infty(W)^K$ and $\mathfrak{X}^\infty_K(W)^K$ have Frechet space structures compatible with their topologies, (1) follows from an application of the open mapping theorem.

Let $A \in \mathfrak{X}^\infty(S)$, and let $y_1, \ldots, y_d$ be co-ordinates on $\mathbf{R}^d$. Then A gives rise to a d-tuple $(a_1, \ldots, a_d)$ where $a_i = A(y_i|_S) \in C^\infty(S)$, $i = 1, \ldots, d$. Clearly the elements of $C^\infty(S)^d$ corresponding to $\mathfrak{X}^\infty(S)$ form a closed subspace of $C^\infty(S)^d$. An application of the open mapping theorem establishes that the quotient topology on $\mathfrak{X}^\infty(S) \simeq \mathfrak{X}^\infty(W)^K/\mathfrak{X}^\infty_K(W)^K$ agrees with the topology induced from $C^\infty(S)^d$. ∎

Using a partition of unity argument and the DST one easily shows:

Corollary **(4.6)**. — *Let* X *be a compact smooth* K*-manifold. Then* $\mathfrak{X}^\infty_K(X)^K$ *is a quotient of a finite free* $C^\infty(X)^K$*-module, and* $\mathfrak{X}^\infty(X/K)$ *embeds as a closed submodule of a finite free* $C^\infty(X)^K$*-module.*

Let (s) denote the space of rapidly decreasing sequences of real numbers ([27], Ch. II, p. 54).

Theorem **(4.7)** (*Vogt-Wagner, see* [75]). — *Let*

(4.8) $$0 \to D \xrightarrow{i} E \xrightarrow{j} F \to 0$$

be an exact sequence of Frechet spaces (so i and j are continuous linear maps). Suppose that D *is a quotient of* (s) *and that* F *embeds in* (s). *Then there is a continuous linear map* $\varphi : F \to E$ *splitting* (4.8). ∎

Proof of (4.3). — As in the proof of (0.2), one can show that (4.3) is true if and only if it is true for representations. Since one already obtains all possible representations as slice representations on compact manifolds, we may reduce to the case where X is compact.

Corollary (4.6) shows that $\mathfrak{X}^\infty_K(X)^K$ is a quotient of finitely many copies of $C^\infty(X)^K$.

Now $C^\infty(X)^K$ is a direct summand of $C^\infty(X)$, and $C^\infty(X)$ is a direct summand of (s) ([27], Ch. II, pp. 129-130). Moreover, a sum of finitely many copies of (s) is isomorphic to (s) ([27], Ch. II, p. 54, n. 4). It follows that $\mathfrak{X}^\infty_K(X)^K$ is a quotient of (s). Similarly, $\mathfrak{X}^\infty(X/K)$ is isomorphic to a closed subspace of (s). Theorem (4.7) then shows that the sequence (4.2) splits, and our proof of theorem (4.3) is complete. ∎

5. Complexification.

In this section we study representations and orbit spaces of reductive complex algebraic groups. To any representation $\rho=(W, K)$ of K there is associated a representation $\rho_{\mathbf{C}}=(W_{\mathbf{C}}, K_{\mathbf{C}})$ of a reductive complex algebraic group $K_{\mathbf{C}}$. We investigate the relations between properties of ρ and $\rho_{\mathbf{C}}$. Our proof of the algebraic lifting theorem (3.7) requires us to switch frequently between considering ρ and considering $\rho_{\mathbf{C}}$. Our results here and in subsequent sections rely heavily on the work of D. Luna ([49-52]).

We now recall some basic properties of reductive complex algebraic groups and their representations. Details can be found in [3], [40], [50] and [58].

All algebraic groups, unless otherwise specified, are linear and defined over **C**. Let G be an algebraic group. A **representation** of G is a finite dimensional complex vector space V (the **representation space**) together with a homomorphism of algebraic groups $\rho: G\to GL(V)$. We denote the representation by ρ or (V, G). As in § 1, if $\rho'=(V', G)$, then the direct sum $\rho+\rho'$ is also denoted $(V+V', G)$, $m\rho$ or (mV, G) denotes the direct sum of m copies of ρ, and θ_m (resp. θ) denotes the trivial representation of G on $\mathbf{C}^m$ (resp. $\mathbf{C}^n$, n unspecified).

An action of G on a complex algebraic variety U is said to be **rational** if the canonical map $G\times U\to U$ is a morphism of varieties. If (V, G) is a representation of G, then V is a rational G-variety. We say that G is **reductive** if every representation of G is completely reducible. One can show that G is reductive if and only if G^0 is reductive. Let H be a reductive algebraic subgroup of G and assume that G is reductive. Then it is well-known (and follows from results quoted below) that the normalizer $N_G(H)$ of H is a reductive algebraic subgroup of G.

When working with subsets of complex algebraic varieties, we use *cl* to denote Zariski closure. If the subset is constructible (e.g. locally closed), then the Zariski closure and closure in the classical topology coincide ([59], Ch. I, § 10).

G *will always denote a reductive algebraic group.* In our references above one finds a proof of

Lemma (5.1). — *Let* U *be a rational affine* G-*variety. Then*

(1) $\mathbf{C}[U]^G$ *is a finitely generated* **C**-*algebra, and it is normal if* U *is normal.*

(2) *If* O *is a* G-*orbit in* U, *then* $cl(O)-O$ *is a union of orbits of lower dimension.*

(3) *If* I *is an ideal of* $\mathbf{C}[U]^G$, *then* $(I\mathbf{C}[U])^G=I$.

(4) $\mathbf{C}[\mathrm{U}]^{\mathrm{G}}$ *separates closed disjoint* G-*invariant algebraic subsets of* U.

(5) *If* $x \in \mathrm{U}$ *and* Gx *is closed, then* G_x *is reductive.* ■

Proposition **(5.2)**. — *Let* U *be a rational affine* G-*variety, let* $q_1, \ldots, q_d$ *be generators of* $\mathbf{C}[\mathrm{U}]^{\mathrm{G}}$, *and let* $q=(q_1, \ldots, q_d) : \mathrm{U} \to \mathbf{C}^d$. *Then*

(1) $q(\mathrm{U})$ *is the variety of relations of* $q_1, \ldots, q_d$.

(2) *If* O *is a* G-*orbit in* U, *then* $cl(\mathrm{O})$ *contains a unique closed orbit.*

(3) q *sets up a bijection between* $q(\mathrm{U})$ *and closed* G-*orbits in* U.

Proof. — Let $\mathrm{Z} \subseteq \mathbf{C}^d$ denote the variety of relations of $q_1, \ldots, q_d$. Clearly $q(\mathrm{U}) \subseteq \mathrm{Z}$. Let $z \in \mathrm{Z}$ and let I denote the ideal of z in $\mathbf{C}[\mathrm{U}]^{\mathrm{G}} \simeq \mathbf{C}[\mathrm{Z}]$. By (5.1.3), $\mathbf{IC}[\mathrm{U}]$ is a proper ideal of $\mathbf{C}[\mathrm{U}]$, hence $q(u)=z$ for any u in the (non-empty) zero set of $\mathbf{IC}[\mathrm{U}]$. Thus $q(\mathrm{U})=\mathrm{Z}$, and (1) is proved.

Let O be as in (2). By (5.1.2), an orbit of minimal dimension in $cl(\mathrm{O})$ is closed. Hence $cl(\mathrm{O})$ contains a closed orbit, and this closed orbit is unique by (5.1.4). We have proved (2), and the proof of (3) is similar. ■

Let U be a rational affine G-variety, and let U′ be a G-invariant subset of U. Let U′/G denote the set of closed G-orbits in U′, and let $\pi_{\mathrm{U,G}}$ denote the map from U to U/G which sends $u \in \mathrm{U}$ to the unique closed orbit in $cl(\mathrm{G}u)$. We say that U′ is a G-**saturated** subset of U if $\pi_{\mathrm{U,G}}^{-1}(\pi_{\mathrm{U,G}}(\mathrm{U}'))=\mathrm{U}'$, in which case $\mathrm{U}'/\mathrm{G} \simeq \pi_{\mathrm{U,G}}(\mathrm{U}')$. We give U/G the quotient structure sheaf, so $\mathbf{C}[\mathrm{U}/\mathrm{G}] \simeq \mathbf{C}[\mathrm{U}]^{\mathrm{G}}$. By (5.2), U/G is an affine variety. We will also consider the quotient holomorphic structures on U/G and U′/G. If H is a reductive algebraic subgroup of G and P is a rational affine H-variety, then (as in § 1) we can construct the **twisted product** $\mathrm{G} \times_{\mathrm{H}} \mathrm{P}$, and $\mathrm{G} \times_{\mathrm{H}} \mathrm{P}$ is a rational affine G-variety. If $\overline{\mathrm{P}}$ is an H-invariant subset of P, then $\mathrm{G} \times_{\mathrm{H}} \overline{\mathrm{P}}$ will denote the image of $\mathrm{G} \times \overline{\mathrm{P}}$ in $\mathrm{G} \times_{\mathrm{H}} \mathrm{P}$.

We now state a version of Luna's slice theorem [50]. Recall that a map φ between smooth complex algebraic varieties is **étale** if the differential of φ is everywhere an isomorphism.

Theorem **(5.3)** ([50]). — *Let* V *be a representation space of* G. *Let* Gx *be a closed orbit,* $x \in \mathrm{V}$. *Choose a* G_x-*splitting of* $\mathrm{V} \simeq \mathrm{T}_x\mathrm{V}$ *as* $\mathrm{T}_x(\mathrm{G}x)+\mathrm{N}_x$ (G_x *is reductive by* (5.1.5)), *and let* φ *denote the canonical equivariant map*

$$\mathrm{G} \times_{\mathrm{G}_x} \mathrm{N}_x \to \mathrm{V}$$
$$[g, n] \mapsto g(x+n).$$

There there is an affine open G-*saturated subset* U *of* V *and an affine open* G_x-*saturated neighborhood* B_x *of* 0 *in* N_x *such that*

(1) $\qquad \varphi : \mathrm{G} \times_{\mathrm{G}_x} \mathrm{B}_x \to \mathrm{U}$

and

(2) $\qquad \bar{\varphi} : (\mathrm{G} \times_{\mathrm{G}_x} \mathrm{B}_x)/\mathrm{G} \to \mathrm{U}/\mathrm{G}$

are étale, where $\overline{\varphi}$ *denotes the map induced by* φ. *Also,* φ *and the natural map* $G\times_{G_x}B_x \to B_x/G_x$ *induce a* G*-isomorphism of* $G\times_{G_x}B_x$ *with the fiber product* $U\times_{U/G}B_x/G_x$. *In particular, the map* φ *of* (1) *is isovariant.* ∎

We have the following immediate

Corollary **(5.4)** ([50]). — *Let* x, φ, U, *and* B_x *be as above. Then*

(1) G_y *is conjugate to a subgroup of* G_x *for* $y\in U$.

Choose a G*-saturated neighborhood* $\overline{B}_x$ *of* 0 *in* B_x *(classical topology) such that the canonical map* $\overline{B}_x/G_x \to \overline{U}/G$ *is a complex analytic isomorphism, where* $\overline{U}=\pi_{V,G}^{-1}(\overline{\varphi}((G\times_{G_x}\overline{B}_x)/G))$. *Then* $\overline{U}$ *is a* G*-saturated neighborhood of* x *and*

(2) $\varphi: G\times_{G_x}\overline{B}_x \to \overline{U}$ *is biholomorphic.* ∎

We will refer to (5.3) as the **algebraic slice theorem** (abbreviated **AST**) and we shall refer to (5.4.2) as the **holomorphic slice theorem** (abbreviated **HST**).

Let V be a representation space of G. Let $q_1, \ldots, q_d$ be homogeneous generators of $\mathbf{C}[V]^G$, let $q=(q_1, \ldots, q_d): V \to \mathbf{C}^d$, and let Z denote $q(V)$. If V, G, q, Z, and d are as above, we call q and the quintuple (V, G, q, Z, d) **orbit maps**. The orbit maps are said to be **minimal** if d is minimal. As before, we confuse q with the associated map $V\to Z$.

Let L be a reductive algebraic subgroup of G. Then $Z_{(L)}$ (resp. $(V/G)_{(L)}$) will denote the points in Z (resp. V/G) whose corresponding closed orbits have isotropy groups in (L). If $Z_{(L)}\neq\emptyset$ (which we now assume), then we say that (L) is an **isotropy class** of (V, G). Let $V^{(L)}$ denote $q^{-1}(Z_{(L)})$, let $V^{\langle L\rangle}$ denote $V^{(L)}\cap V^L$, and let $I(V^{(L)})^G$ denote the ideal in $\mathbf{C}[V]^G$ vanishing on $V^{(L)}$. We stratify Z (resp. V/G) by the collection $\{Z_{(L)}\}$ (resp. $\{(V/G)_{(L)}\}$). If $\zeta=Z_{(L)}\neq\emptyset$ (resp. $\zeta'=(V/G)_{(L)}\neq\emptyset$), then $I(Z_{(L)})$ (resp. $I((V/G)_{(L)})$) or $I(\zeta)$ (resp. $I(\zeta')$) will denote the ideal in $\mathbf{C}[Z]$ (resp. $\mathbf{C}[V/G]$) vanishing on ζ (resp. ζ'). We use $\mathfrak{X}(Z)$ (resp. $\mathfrak{X}(V/G)$) to denote the complex-linear derivations of $\mathbf{C}[Z]$ (resp. $\mathbf{C}[V/G]$) which preserve the ideals $I(\zeta)$ (resp. $I(\zeta')$).

If (W, K, p, S, d) is an orbit map and (L) is a conjugacy class in K, then $S_{(L)}$, $I(W^{(L)})^K$, etc. are defined similarly.

Lemma **(5.5)**. — *Let* (V, G, q, Z, d) *be an orbit map. Then*

(1) $\{Z_{(L)}\}$ *is a finite stratification of* Z *into locally closed irreducible smooth algebraic subvarieties.*

Suppose $Z_{(L)}\neq\emptyset$. *Then*

(2) $V^{\langle L\rangle}$ *is Zariski open in* V^L *and all orbits intersecting* $V^{\langle L\rangle}$ *are closed.*

(3) $cl(Z_{(L)})=\bigcup_{(M)\geq(L)} Z_{(M)}=q(V^L)$.

(4) *If* $g\in G$ *and* $gV^{\langle L\rangle}\cap V^L\neq\emptyset$, *then* $g\in N_G(L)$.

Proof. — The smoothness and local closedness of the $Z_{(L)}$ follows from the AST, and the irreducibility follows from (2). The finiteness of $\{Z_{(L)}\}$ is proved by induction as in the compact case; we omit the details. We have established (1). Let $x \in V$, and let Gy be the closed orbit in $cl(Gx)$. By (5.4.1) we may choose y so that $G_x \subseteq G_y$, and then Gx is closed if and only if $G_x = G_y$. If $x \in V^{\langle L \rangle}$, then $L \subseteq G_x \subseteq G_y$ where G_y is conjugate to L. It follows that $L = G_x = G_y$, hence Gx is closed. The openness of $V^{\langle L \rangle}$ in V^L is an immediate corollary of (5.4.1). Part (2) is proved.

Now $q^{-1}(cl(Z_{(L)})) \supseteq V^L$ by (2), hence $cl(Z_{(L)})$ contains $q(V^L)$. Clearly

$$q(V^L) = \bigcup_{(M) \geq (L)} Z_{(M)},$$

since $V^{\langle M \rangle} \subseteq V^L$ if $M \supseteq L$. If $(L) \not\leq (M)$, then (5.4.1) shows that any $\xi \in Z_{(M)}$ has a neighborhood which misses $Z_{(L)}$. Hence $cl(Z_{(L)}) = q(V^L)$, and (3) is proved.

Let g be as in (4). Then $gx = y \in V^L$ for some $x \in V^{\langle L \rangle}$. It follows that

$$L \subseteq G_y = gG_xg^{-1} = gLg^{-1},$$

and since L and gLg^{-1} have the same dimension and number of components, $L = gLg^{-1}$. Thus $g \in N_G(L)$. ■

Corollary **(5.6)** ([50]). — *Let* V, G, *and* Z *be as above. Then there is a unique isotropy class* (H) *such that*

(1) $Z_{(H)}$ *is Zariski open in* Z.

(2) $Z_{(L)} \neq \emptyset$ *implies* $(H) \leq (L)$. ■

We call (H) the **principal isotropy class** of (V, G), H is called a **principal isotropy group**, and the closed orbits in $V^{(H)}$ are called **principal orbits**. We warn the reader that other authors use the term " principal isotropy group " differently ([63]). The notions coincide if the set of closed orbits contains a non-empty Zariski open subset of V. (We then say that (V, G) has **generically closed orbits**.)

Let K (as always) be a compact Lie group. Then K carries a unique structure of real linear algebraic group such that any representation $K \to GL(W)$ is automatically a morphism of real algebraic groups ([10]). Associated to K is a reductive complex algebraic group $K_{\mathbf{C}}$, the **complexification** of K. If V is a complex representation space of K, then the representation extends uniquely from K to $K_{\mathbf{C}}$, and this property characterizes $K_{\mathbf{C}}$ ([9], [31], [32], [57]). Giving $K_{\mathbf{C}}$ its classical topology, one finds that K is a maximal compact subgroup of $K_{\mathbf{C}}$. If K is a real algebraic subgroup of GL(W), then $K_{\mathbf{C}}$ can be taken to be the Zariski closure of K in $GL(W_{\mathbf{C}} = W \otimes_{\mathbf{R}} \mathbf{C})$ ([9], [10]).

If L is an algebraic group, then every compact subgroup of L (classical topology) is contained in a maximal compact subgroup, all maximal compact subgroups of L are conjugate, and L is reductive if and only if it is isomorphic to the complexification of one of its maximal compact subgroups ([31], [57]).

Let V be a representation space of G. A **slice representation** of (V, G) is a representation $(V/T_x(Gx), G_x)$ where we require that Gx be closed. The slice representation is **proper** if $G_x \neq G$. We say that (V, G) is **orthogonal** (resp. **orthogonalizable**) if V is given (resp. admits) a G-invariant non-degenerate symmetric bilinear form.

Proposition (5.7). — *Let* K *be a maximal compact subgroup of* G, *and let* V *be a representation space of* G. *Then* (V, G) *is orthogonalizable if and only if* $(V, G) \simeq (W_{\mathbf{C}}, K_{\mathbf{C}})$ *for some representation space* W *of* K.

Proof. — Suppose $(V, G) \simeq (W_{\mathbf{C}}, K_{\mathbf{C}})$. Since K is compact, there is an isomorphism $W \simeq \mathbf{R}^n$ such that $\operatorname{Im} K \subseteq O(n)$. Then the image of G lies in $O(n, \mathbf{C})$. We have shown that (V, G) is orthogonalizable. Conversely, if (V, G) is orthogonalizable, then we may assume that $V = \mathbf{C}^n$ and that G has image in $O(n, \mathbf{C})$. If $\overline{K}$ is a maximal compact subgroup of $O(n, \mathbf{C})$ containing the image of K, then $\overline{K}$ is $O(n, \mathbf{C})$-conjugate to the maximal compact subgroup $O(n)$ of $O(n, \mathbf{C})$. Hence we may arrange that $\operatorname{Im} K \subseteq O(n)$, and then $(V, G) \simeq ((\mathbf{R}^n)_{\mathbf{C}}, K_{\mathbf{C}})$. ∎

Proposition (5.8). — *Let* (W, K, p, S, d) *be an orbit map. Let* $V = W_{\mathbf{C}}$, $G = K_{\mathbf{C}}$. *Then*

(1) *The natural isomorphism* $\mathbf{R}[W] \otimes_{\mathbf{R}} \mathbf{C} \simeq \mathbf{C}[V]$ *induces an isomorphism*

$$\mathbf{R}[W]^K \otimes_{\mathbf{R}} \mathbf{C} \simeq \mathbf{C}[V]^K = \mathbf{C}[V]^G,$$

and the natural extension of p *to* $p_{\mathbf{C}}: V \to \mathbf{C}^d$ *is an orbit map for* (V, G).

Let $Z = p_{\mathbf{C}}(V)$, *and let* (H) *be a conjugacy class of reductive algebraic subgroups of* G. *Without changing* (H) *we may arrange that* $H = L_{\mathbf{C}}$ *for some compact subgroup* L *of* K. *Then*

(2) $S_{(L)} \neq \emptyset$ *if and only if* $Z_{(H)} \neq \emptyset$.

If $S_{(L)} \neq \emptyset$, *then*

(3) $S_{(L)} = Z_{(H)} \cap S$, $W^{\langle L \rangle} = V^{\langle H \rangle} \cap W$,

and the isomorphism of (1) *induces isomorphisms*

(4) $I(S_{(L)}) \otimes_{\mathbf{R}} \mathbf{C} \xrightarrow{\sim} I(Z_{(H)})$.

(5) $\mathfrak{X}(S) \otimes_{\mathbf{R}} \mathbf{C} \xrightarrow{\sim} \mathfrak{X}(Z)$.

Proof. — Since K is Zariski dense in G, K-invariant elements of $\mathbf{C}[V]$ are G-invariant, and (1) is immediate. Let L and H be as in (2), and assume that $Z_{(H)} \neq \emptyset$. Since W^L is Zariski dense in V^H, it intersects the Zariski open subset $V^{\langle H \rangle}$ of V^H. If $x \in V^{\langle H \rangle} \cap W^L$, then $L \subseteq K_x \subseteq H$, and since $L_{\mathbf{C}} = H$, K_x must equal L. Hence

$$\emptyset \neq V^{\langle H \rangle} \cap W^L \subseteq W^{\langle L \rangle}, \qquad Z_{(H)} \cap S \subseteq S_{(L)}, \qquad \text{and} \qquad S_{(L)} \neq \emptyset.$$

If $S_{(L)} \neq \emptyset$, then since $S_{(L)} = Z \cap S_{(L)}$, $Z_{(M)} \cap S_{(L)} \neq \emptyset$ for some (M). Our argument above shows that $(M) = (L_{\mathbf{C}})$. Hence $S_{(L)} \neq \emptyset$ implies $Z_{(H)} \neq \emptyset$, $S_{(L)} = Z_{(H)} \cap S$, and $W^{\langle L \rangle} = V^{\langle H \rangle} \cap W$. We have established (2) and (3), and (4) and (5) follow easily. ∎

Corollary (5.9). — *Let* W *be a representation space of* K. *Then*

(1) *The slice representations of* $(W_{\mathbf{C}}, K_{\mathbf{C}})$ *are the complexifications of the slice representations of* (W, K).

(2) $(W_{\mathbf{C}}, K_{\mathbf{C}})$ *has generically closed orbits.*

Proof. — Let $x \in W$. Then $x \in W^{\langle K_x \rangle}$ and (5.8.3) shows that $K_{\mathbf{C}}x$ is closed. Hence $((T_x W/T_x(Kx))_{\mathbf{C}}, (K_x)_{\mathbf{C}}) = (T_x W_{\mathbf{C}}/T_x(K_{\mathbf{C}}x), (K_{\mathbf{C}})_x)$ and (1) follows from (5.8.2). Part (2) follows from (1), (1.4.3), and the AST. Part (2) is also proved in [49] and [50]. ∎

6. Lifting Real Analytic and Complex Analytic Vector Fields.

We analyze coherence properties of orbit spaces and their strata. We show that real analytic and complex analytic versions of the algebraic lifting theorem (3.7) are equivalent to (3.7).

Let Z be a complex affine variety. Let $\mathcal{O}_Z$ denote the structure sheaf of Z, and let $\mathscr{H}_Z$ denote the structure sheaf of the associated complex analytic variety $Z^{(h)}$. At times (as already in the notation $\mathscr{H}_Z$) we will confuse Z with $Z^{(h)}$.

Let (V, G, q, Z, d) be an orbit map, and let ζ be a stratum of Z. The sheaf of ideals corresponding to ζ is denoted $\mathscr{I}_\zeta$, and $\underline{\mathfrak{X}}_Z$ denotes the sheaf of strata preserving derivations of $\mathcal{O}_Z$. We use $\mathscr{I}^h_\zeta$ and $\underline{\mathfrak{X}}^h_Z$ to denote the complex analytic analogues of $\mathscr{I}_\zeta$ and $\underline{\mathfrak{X}}_Z$, respectively. Sheaves $\underline{\mathfrak{X}}_V$, $\underline{\mathfrak{X}}^h_V$, and $\mathscr{H}_V$ are defined similarly. We use $\mathcal{O}_{Z,z}$ (resp. $\mathscr{H}_{Z,z}$, etc.) to denote the stalk of $\mathcal{O}_Z$ (resp. $\mathscr{H}_Z$, etc.) at $z \in Z$. We use $\mathfrak{X}^h(V)$ (resp. $\mathfrak{X}(V)$, resp. $I^h(\zeta)$, etc.) as short-hand for $\underline{\mathfrak{X}}^h_V(V)$ (resp. $\underline{\mathfrak{X}}_V(V)$, resp. $\mathscr{I}^h_\zeta(Z)$, etc.). If $\mathscr{F}$ is a sheaf of $\mathcal{O}_Z$-modules, then there is an associated sheaf of $\mathscr{H}_Z$-modules $\mathscr{F}^{(h)}$, where $\mathscr{F}^{(h)}_z = \mathscr{F}_z \otimes_{\mathcal{O}_{Z,z}} \mathscr{H}_{Z,z}$, $z \in Z$ ([70]). If $\mathscr{F}$ is coherent, then $\mathscr{F}^{(h)}$ equals $\mathscr{F}(Z) \otimes_{\mathbf{C}[Z]} \mathscr{H}_Z$, and $\mathscr{F}^{(h)}$ is coherent.

Let R be a commutative ring with identity. An R-module A is said to be **flat** if tensoring with A takes exact sequences of R-modules to exact sequences of R-modules. Equivalently, if $r_1, \dots, r_n \in R$, then all relations $\sum_i r_i a_i = 0$, $a_1, \dots, a_n \in A$, are generated by relations of the form $\sum_i r_i(r'_i a) = 0$, where the $r'_i \in R$, $a \in A$, and $\sum_i r_i r'_i = 0$ ([54]). We say that A is a **faithfully flat** R-module if A is flat and $A \otimes_R J \neq 0$ for all non-zero ideals J of R. If A is a ring and R a subring of A, then A is a faithfully flat R-module if A is a flat R-module and $JA \cap R = J$ for any ideal J of R ([54]).

Proposition **(6.1)**. — *Let* (V, G, q, Z, d) *be an orbit map, and let* ζ *be a stratum of* Z. *Then*

(1) $\mathscr{H}_{Z,z}$ *is a faithfully flat* $\mathcal{O}_{Z,z}$*-module for all* $z \in Z$.

(2) $\mathscr{I}^h_\zeta = \mathscr{I}^{(h)}_\zeta$.

(3) $\underline{\mathfrak{X}}^h_Z = \underline{\mathfrak{X}}^{(h)}_Z$.

Let U *be a Stein open subset of* $Z^{(h)}$, *let* F *be a finitely generated* $\mathbf{C}[Z]$*-module, and let* $\mathscr{F}$ *denote the corresponding sheaf of* $\mathcal{O}_Z$*-modules. Then*

(4) $\mathscr{F}^{(h)}(U) = F \otimes_{\mathbf{C}[Z]} \mathscr{H}_Z(U)$.

In particular,

(5) $\mathscr{I}^h_\zeta(\mathrm{U}) = \mathrm{I}(\zeta) \otimes_{\mathbf{C}[\mathrm{Z}]} \mathscr{H}_\mathrm{Z}(\mathrm{U})$

and

(6) $\underline{\mathfrak{X}}^h_\mathrm{Z}(\mathrm{U}) = \mathfrak{X}(\mathrm{Z}) \otimes_{\mathbf{C}[\mathrm{Z}]} \mathscr{H}_\mathrm{Z}(\mathrm{U})$.

Proof. — Parts (1) and (2) are in [70]. Now $\underline{\mathfrak{X}}_\mathrm{Z}$ may be thought of as the elements $(a_1, \dots, a_d) \in (\mathcal{O}_\mathrm{Z})^d$ such that $\sum_i a_i \partial/\partial y_i$ preserves the sheaves of ideals of Z and its strata. In other words, there is an exact sequence

(6.2) $$0 \to \underline{\mathfrak{X}}_\mathrm{Z} \to (\mathcal{O}_\mathrm{Z})^d \xrightarrow{\psi} \bigoplus_{j=1}^m \mathcal{O}_\mathrm{Z}/\mathscr{I}_j$$

where each $\mathscr{I}_j$ is zero or the sheaf of ideals of a stratum of Z, ψ maps $(a_1, \dots, a_d) \in (\mathcal{O}_\mathrm{Z})^d$ into $\bigoplus_{j=1}^m (\sum_i a_i \partial f_j/\partial y_i + \mathscr{I}_j)$, and $f_1, \dots, f_m \in \mathbf{C}[y_1, \dots, y_d]$ are the generators of the ideals vanishing on Z and its strata. Replacing $\mathcal{O}_\mathrm{Z}$ by $\mathscr{H}_\mathrm{Z}$ and each $\mathscr{I}_j$ by $\mathscr{I}_j^h$ in (6.2) we obtain an analogous exact sequence involving $\underline{\mathfrak{X}}^h_\mathrm{Z}$, and then $\underline{\mathfrak{X}}^h_\mathrm{Z} = \underline{\mathfrak{X}}^{(h)}_\mathrm{Z}$ by (1) and (2). We have proved (3).

We now prove (4). There is an exact sequence

(6.3) $$0 \to \mathscr{J} \to (\mathcal{O}_\mathrm{Z})^\ell \to \mathscr{F} \to 0$$

for some integer ℓ. From (6.3) we obtain a commutative diagram with exact rows:

(6.4) $$\begin{array}{ccccccc} \mathscr{J}(\mathrm{Z}) \otimes_{\mathbf{C}[\mathrm{Z}]} \mathscr{H}_\mathrm{Z}(\mathrm{U}) & \longrightarrow & \mathscr{H}_\mathrm{Z}(\mathrm{U})^\ell & \longrightarrow & \mathscr{F}(\mathrm{Z}) \otimes_{\mathbf{C}[\mathrm{Z}]} \mathscr{H}_\mathrm{Z}(\mathrm{U}) & \longrightarrow & 0 \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\wr} & & \downarrow{\scriptstyle\beta} & & \\ 0 \longrightarrow \mathscr{J}^{(h)}(\mathrm{U}) & \longrightarrow & \mathscr{H}_\mathrm{Z}(\mathrm{U})^\ell & \longrightarrow & \mathscr{F}^{(h)}(\mathrm{U}) & \longrightarrow & 0. \end{array}$$

(The bottom line of (6.4) is exact since U Stein implies that $\mathrm{H}^1(\mathrm{U}, \mathscr{J}^{(h)}) = 0$.) Clearly the natural map β is surjective. A similar diagram with $\mathscr{F}$ replaced by $\mathscr{J}$ shows that α is surjective. A diagram chase then establishes that β is injective, and we have proved (4) (and (5) and (6)). ∎

One can prove (6.1.4) for arbitrary $\mathbf{C}[\mathrm{Z}]$-modules by taking direct limits over finitely generated submodules. A similar remark applies to (6.11.4) below.

Theorem **(6.5)** (Luna [52]). — *Let* (V, G, q, Z, d) *be an orbit map,* U *an open subset of* $\mathrm{Z}^{(h)}$. *Then*

$$\mathscr{H}_\mathrm{V}(q^{-1}(\mathrm{U}))^\mathrm{G} = q^* \mathscr{H}_\mathrm{Z}(\mathrm{U}). \ \blacksquare$$

Using the above result we prove

Theorem **(6.6)**. — *Let* (V, G, q, Z, d) *be an orbit map,* U *an open subset of* $Z^{(h)}$. *Then*

(1) *The natural map* $q_*: \underline{\mathfrak{X}}^h_V(q^{-1}(U))^G \to \mathrm{Der}(\mathscr{H}_Z(U))$ *has image in* $\underline{\mathfrak{X}}^h_Z(U)$.

Suppose that U *is Stein. If*

(2) $q_*\mathfrak{X}(V)^G = \mathfrak{X}(Z)$

then

(3) $q_*\underline{\mathfrak{X}}^h_V(q^{-1}(U))^G = \underline{\mathfrak{X}}^h_Z(U)$.

Moreover, if $0 \in U$, *then* (3) *implies* (2).

Proof. — Theorem (6.5) shows that the map q_* is defined, and the proof of corollary (1.3) shows that (1) holds. Proposition (6.1) shows that (2) implies (3). Let $A \in \mathfrak{X}(Z)$ and suppose that (3) holds and that $0 \in U$. By (6.1) again, A restricts to an element of $\underline{\mathfrak{X}}^h_Z(U)$, and by (3) there is a $B \in \underline{\mathfrak{X}}^h_V(q^{-1}(U))^G$ such that $q_*B = A|_U$. As in § 3, we may grade $\mathfrak{X}(Z)$ and $\mathfrak{X}(V)^G$ such that q_* is degree preserving. It follows that $q_*\overline{B} = A$ where $\overline{B}$ is the Taylor series of B taken to a sufficiently high degree. Thus (3) implies (2) if $0 \in U$. ∎

From proposition (5.8) one easily obtains

Theorem **(6.7)**. — *Let* (W, K, p, S, d) *be an orbit map, and let* Z *denote* $p_{\mathbf{C}}(W_{\mathbf{C}})$. *Then the following are equivalent:*

(1) $p_*\mathfrak{X}(W)^K = \mathfrak{X}(S)$.

(2) $(p_{\mathbf{C}})_*\mathfrak{X}(W_{\mathbf{C}})^{K_{\mathbf{C}}} = \mathfrak{X}(Z)$. ∎

One might wonder whether $\mathfrak{X}^h(V)^G$ contains vector fields not in the $\mathscr{H}_V(V)^G$-submodule generated by $\mathfrak{X}(V)^G$. Using theorem (6.5) and a trick of Malgrange (as reported in [64]) we see that the answer is "no."

Let U be a complex manifold, V a complex vector space. We use $\mathrm{Map}^h(U, V)$ to denote the $\mathscr{H}_U(U)$-module of holomorphic maps from U to V. If U is a complex affine variety, then $\mathrm{Map}(U, V)$ will denote the $\mathbf{C}[U]$-module of polynomial maps from U to V.

Proposition **(6.8)**. — *Let* V *and* V_1 *be representation spaces of* G, *and let* $U \subseteq V$ *be* G*-saturated and open (classical topology). Then* $\mathrm{Map}(V, V_1)^G$ *generates the* $\mathscr{H}_V(U)^G$*-module* $\mathrm{Map}^h(U, V_1)^G$, *and* $\mathrm{Map}(V, V_1)^G$ *is a noetherian* $\mathbf{C}[V]^G$*-module.*

Proof. — Let $\mathscr{H}$ denote $\mathscr{H}_{V\times V_1^*}$. There is a natural map

$$\psi: \mathrm{Map}^h(U, V_1)^G \to \mathscr{H}(U\times V_1^*)^G$$
$$f \mapsto [(u, v_1^*) \mapsto v_1^*(f(u))].$$

The image of ψ is precisely those functions in $\mathscr{H}(\mathrm{U}\times\mathrm{V}_1^*)^{\mathrm{G}}$ which are linear in V_1^*. Let $\pi_2 : \mathrm{V}^*\times\mathrm{V}_1^{**}\to\mathrm{V}_1$ be projection on the second factor coupled with the standard isomorphism of V_1^{**} with V_1. If $h\in\mathscr{H}(\mathrm{U}\times\mathrm{V}_1^*)$, then $dh(u, v_1^*)\in\mathrm{V}^*\times\mathrm{V}_1^{**}$, and $\pi_2\, dh(u, v_1^*)\in\mathrm{V}_1$. Clearly if $f\in\mathrm{Map}^h(\mathrm{U}, \mathrm{V}_1)^{\mathrm{G}}$, then

$$f=\pi_2\, d(\psi(f))\,|_{\mathrm{U}\times\{0\}}.$$

Let $q_1, \ldots, q_r$ be homogeneous generators of $\mathbf{C}[\mathrm{V}\times\mathrm{V}_1^*]^{\mathrm{G}}$, let $f\in\mathrm{Map}^h(\mathrm{U}, \mathrm{V}_1)^{\mathrm{G}}$, and let $q=(q_1, \ldots, q_r) : \mathrm{V}\times\mathrm{V}_1^*\to\mathbf{C}^r$. By theorem (6.5) there is a holomorphic function α on $q(\mathrm{U}\times\mathrm{V}_1^*)$ such that $\psi(f)=q^*\alpha$. Then

$$f=\sum_{i=1}^{r}\frac{\partial\alpha}{\partial y_i}(q_1, \ldots, q_r)\,|_{\mathrm{U}\times\{0\}}(\pi_2\, dq_i\,|_{\mathrm{U}\times\{0\}}).$$

Hence f lies in $\mathscr{H}_{\mathrm{V}}(\mathrm{U})^{\mathrm{G}}\cdot\mathrm{Map}(\mathrm{V}, \mathrm{V}_1)^{\mathrm{G}}$, and $\mathrm{Map}(\mathrm{V}, \mathrm{V}_1)^{\mathrm{G}}$ is generated by $\pi_2\, dq_i|_{\mathrm{V}\times\{0\}}$, $i=1, \ldots, r$. ∎

Using proposition (6.1) one obtains

Corollary **(6.9)**. — *Let* U, V, V_1 *and* G *be as in* (6.8), *and suppose that the image of* U *in* $(\mathrm{V}/\mathrm{G})^{(h)}$ *is Stein. Then*

(1) $\mathrm{Map}^h(\mathrm{U}, \mathrm{V}_1)^{\mathrm{G}}=\mathrm{Map}(\mathrm{V}, \mathrm{V}_1)^{\mathrm{G}}\otimes_{\mathbf{C}[\mathrm{V}]^{\mathrm{G}}}\mathscr{H}_{\mathrm{V}}(\mathrm{U})^{\mathrm{G}}$.

(2) $\underline{\mathfrak{X}}_{\mathrm{V}}^h(\mathrm{U})^{\mathrm{G}}=\mathfrak{X}(\mathrm{V})^{\mathrm{G}}\otimes_{\mathbf{C}[\mathrm{V}]^{\mathrm{G}}}\mathscr{H}_{\mathrm{V}}(\mathrm{U})^{\mathrm{G}}$. ∎

We now consider real analytic analogues of (3.7). Let $(\mathrm{W}, \mathrm{K}, p, \mathrm{S}, d)$ be an orbit map, and let σ be a stratum of S. Let $\mathscr{A}_{\mathrm{S}}$ denote the sheaf of germs of real analytic functions on S, and let $\underline{\mathfrak{X}}_{\mathrm{S}}^a$, $\mathscr{I}_\sigma^a$, and $\underline{\mathfrak{X}}_{\mathrm{W}}^a$ denote the real analytic analogues of the sheaves $\underline{\mathfrak{X}}_{\mathrm{Z}}^h$, $\mathscr{I}_\zeta^h$, and $\underline{\mathfrak{X}}_{\mathrm{V}}^h$ considered above. Let $\mathbf{R}[\mathrm{S}]_s$ denote the localization of $\mathbf{R}[\mathrm{S}]$ at $s\in\mathrm{S}$. If F is an $\mathbf{R}[\mathrm{S}]$-module, then $\mathscr{F}^{(a)}$ will denote the associated sheaf of $\mathscr{A}_{\mathrm{S}}$-modules $\mathrm{F}\otimes_{\mathbf{R}[\mathrm{S}]}\mathscr{A}_{\mathrm{S}}$. We use $\mathscr{I}_\sigma^{(a)}$ and $\underline{\mathfrak{X}}_{\mathrm{S}}^{(a)}$ to denote the sheaves associated to $\mathrm{I}(\sigma)$ and $\mathfrak{X}(\mathrm{S})$, respectively. The only topology on S referred to in this section is the classical one.

Lemma **(6.10)**. — *Let* $(\mathrm{W}, \mathrm{K}, p, \mathrm{S}, d)$ *be an orbit map, let* σ *be a stratum of* S, *and let* ζ *denote the corresponding stratum of* $\mathrm{Z}=p_{\mathbf{C}}(\mathrm{W}_{\mathbf{C}})$. *Then*

(1) $\mathscr{H}_{\mathrm{Z}}|_{\mathrm{S}}=\mathscr{A}_{\mathrm{S}}\otimes_{\mathbf{R}}\mathbf{C}$.

(2) $\mathscr{I}_\zeta^h|_{\mathrm{S}}=\mathscr{I}_\sigma^a\otimes_{\mathbf{R}}\mathbf{C}$.

(3) $\underline{\mathfrak{X}}_{\mathrm{Z}}^h|_{\mathrm{S}}=\underline{\mathfrak{X}}_{\mathrm{S}}^a\otimes_{\mathbf{R}}\mathbf{C}$.

Proof. — Let $\tilde{f}$ be a germ at $s\in\mathrm{S}$ of a real analytic function on $\mathbf{R}^d$. Let $\tilde{f}_{\mathbf{C}}$ be the corresponding complex analytic germ at $s\in\mathbf{C}^d$. We show that if $\tilde{f}$ vanishes on S, then $\tilde{f}_{\mathbf{C}}$ vanishes on Z; clearly this suffices to prove (1). Using the slice theorems we may reduce to the case $s=0$. Let f (resp. $f_{\mathbf{C}}$) be locally defined representatives of $\tilde{f}$ (resp. $\tilde{f}_{\mathbf{C}}$). Then p^*f vanishes on a K-invariant neighborhood of $0\in\mathrm{W}$, hence $(p_{\mathbf{C}})^*f_{\mathbf{C}}$

vanishes on a $K_{\mathbf{C}}$-saturated neighborhood of 0 in $W_{\mathbf{C}}$. Thus $\tilde{f}_{\mathbf{C}}$ vanishes on Z, and (1) is proved. Using the slice theorems again, we may reduce (2) to the case where σ is the image of W^K. The proof is then trivial. Part (3) follows from (1) and (2) as in proposition (6.1). ■

Proposition **(6.11)**. — *Let* (W, K, p, S, d) *be an orbit map, and let* σ *be a stratum of* S. *Then*

(1) $\mathscr{A}_{S,s}$ *is a faithfully flat* $\mathbf{R}[S]_s$*-module for all* $s \in S$.

(2) $\mathscr{I}_\sigma^a = \mathscr{I}_\sigma^{(a)}$.

(3) $\underline{\mathfrak{X}}_S^a = \underline{\mathfrak{X}}_S^{(a)}$.

Let $U \subseteq S$ *be open, and let* F *be a finitely generated* $\mathbf{R}[S]$*-module. Then*

(4) $\mathscr{F}^{(a)}(U) = F \otimes_{\mathbf{R}[S]} \mathscr{A}_S(U)$.

In particular,

(5) $\mathscr{I}_\sigma^a(U) = I(\sigma) \otimes_{\mathbf{R}[S]} \mathscr{A}_S(U)$.

(6) $\underline{\mathfrak{X}}_S^a(U) = \mathfrak{X}(S) \otimes_{\mathbf{R}[S]} \mathscr{A}_S(U)$.

Proof. — Let $Z = p_{\mathbf{C}}(W_{\mathbf{C}})$ and let ζ be the stratum of Z corresponding to σ. Using (6.10) and (5.8) one sees that (1), (2), and (3) follow from (6.1.1), (6.1.2), and (6.1.3). Let $U \subseteq S$ be open. Then U has a neighborhood basis $\{U_\alpha\}$ in $\mathbf{R}^d$ such that $U_\alpha \cap S = U$ for all α. By Grauert [26], each U_α has a neighborhood basis $\{U_{\alpha,\beta}\}$ of Stein open subsets of $\mathbf{C}^d$ such that $U_{\alpha,\beta} \cap \mathbf{R}^d = U_\alpha$ for all α, β. Then the sets $U_{\alpha,\beta} \cap Z$ are Stein, and $\{U_{\alpha,\beta} \cap Z\}$ is a neighborhood basis of U in Z. Clearly

$$\mathscr{A}_S(U) \otimes_{\mathbf{R}} \mathbf{C} = \operatorname{dir\,lim} \mathscr{H}_Z(U_{\alpha,\beta} \cap Z),$$

$$\mathscr{I}_\sigma^a(U) \otimes_{\mathbf{R}} \mathbf{C} = \operatorname{dir\,lim} \mathscr{I}_\zeta^h(U_{\alpha,\beta} \cap Z), \text{ etc.,}$$

and (4), (5), and (6) follow from (6.1.4), (6.1.5), and (6.1.6). ■

The following theorem follows from the results in [52].

Theorem **(6.12)**. — *Let* (W, K, p, S, d) *be an orbit map,* U *an open subset of* S. *Then*

$$\mathscr{A}_W(p^{-1}(U))^K = p^* \mathscr{A}_S(U). \ ■$$

Using familiar techniques one proves:

Theorem **(6.13)**. — *Let* (W, K, p, S, d) *be an orbit map, and let* U *be an open subset of* S. *Then*

(1) *The natural map* $p_* : \underline{\mathfrak{X}}_W^a(q^{-1}(U))^K \to \operatorname{Der}(\mathscr{A}_S(U))$ *has image in* $\underline{\mathfrak{X}}_S^a(U)$.

(2) $\underline{\mathfrak{X}}_W^a(p^{-1}(U))^K = \mathfrak{X}(W)^K \otimes_{\mathbf{R}[W]^K} \mathscr{A}_W(p^{-1}(U))^K$.

If

(3) $p_* \mathfrak{X}(W)^K = \mathfrak{X}(S)$,

then

(4) $p_* \underline{\mathfrak{X}}^a_W(p^{-1}(U))^K = \underline{\mathfrak{X}}^a_S(U)$,

and if $0 \in U$, *then* (4) *implies* (3). ■

We return to the C^∞ case. Let $\mathscr{C}^\infty_S$, $\mathscr{I}^\infty_\sigma$, $\underline{\mathfrak{X}}^\infty_S$, and $\underline{\mathfrak{X}}^\infty_W$ be the C^∞ analogues of $\mathscr{A}_S$, $\mathscr{I}^a_\sigma$, $\underline{\mathfrak{X}}^a_S$, and $\underline{\mathfrak{X}}^a_W$. If F is an $\mathbf{R}[S]$-module, let $\mathscr{F}^{(\infty)}$ denote the associated sheaf of $\mathscr{C}^\infty_S$-modules $F \otimes_{\mathbf{R}[S]} \mathscr{C}^\infty_S$. We use $\mathscr{I}^{(\infty)}_\sigma$ and $\underline{\mathfrak{X}}^{(\infty)}_S$ to denote the sheaves associated to $I(\sigma)$ and $\mathfrak{X}(S)$, respectively.

Proposition **(6.14)**. — *Let* (W, K, p, S, d) *be an orbit map, and let* σ *be a stratum of* S. *Then*

(1) $\mathscr{C}^\infty_{S,s}$ *is a faithfully flat* $\mathscr{A}_{S,s}$*-module for all* $s \in S$.

(2) $\mathscr{I}^\infty_\sigma = \mathscr{I}^{(\infty)}_\sigma$.

(3) $\underline{\mathfrak{X}}^\infty_S = \underline{\mathfrak{X}}^{(\infty)}_S$.

Let $U \subseteq S$ *be open, and let* F *be an* $\mathbf{R}[S]$*-module. Then*

(4) $\mathscr{F}^{(\infty)}(U) = F \otimes_{\mathbf{R}[S]} \mathscr{C}^\infty_S(U)$.

In particular,

(5) $\mathscr{I}^\infty_\sigma(U) = I(\sigma) \otimes_{\mathbf{R}[S]} \mathscr{C}^\infty_S(U)$.

(6) $\underline{\mathfrak{X}}^\infty_S(U) = \mathfrak{X}(S) \otimes_{\mathbf{R}[S]} \mathscr{C}^\infty_S(U)$.

Proof. — Using the DST (which is a theorem in the real analytic category for representations) we may reduce to the case $s = 0$ in (1). Let $\tilde{a}_1, \ldots, \tilde{a}_n \in \mathscr{A}_{S,0}$, and let B denote the module of relations of the $\tilde{a}_i$. Let $\tilde{b}_j = (\tilde{b}_{1j}, \ldots, \tilde{b}_{nj})$ generate B, $1 \leq j \leq r$, and let U be a neighborhood of 0 in $\mathbf{C}^d$ such that the $\tilde{a}_i$ and $\tilde{b}_{kj}$ have holomorphic representatives a_i and b_{kj} on U. Since $\sum\limits_i a_i b_{ij} = 0$ near $0 \in U \cap S$, $\sum\limits_i a_i b_{ij} = 0$ near $0 \in U \cap p_{\mathbf{C}}(W_{\mathbf{C}})$ by (6.10). By Oka's coherence theorem, we may shrink U so that the $b_j = (b_{kj})$ generate the germs of the relations of the a_i at any point of $U \cap p_{\mathbf{C}}(W_{\mathbf{C}})$ ([30], [73]). Restricting to S we see that the b_j generate the germs of the relations of the a_i on $U' = U \cap S$.

Suppose $\sum\limits_i \tilde{a}_i \tilde{f}_i = 0$ where $\tilde{f}_1, \ldots, \tilde{f}_n \in \mathscr{C}^\infty_{S,0}$. We may assume that the $\tilde{f}_i$ have representatives f_i in $\mathscr{C}^\infty_S(U')$ such that $\sum\limits_i a_i f_i = 0$. Taking Taylor series we obtain a relation $\sum\limits_i a_i\hat{}\, f_i\hat{} = 0$ in $\mathbf{R}[[S]]$. Since the natural map of local rings $\mathscr{A}_{S,0} \to \mathbf{R}[[S]]$ induces an isomorphism on completions, $\mathbf{R}[[S]]$ is a faithfully flat $\mathscr{A}_{S,0}$-module ([70]), and $(f_1\hat{}, \ldots, f_n\hat{})$ is in the module generated by the $b_j\hat{} = (b_{1j}\hat{}, \ldots, b_{nj}\hat{})$. Consequently, the Taylor series of $(p^* f_1, \ldots, p^* f_n)$ at 0 is in the $\mathbf{R}[[W]]$-module (even $\mathbf{R}[[W]]^K$-module) generated by the Taylor series at 0 of the $p^* b_j$. Using the DST, one obtains the analogous result at any point $w \in p^{-1}(U')$. By [53], $(p^* f_1, \ldots, p^* f_n)$ must then be in the $\mathscr{C}^\infty_W(p^{-1}(U'))$-module generated by the p^*b_j. Averaging over K, we obtain that $(p^*f_1, \ldots, p^*f_n)$ is in the $\mathscr{C}^\infty_W(p^{-1}(U'))^K$-module generated by the p^*b_j. Hence $\mathscr{C}^\infty_{S,0}$ is flat over $\mathscr{A}_{S,0}$.

Let J be an ideal in $\mathscr{A}_{S,0}$, and let $\widetilde{f} \in J\mathscr{C}^\infty_{S,0} \cap \mathscr{A}_{S,0}$. As above, the image of $\widetilde{f}$ in $\mathbf{R}[[S]]$ belongs to the ideal generated by J. Thus $\widetilde{f} \in \mathscr{A}_{S,0} \cap J\mathbf{R}[[S]] = J$, and $\mathscr{C}^\infty_{S,0}$ is faithfully flat over $\mathscr{A}_{S,0}$. We have proved (1), and similar techniques prove (2). The proofs of (3), (4), (5), and (6) are as in (6.1), where the vanishing of higher cohomology follows from the fact that $\mathscr{C}^\infty_S$ is a fine sheaf. ∎

The proof of our next result uses no new methods. The statements regarding (2) and (3) below were, of course, already established (by different techniques) in § 3.

Theorem **(6.15)**. — *Let* (W, K, p, S, d) *be an orbit map,* U *an open subset of* S. *Then*

(1) $\underline{\mathfrak{X}}^\infty_W(p^{-1}(U))^K = \mathfrak{X}(W)^K \otimes_{\mathbf{R}[W]^K} \mathscr{C}^\infty_W(p^{-1}(U))^K$.

If

(2) $p_*\mathfrak{X}(W)^K = \mathfrak{X}(S)$,

then

(3) $p_*\underline{\mathfrak{X}}^\infty_W(p^{-1}(U))^K = \underline{\mathfrak{X}}^\infty_S(U)$,

and if $0 \in U$, *then* (3) *implies* (2). ∎

Remark **(6.16)**. — Once we have proven the algebraic lifting theorem (3.7), then we obtain real analytic and complex analytic versions from (6.6), (6.7), and (6.13). This leaves open the question of analogues of (3.7) for non-orthogonalizable actions of reductive algebraic groups. We obtain some (meager) results in this direction. One could also formulate and try to prove analogous results for actions of real reductive groups.

Added in proof. — H. M. Meyer has found an embarrassingly simple example which shows that the algebraic lifting theorem does not generalize to the non-orthogonalizable case: Let G be the group of non-zero complex numbers, and let V be a three dimensional representation space of G with weights -1, 1, and 2. Let x (resp. y) denote the homogeneous generator of $\mathbf{C}[V]^G$ of degree 2 (resp. 3). Then $\mathbf{C}[V]^G \simeq \mathbf{C}[x, y]$, and V/G has strata $\{\pi_{V,G}(0)\}$ and $V/G - \{\pi_{V,G}(0)\}$. It follows that $x\partial/\partial y \in \mathfrak{X}(V/G)$. But degree considerations show that no element of $\mathfrak{X}(V)^G$ can possibly send y to x.

II. — REPRESENTATIONS WITHOUT S^3 STRATA

Let W be a representation space of K, and suppose that (W, K) has finite principal isotropy groups. Then a stratum $(W/K)_{(L)}$ is said to be an **S^3 stratum** if $L^0 \simeq S^3$ and $(W/K)_{(L)}$ is of codimension one in W/K. Equivalently, the slice representation associated to (L) is of the form $(W_L + \theta, L)$ where $(W_L, L^0) \simeq (\mathbf{Q}, S^3 = Sp(1))$. The representations (W, K) and $(W_{\mathbf{C}}, K_{\mathbf{C}})$ are said to have S^3 strata if W/K has S^3 strata.

In this chapter we show that the algebraic lifting theorem (3.7) holds for representations with finite principal isotropy groups and no S^3 strata. Many of our lemmas, etc. will concern general representations of reductive algebraic groups; our main interest and results are in the orthogonal case.

In § 7 we reduce the proof of (3.7) to showing that a certain cohomology group vanishes. This type of reduction was already used in [2]. In § 9 and § 10 we use the Hilbert-Mumford criterion ([58], Ch. 2) to show that the cohomology group vanishes, providing the representation has finite principal isotropy groups, no S^3 strata, and has an orbit space of dimension >2. In § 8 we handle the case of two-dimensional orbit spaces, and we also reduce to proving (3.7) for connected groups.

7. Reduction to a Cohomology Problem.

Let (V, G, q, Z, d) be an orbit map. Let $\mathfrak{X}_G(V)$ denote the elements of $\mathfrak{X}(V)$ annihilating $\mathbf{C}[V]^G$, and let $\mathfrak{X}^h_G(V)$ denote the holomorphic analogue. Let $\underline{\mathfrak{X}}_{Z,G}$ denote the sheaf of $\mathcal{O}_Z$-modules corresponding to $\mathfrak{X}_G(V)^G$, and let $\underline{\mathfrak{X}}^h_{Z,G}$ denote the sheaf of $\mathcal{H}_Z$-modules corresponding to $\mathfrak{X}^h_G(V)^G$. Now $\mathfrak{X}_G(V)^G$ is the kernel of $q_* : \mathfrak{X}(V)^G \to \mathfrak{X}(Z)$, $\mathfrak{X}^h_G(V)^G$ is the kernel of $q_* : \mathfrak{X}^h(V)^G \to \mathfrak{X}^h(Z)$, and our results in § 6 imply that

$$\mathfrak{X}^h_G(V)^G = \mathfrak{X}_G(V)^G \otimes_{\mathbf{C}[V]^G} \mathcal{H}_V(V)^G$$

and that $\underline{\mathfrak{X}}^h_{Z,G} = \underline{\mathfrak{X}}^{(h)}_{Z,G}$.

We say that (V, G) has the **lifting property** (or that **lifting holds** for (V, G)) if $q_* \mathfrak{X}(V)^G = \mathfrak{X}(Z)$. We say that (V, G) is **coregular** if $\mathbf{C}[V]^G$ is a **regular ring**, i.e. isomorphic to $\mathbf{C}[y_1, \ldots, y_d]$ for indeterminants $y_1, \ldots, y_d$. We use $Z_G(V)$ to denote $\pi^{-1}_{V,G}(\pi_{V,G}(0)) = q^{-1}(0)$. We say that a representation (W, K) has the lifting property, is coregular, etc. if its complexification has the lifting property, is coregular, etc. We use $\mathfrak{X}_K(W)$ to denote the elements of $\mathfrak{X}(W)$ annihilating $\mathbf{R}[W]^K$.

We will find the following lemma quite useful.

Lemma **(7.1)**. — *Let* V *be a complex vector space, and suppose that* $G \subseteq G_1$ *are reductive algebraic subgroups of* GL(V).

(1) *Lifting holds for* (V, G) *if and only if lifting holds for* $(V+\theta_m, G)$, $m \in \mathbf{Z}^+$.

(2) *If lifting holds for* (V, G), *then lifting holds for all slice representations of* (V, G).

(3) *If* (V, G) *is coregular, then all slice representations of* (V, G) *are coregular.*

(4) *Suppose* $\mathbf{C}[V]^G = \mathbf{C}[V]^{G_1}$. *Then lifting holds for* (V, G) *if and only if lifting holds for* (V, G_1), *and the two isotropy type stratifications of* $V/G \simeq V/G_1$ *agree.*

Proof. — We leave the proof of (1) to the reader. The HST and (6.6) prove (2). By (5.2), $\mathbf{C}[V]^G$ is regular if and only if $V/G \simeq \mathbf{C}^d$ for some $d \in \mathbf{Z}^+$. If (V', G') is a slice representation of (V, G), then the HST or AST shows that V'/G' is non-singular near $\pi_{V',G'}(0)$. This implies that the graded ring $\mathbf{C}[V']^{G'}$ is regular, and we have proved (3).

We now prove (4). Let $A \in \mathfrak{X}(V)^G$. Since $\mathbf{C}[V]^G = \mathbf{C}[V]^{G_1}$, the projection of A to its G_1-invariant part has the same image as A in $\mathrm{Der}(\mathbf{C}[V]^G)$. Thus all of (4) follows if we can show that the two isotropy type stratifications of $U = V/G \simeq V/G_1$ agree.

We may assume that $V^{G_1} = \{0\}$. Let u_0 denote $\pi_{V,G}(0)$. Then $\mathbf{C}[V]^{G_1}$ contains no non-zero forms of degree 1, hence neither does $\mathbf{C}[V]^G$, and it follows that $V^G = \{0\}$. Thus $\{u_0\}$ is a stratum of both stratifications.

Let $\xi \in U - \{u_0\}$, and let $G_1 x$ denote the corresponding closed G_1-orbit. Then $G_1 x$ must contain a closed G-orbit Gy, and Gy must be the unique closed G-orbit corresponding to ξ. Let N denote a G_y-complement to $T_y(Gy)$ in $T_y V$, and let $N_1 \subset N$ denote a $(G_1)_y$-complement to $T_y(G_1 y)$ in $T_y V$. Choosing a G_y-complement to N_1 in N we obtain an embedding $\mathbf{C}[N_1] \subset \mathbf{C}[N]$ and inclusions

$$\mathbf{C}[N_1]^{(G_1)_y} \subset \mathbf{C}[N_1]^{G_y} \subset \mathbf{C}[N]^{G_y}.$$

Our hypotheses and the AST show that the inclusions above are isomorphisms. Hence $(N_1)^{(G_1)_y} = N^{G_y}$, and it follows that $(V/G)_{(G_y)}$ and $(V/G_1)_{((G_1)_y)}$ agree near ξ. Thus the strata of V/G are open and closed in the strata of V/G_1. But the strata are connected (lemma (5.5)), hence the stratifications agree. ∎

Proposition **(7.2)**. — *Let* V *be a representation space of* G *such that* dim V/G = 1. *Then*

(1) (V, G) *is coregular.*

(2) (V, G) *has the lifting property.*

Proof. — We may assume that G acts non-trivially. Since V/G is normal and of dimension 1, it is non-singular. Thus (V, G) is coregular (see also [78]). Let $f: V \to \mathbf{C}$ be a minimal orbit map. Clearly $\{0\}$ and $\mathbf{C} - \{0\}$ are the strata of $f(V) = \mathbf{C}$. Let z be a co-ordinate on $\mathbf{C}$ and let $z_1, \ldots, z_n$ be co-ordinates on V. Clearly $z\partial/\partial z$ generates $\mathfrak{X}(f(V))$. But $A = \sum_i z_i \partial/\partial z_i \in \mathfrak{X}(V)^G$ and $f_* A = (\deg f) z\partial/\partial z$. Thus (V, G) has the lifting property. ∎

The next results provide some information on the codimension of inverse images of strata of V/G.

Lemma **(7.3).** — *Let* V *be a representation space of the connected semi-simple algebraic group* G, *and let* W *be a representation space of the connected compact Lie group* K.

(1) *If* $f \in \mathbf{C}[V]^G$ *and* $f = f_1 f_2 \dots f_n$ *where the* f_i *are in* $\mathbf{C}[V]$, *then* $f_i \in \mathbf{C}[V]^G$, $i = 1, \dots, n$. *In particular,* $\mathbf{C}[V]^G$ *is a unique factorization domain* (UFD).

(2) *If* $f \in \mathbf{R}[W]^K$ *and* $f = f_1 f_2 \dots f_n$ *where the* f_i *are in* $\mathbf{R}[W]$, *then* $f_i \in \mathbf{R}[W]^K$, $i = 1, \dots, n$. *In particular,* $\mathbf{R}[W]^K$ *is a UFD.*

Proof. — Let f and $f_1, \dots, f_n$ be as in (1) (or (2)). Then the f_i transform by characters of G (or K) ([80], p. 250). But G has no non-trivial complex characters and K has no non-trivial real characters. Hence the f_i are invariants. ■

Corollary **(7.4).** — *Let* V *be a representation space of* G, *and let* $(V/G)_{(L)}$ *be a stratum of* V/G *of codimension* ≥ 2. *If* G^0 *is semi-simple or* (V, G^0) *is orthogonalizable, then* $V^{(L)}$ *has codimension* ≥ 2 *in* V.

Proof. — The image of $V^{(L)}$ in V/G^0 has codimension $\geqslant 2$. Hence if G^0 is semi-simple, then there are $f_1, f_2 \in \mathbf{C}[V]^{G^0}$ which vanish on $V^{(L)}$ and are relatively prime in $\mathbf{C}[V]$. Clearly then $\operatorname{codim} V^{(L)} \geqslant 2$. If (V, G^0) is orthogonalizable, then using the HST we may reduce to the case where $(L) = (G)$, and since $Z_{G^0}(V) = Z_G(V)$ we may further assume that $G = G^0$. Then $(V, G) \simeq (W_{\mathbf{C}}, K_{\mathbf{C}})$ where K is connected and $\dim W/K \geqslant 2$. It follows that there are $f_1, f_2 \in \mathbf{C}[V]^G$ whose restrictions to W are relatively prime forms in $\mathbf{R}[W]$. Thus f_1 and f_2 are relatively prime in $\mathbf{C}[V]$, and $\operatorname{codim} V^{(L)} = \operatorname{codim} Z_G(V) \geqslant 2$. ■

Example **(7.5).** — Let $(V, G) = (2\mathbf{C}^2, O(2, \mathbf{C}))$. Then

$$\mathbf{C}^2 + \mathbf{C}^2 \ni (z_1, z_2, w_1, w_2) \mapsto (z_1^2 + z_2^2, z_1 w_1 + z_2 w_2, w_1^2 + w_2^2) \in \mathbf{C}^3$$

is an orbit map, call it q. Now $\{0\}$ is a codimension 3 stratum of $Z = q(V) = \mathbf{C}^3$. However, $Z_G(V) = q^{-1}(0)$ contains all points of the form

$$(z_1, z_2, w_1, w_2) = (a, a\sqrt{-1}, b, b\sqrt{-1}), \quad a, b \in \mathbf{C},$$

hence $Z_G(V)$ is only of codimension 2. Thus $\operatorname{codim}(V/G)_{(L)} \geq 3$ does not imply $\operatorname{codim} V^{(L)} \geq 3$. We take a closer look at $\operatorname{codim} Z_G(V)$ in § 10.

Proposition **(7.6).** — *Let* (V, G, q, Z, d) *be an orbit map, and let* $X \in \mathfrak{X}^h(Z)$. *Assume that*

(1) $V^G = \{0\}$.

(2) $\operatorname{codim} Z_G(V) \geq 2$.

(3) *Lifting holds for the proper slice representations of* (V, G).

Then there is a cohomology class $\alpha_X \in H^1(Z-\{0\}, \underline{\mathfrak{X}}^h_{Z,G})$ *such that* X *lifts to* $\mathfrak{X}^h(V)^G$ *if and only if* $\alpha_X = 0$.

Proof. — Let x be on a closed G-orbit, $x \neq 0$. Since lifting holds for the slice representation at x, the HST and (6.6) imply that there is a G-invariant holomorphic vector field A_x on a G-saturated neighborhood U_x of x such that $q_* A_x = X|_{q(U_x)}$. The differences $A_x - A_y$ annihilate $\mathscr{H}_V(U_x \cap U_y)^G$, so they give rise to a cohomology class $\alpha_X \in H^1(Z-\{0\}, \underline{\mathfrak{X}}^h_{Z,G})$.

If X lifts to $\mathfrak{X}^h(V)^G$, then clearly $\alpha_X = 0$. If $\alpha_X = 0$, then we may choose a G-invariant holomorphic lift A of X on $V - Z_G(V)$. Since $\operatorname{codim} Z_G(V) \geq 2$, Hartog's extension theorem (see [60] or [81]) shows that A extends to $\bar{A} \in \mathfrak{X}^h(V)^G$, and clearly $q_* \bar{A} = X$. ■

Corollary **(7.7)**. — *Let* V *be a representation space of* G, *and let* H *be a principal isotropy group of* (V, G). *Suppose that*

(1) $N_G(H)/H$ *is finite.*

(2) (V, G) *has generically closed orbits.*

(3) $\operatorname{codim}(V/G)_{(L)} \geq 2$ *implies* $\operatorname{codim} V^{(L)} \geq 2$.

Then (V, G) *has the lifting property. In particular, if* G *is finite, then* (V, G) *has the lifting property* (Bierstone [2]).

Proof. — Since (V, G) has generically closed orbits, $GV^{\langle H \rangle}$ is Zariski open in V. If $x \in V^{\langle H \rangle}$, then H acts trivially on the normal space at x to Gx, so any $A \in \mathfrak{X}^h_G(V)^G$ must be tangent to Gx at x. But $Gx \simeq G/H$ and $(G/H)^H = N_G(H)/H$ is zero-dimensional by (1). Hence $A(x) = 0$, and it follows that $\mathfrak{X}^h_G(V)^G = \{0\}$.

Let $q : V \to \mathbf{C}^d$ be an orbit map, let Z denote $q(V)$, and let U denote the union of all $V^{(L)}$ such that $\operatorname{codim} Z_{(L)} \leq 1$. By (7.1), (7.2), and the argument in (7.6) any $X \in \mathfrak{X}^h(Z)$ has local holomorphic lifts to U. Since $\mathfrak{X}^h_G(V)^G = \{0\}$, these local lifts combine to form $A \in \underline{\mathfrak{X}}^h_V(U)$, and $q_* A = X|_{q(U)}$. By (3), $\operatorname{codim}(V-U) \geq 2$, hence A extends to $\bar{A} \in \mathfrak{X}^h(V)^G$, and $q_* \bar{A} = X$. ■

Example **(7.8)** (cf. [77], Ch. III). — Let K be a connected compact Lie group. Then K acts on $\mathfrak{k}$ via Ad, and a principal isotropy group is a maximal torus T. Now $N_K(T)/T = \mathscr{W}$ is the (finite) Weyl group, so by complexifying and applying (7.4) and (7.7) one sees that $(\mathfrak{k}, K)$ has the lifting property. We determine (using some help from D. Farkas) the structure of $\mathfrak{X}(\mathfrak{k})^K \simeq \mathfrak{X}(\mathfrak{k}/K)$.

Now $\mathbf{R}[\mathfrak{k}]^K \simeq \mathbf{R}[p_1, \ldots, p_d]$ is a regular ring ([39]). Give $\mathfrak{k}$ a K-invariant inner product. We show that $\mathfrak{X}(\mathfrak{k})^K$ is the free $\mathbf{R}[\mathfrak{k}]^K$-module on the gradients $\operatorname{grad} p_1, \ldots, \operatorname{grad} p_d$:

We may assume that the p_i are homogeneous. Let $A \in \mathfrak{X}(\mathfrak{k})^K$, and let $x \in \mathfrak{k}^T = \mathfrak{t}$. Then $A(x)$ is T-invariant, hence $A(x) \in \mathfrak{t}$. Thus there is a natural and injective restriction map from $\mathfrak{X}(\mathfrak{k})^K$ to $\mathfrak{X}(\mathfrak{t})^{\mathscr{W}}$. Clearly the restriction of each $\operatorname{grad} p_i$ to $\mathfrak{t}$ is the gradient of $p'_i = p_i|_{\mathfrak{t}}$ with respect to the induced $\mathscr{W}$-invariant inner product on $\mathfrak{t}$. Since $\mathbf{R}[\mathfrak{k}]^{\mathrm{K}} \simeq \mathbf{R}[\mathfrak{t}]^{\mathscr{W}} \simeq \mathbf{R}[p'_1, \ldots, p'_d]$ ([39] or theorem (11.3) below), it suffices to show that $\mathfrak{X}(\mathfrak{t})^{\mathscr{W}}$ is the free $\mathbf{R}[\mathfrak{t}]^{\mathscr{W}}$-module on the grad p'_i.

Since the action of $\mathscr{W}$ on $\mathfrak{t}$ is generated by reflections, [11] shows that

$$\mathbf{R}[\mathfrak{t}] \simeq \mathbf{R}[\mathfrak{t}]^{\mathscr{W}} \otimes_{\mathbf{R}} \mathbf{R}[\mathscr{W}]$$

as a $\mathscr{W}$-module, where $\mathbf{R}[\mathscr{W}]$ denotes the group ring of $\mathscr{W}$ with the left regular representation. The homogeneous generators of $\mathfrak{X}(\mathfrak{t})^{\mathscr{W}}$ correspond to occurrences of the representation $\mathfrak{t}$ in $\mathbf{R}[\mathscr{W}]$, and this representation occurs $\dim \mathfrak{t} = d$ times ([46]). Thus $\mathfrak{X}(\mathfrak{t})^{\mathscr{W}}$ is a free $\mathbf{R}[\mathfrak{t}]^{\mathscr{W}}$-module on d homogeneous generators $\mathrm{A}_1, \ldots, \mathrm{A}_d$.

Suppose grad $p'_1, \ldots,$ grad p'_d do not generate $\mathfrak{X}(\mathfrak{t})^{\mathscr{W}}$. Then by perhaps rechoosing the p_i and A_j we can find a relation

$$\text{(7.9)} \qquad \operatorname{grad} p'_j = \sum_i f_i \mathrm{A}_i$$

where the $f_i \in \mathbf{R}[\mathfrak{t}]^{\mathscr{W}}$ are homogeneous of positive degree. Applying the vector field in (7.9) to the square of the radius function r^2 and using Euler's identity, we find that

$$(2 \deg p'_j) p'_j = \sum_i f_i \mathrm{A}_i(r^2).$$

But this equation implies that $p'_1, \ldots, p'_{j-1}, p'_{j+1}, \ldots, p'_d$ already generate $\mathbf{R}[\mathfrak{t}]^{\mathscr{W}}$, a contradiction. Hence grad $p_1, \ldots,$ grad p_d generate $\mathfrak{X}(\mathfrak{k})^{\mathscr{W}}$.

8. Two-dimensional Orbit Spaces.

In this section we prove that lifting holds for orthogonal representations with two-dimensional orbit spaces. First we reduce to the case of connected groups.

Lemma **(8.1)**. — *Let* H *be a normal reductive algebraic subgroup of* G, *and let* V *be a representation space of* G. *Then there is a representation space* V_1 *of* G (*and* G/H) *and a* G-*equivariant map* $q : \mathrm{V} \to \mathrm{V}_1$ *which is a minimal orbit map for* (V, H). *If* (V, G) *is orthogonalizable, then so is* $(\mathrm{V}_1, \mathrm{G})$.

Proof. — The space $\mathbf{C}[\mathrm{V}]^{\mathrm{H}}_n$ of forms of degree n in $\mathbf{C}[\mathrm{V}]^{\mathrm{H}}$ is a representation space of G, and the subspace D_n of $\mathbf{C}[\mathrm{V}]^{\mathrm{H}}_n$ generated by products of elements in $\bigoplus_{j<n} \mathbf{C}[\mathrm{V}]^{\mathrm{H}}_j$ is G-invariant. Let E_n be a G-complement to D_n in $\mathbf{C}[\mathrm{V}]^{\mathrm{H}}_n$, and define $q_n : \mathrm{V} \to \mathrm{E}^*_n$ by $q_n(v)(f) = f(v)$; $v \in \mathrm{V}$, $f \in \mathrm{E}_n$. Then q_n is G-equivariant, and since $\mathbf{C}[\mathrm{V}]^{\mathrm{H}}$ is noetherian,

$$q = \bigoplus_{i=1}^{m} q_i : \ \mathrm{V} \to \mathrm{V}_1 = \bigoplus_{i=1}^{m} \mathrm{E}^*_i$$

is a minimal orbit map if m is sufficiently large. If (V, G) is orthogonalizable, then D_n and E_n are orthogonalizable, hence so is $(\mathrm{V}_1, \mathrm{G})$. ∎

Proposition **(8.2)**. — *Let* V *be a representation space of* G. *Suppose that* H *is a normal algebraic subgroup of* G *of finite index (hence* H *is reductive), and suppose that* (V, H) *has the lifting property. Then* (V, G) *has the lifting property.*

Proof. — We may assume that $V^G=\{0\}$ and that $\dim V/G \geq 2$. Let $x \in V$, $x \neq 0$, and suppose that Gx is closed. Then Hx is a union of components of Gx, hence Hx is closed, and H_x is normal of finite index in G_x. By (7.1.2), lifting holds for the slice representation of H_x. Either $\dim H_x < \dim H$, H_x has fewer components than H, or $H=H_x$ and $G_x \neq G$. Hence by induction we can assume that lifting holds for the slice representation of G_x.

Let $q_1: V \to V_1$ be a G-equivariant orbit map for (V, H), and let $q_2: V_1 \to \mathbf{C}^d$ be an orbit map for $(V_1, G/H)$. Then $q=q_2 \circ q_1 : V \to \mathbf{C}^d$ is an orbit map for (V, G). Let $X \in \mathfrak{X}^h(Z=q(V))$. As in the proof of (7.6), X has local G-invariant holomorphic lifts to $V-q^{-1}(0)$. Quotienting by the action of H we obtain local G/H-invariant holomorphic lifts of X to $Z_1-q_2^{-1}(0)$, where $Z_1=q_1(V)$. Since G/H is finite, these local holomorphic lifts are unique, and we obtain X_1 on $Z_1-q_2^{-1}(0)$ covering X. The finiteness of G/H also implies that $q_2^{-1}(0)=\{0\}$, and since Z_1 is normal of dimension $\dim V/H \geq 2$, any holomorphic function on $Z_1-\{0\}$ extends to a holomorphic function on Z_1 ([60]). Hence X_1 has an extension (also called X_1) to all of Z_1. By construction, X_1 is strata preserving on $Z_1-\{0\}$, hence it is clear that $X_1 \in \mathfrak{X}^h(Z_1)$ except perhaps when $\{0\}$ is a stratum of Z_1.

Since $V^G=\{0\}$, $\{0\}$ is a stratum of Z. Suppose $\{0\}$ is a stratum of Z_1, and let $f \in \mathscr{H}_{Z_1,0}$ vanish at 0. Since G/H is finite, $q_2: Z_1 \to Z$ is finite, and there is an $n \in \mathbf{Z}^+$ and $a_1, \ldots, a_n \in \mathscr{H}_{Z,0}$ such that

$$f^n + \sum_{i=1}^{n} (q_2^* a_i) f^{n-i} = 0.$$

Clearly $a_n(0)=0$. Applying X_1 to the above equation and evaluating at 0 we find that $(q_2^* a_{n-1})(0)(X_1 f)(0)=0$. If $(X_1 f)(0) \neq 0$, then $a_{n-1}(0)=0$, and applying X_1 repeatedly we find that $a_n, a_{n-1}, \ldots, a_1$, and 1 all vanish at 0. This is absurd, so $(X_1 f)(0)=0$, and it follows that $X_1 \in \mathfrak{X}^h(Z_1)$. Since (V, H) has the lifting property, X_1 lifts to $\mathfrak{X}^h(V)^H$, hence X lifts to $\mathfrak{X}^h(V)^G$. ∎

Before tackling two-dimensional orbit spaces, we require two preliminary results on stratifications. First, as a corollary of lemma (7.3) we have

Lemma **(8.3)**. — *Let* V *be a representation space for the connected semi-simple algebraic group* G, *and let* W *be a representation space for the compact connected Lie group* K.

(1) *Let* $(V/G)_{(L)}$ *be a codimension* 1 *stratum of* V/G. *Then* $I(V^{(L)})^G$ *is a principal prime ideal.*

(2) *Let* $(W/K)_{(M)}$ *be a codimension* 1 *stratum of* W/K. *Then* $I(W^{(M)})^K$ *is a principal prime ideal. Any form generating* $I(W^{(M)})^K$ *is of even degree, and we can choose a generating form which is a non-negative function on* W.

Proof. — The ideals $I(V^{(L)})^G$ and $I(W^{(M)})^K$ are principal since $\mathbf{C}[V]^G$ and $\mathbf{R}[W]^K$ are UFD's. Strata are irreducible (lemma (5.5)), so their ideals are always prime. Let f be a form generating $I(W^{(M)})^K$. Since the set of principal orbits is connected and dense in W/K, perhaps changing f by a scalar we may arrange that $f>0$ on principal orbits, and then $f\geq 0$. Since $f(x)\geq 0$ and $f(-x)\geq 0$ for all $x\in W$, f is homogeneous of even degree. ∎

Lemma **(8.4)**. — *Let* (W, K, p, S, d) *be an orbit map, and let* U_i *denote the Zariski closure in* $\mathbf{R}^d$ *of the strata of* S *of codimension* $\geq i$. *Then*

(1) $\operatorname{codim} U_i \geq i + \operatorname{codim} S$.

(2) S *is the closure (classical topology) of a component* C *of* $U_0 - U_1$, C *is a manifold without boundary, and* $\dim C = \dim S$.

(3) *If* (W, K) *is coregular, then every non-principal stratum of* S *lies in the closure of a codimension one stratum.*

Proof. — Part (1) is an immediate corollary of (5.8.3); the component C in (2) is clearly the image of the principal orbits. If (3) fails, then by (7.1.3) we may find a case where (W, K) is coregular, $\dim S\geq 2$, and $\{0\}$ is the only non-principal stratum of S. By (2), $S=\mathbf{R}^d$, but this is impossible by (3.4). ∎

Theorem **(8.5)**. — *Let* W *be a faithful orthogonal representation space of* K. *If* $\dim W/K = 2$ *and* $\dim K>0$, *then*

(1) (W, K) *is coregular.*

(2) $\mathfrak{X}(W/K)$ *is generated by the images of the gradients of generators of* $\mathbf{R}[W]^K$.

Proof. — The following demonstration of (1) (improving upon our original one) is due to Th. Vust: Let (V, G) denote $(W_{\mathbf{C}}, K_{\mathbf{C}})$, and let $r^2\in\mathbf{R}[W]^K$ denote the square of the radius function. The complex zero set of r^2 is a G-invariant hypersurface Y of V whose only singularity is at 0. Our hypotheses imply that $\dim V\geq 3$, and it follows that Y is irreducible and non-singular in codimension one, hence normal ([59], p. 391). Thus Y/G is normal (lemma (5.1)) and of dimension one, hence smooth. It follows that $\mathbf{R}[W]^K/(r^2)$ is regular, where (r^2) denotes the ideal generated by r^2. Thus $\mathbf{R}[W]^K$ is regular, and we have proved (1).

By (8.1) and (1), $\mathbf{R}[W]^{K^0}=\mathbf{R}[r^2,f]$ where f is homogeneous and $k^*f=\pm f$ for all $k\in K$. Assume that (2) holds for K^0. Then the images of $\operatorname{grad} r^2$ and $\operatorname{grad} f$ generate $\mathfrak{X}(W/K^0)$. If f is K-invariant, then (7.1.4) shows that (2) holds. If f is not K-invariant, then it follows from (8.2) that the images of $\operatorname{grad} r^2$ and $f\operatorname{grad} f=\frac{1}{2}\operatorname{grad} f^2$ generate $\mathfrak{X}(W/K)$ over $\mathbf{R}[W]^K=\mathbf{R}[r^2,f^2]$. Hence we may reduce to the case that K is connected.

Let $\Sigma(W)$ denote the unit sphere in W. Then ([7], Thm. IV.8.2) either $\Sigma(W)/K$ is diffeomorphic to S^1 and $\Sigma(W)$ fibers over S^1, or $\Sigma(W)/K$ is diffeomorphic to the unit interval (one can also easily derive this result from (1.5)). If $\Sigma(W)$ fibers over S^1, then the exact homotopy sequence of a fibration shows that $\Sigma(W)$ is also a 1-sphere. Then $K=\{id\}$, a case we have ruled out. Thus $\Sigma(W)/K \simeq [0,1]$. Let (K_1) and (K_2) denote the isotropy classes corresponding to $\{0\}$ and $\{1\}$.

Case 1: $(K_1)\neq(K_2)$. — Let $0\neq x\in W^{K_1}$. Then $K_1\subseteq K_x$. Since there are no isotropy classes between (K_1) and (K), we must have $K_1=K_x$. Since the image of W^{K_1} in W/K is homeomorphic to $\mathbf{R}^+$, any two points in the unit sphere $\Sigma(W^{K_1})$ can be joined by an element of K, hence (see (5.5.4)) by an element of $N_K(K_1)$. Thus $N_K(K_1)$ acts transitively on $\Sigma(W^{K_1})$, and $\mathbf{R}[W^{K_1}]^{N_K(K_1)}$ is generated by the restriction of r^2. By (8.3) there is a homogeneous non-negative generator f_1 of $I=I(W^{(K_1)})^K$. We have shown that $\mathbf{R}[W]^K/I$ is generated by r^2, hence $\mathbf{R}[W]^K=\mathbf{R}[f_1, r^2]$. Working with K_2 instead of K_1 we see that $\mathbf{R}[W]^K=\mathbf{R}[f_2, r^2]$ where f_2 is a homogeneous non-negative generator of $I(W^{(K_2)})^K$. Since $\mathbf{R}[f_1, r^2]=\mathbf{R}[f_2, r^2]$, $\deg f_1=\deg f_2=2e$ for some $e\in\mathbf{Z}^+$, and it follows that $f_2=a(r^2)^e-bf_1$ for some $a, b\in\mathbf{R}^+$. Changing f_1 and f_2 by positive scalars we may arrange that

(8.6) $$f_2=(r^2)^e-f_1.$$

By Euler's identity,

(8.7) $$(\operatorname{grad} r^2)(f_i)=(\operatorname{grad} f_i)(r^2)=4ef_i \qquad i=1,2.$$

Since $(\pi_{W,K})_*\operatorname{grad} f_1$ preserves the strata of W/K, $(\operatorname{grad} f_1)(f_1)=\alpha f_1$ for some $\alpha\in\mathbf{R}[W]^K$. Since α is homogeneous of degree $2e-2$, we see that

(8.8) $$(\operatorname{grad} f_1)(f_1)=c(r^2)^{e-1}f_1, \qquad c\in\mathbf{R}.$$

Now

$$(\operatorname{grad} f_1)(f_2)=(\operatorname{grad} f_1)((r^2)^e-f_1)=(4e^2-c)(r^2)^{e-1}f_1$$

by (8.6), (8.7), and (8.8). But $(\operatorname{grad} f_1)(f_2)$ must be divisible by f_2, hence

(8.9) $$c=4e^2.$$

Let $X\in\mathfrak{X}(W/K)$, and consider X as a derivation of $\mathbf{R}[W]^K$. Then $X(f_2)$ is a multiple of f_2, so modifying X by a multiple of grad r^2 we may assume that $X(f_2)=0$. Using (8.6) we find that

$$X(f_1)=e(r^2)^{e-1}X(r^2).$$

But $X(f_1)$ also equals βf_1 for some β, and since $\mathbf{R}[r^2, f_1]$ is a UFD, we find that

$$X(f_1)=\gamma e(r^2)^{e-1}f_1; \qquad X(r^2)=\gamma f_1$$

for some γ. Then (8.7), (8.8), and (8.9) show that $X=(\gamma/4e)\operatorname{grad} f_1$. This completes our proof that (2) holds in case 1.

Case 2: $(K_1)=(K_2)$. — Since the image of $\Sigma(W^{K_1})$ in W/K is two points, we must have $W^{K_1}\simeq\mathbf{R}$ and $N_K(K_1)=K_1$. Let f denote a non-negative homogeneous generator of $I(W^{(K_1)})^K$, and let x be a co-ordinate on W^{K_1} such that r^2 restricts to x^2. If r^2 generates the restriction of $\mathbf{R}[W]^K$ to W^{K_1}, then $\mathbf{R}[W]^K = \mathbf{R}[f, r^2]$ and $p = (f, r^2) : W \to \mathbf{R}^2$ is an orbit map. Lemma (8.4) shows that $p(W) = \{(x_1, x_2) : x_1 \geq 0\}$, yet x_2 must be non-negative since $r^2 \geq 0$. Hence there is an $h \in \mathbf{R}[W]^K$ homogeneous of odd degree m such that h restricts to x^m on W^{K_1}. We may assume that m is minimal, in which case r^2 and h generate the restriction of $\mathbf{R}[W]^K$ to W^{K_1}. Thus $\mathbf{R}[W]^K = \mathbf{R}[f, r^2, h]$. If $m = 1$, then $K = K_1$ and $W \simeq W_1 + \theta_1$ where K acts transitively on $\Sigma(W_1)$. Clearly (2) then holds.

We are reduced to the case $m \geq 3$. Since $\mathbf{R}[W]^K$ is regular and $\mathbf{R}[W]^K/(f)$ is not (being isomorphic to $\mathbf{R}[x^2, x^m]$, $m \geq 3$), the function f cannot be part of a minimal generating set for $\mathbf{R}[W]^K$. Hence $\mathbf{R}[W]^K = \mathbf{R}[r^2, h]$, and then clearly $(r^2)^m - h^2$ generates $I(W^{(K_1)})^K$. Thus we can arrange that

(8.10) $$f = (r^2)^m - h^2.$$

Let $X \in \mathfrak{X}(W/K)$, and consider X as a derivation of $\mathbf{R}[W]^K$. Then X preserves the ideal (f), so modifying X by a multiple of grad r^2 we can arrange that $X(f) = 0$. It follows that

$$0 = X((r^2)^m - h^2) = m(r^2)^{m-1}X(r^2) - 2hX(h).$$

Since $\mathbf{R}[r^2, h]$ is a UFD, there is a $\gamma \in \mathbf{R}[r^2, h]$ such that

(8.11) $$X(h) = \gamma m (r^2)^{m-1}; \quad X(r^2) = 2\gamma h.$$

Now note that $(\operatorname{grad} h)(f) = \alpha f$ where α is homogeneous of degree $m-2$, hence zero. Reasoning as above we find that

(8.12) $$(\operatorname{grad} h)(h) = m^2 (r^2)^{m-1}; \quad (\operatorname{grad} h)(r^2) = 2mh.$$

From (8.11) and (8.12) we see that $X = (\gamma/m) \operatorname{grad} h$. This completes our proof of (2). ∎

Example **(8.13)**. — The tensor product representation of $SO(n) \times SO(2)$ on $\mathbf{R}^n \otimes_{\mathbf{R}} \mathbf{R}^2$ comes under case 1, $n \geq 2$. The adjoint representation of any connected simple compact rank 2 Lie group comes under case 2.

Corollary **(8.14)**. — *Let* W *be a faithful representation space of* K, $\dim W/K = 2$, *and suppose that* (W, K) *is not isomorphic to* $(\mathbf{R}^2, \mathbf{Z}_m)$ *where* $m > 1$ *and* $\mathbf{Z}_m \subseteq SO(2)$ *acts on* $\mathbf{R}^2$ *as rotations. Then* (8.5.1) *and* (8.5.2) *hold.*

Proof. — Theorem (8.5) covers the cases where K is not finite. If (W, K) is isomorphic to some $(\mathbf{R}^2, \mathbf{Z}_m)$, $m > 1$, then one easily sees that both (8.5.1) and (8.5.2) fail. If K is finite and (W, K) is not isomorphic to any $(\mathbf{R}^2, \mathbf{Z}_m)$, $m > 1$, then K is generated by reflections, and the arguments of example (7.8) apply. ∎

Combining (7.2), (7.7), and (8.5) we obtain

Corollary **(8.15).** — *Let* W *be a representation space of* K, $\dim W/K \leq 2$. *Then* (W, K) *has the lifting property.* ■

Remark **(8.16).** — We outline another proof that lifting holds if $\dim W/K = 2$: First one carries out the following two tasks:

Step 1: Classify all representations (W, K) such that $\dim K > 0$, (W, K) has trivial principal isotropy groups, and $\dim W/K = 2$.

Step 2: Prove that the gradients of generators of $\mathbf{R}[W]^K$ generate $\mathfrak{X}(W/K)$ for the representations found in step 1.

Assuming steps 1 and 2 above we can show that (W, K) has the lifting property: If $\dim K = 0$ or (W, K) has trivial principal isotropy groups, then (7.7) or step 1 shows that (W, K) has the lifting property. Suppose that (W, K) has non-trivial principal isotropy class (H). In § 11 we see that the inclusion $W^H \to W$ induces an isomorphism

$$\mathbf{R}[W^H]^N \rightleftarrows \mathbf{R}[W]^K,$$

where $N = N_K(H)/H$ acts on W^H with trivial principal isotropy groups. Moreover, we will see that the natural map $W^H/N \to W/K$ maps strata of W^H/N onto strata of W/K. If N is finite, then lifting for (W, K) follows from (7.7). If $\dim N > 0$, then step 2 shows that $\mathfrak{X}(W^H/N)$ is generated by gradients, and as in example (7.8) it follows that $\mathfrak{X}(W/K)$ is generated by gradients. Thus (W, K) has the lifting property.

Steps 1 and 2 are not as arduous as one might fear. Using the DST and induction one can show that, in general, if (W, K) has finite principal isotropy groups, then $\dim W/K \geq \operatorname{rank} K$ (see [7], Cor. IV.5.4 for generalizations). Thus for the representations considered in step 1, $\operatorname{rank} K \leq 2$. A glance at the list of low-dimensional representations of the connected rank 2 compact Lie groups rules out the case that K^0 is simple. Thus K^0 is a product of at most two rank 1 groups, and the classification of step 1 results in a rather short list.

9. Vector Fields Annihilating the Invariants.

Let V be a representation space of G. The homomorphism $G \to GL(V)$ induces a G-equivariant homomorphism $\eta : \mathfrak{g} \to \operatorname{Der}(\mathbf{C}[V]) = \mathfrak{X}(V)$. Clearly $\eta(\mathfrak{g}) \subseteq \mathfrak{X}_G(V)$. We thus obtain a sequence of maps:

$$\operatorname{Map}(V, \mathfrak{g}) \xrightarrow{\sim} \mathbf{C}[V] \otimes_{\mathbf{C}} \mathfrak{g} \xrightarrow{\mathrm{id} \otimes \eta} \mathbf{C}[V] \otimes_{\mathbf{C}} \mathfrak{X}_G(V) \longrightarrow \mathfrak{X}_G(V),$$

where the last map is the one defining the $\mathbf{C}[V]$-module structure on $\mathfrak{X}_G(V)$. The composition gives a mapping from $\operatorname{Map}(V, \mathfrak{g})$ to $\mathfrak{X}_G(V)$ whose image we denote by $\mathfrak{X}_{\operatorname{Ad} G}(V)$. Now assume that (V, G) is orthogonal and has finite principal isotropy groups. Then $\operatorname{Map}(V, \mathfrak{g}) \xrightarrow{\sim} \mathfrak{X}_{\operatorname{Ad} G}(V)$, and we show that $\mathfrak{X}_{\operatorname{Ad} G}(V) = \mathfrak{X}_G(V)$ if and only if (V, G) has no S^3 strata. In § 10 we show that if $\mathfrak{X}_{\mathrm{Ad}\,G}(V)=\mathfrak{X}_G(V)$, then (V, G) has the lifting property.

Terms $\mathfrak{X}^h_{\mathrm{Ad}\,G}(V)$, $\mathfrak{X}_{\mathrm{Ad}\,K}(W)$, etc. have the obvious definitions.

Example (9.1). — Let $(W, K)=(\mathbf{R}^2, SO(2))$, and let x_1, x_2 be the usual co-ordinates on $\mathbf{R}^2$. If $A=a\partial/\partial x_1+b\partial/\partial x_2\in\mathfrak{X}_K(W)$, then $0=A(x_1^2+x_2^2)=2ax_1+2bx_2$, and $a=-x_2c$, $b=x_1c$ for some c. Hence $A=c(-x_2\partial/\partial x_1+x_1\partial/\partial x_2)$ where $-x_2\partial/\partial x_1+x_1\partial/\partial x_2$ is the image of a generator of $\mathfrak{k}$. Hence $\mathfrak{X}_K(W)=\mathfrak{X}_{\mathrm{Ad}\,K}(W)$ and $\mathfrak{X}_K(W)^K=\mathfrak{X}_{\mathrm{Ad}\,K}(W)^K$.

Example (9.2). — Let $(W, K)=(\mathbf{Q}, S^3)$ where $S^3\subseteq\mathbf{Q}$ acts via left multiplication. Let $L=S^3$ act on $\mathbf{Q}$ by $(q, \ell)\mapsto q\ell^{-1}$, $q\in\mathbf{Q}$, $\ell\in L$. The actions of K and L commute, so the image of $\mathfrak{l}$ lies in $\mathfrak{X}_K(W)^K$. However, $\mathfrak{X}_{\mathrm{Ad}\,K}(W)^K$ contains no vector fields with degree 1 coefficients since $\mathfrak{k}^K=\{0\}$. Thus $\mathfrak{X}_K(W)^K\neq\mathfrak{X}_{\mathrm{Ad}\,K}(W)^K$.

We can be more precise: Let $\langle\ ,\ \rangle$ denote the standard inner product on $\mathbf{Q}\simeq\mathbf{R}^4$, let r^2 denote the square of the radius function, and if $C, D\in\mathfrak{X}(\mathbf{R}^4)$, let $\langle C, D\rangle$ denote the function $x\mapsto\langle C(x), D(x)\rangle$. There is a basis A_1, A_2, A_3 of $\mathfrak{l}\subseteq\mathfrak{X}(\mathbf{R}^4)$ such that $\langle A_i, A_j\rangle=\delta_{ij}r^2$, and there is a similar basis B_1, B_2, B_3 of $\mathfrak{k}$. If $B\in\mathfrak{X}_{\mathrm{Ad}\,K}(W)^K$, then $r^2B=\sum_i\langle B, A_i\rangle A_i$ where each $\langle B, A_i\rangle$ is a polynomial in r^2 with no constant or linear terms, else $\mathfrak{X}_{\mathrm{Ad}\,K}(W)^K$ contains elements with degree 1 coefficients. Thus $B=r^2B'$ where $B'\in\mathfrak{X}_K(W)^K$. If $A\in\mathfrak{X}_K(W)^K$, then $r^2A=\sum_i\langle A, B_i\rangle B_i\in\mathfrak{X}_{\mathrm{Ad}\,K}(W)^K$. Hence $\mathfrak{X}_{\mathrm{Ad}\,K}(W)^K=r^2\mathfrak{X}_K(W)^K$.

Proposition (9.3). — *Let* W *be a representation space of* K, *and let* G *(resp.* V*) denote* $K_{\mathbf{C}}$ *(resp.* $W_{\mathbf{C}}$*). Suppose that the principal isotropy groups of* (W, K) *are finite and that* (W, K) *has no* S^3 *strata. Then*

(1) *The points v in* V *with* $\dim G_v>0$ *are of codimension* ≥ 2.

(2) $\mathfrak{X}^h_G(V)=\mathfrak{X}^h_{\mathrm{Ad}\,G}(V)$.

(3) $\mathfrak{X}_K(W)=\mathfrak{X}_{\mathrm{Ad}\,K}(W)$.

Proof. — Since $\mathfrak{X}^h_G(V)=\mathfrak{X}^h_{G^0}(V)$, we may reduce to the case that G is connected. Choose a basis $A_1, \ldots, A_r$ for $\mathfrak{g}$, and let $\omega=A_1\wedge\ldots\wedge A_r$ denote the corresponding (G-invariant) section of $\Lambda^r(T(V))$. Part (1) is equivalent to showing that ω has zeroes of codimension ≥ 2. Clearly ω never vanishes on the set of principal orbits. Suppose Gx lies on a codimension one stratum of V/G. Let (N_x, G_x) be the slice representation at x. By the HST, a G-saturated neighborhood $\overline{U}$ of x is G-biholomorphic to $G\times_{G_x}\overline{B}_x$ where $\overline{B}_x$ is a G_x-saturated neighborhood of 0 in N_x. There is an isomorphism $(N_x, G_x)\simeq((W_L+\theta)_{\mathbf{C}}, L_{\mathbf{C}})$ where $\dim W_L/L=1$. If L is finite, then $\omega\neq 0$ on $\overline{U}$. If $L^0\neq\{id\}$, then L^0 is a covering space of a sphere in W_L since (W_L, L) has finite principal isotropy groups. Now (W, K) has no S^3 strata, so up to a finite kernel we must have $(W_L, L^0)\simeq(\mathbf{R}^2, SO(2))$. Clearly then the points of $\overline{U}$ with infinite isotropy group have codimension 2. Applying (7.4) we conclude that ω has zeroes of codimension ≥ 2, and (1) is proved.

Let $A \in \mathfrak{X}^h_G(V)$, and let $U \subseteq V$ denote the set of principal orbits. Clearly we may uniquely write $A(x)$ as a sum $\sum_{i=1}^{r} f_i(x) A_i(x)$ for $x \in U$. Then $A \wedge A_2 \wedge \ldots \wedge A_r = f_1 \omega$ on U, and f_1 extends to a well-defined and holomorphic function off the zero set of ω. By Hartog's theorem f_1 extends to a holomorphic function f_1' on V. Similarly $f_2, \ldots, f_r$ extend to $f_2', \ldots, f_r'$ and $A = \sum_i f_i' A_i \in \mathfrak{X}^h_{\mathrm{Ad}\,G}(V)$. We have proved (2), and (3) follows easily. ■

10. Depth Estimates.

Let V be a representation space of G. Using the Hilbert-Mumford criterion ([58], Ch. 2) we obtain estimates for the codimension of $Z_G(V)$. If (V, G) is orthogonal and has finite principal isotropy groups and no S^3 strata, then these estimates establish the vanishing of the cohomology obstructions of proposition (7.6).

In this section we will need to use some of the structure theory of reductive algebraic groups. All the results we use are in [3] or [40].

$\mathbf{C}^*$ will denote the multiplicative group of non-zero complex numbers, and ν_n will denote its one-dimensional representation of weight n, $n \in \mathbf{Z}$. (We will never use the notation $\mathbf{C}^*$ to refer to the dual space of $\mathbf{C}$.)

Proposition (**10.1**). — *Let* V *be a representation space of* G. *Then*

$$\dim Z_G(V) \leq \dim Z_T(V) + \frac{1}{2}(\dim G - \operatorname{rank} G)$$

where $T \simeq (\mathbf{C}^*)^{\operatorname{rank} G}$ *is a maximal torus of* G.

Proof. — Since $Z_G(V) = Z_{G^0}(V)$, we may assume that G is connected. Let $x \in Z_G(V)$. By the Hilbert-Mumford criterion there is a homomorphism $\bar\lambda : \mathbf{C}^* \to G$ such that $\lim_{z\to 0} \bar\lambda(z)x = 0$ (classical topology). Since T is a maximal torus, there is a $g \in G$ such that $z \mapsto (\lambda(z) = g\bar\lambda(z)g^{-1})$ has image in T. Thus $x \in GZ_\lambda$ where

$$Z_\lambda = \{x \in V : \lim_{z\to 0} \lambda(z)x = 0\}.$$

Write $V = V_1 \oplus \ldots \oplus V_n$ where the V_i are 1-dimensional weight spaces of T with weights μ_i, $i = 1, \ldots, n$. If $\lambda : \mathbf{C}^* \to T$ is a homomorphism, then

$$\lambda(z)(v_1, \ldots, v_n) = (z^{\mu_1(\lambda)} v_1, \ldots, z^{\mu_n(\lambda)} v_n)$$

where the $\mu_i(\lambda) \in \mathbf{Z}$. Thus $Z_\lambda = \{(v_1, \ldots, v_n) : v_i \neq 0 \text{ implies } \mu_i(\lambda) > 0\}$. There is a Borel subgroup B containing T and leaving Z_λ invariant ([58], Ch. 2). Thus GZ_λ is the image of the twisted product $G \times_B Z_\lambda$. Now $Z_G(V) = \bigcup_{\operatorname{Im}\lambda \subset T} GZ_\lambda$, and clearly the set of possible Z_λ's is finite. Hence

$$\begin{aligned}\dim Z_G(V) &= \sup_\lambda \dim GZ_\lambda \leq \sup_\lambda \dim G \times_B Z_\lambda \\ &= \sup_\lambda (\dim Z_\lambda + \dim G - \dim B) \\ &= \dim Z_T(V) + \frac{1}{2}(\dim G - \operatorname{rank} G). \ ■\end{aligned}$$

Let V, G, and T be as above. The number of weight spaces of weight 0 is independent of the choice of T, and we denote this number by $\mu_0(V, G)$.

Corollary **(10.2).** — *Let* (V, G) *be a self-dual representation of* G *(e.g. an orthogonal representation). Then*

$$\operatorname{codim} Z_G(V) \geq \frac{1}{2}(\dim V - \dim G + \operatorname{rank} G + \mu_0(V, G)).$$

Hence if (V, G) *has finite principal isotropy groups, then*

$$\operatorname{codim} Z_G(V) \geq \frac{1}{2}(\dim V/G + \operatorname{rank} G + \mu_0(V, G)).$$

Proof. — We continue with the notation used in the proof of (10.1), and we may still assume that G is connected. Since (V, G) is self-dual, the non-zero weights of V occur in pairs $\pm\mu$. Hence $\dim Z_\lambda \leq \frac{1}{2}(n - \mu_0(V, G))$ for any homomorphism $\lambda : \mathbf{C}^* \to T$. Thus $\dim Z_G(V) \leq \frac{1}{2}(n - \mu_0(V, G) + \dim G - \operatorname{rank} G)$. It follows that

$$\operatorname{codim} Z_G(V) \geq \frac{1}{2}(n - \dim G + \operatorname{rank} G + \mu_0(V, G)). \ \blacksquare$$

We need a few results from the theories of depth and local cohomology ([17], [28], [29], [71], [72]). Let R be a commutative ring with identity, A an R-module. A sequence $f_1, \ldots, f_s$ of elements of R is called an A-**sequence** (or A-**regular sequence**) if multiplication by f_{i+1} is injective on $A/(f_1, \ldots, f_i)A$, $0 \leq i \leq s-1$, where $(f_1, \ldots, f_i)$ denotes the ideal in R generated by $f_1, \ldots, f_i$. If I is an ideal of R, we write $\mathbf{depth}_I A \geq s$ (or I-depth $A \geq s$) if there is an A-sequence of length s in I.

Let Z be a complex affine variety, $\mathscr{F}$ a coherent sheaf of $\mathcal{O}_Z$-modules. Let $\operatorname{depth}_z \mathscr{F}$ denote the depth of $\mathscr{F}_z$ with respect to the maximal ideal of $\mathcal{O}_{Z,z}$, $z \in Z$. We say that Z (or $\mathbf{C}[Z]$) is **Cohen-Macaulay** if $\operatorname{depth}_z \mathcal{O}_Z = \dim Z$ for all $z \in Z$. Smooth varieties are Cohen-Macaulay. If V is a representation space of G, then a deep theorem of Hochster and Roberts shows that V/G is Cohen-Macaulay ([35]). (The Hochster-Roberts result has been strengthened by recent work of Boutot.)

Lemma **(10.3)** ([17], [28], [29], [71], [72]). — *Let* Z *be a complex affine variety,* $\mathscr{F}$ *a coherent sheaf of* $\mathcal{O}_Z$*-modules. Let* Y *be a closed subvariety of* Z, *let* I(Y) *denote the ideal of* Y *in* $\mathbf{C}[Z]$, *and let* $\mathscr{I}$ *denote the corresponding sheaf of ideals on* Z. *Then*

(1) $\mathscr{I}_z$-depth $\mathscr{F}_z = \mathscr{I}_z^{(h)}$-depth $\mathscr{F}_z^{(h)}$ *for all* $z \in Y$.

(2) I(Y)-depth $\mathscr{F}(Z) \geq i$ *if and only if* $\mathscr{I}_z$-depth $\mathscr{F}_z \geq i$ *for all* $z \in Y$.

Suppose that Z *is Cohen-Macaulay.*

(3) *Let* $f_1, \ldots, f_s$ *be a sequence in* $\mathbf{C}[Z]$, *and suppose that* $(f_1, \ldots, f_s) \neq \mathbf{C}[Z]$. *Then* $f_1, \ldots, f_s$ *is a regular* $\mathbf{C}[Z]$*-sequence if and only if the zero set of* $(f_1, \ldots, f_s)$ *has codimension* s.

(4) I(Y)-depth $\mathbf{C}[Z] = \operatorname{codim} Y$. $\blacksquare$

Let Y, Z, $\mathscr{F}$, and $\mathscr{I}$ be as above. Then, functorially in $\mathscr{F}$, there are **local cohomology groups** $H^i_Y(Z, \mathscr{F})$ and long exact sequences

$$0 \to H^0_Y(Z, \mathscr{F}) \to H^0(Z, \mathscr{F}) \to H^0(Z-Y, \mathscr{F}) \to H^1_Y(Z, \mathscr{F}) \to \dots$$

Also, $\mathscr{I}_z$-depth $\mathscr{F}_z \geq m+1$ for all $z \in Y$ if and only if $H^i_Y(Z, \mathscr{F}) = 0$ for $0 \leq i \leq m$. There are analogous results for local cohomology groups $H^i_Y(Z, \mathscr{F}^{(h)})$. Since Z is affine, $H^i(Z, \mathscr{F})$ and $H^i(Z, \mathscr{F}^{(h)})$ are zero for $i \geq 1$. From the long exact sequences above we then obtain:

Proposition **(10.4)**. — *Let* Y, Z, I(Y), *and* $\mathscr{F}$ *be as in* (10.3). *Let* $m \in \mathbf{Z}^+$. *Then* I(Y)-depth $\mathscr{F}(Z) \geq m+2$ *implies*

(1) $H^0(Z, \mathscr{F}) \xrightarrow{\sim} H^0(Z-Y, \mathscr{F})$ *and* $H^0(Z, \mathscr{F}^{(h)}) \xrightarrow{\sim} H^0(Z-Y, \mathscr{F}^{(h)})$.

(2) $H^i(Z-Y, \mathscr{F}) = H^i(Z-Y, \mathscr{F}^{(h)}) = 0$, $1 \leq i \leq m$. ■

We will use I(V, G) as shorthand for $I(V^{(G)})^G$, V a representation space of G.

Lemma **(10.5)**. — *Let* V *and* V_1 *be representation spaces of* G. *Then*

$$\operatorname{depth}_{I(V,G)} \operatorname{Map}(V, V_1)^G \geq \operatorname{codim} Z_G(V).$$

Proof. — The argument in ([71], p. 1-3) shows that we can choose $f_1, \dots, f_s$ in I(V, G) whose set of common zeroes has codimension s, where $s = \operatorname{codim} Z_G(V)$. By (10.3.3), $f_1, \dots, f_s$ is a regular sequence for $\mathbf{C}[V]$, hence for

$$\operatorname{Map}(V, V_1) = \mathbf{C}[V] \otimes_{\mathbf{C}} V_1.$$

Since there is a $\mathbf{C}[V]^G$-module projection from Map(V, V_1) onto $\operatorname{Map}(V, V_1)^G$, one sees that $f_1, \dots, f_s$ is a regular sequence for $\operatorname{Map}(V, V_1)^G$, and the lemma is proved. ■

Proposition **(10.6)**. — *Let* V *be an orthogonal representation space of* G. *Assume that the principal isotropy groups of* (V, G) *are finite and that* $\dim V/G \geq 3$. *Then*

$$\operatorname{depth}_{I(V,G)} \mathfrak{X}_{\operatorname{Ad} G}(V)^G \geq 3.$$

Proof. — If G is finite there is nothing to prove. If $\operatorname{rank} G \geq 2$ or $\dim V/G \geq 4$ or $\mu_0(V, G) > 0$, then (10.2) and (10.5) give the required depth estimate. Suppose $\operatorname{rank} G = 1$, $\dim V/G = 3$, and $\mu_0(V, G) = 0$. Examining the low-dimensional representations of the rank 1 compact Lie groups, one easily sees that

$$(V, G^0) \simeq (\nu_r + \nu_{-r} + \nu_s + \nu_{-s}, \mathbf{C}^*)$$

where r and s are strictly positive integers. Since $\mathbf{C}^*$ is abelian and 1-dimensional, $\mathfrak{X}_{\operatorname{Ad} G^0}(V)^{G^0} \simeq \mathbf{C}[V]^{G^0}$. But $\mathbf{C}[V]^{G^0}$ is Cohen-Macaulay ([33] or [35]), so

$$\operatorname{depth}_{I(V, G^0)} \mathbf{C}[V]^{G^0} = 3$$

(it is also easy to establish this fact directly). Since G/G^0 is finite, $\mathbf{C}[V]^{G^0}$ is a finite $\mathbf{C}[V]^G$-module, hence is a Cohen-Macaulay $\mathbf{C}[V]^G$-module ([71]). Thus the direct summand $\mathfrak{X}_{\operatorname{Ad} G}(V)^G$ of $\mathfrak{X}_{\operatorname{Ad} G^0}(V)^{G^0} \simeq \mathbf{C}[V]^{G^0}$ has I(V, G)-depth 3. ■

Combining our results so far we can assert:

Theorem **(10.7).** — *Let* V *be an orthogonal representation space of* G. *Assume that* (V, G) *has finite principal isotropy groups and no* S^3 *strata. Then* (V, G) *has the lifting property.* ■

Remarks **(10.8).** — Let V be an orthogonal representation space of G. Assume that G is connected and that $\dim V/G \geq 4$.

(1) It is not hard to show that the $\mathbf{C}[V]^G$-modules $\mathfrak{X}(V/G)$, $\mathfrak{X}(V)^G$, and $\mathfrak{X}_G(V)^G$ are reflexive. (A module B over $R = \mathbf{C}[V]^G$ is said to be **reflexive** if the canonical map $B \to B^{**}$ is an isomorphism, where $B^* = \mathrm{Hom}_R(B, R)$.) By [65], these reflexive modules have I(V, G)-depth ≥ 2. Unfortunately, we need a depth estimate of 3 for $\mathfrak{X}_G(V)^G$.

(2) Let $E = \mathfrak{X}_G(V)^G/\mathfrak{X}_{\mathrm{Ad}\, G}(V)^G$. Assume that (V, G) has trivial principal isotropy groups and S^3 strata. We have a short exact sequence

$$0 \to \mathfrak{X}_{\mathrm{Ad}\, G}(V)^G \to \mathfrak{X}_G(V)^G \to E \to 0$$

which gives rise to a long exact sequence of local cohomology groups $H^i_{\{\bar{0}\}}(V/G, \cdot)$, where $\bar{0} = \pi_{V,G}(0)$ ([28]). Since we have good depth estimates for $\mathfrak{X}_{\mathrm{Ad}\, G}(V)^G$, depth estimates for E give rise to depth estimates for $\mathfrak{X}_G(V)^G$. Let $f_1, \ldots, f_r$ be the generators of the ideals in $\mathbf{C}[V]^G$ which vanish on the pre-images of the S^3 strata. It follows from (9.2) that $f = f_1 f_2 \ldots f_r$ generates the annihilator of E, so E can be considered as a $\mathbf{C}[V]^G/(f)$-module. Despite all this, we have been unable to say much more about the depth of E without first establishing that (V, G) has the lifting property.

(3) If (V, G) has the lifting property, then we have a short exact sequence

$$0 \to \mathfrak{X}_G(V)^G \to \mathfrak{X}(V)^G \to \mathfrak{X}(V/G) \to 0$$

which, as in (2), gives rise to a long exact sequence of groups $H^i_{\{\bar{0}\}}(V/G, \cdot)$. If (V, G) has finite principal isotropy groups, then from (10.2) and (10.5) we obtain the estimate $\mathrm{depth}_{I(V,G)} \mathfrak{X}(V)^G \geq 3$, and from (1) we obtain $\mathrm{depth}_{I(V,G)} \mathfrak{X}(V/G) \geq 2$. Hence $\mathrm{depth}_{I(V,G)} \mathfrak{X}_G(V)^G \geq 3$. So, for (V, G) satisfying our hypotheses, the lifting problem is equivalent to the estimate $\mathrm{depth}_{I(V,G)} \mathfrak{X}_G(V)^G \geq 3$.

III. — REPRESENTATIONS WITH INFINITE PRINCIPAL ISOTROPY GROUPS

Let V be a representation space of G, H a principal isotropy group. A (special case of a) theorem of Luna and Richardson [51] shows that the inclusion $V^H \to V$ induces an isomorphism $V^H/N_G(H) \xrightarrow{\sim} V/G$, and this isomorphism maps each stratum of $V^H/N_G(H)$ onto a stratum of V/G. Note that if $A \in \mathfrak{X}(V)^G$, then the restriction $\mathrm{res}_H A$ of A to V^H lies in $\mathfrak{X}(V^H)^{N_G(H)}$. If $f \in \mathbf{C}[V]^G$, then

$$\mathrm{res}_H(A(f)) = (\mathrm{res}_H A)(\mathrm{res}_H f),$$

where $\mathrm{res}_H f$ denotes the restriction of f to V^H. Thus if every element of $\mathfrak{X}(V^H)^{N_G(H)}$ extends to $\mathfrak{X}(V)^G$, then we can reduce the lifting problem for (V, G) to the lifting problem for $(V^H, N_G(H)/H)$, where the latter representation has trivial principal isotropy groups.

In § 11 we find conditions which guarantee that every element of $\mathfrak{X}(V^H)^{N_G(H)}$ extends to $\mathfrak{X}(V)^G$. We make several reductions in the proof of the algebraic lifting theorem, including a reduction to the semi-simple case. In § 12 we describe how to calculate principal isotropy classes and isotropy classes of codimension one strata; this information is needed to apply the theorems of § 11. In § 13 we develop a numerical criterion for determining which orthogonal representations have infinite principal isotropy groups or S^3 strata. The results of § 12 and § 13 are slight elaborations on the themes of [1], [21], [22], [37], and [38]. In § 13 we also outline our (inductive) proof of the algebraic lifting theorem. In § 14 we carry out the induction far enough to reduce to the case of representations of the simple groups with trivial principal isotropy groups and S^3 strata.

11. Reductions.

We work towards versions of the Luna-Richardson result.

Let V and V_1 be representation spaces of G, and let H be a reductive algebraic subgroup of G. Then restriction to V^H defines maps

$$\mathrm{Map}^h(V, V_1)^G \to \mathrm{Map}^h(V^H, (V_1)^H)^{N_G(H)}$$

and $$\mathrm{Map}(V, V_1)^G \to \mathrm{Map}(V^H, (V_1)^H)^{N_G(H)}$$

which we call res_H.

If L is a subgroup of our compact Lie group K, then restriction maps res_L are defined similarly.

Lemma (**11.1**). — *Let* V *and* V_1 *be representation spaces of* G. *Let* (H) *and* (L) *be isotropy classes of* (V, G) *where* $H \subseteq L$, *and let* (V_L, L) *denote the slice representation associated to* L. *Suppose that*

(1) *If* Lv *is a closed orbit in* V_L *such that* L_v *is* G-*conjugate to* H, *then* L_v *is* L-*conjugate to* H.

(2) $\mathrm{res}_H : \mathrm{Map}^h(V_L, V_1)^L \to \mathrm{Map}^h((V_L)^H, (V_1)^H)^{N_L(H)}$ *is surjective.*

Then there are G-*saturated open sets* U_α *covering* $V^{(L)}$ *such that any* $f \in \mathrm{Map}^h(V^H, (V_1)^H)^{N_G(H)}$ *has extensions* $f_\alpha \in \mathrm{Map}^h(U_\alpha, V_1)^G$.

Proof. — Let $x \in V^{\langle L\rangle}$. We may identify V_L with an L-complement to $T_x(Gx)$ in $V \simeq T_x(V)$, and as in (5.4) we have a holomorphic slice $\varphi : G \times_L \overline{B} \xrightarrow{\sim} \overline{U} \subseteq V$ where $\overline{B}$ is an L-saturated neighborhood of 0 in V_L, $\overline{U}$ is a G-saturated neighborhood of x, and $\varphi([g, v]) = g(x+v)$; $g \in G$, $v \in \overline{B}$. Let $f \in \mathrm{Map}^h(V^H, (V_1)^H)^{N_G(H)}$ and let $\psi(v) = x + v$, $v \in (V_L)^H$. Then $\psi^* f \in \mathrm{Map}^h((V_L)^H, (V_1)^H)^{N_L(H)}$, and by (2) $\psi^* f$ extends to

$$f' \in \mathrm{Map}^h(V_L, V_1)^L.$$

Restricting f' to $\overline{B}$ we can construct our final extension $\bar{f} \in \mathrm{Map}^h(G \times_L \overline{B}, V_1)^G$.

Let $[g, v] \in \varphi^{-1}(\overline{U} \cap V^{\langle H\rangle})$. Then $HgL = gL$ and $g^{-1}Hg = L_v$. By (1) we can find $\ell \in L$ such that $\ell^{-1}g^{-1}Hg\ell = H$. Thus $[g, v] = [g\ell, \ell^{-1}v]$ where $g\ell \in N_G(H)$, and $\ell^{-1}v \in \overline{B}^{\langle H\rangle} = \overline{B} \cap V_L^{\langle H\rangle}$. Hence $\overline{U} \cap V^{\langle H\rangle} = N_G(H)\psi(\overline{B}^{\langle H\rangle})$. By construction, $(\varphi^{-1})^*\bar{f}$ and f agree on $\psi(\overline{B}^{\langle H\rangle})$, hence on $\overline{U} \cap V^{\langle H\rangle}$. Since $V^{\langle H\rangle}$ is Zariski dense in V^H, $(\varphi^{-1})^*\bar{f}$ and f agree on $\overline{U} \cap V^H$. Thus f has local extensions near every closed orbit in $V^{(L)}$, hence near every point of $V^{(L)}$. ■

Let V be a representation space of G. Let (H) be the principal isotropy class, and let (L) be another isotropy class, $(L) \neq (H)$. We say that (L) is **subprincipal** if there are no isotropy classes (M) with $(H) < (M) < (L)$. We say that (L) is **1-subprincipal** if $(V/G)_{(L)}$ is a codimension one stratum. Clearly 1-subprincipal isotropy classes are subprincipal. A closed orbit and its isotropy group are called subprincipal (resp. 1-subprincipal) if the corresponding isotropy class is subprincipal (resp. 1-subprincipal). The concepts of subprincipal orbits, etc. of representations of compact Lie groups are defined similarly.

Theorem (**11.2**). — *Let* V *and* V_1 *be representation spaces of* G. *Let* (H) *and* $(L_1), \ldots, (L_r)$ *be the principal and* 1-*subprincipal isotropy classes of* (V, G). *Arrange that* H *is a principal isotropy group of the slice representation* (V_i, L_i) *of* L_i, $i = 1, \ldots, r$. *Suppose that*

(1) (V, G) *has generically closed orbits.*

(2) $\mathrm{codim}(V/G)_{(M)} \geq 2$ *implies* $\mathrm{codim}\, V^{(M)} \geq 2$.

(3) $\mathrm{res}_H : \mathrm{Map}^h(V_i, V_1)^{L_i} \to \mathrm{Map}^h((V_i)^H, (V_1)^H)^{N_{L_i}(H)}$ *is surjective,* $i = 1, \ldots, r$.

Then

$$\mathrm{res}_H : \mathrm{Map}^h(V, V_1)^G \to \mathrm{Map}^h(V^H, (V_1)^H)^{N_G(H)}$$

is an isomorphism.

Proof. — Let $f \in \mathrm{Map}(V^H, (V_1)^H)^{N_G(H)}$. By (1), $GV^{\langle H \rangle}$ is open and dense in V, hence local holomorphic extensions of f are uniquely determined. In particular, res_H is injective. Let L denote one of the L_i. Then (11.1.1) holds since H is a principal isotropy group, and (11.1.2) is (3). Hence f extends to a G-saturated neighborhood of $V^{(L)}$. Applying (11.1) with $L = H$ shows that f extends to $V^{(H)}$. Then (2) and Hartog's extension theorem imply that f extends to all of V. ∎

Theorem **(11.3)** (Luna-Richardson [51]). — *Let* V *be a representation space of* G, H *a principal isotropy group. Then*

(1) $\mathrm{res}_H : \mathbf{C}[V]^G \to \mathbf{C}[V^H]^{N_G(H)}$ *is an isomorphism.*

(2) *Suppose that* (L) *is an isotropy class of* (V, G) *and that* $H \subset L$. *Then*

$$(G/L)^H = N_G(H)L/L$$

and H *is a principal isotropy group of the slice representation of* L.

(3) *The stratification of* V/G *agrees with the one induced from* $V^H/N_G(H)$.

Proof. — Let $\pi : V^H/N_G(H) \to V/G$ be the map induced by res_H. By (5.5.4), if $x \in V^{\langle H \rangle}$, then $N_G(H)x = Gx \cap V^H$, and consequently $N_G(H)x$ is closed. Hence π has a (unique) inverse defined on $(V/G)_{(H)} \subseteq V/G$. Clearly then

$$\dim V/G = \dim V^H/N_G(H).$$

Suppose that $\dim V/G = 1$. Then (7.2) shows that (V, G) and $(V^H, N_G(H))$ are coregular, so π is isomorphic to a map of the form $(z \in \mathbf{C}) \mapsto z^m \in \mathbf{C}$, $m \in \mathbf{Z}^+$. Since π is invertible on a Zariski open subset of $\mathbf{C}$, $m = 1$ and π is an isomorphism. Our argument in (11.1) then shows that, for general (V, G), any $f \in \mathscr{H}_{V^H}(V^H)^{N_G(H)}$ extends to $\bar{f} \in \mathscr{H}_V(\pi_{V,G}^{-1}(U))^G$, where $U \subseteq V/G$ is the complement of the strata of codimension ≥ 2. Since V/G is normal, $\mathscr{H}_{V/G}(U) \simeq \mathscr{H}_{V/G}(V/G)$, and $\bar{f}$ extends to $\mathscr{H}_V(V)^G$ ([60]). Thus π is a complex analytic isomorphism, and using a Taylor series argument or the results of [70] we see that π is an algebraic isomorphism. We have established (1).

Let L be as in (2). Then some conjugate $g^{-1}Hg$ of H is a principal isotropy group of the slice representation (V_L, L) of L. To establish (2) it suffices to show that $(G/L)^H = N_G(H)L/L$, for then $g^{-1}Hg = \ell^{-1}H\ell$ where $\ell \in L$, and it follows that H is a principal isotropy group of (V_L, L). Now $L = G_x$ where $x \in V^H$ and Gx is closed. Note that $(G/L)^H \simeq (Gx)^H$ and that $N_G(H)L/L \simeq N_G(H)x$, so we must show that $(Gx)^H = N_G(H)x$: It follows from Luna's slice theorem [50] that $(Gx)^H$ is smooth, and clearly $T_x((Gx)^H) \simeq (\mathfrak{g}/\mathfrak{g}_x)^H$. Since H is reductive, $(\mathfrak{g}/\mathfrak{g}_x)^H$ is generated by the Lie algebra of $N_G(H)$, hence $N_G(H)x$ is open and closed in $(Gx)^H$ (this argument appears in [51]). But by (1), $(Gx)^H$ must contain only one closed $N_G(H)$-orbit. Thus $(Gx)^H = N_G(H)x$, and we have proved (2).

We now prove (3). Let L and (V_L, L) be as above, where $(L) \neq (G)$. Then by (2)

$$(G \times_L V_L)^H \simeq N_G(H) \times_{N_L(H)} (V_L)^H$$

as $N_G(H)$-varieties, hence $(G \times_L V_L)^H / N_G(H) \simeq (V_L)^H / N_L(H)$. By induction we may assume that the stratifications of $V_L/L \simeq (V_L)^H/N_L(H)$ agree, and then the HST shows that the stratifications of $V/G \simeq V^H/N_G(H)$ agree near $(V/G)_{(L)}$. Hence the stratifications agree away from $(V/G)_{(G)}$. But, as in the proof of (7.1.4), the stratifications must agree on $(V/G)_{(G)}$. ■

Using (6.8) we obtain

Corollary **(11.4)**. — *Let* V, G, *and* H *be as in* (11.3), *and let* V_1 *be a representation space of* G. *Then*

$$\mathrm{res}_H : \mathrm{Map}(V, V_1)^G \to \mathrm{Map}(V^H, (V_1)^H)^{N_G(H)}$$

is injective (resp. surjective) if and only if

$$\mathrm{res}_H : \mathrm{Map}^h(V, V_1)^G \to \mathrm{Map}^h(V^H, (V_1)^H)^{N_G(H)}$$

is injective (resp. surjective). ■

Remarks **(11.5)**

(1) By complexifying one obtains analogues of (11.2) and (11.3) for representations of compact Lie groups.

(2) In (11.2) and (11.3) one can replace H by its maximal torus. Such a reduction was considered by Bredon ([6] and [7]).

(3) Let V, G, H, and V_1 be as in (11.4). Suppose that $N_G(H)/H$ is finite and that $V = V_1$. Corollary (7.7) then shows that res_H is often an isomorphism, for example when (V, G) is orthogonal. A result along these lines was obtained earlier by Luna and Vust [77].

It is not difficult to prove the following:

Theorem **(11.6)**. — *Let* X *be a smooth connected* K*-manifold, let* H *be a principal isotropy group, and let* T *be a maximal torus of* H. *Let* Y *(resp.* Y'*) denote the closure of* $X^{\langle H \rangle}$ *in* X^H *(resp. the closure of* $X^{(H)} \cap X^T$ *in* X^T*). Then*

(1) Y *(resp.* Y'*) is a union of components of* X^H *(resp.* X^T*), hence smooth.*

(2) *The natural maps* $Y/N_K(H) \to X/K$ *and* $Y'/N_K(T) \to X/K$ *are diffeomorphisms.* ■

We now give an example where res_H maps $\mathfrak{X}(V)^G$ onto $\mathfrak{X}(V^H)^{N_G(H)}$, and then we give an example where res_H is not surjective.

Example (**11.7**). — Let $(W, K) = (2\mathbf{R}^n, O(n))$, $n > 2$. The principal isotropy class is $(H = O(n-2))$, and the unique 1-subprincipal isotropy class is $(O(n-1))$. The slice representation of $O(n-1)$ is $(\theta_2 + \mathbf{R}^{n-1}, O(n-1))$. Since $(\mathbf{R}^n)^{O(n-2)} \simeq \mathbf{R}^2$ and $N_{O(n)}(O(n-2))/O(n-2) \simeq O(2)$, we have that $(W^H, N_K(H)/H) \simeq (2\mathbf{R}^2, O(2))$. Theorem (11.2) shows that

$$\text{(11.8)} \qquad \operatorname{res}_{O(n-2)} \mathfrak{X}(2\mathbf{R}^n)^{O(n)} = \mathfrak{X}(2\mathbf{R}^2)^{O(2)}$$

if

$$\text{(11.9)} \qquad \operatorname{res}_{O(n-2)} \operatorname{Map}(\theta_2 + \mathbf{R}^{n-1}, \theta_2 + 2\mathbf{R}^{n-1})^{O(n-1)} = \operatorname{Map}(\theta_2 + \mathbf{R}^1, \theta_2 + 2\mathbf{R}^1)^{O(1)},$$

where $\mathbf{R}^1 \simeq (\mathbf{R}^{n-1})^{O(n-2)}$ and $O(1) \simeq N_{O(n-1)}(O(n-2))/O(n-2)$. But (11.9) is quite easy to establish. Using classical invariant theory ([18], [80]) one can, of course, establish (11.8) directly.

Example (**11.10**). — Let $(W, K) = (\mathbf{C}^4 + \mathbf{R}^6, SU(4))$ where $SU(4) = \operatorname{Spin}(6)$ acts on $\mathbf{R}^6$ as $SO(6) = \operatorname{Spin}(6)/\mathbf{Z}_2$. It is not hard to see that $H = SU(2) \simeq Sp(1)$ is a principal isotropy group, and that $L_1 = SU(3)$ and $L_2 = \operatorname{Spin}(5) \simeq Sp(2)$ generate the 1-subprincipal isotropy classes (see example (12.2) below). Note that $W^H \simeq \mathbf{C}^2 + \mathbf{R}^2$ where $N_K(H)/H \simeq U(2)$ acts as usual on $\mathbf{C}^2$ and $S^1 \simeq$ center of $U(2)$ acts with weight 2 on $\mathbf{R}^2$. Now $\mathbf{R}[\mathbf{C}^2 + \mathbf{R}^2]^{U(2)}$ is generated by the squares of the radius functions on $\mathbf{C}^2$ and $\mathbf{R}^2$. Thus the vector field $x\partial/\partial y - y\partial/\partial x \in \mathfrak{X}(\mathbf{R}^2)$ lies in

$$\mathfrak{X}_{U(2)}(\mathbf{C}^2 + \mathbf{R}^2)^{U(2)} \subseteq \mathfrak{X}(\mathbf{C}^2 + \mathbf{R}^2)^{U(2)},$$

but it clearly cannot extend to $\mathfrak{X}(W)^K$. Hence (11.2.3) must fail: The slice representation of L_2 is $(\mathbf{Q}^2 + \theta_1, Sp(2))$, and $\operatorname{Map}(\mathbf{Q}^2 + \theta_1, \mathbf{Q}^2 + \mathbf{R}^5 + \theta_1)^{Sp(2)}$ clearly does not restrict onto $\operatorname{Map}((\mathbf{Q}^2)^{Sp(1)} + \theta_1, (\mathbf{Q}^2)^{Sp(1)} + (\mathbf{R}^5)^{Sp(1)} + \theta_1)^N$ since

$$N = N_{Sp(2)}(Sp(1))/Sp(1) \simeq Sp(1)'$$

acts trivially on $(\mathbf{R}^5)^{Sp(1)} \simeq \mathbf{R}$.

We now give two reductions in the proof of the algebraic lifting theorem.

Proposition (**11.11**). — *Let* V *be an orthogonal representation space for* $G \times (\mathbf{C}^*)^m$, $m \geq 1$. If (V, G) has the lifting property, then $(V, G \times (\mathbf{C}^*)^m)$ *has the lifting property.*

Proof. — Let H be a principal isotropy group of (V, G), and let N denote $N_G(H)/H$. Since (V, G) has the lifting property, $\operatorname{res}_H \mathfrak{X}(V)^G \subseteq \mathfrak{X}(V^H)^N$ maps onto $\mathfrak{X}(V^H/N)$. Thus (V^H, N) has the lifting property, and

$$\mathfrak{X}(V^H)^N = \mathfrak{X}_N(V^H)^N + \operatorname{res}_H \mathfrak{X}(V)^G.$$

Consequently, if $(V^H, N \times (\mathbf{C}^*)^m)$ has the lifting property, then so does $(V, G \times (\mathbf{C}^*)^m)$. Hence we may assume that H is trivial. We may also assume that $m = 1$ (induction on m), $\mathbf{C}^*$ acts non-trivially on V/G (lemma (7.1)), $\dim V/(G \times \mathbf{C}^*) \geq 3$ (corollary (8.15)), $V^{G \times \mathbf{C}^*} = \{0\}$ (lemma (7.1)), and $\operatorname{rank} G \geq 1$ (else theorem (10.7) applies).

Let $0 \neq x \in V$ and suppose that $(G\times\mathbf{C}^*)x$ is closed. All G-orbits in $(G\times\mathbf{C}^*)x$ have conjugate isotropy groups, hence by (5.1.2) all G-orbits in $(G\times\mathbf{C}^*)x$ are closed. Let (N_x, G_x) and $(N'_x, (G\times\mathbf{C}^*)_x)$ denote the slice representations at x for the actions of G and $G\times\mathbf{C}^*$. Either $(G_x)^0 \to ((G\times\mathbf{C}^*)_x)^0$ is an isomorphism and the slice representations restricted to G_x differ by a trivial factor, or $((G\times\mathbf{C}^*)_x)^0$ is a $\mathbf{C}^*$-extension of $(G_x)^0$ and $(N_x, G_x)\simeq(N'_x, G_x)$. Thus using (7.1), (8.2), and induction we may assume the lifting property for all proper slice representations of $(V, G\times\mathbf{C}^*)$.

Let $q_1 : V\to V_1$ be a $\mathbf{C}^*$-equivariant orbit map for the action of G, let $q_2 : V_1\to\mathbf{C}^d$ be an orbit map for $(V_1, \mathbf{C}^*)$, and set $q=q_2\circ q_1$, $Z=q(V)$, $Z_1=q_1(V)$. Let A be a vector field on Z_1 corresponding to a generator of the Lie algebra of $\mathbf{C}^*$. We show that the zeroes of A have codimension ≥ 2 in Z_1:

Let $x\in Z_1$. If x is the image of a principal G-orbit, then the HST and (9.3.1) show that A has zeroes of codimension ≥ 2 in a neighborhood of x. If x is on a codimension one stratum ζ, then a neighborhood U of x is $\mathbf{C}^*$-biholomorphic to a neighborhood of 0 in $\mathbf{C}\times\mathbf{C}^p$, where $\mathbf{C}^*$ preserves $\mathbf{C}\times\{0\}$ and $\{0\}\times\mathbf{C}^p$, $\{0\}\times\mathbf{C}^p\simeq\zeta\cap U$, and $\mathbf{C}^*$ acts orthogonally. Consequently, $\mathbf{C}^*$ acts trivially on $\mathbf{C}\times\{0\}$ and orthogonally on $\{0\}\times\mathbf{C}^p$, and again (9.3.1) shows that A has zeroes of codimension ≥ 2 near x. Thus A has zeroes of codimension ≥ 2 in Z_1.

Let $X\in\mathfrak{X}^h(Z)$. Since lifting holds for the proper slice representations of $(V, G\times\mathbf{C}^*)$, we can find local $G\times\mathbf{C}^*$-invariant holomorphic lifts of X to $V-Z_{G\times\mathbf{C}^*}(V)$. Quotienting by G we obtain $\mathbf{C}^*$-invariant holomorphic lifts X_α of X on an open cover $\{U_\alpha\}$ of $Z_1-Z_{\mathbf{C}^*}(V_1)$. Let $U_{\alpha\beta}$ denote the points of $U_\alpha\cap U_\beta$ which are singular points of Z_1 or where A vanishes. Clearly $X_\alpha-X_\beta=f_{\alpha\beta}A$ on $U_\alpha\cap U_\beta-U_{\alpha\beta}$ where $f_{\alpha\beta}$ is holomorphic. Since Z_1 is normal, $U_\alpha\cap U_\beta$ is a normal analytic space, and since $U_{\alpha\beta}$ has codimension ≥ 2 in $U_\alpha\cap U_\beta$, $f_{\alpha\beta}$ has a unique holomorphic extension to $U_\alpha\cap U_\beta$ ([60]). Thus the obstruction to patching the X_α lies in $H^1(Z_1-Z_{\mathbf{C}^*}(V_1), \mathscr{H}_{Z_1})$. Since $\dim V/(G\times\mathbf{C}^*)\geq 3$ and $\operatorname{rank}(G\times\mathbf{C}^*)\geq 2$, (10.2) shows that $Z_{G\times\mathbf{C}^*}(V)$ has codimension at least 3. Hence there is a regular sequence of length 3 for $\mathbf{C}[V]$ (and $\mathbf{C}[V]^G$) in $I(V, G\times\mathbf{C}^*)$, and $H^1(Z_1-Z_{\mathbf{C}^*}(V_1), \mathscr{H}_{Z_1})=0$ by (10.4). Thus there is a holomorphic vector field X_1 on $Z_1-Z_{\mathbf{C}^*}(V_1)$ covering X, and since $Z_1\cap Z_{\mathbf{C}^*}(V_1)$ has codimension ≥ 3 in Z_1, X_1 extends to a vector field X'_1 on Z_1. Since the X_α and A are tangent to the codimension one strata of Z_1, X'_1 preserves the ideals vanishing on the codimension one strata of Z_1. Now (3.5), (5.8), (6.1), and (6.14) imply that $\underline{\mathfrak{X}}^h_{Z_1}$ is the sheaf of derivations of $\mathscr{H}_{Z_1}$ which preserve the sheaves of ideals of the codimension one strata of Z_1. Thus $X'_1\in\mathfrak{X}^h(Z_1)$, and since (V, G) has the lifting property, X'_1 lifts to $\mathfrak{X}^h(V)^G$, hence X lifts to $\mathfrak{X}^h(V)^{G\times\mathbf{C}^*}$. ∎

Proposition **(11.12).** — *Let* V *be an orthogonal representation space for the reductive algebraic group* $G\times H$, *where* $G\neq\{\mathrm{id}\}$, $H\neq\{\mathrm{id}\}$. *Suppose that lifting holds for orthogonal representations of proper reductive subgroups of* $G\times H$, *and suppose that* (V, G) *is coregular. Then lifting holds for* $(V, G\times H)$.

Proof. — We may assume H is connected, $V^{G\times H}=\{0\}$, and $\dim V/(G\times H)\geq 3$. Let $q_1 : V\to V_1$ be an H-equivariant minimal orbit map for (V, G), and let $q_2 : V_1\to \mathbf{C}^d$ be an orbit map for (V_1, H). Let $q=q_2\circ q_1$, and let Z denote $q(V)$. Since (V, G) is coregular and q_1 is minimal, $q_1(V)=V_1$ and $Z=q_2(V_1)$. We will refer to Z as $q_2(V_1)$ when we give it the stratification induced from V_1/H. Similarly, $q_1(V)$ denotes V_1 with the stratification induced from V/G.

Let $X\in\mathfrak{X}^h(Z)$. Our hypotheses imply that lifting holds for the proper slice representations of $(V, G\times H)$. Consequently, X has local $G\times H$-invariant lifts to $V-Z_{G\times H}(V)$, and they induce local H-invariant holomorphic lifts X_α on $V_1-Z_H(V_1)$. It follows that X preserves the strata of $q_2(V_1)$ on $q_2(V_1)-q_2(0)$, and $X(q_2(0))=0$ since $\{q_2(0)\}=\{q(0)\}$ is a stratum of Z. Thus $X\in\mathfrak{X}^h(q_2(V_1))$. By hypothesis, (V_1, H) has the lifting property, so X lifts to $X_1\in\mathfrak{X}^h(V_1)^H$.

Now $(V, G\times H)$ is the complexification of a representation $(W, K\times L)$. Let $p_1 : W\to W_1$ be an L-equivariant orbit map for (W, K) such that $q_1=(p_1)_{\mathbf{C}}$. Clearly elements of $\mathfrak{X}_L(W_1)$ are tangent to the orbits of L, hence are tangent to the strata of $p_1(W)\subset W_1$. It follows that the elements of $\mathfrak{X}_H(V_1)$ are tangent to the strata of $q_1(V)$. Now X_1 differs from the X_α by multiples of elements of $\mathfrak{X}_H(V_1)^H$, hence X_1 is tangent to the strata of $q_1(V)$ on $q_1(V)-Z_H(V_1)$, where $Z_H(V_1)$ has codimension ≥ 2 in $q_1(V)$ by corollary (7.4). As in the proof of (11.11), it follows that $X_1\in\mathfrak{X}^h(q_1(V))$. By hypothesis, (V, G) has the lifting property, so X_1 lifts to $\mathfrak{X}^h(V)^G$, and X lifts to $\mathfrak{X}^h(V)^{G\times H}$. ∎

12. Calculating Principal and 1-subprincipal Isotropy Groups.

Proposition (**12.1**) (cf. [21]). — *Let* V_1 *and* V_2 *be non-trivial representation spaces of* G, *let* H_i *be a principal isotropy group of* (V_i, G), $i=1, 2$, *and let* $V=V_1+V_2$. *Suppose that* V_1 *and* V_2 *have generically closed orbits. Then*

(1) *The principal (resp.* 1*-subprincipal) isotropy groups of* (V_2, H_1) *are principal (resp.* 1*-subprincipal) isotropy groups of* (V, G).

(2) *If* (L) *is a* 1*-subprincipal isotropy class of* (V, G), *then an element of* (L) *is a* 1*-subprincipal isotropy group of* (V_2, H_1) *or* (V_1, H_2).

Proof. — If $v_1\in V_1^{\langle H_1\rangle}$, then the slice representation at $(v_1, 0)\in V$ is $(V_2+\theta, H_1)$, and (1) is immediate. Let (L) be as in (2), and let $f_i\in\mathbf{C}[V_i]^G$ be a non-zero function vanishing on the non-principal orbits of (V_i, G), $i=1, 2$. If $f_1(v_1)=f_2(v_2)=0$ for all $(v_1, v_2)\in V^{\langle L\rangle}$, then clearly $\operatorname{codim}(V/G)_{(L)}\geqslant 2$. Hence, without loss of generality, we may assume there is a $(v_1, v_2)\in V^{\langle L\rangle}$ such that $f_1(v_1)\neq 0$. Then v_2 lies on a 1-subprincipal orbit of (V_2, H_1), and (2) is proved. ∎

Example (**12.2**). — Let $(W, K)=(W_1+W_2, K)=(\mathbf{C}^4+\mathbf{R}^6, SU(4))$. (See example (11.10).) The principal isotropy class of $(\mathbf{C}^4, SU(4))$ is $(H_1=SU(3))$, and that of $(\mathbf{R}^6, SU(4)\simeq Spin(6))$ is $(H_2=Sp(2))$, where $(W_2, H_1)\simeq(\mathbf{C}^3, SU(3))$ and

$(W_1, H_2) \simeq (\mathbf{Q}^2, \mathrm{Sp}(2))$. Since H_1 and H_2 act transitively on the unit spheres of W_2 and W_1, respectively, the isotropy classes of the codimension one strata are (H_1) and (H_2). Since $\mathrm{SU}(2)$ is a principal isotropy group of $(W_2, H_1) \simeq (\mathbf{C}^3, \mathrm{SU}(3))$, $\mathrm{SU}(2)$ is a principal isotropy group of (W, K).

As a corollary of (11.3) we have:

Proposition **(12.3)**. — *Let* V *be a representation space of* G, H *a principal isotropy group, and let* $L \supseteq H$ *be a* 1*-subprincipal isotropy group. Then there is a* 1*-subprincipal orbit* $N_G(H)x$ *in* $(V^H, N_G(H))$ *such that* $G_x = L$. ■

Example **(12.4)** (cf. [37]). — Let $K = \mathrm{SU}(n) \times \mathrm{SU}(n)$ act on

$$W = \mathrm{Hom}_{\mathbf{C}}(\mathbf{C}^n, \mathbf{C}^n) \simeq (\mathbf{C}^n)^* \otimes_{\mathbf{C}} \mathbf{C}^n$$

by $(g, h)A = gAh^{-1}$; $(g, h) \in K$, $A \in W$. Let $\{z_i\}_{i=1}^n$ be an orthonormal basis for $\mathbf{C}^n$, $\{\xi_i\}_{i=1}^n$ the dual basis. A principal isotropy group of (W, K) is the diagonal maximal torus $\operatorname{diag} T \subseteq T \times T \subseteq \mathrm{SU}(n) \times \mathrm{SU}(n)$, and $W^{\operatorname{diag} T}$ is the set of matrices $\sum_{i=1}^n a_i \xi_i \otimes z_i$. In other words, every element of W can be brought to diagonal form by the action of K, and the subgroup fixing all diagonal elements is diag T. Now

$$(W^{\operatorname{diag} T}, N_K(\operatorname{diag} T)/\operatorname{diag} T) \simeq (\mathbf{C}^n, T \rtimes \Sigma_n)$$

where T acts as usual on $\mathbf{C}^n$ and Σ_n (the symmetric group on n letters) acts by permuting co-ordinates. Thus $x = \Sigma a_i \xi_i \otimes z_i$ lies on a principal orbit if and only if all the a_i are distinct, and x lies on a 1-subprincipal orbit if exactly two of the a_i are equal and these a_i are non-zero. Using (12.3) one sees that the (unique) 1-subprincipal isotropy class of (W, K) is generated by $\operatorname{diag}((U(2) \times U(1)^{n-2}) \cap \mathrm{SU}(n))$. One can calculate the corresponding slice representation (W_L, L) from the equation

$$(W, L) = (W_L, L) + (\mathfrak{k}/\mathfrak{l}, L).$$

One can also calculate (W_L, L) as follows:

Let $U(1)'$ denote the copy of $U(1)$ in $U(2)$ centralizing $\mathrm{SU}(2)$, and let

$$L_0 = \operatorname{diag}((U(1)' \times U(1)^{n-2}) \cap \mathrm{SU}(n)).$$

Then L_0 is the subgroup of $H = \operatorname{diag} T$ acting trivially on $\mathfrak{l}/\mathfrak{h}$. Since H is a principal isotropy group, $(W_L, H) \simeq (\mathfrak{l}/\mathfrak{h} + \theta, H)$. Hence L_0 is the subgroup of L acting trivially on W_L. Thus $(W_L, L/L_0 \simeq \mathrm{SO}(3))$ is a representation with principal isotropy group $H/L_0 \simeq \mathrm{SO}(2)$, and the only possibility is $(W_L, L/L_0) \simeq (\theta + \mathbf{R}^3, \mathrm{SO}(3))$.

Using (12.1) one can more or less reduce the problem of finding principal and 1-subprincipal isotropy groups of representations to the irreducible case. In the irreducible case, (12.3) and the tables of [21] and [37] provide the answers we need. (The tables in [21] only list the Lie algebras of the principal isotropy groups, but in the cases we consider the principal isotropy groups are well-known to be connected.)

13. Indices and an Outline of the Proof of the Algebraic Lifting Theorem.

Let $G = G_1 \times \ldots \times G_s$ be a product of simple algebraic groups, and let ρ be a representation of G. One can assign non-negative rational numbers $\mathrm{ind}_{G_i}\rho$ to ρ such that if $\mathrm{ind}_{G_i}\rho > 1$, $1 \leq i \leq s$, then ρ has finite principal isotropy groups ([1]). (A variant of this criterion was also used in [38].) We find similar criteria for an orthogonal representation to have S^3 strata.

Let V be a complex vector space. We denote by tr_V the trace function on $\mathrm{Hom}_C(V, V)$. Suppose that $\rho = (V, G)$ is a representation of the simple algebraic group G. Then $X \mapsto \mathrm{tr}_V(X^2)$ is an ad $\mathfrak{g}$-invariant bilinear form on $\mathfrak{g}$, hence a multiple of the Cartan-Killing form $X \mapsto \mathrm{tr}_{\mathfrak{g}}(X^2)$. After [1], we call the multiplication factor the **index** of V (or index of ρ), denoted $\mathrm{ind}_G V$ (or $\mathrm{ind}_G \rho$). If ρ is the direct sum of representations ρ_1 and ρ_2 of G, then clearly $\mathrm{ind}_G \rho = \mathrm{ind}_G \rho_1 + \mathrm{ind}_G \rho_2$. The index of a representation of a simple real Lie group is defined similarly.

Proposition **(13.1)**. — *Let* $G = G_1 \times \ldots \times G_s$ *be a product of simple algebraic groups. Let* V *be an orthogonal representation space for* G, H *a principal isotropy group. Suppose that* $j \in \mathbf{Z}^+$, $\mathrm{ind}_{G_i}(V) = 1$ *for* $1 \leq i \leq j$, *and* $\mathrm{ind}_{G_i}(V) > 1$ *for* $j < i \leq s$. *Then* $H^0 \subseteq G_1 \times \ldots \times G_j$ ($H^0 = \{\mathrm{id}\}$ *if* $j = 0$), *and* H^0 *is a torus. Moreover, if* H *is finite, then* (V, G) *has no* S^3 *strata.*

Proof. — Let (L) be an isotropy class of (V, G). Then $(V, L) \simeq (\mathfrak{g}/\mathfrak{l} + V_L, L)$ where (V_L, L) is the slice representation corresponding to L. Let $0 \neq X \in \mathfrak{l}$. Then

$$\mathrm{tr}_V(X^2) = \mathrm{tr}_{\mathfrak{g}}(X^2) - \mathrm{tr}_{\mathfrak{l}}(X^2) + \mathrm{tr}_{V_L}(X^2). \tag{13.2}$$

Let $m(X)$ denote $\mathrm{tr}_V(X^2)/\mathrm{tr}_{\mathfrak{g}}(X^2)$. Decompose X as $X_1 + \ldots + X_s$ where $X_i \in \mathfrak{g}_i$, $i = 1, \ldots, s$. By the associativity of tr_V (see [39], p. 21) we have $\mathrm{tr}_V(X^2) = \sum_i \mathrm{tr}_V(X_i^2)$, hence

$$m(X) = \sum_{X_i \neq 0} \mathrm{tr}_V(X_i^2) \Big/ \sum_{X_i \neq 0} \mathrm{tr}_{\mathfrak{g}_i}(X_i^2). \tag{13.3}$$

Suppose that (L) is principal. Then (V_L, L) is the trivial representation, and (13.2) shows that $m(X) \leq 1$. If $m(X) = 1$, then ad X must act trivially on $\mathfrak{l} = \mathfrak{h}$, i.e. H^0 is abelian. Suppose that $(V/G)_{(L)}$ is an S^3 stratum. Then $\mathrm{tr}_{V_L}(X^2)/\mathrm{tr}_{\mathfrak{l}}(X^2) = \frac{1}{2}$, and $m(X) < 1$. The proposition now follows from (13.3). ∎

Corollary **(13.4)**. — *Let* G *be a simple algebraic group,* V *an orthogonal representation space of* G. *If* G *has infinite principal isotropy groups, then* $\mathrm{ind}_G(V) \leq 1$. *If* G *has finite principal isotropy groups and* S^3 *strata, then* $\mathrm{ind}_G V < 1$. ∎

Example **(13.5)**. — Let $G = \mathrm{Spin}(2n+1, \mathbf{C})$, $n \geq 2$. Then G has basic representations $\varphi_1, \ldots, \varphi_n$ where φ_1 is the usual representation of

$$\mathrm{Spin}(2n+1, \mathbf{C})/\mathbf{Z}_2 \simeq \mathrm{SO}(2n+1, \mathbf{C}) \text{ on } \mathbf{C}^{2n+1}.$$

For $i<n$, φ_i is the exterior power $\Lambda^i\varphi_1$, and φ_n is the spin representation. Let $\pm\mu_1, \ldots, \pm\mu_n$ denote the non-zero weights of φ_1 relative to a maximal torus of G. Then ([20], Supplement) φ_n has the 2^n weights $\frac{1}{2}(\pm\mu_1 \ldots \pm\mu_n)$. Embed $\mathbf{C}^*$ in G by lifting the usual embedding of $\mathbf{C}^*=\mathrm{SO}(2, \mathbf{C})\subseteq \mathrm{SO}(2n+1, \mathbf{C})$. Then, modulo trivial representations, $\varphi_1|_{\mathbf{C}^*}=\nu_2+\nu_{-2}$, $(\mathrm{Ad}\,\mathrm{G}=\Lambda^2\varphi_1)|_{\mathbf{C}^*}=(2n-1)(\nu_2+\nu_{-2})$, and $\varphi_n|_{\mathbf{C}^*}=2^{n-1}(\nu_1+\nu_{-1})$. Hence $\mathrm{ind}_{\mathrm{G}}\varphi_1=\dfrac{1}{2n-1}$, $\mathrm{ind}_{\mathrm{G}}\varphi_n=\dfrac{2^{n-3}}{2n-1}$, and one easily sees that $\mathrm{ind}_{\mathrm{G}}\varphi_i>1$, $2<i<n$.

Let H be a principal isotropy group of $m\varphi_1$. By (13.1), H is finite for $m\geq 2n$, and H^0 is abelian for $m=2n-1$ (in fact, $\mathrm{H}=\mathrm{SO}(2, \mathbf{C})$ in this case).

Example **(13.6)**. — Let $\mathrm{G}=\mathrm{Spin}(2n, \mathbf{C})$, $n\geq 3$. Then G has basic representations $\varphi_1, \ldots, \varphi_n$ where φ_1 is the usual representation of $\mathrm{Spin}(2n, \mathbf{C})/\mathbf{Z}_2=\mathrm{SO}(2n, \mathbf{C})$ on $\mathbf{C}^{2n}$, $\varphi_i=\Lambda^i\varphi_1$ for $1\leq i<n-1$, and φ_{n-1} and φ_n are the half-spin representations. Moreover, $\mathrm{Ad}\,\mathrm{G}=\Lambda^2\varphi_1$. Let $\pm\mu_1, \ldots, \pm\mu_n$ denote the non-zero weights of φ_1 relative to a maximal torus of G. Then ([20], Supplement) φ_{n-1} has the 2^{n-1} weights $\frac{1}{2}(\pm\mu_1 \ldots \pm\mu_n)$ where only an even number of minus signs are allowed, and φ_n has the 2^{n-1} weights whose expressions contain an odd number of minus signs. As above, one can show that $\mathrm{ind}_{\mathrm{G}}\varphi_1=\dfrac{1}{2n-2}$, $\mathrm{ind}_{\mathrm{G}}\varphi_{n-1}=\mathrm{ind}_{\mathrm{G}}\varphi_n=\dfrac{2^{n-4}}{2n-2}$, and $\mathrm{ind}_{\mathrm{G}}\varphi_i>1$ for $2<i<n-1$.

We now indicate how our proof of the algebraic lifting theorem will proceed: Let V be an orthogonal representation space of the reductive algebraic group M. We wish to show that (V, M) has the lifting property. Using (8.2) and (11.11) we may reduce to the case that M is connected and semi-simple, and going to a finite cover we may assume that M is simply connected. Then M is a product of simple factors. Let G_1 be a simple factor of M, and let $\mathrm{G}^\sim$ be the product of the other factors, so $\mathrm{M}=\mathrm{G}_1\times\mathrm{G}^\sim$. We will show that one of the following cases always occurs:

Case **(13.7)**. — There is a reductive algebraic group $\overline{\mathrm{M}}=\overline{\mathrm{G}}_1\times\overline{\mathrm{G}}^\sim$, an inclusion $\eta:\mathrm{M}\to\overline{\mathrm{M}}$ with $\eta(\mathrm{G}_1)\subseteq\overline{\mathrm{G}}_1$, and an extension of the representation of M on V to an orthogonal representation of $\overline{\mathrm{M}}$ such that

(1) $\mathbf{C}[\mathrm{V}]^{\overline{\mathrm{M}}}=\mathbf{C}[\mathrm{V}]^{\mathrm{M}}$.

(2) The principal isotropy class $(\overline{\mathrm{H}}_1)$ of $(\mathrm{V}, \overline{\mathrm{G}}_1)$ is non-trivial, and letting $\overline{\mathrm{N}}$ denote $\mathrm{N}_{\overline{\mathrm{G}}_1}(\overline{\mathrm{H}}_1)/\overline{\mathrm{H}}_1$ we have

$$\mathrm{res}_{\overline{\mathrm{H}}_1}\mathfrak{X}(\mathrm{V})^{\overline{\mathrm{G}}_1}=\mathfrak{X}(\mathrm{V}^{\overline{\mathrm{H}}_1})^{\overline{\mathrm{N}}}$$

and $$\dim\overline{\mathrm{N}}+\dim\overline{\mathrm{G}}^\sim<\dim\mathrm{M}.$$

Case **(13.8)**. — $(\mathrm{V}, \mathrm{G}_1)$ is coregular and has the lifting property.

Case **(13.9)**. — $\mathrm{Ind}_{G_1} V > 1$, or $\mathrm{ind}_{G_1} V = 1$ and any principal isotropy group of (V, M) has finite projection onto G_1.

Assuming that (13.7), (13.8), or (13.9) always holds we can prove that (V, M) has the lifting property: By induction we may suppose that lifting holds for orthogonal representations of reductive algebraic groups of dimension$<\dim M$. If (13.7) holds, then $(V^{\bar{H}_1}, \bar{N} \times \bar{G}^{\sim})$ has the lifting property by induction, hence (V, M) has the lifting property. If (13.8) holds, then proposition (11.12) applies. Finally, suppose that (13.9) holds for each simple factor of M. Then (V, M) has finite principal isotropy groups, and by (13.1) it has no S^3 strata. Theorem (10.7) then shows that (V, M) has the lifting property.

14. Reduction to Representations of Simple Groups with Trivial Principal Isotropy Groups.

Throughout this section, V, G_1, $G^{\sim}$, and $M = G_1 \times G^{\sim}$ are as in § 13. Let V_1 denote an M-complement to V^{G_1} in V.

In order to prove the algebraic lifting theorem, it suffices to show that one of (13.7), (13.8), or (13.9) always applies. If $\mathrm{ind}_{G_1}(V_1) > 1$, then (13.9) applies, and if (V_1, G_1) is the adjoint representation of G_1, then (13.7) and (13.8) apply. Our tables below include the remaining cases, i.e. the non-adjoint representations (V_1, G_1) with $\mathrm{ind}_{G_1}(V_1) \leq 1$. We will use these tables to verify that (13.7), (13.8), or (13.9) always holds. In some cases the verification that a representation satisfies (13.8) is delayed to chapter IV. We used the results of [1] to determine all the irreducible representations of the simple groups with index≤ 1; the computations of [1] are not difficult to verify. See ([20], p. 336) for a determination of which representations of the simple groups are orthogonalizable.

If φ is a representation of G, we will often confuse φ and its representation space $V(\varphi)$, so φ and $(V(\varphi), G)$ both denote the same object. Thus if H is a reductive algebraic subgroup of G, then $\mathfrak{X}(\varphi^H)^{N_G(H)}$ stands for $\mathfrak{X}(V(\varphi)^H)^{N_G(H)}$. If several groups and representations are present in a discussion, we may use the notation $\varphi(G)$ or (φ, G) to emphasize that φ is a representation of G. If φ' is a representation of G', then $\varphi \otimes \varphi'$ will denote the tensor product of $\varphi(G)$ and $\varphi'(G')$, and $\varphi + \varphi'$ will be shorthand for $\varphi \otimes \theta_1(G') + \theta_1(G) \otimes \varphi'$. (Unless otherwise specified, all tensor products are over $\mathbf{C}$.)

Assume that G is connected, simple, and simply connected. Corresponding to an ordering of the simple roots of $\mathfrak{g}$ we obtain an ordering $\varphi_1, \ldots, \varphi_r$ of the basic representations of G, $r = \mathrm{rank}\, G$. (Our ordering of the simple roots is indicated in the tables below.) If φ and ψ are irreducible representations of G, then $\varphi\psi$ (resp. φ^2, φ^3, etc.) will denote the irreducible component of highest weight in $\varphi \otimes \psi$ (resp. $S^2\varphi$, resp. $S^3\varphi$, etc.). We use the standard classification of G into types A_r, B_r, etc. Note the isomorphisms $\mathsf{A}_1 \simeq \mathsf{B}_1 \simeq \mathsf{C}_1$, $\mathsf{B}_2 \simeq \mathsf{C}_2$, and $\mathsf{A}_3 \simeq \mathsf{D}_3$.

In our tables, φ denotes a representation of the given group G, and H denotes a principal isotropy group of φ. Providing G is simple, we indicate how H embeds in G by listing the restriction of $\varphi_1(G)$ to H, denoted $(\varphi_1(G), H)$. Let L be a 1-subprincipal isotropy group of φ, where $H \subseteq L$. A finite cover of L decomposes as $L_1 \times L_0$ where L_0 acts trivially in the slice representation φ_L of L. We list $L_1 \times L_0$. If $L \to GL(\varphi_L)$ has finite kernel, then we always set $L_1 = L$, $L_0 = \{id\}$, and just list L_1.

Let H_1 denote a principal isotropy group of (φ_L, L_1). We wish to apply theorem (11.2) to show that $\mathrm{res}_H \mathfrak{X}(\varphi)^G = \mathfrak{X}(\varphi^H)^{N_G(H)}$, and condition (11.2.3) is equivalent to

$$\text{(14.1)} \qquad \mathrm{res}_{H_1} \mathrm{Map}(\varphi_L, \varphi^{L_0})^{L_1} = \mathrm{Map}((\varphi_L)^{H_1}, \varphi^{L_0 \times H_1})^{N_1},$$

where $N_1 = N_{L_1}(H_1)/H_1$. Thus we list the representation (φ_L, L_1) and the irreducible components of (φ^{L_0}, L_1). It is always a simple matter to check whether res_{H_1} is surjective; most of the time surjectivity follows from classical invariant theory. In the rare cases when (14.1) fails, we place an "x" in the last column. We explain how to handle these cases below.

We find it useful to consider some non-orthogonalizable representations with infinite principal isotropy groups. The corresponding entries are flagged by a "*". It is not necessary to verify that (13.7) holds in these cases, but in fact it does. Representations with S^3 strata are flagged by "**". We denote entries by their table number followed by a dot and their entry number, e.g. I.1 stands for the first entry in table I.

It is useful to add the following assumption to our inductive proof of the algebraic lifting theorem:

$$\text{(14.2)} \qquad \mathrm{ind}_{G_1}(V) \leq \mathrm{ind}_{G_i}(V), \quad i = 1, \ldots, r,$$

where the G_i are the simple factors of M. Let V_2 denote V^{G_1}. Then

$$(V, M) = (V_1, G_1 \times G^{\sim}) + (V_2, G^{\sim}).$$

Let G' denote the simple factors of $G^{\sim}$ which act non-trivially on V_1, and let H_M denote a principal isotropy group of (V, M).

I. $G_1 = \mathsf{A}_r$: The representation φ_1 is the standard action of $\mathsf{A}_r = SL(r+1, \mathbf{C})$ on $\mathbf{C}^{r+1}$; $\varphi_i = \Lambda^i \varphi_1$ has the induced action, $2 \leq i \leq r$. Complex conjugation on $\mathbf{C}^{r+1}$ induces an automorphism of A_r which interchanges φ_i and its dual φ_{r-i}, $1 \leq i \leq r$, and $\mathrm{Ad}\, \mathsf{A}_r = \varphi_1 \varphi_r$. We show that (13.7), (13.8), or (13.9) holds when $G_1 = \mathsf{A}_r$, $r = 2$ or $r \geq 4$, and we also handle some of the representations of A_1 and A_3. We complete the arguments for $\mathsf{A}_1 \simeq \mathsf{C}_1$ in **III** and for $\mathsf{A}_3 \simeq \mathsf{D}_3$ in **IV**.

Table I is constructed inductively using the results in [21], [37], and § 12. The contents of I.11 follow from example (12.4); I.4 and I.6 are established using similar techniques. The contents of I.10 follow from those of I.9 and I.11 via (12.1). The derivation of the contents of the other entries is routine.

TABLE I

$G = A_r, \quad r \geq 2 : \quad \varphi_1 - \varphi_2 - \varphi_3 - \cdots - \varphi_{r-1} - \varphi_r$

	G	φ	$\mathrm{ind}_G \varphi$	H	$(\varphi_1(G), H)$	$L_1 \times L_0$	(φ_L, L_1)	comps. of (φ^{L_0}, L_1)	
1	A_r	$k(\varphi_1 + \varphi_r)$ $1 \leq k < r$	$\frac{k}{r+1}$	A_{r-k}	$\varphi_1 + \theta$	A_{r-k+1}	$\varphi_1 + \varphi_{r-k+1} + \theta$	$\varphi_1, \varphi_{r-k+1}, \theta$	
2		$r(\varphi_1 + \varphi_r)$	$\frac{r}{r+1}$	$\{id\}$		A_1	$2\varphi_1 + \theta$	φ_1, θ	**
2′	$A_r \times C^*$	$r(\varphi_1 \otimes \nu_1 + \varphi_r \otimes \nu_{-1})$		C^*		$(A_1 \times C^*)$	$\varphi_1 \otimes (\nu_1 + \nu_{-1}) + \theta$	$\varphi_1 \otimes \nu_1$, $\varphi_1 \otimes \nu_{-1}, \theta$	
3	A_r	$(r+1)(\varphi_1 + \varphi_r)$	1	$\{id\}$		NONE			
4	$A_{2k}, \quad k \geq 2$	$\varphi_2 + \varphi_{2k-1}$	$\frac{2k-1}{2k+1}$	$A_1^{(1)} \times \ldots \times A_1^{(k)}$	$\varphi_1^{(1)} + \ldots + \varphi_1^{(k)} + \theta$	$A_2 \times (A_1^{(2)} \times \ldots \times A_1^{(k)})$ $C_2 \times (A_1^{(3)} \times \ldots \times A_1^{(k)})$	$\varphi_1 + \varphi_2 + \theta$ $\varphi_2 + \theta$	$\varphi_1, \varphi_2, \theta$ $\varphi_1, \varphi_2, \theta$	
5		$\varphi_1 + \varphi_2 + \varphi_{2k-1} + \varphi_{2k}$	$\frac{2k}{2k+1}$	$\{id\}$		A_1	$2\varphi_1 + \theta$	φ_1, θ	**
5′	$A_{2k} \times C^*$	$\varphi_1 \otimes \nu_k + \varphi_{2k} \otimes \nu_{-k}$ $+ \varphi_2 \otimes \nu_{-1} + \varphi_{2k-1} \otimes \nu_1$		C^*		$(A_1 \times C^*)$	$\varphi_1 \otimes (\nu_1 + \nu_{-1}) + \theta$	$\varphi_1 \otimes \nu_1, \varphi_1 \otimes \nu_{-1}$, ν_2, ν_{-2}, θ	
6	$A_{2k-1}, \quad k \geq 3$	$\varphi_2 + \varphi_{2k-2}$	$\frac{2k-2}{2k}$	$A_1^{(1)} \times \ldots \times A_1^{(k)}$	$\varphi_1^{(1)} + \ldots + \varphi_1^{(k)}$	$C_2 \times (A_1^{(3)} \times \ldots \times A_1^{(k)})$	$\varphi_2 + \theta$	φ_2, θ	
7		$\varphi_1 + \varphi_2 + \varphi_{2k-2} + \varphi_{2k-1}$	$\frac{2k-1}{2k}$	$\{id\}$		A_1	$2\varphi_1 + \theta$	φ_1, θ	**
7′	$A_{2k-1} \times C^*$	$\varphi_1 \otimes \nu_1 + \varphi_{2k-1} \otimes \nu_{-1}$ $+ \varphi_2 + \varphi_{2k-2}$		C^*		$(A_1 \times C^*)$	$\varphi_1 \otimes (\nu_1 + \nu_{-1}) + \theta$	$\varphi_1 \otimes \nu_1, \varphi_1 \otimes \nu_{-1}$, ν_2, ν_{-2}, θ	
8	$A_r, \quad r \geq 4$	$2\varphi_1 + \varphi_2 + \varphi_{r-1} + 2\varphi_r$	1	$\{id\}$		NONE			
9	A_5	φ_3	$\frac{1}{2}$	$A_2 \times A_2'$	$\varphi_1 + \varphi_1'$				*
10		$2\varphi_3$	1	$C^* \times (C^*)'$	$2(\nu_1 + \nu_1' + \nu_{-1} \otimes \nu_{-1}')$	$A_1 \times C^*$	$\varphi_1^2 + \theta$	φ_1^2, θ	
11	$A_r \times A_r'$	$\varphi_1 \otimes \varphi_r' + \varphi_r \otimes \varphi_1'$		$(C^*)^{(1)} \times \ldots \times (C^*)^{(r)}$		$A_1 \times ((C^*)^{(2)}$ $\times \ldots \times (C^*)^{(r)})$	$\varphi_1^2 + \theta$	φ_1^2, θ	

TABLE II

$$G = \mathsf{B}_r,\ r \geq 2:\quad \underset{\varphi_1}{\circ} - \underset{\varphi_2}{\circ} - \underset{\varphi_3}{\circ} - \cdots - \underset{\varphi_{r-1}}{\circ} \Rightarrow \underset{\varphi_r}{\circ}$$

	G	φ	$\mathrm{ind}_G\varphi$	H	$(\varphi_1(G), H)$	$L_1 \times L_0$	(φ_L, L_1)	comps. of (φ^{L_0}, L_1)	
1	B_r	$2k\varphi_1,\ 1 \leq k < r-1$	$\frac{2k}{2r-1}$	B_{r-k}	$\varphi_1 + \theta$	D_{r-k+1}	$\varphi_1 + \theta$	φ_1, θ	
2		$(2k-1)\varphi_1,\ 1 \leq k < r-1$	$\frac{2k-1}{2r-1}$	D_{r-k+1}	$\varphi_1 + \theta$	B_{r-k+1}	$\varphi_1 + \theta$	φ_1, θ	
3		$(2r-3)\varphi_1$	$\frac{2r-3}{2r-1}$	$\mathsf{A}_1 \times \mathsf{A}_1'$	$\varphi_1 \otimes \varphi_1' + \theta$	B_2	$\varphi_1 + \theta$	φ_1, θ	
4		$(2r-2)\varphi_1$	$\frac{2r-2}{2r-1}$	A_1	$\varphi_1^2 + \theta$	$(\mathsf{A}_1 \times \mathsf{A}_1')$	$\varphi_1 \otimes \varphi_1' + \theta$	$\varphi_1 \otimes \varphi_1', \theta$	
5		$(2r-1)\varphi_1$	1	C^*	$\nu_2 + \nu_{-2} + \theta$	A_1	$\varphi_1^2 + \theta$	φ_1^2, θ	
6	B_3	φ_3	$\frac{1}{5}$	G_2	φ_1	B_3	φ_3	φ_3	
7		$\varphi_1 + \varphi_3$	$\frac{2}{5}$	A_2	$\varphi_1 + \varphi_2 + \theta$	G_2	$\varphi_1 + \theta$	φ_1, θ	
						D_3	$\varphi_2 + \varphi_3 + \theta$	$\varphi_1, \varphi_2, \varphi_3, \theta$	
8		$2\varphi_3$	$\frac{2}{5}$	A_2	$\varphi_1 + \varphi_2 + \theta$	G_2	$\varphi_1 + \theta$	φ_1, θ	
9		$2\varphi_1 + \varphi_3$	$\frac{3}{5}$	A_1	$2\varphi_1 + \theta$	A_2	$\varphi_1 + \varphi_2 + \theta$	$\varphi_1, \varphi_2, \theta$	
						B_2	$2\varphi_2 + \theta$	$\varphi_1, \varphi_2, \theta$	$\times$
10		$\varphi_1 + 2\varphi_3$	$\frac{3}{5}$	A_1	$2\varphi_1 + \theta$	A_2	$\varphi_1 + \varphi_2 + \theta$	$\varphi_1, \varphi_2, \theta$	

11		$3\varphi_3$	$\frac{3}{5}$	A_1	$2\varphi_1+\theta$	A_2	$\varphi_1+\varphi_2+\theta$	$\varphi_1, \varphi_2, \theta$	
12		$a\varphi_1+b\varphi_3$ $a+b=4,\ b>0$	$\frac{4}{5}$	$\{\mathrm{id}\}$		A_1	$2\varphi_1+\theta$	φ_1, θ	**
13		$a\varphi_1+b\varphi_3$ $a+b=5,\ b>0$	1	$\{\mathrm{id}\}$		NONE			
14	B_4	φ_4	$\frac{2}{7}$	B_3	$\varphi_3+\theta$	B_4	φ_4	φ_4	
15		$\varphi_1+\varphi_4$	$\frac{3}{7}$	G_2	$\varphi_1+\theta$	B_3	$\varphi_3+\theta$	$\varphi_1, \varphi_3, \theta$	
16		$2\varphi_1+\varphi_4$	$\frac{4}{7}$	A_2	$\varphi_1+\varphi_2+\theta$	G_2	$\varphi_1+\theta$	φ_1, θ	
17		$2\varphi_4$	$\frac{4}{7}$	A_2	$\varphi_1+\varphi_2+\theta$	G_2	$\varphi_1+\theta$	φ_1, θ	
						D_3	$\varphi_2+\varphi_3+\theta$	$\varphi_1, \varphi_2, \varphi_3, \theta$	
18		$3\varphi_1+\varphi_4$	$\frac{5}{7}$	A_1	$2\varphi_1+\theta$	A_2	$\varphi_1+\varphi_2+\theta$	$\varphi_1, \varphi_2, \theta$	
19		$\varphi_1+2\varphi_4$	$\frac{5}{7}$	A_1	$2\varphi_1+\theta$	A_2	$\varphi_1+\varphi_2+\theta$	$\varphi_1, \varphi_2, \theta$	
20		$a\varphi_1+b\varphi_4$ $a+2b=6,\ b>0$	$\frac{6}{7}$	$\{\mathrm{id}\}$		A_1	$2\varphi_1+\theta$	φ_1, θ	**
21		$a\varphi_1+b\varphi_4$ $a+2b=7,\ b>0$	1	$\{\mathrm{id}\}$		NONE			
22	B_5	φ_5	$\frac{4}{9}$	A_4	$\varphi_1+\varphi_4+\theta$				*
23		$2\varphi_5$	$\frac{8}{9}$	$\{\mathrm{id}\}$		A_1	$2\varphi_1+\theta$	φ_1, θ	**
24		$\varphi_1+2\varphi_5$	1	$\{\mathrm{id}\}$		NONE			

TABLE III

$$G = \mathbf{C}_r,\ r \geq 1 : \quad \underset{\varphi_1}{\circ} - \underset{\varphi_2}{\circ} - \underset{\varphi_3}{\circ} - \cdots - \underset{\varphi_{r-1}}{\circ} \Leftarrow \underset{\varphi_r}{\circ}$$

	G	φ	$\mathrm{ind}_G\,\varphi$	H	$(\varphi_1(G), H)$	$L_1 \times L_0$	(φ_L, L_1)	comps. of (φ^{L_0}, L_1)	
1	$\mathbf{C}_r$	$2k\varphi_1,\ 1 \leq k < r$	$\frac{k}{r+1}$	$\mathbf{C}_{r-k}$	$\varphi_1 + \theta$	$\mathbf{C}_{r-k+1}$	$2\varphi_1 + \theta$	φ_1, θ	
2		$2r\varphi_1$	$\frac{r}{r+1}$	$\{\mathrm{id}\}$		$\mathbf{C}_1$	$2\varphi_1 + \theta$	φ_1, θ	**
3		$(2r+2)\varphi_1$	1	$\{\mathrm{id}\}$		NONE			
4	$\mathbf{C}_r,\ r \geq 2$	φ_2	$\frac{r-1}{r+1}$	$\mathbf{C}_1^{(1)} \times \ldots \times \mathbf{C}_1^{(r)}$	$\varphi_1^{(1)} + \ldots + \varphi_1^{(r)}$	$\mathbf{C}_2 \times (\mathbf{C}_1^{(3)} \times \ldots \times \mathbf{C}_1^{(r)})$	$\varphi_2 + \theta$	φ_2, θ	
5		$2\varphi_1 + \varphi_2$	$\frac{r}{r+1}$	$\{\mathrm{id}\}$		$\mathbf{C}_1$	$2\varphi_1 + \theta$	φ_1, θ	**
5′	$\mathbf{C}_r \times \mathbf{C}_1'$	$\varphi_1 \otimes \varphi_1' + \varphi_2$		$\mathbf{C}_1$		$(\mathbf{C}_1 \times \mathbf{C}_1')$	$\varphi_1 \otimes \varphi_1' + \theta$	$\varphi_1 \otimes \varphi_1', \varphi_1^2, \theta$	
6	$\mathbf{C}_r,\ r \geq 2$	$4\varphi_1 + \varphi_2$	1	$\{\mathrm{id}\}$		NONE			
7	$\mathbf{C}_2$	$k\varphi_2,\ 1 \leq k \leq 3$	$\frac{k}{3}$	see II.3-II.5					
8		$2\varphi_1 + 2\varphi_2$	1	$\{\mathrm{id}\}$		NONE			
9	$\mathbf{C}_3$	$2\varphi_2$	1	$\mathbf{C}^*$	$3\nu_1 + 3\nu_{-1}$	$\mathbf{C}_1$	$\varphi_1^2 + \theta$	φ_1^2, θ	
10	$\mathbf{C}_3 \times \mathbf{C}_2'$	$\varphi_1 \otimes \varphi_1' + \varphi_2$		$\mathbf{C}^*$		$\mathbf{C}_1$	$\varphi_1^2 + \theta$	φ_1^2, θ	
11	$\mathbf{C}_2 \times \mathbf{C}_1' \times \mathbf{C}_1''$	$\varphi_1 \otimes \varphi_1' + \varphi_1 \otimes \varphi_1'' + \varphi_2$		$\mathbf{C}^*$		$\mathbf{C}_1$	$\varphi_1^2 + \theta$	φ_1^2, θ	
12	$\mathbf{C}_2 \times \mathbf{C}_2'$	$\varphi_1 \otimes \varphi_1' + \varphi_2 + \varphi_2'$		$\mathbf{C}^*$		$\mathbf{C}_1$	$\varphi_1^2 + \theta$	φ_1^2, θ	
13	$\mathbf{C}_2 \times \mathbf{C}_2' \times \mathbf{C}_1''$	$\varphi_1 \otimes \varphi_1' + \varphi_1' \otimes \varphi_1'' + \varphi_2$		$\mathbf{C}^*$		$\mathbf{C}_1$	$\varphi_1^2 + \theta$	φ_1^2, θ	
14	$\mathbf{C}_2 \times \mathbf{C}_1'$	$\varphi_1 \otimes \varphi_1' + 2\varphi_2$		$\mathbf{C}^*$		$\mathbf{C}_1$	$\varphi_1^2 + \theta$	φ_1^2, θ	
15	$\mathbf{C}_2 \times \mathbf{C}_2' \times \mathbf{C}_1'' \times \mathbf{C}_1'''$	$\varphi_1 \otimes \varphi_1' + \varphi_1 \otimes \varphi_1'' + \varphi_1' \otimes \varphi_1'''$		$\mathbf{C}^*$		$\mathbf{C}_1$	$\varphi_1^2 + \theta$	φ_1^2, θ	
16	$\mathbf{C}_3 \times \mathbf{C}_2' \times \mathbf{C}_2''$	$\varphi_1 \otimes \varphi_1' + \varphi_1 \otimes \varphi_1''$		$\mathbf{C}^*$		$\mathbf{C}_1$	$\varphi_1^2 + \theta$	φ_1^2, θ	
17	$\mathbf{C}_2 \times \mathbf{C}_1' \times \mathbf{C}_1'' \times \mathbf{C}_1'''$	$\varphi_1 \otimes \varphi_1' + \varphi_1 \otimes \varphi_1'' + \varphi_1 \otimes \varphi_1'''$		$\mathbf{C}^*$		$\mathbf{C}_1$	$\varphi_1^2 + \theta$	φ_1^2, θ	
18	$\mathbf{C}_1^{(1)} \times \ldots \times \mathbf{C}_1^{(m)},\ m \geq 2$	$\varphi_1^{(1)} \otimes \varphi_1^{(2)} + \varphi_1^{(2)} \otimes \varphi_1^{(3)} + \ldots + \varphi_1^{(m)} \otimes \varphi_1^{(1)}$		$\mathbf{C}^*$		$\mathbf{C}_1$	$\varphi_1^2 + \theta$	φ_1^2, θ	

Suppose $(V_1, G_1) = (r(\varphi_1 + \varphi_r), \mathsf{A}_r) = \mathrm{I}.2$. Then $(V_1, \mathsf{A}_r \times G^{\sim})$ is of the form $\varphi_1 \otimes \psi + \varphi_r \otimes \psi^*$ where ψ is a representation of $G^{\sim}$. There is an embedding of (V, M) in $(V, \bar{M}) = (\varphi_1 \otimes \nu_1 \otimes \psi + \varphi_r \otimes \nu_{-1} \otimes \psi^*, \mathsf{A}_r \times \mathbf{C}^* \times G^{\sim}) + (V_2, G^{\sim})$. But

$$(r\varphi_1 \otimes \nu_1 + r\varphi_r \otimes \nu_{-1}, \mathsf{A}_r \times \mathbf{C}^*) = \mathrm{I}.2'$$

has principal isotropy groups of dimension one, hence $\mathbf{C}^*$ acts trivially on the orbit space V/M, and $\mathbf{C}[V]^M = \mathbf{C}[V]^{\bar{M}}$. Since theorem (11.2) applies to $\mathrm{I}.2'$, we see that $\mathrm{I}.2$ satisfies (13.7). Similarly, $\mathrm{I}.5$ and $\mathrm{I}.7$ satisfy (13.7).

At this point we know that (13.7) applies if $(V_1, G_1) = (\varphi, \mathsf{A}_r)$ and $\mathrm{ind}_{\mathsf{A}_r}\varphi < 1$, or if $(\varphi, \mathsf{A}_r) = (2\varphi_3, \mathsf{A}_5)$. If $r \geq 4$ and $(\varphi, \mathsf{A}_r) = (2\varphi_1 + \varphi_2 + \varphi_{r-1} + 2\varphi_r, \mathsf{A}_r) = \mathrm{I}.8$, then $(V_1, \mathsf{A}_r \times G')$ embeds in $\rho = (\varphi_1 \otimes \varphi_1' + \varphi_r \otimes \varphi_1' + \varphi_2 + \varphi_{r-1}, \mathsf{A}_r \times \mathsf{A}_1')$. But the index of ρ with respect to A_1' is $2r/4 > 1$, and it follows from (13.1) that the principal isotropy groups of ρ are finite. Thus H_M has finite projection onto A_r, and $\mathrm{I}.8$ satisfies (13.9).

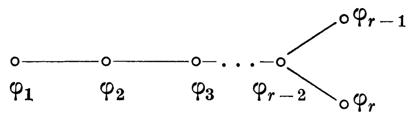
It remains to consider the case $(V_1, G_1) = ((r+1)(\varphi_1 + \varphi_r), \mathsf{A}_r)$, $r \geq 2$. Suppose that a principal isotropy group of $(V_1, \mathsf{A}_r \times G')$ has infinite projection onto A_r. As before, $(V_1, \mathsf{A}_r \times G') = \varphi_1 \otimes \psi + \varphi_r \otimes \psi^*$ for some representation ψ of G', and we enlarge $(V, \mathsf{A}_r \times G')$ to $(\varphi_1 \otimes \nu_1 \otimes \psi + \varphi_r \otimes \nu_{-1} \otimes \psi^*, \mathsf{A}_r \times \mathbf{C}^* \times G')$. Then the action of G' on $V_1/(\mathsf{A}_r \times \mathbf{C}^*)$ has only infinite isotropy groups. By classical invariant theory, $(V_1, \mathsf{A}_r \times \mathbf{C}^*)$ is coregular, and a minimal G'-equivariant orbit map is the composition

$$\varphi_1 \otimes \nu_1 \otimes \psi + \varphi_r \otimes \nu_{-1} \otimes \psi^* \longrightarrow \varphi_1 \otimes \nu_1 \otimes \psi \otimes \varphi_r \otimes \nu_{-1} \otimes \psi^* \xrightarrow{j} \psi \otimes \psi^*$$

where j contracts $\varphi_1 \otimes \nu_1$ with $\varphi_r \otimes \nu_{-1} = (\varphi_1 \otimes \nu_1)^*$. Let ψ_1 and ψ_2 be non-zero representations of G'. If $\psi = \psi_1 + \psi_2$, then $\psi \otimes \psi^* = \psi_1 \otimes \psi_1^* + \psi_2 \otimes \psi_2^* + \psi_1 \otimes \psi_2^* + \psi_1^* \otimes \psi_2$. But $\psi_i \otimes \psi_i^* = (\mathrm{Ad}\, GL(V(\psi_i)), G')$, so $\psi_1 \otimes \psi_1^* + \psi_2 \otimes \psi_2^*$ contains a copy of $\mathrm{Ad}\, G'$. It follows that $\psi \otimes \psi^*$ has index > 1 with respect to each simple component of G', hence $\psi \otimes \psi^*$ has finite principal isotropy groups. Suppose that $\psi = \psi_1 \otimes \psi_2$ where no ψ_i is θ_1. Let $\psi_i \otimes \psi_i^* - \theta_1$ denote a G'-complement to the copy of θ_1 in $\psi_i \otimes \psi_i^*$, $i = 1, 2$. Then $\psi \otimes \psi^* = (\psi_1 \otimes \psi_1^* - \theta_1) + (\psi_2 \otimes \psi_2^* - \theta_1) + \theta_1 + (\psi_1 \otimes \psi_1^* - \theta_1) \otimes (\psi_2 \otimes \psi_2^* - \theta_1)$ where the first two representations contain a copy of $\mathrm{Ad}\, G'$. Again, $\psi \otimes \psi^*$ has finite principal isotropy groups. Thus ψ must be an irreducible $(r+1)$-dimensional representation of a simple group with index $\leq 1/(2r+2)$. The tables show that the only possibility is $\psi = \varphi_1'$ (or φ_r') for another copy A_r' of A_r. If A_r' acts non-trivially on V_2, then (14.2) and proposition (13.1) imply that H_M has finite projection onto A_r', hence finite projection onto A_r. If A_r' acts trivially on V_2, then we are in case $\mathrm{I}.11$. We have completed the proof that (13.7), (13.8), or (13.9) holds when $G_1 = \mathsf{A}_r$, $r = 2$ or $r \geq 4$.

II. $G_1 = \mathsf{B}_r$: Note that $\mathsf{B}_r = \mathrm{Spin}(2r+1, \mathbf{C})$ (see example (13.5)). We show that (13.7), (13.8), or (13.9) holds when $G_1 = \mathsf{B}_r$, $r \geq 3$. Our arguments for $\mathsf{B}_1 \simeq \mathsf{C}_1$ and $\mathsf{B}_2 \simeq \mathsf{C}_2$ are completed in **III**.

TABLE IV

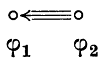
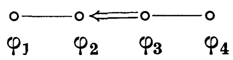
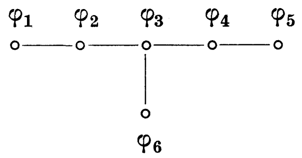
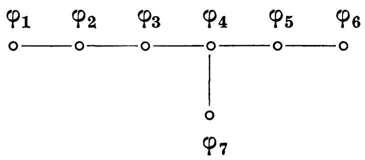
$G = \mathsf{D}_r,\ r \geq 3$: 

	G	φ	$\mathrm{ind}_G\varphi$	H	$(\varphi_1(G), H)$	$L_1 \times L_0$	(φ_L, L_1)	comps. of (φ^{L_0}, L_1)	
1	D_r	$2k\varphi_1,\ 1 \leq k < r-2$	$\frac{2k}{2r-2}$	D_{r-k}	$\varphi_1 + \theta$	B_{r-k}	$\varphi_1 + \theta$	φ_1, θ	
2		$(2k-1)\varphi_1,\ 1 \leq k < r-1$	$\frac{2k-1}{2r-2}$	B_{r-k}	$\varphi_1 + \theta$	D_{r-k+1}	$\varphi_1 + \theta$	φ_1, θ	
3		$(2r-4)\varphi_1$	$\frac{2r-4}{2r-2}$	$\mathsf{A}_1 \times \mathsf{A}_1'$	$\varphi_1 \otimes \varphi_1' + \theta$	B_2	$\varphi_1 + \theta$	φ_1, θ	
4		$(2r-3)\varphi_1$	$\frac{2r-3}{2r-2}$	A_1	$\varphi_1^2 + \theta$	$(\mathsf{A}_1 \times \mathsf{A}_1')$	$\varphi_1 \otimes \varphi_1' + \theta$	$\varphi_1 \otimes \varphi_1', \theta$	
5		$(2r-2)\varphi_1$	1	C^*	$\nu_2 + \nu_{-2} + \theta$	A_1	$\varphi_1^2 + \theta$	φ_1^2, θ	
6	D_3	$k(\varphi_2 + \varphi_3),\ 1 \leq k \leq 4$	$\frac{k}{4}$	see I. 1-I. 3					
7		$\varphi_1 + \varphi_2 + \varphi_3$	$\frac{2}{4}$	A_1	$2\varphi_1 + \theta$	A_2	$\varphi_1 + \varphi_2 + \theta$	$\varphi_1, \varphi_2, \theta$	
						B_2	$2\varphi_2 + \theta$	$\varphi_1, \varphi_2, \theta$	$\times$
8		$2\varphi_1 + \varphi_2 + \varphi_3$	$\frac{3}{4}$	{id}		A_1	$2\varphi_1 + \theta$	φ_1, θ	**
9		$\varphi_1 + 2\varphi_2 + 2\varphi_3$	$\frac{3}{4}$	{id}		A_1	$2\varphi_1 + \theta$	φ_1, θ	**
10		$a\varphi_1 + b(\varphi_2 + \varphi_3)$ $a + b = 4,\ ab \neq 0$	1	{id}		NONE			
11	D_4	$\varphi_1 + \varphi_3$	$\frac{2}{6}$	G_2	$\varphi_1 + \theta$	B_3	$\varphi_3 + \theta$	$\varphi_1, \varphi_3, \theta$	
12		$2\varphi_1 + \varphi_3$	$\frac{3}{6}$	A_2	$\varphi_1 + \varphi_2 + \theta$	G_2	$\varphi_1 + \theta$	φ_1, θ	
						D_3	$\varphi_2 + \varphi_3 + \theta$	$\varphi_1, \varphi_2, \varphi_3, \theta$	

13		$\varphi_1+\varphi_3+\varphi_4$	$\frac{3}{6}$	A_2	$\varphi_1+\varphi_2+\theta$	G_2	$\varphi_1+\theta$	φ_1, θ	
14		$3\varphi_1+\varphi_3$	$\frac{4}{6}$	A_1	$2\varphi_1+\theta$	A_2	$\varphi_1+\varphi_2+\theta$	$\varphi_1, \varphi_2, \theta$	
						B_2	$2\varphi_2+\theta$	$\varphi_1, \varphi_2, \theta$	$\times$
15		$2\varphi_1+2\varphi_3$	$\frac{4}{6}$	A_1	$2\varphi_1+\theta$	A_2	$\varphi_1+\varphi_2+\theta$	$\varphi_1, \varphi_2, \theta$	
16		$2\varphi_1+\varphi_3+\varphi_4$	$\frac{4}{6}$	A_1	$2\varphi_1+\theta$	A_2	$\varphi_1+\varphi_2+\theta$	$\varphi_1, \varphi_2, \theta$	
17		$a\varphi_1+b\varphi_3+c\varphi_4$ $a+b+c=5,\ ab\neq 0$	$\frac{5}{6}$	$\{\mathrm{id}\}$		A_1	$2\varphi_1+\theta$	φ_1, θ	**
18		$a\varphi_1+b\varphi_3+c\varphi_4$ $a+b+c=6,\ ab\neq 0$	1	$\{\mathrm{id}\}$		NONE			
19	D_5	$\varphi_4+\varphi_5$	$\frac{4}{8}$	A_3	$\varphi_1+\varphi_3+\theta$	A_4	$\varphi_1+\varphi_4+\theta$	$\varphi_1, \varphi_2, \varphi_3, \varphi_4, \theta$	
						B_3	$\varphi_1+\theta$	$\varphi_1, \varphi_3, \theta$	
20		$\varphi_1+\varphi_4+\varphi_5$	$\frac{5}{8}$	A_2	$\varphi_1+\varphi_2+\theta$	G_2	$\varphi_1+\theta$	φ_1, θ	
						D_3	$\varphi_2+\varphi_3+\theta$	$\varphi_1, \varphi_2, \varphi_3, \theta$	
21		$2\varphi_1+\varphi_4+\varphi_5$	$\frac{6}{8}$	A_1	$2\varphi_1+\theta$	A_2	$\varphi_1+\varphi_2+\theta$	$\varphi_1, \varphi_2, \theta$	
22		$3\varphi_1+\varphi_4+\varphi_5$	$\frac{7}{8}$	$\{\mathrm{id}\}$		A_1	$2\varphi_1+\theta$	φ_1, θ	**
23		$2\varphi_4+2\varphi_5$	1	$\{\mathrm{id}\}$		NONE			
24		$4\varphi_1+\varphi_4+\varphi_5$	1	$\{\mathrm{id}\}$		NONE			
25	D_6	φ_5	$\frac{4}{10}$	A_5	$\varphi_1+\varphi_5$				*
26		$2\varphi_5$	$\frac{8}{10}$	$A_1\times A_1'\times A_1''$	$2\varphi_1+2\varphi_1'+2\varphi_1''$	$C_2\times A_1''$	$\varphi_2+\theta$	φ_2, θ	
27		$\varphi_1+2\varphi_5$	$\frac{9}{10}$	$\{\mathrm{id}\}$		A_1	$2\varphi_1+\theta$	φ_1, θ	**
28		$2\varphi_1+2\varphi_5$	1	$\{\mathrm{id}\}$		NONE			

TABLE V

EXCEPTIONAL GROUPS

	G	φ	$\mathrm{ind}_G \varphi$	H	$(\varphi_1(G), H)$	$L_1 \times L_0$	(φ_L, L_1)	comps. of (φ^{L_0}, L_1)	
1	G_2 	φ_1	$\frac{1}{4}$	A_2	$\varphi_1+\varphi_2+\theta$	G_2	φ_1	φ_1	
2		$2\varphi_1$	$\frac{2}{4}$	A_1	$2\varphi_1+\theta$	A_2	$\varphi_1+\varphi_2+\theta$	$\varphi_1, \varphi_2, \theta$	
3		$3\varphi_1$	$\frac{3}{4}$	$\{id\}$		A_1	$2\varphi_1+\theta$	φ_1, θ	**
4		$4\varphi_1$	1	$\{id\}$		NONE			
5	F_4 	φ_1	$\frac{1}{3}$	D_4	$\varphi_1+\varphi_3+\varphi_4+\theta$	B_4	$\varphi_1+\theta$	$\varphi_1, \varphi_4, \theta$	
6		$2\varphi_1$	$\frac{2}{3}$	A_2	$3\varphi_1+3\varphi_2+\theta$	G_2	$\varphi_1+\theta$	φ_1, θ	
7		$3\varphi_1$	1	$\{id\}$		NONE			
8	E_6 	φ_1 or φ_5	$\frac{1}{4}$	F_4	$\varphi_1+\theta$				*
9		$\varphi_1+\varphi_5$	$\frac{1}{2}$	D_4	$\varphi_1+\varphi_3+\varphi_4+\theta$	B_4	$\varphi_1+\theta$	$\varphi_1, \varphi_4, \theta$	
10		$2\varphi_1+2\varphi_5$	1	$\{id\}$		NONE			
11	E_7 	φ_1	$\frac{1}{3}$	E_6	$\varphi_1+\varphi_5+\theta$				*
12		$2\varphi_1$	$\frac{2}{3}$	D_4	$2\varphi_1+2\varphi_3+2\varphi_4+\theta$	B_4	$\varphi_1+\theta$	$\varphi_1, \varphi_4, \theta$	

The determination of the contents of the entries of table II (and succeeding tables) requires no new techniques. In II.9, property (14.1) fails to hold. However, we may embed $\mathrm{II}.9=(2\varphi_1+\varphi_3, \mathsf{B}_3)$ into the representation $(2\varphi_1'+\varphi_3'', \mathsf{B}_3'\times\mathsf{B}_3'')$ without changing the invariants or commutant (=the subgroup of $\mathrm{GL}(\mathrm{V}(2\varphi_1+\varphi_3))$ commuting with the group action). Using II.1 and II.6 we then see that II.9 satisfies (13.7) (and (13.8)).

Entries 12, 20, and 23 have S^3 strata, and in chapter IV we show that they satisfy (13.8). An index argument akin to the one we used for I.8 shows that II.13, II.21, and II.24 satisfy (13.9). All entries not discussed so far satisfy (13.7). Thus (13.7), (13.8), or (13.9) holds when $\mathrm{G}_1=\mathsf{B}_r$, $r\geq 3$.

III. $\mathrm{G}_1=\mathsf{C}_r$: The representation φ_1 is the usual action of $\mathsf{C}_r=\mathrm{Sp}(r, \mathbf{C})$ on $\mathbf{C}^{2r}$, φ_i is the irreducible component of $\Lambda^i\varphi_1$ of highest weight, $1\leq i\leq r$, and $\mathrm{Ad}\,\mathsf{C}_r=\varphi_1^2$. We show that (13.7), (13.8), or (13.9) holds when $\mathrm{G}_1=\mathsf{C}_r$, $r\geq 1$.

In Chapter IV we show that III.2 satisfies (13.8). If

$$(\mathrm{V}_1, \mathrm{G}_1)=(2\varphi_1+\varphi_2, \mathsf{C}_r)=\mathrm{III}.5,$$

then clearly $(\mathrm{V}_1, \mathrm{G}_1\times\mathrm{G}')$ embeds in $(\varphi_1\otimes\varphi_1'+\varphi_2, \mathsf{C}_r\times\mathsf{C}_1')$. Enlarging (V, M) to $(\mathrm{V}, \overline{\mathrm{M}})=(\varphi_1\otimes\varphi_1'+\varphi_2, \mathsf{C}_r\times\mathsf{C}_1')+(\mathrm{V}_2, \mathrm{G}^{\sim})$ and using III.5′, we see that III.5 satisfies (13.7). We have handled all cases of index<1, and III.9 satisfies (13.7).

Suppose $(\mathrm{V}_1, \mathrm{G}_1)=(4\varphi_1+\varphi_2, \mathsf{C}_r)=\mathrm{III}.6$. Then $(\mathrm{V}_1, \mathsf{C}_r\times\mathrm{G}')$ embeds in $(\varphi_1\otimes\varphi_1'+\varphi_2, \mathsf{C}_r\times\mathsf{C}_2')$. If $r\geq 4$, then $\mathrm{ind}_{\mathsf{C}_2'}\mathrm{V}_1=r/3>1$, and using (14.2) and proposition (13.1) we see that (13.9) applies. Suppose that $r=3$. If $\mathrm{G}'=\mathsf{C}_2'$, then one is in the case of III.10 if $(\mathrm{V}_2, \mathrm{G}')$ is trivial, else (13.9) applies. If G' is a proper subgroup of C_2', then an index argument shows that (13.9) applies. If $r=2$, we leave it to the reader to verify that the only new representations to be considered are III.11, III.12, and III.13. Similarly, III.8 leads only to III.14.

It remains to consider III.3. Now our arguments in **IV** and **V** below are independent of our results concerning III.3, so we may assume that H_{M} has finite projection onto all simple components of M not of type C, and we may also assume that H_{M} has finite projection onto all factors of type C which do not correspond to case III.3. We are going to show that (13.7) applies (with $\mathrm{M}=\overline{\mathrm{M}}$) or that (13.9) applies, so we may reduce to the case that every simple factor of M corresponds to III.3.

Suppose that C_s and C_t' are factors of M and that $(\mathrm{V}, \mathsf{C}_s\times\mathsf{C}_t')$ contains k copies of $(\varphi_1\otimes\varphi_1', \mathsf{C}_s\times\mathsf{C}_t')$, $k\geq 1$. Then $|s-t|\leq 1$, and $k=1$ if $s\geq 2$ or $t\geq 2$. We may assume that $\mathrm{G}_1=\mathsf{C}_r$ has maximal rank among the simple components of M, and for now assume that $r\geq 2$. If $(\mathrm{V}_1, \mathsf{C}_r\times\mathrm{G}')$ contains the factor $(\varphi_1\otimes\varphi_1', \mathsf{C}_r\times\mathsf{C}_r')$, then $(\mathrm{V}, \mathsf{C}_r\times\mathrm{G}')$ must be $\rho=(\varphi_1\otimes\varphi_1'+2\varphi_1+2\varphi_1'+\theta, \mathsf{C}_r\times\mathsf{C}_r')$ if $r\geq 3$, and must embed in

$$\rho'=(\varphi_1\otimes\varphi_1'+\varphi_1\otimes\varphi_1''+\varphi_1'\otimes\varphi_1'''+\theta, \mathsf{C}_2\times\mathsf{C}_2'\times\mathsf{C}_1''\times\mathsf{C}_1''')$$

if $r=2$. The principal isotropy groups of ρ are finite, and eliminating any group factor in ρ' leads to a representation with finite principal isotropy groups.

Entry III.15 shows that ρ' itself comes under (13.7). If $(V_1, \mathsf{C}_r \times G')$ contains a factor $(\varphi_1 \otimes \varphi_1', \mathsf{C}_r \times \mathsf{C}_{r-1}')$, then one can show that III.16 and III.17 are the only new cases to consider. Entry III.18 is the only new case when all the groups are of rank 1. Thus one of (13.7), (13.8), or (13.9) holds when $G_1 = \mathsf{C}_r$, $r \geq 1$.

IV. $G_1 = \mathsf{D}_r$: Note that $\mathsf{D}_r = \mathrm{Spin}(2r, \mathbf{C})$ (see example (13.6)). There is an automorphism of D_r which interchanges φ_{r-1} and φ_r, and there are automorphisms of D_4 inducing all permutations of φ_1, φ_3, and φ_4. We do not need to consider both a representation of D_r and a representation derived from it by an automorphism of D_r, and this simplification is incorporated in table IV. We show that (13.7), (13.8), or (13.9) holds when $G_1 = \mathsf{D}_r$, $r \geq 3$.

In IV.7 and IV.14 property (14.1) fails to hold. Now IV.7 (= examples (11.10) and (12.2)) satisfies (13.8) by theorem (8.5), and IV.14 $= (3\varphi_1 + \varphi_3, \mathsf{D}_4)$ embeds in $(3\varphi_1' + \varphi_3'', \mathsf{D}_4' \times \mathsf{D}_4'')$ where the two representations have the same invariants and commutant. Using IV.2, we see that IV.14 satisfies (13.7) (and (13.8)).

In Chapter IV we show that IV.8, IV.9, IV.17, IV.22, and IV.27 satisfy (13.8). An index argument shows that IV.10, IV.18, IV.23, IV.24, and IV.28 satisfy (13.9). The cases we have not discussed satisfy (13.7). Thus (13.7), (13.8), or (13.9) holds when $G_1 = \mathsf{D}_r$, $r \geq 3$.

V. G_1 is exceptional: The group E_8 is not in table V since its only irreducible non-trivial representation of index ≤ 1 is Ad E_8. In Chapter IV we show that V.3 satisfies (13.8). An index argument shows that V.4, V.7, and V.10 satisfy (13.9). All other orthogonal representations satisfy (13.7), so (13.7), (13.8), or (13.9) is satisfied for representations of the exceptional groups.

IV. — REPRESENTATIONS OF SIMPLE GROUPS WITH S^3 STRATA

Let (V, G) be one of the remaining cases in our inductive proof of the algebraic lifting theorem. Then (V, G) has trivial principal isotropy class and a unique 1-subprincipal isotropy class ($L = \mathbf{A}_1$). In § 15 we develop techniques for showing that (V, G) satisfies (13.8). In § 16 we apply these techniques to the cases where the closure of $(V/G)_{(L)}$ in V/G is normal. (We say, rather imprecisely, that (V, G) has **normal codimension one strata.**) In § 17 we handle the cases with non-normal codimension one strata. Our proofs involve much use of classical invariant theory (abbreviated **CIT** henceforth); all the facts we need can be found in [18] or [80].

15. The Method.

Let (V, G) be a representation of G, let (L) be an isotropy class of (V, G), and let N denote $N_G(L)/L$. We denote by $\mathfrak{X}^+(V^L)^N$ (resp. $\mathfrak{X}^+(V^L/N)$) the elements of $\mathfrak{X}(V^L)^N$ (resp. $\mathfrak{X}(V^L/N)$) which preserve the image of $\mathbf{C}[V]^G$ in $\mathbf{C}[V^L]^N$ (resp. $\mathbf{C}[V^L/N]$).

Lemma **(15.1)** (*Luna* [51]). — *Let* V *be a representation space of* G, *and let* (L) *be an isotropy class of* (V, G). *Then the canonical map*

$$V^L/N_G(L) \to \Sigma = cl((V/G)_{(L)}) \subseteq V/G$$

is a normalization of Σ. *Moreover, if* $v \in V^L$, *then* Gv *is closed if and only if* $N_G(L)v$ *is closed.* ■

Proposition **(15.2)**. — *Let* V *be an orthogonal representation space of the connected reductive algebraic group* G. *Suppose that* (V, G) *has a unique* 1*-subprincipal isotropy class* (L). *Let* N *denote* $N_G(L)/L$ *and assume that*

(1) (V, G) *is coregular.*
(2) $\mathfrak{X}^+(V^L)^N \subseteq \mathrm{res}_L \mathfrak{X}(V)^G + \mathfrak{X}_N(V^L)^N$.
(3) (V^L, N) *has the lifting property.*

Then (V, G) *has the lifting property.*

Proof. — We may assume that $V^G = \{0\}$. Let $X \in \mathfrak{X}(V/G)$. Since X is strata preserving, X gives rise to a derivation of $\mathrm{res}_L \mathbf{C}[V]^G$. Since $\mathbf{C}[V^L]^N$ is the normalization of $\mathrm{res}_L\, \mathbf{C}[V]^G$ (by (15.1)), X lifts to a derivation X_L of $\mathbf{C}[V^L]^N \simeq \mathbf{C}[V^L/N]$. We show that $X_L \in \mathfrak{X}^+(V^L/N)$:

Let $\pi : V^L/N \to V/G$ denote the canonical map. Let (M) be an isotropy class of (V^L, N), and let $x \in V^L$ be such that Nx is closed, $N_x = M$. Then Gx is closed (by (15.1)), and $M = N_{G_x}(L)/L$. It follows that π maps $\sigma = (V^L/N)_{(M)}$ onto $(V/G)_{(G_x)}$, and then one easily sees that $\pi(\sigma)$ is a union $\tau_1 \cup \ldots \cup \tau_r$ of strata of V/G.

Let I denote the ideal in $\mathbf{C}[V^L/N]$ vanishing on σ, and let J denote the ideal in $\mathbf{C}[V/G]$ vanishing on $\tau_1 \cup \ldots \cup \tau_r$. Then X_L preserves $\pi^* J$, π is finite (being a normalization), and I is prime. Arguing as in (8.2) one sees that X_L preserves I. Thus X_L preserves the strata of V^L/N, i.e. $X_L \in \mathfrak{X}^+(V^L/N)$.

Let $q : V \to \mathbf{C}^d$ be a minimal orbit map, and consider X as an element of $\mathfrak{X}(Z = q(V))$. Then (2) and (3) show that there is an $A \in \mathfrak{X}(V)^G$ such that $X' = X - q_* A$ vanishes on $q(V^L)$. Below we show that

(15.3) $$X' \in q_* \mathfrak{X}(V)^G + I(Z_{(G)}) \mathfrak{X}(Z).$$

Since $X \in \mathfrak{X}(V/G)$ was arbitrary, it follows that $I(Z_{(G)})$ annihilates $\mathfrak{X}(Z)/q_*\mathfrak{X}(V)^G$. Now $\mathbf{C}[Z]$ and the $\mathbf{C}[Z]$-modules $\mathfrak{X}(Z)$ and $q_*\mathfrak{X}(V)^G$ can be graded (as in § 3) such that $I(Z_{(G)})$ consists of elements of strictly positive degree. An induction on degree then shows that $\mathfrak{X}(Z)/q_*\mathfrak{X}(V)^G = \{0\}$, i.e. shows that (V, G) has the lifting property.

Let $y_1, \ldots, y_d$ be co-ordinates on $\mathbf{C}^d$, and set $\deg y_i = e_i = \deg p_i$, $i = 1, \ldots, d$. By (1), $\mathbf{C}[y_1, \ldots, y_d] \simeq \mathbf{C}[Z]$, and (8.3) shows that $I(Z_{(L)})$ has a homogeneous generator $f \in \mathbf{C}[y_1, \ldots, y_d]$. Our vector field X' vanishes on $Z_{(L)}$, hence f divides the coefficients of X', and it suffices to establish (15.3) for $X' = f\partial/\partial y_i$, $i = 1, \ldots, d$.

Let $z_1, \ldots, z_n$ be co-ordinates on V, and set $B = q_*(\sum_{j=1}^{n} z_j \, \partial/\partial z_j) \in q_*\mathfrak{X}(V)^G$. Then $B = \sum_{j=1}^{d} e_j y_j \, \partial/\partial y_j$. Let $B_{ij} = (\partial f/\partial y_j)\,\partial/\partial y_i - (\partial f/\partial y_i)\,\partial/\partial y_j$, $1 \leq i, j \leq d$. Since each B_{ij} annihilates f, each B_{ij} is in $\mathfrak{X}(Z)$ by (3.5), (5.8), and (6.14). One easily computes that

$$\frac{\partial f}{\partial y_i} B + \sum_{j=1}^{d} e_j y_j B_{ij} = (\deg f) f \frac{\partial}{\partial y_i}.$$

This establishes (15.3) and the proposition. ∎

Remark **(15.4)**. — Let V, G, etc. be as above. Then calculating $\mathfrak{X}(Z)$ is tantamount to calculating the relations of the partial derivatives $\partial f/\partial y_i$. Hence knowing generators of $\mathfrak{X}(V)^G$ (hence $\mathfrak{X}(Z)$) allows one to calculate these relations. Conversely, in any particular case, a knowledge of these relations would be a great aid in establishing the lifting property for (V, G). However, in most cases we found it difficult to say anything about $\mathfrak{X}(Z)$ solely by contemplating the $\partial f/\partial y_i$.

Example **(15.5)**. — Let $(V, G) = (2n\varphi_1, \mathbf{C}_n) =$ entry III.2. We use (15.2) to show that (V, G) has the lifting property: By CIT, the invariants of $\mathbf{C}_n$ acting on any number of copies of φ_1 are generated by " skew products " of pairs of copies of φ_1, i.e. by the invariants corresponding to $\theta_1 \subseteq \theta_1 + \varphi_2 = \Lambda^2\varphi_1 \subseteq S^2(\varphi_1 + \varphi_1)$. Recall that the

1-subprincipal isotropy class of (V, G) is $(\mathrm{L}=\mathbf{C}_1')$, where we may choose L to be the standardly embedded copy of $\mathbf{C}_1'$ in $\mathbf{C}_n$ (see III.1). Thus

$$(\varphi_1(\mathbf{C}_n)^{\mathbf{C}_1'}, \mathrm{N}_{\mathbf{C}_n}(\mathbf{C}_1')/\mathbf{C}_1') \simeq (\varphi_1, \mathbf{C}_{n-1}).$$

Clearly then skew products of copies of $(\varphi_1, \mathbf{C}_n)$ restrict to skew products of copies of $(\varphi_1, \mathbf{C}_{n-1})$. It follows that $\mathfrak{X}(\mathrm{V}^{\mathrm{L}})^{\mathrm{N}} = \mathrm{res}_{\mathrm{L}}\mathfrak{X}(\mathrm{V})^{\mathrm{G}}$. Counting shows that $\mathbf{C}[\mathrm{V}]^{\mathrm{G}}$ is a regular ring, and (13.1) and (10.7) show that $(\mathrm{V}^{\mathrm{L}}, \mathrm{N})$ has the lifting property. The hypotheses of (15.2) are satisfied, so $(\mathrm{V}, \mathrm{G}) = (2n\varphi_1, \mathbf{C}_n)$ has the lifting property.

We develop two techniques for proving that rings of invariants $\mathbf{C}[\mathrm{V}]^{\mathrm{G}}$ are regular. One method assumes that we know $\mathbf{C}[\mathrm{V}^{\mathrm{L}}]^{\mathrm{N}}$ where L and N are as in (15.2); the other method relies on knowledge of the invariants of slice representations. The following proposition contains the essence of our first method.

Proposition **(15.6)**. — *Let* V *be a representation space of* G. *Let* (L) *be a* 1*-subprincipal isotropy class of* (V, G), *and let* $d = \dim \mathrm{V}/\mathrm{G}$. *Assume that*

(1) (L) *is the unique subprincipal isotropy class of* (V, G).

(2) *There are forms* $p_1, \ldots, p_d \in \mathbf{C}[\mathrm{V}]^{\mathrm{G}}$ *whose restrictions* $p_1', \ldots, p_d'$ *to* V^{L} *are a minimal generating set of* $\mathrm{res}_{\mathrm{L}}\mathbf{C}[\mathrm{V}]^{\mathrm{G}}$.

(3) *The relations of* $p_1', \ldots, p_d'$ *are generated by a polynomial* $f(y_1, \ldots, y_d)$, *where* $f(p_1, \ldots, p_d)$ *is homogeneous of degree* e.

(4) $\mathrm{I}(\mathrm{V}^{(\mathrm{L})})^{\mathrm{G}}$ *is generated by a form* f_{L}, *where* $\deg f_{\mathrm{L}} \geq e$.

Then $\mathbf{C}[\mathrm{V}]^{\mathrm{G}}$ *is a regular ring with generators* $p_1, \ldots, p_d$, *and* f_{L} *is a constant multiple of* $f(p_1, \ldots, p_d)$.

Proof. — If $h = f(p_1, \ldots, p_d) \neq 0$, then f_{L} must be a constant multiple of h by (3) and (4), and the proposition follows easily. Suppose that $h = 0$. Then

$$\mathbf{C}[\mathrm{V}/\mathrm{G}] \simeq \mathbf{C}[y_1, \ldots, y_d, y]/(f)$$

where y is a generator corresponding to f_{L}. Since the p_i' are a minimal generating set of $\mathrm{res}_{\mathrm{L}}\mathbf{C}[\mathrm{V}]^{\mathrm{G}}$, f contains no constant or linear terms, and it follows that V/G is singular along the set $y_1 = \ldots = y_d = 0$. But (1) implies that V/G is non-singular off the zero set of y, and we have a contradiction. ∎

Establishing (15.6.2) is much easier if we know that $\mathrm{res}_{\mathrm{L}}\mathbf{C}[\mathrm{V}]^{\mathrm{G}} = \mathbf{C}[\mathrm{V}^{\mathrm{L}}]^{\mathrm{N}}$:

Proposition **(15.7)**. — *Let* V *be a representation space of* G, *let* (L) *be a* 1*-subprincipal isotropy class of* (V, G), *and let* Σ *denote the closure of* $(\mathrm{V}/\mathrm{G})_{(\mathrm{L})}$ *in* V/G. *Assume that* $\mathrm{I}(\mathrm{V}^{(\mathrm{L})})^{\mathrm{G}}$ *is a principal ideal. Then the following are equivalent:*

(1) Σ *is normal.*

(2) $\mathrm{res}_{\mathrm{L}}\mathbf{C}[\mathrm{V}]^{\mathrm{G}} = \mathbf{C}[\mathrm{V}^{\mathrm{L}}]^{\mathrm{N}_{\mathrm{G}}(\mathrm{L})}$.

(3) *If* (M) *is the isotropy class of a codimension two stratum of* V/G *and* $L \subset M$, *then*

(i) $(G/M)^L = N_G(L)M/M$, *and*

(ii) $\mathrm{res}_L \mathbf{C}[V_M]^M = \mathbf{C}[(V_M)^L]^{N_M(L)}$,

where (V_M, M) *is the slice representation of* M.

Proof. — Lemma (15.1) implies the equivalence of (1) and (2). Since V/G is Cohen-Macaulay ([35]) and $I(V^{(L)})^G$ is principal, Σ is Cohen-Macaulay ([17], p. 52; [71], p. IV-19). Thus Σ is normal if and only if it is non-singular in codimension one ([17], p. 74; [71], p. IV-44). Let Σ' denote the points of Σ not lying on $(V/G)_{(L)}$ or on a codimension two stratum of V/G, and let $\pi : V^L/N_G(L) \to \Sigma$ denote the canonical map. If (3) holds, then lemma (11.1) shows that π induces a complex analytic isomorphism of $\pi^{-1}(\Sigma - \Sigma')$ with $\Sigma - \Sigma'$. Since $V^L/N_G(L)$ is normal, $\Sigma - \Sigma'$ and hence Σ are non-singular in codimension one. It follows that Σ is normal, i.e. (1) holds.

Suppose that (2) holds. Just as in the proof of (11.3.2) one can show that (3i) holds and that

$$(G \times_M V_M)^L \simeq N_G(L) \times_{N_M(L)} (V_M)^L.$$

Using the HST one can then see that (2) also implies (3ii). ■

Remarks (**15.8**). — Let V, G, and L be as above.

(1) Suppose that (L) is the unique 1-subprincipal isotropy class of (V, G). Then proposition (15.7) remains true if one replaces condition (3i) by the requirement that (V_M, M) has a unique 1-subprincipal isotropy class generated by L. The proof is the same.

(2) In § 17 the examples of non-normal codimension one strata are all of the following type: The group G is simple and connected, and $(L) = (\mathbf{A}_1)$ and $(M = L \times L') = (\mathbf{A}_1 \times \mathbf{A}_1')$ are the isotropy classes of the unique codimension one and codimension two strata, respectively. The associated slice representations are $(2\varphi_1 + \theta, \mathbf{A}_1)$ and $(2\varphi_1 + 2\varphi_1' + \theta, \mathbf{A}_1 \times \mathbf{A}_1')$. Note that $M \subset N_G(L)$. It turns out that

$$(G/M)^L = N_G(L)/M \cup N_G(L)n/M$$

where $n \in N_G(M)$ and $n^{-1}Ln = L'$. Thus hypothesis (3ii) of (15.7) holds while (3i) fails.

In the simple situation above one can characterize $\mathrm{res}_L \mathbf{C}[V]^G$: Let $f \in \mathbf{C}[V^L]^{N_G(L)}$. If $f \in \mathrm{res}_L \mathbf{C}[V]^G$, then $f|_{V^M} \in \mathbf{C}[V^M]^{N_G(M)}$. Conversely, suppose that $f|_{V^M} \in \mathbf{C}[V^M]^{N_G(M)}$. Then there is a well-defined polynomial function $\bar{f}$ on $V^L \cup V^{L'}$ which equals f on V^L and equals $f \circ n$ on $V^{L'}$. A slight modification of the proof of lemma (11.1) shows that $\bar{f}$ has local G-invariant holomorphic extensions along $V^{(M)}$. It follows that the holomorphic function h on $(V/G)_{(L)}$ corresponding to f extends to $\Sigma - \Sigma'$, where $\Sigma = cl((V/G)_{(L)})$ and Σ' has codimension ≥ 2 in Σ. As in the proof of (15.7), Σ is Cohen-Macaulay, and (10.3) and (10.4) then show that h extends to a holomorphic function on Σ, hence to a holomorphic function on V/G. It follows that $f \in \mathrm{res}_L \mathbf{C}[V]^G$.

The following result allows us to prove estimates like (15.6.4):

Proposition **(15.9).** — *Let* $V_1, \ldots, V_n$, *and* V' *be representation spaces of* G. *Let* V *denote* $V_1 + \ldots + V_n$, *let* L *be a subgroup of* G, *and let* E *denote the kernel of*

$$\mathrm{res}_L : \mathrm{Map}(V, V')^G \to \mathrm{Map}(V^L, (V')^L)^{N_G(L)}.$$

Then

(1) E *has generators (as a* $\mathbf{C}[V]^G$*-module) which are simultaneously homogeneous in each* V_i.

(2) *If* E *is generated by a single element* α *and if* $V_1 \simeq V_2$ *as* G*-representations, then* α *is homogeneous of the same degree in* V_1 *and* V_2.

(3) *Suppose that* (V_1, G) *has generically closed orbits and has a principal isotropy group* H *containing* L. *Let* $\overline{V}$ *denote* $V_2 + \ldots + V_n$ *and let* e *be the minimal degree (in* $\overline{V}$*) of a non-zero element of* $\mathrm{Map}(\overline{V}, V')^H$ *which vanishes on* $\overline{V}^L$. *Suppose that* α *is a non-zero element of* E *which is homogeneous in* $\overline{V}$, *say of degree* a. *Then* $a \geq e$.

Proof. — Parts (1) and (2) are quite easy, and one can see that (3) holds by considering the restrictions of α to sets of the form $\{v_1\} \times \overline{V}$, $v_1 \in V_1^{\langle H \rangle}$. ∎

The next two lemmas are the basis of another method of showing that representations are coregular.

Let R be a commutative ring with identity, A an R-module. If $f \in R$, then R_f (resp. A_f) will denote the localization of R (resp. A) at f. Suppose that V is a complex vector space, Z a complex affine variety, and $f \in \mathbf{C}[Z]$. Then $\mathbf{C}[Z]_f \simeq \mathbf{C}[Z_f]$ and $\mathrm{Map}(Z, V)_f \simeq \mathrm{Map}(Z_f, V)$ where Z_f denotes the points of Z at which f does not vanish.

Lemma **(15.10).** — *Let* V, V_1, *and* V_2 *be representation spaces of* G. *Let* H *be a principal isotropy group of* (V, G). *Assume that* (V, G) *has generically closed orbits, and let* f *be a non-zero element of* $\mathbf{C}[V]^G$ *vanishing on the non-principal orbits. Then restriction induces an isomorphism*

$$\mathrm{Map}(V + V_1, V_2)_f^G \xrightarrow{\sim} \mathrm{Map}(V^H + V_1, V_2)_{f'}^{N_G(H)},$$

where f' *denotes* f *restricted to* V^H.

Proof. — Consider the natural map

$$\eta : G \times_{N_G(H)} (V_{f'}^H) \to V_f.$$

Our hypotheses imply that $GV_{f'}^H = V_f$. Thus η is onto. By (5.5.4), η is one-to-one. Since the domain space and range space of η are smooth, η is a complex analytic isomorphism ([81], p. 106), hence an algebraic isomorphism ([70]), and the lemma follows easily. ∎

Lemma **(15.11).** — *Let* B *be a finitely generated domain over* $\mathbf{C}$, *and let* A *be a subalgebra generated by non-zero elements* $f_1, \ldots, f_d \in B$. *Suppose that* $A_{f_1} = B_{f_1}$, $A_{f_2} = B_{f_2}$, *and that* $\dim B = d$. *Then* $A = B$.

Proof. — Clearly $\dim A = \dim A_{f_1} = \dim B_{f_1} = d$, so A is a regular ring. Let $b \in B$. Then $b = a_1/f_1^i = a_2/f_2^j$ for some $a_1, a_2 \in A$ and $i, j \in \mathbf{Z}^+$, and $f_2^j a_1 = f_1^i a_2$. Since f_1 and f_2 are primes in A, f_2^j divides a_2 in A. Hence $b \in A$. ■

Example (**15.12**). — Let $(V, G) = (k\mathbf{C}^n, O(n, \mathbf{C}))$ where $1 \leq k \leq n$. Using the fact that $\mathbf{C}[\mathbf{C}^n]^{O(n,\mathbf{C})}$ is generated by $z_1^2 + \ldots + z_n^2$, we prove that $\mathbf{C}[V]^G$ is generated by inner products. Now a principal isotropy group of $(\mathbf{C}^n, O(n, \mathbf{C}))$ is a standardly embedded $O(n-1, \mathbf{C})$, and applying (15.10) we see that

$$\mathbf{C}[V]_f^G \simeq \mathbf{C}[(\mathbf{C}^1)_{f'} \times (k-1)(\mathbf{C}^1 + \mathbf{C}^{n-1})]^{O(1,\mathbf{C}) \times O(n-1,\mathbf{C})},$$

where $f = z_1^2 + \ldots + z_n^2$ generates the invariants on the first copy of $\mathbf{C}^n$ and f' denotes the restriction of f to $(\mathbf{C}^n)^{O(n-1,\mathbf{C})} = \mathbf{C}^1$. Let $y_2, \ldots, y_k$ denote typical points in the copies of $\mathbf{C}^{n-1}$, and let $x_1, \ldots, x_k$ denote co-ordinate functions on the copies of $\mathbf{C}^1$. By induction we may assume that the invariants of $\mathbf{C}[(k-1)\mathbf{C}^{n-1}]^{O(n-1,\mathbf{C})}$ are generated by inner products. Then the invariants $x_i x_j + y_i \cdot y_j$, $2 \leq i \leq j \leq k$, and the invariants $x_i x_j$, $1 \leq i \leq j \leq k$, generate $R = \mathbf{C}[\mathbf{C}^1 + (k-1)(\mathbf{C}^1 + \mathbf{C}^{n-1})]^{O(1,\mathbf{C}) \times O(n-1,\mathbf{C})}$. At points where $f' = x_1^2$ does not vanish, we see that $x_i x_j = (x_1 x_i)(x_1 x_j)/x_1^2$, so the $x_i x_j + y_i \cdot y_j$ and $x_1 x_j$ generate $R_{f'}$. These latter invariants are the restrictions of inner product invariants of $k\mathbf{C}^n$, so inner products generate $\mathbf{C}[V]_f^G$. Inverting a generator of the invariants on a second copy of $\mathbf{C}^n$ gives a similar result, and then (15.11) shows that inner products generate $\mathbf{C}[V]^G$.

We conclude our preliminaries with the following exercise in CIT.

Lemma (**15.13**). — *Let $m, n \in \mathbf{Z}^+$, $n \geq 3$, $m \geq n-2$. Then $\mathbf{C}[m\mathbf{C}^n + \Lambda^2\mathbf{C}^n]^{SO(n,\mathbf{C})}$ has a generating set consisting of*

(1) *elements of $\mathbf{C}[m\mathbf{C}^n]^{SO(n,\mathbf{C})}$ and $\mathbf{C}[\Lambda^2\mathbf{C}^n]^{SO(n,\mathbf{C})}$.*

(2) *elements of degree ≤ 2 in the $\mathbf{C}^n$ variables.*

(3) *invariants obtained by wedging copies of $\Lambda^{n-2i}(\mathbf{C}^n)$ in $S^{n-2i}(m\mathbf{C}^n)$ with the copy of $\Lambda^{2i}\mathbf{C}^n \subseteq S^i(\Lambda^2\mathbf{C}^n)$, $1 \leq i \leq \left[\dfrac{n-1}{2}\right]$.*

Proof. — Let f be a generator of $\mathbf{C}[m\mathbf{C}^n + \Lambda^2\mathbf{C}^n]^{SO(n,\mathbf{C})}$. We may suppose that f is homogeneous in both the $\mathbf{C}^n$ and $\Lambda^2\mathbf{C}^n$ variables. Polarizing f we obtain a multilinear invariant $f'(x_1, \ldots, \omega_1, \omega_2, \ldots)$ which is symmetric in the 2-forms ω_i. It suffices to determine f' on decomposable forms, so we replace each ω_i by $y_{i1} \wedge y_{i2}$ where the y_{ij} are elements of new copies of $\mathbf{C}^n$. We thus obtain a multilinear invariant $\bar{f}(x_1, \ldots, y_{11}, y_{12}, \ldots, y_{ij}, \ldots)$ which is symmetric in i and, for any fixed i, skew in j.

By CIT, $\bar{f}$ is a sum of terms which are either products of inner products of the x_k and y_{ij}, or such products multiplied by a determinant of n of the x_k and y_{ij}. First suppose that $\bar{f}$ contains a term h of the form

$$h_0 \det(y_{11}, y_{12}, \ldots, y_{r1}, y_{r2}, y_{(r+1)1}, \alpha_1, \ldots, \alpha_{n-2r-1})(y_{(r+1)2} \cdot \alpha_0)$$

where $r \geq 0$, h_0 is a product of inner products, and the α_ℓ are x_k's or y_{ij}'s. Averaging h over the symmetries of the y_{ij}, $1 \leq i \leq r+1$, we obtain $\frac{1}{2r+2}$ times the expression

$$h - h_0 \det(y_{(r+1)2}, y_{12}, \ldots, \alpha_{n-2r-1})(y_{11}\cdot\alpha_0) - \ldots$$
$$- h_0 \det(y_{11}, \ldots, y_{r2}, y_{(r+1)2}, \alpha_1, \ldots)(y_{(r+1)1}\cdot\alpha_0),$$

which by the Second Main Theorem for $SO(n, \mathbf{C})$ equals

$$h_0(\det(y_{11}, y_{12}, \ldots, y_{(r+1)1}, y_{(r+1)2}, \alpha_2, \alpha_3, \ldots)(\alpha_1\cdot\alpha_0) + \ldots$$
$$+ \det(y_{11}, y_{12}, \ldots, y_{(r+1)1}, \alpha_1, \ldots, \alpha_{n-2r-2}, y_{(r+1)2})(\alpha_{n-2r-1}\cdot\alpha_0)).$$

(The expression is zero if $n = 2r+1$.) Continuing inductively we can change all terms containing determinants to expressions divisible by terms corresponding to generators of type (1) or (3).

Suppose that $\bar{f}$ contains a term h of the form $h_0(y_{11}\cdot x_1)$ where h_0 is a product of inner products. Then $h_0(y_{11}\cdot x_1)$ is of the form $h_1(y_{11}\cdot x_1)(y_{12}\cdot y_{21})\ldots(y_{(r-1)2}\cdot y_{r1})(y_{r2}\cdot x_2)$. Averaging over the symmetries of the y_{ij}, $1 \leq i \leq r$, we obtain an expression divisible by a term corresponding to a generator of type (2). Similarly, terms involving no inner products between the x_k and y_{ij} can be transformed to expressions divisible by terms corresponding to generators of type (1).

We have transformed $\bar{f}$ to a multilinear form $\bar{f}_1$, where $\bar{f}_1$ is a sum of expressions each of which is divisible by a term corresponding to a generator of type (1), (2), or (3). Averaging $\bar{f}_1$ over the symmetries of all the y_{ij} we recover $\bar{f}$, and it follows that f itself is in the ideal of the generators of type (1), (2), or (3). This establishes the proposition. ∎

16. Normal Codimension One Strata.

In table VI appear representations with S^3 strata. Included are most of the orthogonal representations of the simple Lie groups for which lifting is still in doubt (missing are II.23 and IV.27). In this section we establish coregularity and the lifting property for the representations in table VI which have normal codimension one strata. All remaining cases are handled in § 17.

First some preliminaries on notation: Let V be a representation space of G. The properties of (V, G) we are interested in only depend upon the image of G in GL(V), and we will often find it convenient to replace (V, G) by a representation $(\overline{V}, \overline{G})$ such that $V \simeq \overline{V}$ and $\mathrm{Im}(G \to GL(V)) = \mathrm{Im}(\overline{G} \to GL(V))$. In such a case we write $(V, G) \sim (\overline{V}, \overline{G})$. For example, $(\mathbf{C}^4, SO(4, \mathbf{C})) \sim (\varphi_1 \otimes \varphi_1', \mathsf{A}_1 \times \mathsf{A}_1')$.

In the cases to be considered, (V, G) is orthogonal, hence there are natural isomorphisms $S^*(V) \simeq S^*(V^*) \simeq \mathbf{C}[V]$. We will frequently identify $\mathbf{C}[V]$ with $S^*(V)$ and exhibit generators of $\mathbf{C}[V]^G$ as elements of $S^*(V)^G$. Our calculations of $S^*(V)^G$ will require knowledge of the irreducible factors of tensor products of certain represen-

TABLE VI

	G	V=V(φ)	dim V/G	N	V^L	Max. Proper Slice Reps.	Lower Bound for deg f_L
1	D_5	$3\varphi_1+\varphi_4+\varphi_5$	17	$\mathsf{D}_3\times\mathsf{A}_1'$	$3\varphi_1+\varphi_2\otimes\varphi_1'+\varphi_3\otimes\varphi_1'$	3,9	14
2	B_4	$4\varphi_1+\varphi_4$	16	$\mathsf{B}_2\times\mathsf{A}_1'$	$4\varphi_1+\varphi_2\otimes\varphi_1'$	7	12
3		$2\varphi_1+2\varphi_4$	14		$2\varphi_1+2\varphi_2\otimes\varphi_1'$	8	12
4		$3\varphi_4$	12		$3\varphi_2\otimes\varphi_1'$	11	12
5	D_4	$4\varphi_1+\varphi_3$	12	$\mathsf{A}_1\times\mathsf{A}_1'\times\mathsf{A}_1''$	$4\varphi_1\otimes\varphi_1'+\varphi_1\otimes\varphi_1''$	10,13	12
6		$3\varphi_1+2\varphi_3$	12		$3\varphi_1\otimes\varphi_1'+2\varphi_1\otimes\varphi_1''$	11,12	10
7		$3\varphi_1+\varphi_3+\varphi_4$	12		$3\varphi_1\otimes\varphi_1'+\varphi_1\otimes\varphi_1''+\varphi_1'\otimes\varphi_1''$	11,13	10
8		$2\varphi_1+2\varphi_3+\varphi_4$	12		$2\varphi_1\otimes\varphi_1'+2\varphi_1\otimes\varphi_1''+\varphi_1'\otimes\varphi_1''$	12,13	10
9	A_4	$4\varphi_1+4\varphi_4$	16	$\mathsf{A}_2\times\mathbf{C}^*$	$4\varphi_1\otimes\nu_1+4\varphi_2\otimes\nu_{-1}$	16	8
10	B_3	$3\varphi_1+\varphi_3$	8	$\mathsf{A}_1\times\mathsf{A}_1'$	$3\varphi_1^2+\varphi_1\otimes\varphi_1'$	14,17	10
11		$2\varphi_1+2\varphi_3$	9		$2\varphi_1^2+2\varphi_1\otimes\varphi_1'$	15,17	8
12		$\varphi_1+3\varphi_3$	10		$\varphi_1^2+3\varphi_1\otimes\varphi_1'$	16,17	8
13		$4\varphi_3$	11		$4\varphi_1\otimes\varphi_1'$	17	8
14	D_3	$2\varphi_1+\varphi_2+\varphi_3$	5	$\mathsf{A}_1\times\mathbf{C}^*$	$2(\nu_2+\nu_{-2})+\varphi_1\otimes(\nu_1+\nu_{-1})$	18,20	8
15		$\varphi_1+2\varphi_2+2\varphi_3$	7		$\nu_2+\nu_{-2}+2\varphi_1\otimes(\nu_1+\nu_{-1})$	19,20	6
16		$3\varphi_2+3\varphi_3$	9		$3\varphi_1\otimes(\nu_1+\nu_{-1})$	20	6
17	G_2	$3\varphi_1$	7	A_1	$3\varphi_1^2$	20	6
18	B_2	$\varphi_1+2\varphi_2$	3	A_1	$\theta_1+2\varphi_1$	21	6
19		$4\varphi_2$	6		$4\varphi_1$	22	4
20	A_2	$2\varphi_1+2\varphi_2$	4	$\mathbf{C}^*$	$2(\nu_1+\nu_{-1})$	22	4
21	$\mathsf{A}_1\times\mathsf{A}_1'$	$2\varphi_1+2\varphi_1'$	2			22	4
22	A_1	$2\varphi_1$	1	$\{id\}$	$\{0\}$		2

tations of G. The results we will need are in ([20], Supplement), [44], or follow from the techniques in [45].

Suppose that $V=V_1+\ldots+V_m$ is a direct sum of representation spaces of G. We say that a form $f\in\mathbf{C}[V]^G$ has **degree** $(a_1, \ldots, a_m)$ if f is homogeneous of degree a_i in V_i, $i=1, \ldots, m$. We call $a_1+\ldots+a_m$ the **total degree** (or also just degree) of f. We will confuse $\mathfrak{X}(V)^G$ with $\mathrm{Map}(V, V)^G$ and with the elements of $\mathbf{C}[V+V^*]^G$ linear in V^*. We define the **degree** (resp. **total degree**) of an element of $\mathrm{Map}(V, V)^G$ or $\mathfrak{X}(V)^G$ to be the degree (resp. total degree) of the corresponding element of $\mathbf{C}[V_1+\ldots+V_m+V_1^*+\ldots+V_m^*]^G$ (providing this degree is defined).

Suppose that V_i and V_j are copies of the same representation of G for some i and j (we allow $i=j$). Then there is a sequence of G-equivariant linear maps:

$$\mathfrak{X}(V) \xrightarrow{\text{proj.}} \mathrm{Map}(V, V_i) \xrightarrow{\sim} \mathrm{Map}(V, V_j) \xrightarrow{\text{incl.}} \mathrm{Map}(V, V).$$

If (V, G) is orthogonal and $f\in\mathbf{C}[V]^G$, then this sequence of maps applied to grad f gives rise to elements of $\mathrm{Map}(V, V_j)^G$ and $\mathrm{Map}(V, V)^G$. We call these elements **generalized gradients** of f.

We now remark on the contents and construction of table VI: Each entry is a representation $(V, G)=(V(\varphi), G)$ with a unique (except in VI.21) 1-subprincipal isotropy class $(L=\mathsf{A}_1)$. We use N to denote a finite cover of $N_G(L)/L$, and we list its canonical representation on V^L. We list the **maximal proper slice representations** of (V, G), i.e. those slice representations (V_i, G_i) such that no isotropy classes lie strictly between (G_i) and (G). Clearly every proper slice representation of (V, G) is a slice representation of some (V_i, G_i). We indicate the entry numbers of the (V_i, G_i); trivial factors are omitted. Chasing down these slice representations one easily sees that (L) is the unique subprincipal isotropy class of (V, G) (except in VI.21). The (V_i, G_i) can be determined using tables I-V and the following observation: Suppose that $W=W_1+W_2$ is a direct sum of representation spaces of the compact Lie groupe K. Then the slice representation at (w_1, w_2) is a slice representation of the slice representations at $(w_1, 0)$ and $(0, w_2)$; $w_1\in W_1$, $w_2\in W_2$.

In each case G is connected and (V, G) is orthogonal, so $I(V^{(L)})^G$ has a homogeneous generator f_L, and $\deg f_L$ is even. We list a lower bound for $\deg f_L$. In VI.22 we know $\deg f_L=2$, and the other estimates are obtained using (15.9). For example, suppose that $(V, G)=(\varphi_1+3\varphi_3, \mathsf{B}_3)$. Since $(3\varphi_3, \mathsf{B}_3)$ contains no principal orbits of (V, G), f_L must be positively homogeneous in φ_1. From (15.9) we see that $\deg f_L>6$ (hence $\deg f_L\geq 8$), where $\deg f_L\geq 6$ is the estimate for the slice representation $(\theta_1+3\varphi_2+3\varphi_3, \mathsf{D}_3)$.

Only entries 1, 2, 4, 5, and 6 of table VI are not slice representations of other entries, so by (7.1) it suffices to verify (13.8) in these cases. Using (15.7) one sees that the representations which do not have VI.22 as slice representation have normal codimension one strata. In particular, VI.1, VI.2, VI.4, and VI.6 have normal codimension one strata, and these are the four cases we handle in this section. We apply proposition (15.2). Note that we may assume hypothesis (15.2.3) by our general induction scheme, and (15.2.2) reduces to

(16.1) $$\mathfrak{X}(V^L)^N = \mathrm{res}_L \mathfrak{X}(V)^G + \mathfrak{X}_N(V^L)^N$$

since the codimension one stratum of (V, G) is normal. Thus we need only verify (15.2.1) (coregularity) and (16.1).

Entry **VI.2.** — $(V, G) = (4\varphi_1 + \varphi_4, \mathsf{B}_4)$, $(V^L, N) = (4\varphi_1 + \varphi_2 \otimes \varphi_1', \mathsf{B}_2 \times \mathsf{A}_1')$. By CIT, the A_1'-invariants of $\varphi_2 \otimes \varphi_1'$ have degree two generators which transform by $(\varphi_1 + \theta_1, \mathsf{B}_2)$, and the relations are generated by the equality $x^2 = r^2$ where x is a generator of $\mathbf{C}[\theta_1]$ and r^2 is the square of the radius function of $\varphi_1(\mathsf{B}_2)$. We quotient by the action of A_1' to obtain a B_2-equivariant orbit map

$$4\varphi_1 + \varphi_2 \otimes \varphi_1' \to (4\varphi_1 + \varphi_1 + \theta_1) = \psi.$$

The invariants of ψ are generated by inner products $x_i \cdot x_j$, $1 \leq i \leq j \leq 5$, by a determinant d_6, and by x. The relations are generated by

(16.2) $$d_6^2 = \det(x_i \cdot x_j).$$

Pulling back these generators to $\mathbf{C}[V^L]^N$ amounts to adding the relation $x^2 = x_5 \cdot x_5$. Thus $\mathbf{C}[V^L]^N$ has 16 generators and the degree 12 relation (16.2). By (15.6), (V, G) is coregular. The elements of $\mathbf{C}[4\varphi_1 + \varphi_4]^{\mathsf{B}_4}$ restricting to x and the $x_i \cdot x_j$ are obvious, and the invariant which is the contraction of $\Lambda^4 \varphi_1 \subseteq S^4(4\varphi_1)$ with

$$\Lambda^4 \varphi_1 \subseteq S^2(\varphi_4) \simeq \Lambda^4 \varphi_1 + \varphi_1 + \theta_1$$

restricts to a multiple of d_6.

We verify (16.1) in two parts, corresponding to the splitting

$$\mathfrak{X}(V^L)^N = (E = \mathrm{Map}(V^L, \varphi_2 \otimes \varphi_1')^N) + (F = \mathrm{Map}(V^L, 4\varphi_1)^N).$$

By CIT, $\mathrm{Map}(\varphi_2 \otimes \varphi_1', \varphi_2 \otimes \varphi_1')^{\mathsf{A}_1'}$ has degree two generators (as a $\mathbf{C}[\varphi_2 \otimes \varphi_1']^{\mathsf{A}_1'}$-module) which transform by $(\varphi_2 \otimes \varphi_2, \mathsf{B}_2) \simeq (\varphi_1 + \theta_1 + \Lambda^2 \varphi_1, \mathsf{B}_2)$. Thus there is an embedding

$$E \to \overline{E} = \mathrm{Map}(4\varphi_1 + \varphi_1 + \theta_1, \varphi_1 + \theta_1 + \Lambda^2 \varphi_1)^{\mathsf{B}_2}.$$

The determinant in $\mathrm{Map}(4\varphi_1, \varphi_1)^{\mathsf{B}_2} \subseteq \overline{E}$ pulls back to a generalized gradient of d_6. Lemma (15.13) shows that we may choose all other generators of $\overline{E}$ to have degree 0 with respect to one of the first four copies of φ_1. Thus the corresponding elements of E can be considered as elements of copies of $\mathfrak{X}(3\varphi_1 + \varphi_2 \otimes \varphi_1')^N$. By II.18 these elements are in $\mathrm{res}_L \mathfrak{X}(V)^G$. Hence $E \subseteq \mathrm{res}_L \mathfrak{X}(V)^G$.

Note that F embeds in $\overline{F} = \mathrm{Map}(4\varphi_1 + \varphi_1 + \theta_1, 4\varphi_1)^{\mathsf{B}_2}$. As above, all generators of $\overline{F}$ which do not involve all of the first four copies of φ_1 pull back to elements of F which lie in $\mathrm{res}_L \mathfrak{X}(V)^G$. The only problems are the four determinants in $\mathrm{Map}(4\varphi_1, 4\varphi_1)^{\mathsf{B}_2} \subseteq F$. Now all elements of $\mathbf{C}[4\varphi_1]^{\mathsf{B}_2} \simeq \mathbf{C}[4\mathbf{C}^5]^{SO(5,\mathbf{C})}$ are $O(5, \mathbf{C})$-invariant, while the vector fields $X_1, \ldots, X_4$ corresponding to the determinants transform by the determinant representation of $O(5, \mathbf{C})$. Thus these vector fields act trivially on $\mathbf{C}[4\mathbf{C}^5]^{SO(5,\mathbf{C})}$. By (9.3),

$$\mathfrak{X}_{SO(5,\mathbf{C})}(4\mathbf{C}^5) = \mathfrak{X}_{\mathrm{Ad}\, SO(5,\mathbf{C})}(4\mathbf{C}^5),$$

so the X_i are projections to F of elements of $\mathfrak{X}_{\mathrm{Ad}\,N}(V^L)^N$. Thus each X_i belongs to $\mathfrak{X}_{\mathrm{Ad}\,N}(V^L)^N + E$. Hence (16.1) holds, and (V, G) has the lifting property.

Entry **VI.6**

$$(V, G) = (3\varphi_1 + 2\varphi_3, \mathsf{D}_4),$$
$$(V^L, N) = (3\varphi_1 \otimes \varphi_1' + 2\varphi_1 \otimes \varphi_1'', \mathsf{A}_1 \times \mathsf{A}_1' \times \mathsf{A}_1'').$$

The invariants of $(3\varphi_1 \otimes \varphi_1', \mathsf{A}_1 \times \mathsf{A}_1') \sim (3\mathbf{C}^4, SO(4, \mathbf{C}))$ are generated by inner products $x_i \cdot x_j$, $1 \leq i \leq j \leq 3$, and $(2\varphi_1 \otimes \varphi_1'', \mathsf{A}_1 \times \mathsf{A}_1'')$ has generators we denote by $y_k \cdot y_\ell$, $1 \leq k \leq \ell \leq 2$. Quotienting by the action of A_1'' we obtain an $\mathsf{A}_1 \times \mathsf{A}_1'$-equivariant orbit map

$$3\varphi_1 \otimes \varphi_1' + (2\varphi_1 \otimes \varphi_1'') \to 3\varphi_1 \otimes \varphi_1' + (\varphi_1^2 + \theta_3) = (V_0, \mathsf{A}_1 \times \mathsf{A}_1').$$

Let y denote a typical point in the copy of φ_1^2. Now $\mathrm{Ad}(\mathsf{A}_1 \times \mathsf{A}_1') \simeq \Lambda^2(\varphi_1 \otimes \varphi_1') \simeq \varphi_1^2 + (\varphi_1')^2$, so for each m and n with $1 \leq m < n \leq 3$ we obtain a copy of φ_1^2 (typical point denoted x_{mn}) in the tensor product of the corresponding copies of $\varphi_1 \otimes \varphi_1'$. Hence we obtain invariants $\alpha_{mn} = x_{mn} \cdot y$ of $(V_0, \mathsf{A}_1 \times \mathsf{A}_1')$, and by lemma (15.13) the invariants of $(V_0, \mathsf{A}_1 \times \mathsf{A}_1')$ are generated by the $x_i \cdot x_j$, $y_k \cdot y_\ell$, α_{mn}, and $y \cdot y$. Pulling back to $\mathbf{C}[V^L]^N$ we lose $y \cdot y$ as a generator $(y \cdot y = (y_1 \cdot y_1)(y_2 \cdot y_2) - (y_1 \cdot y_2)^2)$, so $\mathbf{C}[V^L]^N$ has 12 generators. The Second Main Theorem for $SO(3, \mathbf{C})$ shows that

$$\textbf{(16.3)} \qquad \det(x_{12}, x_{13}, x_{23})(y \cdot y) = \det(y, x_{13}, x_{23})(x_{12} \cdot y) + \det(x_{12}, y, x_{23})(x_{13} \cdot y) + \det(x_{12}, x_{13}, y)(x_{23} \cdot y).$$

Re-expressing (16.3) in terms of our generators above, we see that the left hand side of (16.3) is a multiple of $\det(x_i \cdot x_j)\det(y_k \cdot y_\ell)$ and that the right hand side is quadratic in the α_{mn}. In particular, the relation is non-trivial and of degree 10. By (15.6), (V, G) is coregular. The elements of $\mathbf{C}[3\varphi_1 + 2\varphi_3]^{\mathsf{D}_4}$ restricting to the $x_i \cdot x_j$ and $y_k \cdot y_\ell$ are obvious, while the α_{mn} are obtained by restricting the contractions of the 3 copies of $\Lambda^2\varphi_1 \subseteq S^2(3\varphi_1)$ with $\Lambda^2\varphi_1 \simeq \Lambda^2\varphi_3 \subseteq S^2(2\varphi_3)$. The verification of (16.1) proceeds by the same type of argument used for VI.2, so (V, G) has the lifting property.

Entry **VI.4.** — $(V, G) = (3\varphi_4, \mathsf{B}_4)$, $(V^L, N) = (3\varphi_2 \otimes \varphi_1', \mathsf{B}_2 \times \mathsf{A}_1') \simeq (3\varphi_1 \otimes \varphi_1', \mathsf{C}_2 \times \mathsf{A}_1')$. By CIT, $\mathbf{C}[3\varphi_1 \otimes \varphi_1']^{\mathsf{C}_2}$ has degree 2 generators which transform by $(\theta_6 + 3(\varphi_1')^2, \mathsf{A}_1')$. We denote the invariants corresponding to θ_6 by $x_i \cdot x_j$, $1 \leq i \leq j \leq 3$, and we denote typical points in the 3 copies of $(\varphi_1')^2$ by y_k, $1 \leq k \leq 3$. Then $\mathbf{C}[V^L]^N$ is generated by the determinant d_6 of the y_k, the inner products $y_k \cdot y_\ell$, $1 \leq k \leq \ell \leq 3$, and the $x_i \cdot x_j$. CIT for $SO(3, \mathbf{C})$ gives us the relation

$$\textbf{(16.4)} \qquad d_6^2 = \det(y_k \cdot y_\ell),$$

and CIT for $\mathbf{C}_2$ indicates that the relations of $\mathbf{C}[V^L]^{\mathbf{C}_2}$ have a generator of degree 6. This latter relation must be $\mathbf{A}_1'$-invariant, so we obtain a relation

$$(16.5) \qquad ad_6 = f(x_i \cdot x_j, y_k \cdot y_\ell)$$

where $a=0$ or 1. We show below that $a=1$. Then substituting the right hand side of (16.5) for d_6 in (16.4) yields a relation whose right hand side is cubic in the $(y_k \cdot y_\ell)$ and whose left hand side is at most quadratic in the $y_k \cdot y_\ell$. Thus this relation is non-trivial. Since its degree is 12, (15.6) shows that (V, G) is coregular. Using the fact that $S^2(\varphi_4(\mathbf{B}_4)) \simeq \varphi_4^2 + \varphi_1 + \theta_1$, one easily exhibits generators of $\mathbf{C}[V]^G$ which restrict to our generators of $\mathbf{C}[V^L]^N$.

By carefully looking at the degree 6 relation among the generators of $\mathbf{C}[V^L]^{\mathbf{C}_2}$, one could show that $a \neq 0$. The following demonstration is more in the spirit of this paper: Suppose that $a=0$. Since $\deg f_L \geq 12$, the $x_i \cdot x_j$, $y_k \cdot y_\ell$, and d_6 lift to unique homogeneous elements α_{ij}, $\beta_{k\ell}$, and δ_6 of $\mathbf{C}[V]^G$. Equation (16.5) lifts to the relation $f(\alpha_{ij}, \beta_{k\ell}) = 0$. Let $h = \delta_6^2 - \det(\beta_{k\ell})$. If f_L is a multiple of h, then

$$\mathbf{C}[V]^G \simeq \mathbf{C}[\alpha_{ij}, \beta_{k\ell}, \delta_6]/(f),$$

and if f_L is not a multiple of h, then $\mathbf{C}[V]^G \simeq \mathbf{C}[\alpha_{ij}, \beta_{k\ell}, \delta_6, f_L]/(f, h)$. In either case, $V/G - \{\pi_{V,G}(0)\}$ has singularities. But, ignoring trivial factors, all proper slice representations of (V, G) are slice representations of VI.11, and VI.11 is a slice representation of VI.6. We have shown that VI.6 is coregular, hence VI.11 is coregular and $V/G - \{\pi_{V,G}(0)\}$ is smooth. We have a contradiction. Thus $a=1$ and $\mathbf{C}[V]^G$ is the regular ring described above.

We now turn to the proof of (16.1): Note that $\mathfrak{X}(V^L)^N$ is isomorphic to 3 copies of $E' = \mathrm{Map}(3\varphi_1 \otimes \varphi_1', \varphi_1 \otimes \varphi_1')^{\mathbf{C}_2 \times \mathbf{A}_1'}$. As above, E' has generators of degree ≤ 6. Using (15.9) and the estimates of $\deg f_L$ in table VI, one shows that $\mathrm{res}_L : \mathfrak{X}(V)^G \to \mathfrak{X}(V^L)^N$ has zero kernel in degree ≤ 6. Hence to establish (16.1) it would suffice to show that E' and $E = \mathrm{Map}(3\varphi_4, \varphi_4)^{\mathbf{B}_4}$ have the same number of generators of degree ≤ 6. However, there is a 5 (resp. 6)-dimensional space of invariants of degree (1, 1, 1, 1) in E (resp. E'). We must show that $E'/\mathrm{res}_L E$ lies in the image of $\mathfrak{X}_N(V^L)^N$:

Let L_1 be another copy of $\mathbf{A}_1$. The standard embedding

$$SU(2) \times SU(2) \subseteq SO(4) \times SO(4) \subseteq SO(8) \subseteq SO(9)$$

lifts to an embedding of $SU(2) \times SU(2)$ in Spin(9). Complexifying we obtain an embedding of $M = L \times L_1$ into $\mathbf{B}_4$ such that the image of L_1 in N lies in $\mathbf{C}_2$ and is a principal isotropy group of $(2\varphi_1, \mathbf{C}_2)$. Now $N_{\mathbf{B}_4}(M)/M \simeq O(4, \mathbf{C})$, and we have a commutative diagram

$$\begin{tikzcd} E \arrow[r, "\mathrm{res}_M"] \arrow[d, "\mathrm{res}_L"'] & E'' = \mathrm{Map}(3\mathbf{C}^4, \mathbf{C}^4)^{SO(4,\mathbf{C})} \\ E' \arrow[ur, "\mathrm{res}_{L_1}"'] & \end{tikzcd}$$

where the image of res_M is $O(4, \mathbf{C})$-invariant. Thus the determinant $X'' \in E''$ is not in $\mathrm{Im}\,\mathrm{res}_M$. Now X'' corresponds to an element of $\mathfrak{X}_{\mathrm{Ad}\,SO(4,\mathbf{C})}(3\mathbf{C}^4)^{SO(4,\mathbf{C})}$, and using (11.2) or (15.13) one can see that X'' has a pre-image X' in $E' \cap \mathfrak{X}_{\mathrm{Ad}\,N}(V^L)^N$. Then X' generates $E'/\mathrm{res}_L E$ in degree $(1, 1, 1, 1)$. In degrees $(2, 2, 1, 1)$ (or $(2, 1, 2, 1)$, etc.) one finds that E' has dimension 17, E has dimension 16, and hence $(x_1 \cdot x_2)X'$ (or $(x_1 \cdot x_3)X'$, etc.) generates $E'/\mathrm{res}_L E$. In degree $(3, 1, 1, 1)$ (or $(1, 3, 1, 1)$, etc.) the respective dimensions are 11 and 10, and $(x_1 \cdot x_1)X'$ (or $(x_2 \cdot x_2)X'$, etc.) generates the cokernel. For degrees $(3, 2, 0, 1)$, etc. we already know res_L is surjective. Thus (16.1) holds, and (V, G) has the lifting property.

Entry **VI.1**:

$$(V, G) = (3\varphi_1 + \varphi_4 + \varphi_5, \mathsf{D}_5),$$
$$(V^L, N) = (3\varphi_1 + \varphi_2 \otimes \varphi_1' + \varphi_3 \otimes \varphi_1', \mathsf{D}_3 \times \mathsf{A}_1').$$

We found it difficult to compute $\mathbf{C}[V^L]^N$ using techniques similar to those used above. Instead we used the method of example (15.12) (inductively) to compute $\mathbf{C}[V]^G$. While writing [67] we discovered that, paradoxically, it is not difficult to apply our usual method to the "larger" representation $(5\varphi_1 + \varphi_5, \mathsf{D}_6)$ which has $(V + \theta, G)$ as a slice representation. We omit the details of either computation.

To describe the generators of $\mathbf{C}[V]^G$ we need to note the following identities for representations of D_5:

$$S^2\varphi_4 + S^2\varphi_5 \simeq \Lambda^5\varphi_1 + 2\varphi_1; \qquad \varphi_4 \otimes \varphi_5 \simeq \theta_1 + \Lambda^4\varphi_1 + \Lambda^2\varphi_1.$$

The generators of $\mathbf{C}[V]^G$ are then the 6 inner product invariants in $(3\varphi_1, \mathsf{D}_5)$, the invariant in $\varphi_4 \otimes \varphi_5$, the contraction of the two copies of φ_1 in $S^2\varphi_4 + S^2\varphi_5$, the contractions of the 3 original copies of φ_1 with the 2 copies in $S^2\varphi_4 + S^2\varphi_5$, and the contractions of the 3 copies of $\Lambda^2\varphi_1$ in $S^2(3\varphi_1)$ with the copy in $\varphi_4 \otimes \varphi_5$. One can show that f_L is indeed of degree 14, as estimated in table VI.

We need to note the following: The restriction map from $\rho = (\varphi_4 + \varphi_5, \mathsf{D}_5)$ to $\rho^L = (\varphi_2 \otimes \varphi_1' + \varphi_3 \otimes \varphi_1', \mathsf{D}_3 \times \mathsf{A}_1')$ induces a map from $\rho \otimes \rho$ to $\rho^L \otimes \rho^L$. By examining the action of $N_G(L)$ and its normalizer on φ_1, φ_4, and φ_5 one can see that

(16.6) $\varphi_4^2 \subseteq S^2\varphi_4$ projects onto each irreducible factor of $S^2(\varphi_2 \otimes \varphi_1') \simeq \varphi_1 + \varphi_2^2 \otimes (\varphi_1')^2$.

(16.7) $\varphi_1 \subseteq S^2\varphi_4$ projects onto $\varphi_1 \subseteq S^2(\varphi_2 \otimes \varphi_1')$.

(16.8) $\Lambda^2\varphi_4 \simeq \Lambda^3\varphi_1$ projects onto each irreducible factor of $\Lambda^2(\varphi_2 \otimes \varphi_1') \simeq \varphi_1 \otimes (\varphi_1')^2 + \varphi_2^2$.

The analogues of (16.6), (16.7), and (16.8) hold for the components of $S^2\varphi_5$ and $\Lambda^2\varphi_5$. Also

(16.9) $(\varphi_2 \otimes \varphi_1' \otimes \varphi_3 \otimes \varphi_1', \mathsf{D}_3 \times \mathsf{A}_1') \simeq (\Lambda^2\varphi_1 \otimes (\varphi_1')^2 + \Lambda^2\varphi_1 + (\varphi_1')^2 + \theta_1, \mathsf{D}_3 \times \mathsf{A}_1')$,

(16.10) $\Lambda^4\varphi_1 \simeq \varphi_4\varphi_5 \subseteq \varphi_4 \otimes \varphi_5$ projects onto $\Lambda^2\varphi_1 \otimes (\varphi_1')^2$, $\Lambda^2\varphi_1$, and θ_1,

and

(16.11) $\Lambda^2\varphi_1 \simeq \varphi_2 \subseteq \varphi_4 \otimes \varphi_5$ projects onto $\Lambda^2\varphi_1$ and $(\varphi_1')^2$.

Now $\mathrm{Map}(V^L, V^L)^N$ is isomorphic to 3 copies of $E = \mathrm{Map}(V^L, \varphi_1)^N$ plus $F_2 = \mathrm{Map}(V^L, \varphi_2 \otimes \varphi_1')^N$ plus $F_3 = \mathrm{Map}(V^L, \varphi_3 \otimes \varphi_1')^N$. We first show that $E \subseteq \mathrm{res}_L \mathfrak{X}(V)^G$: Since

$$\mathbf{C}[\varphi_2 \otimes \varphi_1' + \varphi_3 \otimes \varphi_1']^{\mathsf{A}_1'}$$

has degree 2 generators which transform by $(2\varphi_1 + \Lambda^2\varphi_1 + \theta_1, \mathsf{D}_3)$, we obtain an embedding

$$E \to \overline{E} = \mathrm{Map}(3\varphi_1 + (2\varphi_1 + \Lambda^2\varphi_1 + \theta_1), \varphi_1)^{\mathsf{D}_3}.$$

By IV.21, generators of $\overline{E}$ which do not involve all of the first 3 copies of φ_1 pull back to elements of E which are in $\mathrm{res}_L \mathfrak{X}(V)^G$. By lemma (15.13), the only generators we need worry about are the determinant invariant α of the 6 copies of φ_1 and the invariant β corresponding to the wedge product of $\Lambda^2\varphi_1$ with the first 3 copies of φ_1 and the last copy of φ_1. Now

(16.12) $$((\Lambda^6\varphi_1(\mathsf{D}_5))^L, \mathsf{D}_3 \times \mathsf{A}_1') \simeq (\theta_1 + \Lambda^4\varphi_1 \otimes (\varphi_1')^2 + \Lambda^2\varphi_1, \mathsf{D}_3 \times \mathsf{A}_1'),$$

and under res_L the element of $\mathrm{Map}(3\varphi_1 + \Lambda^6\varphi_1, \varphi_1)^{\mathsf{D}_5}$ obtained by wedging the copies of φ_1 and $\Lambda^6\varphi_1$ restricts to the element in $\mathrm{Map}(3\varphi_1 + \theta_1 + \Lambda^4\varphi_1 \otimes (\varphi_1')^2 + \Lambda^2\varphi_1, \varphi_1)^{\mathsf{D}_3 \times \mathsf{A}_1'}$ obtained by wedging the copies of φ_1 and $\Lambda^2\varphi_1$. Using (16.10) and the isomorphism $\Lambda^6\varphi_1(\mathsf{D}_5) \simeq \Lambda^4\varphi_1(\mathsf{D}_5)$ one can then see that $\beta \in \mathrm{res}_L \mathfrak{X}(V)^G$. Similarly, using (16.6) and (16.7) one can see that $\alpha \in \mathrm{res}_L \mathfrak{X}(V)^G$. Hence $E \subseteq \mathrm{res}_L \mathfrak{X}(V)^G$.

Taking A_1'-invariants again, we obtain an embedding

$$F_2 + F_3 \to \mathrm{Map}(3\varphi_1 + (2\varphi_1 + \Lambda^2\varphi_1 + \theta_1), 2\Lambda^2\varphi_1 + 2\theta_1 + \Lambda^3\varphi_1 + 2\varphi_1)^{\mathsf{D}_3}.$$

The techniques used in the proof of lemma (15.13) show that the latter module has generators of degree ≤ 4 with respect to the first 3 copies of φ_1, but it is difficult to pass from just this information to a proof that res_L is surjective. Our approach will be to find the D_3-invariants and then find the A_1'-invariants of these, rather than the reverse:

Let $\rho_3 = (3\varphi_1 + \varphi_2 \otimes \varphi_1' + \varphi_3 \otimes \varphi_1' + \varphi_2 \otimes \varphi_1', \mathsf{D}_3 \times \mathsf{A}_1')$. Then F_3 is isomorphic to the invariants of ρ_3 which are of degree 1 in the last factor. We reduce the calculation of the invariants of ρ_3 to CIT by the following device: Since $\varphi_1 \simeq \Lambda^2\varphi_2 \simeq \Lambda^2\varphi_3$, the degree 2 generators of the A_1''-invariants of $\varphi_2 \otimes \varphi_1''$ or $\varphi_3 \otimes \varphi_1''$ transform by φ_1. Now $\dim V(\varphi_2 \otimes \varphi_1'')/\mathsf{A}_1'' = 5$, and $\mathbf{C}[\varphi_2 \otimes \varphi_1'']^{\mathsf{D}_3 \times \mathsf{A}_1''} = \mathbf{C}$ since $\mathbf{C}[2\varphi_2]^{\mathsf{D}_3} = \mathbf{C}$. It follows that the image of $V(\varphi_2 \otimes \varphi_1'')/\mathsf{A}_1''$ in φ_1 is the zero set of the square of the radius function. Let M_1, M_2, M_3 represent copies of A_1, and let λ_1, λ_2, λ_3, respectively, denote their basic 2-dimensional representations. Our argument above shows that the invariants of $\rho_3' = (\varphi_3 \otimes \lambda_1 + \varphi_2 \otimes \lambda_2 + \varphi_3 \otimes \lambda_3 + \varphi_2 \otimes \varphi_1' + \varphi_3 \otimes \varphi_1' + \varphi_2 \otimes \varphi_1', \mathsf{D}_3 \times \mathsf{A}_1' \times M_1 \times M_2 \times M_3)$ correspond in a one-to-one manner to the invariants of ρ_3 restricted to the zero set of the squares of the radius functions of the first 3 copies of φ_1. Thus it suffices to calculate the invariants of ρ_3'.

By CIT, the invariants of (ρ_3', D_3) are generated by the contractions of copies of φ_2 with $\varphi_3 = \varphi_2^*$ and by copies of $\theta_1 \simeq \Lambda^4\varphi_2$ and $\theta_1 \simeq \Lambda^4\varphi_3$ in S^4 (the copies of φ_2 and φ_3

in ρ_3'). Relative to the action of M_1, M_2, and M_3 these generators span a representation consisting of one copy of $\lambda_1\otimes\lambda_2+\lambda_1\otimes\lambda_3+\lambda_2\otimes\lambda_3$ plus several copies of θ_1, λ_1, λ_2, and λ_3. Quotienting by the action of M_1 we obtain generators of the $(\mathsf{D}_3\times M_1)$-invariants. Since we have ignored relations, our generating set is not minimal. However, the following simple observation weeds out the spurious generators: If an element of, say, degree $(2, 1, 2, 1, 0, 0)$ is a generator of the $(\mathsf{D}_3\times M_1)$-invariants, then symmetry arguments force there to be associated generators of degrees $(2, 2, 1, 0, 1, 0)$, $(2, 1, 0, 1, 2, 0)$, etc. Thus we may throw away a generator if any of its associated generators does not appear. After weeding out spurious generators, we divide by the action of M_2 and exploit symmetries to prune any new superfluous generators. Finally, we find the M_3-invariants and prune again. A similar calculation finds the D_3-invariants of $\rho_2=3\varphi_1+\varphi_2\otimes\varphi_1'+\varphi_3\otimes\varphi_1'+\varphi_3\otimes\varphi_1'$. As a result one finds that

$$\mathbf{C}[3\varphi_1+\varphi_2\otimes\varphi_1'+\varphi_3\otimes\varphi_1'+(\varphi_2\otimes\varphi_1'+\varphi_3\otimes\varphi_1')]^{\mathsf{D}_3}$$

has the following types of generators which are of degree ≤ 1 in the last two factors:

(i) Contractions of copies of $\Lambda^2\varphi_1\subseteq S^2(3\varphi_1)$ with copies of $\Lambda^2\varphi_1\subseteq\varphi_2\otimes\varphi_3=\Lambda^2\varphi_1+\theta_1$.

(ii) Contractions of copies of φ_1 with copies of $\Lambda^2\varphi_2\simeq\varphi_1$ or $\Lambda^2\varphi_3\simeq\varphi_1$.

(iii) Contractions of φ_2^2 and $\varphi_3^2\subseteq\varphi_2^2+\varphi_3^2=\Lambda^3\varphi_1\subseteq S^3(3\varphi_1)$ with copies of $\varphi_3^2=S^2\varphi_3$ and $\varphi_2^2=S^2\varphi_2$.

(iv) $\theta_1\subseteq$ copies of $\varphi_2\otimes\varphi_3$.

(v) Inner product invariants of copies of φ_1.

The D_3-invariants of type (v) are A_1'-invariant, but the types (i) through (iv) are A_1'-invariant or transform by $((\varphi_1')^2, \mathsf{A}_1')$. We leave it to the reader to show that A_1'-invariant elements of type (iii) are in $\operatorname{res}_L\mathfrak{X}(V)^G$; the analogous fact for types other than (iii) follows from IV.21.

It remains to show that the A_1'-invariants formed by taking contractions and determinant invariants of copies of $(\varphi_1')^2$ of types (i) through (iv) are all in $\operatorname{res}_L\mathfrak{X}(V)^G$. We can immediately eliminate several cases: We know that the $(\mathsf{D}_3\times\mathsf{A}_1')$-invariants have generators of degree ≤ 4 in the 3 copies of φ_1, and generators of degree ≤ 2 in the copies of φ_1 are in $\operatorname{res}_L\mathfrak{X}(V)^G$ by IV.21. Thus, for example, we need not consider the contraction of two copies of $(\varphi_1')^2$ of type (iii) or of two copies of $(\varphi_1')^2$ of type (iv). We are only concerned with invariants of degree exactly 1 with respect to the last copy of $\varphi_2\otimes\varphi_1'+\varphi_3\otimes\varphi_1'$, so, for example, contractions of representations $(\varphi_1')^2$ of type (ii) need not be considered. We give two examples and some remarks on how to handle the remaining contractions and determinant invariants.

Let σ_1 and σ_2 be copies of $(\varphi_1')^2$ of type (i). We show that their contraction is in $\operatorname{res}_L\mathfrak{X}(V)^G$: If $\varphi_1=\varphi_1(\mathsf{D}_3)$ or $\varphi_1(\mathsf{D}_5)$, we indicate copies of $\Lambda^2\varphi_1$ lying in $S^2(3\varphi_1)$ by $\underline{\Lambda^2\varphi_1}$. It follows from (16.12) that the restriction

$$\operatorname{res}_L:\ \underline{\Lambda^2\varphi_1(\mathsf{D}_5)}\wedge\Lambda^6\varphi_1(\mathsf{D}_5)\to\underline{\Lambda^2\varphi_1(\mathsf{D}_3)}\wedge(\Lambda^6\varphi_1(\mathsf{D}_5))^L$$

has image the representations $\underline{\Lambda^2\varphi_1}\wedge\Lambda^4\varphi_1\otimes(\varphi_1')^2\simeq(\varphi_1')^2$ and $\underline{\Lambda^2\varphi_1}\wedge\Lambda^2\varphi_1$. Using (16.9) and (16.10) we then see that by wedging a copy of $\underline{\Lambda^2\varphi_1(\mathsf{D}_5)}$ with $\Lambda^6\varphi_1\simeq\varphi_4\varphi_5\subset\varphi_4\otimes\varphi_5$ we obtain a copy of $\Lambda^8\varphi_1$ which restricts onto the copies of $(\varphi_1')^2$ and $\underline{\Lambda^2\varphi_1}\wedge\Lambda^2\varphi_1$ in $\underline{\Lambda^2\varphi_1}\otimes(\varphi_2\otimes\varphi_1'\otimes\varphi_3\otimes\varphi_1')$. The copy of $(\varphi_1')^2$ is of type (i). Thus the contraction α of σ_1 and σ_2 differs from restricting the contraction of the corresponding copies of $\Lambda^8\varphi_1$ by a contraction of representations of the form $\underline{\Lambda^2\varphi_1}\wedge\Lambda^2\varphi_1$. Using the formula for the inner product of two 4-forms, one can see that a contraction of such representations is a sum of products of lower degree invariants. Thus α lies in $\mathrm{res}_{\mathrm{L}}\mathfrak{X}(\mathrm{V})^{\mathrm{G}}$, modulo generators of lower degree.

Let σ_3 be a copy of $(\varphi_1')^2$ of type (iv). We show that the determinant invariant $\det(\sigma_1, \sigma_2, \sigma_3)$ is in $\mathrm{res}_{\mathrm{L}}\mathfrak{X}(\mathrm{V})^{\mathrm{G}}$: Note that, in general, there is a non-zero $\mathrm{O}(n, \mathbf{C})$-equivariant map:

(16.13) $$\Lambda^k\mathbf{C}^n\otimes\Lambda^k\mathbf{C}^n\to\mathbf{C}^n\otimes\Lambda^{k-1}\mathbf{C}^n\otimes\Lambda^{k-1}\mathbf{C}^n\otimes\mathbf{C}^n\to\mathbf{C}^n\otimes\mathbf{C}^n\to\Lambda^2\mathbf{C}^n,\quad 1\le k\le n-1,$$

where the second arrow is contraction of the two copies of $\Lambda^{k-1}\mathbf{C}^n$. Applying (16.13) to the copies of $\Lambda^8\varphi_1$ corresponding to σ_1 and σ_2 we obtain a copy ρ of $\Lambda^2\varphi_1$ which restricts onto the copy $\sigma_1\wedge\sigma_2$ of $(\varphi_1')^2$ in $\sigma_1\otimes\sigma_2$ and also onto a representation τ_1 which results from applying (16.13) to representations of the form $\underline{\Lambda^2\varphi_1}\wedge\Lambda^2\varphi_1$. It is easy to see that τ_1 is a sum of products of invariants and copies of $\Lambda^2\varphi_1$ occurring in lower degree. Then contracting ρ with the copy of $\Lambda^2\varphi_1\subset\varphi_4\otimes\varphi_5$ corresponding to σ_3 (see (16.11)) we obtain an element of $\mathfrak{X}(\mathrm{V})^{\mathrm{G}}$ which restricts onto the contraction of $\sigma_1\wedge\sigma_2$ and $\sigma_3(=\det(\sigma_1, \sigma_2, \sigma_3))$, modulo generators of lower degree.

Along with the copy of $\Lambda^8\varphi_1\simeq\Lambda^2\varphi_1$ associated to σ_1 there is also a copy of $\Lambda^4\varphi_1$ obtained by wedging the appropriate copies of $\underline{\Lambda^2\varphi_1(\mathsf{D}_5)}$ and $\Lambda^2\varphi_1\subset\varphi_4\otimes\varphi_5$. Similarly, to every copy of $(\varphi_1')^2$ of types (ii) through (iv) there are associated copies of $\Lambda^2\varphi_1$ and $\Lambda^4\varphi_1$. In the examples above we obtained our desired results using only the associated copies of $\Lambda^2\varphi_1$. In some cases one also has to apply (16.13) to (or contract) the associated copies of $\Lambda^4\varphi_1$.

The techniques we have mentioned are sufficient to establish that

$$\mathrm{F}_2+\mathrm{F}_3\subset\mathrm{res}_{\mathrm{L}}\mathfrak{X}(\mathrm{V})^{\mathrm{G}}.$$

Thus (V, G) has the lifting property.

17. Non-normal Codimension One Strata.

The remaining cases are $(4\varphi_1+\varphi_3, \mathsf{D}_4)$, $(2\varphi_5, \mathsf{B}_5)$, and $(\varphi_1+2\varphi_5, \mathsf{D}_6)$. Since the second representation is a slice representation of the third, we need only consider the first and third cases. Our techniques in § 16 are sufficient to treat the case of $(4\varphi_1+\varphi_3, \mathsf{D}_4)$; we need different methods for $(\varphi_1+2\varphi_5, \mathsf{D}_6)$. As before, we employ proposition (15.2) and we may assume that (15.2.3) holds.

Entry **VI.5.** — $(V, G)=(4\varphi_1+\varphi_3, \mathsf{D}_4)$, $(V^L, N)=(4\varphi_1\otimes\varphi_1'+\varphi_1\otimes\varphi_1'', \mathsf{A}_1\times\mathsf{A}_1'\times\mathsf{A}_1'')$. Let d_4 and $x_i\cdot x_j$, $1\leq i, j\leq 4$ denote the determinant and inner product invariants of $(4\varphi_1\otimes\varphi_1', \mathsf{A}_1\times\mathsf{A}_1')\sim(4\mathbf{C}^4, SO(4, \mathbf{C}))$, and let β denote the usual generator of the invariants of $(\varphi_1\otimes\varphi_1'', \mathsf{A}_1\times\mathsf{A}_1'')$. Then $\mathbf{C}[V^L]^N$ has generators the $x_i\cdot x_j$, d_4, and β with the relation

$$d_4^2=\det(x_i\cdot x_j).$$

From the standard embedding $SO(4, \mathbf{C})\subseteq SO(8, \mathbf{C})$ we obtain an embedding

$$M=\mathrm{Spin}(4, \mathbf{C})\subseteq \mathrm{Spin}(8, \mathbf{C})=G,$$

and

$$(V^M, N_G(M)/M)\simeq(4\mathbf{C}^4, O(4, \mathbf{C})).$$

It follows that $\mathrm{res}_M\mathbf{C}[V]^G$ is generated by the $x_i\cdot x_j$. Since β generates the ideal in $\mathbf{C}[V^L]^N$ vanishing on V^M, $\mathrm{res}_L\mathbf{C}[V]^G$ is contained in the subalgebra of $\mathbf{C}[V^L]^N$ generated by the $x_i\cdot x_j$, β, and $\alpha=\beta d_4$. Now the representation $\Lambda^4\varphi_1$ of D_4 is not irreducible, in fact $\Lambda^4\varphi_1\simeq\varphi_3^2+\varphi_4^2$. Then contracting $\varphi_3^2\subseteq\Lambda^4\varphi_1\subseteq S^4(4\varphi_1)$ with $\varphi_3^2\subseteq S^2(\varphi_3)$ we obtain an invariant in $\mathbf{C}[V]^G$ whose restriction to V^L is a non-zero multiple of α. (Note that remark (15.8) also shows that $\alpha\in\mathrm{res}_L\mathbf{C}[V]^G$.) Thus $\mathrm{res}_L\mathbf{C}[V]^G$ is generated by the $x_i\cdot x_j$, α, and β, and there is a degree 12 relation

(17.1) $$\alpha^2-\beta^2\det(x_i\cdot x_j)=0.$$

By (15.6), (V, G) is coregular.

Let $$X\in\mathfrak{X}^+(V^L)^N,$$

and let $\overline{X}$ denote the projection of X to $\mathrm{Map}(V^L, 4\varphi_1\otimes\varphi_1')^{\mathsf{A}_1\times\mathsf{A}_1'\times\mathsf{A}_1''}$. It is easy to see that if $\overline{X}$ is divisible by β, then $\overline{X}$ lies in the $\mathrm{res}_L\mathbf{C}[V]^G$-module generated by generalized gradients of elements of $\mathrm{res}_L\mathbf{C}[V]^G$. If $\overline{X}$ is independent of $\varphi_1\otimes\varphi_1''$, i.e. lies in the image of $\mathfrak{X}(4\varphi_1\otimes\varphi_1')^{\mathsf{A}_1\times\mathsf{A}_1'}\simeq\mathfrak{X}(4\mathbf{C}^4)^{SO(4,\mathbf{C})}$, then we may write $\overline{X}=\overline{X}_++\overline{X}_-$ where $\overline{X}_+$ is the $O(4, \mathbf{C})$-invariant part of $\overline{X}$. Clearly $\overline{X}_+\in\mathrm{Im}\,\mathrm{res}_L$. $\overline{X}_-$ must act trivially on $\mathbf{C}[4\mathbf{C}^4]^{O(4,\mathbf{C})}$, hence trivially on the finite extension $\mathbf{C}[4\mathbf{C}^4]^{SO(4,\mathbf{C})}$. Thus we may reduce to the case that X annihilates the $x_i\cdot x_j$.

Clearly $X(\beta)$ is a multiple of α and β, so modulo multiples of grad β and a generalized gradient of α we may further assume that $X(\beta)=0$. Then $X(\alpha)=0$ by (17.1), so $X\in\mathfrak{X}_N(V^L)^N$. We have established (15.2.1) and (15.2.2), and we may assume (15.2.3). Thus (V, G) has the lifting property.

Our computations and the proof of proposition (15.2) also establish the following:

Lemma **(17.2).** — *Let* V, G, α, β, *etc. be as above.*

(1) *All elements of* $\mathrm{Map}(4\varphi_1\otimes\varphi_1'+\varphi_1\otimes\varphi_1'', 4\varphi_1\otimes\varphi_1')^{\mathsf{A}_1\times\mathsf{A}_1'\times\mathsf{A}_1''}$ *which are homogeneous of degree* ≥ 2 *in* $\varphi_1\otimes\varphi_1''$ *lie in the* $\mathbf{C}[\alpha, \beta]$*-module generated by the elements of* $\mathrm{res}_L\mathrm{Map}(4\varphi_1+\varphi_3, 4\varphi_1)^{\mathsf{D}_4}$ *which are of degree* ≤ 2 *in* φ_3.

(2) *The* $\mathbf{C}[V]^G$*-module* $\mathfrak{X}(V/G)$ *is generated by the elements of*

$$\mathfrak{X}(V)^G=\mathrm{Map}(4\varphi_1+\varphi_3, 4\varphi_1+\varphi_3)^{\mathsf{D}_4}$$

which are of degree ≤ 2 *in the copies of* φ_3. *In particular, the image of* $\mathrm{Map}(4\varphi_1+\varphi_3, 4\varphi_1)^{\mathbf{D}_4}$ *in* $\mathfrak{X}(V/G)$ *is generated (as a* $\mathbf{C}[\alpha, \beta]$*-module) by elements of degree* ≤ 2 *in* φ_3. ■

We now consider the

Final Case. — $(V, G)=(2\varphi_5+\varphi_1, \mathbf{D}_6)$. We need some notation: Let $M_1, \overline{M}_1, M_2, \ldots$ denote copies of $\mathbf{A}_1$, and let $\lambda_1, \overline{\lambda}_1, \lambda_2, \ldots$ denote their corresponding basic representations. Let M denote $M_1\times M_2\times M_3$, let $\overline{M}$ denote $\overline{M}_1\times\overline{M}_2\times\overline{M}_3$, and let Σ_3 denote the symmetric group on 3 letters. We identify $\overline{M}$ with a principal isotropy group of $(2\varphi_5, \mathbf{D}_6)$ (see IV.26). The universal cover of $N_{\mathbf{D}_6}(\overline{M})$ is isomorphic to the semi-direct product $(M\times\overline{M})\rtimes\Sigma_3$ where Σ_3 permutes indices in the usual way, and

$$(17.3)\qquad (\varphi_1(\mathbf{D}_6), M\times\overline{M})\simeq\lambda_1\otimes\overline{\lambda}_1+\lambda_2\otimes\overline{\lambda}_2+\lambda_3\otimes\overline{\lambda}_3.$$

From example (13.6) we see that

$$(17.4)\qquad (\varphi_5(\mathbf{D}_6), M\times\overline{M})\simeq\lambda_1\otimes\lambda_2\otimes\lambda_3+\lambda_1\otimes\overline{\lambda}_2\otimes\overline{\lambda}_3+\overline{\lambda}_1\otimes\lambda_2\otimes\overline{\lambda}_3+\overline{\lambda}_1\otimes\overline{\lambda}_2\otimes\lambda_3,$$

and the decomposition of φ_6 is obtained by interchanging λ_i and $\overline{\lambda}_i$ in (17.4), $i=1, 2, 3$. Let λ denote $\lambda_1\otimes\lambda_2\otimes\lambda_3$. Let ρ_0 (resp. ρ_{-1}) denote the trivial (resp. non-trivial) one-dimensional representation of Σ_3, and let ρ_2 denote the irreducible two-dimensional representation. We denote the element of Σ_3 interchanging i and j by σ_{ij}.

The isotropy class of the (unique) codimension one stratum of V/G is $(L=\overline{M}_3)$, and $(V^{\overline{M}_3}, N_G(\overline{M}_3)/\overline{M}_3)\sim(V_1, G_1)=(2\varphi_3\otimes\lambda_3+\varphi_1, \mathbf{D}_4\times M_3)$. Via an outer automorphism, $(4\varphi_3+\varphi_1, \mathbf{D}_4)\simeq(4\varphi_1+\varphi_3, \mathbf{D}_4)$, the case we just considered. Now $(\overline{M}_2\times\overline{M}_3)$ is the isotropy class of a codimension 2 stratum, and we obtain an associated representation $(V_2, G_2)=(2\lambda+\lambda_1\otimes\overline{\lambda}_1, (M\times\overline{M}_1)\rtimes\{\mathrm{id}, \sigma_{23}\})$.

We now construct a diagram relating the $\mathbf{C}[V_i]^{G_i}$ and associated representations: It follows from lemma (15.10) that the invariants of (V, G) restrict injectively into those of $(2\varphi_5^{\overline{M}}+\varphi_1, N_G(\overline{M}))\sim(2\lambda+\lambda_1\otimes\overline{\lambda}_1+\lambda_2\otimes\overline{\lambda}_2+\lambda_3\otimes\overline{\lambda}_3, (M\times\overline{M})\rtimes\Sigma_3)$. Quotienting by the action of $\overline{M}$ we obtain an injection $\psi:\mathbf{C}[V]^G\to\mathbf{C}[U]^N$ where

$$(U, N)=(2\lambda+\xi_1+\xi_2+\xi_3, M\rtimes\Sigma_3).$$

Each ξ_i is a copy of $\mathbf{C}$ with co-ordinate β_i, Σ_3 acts on the β_i by permuting indices, and $\beta=\beta_1+\beta_2+\beta_3$ is the restriction of the square of the radius function of $(\varphi_1, \mathbf{D}_6)$. Similarly, the invariants of (V_1, G_1) restrict injectively into those of

$$(2(\varphi_3\otimes\lambda_3)^{\overline{M}_1\times\overline{M}_2}+\varphi_1, N_{\mathbf{D}_4}(\overline{M}_1\times\overline{M}_2)\times M_3),$$

and quotienting by the action of $\overline{M}_1\times\overline{M}_2$ we obtain an injection $\psi_1:\mathbf{C}[V_1]^{G_1}\to\mathbf{C}[U_1]^{N_1}$ where $(U_1, N_1)=(2\lambda+\xi_1+\xi_2, M\rtimes\{\mathrm{id}, \sigma_{12}\})$. In the same manner, there is an injection $\psi_2:\mathbf{C}[V_2]^{G_2}\to\mathbf{C}[U_2]^{N_2}$ where $(U_2, N_2)=(2\lambda+\xi_1, M\rtimes\{\mathrm{id}, \sigma_{23}\})$. We have a commutative diagram

$$
(17.5)\qquad
\begin{array}{ccc}
\mathbf{C}[V]^{G} & \xrightarrow{\psi} & \mathbf{C}[U]^{N} \\
\downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_1'} \\
\mathbf{C}[V_1]^{G_1} & \xrightarrow{\psi_1} & \mathbf{C}[U_1]^{N_1} \\
\vdots{\scriptstyle \pi_2} & & \vdots{\scriptstyle \pi_2'} \\
\mathbf{C}[V_2]^{G_2} & \xrightarrow{\psi_2} & \mathbf{C}[U_2]^{N_2}
\end{array}
$$

where the π_i and π_i' are the obvious restriction maps. Restriction from $\mathbf{C}[V_1]$ to $\mathbf{C}[V_2]$ does not carry all G_1-invariants to G_2-invariants, so π_2 is only partially defined. Similarly, π_2' is only partially defined. We use (17.5) to determine $\mathbf{C}[V]^G$. First we need to determine $\mathbf{C}[2\lambda]^M$ and $\mathbf{C}[2\lambda]^N$.

Lemma **(17.6)**

(1) $\mathbf{C}[2\lambda]^N$ *is a regular ring with generators* α_{40}, α_{31}, α_{11}, α_{22}, α_{33}, α_{13}, *and* α_{04} *where degree* $\alpha_{ij}=(i,j)$.

(2) $\mathbf{C}[2\lambda]^M$ *is generated by the* α_{ij} *and* x_1, x_2, *and* x_3 *where*

$$x_1+x_2+x_3=0,$$

and Σ_3 *acts on the* x_i *by permuting indices. The* x_i *have degree* $(2, 2)$ *and satisfy the equations*

$$x_i^3-3sx_i-2t=0$$

where

$$s=\frac{1}{6}(x_1^2+x_2^2+x_3^2), \qquad t=\frac{1}{6}(x_1^3+x_2^3+x_3^3).$$

(3) *As* $\mathbf{C}[2\lambda]^N$*-module,*

$$\mathbf{C}[2\lambda]^M\simeq \mathbf{C}[2\lambda]^N\otimes_{\mathbf{C}}(\rho_0+2\rho_2+\rho_{-1})$$

where ρ_0 *has generator* 1; *the copies of* ρ_2 *have generators* x_1, x_2 *and* x_1^2-2s, x_2^2-2s; *and* ρ_{-1} *has generator*

$$\varepsilon=(x_1-x_2)(x_3-x_1)(x_2-x_3).$$

We give the proof of (17.6) below. Now the hypotheses of theorem (11.2) can be verified for

$$\mathrm{res}_{\overline{M}}:\ \mathrm{Map}(2\varphi_5, S^2\varphi_1)^{D_4}\to \mathrm{Map}(2\lambda, \xi_1+\xi_2+\xi_3)^N,$$

so $\mathrm{res}_{\overline{M}}$ is surjective and there are invariants

$$(17.7)\qquad \gamma=x_1\beta_1+x_2\beta_2+x_3\beta_3$$

and

$$(17.8)\qquad \delta=x_1^2\beta_1+x_2^2\beta_2+x_3^2\beta_3$$

in $\psi(\mathbf{C}[V]^G)$. We thus have obtained elements s, t, α_{ij}, β, γ, and δ in $\psi(\mathbf{C}[V]^G)$. We denote also by s, t, ... the pre-images of these elements in $\mathbf{C}[V]^G$ and also their images in any of the $\mathbf{C}[V_i]^{G_i}$ or $\mathbf{C}[U_i]^{N_i}$. Note that σ_{12}-invariant elements of $\mathbf{C}[2\lambda]^M$ (e.g. x_3, x_1x_2, etc.) have pre-images in $\mathbf{C}[V_1]^{G_1}$ (also to be denoted x_3, x_1x_2, etc.). With these conventions one computes that

$$\textbf{(17.9)} \qquad \begin{aligned} &\delta + x_3\gamma + x_1x_2\beta = (x_3 - x_2)(x_3 - x_1)\beta_3 \text{ on } U \\ &\delta + x_3\gamma + x_1x_2\beta = 0 \text{ on } U_1 \text{ and } V_1. \end{aligned}$$

The following shows that the α_{ij}, β, γ, and δ generate $\mathrm{res}_L\mathbf{C}[V]^G$.

Lemma **(17.10)**. — *Let* $a \in \mathbf{C}[V_1]^{G_1}$ *and suppose that* $\pi_2 a$ *is defined, i.e.* a *restricts to an element of* $\mathbf{C}[V_2]^{G_2}$. *Then* $a \in \mathbf{C}[\alpha_{ij}, \beta, \gamma, \delta]$.

Proof. — First suppose that $\pi_2(a) = 0$. Now

$$\mathbf{C}[V_1]^{G_1} \subseteq \mathbf{C}[4\varphi_3 + \varphi_1]^{\mathbf{D}_4} \simeq \mathbf{C}[4\varphi_1 + \varphi_3]^{\mathbf{D}_4},$$

and the generator α of degree $(1, 1, 1, 1, 2)$ of the latter ring corresponds to an element of $\mathbf{C}[V_1]^{G_1}$ whose image in $\mathbf{C}[U_1]^{N_1}$ is of the form $(\beta_1 - \beta_2)b$, where $0 \neq b \in \mathbf{C}[2\lambda]^M$, $\sigma_{12}b = -b$, and degree $b = (2, 2)$. The space of such b's is one-dimensional (generated by $x_1 - x_2$), and since $\gamma + \frac{1}{2}x_3\beta$ has image $\frac{1}{2}(\beta_1 - \beta_2)(x_1 - x_2)$ in $\mathbf{C}[U_1]^{N_1}$, the functions α and $\gamma + \frac{1}{2}x_3\beta$ are multiples of each other. Then our analyses of $\mathbf{C}[4\varphi_1 + \varphi_3]^{\mathbf{D}_4}$ and $\mathbf{C}[2\lambda]^M$ show that $\mathbf{C}[V_1]^{G_1}$ is generated over $\mathbf{C}[\alpha_{ij}, \beta, \gamma]$ by 1, x_3, and $x_3^2 - 2s$. Now the ideal in $\mathbf{C}[V_1]^{\mathbf{D}_4}$ vanishing on V_2 is principal with a degree 12 generator (see (17.1)). But then

$$\textbf{(17.11)} \qquad \gamma^2 - \beta\delta$$

must also generate the ideal since it projects to the non-zero function $-(x_1 - x_2)^2\beta_1\beta_2$ on U_1 and projects to zero on U_2. Thus the kernel of π_2 is generated over $\mathbf{C}[\alpha_{ij}, \beta, \gamma]$ by $\gamma^2 - \beta\delta$,

$$\textbf{(17.12)} \qquad \gamma\delta - 3s\beta\gamma - 2t\beta^2 = -x_3(\gamma^2 - \beta\delta)$$

and

$$\textbf{(17.13)} \qquad \delta^2 - 2t\beta\gamma - 3s\beta\delta = x_3^2(\gamma^2 - \beta\gamma),$$

where the above equations hold on U_1. Hence $\pi_2(a) = 0$ implies that $a \in \mathbf{C}[\alpha_{ij}, \beta, \gamma, \delta]$.

In general, we may reduce to the case where a is homogeneous in each irreducible factor of V_1. Then $(\psi_2 \circ \pi_2)(a)$ is of the form $\beta_1^m b$ where $m \in \mathbf{Z}^+$ and $b \in \mathbf{C}[2\lambda]^{N_2}$. If $m = 0$, then b is σ_{12} and σ_{23} invariant, i.e. $b \in \mathbf{C}[2\lambda]^N$ and $a \in \mathbf{C}[\alpha_{ij}]$. Now $\mathbf{C}[2\lambda]^{N_2}$ is generated over $\mathbf{C}[\alpha_{ij}]$ by 1, x_1, and $x_1^2 - 2s$. For $m \geq 1$, $\beta^{m-1}\gamma$ (resp. $\beta^{m-1}(\delta - 2s\beta)$) restricts to $\beta_1^m x_1$ (resp. $\beta_1^m(x_1^2 - 2s)$) on U_2. Thus we may reduce to the case $\pi_2(a) = 0$, and our proof of the lemma is complete. ■

Let f_L be a form generating $I(V^{(L)})^G$. Applying (15.9) to the sequence of slice representations

$$(2\varphi_5+\varphi_1, \mathsf{D}_6) \to (\theta_1+2\varphi_5, \mathsf{B}_5) \to (\theta_4+(\varphi_1+\varphi_2)+(\varphi_3+\varphi_4), \mathsf{A}_4) \to (\theta_7+\varphi_1+2\varphi_2, \mathsf{B}_2)$$

we see that $\deg f_L \geq 24$ in $2\varphi_5$, and similarly $\deg f_L \geq 6$ in φ_1. Let a (resp. b) denote the left (resp. right) hand side of (17.9). Then $a(\sigma_{13}a)(\sigma_{23}a)=b(\sigma_{13}b)(\sigma_{23}b)$ and expanding we see that

$$(17.14) \qquad h=\delta^3-3s\beta\,\delta^2-3s\gamma^2\,\delta-6t\beta\gamma\,\delta+2t\gamma^3+9s^2\beta\gamma^2+12st\beta^2\gamma+4t^2\beta^3$$

equals $-\varepsilon^2\beta_1\beta_2\beta_3$ on U. Hence $h\in I(V^{(L)})^G$, and $\deg f_L \geq \deg h$. It is easy to show that (L) is the unique subprincipal isotropy class of (V, G), and then (15.6) shows that $\mathbf{C}[V]^G$ is the regular ring $\mathbf{C}[\alpha_{ij}, \beta, \gamma, \delta]$.

Using the methods of [45] one can obtain the following identities for D_6-representations: $S^2\varphi_5 \simeq \varphi_5^2+\varphi_2$, $S^3\varphi_5 \simeq \varphi_5^3+\varphi_2\varphi_5+\varphi_5$, and $S^4\varphi_5 \simeq \varphi_5^4+\varphi_2\varphi_5^2+\varphi_2^2+\varphi_5^2+\varphi_4+\theta_1$. By counting one sees (as one must) that $\mathbf{C}[2\varphi_5]^{\mathsf{D}_6}$ has generators corresponding to the α_{ij}. Now $S^2\varphi_2=\varphi_2^2+\varphi_4+\varphi_1^2+\theta_1$, so γ is obtained by contracting the copy of

$$\varphi_1^2 \subseteq \varphi_2 \otimes \varphi_2 \subseteq S^2\varphi_5 \otimes S^2\varphi_5 \quad \text{with} \quad \varphi_1^2 \subseteq S^2\varphi_1.$$

The multiples of γ by α_{11}^2 and α_{22} are of degree (4, 4, 2), so $S^4\varphi_5 \otimes S^4\varphi_5$ must contain at least two copies of φ_1^2. But $\varphi_4\otimes\varphi_4$, $\varphi_2^2\otimes\varphi_2^2$, and $\varphi_2\varphi_5^2\otimes\varphi_2\varphi_5^2$ contain copies of φ_1^2, so there is a third generator of degree (4, 4, 2) corresponding to δ.

We now establish (15.2.2): Let $X\in\mathfrak{X}^+(V_1)^{G_1}$ (i.e. $X\in\mathfrak{X}(V_1)^{G_1}$ and X preserves $\pi_1(\mathbf{C}[V]^G)$). Clearly X maps β to a multiple of β, γ, and δ. Let grad_2 denote projection of gradients of elements of $\mathbf{C}[V_1]^{G_1}=\mathbf{C}[2\varphi_3\otimes\lambda_3+\varphi_1]^{\mathsf{D}_4\times \mathsf{M}_3}$ onto the φ_1 factor. Then modifying X by multiples of grad β, $\mathrm{grad}_2\,\gamma$, and $\mathrm{grad}_2\,\delta$ we obtain X′ such that $X'(\beta)=0$. Suppose that X annihilated the α_{ij}. Then X′ also annihilates the α_{ij}, hence annihilates the finite extension $\mathbf{C}[2\lambda]^M$ of $\mathbf{C}[2\lambda]^N=\mathbf{C}[\alpha_{ij}]$, and from (17.9) we obtain the relation

$$X'(\delta)+x_3X'(\gamma)=0.$$

Since $\pi_2(x_3)$ is not defined (i.e. x_3 does not restrict to an element of $\mathbf{C}[V_2]^{G_2}$) while $\pi_2(X'(\delta))$ and $\pi_2(X'(\gamma))$ are defined, we must have that $\pi_2(X'(\gamma))=0$. Thus $X'(\gamma)$ is a $\pi_1(\mathbf{C}[V])^G$-multiple of the expressions in (17.11), (17.12), and (17.13). Now by restricting to (V_2, G_2) one can see that $(\mathrm{grad}_2\gamma)(\gamma)=4\delta$, $(\mathrm{grad}_2\gamma)(\beta)=4\gamma$, and clearly grad β applied to β, γ, and δ is multiplication by 4. Then $A=\frac{1}{4}(\gamma\,\mathrm{grad}\,\beta-\beta\,\mathrm{grad}_2\gamma)$ annihilates $\mathbf{C}[\alpha_{ij}, \beta]$, and $A(\gamma)=\gamma^2-\beta\,\delta=(17.11)$. Similarly,

$$-x_3A=\frac{1}{4}(\delta\,\mathrm{grad}\,\beta-\beta\,\mathrm{grad}_2\,\delta)$$

and
$$x_3^2A=3sA+\frac{1}{4}(\delta\,\mathrm{grad}_2\,\gamma-\gamma\,\mathrm{grad}_2\,\delta)$$

map γ into the expressions in (17.12) and (17.13), respectively. Thus modulo multiples of A, x_3A, and x_3^2A we can arrange that $X' \in \mathfrak{X}_{G_1}(V_1)^{G_1}$. Hence to establish (15.2.2) it suffices to be able to reduce to the case that X annihilates the α_{ij}. This last reduction requires the following preliminaries.

Let $Y \in \mathfrak{X}(V)^G$. Then Y gives rise to a $\mathbf{C}[V]^G$-valued derivation on $\mathbf{C}[2\varphi_5]^{\mathbf{D}_4}$. Let $a \in \mathbf{C}[2\varphi_5]^{\mathbf{D}_4}$, and let $v \in ((2\varphi_5)^{\overline{M}} + \varphi_1)$. The exterior derivative of a evaluated at v, $da(v)$, is $\overline{M}$-invariant, so the contraction of $Y(v)$ and $da(v)$ only depends on the $\overline{M}$-invariant part of $Y(v)$. Thus Y gives rise to $Y' \in \mathrm{Map}(2\varphi_5^{\overline{M}} + \varphi_1, 2\varphi_5^{\overline{M}})^{N_G(\overline{M})}$ such that

$$Y'(\mathrm{res}(a)) = \mathrm{res}(Y(a)),$$

where res denotes restriction to $2\varphi_5^{\overline{M}} + \varphi_1$. Quotienting by the action of $\overline{M}$ on φ_1 we obtain an element $\eta(Y) \in \mathrm{Map}(U, 2\lambda)^N$ such that

$$\eta(Y)(\psi(a)) = \psi(Y(a)), \quad a \in \mathbf{C}[\alpha_{ij}].$$

Similarly we obtain $\eta_1 : \mathfrak{X}(V_1)^{G_1} \to \mathrm{Map}(U_1, 2\lambda)^{N_1}$ such that

$$\eta_1(Y)(\psi_1(a)) = \psi_1(Y(a)), \quad Y \in \mathfrak{X}(V_1)^{G_1}, \quad a \in \mathbf{C}[\alpha_{ij}].$$

There is a commutative diagram

$$\textbf{(17.15)} \qquad \begin{array}{ccc} \mathfrak{X}(V)^G & \xrightarrow{\eta} & \mathrm{Map}(U, 2\lambda)^N \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_1'} \\ \mathfrak{X}(V_1)^{G_1} & \xrightarrow{\eta_1} & \mathrm{Map}(U_1, 2\lambda)^{N_1} \end{array}$$

where p_1 and p_1' are induced by restriction.

Let Δ denote the $\mathbf{C}[\beta, \gamma]$-submodule of $\mathrm{Map}(U_1, 2\lambda)^{N_1}$ generated by elements of the form

$$\textbf{(17.16)} \qquad A + \sigma_{12}A + \beta_1 B + \beta_2\sigma_{12}B; \quad A, B \in \mathfrak{X}(2\lambda)^M.$$

We show that $\Delta \subset \eta_1(\mathfrak{X}(V_1)^{G_1})$: It follows from IV.1 that the $\mathbf{D}_4$-invariant vector fields on $(2\varphi_3 \otimes \lambda_3, \mathbf{D}_4 \times M_3)$ restrict onto the $(M_1 \times M_2) \rtimes \{\mathrm{id}, \sigma_{12}\}$-invariant vector fields on $(2\lambda, N_1)$. Hence, averaging over M_3, we find that $\mathfrak{X}(2\varphi_3 \otimes \lambda_3)^{\mathbf{D}_4 \times M_3}$ restricts onto $\mathfrak{X}(2\lambda)^{N_1}$. Thus the elements of the form $A + \sigma_{12}A$ in (17.16) are in Im η_1. Similarly, by "averaging" the result in (17.2.1) (with some care) one sees that Im η_1 contains terms $\beta_1 B + \beta_2\sigma_{12}B$ where the coefficient B of β_1 is an arbitrary element of $\mathfrak{X}(2\lambda)^M$. Thus $\Delta \subset \mathrm{Im}\, \eta_1$.

Let Δ_0 denote the elements of Im η_1 which act trivially on $\mathbf{C}[2\lambda]^N$, hence also trivially on $\mathbf{C}[2\lambda]^M$. By "averaging" the results in (17.2.2) one can see that

$\operatorname{Im}\eta_1 \subset \Delta + \Delta_0$. Let $\mathscr{D}_1$ denote the elements of Δ which map $\mathbf{C}[2\lambda]^N = \mathbf{C}[\alpha_{ij}]$ into $\mathbf{C}[\alpha_{ij}, \beta, \gamma, \delta]$, and let $\mathscr{D}$ denote $(p_1' \circ \eta)(\mathfrak{X}(V)^G) + \Delta_0$. If $X \in \mathfrak{X}^+(V_1)^{G_1}$, then

$$\eta_1(X) \in \mathscr{D}_1 + \Delta_0.$$

Hence proving that $\mathscr{D}_1 \subset \mathscr{D}$ is sufficient to establish (15.2.2).

To prove that $\mathscr{D}_1 \subset \mathscr{D}$ we will need to deal with several Σ_3-modules. We will denote elements of such modules by subscripted capital Roman letters A_{-1}, B_0, C_1, D_2, E_3, etc. A subscript of 0 (resp. -1) indicates elements transforming by ρ_0 (resp. ρ_{-1}). An element denoted C_1 will indicate that there are also C_2 and C_3 such that $C_1 + C_2 + C_3 = 0$ and that Σ_3 acts on the C_i by permuting indices. The analogous convention applies to elements D_2, E_3, etc.

For now we assume

Lemma **(17.17)**. — *Let* $A_1, A_2, A_3 \in \mathfrak{X}(2\lambda)^M$ *transform by* ρ_2. *Then for* $i \geq 0$,

$$B^i = x_1^i \beta_1 A_1 + x_2^i \beta_2 A_2 \in (p_1' \circ \eta)(\mathfrak{X}(V)^G).$$

Now let $X \in \mathscr{D}_1$. To prove that $X \in \mathscr{D}$ we may clearly reduce to the case that X has fixed total degree of homogeneity with respect to β_1 and β_2, say m. If $m = 0$, then $X \in \mathfrak{X}(2\lambda)^{N_1}$ and X preserves $\mathbf{C}[2\lambda]^N$. In $\mathfrak{X}(2\lambda)^M$ we may decompose X as $Y + Z$ where Y is the Σ_3-invariant part of X. By IV.26, $Y \in \mathscr{D}$. Since Z must act trivially on $\mathbf{C}[2\lambda]^N$, $Z \in \Delta_0$, hence $X \in \mathscr{D}$.

Suppose $m \geq 1$. Then $X = \beta_1^m A + \beta_1^{m-1}\beta_2 B + \ldots$, and $\pi_2(X(\alpha_{ij})) = \beta_1^m A(\alpha_{ij})$ is invariant under σ_{23}. Thus $A = A_0 + B_1 + C$ where $C \in \mathfrak{X}_M(2\lambda)^M$. Now

$$\beta^{m-1}(\beta_1 C + \beta_2 \sigma_{12} C) \in \Delta_0, \qquad \beta^m A_0 \in \mathscr{D},$$

and lemma (17.17) implies that $\beta^{m-1}(\beta_1 B_1 + \beta_2 B_2) \in \mathscr{D}$. Thus X has the same coefficient of β_1^m as an element of $\mathscr{D}$. If $m = 1$, then it follows that $X \in \mathscr{D}$. If $m = 2$, then modulo elements obviously in $\mathscr{D}$ we may reduce to the case that

$$X = \beta^2 A_3 + \beta\gamma B_3 + \gamma^2 C_3 + \beta(\beta_1 - \beta_2) D_{-1} + \gamma(\beta_1 - \beta_2) E_{-1},$$

and, as above, X has the same coefficient of β_1^2 as an element $Y \in \mathscr{D}$. Moreover, Y has the form $\beta^2 F_0 + \beta(\beta_1 H_1 + \beta_2 H_2) + \beta(\beta_1 H + \beta_2 \sigma_{12} H)$ where $H \in \mathfrak{X}_M(2\lambda)^M$. Comparing coefficients of β_1^2, $\beta_1\beta_2$, and β_2^2 one sees that

(17.18) $$X - Y = -(x_1 - x_2)^2 \beta_1\beta_2 C_3 - 2(x_1 - x_2)\beta_1\beta_2 E_{-1}.$$

But
$$-(x_1 - x_2)^2 \beta_1\beta_2 C_3 = \beta(x_1^2 \beta_1 C_1 + x_2^2 \beta_2 C_2) + \delta(\beta_1 C_1 + \beta_2 C_2) - 2\gamma(x_1\beta_1 C_1 + x_2 \beta_2 C_2) \in \mathscr{D}.$$

To handle the other term in (17.18) we assume for now

Lemma **(17.19)**. — *Let* $E_{-1} \in \mathfrak{X}(2\lambda)^M$. *Then*

$$E_{-1} = B + (x_1 A_2 - x_2 A_1)$$

where $B \in \mathfrak{X}_M(2\lambda)^M$.

Now $$(x_1 - x_2)\beta_1\beta_2(x_1 A_2 - x_2 A_1) = \delta(\beta_1 A_1 + \beta_2 A_2) - \gamma(x_1 \beta_1 A_1 + x_2 \beta_2 A_2) \in \mathscr{D}.$$

Thus $X \in \mathscr{D}$ if $m \leq 2$.

Finally, suppose that $m \geq 3$. Then we may write $X = \gamma^2 Y + \beta Z$ where $Y, Z \in \Delta$. Computations as above show that multiplication by $\gamma^2 - \beta\delta$ maps Δ into $\mathscr{D}$. Thus we may reduce to the case $Y = 0$, i.e. we may assume $X = \beta Z$. Let $a \in \mathbf{C}[\alpha_{ij}]$. Then $X(a) = \beta Z(a) = \psi_1(\beta b)$ where $b \in \mathbf{C}[V_1]^{G_1}$ and $\beta b \in \pi_1(\mathbf{C}[V]^G)$. Thus $\pi_2(\beta b)$ is defined, and since β does not vanish identically on V_2, $\pi_2(b)$ must also be defined. By (17.10), $b \in \pi_1(\mathbf{C}[V]^G)$, and it follows that $Z \in \mathscr{D}_1$. By induction on m, $Z \in \mathscr{D}$, hence $X \in \mathscr{D}$. Our proof of (15.2.2) is complete. We may assume (15.2.3), so (V, G) has the lifting property.

Proof of (17.6). — Note that $(2\lambda, M) \sim (2\mathbf{C}^4 \otimes \lambda_3, SO(4, \mathbf{C}) \times M_3)$. CIT shows that $\mathbf{C}[2\mathbf{C}^4 \otimes \lambda_3]^{SO(4, \mathbf{C})}$ has generators α_{11} (of degree (1, 1)) and a_{12} (the determinant, degree = (2, 2)) which are M_3-invariant, and 3 generators each of degrees (2, 0), (0, 2), and (1, 1) which transform by (λ_3^2, M_3). Let y_1, y_2, and y_3 denote typical points in these 3 copies of λ_3^2. We obtain generators of $\mathbf{C}[2\lambda]^M$ from the inner products and determinant of the y_i with the relation

(17.20) $$\det(y_1, y_2, y_3)^2 = \det(y_i \cdot y_j).$$

Thus $\mathbf{C}[2\lambda]^M$ has one generator in degrees (4, 0), (3, 1), (1, 1), (3, 3), (1, 3), and (0, 4), and it has 3 generators in degree (2, 2). We may assume that the generators transform by representations of Σ_3 (lemma (8.1)), so the generators in degrees $(i, j) \neq (2, 2)$ transform by ρ_0 or ρ_{-1}. Repeating the calculation above with $SO(4, \mathbf{C})$ replaced by $O(4, \mathbf{C})$ = image of $(M_1 \times M_2) \rtimes \{\mathrm{id}, \sigma_{12}\}$, we obtain the same set of generators, except that a_{12} drops out. Thus we have the advertised generator α_{ij} for $(i, j) \neq (2, 2)$. Let a_{ij} be the determinant invariant of $\mathbf{C}[2\lambda]^{M_i \times M_j}$ for $i \neq j$. The a_{ij} span at least a copy of ρ_2, and then one can see that the $\mathbf{C}[2\lambda]^M$ generators of degree (2, 2) transform by $\rho_2 + \rho_0$. Corresponding to ρ_0 we obtain $\alpha_{22} \in \mathbf{C}[2\lambda]^N$, and the copy of ρ_2 must be spanned by elements x_i as described in (17.6.2).

We now establish (17.6.3): Consider the copy τ of ρ_2 spanned by the x_i. It is well-known that $\mathbf{C}[\tau]$ is the free $\mathbf{C}[\tau]^{\Sigma_3}$-module generated by

$$P = \{1, x_1, x_2, x_1^2 - 2s, x_2^2 - 2s, \varepsilon\},$$

where $\mathbf{C}[\tau]^{\Sigma_3}$ is generated by s and t. Thus $\mathbf{C}[2\lambda]^M$ is generated over $\mathbf{C}[2\lambda]^N$ by P. By Galois theory (see [46]), the total quotient field Q_1 of $\mathbf{C}[2\lambda]^M$ has dimension 6 over the total quotient field Q_2 of $\mathbf{C}[2\lambda]^N$. Since any element of Q_1 can be written with denominator in $\mathbf{C}[2\lambda]^N$, Q_1 is generated over Q_2 by P. Hence the elements of P are linearly independent over Q_2, hence over $\mathbf{C}[2\lambda]^N$, and we have established (17.6.3).

We now show that s and t are in $\mathbf{C}[\alpha_{ij}]$: The $y_k \cdot y_\ell$ and $\det(y_1, y_2, y_3)$ are σ_{12}-invariant, and checking degrees one sees that they all lie in $\mathbf{C}[\alpha_{ij}]+\mathbf{C}[\alpha_{ij}]x_3$. By our construction of the α_{ij} (the method of proof of lemma (8.1)) one sees that

(17.21) $$\det(y_1, y_2, y_3) = a\alpha_{33} + bx_3 + c$$

where a is a non-zero constant and $b, c \in \mathbf{C}[\alpha_{ij}]$ do not involve α_{33}. Similarly one sees that the inner products $y_k \cdot y_\ell$ are Σ_3-invariant, except perhaps for $y_1 \cdot y_2$ and $y_3 \cdot y_3$. Both $y_1 \cdot y_2$ and $y_3 \cdot y_3$ cannot be Σ_3-invariant since $\mathbf{C}[2\lambda]^N$ has only one generator in degree $(2, 2)$. Then from (17.20) we obtain a relation of the form

(17.22) $$dx_3^3 + ex_3^2 + fx_3 + g = 0$$

where $d, e, f, g \in \mathbf{C}[\alpha_{ij}]$ are not all zero. If $d \neq 0$, then comparing coefficients with the equation $x_3^3 - 3sx_3 - 2t = 0$ and using (17.6.3) we see that $s, t \in \mathbf{C}[\alpha_{ij}]$. If $d=0$, then applying σ_{23} to (17.22) and subtracting the result from (17.22) we obtain

$$0 = e(x_3^2 - x_2^2) + f(x_3 - x_2) = (x_3 - x_2)(e(x_3 + x_2) + f).$$

Then $-ex_1 + f = 0$, and (17.6.3) shows that $e = f = 0$. Thus all the coefficients of (17.22) are zero, a contradiction. We have established (17.6.1), (17.6.2), and (17.6.3). ■

Proof of (17.17). — Define

$$C = x_1^i A_1 + x_2^i A_2 + x_3^i A_3,$$
$$D = \sum_{1 \leq j < k \leq 3} (\beta_j - \beta_k)(x_j^i A_j - x_k^i A_k).$$

Then $B^i = \frac{1}{3} p_1'(D + \beta C)$. Note that $C \in \mathfrak{X}(2\lambda)^N \subseteq \eta(\mathfrak{X}(V)^G)$ and that $D \in \mathrm{Map}(2\lambda, \rho_2 \otimes \lambda)^N$, where the copy of ρ_2 is spanned by $\beta_j - \beta_k$, $1 \leq j < k \leq 3$. Thus it suffices to show that the restriction

(17.23) $$\mathrm{Map}(2\varphi_5, \varphi_1^2 \otimes \varphi_5)^{D_6} \to \mathrm{Map}(2\lambda, (\varphi_1^2)^{\overline{M}} \otimes \varphi_5^{\overline{M}})^N \simeq \mathrm{Map}(2\lambda, \rho_2 \otimes \lambda)^N$$

is surjective. Now

(17.24) $$\varphi_1^2 \otimes \varphi_5 = \varphi_1^2 \varphi_5 + \varphi_1 \varphi_6$$

and

(17.25) $$\varphi_1 \otimes \varphi_6 = \varphi_1 \varphi_6 + \varphi_5.$$

Restricting the terms in (17.25) to the subspaces fixed by $\overline{M}$ shows that

(17.26) $$((\varphi_1 \varphi_6)^{\overline{M}}, N) = (\rho_2 \otimes \lambda, N).$$

Suppose that the composition

$$\omega : \varphi_1 \varphi_6 \to \varphi_1^2 \otimes \varphi_5 \to (\varphi_1^2)^{\overline{M}} \otimes \varphi_5^{M} \simeq \rho_2 \otimes \lambda$$

is not the zero map. Using theorem (11.2) one verifies that

$$\mathrm{res}_{\overline{M}}: \mathrm{Map}(2\varphi_5, \varphi_1\varphi_6)^{D_4} \to \mathrm{Map}(2\lambda, (\varphi_1\varphi_6)^{\overline{M}})^N$$

is surjective, hence, since $\omega \neq 0$, the map in (17.23) is surjective. Surjectivity fails for

$$\mathrm{res}_{\overline{M}}: \mathrm{Map}(2\varphi_5, \varphi_1^2 \otimes \varphi_5)^{D_4} \to \mathrm{Map}(2\lambda, (\varphi_1^2 \otimes \varphi_5)^{\overline{M}})^N,$$

which explains our less than straightforward methods.

It remains to show that $\omega \neq 0$. Now the invariant γ lies in

$$\varphi_2 \otimes \varphi_2 \otimes \varphi_1^2 \subseteq S^2\varphi_5 \otimes S^2\varphi_5 \otimes \varphi_1^2.$$

There is a generalized gradient A of γ lying in $(\varphi_2 \otimes \varphi_5 \otimes \varphi_1^2 \otimes \varphi_5)^{D_4} \subseteq \mathrm{Map}(2\varphi_5 + \varphi_1, 2\varphi_5)^{D_4}$ such that $A(\alpha_{11})$ is a non-zero multiple of γ. Thus $\eta(A) \neq 0$. But

$$\varphi_2 \otimes \varphi_5 = \varphi_2\varphi_5 + \varphi_1\varphi_6 + \varphi_5,$$

so by (17.24) the vector field A lies in

$$\varphi_2 \otimes \varphi_5 \otimes \varphi_1\varphi_6 \subseteq \varphi_2 \otimes \varphi_5 \otimes \varphi_1^2 \otimes \varphi_5.$$

Hence if ω is the zero map, then $\eta(A) = 0$. Thus $\omega \neq 0$, and our proof of (17.17) is complete. ∎

Proof of (17.19). — As in the proof of (17.6), CIT tells us that $\mathfrak{X}(2\lambda)^M$ has generators of total degree ≤ 8 as a $\mathbf{C}[2\lambda]^M$-module. Let $A \in \mathfrak{X}(2\lambda)^M$ transform by ρ_{-1} and have degree $\ell \leq 8$, and let a be one of the α_{ij}, $\deg a = m \leq 6$. If $A(a) \neq 0$, then $A(a)$ is a non-zero multiple of ε, hence $12 \leq \deg A(a) = \ell + m - 2$. Thus $A(a) = 0$ if $\ell < 8$ or $m < 6$, and if $A \notin \mathfrak{X}_M(2\lambda)^M$, then $\ell = 8$ and $A(\alpha_{33}) = b\varepsilon$ for some constant $b \neq 0$.

We construct such an A. Observe that

$$0 \neq (\operatorname{grad} x_i)(x_i) = c + dx_i, \qquad i = 1, 2, 3$$

where $c, d \in \mathbf{C}[\alpha_{ij}]$. Using symmetry and the equation $x_1 + x_2 + x_3 = 0$ we see that

$$(\operatorname{grad} x_1)(x_2) = (\operatorname{grad} x_2)(x_1) = -\frac{1}{2}c + dx_3.$$

Thus

$$\textbf{(17.27)} \qquad (x_1 \operatorname{grad} x_2 - x_2 \operatorname{grad} x_1)(x_1) = -c\left(x_2 + \frac{1}{2}x_1\right) + d(x_1^2 - 2s) + 2d(x_2^2 - 2s).$$

By (17.6.3), the expression in (17.27) is not zero. Thus

$$A = x_1 \operatorname{grad} x_2 - x_2 \operatorname{grad} x_1 \notin \mathfrak{X}_M(2\lambda)^M.$$

Now A transforms by ρ_{-1}, so $A(\alpha_{33}) = b\varepsilon$ for some $b \neq 0$. Hence the $\mathbf{C}[2\lambda]^M$-module $\mathfrak{X}(2\lambda)^M/\mathfrak{X}_M(2\lambda)^M$ has a generating set which transforms only by ρ_0 and ρ_2. Using (17.6.3) and the fact that $\rho_{-1} \otimes \rho_2 \simeq \rho_2$ and that $\rho_2 \otimes \rho_2 \simeq \rho_0 + \rho_2 + \rho_{-1}$, we then see that any E_{-1} is a sum

$$B + \varepsilon A_0 + (x_1A_2 - x_2A_1) + (x_1^2 - 2s)C_2 - (x_2^2 - 2s)C_1$$

where $B\in\mathfrak{X}_M(2\lambda)^M$. But

$$\varepsilon A_0 = x_1 D_2 - x_2 D_1$$

where $D_i = 3(x_i^2 - 2s)A_0$, $i=1, 2$, and

$$(x_1^2-2s)C_2-(x_2^2-2s)C_1 = x_1 E_2 - x_2 E_1$$

where $E_i = -x_i C_i + \frac{1}{3}\sum_j x_j C_j$, $i=1, 2$. This completes our proof of (17.19) and the algebraic lifting theorem. ∎

Remark **(17.28)**. — We say that (V, G) is **cofree** if $\mathbf{C}[V]$ is a free $\mathbf{C}[V]^G$-module. (Freeness, graded freeness, and flatness are all equivalent in this case; see [4], Ch. 2, § 11, no. 4, Prop. 7.) In several of the cases we have considered, (V, G) is coregular and $\operatorname{codim} Z_G(V) = \dim V/G$ (e.g. apply (10.2) to $(2\varphi_1+\varphi_2, \mathbf{C}_2)$). These conditions imply cofreeness:

Proposition **(17.29)**. — *Let $p_1, \ldots, p_d$ be a minimal set of forms generating* $\mathbf{C}[V]^G$. *Then the following are equivalent*:

(1) (V, G) *is cofree.*

(2) (V, G) *is coregular and* $\operatorname{codim} Z_G(V) = \dim V/G$.

(3) *The p_i are a regular sequence in* $\mathbf{C}[V]$.

Proof. — Let R denote $\mathbf{C}[V]^G$, let S denote $\mathbf{C}[V]$, and let R_0 (resp. S_0) denote the localization of R (resp. S) at the ideal of functions vanishing at $0\in V$. If S is R-flat, then S_0 is R_0-flat, and it follows that R_0 is a regular local ring ([17], p. 94). Thus $R=\mathbf{C}[p_1, \ldots, p_d]$ is regular, and S being R-free implies that $p_1, \ldots, p_d$ is a regular sequence in $S=\mathbf{C}[V]$. By lemma (10.3), $\operatorname{codim} Z_G(V) = d = \dim \mathbf{C}[V]^G$. Hence (1) implies (2) and (3). Lemma (10.3) shows that (2) implies (3). If (3) holds, then there is a regular sequence $p_1, \ldots, p_d, f_{d+1}, \ldots, f_n$ for $\mathbf{C}[V]$ consisting of forms ([34], p. 1036) where $n=\dim V$. It follows that $\mathbf{C}[V]$ is a free $\mathbf{C}[p_1, \ldots, p_d, f_{d+1}, \ldots, f_n]$-module ([34], p. 1036), hence is a free module over $\mathbf{C}[p_1, \ldots, p_d]=\mathbf{C}[V]^G$. ∎

There is no doubt that (17.29) has been noticed before. Several authors have referred to ([5], Ch. V, § 5, no. 5, Lemma 5) which states the equivalence of (1) and (3) above, but it does not mention the more computable condition (2). In [68] we use (17.29) to classify the cofree representations of the connected simple complex algebraic groups.

Remark **(17.30)**. — Let V be a representation space of the finite group G. Chevalley's theorem says that (V, G) is coregular if (and only if) G is generated by generalized reflections (elements fixing a hyperplane of V). Let L be a subprincipal isotropy group of (V, G). If (V, G) is coregular, then the slice representation of L is coregular, and it follows that L is 1-subprincipal and is generated by generalized reflections. Conversely, one easily shows that every generalized reflection lies in some 1-subprincipal L. Hence Chevalley's theorem may be reformulated as: (V, G) is coregular if and only if

(1) The slice representations of subprincipal isotropy groups are coregular.

(2) G is generated by the subprincipal isotropy groups.

It would be nice if conditions (1) and (2) guaranteed that (V, G) is coregular for general reductive G. (Note that if (V, G) is orthogonal and satisfies (1), then (8.4.3) shows that the subprincipal isotropy groups are 1-subprincipal.) We know of no orthogonal representation which satisfies (1) and (2) yet is not coregular. In the non-orthogonal case, (1) and (2) are not sufficient for coregularity. Examples are the representations $(4\varphi_1+\varphi_2, \mathbf{A}_r)$ and $(2\varphi_1+\varphi_1^2, \mathbf{A}_r)$, $r \geqslant 2$.

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