PUBLICATIONS MATHÉMATIQUES DE L'I.H.É.S.

JOHN W. MORGAN

The algebraic topology of smooth algebraic varieties

Publications mathématiques de l'I.H.É.S., tome 48 (1978), p. 137-204
<http://www.numdam.org/item?id=PMIHES_1978__48__137_0>

THE ALGEBRAIC TOPOLOGY OF SMOOTH ALGEBRAIC VARIETIES

by JOHN W. MORGAN

Introduction.

Let V^k be a smooth, algebraic variety in $\mathbf{C}^N$. Any point p in $\mathbf{C}^N - V^k$ can be used to define a smooth, real-valued function $d_p : V \to \mathbf{R}$, $d_p(v) = ||v - p||$. According to ([8], page 39), d_p is a nondegenerate Morse function for an open dense set of $p \in \mathbf{C}^N$. The number of critical points of d_p can be bounded above by the degrees of the polynomials used to define V. In particular there are only finitely many critical points. Each critical point has index $\leqq k$. Thus V^k, which is an open smooth manifold of real dimension $2k$, is homotopy equivalent to a finite CW-complex of dimension k. In this paper we study which CW-complexes arise in this manner up to homotopy equivalence. A subsidiary question to the question about the possible homotopy types for smooth varieties is Serre's question: which finitely presented groups appear as the fundamental group of a smooth variety?

We by no means give complete answers to these questions. Rather we study conditions imposed on certain algebraic topological invariants by supposing that the space under consideration is a smooth algebraic variety. The algebraic topological invariants that we consider are *rational invariants* in the sense that they are functors from CW-complexes to algebraic structures on rational vector spaces. Examples are the rational cohomology ring $H^*(V; \mathbf{Q})$, the tower of nilpotent quotients of the fundamental group, the rational homotopy groups with the Whitehead product, considered as a graded Lie algebra, and the rational cohomology rings of various stages in the Postnikov system for V. We find that the assumption that V is a smooth algebraic variety implies that these invariants have, in a natural way, enhanced algebraic structure. They become algebraic objects (algebras, Lie algebra, etc.) in the category of rational vector spaces with *mixed Hodge structures*.

Generalizing the classical notion of Hodge structures, Deligne introduced mixed Hodge structures in [3]. In [4] he expanded the proof that the cohomology of a smooth projective variety carries a Hodge structure by showing that the cohomology of a smooth, open variety carries a mixed Hodge structure. In this paper we generalize Deligne's results to include further algebraic topological invariants. We do this using his argument as a model. He proceeds by defining certain filtrations on the differential forms of an open variety, showing a spectral sequence degenerates, and then deducing the existence of a mixed Hodge structure on the cohomology. We examine the steps in his argument and show that his filtrations behave nicely with respect to the wedge product and differentiation of forms. These operations are interesting for the following reason. Sullivan [11] showed that from the differential algebra of forms on a manifold one can algebraically recover all the real (or rational) algebraic topology (including all the invariants mentioned before). Once we know the relation of Deligne's filtrations to the differential algebra structure of forms, we are in a position to carry the filtrations through Sullivan's theory.

The central object in Sullivan's theory is the minimal model. This is a differential algebra constructed from the differential algebra of forms on the manifold. It is unique up to isomorphism (1) and is equivalent to the real form of the rational homotopy type of the manifold. (Two simply connected spaces have the same rational homotopy type if and only if there is a third space to which they both map by maps inducing isomorphisms on rational cohomology. In the extension of this notion to non-simply connected spaces, one must allow the third space to be replaced by an inverse system of spaces, see [1].) Deligne's filtrations produce a family of mixed Hodge structures on the minimal model. This family is parameterized by the automorphisms of the minimal model homotopic to the identity. Thus, when we take an algebraically derived invariant of the minimal model which is unique up to canonical isomorphism, that invariant carries a canonical mixed Hodge structure. The four examples of algebraic invariants given earlier all receive their mixed Hodge structures in this manner.

In the case of the cohomology ring the existence of a mixed Hodge structure, though extremely important for other questions, does not impose any conditions on the underlying ring. But in the case of the Lie algebras associated either with the fundamental group or the higher homotopy groups, it does. For example, the tower of rational Lie algebras associated to the nilpotent quotients of the fundamental group is isomorphic to the tower of nilpotent quotients of a graded Lie algebra. Moreover, the indexing set for the grading is the negative integers. The existence of this grading with negative "weights" is a non-trivial homogeneity condition on the relations in the Lie algebra.

All these results concerning the existence of mixed Hodge structures are derived from the theory of differential forms on a compact Kähler manifold. This theory, of course, applies directly to a smooth projective variety. Beginning with an affine variety we can form its projective version. This projective variety need not be smooth, even if the original affine variety is. However, using Hironaka's resolution of singularities [6], we can find another projective variety which is smooth, and which

(1) This isomorphism is not a canonical one but is well defined up to homotopy.

contains the given affine variety as the complement of a union of smooth divisors with normal crossings. This is the requisite connection between compact Kähler manifolds and smooth affine varieties. Once we have this, it is simply a question of linear algebra to deduce from the Hodge theory on compact Kähler manifolds the stated results about mixed Hodge structures.

From this point of view, there is no reason to restrict attention to affine varieties. Any abstract smooth variety can be found as the complement of a union of divisors with normal crossings in a smooth compact variety, and Hodge theory is valid for compact varieties, not just projective ones. Consequently, our results are equally valid for any open, smooth variety.

This paper may be viewed, not only as a generalization of Deligne's work on the cohomology of open smooth varieties, but also as a generalization of [5]. There it was proved that the rational homotopy type of a compact Kähler manifold is determined by its rational cohomology ring. Such a statement is not true in general for open varieties. The correct generalization of this result is given in terms of the Hironaka completion. Let $V \subset \overline{V}$ be the complement of a divisor with normal crossings in a non-singular, compact variety. Let D be the divisor and D^p its subvariety of points of multiplicity p. Define $\widetilde{D}^p$ to be the normalization of D^p for $p \geqq 1$, and to be $\overline{V}$ for $p=0$. The rational homotopy type of V is determined by:

1) the cohomology groups of the $\widetilde{D}^p$, $p \geqq 0$,
2) the multiplication maps $H(\widetilde{D}^p) \otimes H(\widetilde{D}^q) \to H(\widetilde{D}^{p+q})$ for $p, q \geqq 0$, and
3) the Gysin maps $H(\widetilde{D}^p) \to H(\widetilde{D}^{p-1})$.

Deligne, in [4], showed that if one takes the Gysin spectral sequence associated to $\overline{V}-D$, then the E_1 term is the cohomology of the various "pieces", d_1 is the Gysin map, and $E_2 = E_\infty$. Furthermore, he produced a natural isomorphism between E_∞ and $H^*(V)$ (over the complex numbers). Thus we can regard $\{E_1, d_1\}$ as a differential graded algebra whose cohomology coincides with that of X. In this paper we prove that the minimal models of $\{E_1, d_1\}$ and of the differential forms on V are isomorphic. This isomorphism is canonical (up to homotopy) over $\mathbf{C}$, but also exists over $\mathbf{Q}$.

If we consider this theorem for the special case $V=\overline{V}$, then we find the main result of [5]: for a compact smooth variety the minimal models of the cohomology ring and of the differential forms are isomorphic.

In general we see that the homotopy theoretic complexity of an open smooth variety is no greater than the homological complexity of any Hironaka completion for it. As an example of this, let V be an affine variety which is the complement of a smooth hyperplane section of a smooth projective variety. The rational cohomology ring of V determines its rational homotopy type.

The equivalence of minimal models above imposes further restrictions on the possible rational homotopy types of smooth varieties beyond the homogeneity conditions already mentioned. Let us use the tower of rational nilpotent Lie algebras associated with the fundamental group to illustrate. We already know that the tower is isomorphic to the nilpotent quotients of a graded Lie algebra with negative weights. The new results tell us that we can take this graded Lie algebra to have generators of weights -1 and -2 and relations of weights -2, -3, and -4. That these are the only possible weights corresponds to the fact that the only possible weights of the mixed Hodge structures on $H^1(V)$ and $H^2(V)$ are respectively $\{1, 2\}$ and $\{2, 3, 4\}$. Consequently, once we know the graded Lie algebra modulo its fifth order commutators, we can construct the complete tower of graded, nilpotent Lie algebras. This contrasts sharply with Serre's result that any finite group is the fundamental group of a smooth variety ([10], § 20).

The paper is divided into two parts. The first comprises sections 1, 2, and 3. It deals with various filtered algebras of differential forms associated to a smooth variety. The main results are amalgamated into a mixed Hodge diagram of differential algebras in section 3. The second half of the paper, sections 4-10, deals with the minimal model of a mixed Hodge diagram. Here we study the various mixed Hodge structures that result on invariants derived from the minimal model. This discussion is valid for any mixed Hodge diagram, not just those which arise from forms on a smooth variety. These sections are really just exercises in the complicated linear algebra of certain filtered differential algebras.

Section 1 is a review of filtrations, gradings, differential algebras, and spectral sequences. We also give Deligne's definition of a mixed Hodge structure there. Section 2 and 3 produce the various algebras of differential forms associated to a smooth variety and study some of their basic properties. In section 2 we work topologically and construct a rational algebra with a filtration associated to a union of divisors with normal crossings $D \subset V$. We show that the algebra calculates the cohomology (and in fact the homotopy type) of the complement $X = V - D$. The filtration leads to a spectral sequence generalizing the Gysin long exact sequence for $X = V - D$. The E_1 term is the cohomology of the various "pieces" and d_1 is the Gysin map. In section 3 we recall Deligne's work in case V and the D_i are compact, smooth algebraic varieties. He found a complex algebra of forms which not only has the complex analogue of the topological filtration in section 2, but also has a Hodge filtration. Using both, and Hodge theory for compact varieties, he was able to show that the Gysin sequence degenerates at E_2 in the algebraic case.

Section 4 extracts certain properties that Deligne's filtered algebras have with respect to the differential and wedge product. It is these technical results that eventually let us pass to homotopy theory. Section 5 is a review of Sullivan's theory relating homotopy types and differential algebras. In particular we consider the existence and uniqueness for his minimal model.

Sections 6, 7, 8 contain the heart of the matter. In them we use the technical results of 4 to pass from the differential algebras of forms to the minimal model (*i.e.* homotopy theory) *carrying along the filtrations* of sections 2 and 3. Section 6 is the purely complex discussion. We show that the natural bigrading on the complex cohomology of a smooth variety, coming from its mixed Hodge structure, extends to a bigrading of the complex minimal model. The resulting bigraded minimal model is unique up to isomorphism and functorial up to homotopy preserving the bigradings. Section 7 is a rational discussion. There we show that the filtration on the $\mathbf{Q}$-forms of a smooth variety passes to a nice filtration on the minimal model. This filtration on the minimal model is characterized, up to isomorphism homotopic to the identity, by certain internal properties and its effect on cohomology. This filtered minimal model is also functorial up to homotopy compatible with the filtrations. Section 8 pieces the $\mathbf{Q}$-filtration and the $\mathbf{C}$-bigrading of the minimal model together to form a mixed Hodge structure. Even though both ingredients are unique up to isomorphism, the resulting mixed Hodge structure will change as we change the isomorphism between the complexification of the $\mathbf{Q}$-minimal model and the $\mathbf{C}$-minimal model. Thus we have not one mixed Hodge structure on the minimal model, but a family of them.

Sections 9 and 10 translate the algebraic results of sections 6, 7 and 8 into more classical homotopy theoretic language. In section 9 we deduce the existence of mixed Hodge structures on many algebraic topological invariants, including the ones mentioned at the beginning of the introduction. We also consider the resulting bigradings these mixed Hodge structures give on the complex invariants. This section contains a proof of the equivalence of the complex minimal model for X and the minimal model of the E_1 term of the complex Gysin spectral sequence for $V-D=X$. In section 10 we turn to the rational homotopy theory and show that this equivalence of minimal models also exists (unnaturally) over $\mathbf{Q}$. We also give the proof that the complement of a smooth hyperplane section has the rational homotopy type determined by its cohomology ring. This is a consequence of the Lefschetz theorems for hyperplane sections.

1. Filtrations and Mixed Hodge Structures - Generalities.

This section outlines some general results about vector spaces and differential algebras with filtrations. We begin by introducing those definitions and elementary lemmas required in the sequel, and then turn to Deligne's theory of mixed Hodge structures. Most of the results in this section are contained in either ([4] chapters 1 and 2), or ([5] chapter 1).

For us, a *differential graded algebra*, or *differential algebra* for short, over a field k ($k=\mathbf{Q}$, $\mathbf{R}$, or $\mathbf{C}$ almost always) is a graded vector space over k, $A=\bigoplus_{i\geq 0} A^i$, with a differential, $d: A^i \to A^{i+1}$, and a product $A^i \otimes A^j \xrightarrow{\wedge} A^{i+j}$, satisfying:

$$(1.1)\quad \begin{cases} a)\ d^2=0, \\ b)\ d(x\wedge y)=dx\wedge y+(-1)^i x\wedge dy \quad \text{for} \quad x\in A^i, \\ c)\ x\wedge y=(-1)^{i\cdot j} y\wedge x \quad \text{for} \quad x\in A^i \quad \text{and} \quad y\in A^j, \\ d)\ \wedge \text{ makes A an associative algebra with unit, } 1\in A^0. \end{cases}$$

A^i is the component of degree i in A.

A is *connected* if A^0 is the ground field. It is 1-*connected* if, in addition, $A^1=0$. The cohomology of A, H(A), is a graded algebra which can be made a differential algebra by defining d to be zero. We will always assume that $H^0(A)$ is the ground field. Unless explicitly stated to the contrary, we also assume $H^i(A)$ is finitely generated for each i.

If V is a graded vector space, then the free graded-commutative algebra generated by V is denoted $\Lambda(V)$. If V is homogeneous of degree r, then $\Lambda(V)$ (also denoted $\Lambda(V^r)$) is the symmetric algebra S(V) when r is even and is the exterior algebra when r is odd.

The algebra $\Lambda(V)$ is *generated in positive degrees* when V is non-zero only in the positive degrees. The augmentation ideal of a connected algebra A, $\mathscr{I}(A)$, is $\bigoplus_{i>0} A^i$. The indecomposables are the quotient $\mathscr{I}(A)/(\mathscr{I}(A)\wedge\mathscr{I}(A))$. We denote the indecomposables by I(A). For $A=\Lambda(V)$ we have a natural identification of graded vector spaces $I(A)=V$.

A *decreasing filtration* on V, F(V), is a sequence of subspaces:

$$V=F^n(V)\supset F^{n+1}(V)\supset\ldots\supset F^m(V)=0.$$

An *increasing filtration*, W(V), is a sequence:

$$0=W_a(V)\subset W_{a+1}(V)\subset\ldots\subset W_b(V)=V.$$

Throughout this paper F will be a decreasing filtration and W will be an increasing one. Notice that we assume all filtrations to be of finite length, unless otherwise specified. In the case of an infinite filtration W(V) we always require that $\bigcup_i W_i(V)=V$. We state results here for decreasing filtrations; there are obvious analogues for increasing filtrations.

The *associated graded object* to F(V), $\mathrm{Gr}^n_F(V)$, is $F^n(V)/F^{n+1}(V)$. Given two filtered vector spaces (X, F(X)) and (V, F(V)), or (X, F) and (V, F) for short, a homomorphism $f:X\to V$ is *compatible* with the filtrations if $f(F^i(X))\subset F^i(V)$. The map is *strictly compatible*, if in addition, $f(X)\cap F^i(V)=f(F^i(X))$. In terms of elements, the extra condition for strictness is the following:

$$(v\in F^i(V)\ \text{and}\ v=f(x))\Rightarrow(v=f(x')\ \text{for some}\ x'\in F^i(X)).$$

If V is a vector space over k and k' is a field extension of k, then any filtration F(V) defines a filtration on $V_{k'}$ by $F^i(V_{k'})=(F^i(V))_{k'}$. Here $V_{k'}=V\otimes_k k'$. For a map $f:X\to V$ to be compatible (respectively strictly compatible) with filtrations, it is necessary and sufficient that $f\otimes_k \mathrm{Id}_{k'}$ be compatible (respectively strictly compatible) with the extended filtrations.

If $X\subset V$, then any filtration on V, F(V), induces a filtration on X and on the quotient V/X by:

$$F^i(X)=F^i(V)\cap X \quad\text{and}\quad F^i(V/X)=\mathrm{Im}(F^i(V)).$$

It is an easy lemma ([4], (1.1.9)) that if $X_1\subset X_2\subset V$ and if V has a decreasing filtration, then the two naturally induced filtrations on X_2/X_1 agree. (First, induce a filtration on X_2 and then take its quotient, or induce a filtration on V/X_1 and restrict it to X_2/X_1.)

(1.2) In particular, a filtration on a cochain complex induces a unique filtration on the cohomology.

Given filtrations $F(V_1)$ and $F(V_2)$, the *multiplicative extension* to $V_1 \otimes V_2$ is defined by $F^i(V_1 \otimes V_2) = \sum_a F^{i-a}(V_1) \otimes F^a(V_2)$. We also have the multiplicative extension of $F(V)$ to the tensor algebra of V, $T(V)$ by:

$$F^i(T^j(V)) = \sum_{a_1 + \ldots + a_j = i} F^{a_1}(V) \otimes \ldots \otimes F^{a_j}(V).$$

This induces a filtration on the quotient $\Lambda(V)$. In either case, it is the unique filtration on $S(V)$ or $\Lambda(V)$ extending $F(V)$, such that multiplication $S^i(V) \otimes S^j(V) \to S^{i+j}(V)$ or $\Lambda^i(V) \otimes \Lambda^j(V) \to \Lambda^{i+j}(V)$ is strictly compatible with the filtration. More generally, given a free algebra $A = \Lambda(V)$ generated in degrees > 0, a *multiplicative* filtration on A, $F(A)$, is a filtration on each A^i such that wedge product $A^i \otimes A^j \xrightarrow{\wedge} A^{i+j}$ is strictly compatible with the filtration.

Lemma **(1.3)**. — *Let* $A = \Lambda(V)$, *where* V *is a graded vector space non-zero only in positive degrees.*

a) *Given a filtration* $F(V)$, *we form the multiplicative extension* $F(A)$. *This is a multiplicative filtration.*

b) *If* $F(A)$ *is a multiplicative filtration, then it induces a filtration on* $I(A) = V$. *By restricting this to* V^i *we get a filtration* $F(V^i)$. *The filtration* $F(A)$ *is isomorphic to the multiplicative extension of* $F(V)$ *by an automorphism of* A *which induces the identity on* $I(A)$.

c) *If* $F(A)$ *and* $F'(A)$ *are multiplicative filtrations which induce the same filtration on* $I(A)$ *and if* $F^i(A) \subset F'^i(A)$, *then* $F(A) = F'(A)$.

Proof. — *a)* is straightforward.

c) ⇒ *b)*: Given $F(A)$ and the induced filtrations on $I(A)$ and V^i, it is possible to choose maps $\varphi_i : V^i \to A$ such that $V^i \xrightarrow{\varphi_i} A \to I(A)$ is the inclusion of $V^i \subset I(A)$, and such that $\varphi_i(F^\ell(V^i)) \subset F^\ell(A)$. We use the $\{\varphi_i\}$ to define $\varphi : \Lambda(V) \to A$. The map φ is an isomorphism of algebras. If we let $F(\Lambda(V))$ be the multiplicative extension of the $F(V^i)$, then φ is compatible with the filtrations. Applying *c)*, we conclude that φ is an isomorphism of filtered algebras.

c) We prove by induction on i that $F(A^i) = F'(A^i)$. Suppose we know this for $i \leq n-1$. We have an exact sequence:

$$\bigoplus_{\substack{i,j \leq n-1 \\ i+j=n}} (A^i \otimes A^j) \xrightarrow{\wedge} A^n \xrightarrow{\rho} \bigoplus_{\{i \mid k_i = n\}} V_i \to 0.$$

The filtrations agree on $\bigoplus_{i,j \leq n-1} (A^i \otimes A^j)$ and on V^n and wedge product as well as ρ is strictly compatible with both filtrations. It follows easily that if $F^i(A^n) \subset F'^i(A^n)$ for all i, then $F^i(A^n) = F'^i(A^n)$.

A decreasing filtration $F(V)$ yields an increasing filtration $F_*(V^*)$, $(V^* = \mathrm{Hom}_k(V, k))$, by $F_i(V^*)=\{\varphi : V \to k \mid \varphi(F^{i+1}(V))=0\}$. We change this to a decreasing filtration by setting $F^i(V^*)=F_{-i}(V^*)$. This is the *dual filtration*. If V_1 and V_2 have decreasing filtrations, then $\mathrm{Hom}(V_1, V_2)=V_1^*\otimes V_2$ receives the multiplicative extension of the dual filtration on V_1^* and the filtration on V_2. More directly:

$$F^i(\mathrm{Hom}(V_1, V_2))=\{\varphi : V_1 \to V_2 \mid \forall a,\ \varphi(F^a(V_1))\subset F^{a+i}(V_2)\}.$$

A *decreasing filtration of a differential algebra* A, F(A), is a decreasing filtration of each component A^i such that both $d : A^i \to A^{i+1}$ and $\wedge : A^i\otimes A^j \to A^{i+j}$ are compatible with the filtrations. By (1.2), H(A) receives a filtration induced from F(A), F(H(A)). Such a filtration gives rise to a spectral sequence, $\{E_r^{p,q}(A), d_r\}_{r\geq 0}$:

(1.4)

a) $$E_r^{p,q}(A)=\frac{\{x\in F^p(A^{p+q}) \mid dx\in F^{p+r}(A^{p+q+1})\}}{\{x\in F^{p+1}(A^{p+q}) \mid dx\in F^{p+r}(A^{p+q+1})\}+dF^{p-r+1}\cap F^p(A^{p+q})}.$$

b) $d_r : E_r^{p,q}(A) \to E_r^{p+r,q-r+1}(A)$ is the map induced by d; it gives $E_r(A)$ the structure of a differential algebra.

c) $E_0^{p,q}=\mathrm{Gr}_F^p(A^{p+q})$ and d_0 is the map induced on associated graded objects.

d) $E_{r+1}^{p,q}(A)=H(E_r^{p,q})$ with respect to the differential operator d_r.

e) $E_\infty^{p,q}$ is naturally identified with $F^p(H^{p+q}(A))/F^{p+1}(H^{p+q}(A))$.

Lemma **(1.5)** ([4], (1.3.2) and (1.3.4)). — *Let* F(A) *be a decreasing filtration on a differential algebra, and let* $\{E_r^{p,q}(A)\}$ *be the resulting spectral sequence.* $\{E_r(A)\}$ *degenerates at* E_k, *that is* $E_k(A)=E_{k+1}(A)=\ldots=E_\infty(A)$, *if and only if* $F^p(A)\cap dA\subset dF^{p-k+1}(A)$ *for all p. In particular* $E_1=E_\infty$ *if and only if d is strictly compatible with* F(A).

Example. — Let $\mathscr{E}$ be the complex valued C^∞-forms on a complex manifold. Let $F^p(\mathscr{E})$ be all forms which can be written locally near any point as $\sum dz_{i_1}\wedge\ldots\wedge dz_{i_p}\wedge\omega$ in a local holomorphic coordinate system. The filtration $F(\mathscr{E})$ is the *Hodge filtration*, and $\mathrm{Gr}_F^p(\mathscr{E})\cong\bigoplus_{q\geq 0}\mathscr{E}^{p,q}$, where $\mathscr{E}^{p,q}$ is the space of forms of type (p, q). We have $d=\partial+\bar\partial$ where $\partial : \mathscr{E}^{p,q}\to\mathscr{E}^{p+1,q}$ and $\bar\partial : \mathscr{E}^{p,q}\to\mathscr{E}^{p,q+1}$. In the associated spectral sequence $E_0^{p,q}(\mathscr{E})=\mathscr{E}^{p,q}$, and $d_0=\bar\partial$. Thus $E_1=H_{\bar\partial}(\mathscr{E})$, the $\bar\partial$-cohomology of $\mathscr{E}$. If the complex manifold is a compact Kähler manifold, then $E_1=E_\infty$ ([12], [5]). This means that $d : \mathscr{E}\to\mathscr{E}$ is strictly compatible with $F(\mathscr{E})$. As a special case of this, a closed $(p, 0)$-form (*i.e.* a global holomorphic p-form) is exact if and only if it is 0.

Definition. — Let (A, F) and (B, F) be filtered chain complexes. An *elementary quasi-isomorphism* from (A, F) to (B, F) is a map $\rho : A\to B$ which is compatible with the filtrations and which induces an isomorphism $E_1^{*,*}(\rho) : E_1^{*,*}(A)\to E_1^{*,*}(B)$. More generally (A, F) and (C, F) are *quasi-isomorphic* if there is a finite chain:

$$(A, F)=(A_0, F), (A_1, F), \ldots, (A_r, F)=(C, F)$$

and elementary quasi-isomorphisms from each (A_i, F) to either its predecessor or its successor.

Definition. — If A is a filtered chain complex and W is an increasing filtration then Dec W is another increasing filtration on A defined by:

$$\mathrm{Dec}\,W_i(A^n)=\{x\in W_{i-n}(A^n)\mid dx\in W_{i-n-1}(A^{n+1})\}.$$

Clearly $d:\mathrm{Dec}\,W_i\to\mathrm{Dec}\,W_i$. The induced filtration on H(A) is given by:

$$\mathrm{Dec}\,W_i(H^n(A))=W_{i-n}(H^n(A)).$$

The spectral sequences for (A, W) and (A, Dec W) are related by a shift of indexing. We have ${}_{W}E_i^{a,b}(A)={}_{\mathrm{Dec}\,W}E_{i-1}^{-b,2b+a}(A)$ for all $i\geqq 2$, and all a and b.

A *splitting* of a filtration F(V) is a direct sum decomposition $V=\bigoplus_i V_i$ with $F^p(V)=\bigoplus_{i\geqq p} V_i$. This is equivalent to an isomorphism $V\to \mathrm{Gr}_F(V)$. A map $\varphi: X\to V$, which is compatible with filtrations F(X) and F(V), is *strictly compatible* with them if there are splittings of the filtrations, $V=\bigoplus_i V_i$ and $X=\bigoplus_i X_i$, such that $\varphi\mid X_i\to V_i$. If F(A) is a filtration of a differential algebra, then a *splitting* for it is a decomposition of A, $A^*=\bigoplus(A^*)_i$, with $A_i\otimes A_j\xrightarrow{\wedge}A_{i+j}$, with $d:A_i\to A_i$, and with $F^p(A)=\bigoplus_{i\geqq p}A_i$. Such a splitting identifies A with $E_0(A)$ and d with d_0.

We now recall the basic definitions and results in Deligne's theory of mixed Hodge structures. The basic reference for this is [4].

Definition **(1.6)**. — A *Hodge structure of weight n* on a vector space V defined over k, $k\subset \mathbf{R}$, is a finite bi-grading:

$$V_{\mathbf{C}}=\bigoplus_{p+q=n}V^{p,q},\qquad \text{with}\qquad \overline{V^{p,q}}=V^{q,p}.$$

Equivalently, we could give a finite decreasing filtration $F(V_{\mathbf{C}})$ such that $F(V_{\mathbf{C}})$ is *n-opposed to its complex conjugate.* This means that:

$$F^p(V_{\mathbf{C}})\oplus\overline{F}^{n+1-p}(V_{\mathbf{C}})=V_{\mathbf{C}}\qquad\text{for every } p.$$

Then we have $V_{\mathbf{C}}=\bigoplus_{p+q=n}(F^p(V_{\mathbf{C}})\cap\overline{F}^q(V_{\mathbf{C}}))$.

Examples. — The primary examples of Hodge structures of weight n are those on $H^n(V;\mathbf{Q})$ for V a nonsingular, complex projective variety (or more generally, a compact Kähler manifold) [12]. The filtration on $H^n(V;\mathbf{C})$ is the Hodge filtration.

Definition **(1.7)** ([4]). — A *mixed Hodge structure defined over k*, $k\subset\mathbf{R}$, is a triple $\{V, W(V), F(V_{\mathbf{C}})\}$ (or $\{(V, W, F)\}$ for short) with:

a) V a vector space over k,

b) W(V) an increasing filtration (which is allowed to be infinite but must be bounded below, *i.e.* $W_N=0$ for N sufficiently small), and

c) $F(V_{\mathbf{C}})$ a decreasing filtration (which is possibly infinite) such that on $\mathrm{Gr}_n^W(V_{\mathbf{C}})$ the filtration induced by F (1.2) is finite and n-opposed to its complex conjugate.

W is the *weight filtration*, and F is the *Hodge filtration*. In case W and F are finite $\{V, W, F\}$ is called a *finite* mixed Hodge structure. If $W_0(V)=0$, then $\{V, W, F\}$ is a mixed Hodge structure *with positive weights*.

If $\{V, W, F\}$ is a mixed Hodge structure, then on $\mathrm{Gr}_n^W(V)$ we have the Hodge structure of weight n. Thus $\mathrm{Gr}_n^W(V)_{\mathbf{C}} \cong \bigoplus_{p+q=n} \mathscr{A}^{p,q}$. Define $h^{p,q}=\dim_{\mathbf{C}}(\mathscr{A}^{p,q})$. These $h^{p,q}$ are the *Hodge numbers* of the mixed Hodge structure. A mixed Hodge structure on V whose only non zero Hodge numbers are $h^{p,q}$ for $p+q=n$ is identical to a Hodge structure of weight n on V.

A *morphism* of mixed Hodge structures, $\varphi:\{X, W, F\}\to\{V, W, F\}$ is a k-linear map $\varphi: X\to V$ which is compatible with W and F (and hence automatically $\overline{F}$). If V_1 and V_2 have mixed Hodge structures, then the direct sum filtrations induce one on $V_1 \oplus V_2$.

For any mixed Hodge structure (V, W, F) define:

$$
(\mathbf{1.8})\begin{cases} a) & R^{p,q}=W_{p+q}(V_{\mathbf{C}})\cap F^p(V_{\mathbf{C}}) \\ b) & L^{p,q}=W_{p+q}(V_{\mathbf{C}})\cap \overline{F}^q(V_{\mathbf{C}})+\sum_{i\geqq 2} W_{p+q-i}(V_{\mathbf{C}})\cap \overline{F}^{q-i+1}(V_{\mathbf{C}}) \\ c) & A^{p,q}=R^{p,q}\cap L^{p,q}. \end{cases}
$$

Proposition (**1.9**) ([4], (1.2.8)). — *Let* (V, W, F) *be a mixed Hodge structure. Then* $V_{\mathbf{C}}=\bigoplus_{p,q} A^{p,q}$ *gives a functorial bigrading compatible with tensor products and duals*:

1) $W_i(V_{\mathbf{C}})=\bigoplus_{p+q\leqq i} A^{p,q}$,

2) $F^j(V_{\mathbf{C}})=\bigoplus_{p\geqq j} A^{p,q}$, *and*

3) *the composition* $A^{p,q}\hookrightarrow W_{p+q}(V_{\mathbf{C}})\to \mathrm{Gr}^W_{p+q}(V_{\mathbf{C}})=\bigoplus_{p'+q'=p+q}\mathscr{A}^{p,q}$ *sends* $A^{p,q}$ *isomorphically onto* $\mathscr{A}^{p,q}$.

Proof. — Functoriality is clear from the definition of the $A^{p,q}$'s. From 3) parts 1) and 2) follow immediately. To prove 3) we first note that the map $A^{p,q}\to\bigoplus_{p',q'}\mathscr{A}^{p',q'}$ has its image contained in $\mathscr{A}^{p,q}$. The next proposition is then used to show that $A^{p,q}\to\mathscr{A}^{p,q}$ is an isomorphism.

Proposition (**1.10**). — *Let* (V, W, F) *be a mixed Hodge structure. Then*

$$W_{p+q}(V_{\mathbf{C}})=R^{p,q}+L^{p,q}.$$

Proof. — Clearly $R^{p,q}+L^{p,q}\hookrightarrow W_{p+q}$, and $\mathrm{Gr}_n^W(R^{p,q}+L^{p,q})\to \mathrm{Gr}_n^W(V_{\mathbf{C}})$ is onto for $n\leqq p+q$. Since $W_N=0$ for N sufficiently small it follows that:

$$R^{p,q}+L^{p,q}=W_{p+q}(V_{\mathbf{C}}).$$

Note. — It is not always true that $\overline{A^{p,q}}=A^{q,p}$. However, modulo $W_{p+q-1}(V_{\mathbf{C}})$ this is true.

Proposition **(1.11)**. — *Let* V *be a k-vector space and suppose given a decomposition*:

$$V_{\mathbf{C}} \cong \bigoplus_{p,q} A^{p,q}$$

such that:

1) $\bigoplus_{p+q\leqq n} A^{p,q} \subset V_{\mathbf{C}}$ *is a k-subspace*,

2) $\overline{A^{p,q}} = A^{q,p}$ *modulo* $\bigoplus_{i+j<p+q} A^{i,j}$, *and*

3) $A^{p,q}$ *is nonzero only for* $p+q \geqq -N$ *for some* N.

Then there is a unique mixed Hodge structure on V *so that* (1.9) *parts* 1 *and* 2 *hold.*

(Note that the $A^{p,q}$ of this proposition are not required *a priori* to be related to any mixed Hodge structure.)

Proof. — Define $W_n(V)$ to be the k-subspace determined by $\bigoplus_{p+q\leqq n} A^{p,q}$. Define $F^j(V_{\mathbf{C}})$ to be $\bigoplus_{p\geqq j} A^{p,q}$. Then:

a) $\qquad W_n = 0 \quad \text{for} \quad n \leqq -N,$

b) $\qquad \mathrm{Gr}_n^W(V_{\mathbf{C}}) = \bigoplus_{p+q=n} A^{p,q}, \quad$ and

c) $\qquad F^j(\mathrm{Gr}_n^W(V_{\mathbf{C}})) = \bigoplus_{\substack{p\geqq j\\ p+q=n}} A^{p,q}.$

Since $\overline{A^{p,q}} = A^{q,p}$ modulo W_{p+q-1}, we have that:

d) $\qquad \overline{F}^j(\mathrm{Gr}_n^W(V_{\mathbf{C}})) = \bigoplus_{\substack{q\geqq j\\ p+q=n}} A^{p,q}.$

Thus F and $\overline{F}$ are n-opposed on $\mathrm{Gr}_n^W(V_{\mathbf{C}})$.

Theorem **(1.12)**. — *Let* V *have a finite mixed Hodge structure, and* V_1 *and* V_2 *have mixed Hodge structures.*

1) *The dual filtrations on* V^* *define a mixed Hodge structure.*
2) *The multiplicative extensions of the filtrations to* $V_1 \otimes V_2$, $\mathrm{Hom}(V, V_1)$, $S(V_1)$, *and* $\Lambda(V_1)$ *define mixed Hodge structures.*
3) ([4]) *Any morphism of mixed Hodge structures is strictly compatible with* W, F, $\overline{F}$, *and the filtrations induced by* F *and* $\overline{F}$ *on* Gr^W.
4) ([4]) *The kernel and cokernel of a morphism of mixed Hodge structures, with their induced filtrations, are mixed Hodge structures.*

Definition **(1.13)**. — Let A be a differential algebra. A *mixed Hodge structure on* A is a mixed Hodge structure on each A^i such that $d: A^i \to A^{i+1}$ and $\wedge : A^i \otimes A^j \to A^{i+j}$ are morphisms of mixed Hodge structures. If A is connected, then the mixed Hodge structure has *positive weights* if the one on A^i, for all $i > 0$, is positive.

2. A Generalization of the Gysin Sequence.

We consider a divisor with normal crossings D in a complex manifold V and find a differential algebra E, defined over **Q**, associated to this situation. E is appropriate for calculating the rational cohomology (and even the rational homotopy type) of the complement $V-D$. This we prove by mapping E to the **Q**-polynomial forms on some C^1-triangulation for $V-D$. In addition to this, however, E has an increasing filtration W(E). The spectral sequence associated to this filtration is the Gysin spectral sequence. The term $E_1^{-p,q}$ is a sum of the cohomology in degree $q-2p$ of the various p-fold intersections of the divisors. The differential d_1 is a sum of Gysin maps. Just as E is based on the **Q**-polynomial forms on some C^1-triangulation, there are algebras based on the C^∞-forms and on piecewise C^∞-forms. We compare these various algebras and the resulting spectral sequences. First let us recall the algebras used by Sullivan in the case of an arbitrary C^1-triangulated, C^∞-manifold, and arbitrary simplicial complex.

Definition **(2.1)** (Sullivan [11], [5]). — The **Q**-*polynomial* forms on a simplicial complex K, $\mathscr{E}(|K|)$, are collections of forms, one on each simplex, ω_σ on σ, such that $\omega_\sigma|\tau=\omega_\tau$ for τ a face of σ (denoted $\tau<\sigma$). Each ω_σ can be written as:

$$\sum p(x_0, \ldots, x_k)\, dx_{i_1}\wedge \ldots \wedge dx_{i_j}$$

where $x_0, \ldots, x_k$ are the barycentric coordinates for σ and p is a polynomial with rational coefficients. Wedge product and d are defined by the usual operations in each simplex.

If K is a C^1-triangulation of a C^∞-manifold, then $\mathscr{E}_{p.C^\infty}(|K|)$, the *piecewise* C^∞-*forms* on K, are collections $\{\omega_\sigma \text{ on } \sigma\}$ such that ω_σ is a C^∞-form on σ, and $\omega_\sigma|\tau=\omega_\tau$ if $\tau<\sigma$.

One of the main reasons for studying these forms is the following theorem.

Theorem **(2.2)** (Sullivan) ([11], [13]).

a) *Integration induces a map of cochain complexes*:

$$\mathscr{E}(|K|) \xrightarrow{\int} (\textbf{Q}\textit{-simplicial cochains on } K)$$

which induces an isomorphism on cohomology rings.

b) *If* K *is a* C^1-*triangulation of a* C^∞-*manifold* M, *and if* $\mathscr{E}_{C^\infty}(M)$ *is the differential algebra of* C^∞-*forms, then*:

$$\mathscr{E}(|K|)_{\mathbf{R}} \hookrightarrow \mathscr{E}_{p.C^\infty}(|K|) \hookleftarrow \mathscr{E}_{C^\infty}(M)$$

are inclusions of differential algebras which induce isomorphisms on the cohomology rings.

If K′ is a subdivision of a complex K so that every vertex of K′ has *rational* coordinates in the barycentric coordinates of K, then a **Q**-polynomial form on K induces one on K′ by restriction. The resulting map $\mathscr{E}(K)\to\mathscr{E}(K')$ induces an isomorphism on cohomology.

Let V be a complex manifold; $D \subset V$ is a *divisor with normal crossings if* $D = \bigcup_i D_i$ where each $D_i \subset V$ is a nonsingular divisor and if locally the D_i cross like the coordinate hyperplanes in $\mathbf{C}^n$. Let $X = V - D$. Let D^p be the points of multiplicity at least p, and $\widetilde{D}^p$ be the normalization of D^p. A point in $\widetilde{D}^p$ is a point in D^p together with a choice of exactly p sheets which intersect at that point. $\widetilde{D}^1$ is $\coprod_i D_i$. There is a bundle of coefficients of rank one on $\widetilde{D}^p$. Let $(x, D_1, \ldots, D_p)$ be a point in $\widetilde{D}^p$. The fiber of the coefficient system over this point is $\Lambda^p(\mathbf{Z}^{\{D_1, \ldots, D_p\}})$. Thus an isomorphism of the fiber with $\mathbf{Z}$ is just an orientation of the set of divisors $\{D_1, \ldots, D_p\}$. We denote this system of coefficients by ε^p. The associated coefficients with fibers $\mathbf{Q}$, $\mathbf{R}$, or $\mathbf{C}$ are denoted by $\varepsilon^p_{\mathbf{Q}}$, $\varepsilon^p_{\mathbf{R}}$, or $\varepsilon^p_{\mathbf{C}}$.

Each D_i has a tubular neighborhood N_i in V. If we choose these sufficiently small the various N_i will all cross transversally. Let $N^p \subset N$ be those points in at least p of the N_i. Then N^p is a regular neighborhood for D^p. We can separate the sheets of N^p to obtain $\widetilde{N}^p$ which contains $\widetilde{D}^p$ as a deformation retract. Take a C^1-triangulation of V so that all the N_i become subcomplexes. We will define a rational differential algebra supported on this triangulation, which calculates the cohomology of $X = V - D$. Begin by choosing a $\mathbf{Q}$-polynomial 2-form, ω_i, which represents the Thom class in $H^2(N_i, \partial N_i)$. The differential algebra $E(X)$ consists of compatible collections of forms, one on each simplex. The forms on a simplex $\sigma \subset \overline{V - N}$ are the $\mathbf{Q}$-polynomial forms on σ. The forms on a simplex $\sigma \subset \overline{N^p - N^{p+1}}$ are the tensor product of the $\mathbf{Q}$-polynomial forms on σ, with an exterior algebra on p one-dimensional generators, $\Lambda(\theta_{i_1}, \ldots, \theta_{i_p})$. These forms are indexed by the p divisors that cross near σ. We define $d\theta_{i_j} = \omega_{i_j}|\sigma$. There is an analogous real differential algebra $E_{p.C^\infty}(X)$ built using the $\{\theta_i\}$ with $\mathscr{E}_{p.C^\infty}(\sigma)$ replacing $\mathscr{E}(\sigma)$. There is an inclusion map $E(X)_{\mathbf{R}} \hookrightarrow E_{p.C^\infty}(X)$.

We also need a C^∞ version of this construction. For this it is necessary to choose C^∞-Thom forms $\mu_i \in \mathscr{E}_{C^\infty}(N_i, \partial N_i)$. $E_{C^\infty}(X)$ is defined as the global sections of a sheaf. The value of this sheaf of an open set $U \subset V - N$ is $\mathscr{E}_{C^\infty}(U)$. Its value on any open set U meeting N^p and missing N^{p+1} is $\mathscr{E}_{C^\infty}(U) \otimes \Lambda(\tau_{i_1}, \ldots, \tau_{i_p})$ where the τ_i are indexed by the p divisors near U; $d\tau_{i_j} = \mu_{i_j}|\sigma$.

To compare $E_{p.C^\infty}(X)$ and $E_{C^\infty}(X)$ it is necessary to choose one-forms:

$$\lambda_i \in \mathscr{E}_{p.C^\infty}(N_i, \partial N_i)$$

such that $d\lambda_i = \omega_i - \mu_i$. This is possible because both ω_i and μ_i represent the same relative cohomology class. Once we have chosen the λ_i we define a map:

$$E_{C^\infty}(X) \hookrightarrow E_{p.C^\infty}(X).$$

It is the extension of the inclusion of C^∞-forms on V to piecewise C^∞-forms on V, given by sending τ_i to $\theta_i - \lambda_i$. One checks easily that it commutes with d and defines a map of differential algebras.

We filter these three algebras so as to get the Gysin spectral sequence. Define

$W_j(E(X))$ to be all $\sum_a \alpha_a$ where $\alpha_a = \beta_a \wedge \theta_{i_1} \wedge \ldots \wedge \theta_{i_t}$ with $t \geqq j$ and $\beta_a \in \mathscr{E}(V)$. Likewise filter $E_{p.C^\infty}(X)$ and $E_{C^\infty}(X)$. The differential and wedge product are compatible with the filtrations as are the maps $E(X)_{\mathbf{R}} \hookrightarrow E_{p.C^\infty}(X) \hookleftarrow E_{C^\infty}(X)$.

Theorem **(2.3)**. — 1) *The* ${}_{W}E_1$*-term of the spectral sequence associated to* $W(E(X))$ *is given by*:

$$ {}_{W}E_1^{-p,q}(E(X)) = \begin{cases} H^{q-2p}(\widetilde{D}^p;\, \varepsilon^p_{\mathbf{Q}}), & p>0 \\ H^q(V;\, \mathbf{Q}), & p=0 \\ 0, & p<0. \end{cases} $$

2) *The inclusion maps* $E(X)_{\mathbf{R}} \hookrightarrow E_{p.C^\infty}(X) \hookleftarrow E_{C^\infty}(X)$ *are quasi-isomorphisms.*

Proof. — Suppose that $p>0$. For $\sigma \subset (\overline{V-N^p})$ the associated graded $\mathrm{Gr}^W_p(E(\sigma))$ is zero. For $\sigma \subset N^p$ there are one or more simplices $\widetilde{\sigma} \subset \widetilde{N}^p$ lying above σ. These are indexed by all sets of p elements chosen from the divisors which lie near σ. Denote the simplex corresponding to $D_{i_1}, \ldots, D_{i_p}$ by $\widetilde{\sigma}_{i_1 \ldots i_p}$. Define an isomorphism:

$$ \mathrm{Gr}^W_p(E(\sigma)) \to \mathscr{E}(\widetilde{N}^p;\, \varepsilon^p_{\mathbf{Q}}) $$

by sending:

$$ \alpha \wedge \theta_{i_1} \wedge \ldots \wedge \theta_{i_p} \mapsto (\alpha \text{ on } \widetilde{\sigma}_{i_1 \ldots i_p}) \otimes (\text{orientation}(i_1, \ldots, i_p)). $$

This is compatible with wedge products:

$$ \mathrm{Gr}^W_p \wedge \mathrm{Gr}^W_q \to \mathrm{Gr}^W_{p+q}, \quad \text{and} \quad \mathscr{E}(\widetilde{N}^p;\, \varepsilon^p_{\mathbf{Q}}) \wedge \mathscr{E}(\widetilde{N}^q;\, \varepsilon^q_{\mathbf{Q}}) \to \mathscr{E}(\widetilde{N}^{p+q};\, \varepsilon^{p+q}_{\mathbf{Q}}), $$

and also commutes with d. It is compatible with the restriction maps on simplices. Thus it induces an isomorphism $\mathrm{Gr}^W_p(E(X)) \xrightarrow{\sim} \mathscr{E}(\widetilde{N}^p;\, \varepsilon^p_{\mathbf{Q}})$. Hence:

$$ {}_{W}E_1^{-p,q}(E(X)) \cong H^{q-2p}(\widetilde{N}^p;\, \varepsilon^p_{\mathbf{Q}}). $$

Since $\widetilde{N}^p$ deforms onto $\widetilde{D}^p$, this last term is identified with $H^{q-2p}(\widetilde{D}^p;\, \varepsilon^p_{\mathbf{Q}})$. One sees immediately that $\mathrm{Gr}^W_0(E(X)) = \mathscr{E}(V)$. Thus ${}_{W}E_1^{0,q} \cong H^q(V)$.

The same argument calculates ${}_{W}E_1^{-p,q}(E_{p.C^\infty}(X))$ and ${}_{W}E_1^{-p,q}(E_{C^\infty}(X))$ to be the same cohomology groups with $\varepsilon^p_{\mathbf{Q}}$ replaced with $\varepsilon^p_{\mathbf{R}}$. The inclusion maps induce the usual inclusions:

$$ \mathscr{E}(\widetilde{N}^p;\, \varepsilon^p_{\mathbf{R}}) \hookrightarrow \mathscr{E}_{p.C^\infty}(\widetilde{N}^p;\, \varepsilon^p_{\mathbf{R}}) \hookleftarrow \mathscr{E}_{C^\infty}(\widetilde{N}^p;\, \varepsilon^p_{\mathbf{R}}) $$

on Gr^W_p. Hence on ${}_{W}E_1$ they induce isomorphisms of cohomology.

Corollary **(2.4)**. — $E(X)_{\mathbf{R}} \hookrightarrow E_{p.C^\infty}(X) \hookleftarrow E_{C^\infty}(X)$ *induce filtered isomorphisms on cohomology.*

Next we wish to compare $E(X)$ with the usual $\mathbf{Q}$-polynomial forms on X, $\mathscr{E}(X)$. For this it is necessary to triangulate $X \subset V$ so that every simplex of X lies rectilinearly in some simplex of V with its vertices rational in the barycentric coordinates of that simplex. Now choose forms $\beta_i \in \mathscr{E}(N_i \cap X, \partial N_i)$ such that $d\beta_i = \omega_i | N_i \cap X$. This is possible since $N_i \cap X = N_i - D_i$ and $H(N_i - D_i, \partial N_i) = 0$. Once we have such forms we define $E(\sigma) \to \mathscr{E}(\sigma \cap X)$ to extend the restriction map on $\mathbf{Q}$-polynomial forms and to send $\theta_i | \sigma$ to $\beta_i | \sigma \cap X$. The choice of the β_i also defines a map $E_{p.C^\infty}(X) \to \mathscr{E}_{p.C^\infty}(X)$ so that:

$$\begin{tikzcd} E(X) \arrow[r, hook] \arrow[d, "\psi"] & E_{p.C^\infty}(X) \arrow[d, "\psi_{p.C^\infty}"] \\ \mathscr{E}(X) \arrow[r, hook] & \mathscr{E}_{p.C^\infty}(X) \end{tikzcd}$$

commutes.

To define $E_{C^\infty}(X) \xrightarrow{\psi_{C^\infty}} \mathscr{E}_{C^\infty}(X)$ it is necessary to choose forms $\alpha_i \in \mathscr{E}_{C^\infty}(N_i - D_i, \partial N_i)$ so that $d\alpha_i = \mu_i$. One extends the restriction map on $\mathscr{E}_{C^\infty}(V) \to \mathscr{E}_{C^\infty}(X)$ by sending $\tau_i \mapsto \alpha_i$. The diagram comparing the C^∞-situation with the piecewise C^∞-situation does not commute. What is true is that it commutes up to homotopy.

Definition **(2.6)**. — Let f_0 and f_1 be maps of differential algebras $A \to B$. We say that f_0 and f_1 are *homotopic* if there is an $H : A \to B \otimes \Lambda(t, dt)$ such that $H|_{t=i}$ is f_i for $i = 0, 1$. Here t is of degree 0. If $H(a) = \sum_r (b_r t^r + \beta_r t^r\, dt)$ then $H|_{t=i}(a) = \sum_r b_r i^r \in B$. Since $\Lambda(t, dt)$ has the cohomology of a point, homotopic maps are the same on cohomology.

Theorem **(2.7)**. — *For any choices of* $\psi_{p.C^\infty}$ *and* ψ_{C^∞} *as above the diagram:*

$$\begin{tikzcd} E_{C^\infty}(X) \arrow[r, hook, "i"] \arrow[d, "\psi_{C^\infty}"] & E_{p.C^\infty}(X) \arrow[d, "\psi_{p.C^\infty}"] \\ \mathscr{E}_{C^\infty}(X) \arrow[r, hook, "i"'] & \mathscr{E}_{p.C^\infty}(X) \end{tikzcd}$$

commutes up to homotopy.

Proof. — The diagram actually commutes on the C^∞-forms on V. The homotopy is taken to be the constant homotopy, $H(a) = a$ for these forms. We extend H to all of $E_{C^\infty}(X)$ by giving its value on the τ_j. Recall that the image of τ_j in $E_{p.C^\infty}(X)$ is $\theta_j - \lambda_j$ where λ_j is a relative one form in N_j with $d\lambda_j = \omega_j - \mu_j$. Thus:

$$\psi_{p.C^\infty} \circ i(\tau_j) = \psi_{p.C^\infty}(\theta_j) - \lambda_j = \beta_j - \lambda_j.$$

On the other hand $i \circ \psi_{C^\infty}(\tau_j) = \alpha_j$. Both α_j and $\beta_j - \lambda_j$ are relative one-forms in N_j. They have the same image under d. Thus there is a relative function f_j so that $df_j = (\beta_j - \lambda_j) - \alpha_j$. We let $H(\tau_j) = \beta_j - \lambda_j - d(f_j \otimes t)$. This defines the required homotopy.

We have made several choices in defining E(X), W(E(X)) and the map $E(X) \xrightarrow{\psi} \mathscr{E}(X)$. We could give a direct argument that in an appropriate derived category these choices do not effect the results up to isomorphism. These arguments are somewhat cumbersome however. Fortunately, it turns out that in the case of interest, when V is a complete variety, these results follow for free from the analysis in the rest of this paper. For this reason we do not give the direct argument.

3. Cohomology of Nonsingular Varieties.

In this section we recall the work of Deligne concerning the cohomology of open, nonsingular, complex algebraic varieties. By viewing such a variety as the complement in a nonsingular, compact variety of a divisor with normal crossings, and by using the complex of meromorphic forms with logarithmic singularities along the divisor, Deligne showed that a natural mixed Hodge structure exists on its cohomology. We review his main results, as well as some technical statements about this complex of forms. It is important that the weight spectral sequence for the log complex has a real (or rational) structure. The log complex itself does not, however, have a real structure. Deligne equates the weight spectral sequence with the spectral sequence for $R_{j_*(\mathbf{C})}$, j being the inclusion of the open variety into the compact one and $\mathbf{C}$ being the constant sheaf of complex numbers on the open variety. He then imposes the rational structure by using $R_{j_*(\mathbf{Q})} \subset R_{j_*(\mathbf{C})}$. We impose the rational structure by comparing the log complex to the differential algebras of the previous sections. With the exception of these comparisons, all the work in this section is a summary of results in [4].

By variety we will always mean nonsingular, complex, algebraic variety. A study of the homotopy theory of compact varieties was carried out in [5]. We are mainly concerned here with non-compact varieties, though all results are valid in general. Our results come from reducing to the study of compact varieties and applying Hodge theory.

We can embed any variety X in a possibly singular compact variety V′, $X \rightarrow V'$, as a Zariski open set [9]. If X is an affine variety, we take V′ to be the projective variety which is the solution set of the homogenous polynomials associated to the polynomials defining X. In any case, by the fundamental theorem of Hironaka on resolution of singularities [6], we can replace V′ by another variety V which is nonsingular and compact, with $V-X$ a divisor with normal crossings, $V-X=D$.

$\mathscr{E}(\log D)$, *the log complex*, is a differential algebra of C^∞-forms on V with certain controlled singularities along D. It is defined as the global sections of a C^∞-sheaf over V, or more precisely, as a subsheaf of $j_*(\mathscr{E}_{V-D})$ where $j: V-D \hookrightarrow V$. This means that we take all forms on $V-D$ which have given types of local expressions in a neighborhood of every point of V. Away from D the local condition is simply that the form be a C^∞-complex valued form. Locally near $x \in D_{i_1} \cap \ldots \cap D_{i_t}$ we choose an analytic coordinate system U, where D is given by $\prod_{i=1}^{p} z_i = 0$. In U the form must be expressible as:

$$\Sigma\omega_J \wedge \left(\frac{dz_{j_1}}{z_{j_1}} \wedge \ldots \wedge \frac{dz_{j_\ell}}{z_{j_\ell}}\right)$$

with $j_t \in (1, 2, \ldots, p)$ and ω_J a C^∞-form throughout U. (This condition is independent of the particular analytic chart closen.) The form $\dfrac{dz_i}{z_i}$ is meromorphic with a pole along D_i. It is the unique C^∞-form on $U - \{z_i = 0\}$ such that when multiplied by z_i it is equal to dz_i. We *define* $z_i . \dfrac{dz_i}{z_i}$ to be equal to dz_i on all of U. The obvious differential and wedge product operations (defined locally) on $\mathscr{E}(\log D)$ make it a sub-differential algebra of $\mathscr{E}_{C^\infty}(X)$.

Equivalently, we could form the holomorphic log complex $\Omega(D)$. In $\Omega(D)$, the only local relationships between the various expressions are $z_i . \dfrac{dz_i}{z_i} = dz_i$ and its derived equation $dz_i \wedge \dfrac{dz_i}{z_i} = 0$. The algebra $\mathscr{E}(\log D)$ is then the global sections of $\Omega(D) \otimes_{\mathcal{O}} \Omega^{(0,*)}_{C^\infty}$, where $\mathcal{O}$ is the sheaf of germs of holomorphic functions and $\Omega^{0,q}_{C^\infty}$ is the sheaf of germs of C^∞-forms of type $(0, q)$.

We define filtrations $W(\mathscr{E}(\log D))$ and $F(\mathscr{E}(\log D))$ by:

$$W_\ell = \left\{\omega \mid \text{locally } \omega = \Sigma\omega_J \wedge \left(\frac{dz_{j_1}}{z_{j_1}} \wedge \ldots \wedge \frac{dz_{j_t}}{j_t}\right) \text{ with } \omega_J \text{ a } C^\infty\text{-form and } t \leq \ell\right\},$$

$$F^p = \left\{\omega \mid \text{locally } \omega = \Sigma\omega_J \wedge \left(dz_{j_1} \wedge \ldots \wedge dz_{j_s} \wedge \frac{dz_{j_{s+1}}}{z_{j_{s+1}}} \wedge \ldots \wedge \frac{dz_{j_t}}{z_{j_t}}\right) \text{ with } t \geq p\right\}.$$

Thus, the weight filtration comes from allowing no more than a fixed number of $\dfrac{dz_i}{z_i}$'s in each monomial, and the Hodge filtration comes from requiring at least so many dz_i's $\left(\text{including } \dfrac{dz_i}{z_i}\text{'s}\right)$ in each monomial. We note that wedge product and d are compatible with both filtrations.

(3.1) Suppose that $D \subset V$ is a divisor with normal crossings and that $X = V - D$. Suppose that we have made choices of neighborhoods N_i of D_i and of 2-forms:

$$\mu_i \in \mathscr{E}_{C^\infty}(N_i, \partial N_i)$$

necessary to define $E_{C^\infty}(X)$. We can compare $E_{C^\infty}(X)_{\mathbf{C}}$ with $\mathscr{E}(\log D)$. For this it is necessary to choose γ_i, one-forms in $\mathscr{E}(\log D)$ with support in N_i such that $d\gamma_i = \mu_i$. Given these, extend the identity on $\mathscr{E}_{C^\infty}(V)$ to a map $\rho : E_{C^\infty}(X)_{\mathbf{C}} \to \mathscr{E}(\log D)$ by defining $\rho(\tau_i) = \gamma_i$.

Lemma **(3.2)**. — *Let* U *be an open set in which* D_j *is given by* $\{z_j=0\}$. *Then* $\gamma_j|U$ *is of the form* $\frac{-1}{2\pi i}\cdot\frac{dz_j}{z_j}+\chi$ *where* χ *is a* C^∞*-form throughout* U.

Proof. — Suppose that $\gamma_j|U$ vanishes for $|z|>\varepsilon$. Let $\{|z|>\varepsilon\}$ be U_0. Let $f(x)$ be a C^∞-function which is identically 1 near $x=0$ and 0 for $x>\varepsilon$. If we integrate $d\left(\frac{-1}{2\pi i}f(|z_j|)\frac{dz_j}{z_j}\right)$ over a 2-disk in the z_j-direction we find:

$$\int_0^1\int_0^{2\pi} d\left(\frac{-1}{2\pi i}f(|z_j|)\frac{dz_j}{z_j}\right)=\int_0^1 d\left(\frac{-1}{2\pi i}f(|z_j|)\right).2\pi i=\frac{-f(1)+f(0)}{2\pi i}.2\pi i=1.$$

Thus $d\left(\frac{-1}{2\pi i}f(|z_j|)\frac{dz_j}{z_j}\right)$ is a local Thom form. The difference of this and $\mu_j|U$ will be an exact relative form. Thus there is a C^∞-form λ_j in U, vanishing in U_0, such that $d\lambda_j=d\left(\frac{-1}{2\pi i}f(|z_j|)\frac{dz_j}{z_j}\right)-\mu_j|U$. Also we have $\gamma_j|U$, a one-form in U vanishing in U_0 with $d\gamma_j|U=\mu_j|U$. Hence $\gamma_j|U-\lambda_j+\frac{1}{2\pi i}f(|z_j|)\frac{dz_j}{z_j}$ is a closed one form in $\mathscr{E}(\log D)$ defined in U and vanishing in U_0. The first cohomology of (U, U_0) calculated using the log complex is zero. Thus this form is $d(\varphi)$ for some function φ in $\mathscr{E}(\log D)$. Since all functions in $\mathscr{E}(\log D)$ are C^∞, this proves (3.2).

The main results of [4] are summarized in the following theorem.

Theorem **(3.3)** ([4]). — a) *Any such* ρ *as defined above is a quasi-isomorphism with respect to the weight filtrations. This gives* $\{{}_WE_r(\mathscr{E}(\log D))\}_{r\geqq 1}$ *a real or even rational structure* :

$${}_WE_1^{-p,q}\cong\begin{cases}H^{q-2p}(\widetilde{D}^p;\varepsilon_{\mathbf{C}}^p) & p>0\\ H^q(V;\mathbf{C}) & p=0\\ 0 & p<0\end{cases}$$

as spaces with rational structure.

b) *The filtration* $F(\mathscr{E}(\log D))$ *induces one on* ${}_WE_0(\mathscr{E}(\log D))$. *The differential* d_0 *is strictly compatible with it. The homology of* ${}_WE_0(\mathscr{E}(\log D))$, *being a subquotient of* ${}_WE_0(\mathscr{E}(\log D))$, *receives an induced filtration* F_r. *Under the identifications of part* a) $F_r^n({}_WE_1^{-p,q})$ *becomes* $F^{n-p}(H^{q-2p}(\widetilde{D}^p;\varepsilon_{\mathbf{C}}^p))$ *for* $p>0$, *and becomes* $F^n(H^q(V;\mathbf{C}))$ *for* $p=0$. *(In both cases the filtration on cohomology is the usual Hodge filtration.) Thus the induced filtration on* ${}_WE_1^{-p,q}$ *is q-opposed to its complex conjugate.*

c) $\mathscr{E}(\log D)\hookrightarrow\mathscr{E}_{C^\infty}(X)_{\mathbf{C}}$ *induces an isomorphism on cohomology.*

Corollary **(3.4)**. — *Let* $X=V-D$ *and suppose we have made choices as in section* 2 *to define* $E(X)\xrightarrow{\psi}\mathscr{E}(X)$. *Then* ψ *is an isomorphism on cohomology.*

Proof. — We make choices to define $E_{C^\infty}(X)$, a map $\rho : E_{C^\infty}(X)_{\mathbf{C}} \to \mathscr{E}(\log D)$, and $\psi_{C^\infty} : E_{C^\infty}(X) \to \mathscr{E}_{C^\infty}(X)$. Results (2.4) and (2.7) show that, up to homotopy commutative diagrams, $\psi_{\mathbf{R}} : E(X)_{\mathbf{R}} \to \mathscr{E}(X)_{\mathbf{R}}$ can be identified with

$$\psi_{C^\infty} : E_{C^\infty}(X) \to \mathscr{E}_{C^\infty}(X).$$

The complex form of this is easily seen to be homotopic to $r \circ \rho$, where

$$r : \mathscr{E}(\log D) \hookrightarrow \mathscr{E}_{C^\infty}(X; \mathbf{C})$$

is the restriction map. By (3.3) a) ρ induces an isomorphism on cohomology; by (3.3) c) r does also. Thus ψ_{C^∞} and ψ must be isomorphisms on cohomology.

Note **(3.5)**. — Results (2.4), (2.7), and (3.3) give a quasi-isomorphism between $(E(X), W)_{\mathbf{C}}$ and $(\mathscr{E}(\log D), W)$. We have the Sullivan equivalence between $\mathscr{E}(X)_{\mathbf{C}}$ and $\mathscr{E}_{C^\infty}(X; \mathbf{C})$. The content of the above argument is that under these quasi-isomorphisms $\psi_{\mathbf{C}}$ becomes homotopic to the restriction map $\mathscr{E}(\log D) \to \mathscr{E}_{C^\infty}(X; \mathbf{C})$.

Sketch of proof of (3.3). — The filtrations on $\mathscr{E}(\log D)$ are induced from the obvious ones on the associated sheaf. On the sheaf level the filtration F on Gr^W has a splitting associated with Hodge type $\mathrm{Gr}^W_n = \bigoplus_{p+q=n} (\mathrm{Gr}^W_n)^{p,q}$. Each of these components is a module over the C^∞-functions. Since the C^∞-functions are flat over the complex analytic functions the usual ∂- and $\bar\partial$-Poincaré lemmas can be generalized to prove that $\bigoplus_{p+q=n} (\mathrm{Gr}^W_n)^{p,q}$ is a fine double complex which resolves the complex:

$$(*) \qquad 0 \to \Omega^0_{\tilde{D}^n} \to \Omega^1_{\tilde{D}^n} \to \ldots \to \Omega^k_{\tilde{D}^n} \to 0.$$

On the level of forms define a map:

$$\bigoplus_{p+q=n} (\mathrm{Gr}^W_n)^{p,q} \xrightarrow{R} \bigoplus_{p,q} \mathscr{E}^{p-n,q}_{C^\infty}(\tilde{D}^n; \mathscr{E}^n_{\mathbf{C}})$$

as follows:

$$\left[\varphi \wedge \frac{dz_{j_1}}{z_{j_1}} \wedge \ldots \wedge \frac{dz_{j_n}}{z_{j_n}}\right] \mapsto \left(\frac{-1}{2\pi i}\right)^n \varphi \,|\, (D_{j_1} \cap \ldots \cap D_{j_n}) \otimes (\text{orient.}(j_1, \ldots, j_n)).$$

Since $\mathscr{E}^{*,*}(\tilde{D}^n; \varepsilon^p_{\mathbf{C}})$ is also a fine resolution of $(*)$, general sheaf theory tells us that R must induce an isomorphism on cohomology and, in fact, an isomorphism on the E_1-terms of the spectral sequences induced by filtering by the first index. The basic results for compact Kähler manifolds (or more generally complete varieties) [12], imply that this spectral sequence degenerates at E_1 and that the induced filtration of H^r gives a Hodge structure of weight r. (See [4] and [5].) This implies that $d_0 : {}_WE_0 \to {}_WE_0$ is strictly compatible with F, and that the induced filtration on ${}_WE_1^{-p,q}$ defines a Hodge structure of weight q. It is given by $F^j \cap \bar{F}^{q-j}({}_WE_1^{-p,q}) = H^{j-q,q-j-p}(\tilde{D}^p; \varepsilon^p_{\mathbf{C}})$, where the right hand side is the usual Hodge structure.

Lemma (3.2) implies that the following diagram commutes:

$$\begin{array}{ccccc} \mathrm{Gr}_p^{\mathrm{W}}(\mathrm{E}_{\mathrm{C}^\infty}(\mathrm{X})_{\mathbf{C}}) & \xrightarrow{\mathrm{E}_0(\rho)} & \mathrm{Gr}_p^{\mathrm{W}}(\mathscr{E}(\log \mathrm{D})) & \xrightarrow{\mathrm{R}} & \mathscr{E}_{\mathrm{C}^\infty}(\widetilde{\mathrm{D}}^p; \varepsilon_{\mathbf{C}}^p) \\ \| & & & & \| \\ \mathscr{E}_{\mathrm{C}^\infty}(\widetilde{\mathrm{N}}^p; \varepsilon_{\mathbf{C}}^p) & & \xrightarrow{\quad\text{restriction}\quad} & & \mathscr{E}_{\mathrm{C}^\infty}(\widetilde{\mathrm{D}}^p; \varepsilon_{\mathbf{C}}^p) \end{array}$$

Thus the map induced on ${}_{\mathrm{W}}\mathrm{E}_1$ by ρ becomes the identity when we make the identifications of the ${}_{\mathrm{W}}\mathrm{E}_1$-terms given here and in (2.3).

Definition **(3.5)**. — A *mixed Hodge diagram defined over a field* $k \subset \mathbf{R}$ is a pair of filtered differential algebras (A, W), defined over k, and ($\mathscr{E}$, W, F), defined over $\mathbf{C}$, and a filtered map of differential algebras $\varphi : (\mathrm{A}, \mathrm{W})_{\mathbf{C}} \to (\mathscr{E}, \mathrm{W})$ satisfying the following two conditions:

1) φ induces an isomorphism on ${}_{\mathrm{W}}\mathrm{E}_1$ (*i.e.* φ is a quasi-isomorphism).
2) The differential d_0 is strictly compatible with the filtration on ${}_{\mathrm{W}}\mathrm{E}_0(\mathscr{E})$ induced by $\mathrm{F}(\mathscr{E})$.

Consider ${}_{\mathrm{W}}\mathrm{E}_1(\mathscr{E})$ as the homology of ${}_{\mathrm{W}}\mathrm{E}_0(\mathscr{E})$ and let $\mathrm{F}({}_{\mathrm{W}}\mathrm{E}_0)$ define a filtration F_r on ${}_{\mathrm{W}}\mathrm{E}_1(\mathscr{E})$. Let φ^* induce a real structure on ${}_{\mathrm{W}}\mathrm{E}_1(\mathscr{E})$. The filtration F_r on ${}_{\mathrm{W}}\mathrm{E}_1^{p,q}(\mathscr{E})$ is q-opposed to its complex conjugate.

The following proposition is an immediate corollary of (3.3) and this definition.

Proposition **(3.6)**. — *If* $\mathrm{X} = \mathrm{V} - \mathrm{D}$ *is the complement in a compact variety of a divisor with normal crossings, then any diagram as in* (3.1) *is a mixed Hodge diagram.*

We turn now to the question of the dependence of the mixed Hodge diagram in (3.1) upon the choices made in defining it.

Definition **(3.7)**. — A *morphism* between mixed Hodge diagrams is a diagram:

$$\begin{array}{ccc} \mathrm{A}_{\mathbf{C}} & \xrightarrow{\rho} & \mathscr{E} \\ \Big\downarrow {\scriptstyle f_{\mathbf{C}}} & & \Big\downarrow {\scriptstyle g} \\ \mathrm{A}'_{\mathbf{C}} & \xrightarrow{\rho'} & \mathscr{E}' \end{array}$$

and a homotopy H from $g \circ \rho$ to $\rho' \circ f_{\mathbf{C}}$ such that:

1) f and H are compatible with the weight filtration, and
2) g is compatible with both W and F.

Note. — We do not claim that morphisms can be composed.

An *elementary equivalence* is a morphism which induces an isomorphism on cohomology. An *equivalence* is a finite string of elementary equivalences, possibly with arrows in both directions.

Theorem **(3.8)**. — *Any two mixed Hodge diagrams as constructed in* (3.1) *for* $X=V-D$ *are equivalent.*

Proof. — In the construction of the mixed Hodge diagram for $X=V-D$ we made 3 choices:

1) the neighborhoods N_i of D_i,
2) closed, relative 2-forms μ_i in $\mathscr{E}_{C^\infty}(N_i, \partial N_i)$ representing the Thom class, and
3) relative 1-forms γ_i in $\mathscr{E}(\log D)$ supported in N_i with $d\gamma_i=\mu_i$.

Let us consider the choices in reverse order.

I. *Neighborhoods and 2-forms are fixed; we have two sets of 1-forms.*

Let these two sets be $\{\gamma_i\}$ and $\{\gamma_i'\}$, and let ρ, ρ' be the two maps $E_{C^\infty}(X)_{\mathbf{C}} \to \mathscr{E}(\log D)$ induced by these forms. Since $d\gamma_i=d\gamma_i'$ the difference is a closed 1-form in $\mathscr{E}(\log D)$ supported in N_i. Consequently, $\gamma_i-\gamma_i'$ is df_i for some C^∞-function f_i supported in N_i. To show that the identity maps on $E_{C^\infty}(X)$ and $\mathscr{E}(\log D)$ constitute an elementary equivalence we need a homotopy from ρ to ρ' which is compatible with the filtrations. On the C^∞-forms we define the homotopy to be constant: $H(a)=\rho(a)$. This is extended to all of $E_{C^\infty}(X)_{\mathbf{C}}$ by defining $H(\tau_i)=\rho(\tau_i)-d(f_i\otimes t)$. This is easily seen to provide the requisite homotopy.

II. *Neighborhoods are fixed; we have two sets of 2-forms* $\{\mu_i\}$ *and* $\{\mu_i'\}$; *we are free to pick the 1-forms in each case.*

Choose 1-forms $\{\gamma_i\}$ to define $\rho : E_{C^\infty}(X)_{\mathbf{C}} \to \mathscr{E}(\log D)$. Since μ_i and μ_i' represent the same relative class, their difference is exact: $\mu_i-\mu_i'=d\alpha_i$ for some $\alpha_i \in \mathscr{E}_{C^\infty}(N_i, \partial N_i)$. Let $\{\gamma_i'\}$ be $\{\gamma_i-\alpha_i\}$. This defines $\rho' : E_{C^\infty}(X)_{\mathbf{C}} \to \mathscr{E}(\log D)$. Define:

$$f : E_{C^\infty}(X) \to E_{C^\infty}(X)'$$

to be the identity on C^∞-forms and to send τ_i to $\tau_i'+\alpha_i$. This defines a quasi-isomorphism of $E_{C^\infty}(X) \to E_{C^\infty}(X)'$ such that $\rho' \circ f_{\mathbf{C}}=\rho$.

III. *We have two sets of neighborhoods* $\{N_i\}$ *and* $\{N_i'\}$ *but are free to choose the forms.*

Given two sets of neighborhoods we can find a third set contained in both. Thus it suffices to consider the case when $N_i' \subset \operatorname{int} N_i$. Choose 2-forms $\mu_i' \in \mathscr{E}_{C^\infty}(N_i', \partial N_i')$ representing the Thom classes, and 1-forms $\gamma_i' \in \mathscr{E}(\log D)$ supported over N_i', such that $d\gamma_i'=\mu_i'$. Extend these to forms on all of N_i by letting them be zero outside of N_i'. In this case the map $E_{C^\infty}(X) \to E_{C^\infty}(X)'$ which sends τ_i to τ_i' commutes with the maps to $\mathscr{E}(\log D)$. This is the required elementary equivalence.

Proposition **(3.9)** *(Naturality).* — *Let* $f : V \to V'$ *be an algebraic map between smooth compact varieties. Let* $X=V-D$ *and* $X'=V'-D'$ *be the complements of divisors with* *normal crossings. Suppose that f induces a map $f: X \to X'$. Then f induces a morphism of appropriately chosen mixed Hodge diagrams for* (V', D') *and* (V, D).

Proof. — The requirement that f induce a map $f: X \to X'$ is equivalent to the requirement that $f^{-1}(D') \subset D$. The log complex is functorial for such maps (see (3.2.11) of [4]). We choose neighborhoods N_i of D_i and N'_i of D'_i so that $f^{-1}(N'_i) \subset \bigcup_j N_j$. Take any choices of 2-forms and 1-forms supported in these neighborhoods — $\{\mu_i\}$, $\{\mu'_i\}$, $\{\gamma_i\}$, $\{\gamma'_i\}$. The form $f^*(\mu'_i)$ is a relative 2-form in $(\bigcup_j N_j, \partial(\bigcup_j N_j))$. As such it is homologous to a linear combination $\sum_j a_{ij} \mu_j$. The a_{ij} are non-negative integers. If $f^{-1}(D'_i)$ is a union of the divisors D_i counted with multiplicity α_{ij}, then $\alpha_{ij} = a_{ij}$.

Thus for each index i there are integers a_{ij} and a 1-form supported in $\bigcup_j N_j$, λ_i, such that $f^*(\mu'_i) = (\sum_j a_{ij} \mu_j) + \lambda_i$. Define $f^*: E_{C^\infty}(X') \to E_{C^\infty}(X)$ to be the usual induced map on C^∞-forms, and to send τ'_i to $(\sum_j a_{ij} \tau_j) + \lambda_i$. It remains to check that the following diagram commutes up to a homotopy compatible with the filtrations:

$$\begin{array}{ccc} E_{C^\infty}(X')_{\mathbf{C}} & \xrightarrow{f^*} & E_{C^\infty}(X)_{\mathbf{C}} \\ \Big\downarrow{\scriptstyle \rho'} & & \Big\downarrow{\scriptstyle \rho} \\ \mathscr{E}(\log D') & \xrightarrow{f^*} & \mathscr{E}(\log D) \end{array}$$

First observe that on the C^∞-forms the diagram actually commutes. Thus the homotopy $H: E_{C^\infty}(X')_{\mathbf{C}} \to \mathscr{E}(\log D)$ will be constant on the C^∞-forms. We must give its value on the τ'_i. We claim that $\rho \circ f^*(\tau'_i) - f^* \circ \rho'(\tau'_i)$ is a closed C^∞-form on V supported in $\bigcup_j N_j$. Since $\rho \circ f^*(\tau'_i)$ and $f^* \circ \rho'(\tau'_i)$ are both relative 1-forms whose differentials are the same (namely $f^* \mu'_i$), their difference is a closed relative 1-form on $\bigcup_j N_j$. The relative cohomology of $(\bigcup_j N_j, \partial(\bigcup_j N_j))$ in dimension 1, calculated using the log complex, is zero. Consequently, there is a function in $\mathscr{E}(\log D)$ supported on $\bigcup_j N_j$, χ_i, such that $d\chi_i = \rho \circ f^*(\tau'_i) - f^* \circ \rho'(\tau'_i)$. Such a function is automatically C^∞. We define $H(\tau'_i) = \rho \circ f^*(\tau'_i) - d(\chi_i \otimes t)$. One sees easily that this defines the required homotopy.

Corollary **(3.10)**. — *Given a nonsingular variety* X, *all possible mixed Hodge diagrams associated to all possible completions of* X, $X = V - D$, *where* V *is complete and* D *is a divisor with normal crossings, are equivalent.*

Proof. — Given X and two completions $X = V - D$ and $X = V' - D'$, there is a third $X = V'' - D''$ dominating each. The map $f: V'' \to V$ induces a morphism of appropriately chosen mixed Hodge diagrams for $X \subset V$ and $X \subset V''$. This morphism induces an isomorphism on cohomology, since, when restricted to X, it is the identity.

Thus appropriately chosen mixed Hodge diagrams for $X \subset V$ and $X \subset V''$ (and likewise $X \subset V'$) are equivalent. The result now follows from (3.8).

If we were willing to ignore the multiplicative structures, then the filtered complexes (A, W) and $(\mathscr{E}, W, F)$ and the quasi-isomorphism $\rho : (A, W)_{\mathbf{C}} \to (\mathscr{E}, W)$ could be obtained from general sheaf theory as is done in ([4], section 3). If $X \underset{j}{\hookrightarrow} V$ is the complement of a divisor with normal crossings, one obtains (A, W) as follows. Let $\mathscr{E}(X)$ be the sheaf of C^∞-forms on X, then $H^*(X; R) = H^*(X, \mathscr{E}(X)) = H^*(V, j_*(\mathscr{E}(X)))$ and $j_*(\mathscr{E}(X))$ has the usual " bête " filtration. Resolving the filtered sheaf $j_*(\mathscr{E}(X))$ gives a filtered complex (A, W) which is quasi-isomorphic to $(E_{C^\infty}(X), W)$. To obtain $(\mathscr{E}, W, F)$ one takes the log complex $\Omega(D)$ on V with two filtrations: W from the number of $\dfrac{dz_i}{z_i}$ and F from the " bête " filtration. Resolving this bifiltered sheaf gives a bifiltered complex quasi-isomorphic to $(\mathscr{E}(\log D), W, F)$, (see [4], section (3.1)). It is because we need filtered differential algebras rather than filtered complexes that we make the explicit constructions in sections 2 and 3 instead of appealing to abstract sheaf theory.

4. Principle of two types

This section is a further study of mixed Hodge diagrams. We show that the cohomology of a mixed Hodge diagram has a mixed Hodge structure. Then we examine the relationship of the filtrations and the differential. This leads to results which form the basis of the multiplicative study carried out in section 6. Throughout this section $(E, W)_{\mathbf{C}} \xrightarrow{\varphi} (\mathscr{E}, W, F)$ is a mixed Hodge diagram. We continually identify $H(E; \mathbf{C})$ with $H(\mathscr{E})$ via φ^*.

Lemma **(4.1)** (Deligne):

a) $${}_W E_2(\mathscr{E}) = {}_W E_\infty(\mathscr{E}).$$

b) $$0 \to F^{p+1}(\mathscr{E}) \to F^p(\mathscr{E}) \to \mathrm{Gr}^p_F(\mathscr{E}) \to 0$$

induces short exact sequences:

$$0 \to {}_W E_r(F^{p+1}(\mathscr{E})) \to {}_W E_r(F^p(\mathscr{E})) \to {}_W E_r(\mathrm{Gr}^p_F(\mathscr{E})) \to 0.$$

Proof. — Let $F_r({}_W E_0(\mathscr{E}))$ be the filtration induced from $F(\mathscr{E})$. By induction let $F_r({}_W E_{i+1}(\mathscr{E}))$ be the filtration induced from $F_r({}_W E_i(\mathscr{E}))$ by considering ${}_W E_{i+1}$ as the homology of ${}_W E_i$. One sees easily that if the differentials $d_0, \ldots, d_i$ are all strictly compatible with the F_r, then d_{i+1} preserves the filtration. In our case d_0 is strictly compatible with F_r, the spectral sequence has a real structure from ${}_W E_1$ on, and F_r together with the real structure defines a Hodge structure on each ${}_W E_1^{p,q}$. By induction on i we shall verify that each d_i is strictly compatible with F_r and that F_r defines a Hodge structure on ${}_W E_i$. Suppose we have proven these statements for ${}_W E_{i-1}$ and d_i. Then d_{i-1} is a map of Hodge structures and hence ${}_W E_i$ receives a Hodge structure. By the above discussion d_i is then compatible with F_r. Since d_i is real and F_r defines a Hodge structure on ${}_W E_i$, d_i must be strictly compatible with F_r. The Hodge structure on ${}_W E_i^{p,q}$ has weight q, and $d_i : {}_W E_i^{p,q} \to {}_W E_i^{p+i,q-i+1}$. Thus for $i > 1$ the Hodge structure on the range of d_i has weight less than that of the Hodge structure on the domain of d_i. Whenever this is true for a morphism of Hodge structures that morphism must be zero. Thus $d_i = 0$ for all $i > 1$, and thus ${}_W E_2(\mathscr{E}) = {}_W E_\infty(\mathscr{E})$.

The condition that the d_i all be strictly compatible with the filtrations F_r is exactly condition (7.2.2) of [14]. Applying result (7.2.5) of [14] gives the result in part b) of this lemma. (Section (7.2) of [14] can be read independently of the rest of [14].)

Proposition **(4.2)** (Deligne). — *Let* $(E, W)_{\mathbf{C}} \xrightarrow{\varphi} (\mathscr{E}, W, F)$ *be a mixed Hodge diagram. Then* $\mathscr{E} = \bigoplus_{i,j} \mathscr{E}_{i,j}$ *with* $\operatorname{Dec} W_k(\mathscr{E}) = \bigoplus_{i+j \leqq k} \mathscr{E}_{i,j}$ *and* $F^p(\mathscr{E}) = \bigoplus_{i \geqq p} \mathscr{E}_{i,j}$.

Proof. — For a filtered complex (L, W) with W bounded below, choose, for all integers n and i and for all r, $0 \leqq r \leqq \infty$, a subspace $A_r^{i,n-i} \hookrightarrow W_i(L^n)$ such that:

1) $d(A_r^{i,n-i}) \hookrightarrow W_{i-r}(L^{n+1})$, and
2) $A_r^{i,n-i} \to {}_W E_r^{i,n-i}(L)/\operatorname{Ker} d_r$ is an isomorphism.

(For $r = \infty$ we interpret d_r to be zero.)

One proves inductively on k that:

$$W_k(L^n) = \sum_{0 \leqq r \leqq \infty} \Big(\sum_{i \leqq k} A_r^{i,n-i} \oplus \sum_{i \leqq r+k} dA_r^{i,n-i-1} \Big).$$

Choose such a decomposition of $\operatorname{Gr}_F^p(\mathscr{E})$ with respect to W. Using (4.1) part b) we can lift these $A_r^{i,n-i}$ for $\operatorname{Gr}_F^p(\mathscr{E})$ to subspaces $\widetilde{A}_r^{i,n-i} \hookrightarrow Z_r^{i,n-i}(F^p(\mathscr{E}))$. The $\widetilde{A}_r^{i,n-i}$ and $d(\widetilde{A}_r^{i,n-i})$ provide a splitting of:

$$\begin{tikzcd} F^p(\mathscr{E}) \arrow[r] & \operatorname{Gr}_F^p(\mathscr{E}) \arrow[r] \arrow[l, bend right, dashed] & 0 \end{tikzcd}$$

which is compatible with W. By induction on p we prove that this implies that $(\mathscr{E}, W) = \bigoplus_i (\mathscr{E}_i, W)$ with $F^p(\mathscr{E}) = \bigoplus_{i \geqq p} \mathscr{E}_i$.

Since ${}_W E_*(\mathscr{E})$ degenerates at E_2, ${}_{\operatorname{Dec} W}E_*(\mathscr{E})$ degenerates at E_1. Thus for every i ${}_{\operatorname{Dec} W}E_*(\mathscr{E}_i)$ degenerates at E_1, *i.e.* $d : \mathscr{E}_i \to \mathscr{E}_i$ is strictly compatible with $\operatorname{Dec} W(\mathscr{E}_i)$. Thus we can write $\mathscr{E}_i = \bigoplus_j \mathscr{E}'_{i,j}$ with $\operatorname{Dec} W_k(\mathscr{E}_i) = \bigoplus_{j \leq k} \mathscr{E}'_{i,j}$. Letting $\mathscr{E}_{i,j}$ be $\mathscr{E}'_{i,j-i}$ gives the decomposition required.

Theorem **(4.3)**. — *The filtrations* Dec W *and* F *on* $H(\mathscr{E})$ *define a mixed Hodge structure when we use* φ^* *to give* $H(\mathscr{E})$ *a real structure.* $\operatorname{Dec} W_{n-1}(H^n(\mathscr{E})) = 0$.

Proof. — By definition the Hodge filtration F_r induces on ${}_W E_1^{p,q}(\mathcal{E})$ a Hodge structure of weight q. According to (4.1) the induced filtration F_r on ${}_W E_\infty$ also defines a Hodge structure. We claim that the filtration induced by $F(H(\mathcal{E}))$ on ${}_W E_\infty(\mathcal{E})$ also gives a Hodge structure of weight q. (Here we are viewing ${}_W E_\infty$ as a subquotient of $H(\mathcal{E})$.) The reason for this is that in the presence of the splitting of (4.2) the two filtrations which F induces on ${}_W E_\infty(\mathcal{E})$ agree. But ${}_W E_\infty^{p,q}(\mathcal{E}) = {}_{\mathrm{Dec}\,W} E_\infty^{-q,p+2q}$. Thus $F(H(\mathcal{E}))$ induces a Hodge structure of weight q on ${}_{\mathrm{Dec}\,W} E^{-q,*} = \mathrm{Gr}_q^W(H(\mathcal{E}))$. This is the definition of $(H(\mathcal{E}), \mathrm{Dec}\,W, F)$ being a mixed Hodge structure. Since $W_{-1}(\mathcal{E}) = 0$, we have $\mathrm{Dec}\,W_{n-1}(\mathcal{E}^n) = 0$, and hence $\mathrm{Dec}\,W_{n-1}(H^n(\mathcal{E})) = 0$.

Corollary **(4.4)**. — *An elementary equivalence between mixed Hodge diagrams induces an isomorphism of the Hodge structures induced by the Hodge filtration on* ${}_{\mathrm{Dec}\,W} E_1$, *and, in particular, equivalent mixed Hodge diagrams are quasi-isomorphic with respect to the weight filtration.*

Proof. — If $(V, W, F) \xrightarrow{\varphi} (V', W, F)$ is a map between mixed Hodge structures (*i.e.* φ is compatible with W and F) which is an isomorphism of underlying vector spaces, then φ is an isomorphism of mixed Hodge structures. Consider now an elementary equivalence between mixed Hodge diagrams. The map it induces on cohomology is an isomorphism and is compatible with Dec W and F. Thus it is an isomorphism of bifiltered cohomology. This means that the induced map on ${}_{\mathrm{Dec}\,W} E_\infty$ is an isomorphism of Hodge structures. Since ${}_{\mathrm{Dec}\,W} E_1 = {}_{\mathrm{Dec}\,W} E_\infty$, the map is also an isomorphism of the Hodge structures on the ${}_{\mathrm{Dec}\,W} E_1$-terms.

The principle of two types is a further exploitation of the splitting in (4.2) for a mixed Hodge diagram. It says that if, given a cohomology class, we can find representatives for it which are in good position with respect to the filtrations, then we have a hold over the position of the class vis-à-vis the splitting (1.8) associated with the mixed Hodge structure on cohomology. This is the result that allows us to restrict the possible homotopy types of smooth varieties.

Let $(A, W)_{\mathbf{C}} \xrightarrow{\varphi} (\mathcal{E}, W, F)$ be a mixed Hodge diagram. Define $\bar{\mathcal{E}}$ to be the differential algebra $\mathcal{E}$ with the opposite complex structure. The filtrations $W(\mathcal{E})$ and $F(\mathcal{E})$ of course define filtrations $W(\bar{\mathcal{E}})$ and $\bar{F}(\bar{\mathcal{E}})$. The map φ defines:

$$\bar{\varphi} : (A, W)_{\mathbf{C}} \to (\bar{\mathcal{E}}, W)$$

which is a quasi-isomorphism.

Definition **(4.5)**. — *a)* $R^{p,q}(\mathcal{E}) = \mathrm{Dec}\,W_{p+q}(\mathcal{E}) \cap F^p(\mathcal{E})$.

b) $L^{p,q}(\bar{\mathcal{E}}) = \mathrm{Dec}\,W_{p+q}(\bar{\mathcal{E}}) \cap \bar{F}^q(\bar{\mathcal{E}}) + \sum_{i \geq 2} \mathrm{Dec}\,W_{p+q-i}(\bar{\mathcal{E}}) \cap \bar{F}^{q-i+1}(\bar{\mathcal{E}})$.

These two subspaces are the ones refered to in the principle of two types. To be able to use the principle systematically throughout the construction of the minimal model we must understand the relation of these subspaces to wedge product and d. Also we need to understand when a cohomology class has a representative in these spaces.

Proposition **(4.6)**. — *Let* $R^{p,q}(H(\mathscr{E}))$ *be the subspace defined in* (1.8) *a) for the mixed Hodge structure induced on cohomology by* $\mathrm{Dec}\, W(\mathscr{E})$ *and* $F(\mathscr{E})$.

a) *A class* x *is in* $R^{p,q}(H(\mathscr{E}))$ *if and only if* x *has a closed representative in* $R^{p,q}(\mathscr{E})$.

b) $d : R^{p,q}(\mathscr{E}) \to R^{p,q}(\mathscr{E})$.

c) *If* $x \in R^{p,q}(\mathscr{E})$ *is exact, then* $x = dy$ *for some* $y \in R^{p,q}(\mathscr{E})$.

d) $R^{p,q}(\mathscr{E}) \otimes R^{p',q'}(\mathscr{E}') \xrightarrow{\wedge} R^{p+p',q+q'}(\mathscr{E})$.

Proposition **(4.7)**. — *Let* $L^{p,q}(H(\mathscr{E}))$ *be the subspace defined in* (1.8) *b) for the mixed Hodge structure induced on cohomology by* $\mathrm{Dec}\, W(\mathscr{E})$ *and* $F(\mathscr{E})$.

a) *A class* x *is in* $L^{p,q}(H(\mathscr{E}))$ *if and only if* x *has a closed representative in* $L^{p,q}(\bar{\mathscr{E}})$.

b) $d : L^{p,q}(\bar{\mathscr{E}}) \to L^{p,q}(\bar{\mathscr{E}})$.

c) *If* x *is in* $L^{p,q}(\bar{\mathscr{E}})$ *and* x *is exact, then* $x = dy$ *for some* $y \in L^{p,q}(\bar{\mathscr{E}})$.

d) $L^{p,q}(\bar{\mathscr{E}}) \otimes L^{p',q'}(\bar{\mathscr{E}}) \xrightarrow{\wedge} L^{p+p',q+q'}(\bar{\mathscr{E}})$.

Proof. — Let $\mathscr{E} = \bigoplus_{i,j} \mathscr{E}_{i,j}$ be a splitting as in (4.2). Then:

$$R^{p,q}(H(\mathscr{E})) = \bigoplus_{\substack{i+j \leqq p+q \\ i \geqq p}} H(\mathscr{E}_{i,j}).$$

Thus $x \in R^{p,q}(H(\mathscr{E}))$ if and only if x has a closed representative in $\bigoplus_{\substack{i+j \leqq p+q \\ i \geqq p}} \mathscr{E}_{i,j}$. Clearly this subspace is $\mathrm{Dec}\, W_{p+q}(\mathscr{E}) \cap F^p(\mathscr{E})$. Likewise, since $d : \mathscr{E} \to \mathscr{E}$ is of type (0, 0) in this bigrading, part c) of (4.2) follows immediately. Since d and $\wedge$ are compatible with both Dec W and F, parts b) and d) hold.

In proposition (4.7) we are identifying $H(\mathscr{E})$ with $H(\bar{\mathscr{E}})$ via $\varphi^* \circ (\varphi^*)^{-1}$. This result is proved by an argument similar to the one above but based on the splitting $\bar{\mathscr{E}} = \bigoplus_{i,j} \bar{\mathscr{E}}_{i,j}$.

Proposition **(4.8)**. — *If* $x \in \mathrm{Dec}\, W_{p+q}(A)$ *is exact, then* $x = dy$ *for some* $y \in \mathrm{Dec}\, W_{p+q}(A)$.

Proof. — This statement is equivalent to the statement that ${}_{\mathrm{Dec}\,W}E(A)$ degenerates at E_1, or that ${}_W E(A)$ degenerates at E_2. The map $\varphi : (A, W)_{\mathbf{C}} \to (\mathscr{E}, W)$ induces an isomorphism of spectral sequences beginning at E_1. As we have already seen:

$${}_W E_2(\mathscr{E}) = {}_W E_\infty(\mathscr{E}).$$

Corollary **(4.9)** *(The principle of two types)*:

a) *If* $\alpha \in R^{p,q}(\mathscr{E})$ *and* $\alpha' \in L^{p,q}(\bar{\mathscr{E}})$ *are closed and represent the same class in* $H(A_{\mathbf{C}})$ *(when pulled back via* $(\varphi^*)^{-1}$ *and* $(\bar{\varphi}^*)^{-1}$ *respectively), then the class that they represent is in* $A^{p,q} \subset H^n(A_{\mathbf{C}})$.

b) *Any class* $x \in \mathrm{Dec}\, W_n(H(A_{\mathbf{C}}))$ *is equal to* $(\varphi^*)^{-1}([\alpha]) + (\bar{\varphi}^*)^{-1}([\alpha'])$ *for appropriate closed forms* $\alpha \in R^{p,q}(\mathscr{E})$ *and* $\alpha' \in L^{p,q}(\bar{\mathscr{E}})$, *provided that* $p + q \geqq n$.

5. Homotopy Theory of Differential Algebras.

In this section we will give an outline of Sullivan's theory of homotopy type for differential algebras and its connection with usual homotopy theory for spaces. For other accounts of this theory, see [11] and [5].

Definition (5.1). — A *Hirsch extension* of a differential algebra $\mathscr{A}$ is an inclusion $\mathscr{A} \hookrightarrow \mathscr{B}$ of differential algebras which, when we ignore the differentials, is isomorphic to $\mathscr{A} \hookrightarrow \mathscr{A} \otimes \Lambda(\mathrm{V})_k$, and such that the differential of $\mathscr{B}$ sends $\mathrm{V} \to \mathscr{A}^{k+1}$. The integer k is the *degree* of the extension. A Hirsch extension is of *finite dimension* if V is of finite dimension.

Note. — The differential of $\mathscr{B}$, d, is determined by the differential of $\mathscr{A}$ and by $d|\mathrm{V}$. If $\mathscr{A}$ is a free algebra, then so is $\mathscr{B}$.

Definition (5.2). — A differential algebra $\mathscr{M}$ is a *minimal algebra* if:

a) it is connected,

b) it is an increasing union of sub-differential algebras:

$$\text{ground field} = \mathscr{M}_0 \subset \mathscr{M}_1 \subset \mathscr{M}_2 \subset \ldots$$

with $\mathscr{M}_i \subset \mathscr{M}_{i+1}$ a Hirsch extension, and

c) the differential of $\mathscr{M}$, d, is decomposable, *i.e.* $d : \mathrm{I}(\mathscr{M}) \to \mathrm{I}(\mathscr{M})$ is zero.

The sequence of subalgebras in *b)* is called a *series* for $\mathscr{M}$. It is a *finite dimensional series* if each Hirsch extension is of finite dimension.

Condition *a)* is equivalent to requiring that each extension be of positive degree.

Note. — If $\mathscr{M} \subset \mathscr{M} \otimes_d \Lambda(\mathrm{V})$ is a Hirsch extension with $\mathscr{M}$ minimal, and V is homogeneous, then $\mathscr{M} \otimes_d \Lambda(\mathrm{V})$ is minimal if and only if $d : \mathrm{V} \to \mathscr{M}$ sends V to decomposable elements in $\mathscr{M}$.

Definition (5.3). — Let $\mathscr{A}$ be a differential algebra. An *i-minimal model for* $\mathscr{A}$ is a map $\rho : \mathscr{M} \to \mathscr{A}$ of differential algebras such that:

a) $\mathscr{M}$ is minimal,

b) $\mathrm{I}(\mathscr{M}) = 0$ in degree $\geq i+1$, *i.e.* each Hirsch extension in a series for $\mathscr{M}$ has degree $\leq i$, and

c) $\rho^* : \mathrm{H}(\mathscr{M}) \to \mathrm{H}(\mathscr{A})$ is an isomorphism in degrees $\leq i$ and injective in degree $(i+1)$.

In case $i = \infty$, $\rho : \mathscr{M} \to \mathscr{A}$ is a *minimal model for* $\mathscr{A}$.

Recall from section 2 that a homotopy from $f_0 : \mathscr{A} \to \mathscr{B}$ to $f_1 : \mathscr{A} \to \mathscr{B}$ is a map $\mathrm{H} : \mathscr{A} \to \mathscr{B} \otimes (t, dt)$ with $\mathrm{H}|_i = f_i$ for $i = 0$ and 1. From the homotopy H we can construct a chain homotopy between the f_0 and f_1 on the underlying cochain complexes. If $b \in \mathscr{B} \otimes (t, dt)$ is of degree n:

$$b = \sum_{i \geqq 0} (\beta_i \otimes t^i + \gamma_i \otimes t^i \, dt,)$$

then define $\int_0^t b$ to be $\sum_{i \geqq 0} (-1)^{n-1} \frac{\gamma_i}{(i+1)} t^{i+1}$. It is an element of $\mathscr{B}$. The sign $(-1)^{n-1}$ enters because we are moving a degree -1 operator passed a form of degree $(n-1)$.

Proposition (5.5). — *Let* $\mathrm{H} : \mathscr{A} \rightarrow \mathscr{B} \otimes (t, dt)$ *be a homotopy from* f_0 *to* f_1. *Then*:

$$\int_0^1 d\mathrm{H}(\alpha) + d\left(\int_0^1 \mathrm{H}(\alpha)\right) = f_1(\alpha) - f_0(\alpha).$$

Proof. — Let $\alpha \in \mathscr{A}^n$, and $\mathrm{H}(\alpha) = \sum_{i \geqq 0} (\beta_i \otimes t^i + \gamma_i \otimes t^i \, dt)$.

$$\int_0^1 d\mathrm{H}(\alpha) = \int_0^1 \sum_{i \geqq 0} (d\beta_i \otimes t^i + (-1)^n \beta_i \otimes i t^{i-1} \, dt + d\gamma_i \otimes t^i \, dt)$$

$$= \sum_{i \geqq 0} \left\{ \int_0^1 ((-1)^n i \beta_i \otimes t^{i-1} \, dt) + \int_0^1 (d\gamma_i \otimes t^i \, dt) \right\}$$

$$= \sum_{i \geqq 1} \beta_i + \sum_{i \geqq 0} (-1)^n \frac{d\gamma_i}{i+1}.$$

$$d\int_0^1 \mathrm{H}(\alpha) = d\left(\sum_{i \geqq 0} (-1)^{n-1} \frac{\gamma_i}{i+1}\right) = \sum_{i \geqq 0} (-1)^{n-1} \frac{d\gamma_i}{i+1}.$$

Thus $\int_0^1 d\mathrm{H}(\alpha) + d\int_0^1 d\mathrm{H}(\alpha) = \sum_{i \geqq 1} \beta_i$.

Since $f_1(\alpha) = \sum_{i \geqq 0} \beta_i$ and $f_0(\alpha) = \beta_0$, this proves the proposition.

Theorem (5.6) ([11]). — *If* $\mathscr{A}$ *is a differential algebra, then* $\mathscr{A}$ *has an i-minimal model for any* $i \geq 0$, $\rho : \mathscr{M} \rightarrow \mathscr{A}$. *Furthermore, given two i-minimal models*:

$$\begin{tikzcd} \mathscr{M} \arrow[d, dashed, "\mathrm{I}"'] \arrow[dr, "\rho"] & \\ \mathscr{M}' \arrow[r, "\rho'"'] & \mathscr{A} \end{tikzcd}$$

then there is an isomorphism $\mathrm{I} : \mathscr{M} \xrightarrow{\cong} \mathscr{M}'$ *and a homotopy from* ρ *to* $\rho' \circ \mathrm{I}$. *The isomorphism* I *is itself unique up to homotopy.*

We will give a sketch of the main steps in both the construction and in the proof of uniqueness. We begin with a construction which is based on the idea of relative cochains and relative cohomology for a map.

Given a map $f : \mathscr{A} \to \mathscr{B}$ of differential algebras we can form the cochain complex which is the mapping cone complex for f:

$$C^n = \mathscr{A}^n \oplus \mathscr{B}^{n-1}$$

with $d : C^n \to C^{n+1}$ defined by $d(a, b) = (-d(a), d(b) + f(a))$. Its cohomology is by definition $H(\mathscr{A}, \mathscr{B})$. The maps of chain complexes $\mathscr{B}^{*-1} \overset{i_2}{\hookrightarrow} C^*$, and $C^* \xrightarrow{-\pi_1} \mathscr{A}^*$ induce maps on cohomology which fit into the long exact sequence for the pair:

$$\ldots \to H^i(\mathscr{A}) \to H^i(\mathscr{B}) \xrightarrow{i_2^*} H^{i+1}(\mathscr{A}, \mathscr{B}) \xrightarrow{-\pi_1^*} H^{i+1}(\mathscr{A}) \xrightarrow{f^*} H^{i+1}(\mathscr{B}) \to \ldots$$

Suppose we have an $(i-1)$-minimal model for $\mathscr{A}$, $u : \mathscr{N} \to \mathscr{A}$; then we will construct an i-minimal model $\rho : \mathscr{M} \to \mathscr{A}$ which contains $\mathscr{N}$.

The steps are the following:

I: Let $V = H^{i+1}(\mathscr{N}, \mathscr{A})$. ($H^j(\mathscr{N}, \mathscr{A}) = 0$ for $j < i+1$.)

II: *Choose* a splitting, $V \overset{s}{\dashrightarrow} \mathscr{Z}^{i+1}(\mathscr{N}, \mathscr{A})$, for the natural map:

$$\mathscr{Z}^{i+1}(\mathscr{N}, \mathscr{A}) \to H^{i+1}(\mathscr{N}, \mathscr{A}) = V.$$

($\mathscr{Z}^*(\mathscr{N}, \mathscr{A})$ is the space of cocycles.)

III: Define $\mathscr{M}_1 = \mathscr{N} \otimes \Lambda(V)_i$ with $d|V : V \to \mathscr{N}$ given by:

$$V \xrightarrow{s} \mathscr{Z}^{i+1}(\mathscr{N}, \mathscr{A}) \xrightarrow{-\pi_1} \mathscr{Z}^{i+1}(\mathscr{N}).$$

IV: Define $\rho_1 : \mathscr{M}_1 \to \mathscr{A}$ by setting $\rho|N$ equal to u and $\rho|V$ equal to the composition:

$$V \xrightarrow{s} \mathscr{Z}^{(i+1)}(\mathscr{N}, \mathscr{A}) \xrightarrow{\pi_2} \mathscr{A}.$$

Then, in $\mathscr{M}_1$, d is decomposable, since d maps V into $\mathscr{N}^{i+1}$ and all the indecomposables for $\mathscr{N}$ are in degrees $\leq i$ (actually $< i$). By construction $\rho_1^* : H^{i+1}(\mathscr{N}, \mathscr{M}_1) \to H^{i+1}(\mathscr{N}, \mathscr{A})$ is an isomorphism; also $H^j(\mathscr{N}, \mathscr{M}_1) = 0$ for $j < i+1$. One shows easily that

$$\rho_1^* : H^i(\mathscr{M}_1) \to H^i(\mathscr{A})$$

is an isomorphism and that $\operatorname{kernel}(u^*) \subset H^{i+1}(\mathscr{N})$ goes to zero in $H^{i+1}(\mathscr{M}_1)$. I $H^1(\mathscr{A}) = 0$, then $\mathscr{N}$ will have no generators in degree 1. As a consequence the forms of degree $i+1$ in $\mathscr{M}_1$ are the same as those of degree $i+1$ in $\mathscr{N}$. Thus

$$\rho_1^* : H^{i+1}(\mathscr{M}_1) \to H^{i+1}(\mathscr{A})$$

will be an injection in this case. $\mathscr{M}_1$ is then an i-minimal model for $\mathscr{A}$.

If $\mathscr{N}$ has generators in degree one, then $\rho_1^* : H^{i+1}(\mathscr{M}_1) \to H^{i+1}(\mathscr{A})$ may not be injective. We repeat the above argument with $\mathscr{M}_1$ replacing $\mathscr{N}$. This constructs $\mathscr{M}_2 \xrightarrow{\rho_2} \mathscr{A}$. $\mathscr{M}_2$ still has a decomposable d, and $\operatorname{kernel}(\rho_1^*) \subset H^{i+1}(\mathscr{M}_1)$ goes to zero in $H^{i+1}(\mathscr{M}_2)$. We continue in this fashion building $\mathscr{M}_{j+1}$ from $\mathscr{M}_j$. Let $\mathscr{M}$ be the limit (*i.e.* union) of the $\mathscr{M}_j$, and let $\rho : \mathscr{M} \to \mathscr{A}$ be defined by $\rho|\mathscr{M}_j = \rho_j$. Then $\rho : \mathscr{M} \to \mathscr{A}$ satisfies all the properties required of an i-minimal model except that $\mathscr{M}$ is a union of Hirsch extensions indexed by the ordinal of order type 2ω rather than one of order type ω (*i.e.* a sequence). The algebra can, however, be represented as a sequence of Hirsch extensions. Unfortunately, the latter representation is non-canonical, but nevertheless it does show that $\rho:\mathscr{M}\to\mathscr{A}$ is an i-minimal model. For $i=\infty$, we take the union of the i-minimal models constructed above for each $i<\infty$. Again, in case $H^1(\mathscr{A})\neq 0$, we must rearrange to get a sequence of Hirsch extensions.

Let $\rho:\mathscr{M}\to\mathscr{A}$ and $\rho':\mathscr{M}'\to\mathscr{A}$ be i-minimal models for $i\leqq\infty$. The proof of uniqueness proceeds by induction on some series for $\mathscr{M}$. For the inductive step we need an obstruction theory for lifting up to homotopy.

Theorem **(5.7)** ([5]). — *Let $\mathscr{M}\subset\mathscr{M}'$ be a Hirsch extension,* i.e. $\mathscr{M}'\cong\mathscr{M}\otimes_d\Lambda(V)_n$. *Let*:

$$\begin{array}{ccc}\mathscr{M} & \xrightarrow{f} & \mathscr{A}\\ \Big\downarrow & & \Big\downarrow{\scriptstyle\varphi}\\ \mathscr{M}' & \xrightarrow{f'} & \mathscr{B}\end{array}$$

be a homotopy commutative diagram with H *a homotopy from $\varphi\circ f$ to $f'|\mathscr{M}$.*

a) *There is one obstruction* $\mathcal{O}\in\mathrm{Hom}(V, H^{n+1}(\mathscr{A},\mathscr{B}))=H^{n+1}(\mathscr{A},\mathscr{B};V^*)$ *to extending f to a map $\tilde f:\mathscr{M}\to\mathscr{A}$, with $\varphi\circ\tilde f$ homotopic to f' by a homotopy extending* H.

b) *The obstruction is given by*:

$$\mathcal{O}(v)=\Big[f(dv),\ -f'(v)+\int_0^1 H(dv)\Big]\in H^{n+1}(\mathscr{A},\mathscr{B}).$$

c) *If the original diagram commutes, and if φ is onto, then $\mathcal{O}$ is the obstruction to finding $\tilde f:\mathscr{M}'\to\mathscr{A}$ extending f and with $\varphi\circ\tilde f=f'$.*

Proof. — Let $\alpha(v)$ denote the relative cochain in $C(\mathscr{A},\mathscr{B})$:

$$\Big(f(dv),\ -f'(v)+\int_0^1 H(dv)\Big).$$

Then:

$$\begin{aligned} d\alpha(v)&=\Big(-df(dv),\ -f'(dv)+d\int_0^1 H(dv)+\varphi f(dv)\Big)\\ &=\Big(0,\ -f'(dv)+\varphi f(dv)+d\int_0^1 H(dv)\Big)\\ &=\Big(0,\ -\int_0^1 dH(dv)\Big)\\ &=(0,0).\end{aligned}$$

Thus $\alpha(v)$ is a relative cocycle.

If the cohomology class of $\alpha(v)$ is zero for all $v\in V$, then pick (linearly in v) elements ω_v with $d\omega_v=\alpha(v)$. Let $\omega_v=(a_v, b_v)$, with $a_v\in\mathscr{A}^n$ and $b_v\in\mathscr{B}^{n-1}$.

We define extensions $\tilde f:\mathscr{M}'\to\mathscr{A}$ of f and $\tilde H:\mathscr{M}'\to\mathscr{B}\otimes(t, dt)$ of H by:

(5.8) $$f(v)=-a_v \quad\text{and}\quad \tilde H(v)=\varphi\tilde f(v)+\int_0^t H(dv)-d(b_v\otimes t).$$

Then:

$$\begin{aligned}
d\widetilde{f}(v) &= -da_v = f(dv)\,; \\
\widetilde{\mathrm{H}}(v)|_{t=0} &= \varphi\widetilde{f}(v)\,; \\
\widetilde{\mathrm{H}}(v)|_{t=1} &= \varphi\widetilde{f}(v) + \int_0^1 \mathrm{H}(dv) - db_v \\
&= \varphi\widetilde{f}(v) + \int_0^1 \mathrm{H}(dv) + f'(v) - \int_0^1 \mathrm{H}(dv) - \varphi\widetilde{f}(v) \\
&= f'(v)\,; \\
d\widetilde{\mathrm{H}}(v) &= \varphi\widetilde{f}(dv) + d\int_0^1 \mathrm{H}(dv) = \mathrm{H}(dv).
\end{aligned}$$

Thus $\widetilde{f}$ and $\widetilde{\mathrm{H}}$ are maps of differential algebras extending f and H. In addition $\widetilde{\mathrm{H}}$ is a homotopy from $\varphi\circ\widetilde{f}$ to f'. If the original diagram commutes, then we take $\mathrm{H} : \mathcal{M} \to \mathcal{B}\otimes(t, dt)$ to be the constant homotopy, $\mathrm{H}(x) = \varphi\circ\widetilde{f}(x)\otimes 1$. If $\alpha(v)$ is exact for all $v\in\mathrm{V}$, and φ is onto, then we can pick $\omega_v = (a_v, 0)$ with $d\omega_v = \alpha(v)$. With these choices (5.8) defines $\widetilde{f}$ with $\varphi\circ\widetilde{f} = f$.

If we have two i-minimal models for $\mathcal{A}$:

$$\begin{tikzcd}
\mathcal{M}' \arrow[dr, "\rho'"] & \\
& \mathcal{A} \\
\mathcal{M} \arrow[ur, "\rho"'] &
\end{tikzcd}$$

we apply (5.7) to prove (5.6). Since the relative cohomology of $(\mathcal{M}', \mathcal{A})$ vanishes in degrees less than or equal to $i+1$, we meet no obstruction to lifting $\rho : \mathcal{M} \to \mathcal{A}$ to a map $\mathrm{I} : \mathcal{M} \to \mathcal{M}'$ with $\rho'\circ\mathrm{I}$ homotopic to ρ until we get to the generators of $\mathcal{M}$ in degrees $>i$. Since $\mathcal{M} \xrightarrow{\rho} \mathcal{A}$ is an i-minimal model it has no generators in degrees $>i$. Thus we can construct a map $\mathrm{I} : \mathcal{M} \to \mathcal{M}'$ with $\rho'\circ\mathrm{I}$ homotopic to ρ. To prove that I is well defined up to homotopy one applies the appropriate relative version of (5.7). To prove that I is an isomorphism we use the following

Proposition **(5.9)** ([11]). — *Let $\mathcal{M}$ and $\mathcal{M}'$ be minimal algebras generated in degrees $\leqq i$. If $\mathrm{I} : \mathcal{M} \to \mathcal{M}'$ is an isomorphism on cohomology in degrees $\leqq i$ and an injection in degree $i+1$, then* I *is an isomorphism of minimal algebras.*

This result is proved by a straightforward induction on dimension.

If a minimal model is 1-connected, then it has no generators in degree 1. If in addition its cohomology in each degree is finite dimensional, then it has a canonical series:

$$\text{ground field} \hookrightarrow \mathcal{M}_1 \hookrightarrow \mathcal{M}_2 \hookrightarrow \dots$$

with $\mathcal{M}_i \hookrightarrow \mathcal{M}_{i+1}$ a finite Hirsch extension of degree $i+1$. In fact $\mathcal{M}_i$ is defined as the subalgebra generated in degrees $\leqq i$. It is easily seen to be a free algebra and closed under d. The second condition requires the decomposability of d.

If the minimal model $\mathcal{M}$ is not 1-connected, then its sub-differential algebra generated by all the elements in degree 1 has a canonical series. This series is a series of finite Hirsch extensions all of degree 1 if $H^1(\mathcal{M})$ is finite dimensional:

$$\text{ground field} \hookrightarrow \mathcal{M}_1 \hookrightarrow \mathcal{M}_2 \hookrightarrow \dots$$

This time $\mathcal{M}_1$ is defined to be the sub-differential algebra generated by closed 1-forms; $\mathcal{M}_{i+1}$ is the sub-differential algebra generated by $\mathcal{M}_i$ and all 1-forms x such that $dx \in \mathcal{M}_i$. We can extend this to a canonical series for all of $\mathcal{M}$. There are two drawbacks, however. One is that the indexing set is the ordinal ω^2, and the other ist he fact that, even if the cohomology of $\mathcal{M}$ is finite dimensional, the Hirsch extensions may become infinite dimensional after we pass degree 1. The subalgebras are defined by:

$$\mathcal{M}_{k\omega} = \text{sub-differential algebra generated in degrees} \leq k,$$

and: $$\mathcal{M}_{k\omega+\ell} = \text{sub-differential algebra generated by } \mathcal{M}_{k\omega+(\ell-1)} \text{ and all } x \text{ of degree } k+1 \text{ such that } dx \in \mathcal{M}_{k\omega+(\ell-1)}.$$

Then $\mathcal{M}_\alpha \subset \mathcal{M}_{\alpha+1}$ is a Hirsch extension, and $\mathcal{M}_{r\omega} = \bigcup_{\alpha < r\omega} \mathcal{M}_\alpha$.

This completes our discussion of the purely algebraic side of Sullivan's theory. The bridge from this to homotopy theory lies in the connection between finite Hirsch extensions and principal, rational fibrations. Let X be a simplicial complex, and let $f : \mathcal{A} \to \mathcal{E}(X)$ induce an isomorphism on cohomology. There is a natural one-to-one correspondence between finite dimensional Hirsch extensions of $\mathcal{A}$ and principal fibrations having X as base and an Eilenberg-Mac-Lane space $K(\pi, n)$, with π a rational vector space, as fiber. Under this correspondence $\mathcal{A} \otimes_d \Lambda(V)_k$ is associated to the fibration $K(V^*, k) \to E \xrightarrow{p} X$ with k-invariant in $H^{k+1}(X, V^*)$ given by the homomorphism $d : V \to H^{k+1}(\mathcal{A})$ under the natural identifications:

$$\mathrm{Hom}(V, H^{k+1}(\mathcal{A})) = H^{k+1}(\mathcal{A}; V^*) \xrightarrow{f^*} H^{k+1}(X; V^*).$$

Furthermore there is a map $\mathcal{A} \otimes_d \Lambda(V) \to \mathcal{E}(E)$ which extends the map $\mathcal{A} \xrightarrow{p^* \circ f} E(\mathcal{E})$ and which induces an isomorphism on cohomology. Thus a series of finite Hirsch extensions corresponds to a tower of rational, principal fibrations. This connection is more than a formal one, as the next series of results show. The reference for these is [11].

(5.10) *Let* X *be a polyhedron with* $\pi_1(X) = \{e\}$ *and* $H^i(X; \mathbf{Q})$ *finite dimensional for every i. Let* $\mathcal{E}(X)$ *be the* $\mathbf{Q}$*-polynomial forms and* $\mathcal{M}_X$ *its minimal model. Let* $\mathcal{M}_2 \subset \mathcal{M}_3 \subset \dots$ *be the canonical series for* $\mathcal{M}$. *Then*:

a) *the tower of principal, rational fibrations determined by this series is the rational Postnikov tower for* X. *In particular*:
b) *the indecomposables,* $I(\mathcal{M})$, *are dual to the rational homotopy groups of* X;
c) *the cohomology of* $\mathcal{M}_i$ *is equal the cohomology of the i-th stage in the rational Postnikov tower for* X, $H^*(X_i)$;

d) *the rational k-invariants of* X, $k^{i+1} \in \mathrm{H}^{i+1}(\mathrm{X}_{i-1}; \pi_i(\mathrm{X}) \otimes \mathbf{Q})$ *are the elements given by* $d : \mathrm{I}^i(\mathscr{M}) \to \mathrm{H}^{i+1}(\mathscr{M}_{i-1})$ *under the equality*:

$$\begin{aligned}\mathrm{Hom}(\mathrm{I}^i(\mathscr{M}), \mathrm{H}^{i+1}(\mathscr{M}_{i-1})) &= \mathrm{Hom}((\pi_i(\mathrm{X}) \otimes \mathbf{Q})^*, \mathrm{H}^{i+1}(\mathrm{X}_{i-1})) \\ &= \mathrm{Hom}(\mathrm{H}_{i+1}(\mathrm{X}_{i-1}), \pi_i(\mathrm{X}) \otimes \mathbf{Q}) = \mathrm{H}^{i+1}(\mathrm{X}_{i-1}, \pi_i(\mathrm{X}) \otimes \mathbf{Q}),\end{aligned}$$

and

e) *the Whitehead product*:

$$\sum_{i+j=k} (\pi_i(\mathrm{X}) \otimes \mathbf{Q}) \otimes (\pi_j(\mathrm{X}) \otimes \mathbf{Q}) \to \pi_{k-1}(\mathrm{X}) \otimes \mathbf{Q}$$

is dual to the map induced by d, $d : \mathrm{I}^{k-1}(\mathscr{M}) \to (\mathrm{I}(\mathscr{M}) \wedge \mathrm{I}(\mathscr{M}))^k$.

In the case that X is non-simply connected the results are more complicated, but again of the same nature. The statement is that the minimal model for the $\mathbf{Q}$-polynomial forms determines the rational nilpotent completion of X. Before describing what this means in general, let us concentrate on the fundamental group. We form the lower central series for $\pi_1(\mathrm{X})$:

$$\ldots \subset \Gamma_3 \subset \Gamma_2 \subset \pi_1(\mathrm{X})$$

where $\Gamma_2 = [\pi_1(\mathrm{X}), \pi_1(\mathrm{X})]$ and $\Gamma_{i+1} = [\pi_1(\mathrm{X}), \Gamma_i]$. Taking quotients defines the nilpotent completion of $\pi_1(\mathrm{X})$. It is the tower of nilpotent groups:

$$\ldots \to \pi_1(\mathrm{X})/\Gamma_3 \to \pi_1(\mathrm{X})/\Gamma_2 \to \{e\}.$$

Each $\pi_1(\mathrm{X})/\Gamma_n$ is a nilpotent group of index n. It is a central extension of $\pi_1(\mathrm{X})/\Gamma_{n-1}$ by the abelian group Γ_{n+1}/Γ_n. It is possible to " tensor " these nilpotent groups with $\mathbf{Q}$. This gives a tower of rational nilpotent Lie groups called the rational nilpotent completion of $\pi_1(\mathrm{X})$ ([7], [1] and [5]). The 1-minimal model for $\mathscr{E}(\mathrm{X})$ has a canonical series, $\mathbf{Q} \subset \mathscr{M}_1 \subset \mathscr{M}_2 \subset \ldots$, with each $\mathscr{M}_i$ generated in degree 1. By dualizing we get a tower of $\mathbf{Q}$-Lie algebras:

$$\ldots \to \mathscr{L}_2 \to \mathscr{L}_1 \to 0.$$

Each $\mathscr{L}_{i+1}$ is a central extension of $\mathscr{L}_i$. Sullivan's result is that:

(5.11) *This tower of rational Lie algebras is the tower of nilpotent Lie algebras associated to the rational nilpotent completion of* $\pi_1(\mathrm{X})$.

Note that since the Lie algebras and Lie groups are nilpotent, the Campbell-Hausdorff formula and its inverse for determining one from the other become $\mathbf{Q}$-polynomials. Thus, knowing the tower of rational Lie algebras is equivalent to knowing the tower of rational Lie groups.

In general the minimal model for $\mathscr{E}(\mathrm{X})$ has a series of finite dimensional extensions; in fact it has many. Each such series determines a tower of rational principal fibrations to which X maps, $\mathrm{X} \xrightarrow{f} \{\mathrm{Y}_1 \leftarrow \mathrm{Y}_2 \leftarrow \ldots\}$ with $f^* : \mathrm{H}(\mathrm{X}; \mathbf{Q}) \to \varinjlim \{\mathrm{H}(\mathrm{Y}_i; \mathbf{Q})\}$ an isomorphism. The Kan rational nilpotent completion of X [1] is the category of all rational towers to which X maps inducing an isomorphism on cohomology. The ones which can be constructed from series for the minimal model of $\mathscr{E}(X)$ form a cofinal subcategory. Thus:

(5.12) *The minimal model for* $\mathscr{E}(X)$ *determines the Kan rational nilpotent completion of* X.

6. A Bigrading in Complex Homotopy Theory.

Any mixed Hodge structure (V, W, F) induces a bigrading on $V_{\mathbf{C}}$ (1.8). Thus the cohomology of any mixed Hodge diagram has a bigrading. We use the principle of two types to extend this to a bigrading of the complex minimal model of a mixed Hodge diagram. This bigrading will be stronger than the one in section 4 because it will be a multiplicative one.

Throughout this section $(A, W) \xrightarrow{\varphi} (\mathscr{E}, W, F)$ is a mixed Hodge diagram, and $(\bar{\mathscr{E}}, W, \bar{F})$ and $\bar{\varphi}$ are the same real objects with the conjugate complex structures.

A bigraded differential algebra is a decomposition:

$$\mathscr{M} = \bigoplus_{0 \leq r, s} \mathscr{M}^{r,s}$$

with $\mathscr{M}^{0,0}$ = ground field, and with d and wedge product of type (0, 0). A morphism from a bigraded algebra to $\mathscr{E} \xleftarrow{\varphi} A_{\mathbf{C}} \xrightarrow{\bar{\varphi}} \bar{\mathscr{E}}$ is a diagram:

$$\begin{tikzcd}
\mathscr{E} & A_{\mathbf{C}} \arrow[l, "\varphi"'] \arrow[r, "\bar{\varphi}"] & \bar{\mathscr{E}} \\
 & \mathscr{M} \arrow[ul, "\psi"] \arrow[u, "\rho"] \arrow[ur, "\psi'"'] &
\end{tikzcd}$$

and homotopies $H : \mathscr{M} \to \mathscr{E} \otimes (t, dt)$ and $H' : \mathscr{M} \to \bar{\mathscr{E}} \otimes (t, dt)$ from $\varphi \circ \rho$ to ψ and $\bar{\varphi} \circ \rho$ to ψ' respectively such that:

(6.0)
$$\begin{cases} 1)\ \rho(\mathscr{M}^{r,s}) \subset \operatorname{Dec} W_{r+s}(A),\ \psi(\mathscr{M}^{r,s}) \subset R^{r,s}(\mathscr{E}), \text{ and } \psi'(\mathscr{M}^{r,s}) \subset L^{r,s}(\bar{\mathscr{E}}), \\ \text{and} \\ 2)\ H(\mathscr{M}^{r,s}) \subset \operatorname{Dec} W_{r+s}(\mathscr{E} \otimes (t, dt)), \text{ and } H'(\mathscr{M}^{r,s}) \subset \operatorname{Dec} W_{r+s}(\bar{\mathscr{E}} \otimes (t, dt)). \end{cases}$$

(Here we extend $W(\mathscr{E})$ to $\mathscr{E} \otimes (t, dt)$ by defining $W_k(\mathscr{E} \otimes (t, dt)) = W_k(\mathscr{E}) \otimes (t, dt)$.)

(6.1) If in addition $\rho : \mathscr{M} \to A_{\mathbf{C}}$ is a minimal model for $A_{\mathbf{C}}$, then $(\mathscr{M}, \psi, \rho, \psi', H, H')$ *is a bigraded minimal model* for the mixed Hodge diagram. A *homotopy* between two morphisms is a morphism from the bigraded algebra to the mixed Hodge diagram $\mathscr{E} \otimes (s, ds) \leftarrow A_{\mathbf{C}} \otimes (s, ds) \to \bar{\mathscr{E}} \otimes (s, ds)$ which restricts at $s = 0$ and $s = 1$ to the two morphisms in question. An *equivalence* between bigraded minimal models is an isomorphism between the bigraded minimal algebras (preserving the bigrading) and a homotopy between the two morphisms of the source of this isomorphism into the mixed Hodge diagram. The main result of this section is that any mixed Hodge diagram has a functorial bigraded minimal model unique up to equivalence. The proof proceeds by induction over the canonical series for the minimal model. If $H^1(A) \neq 0$, then this series is indexed by the ordinal ω^2. To sustain the induction we must, upon reaching a limit ordinal, show the inductive properties hold for the union of what has preceeded. This will always be clear and left unmentioned. The proof of existence, uniqueness, and functoriality occupies the rest of this section.

We begin by considering a bigraded differential algebra $\mathscr{M}_\alpha = \bigoplus_{r,s} \mathscr{M}_\alpha^{r,s}$ and a morphism, $\mathscr{M}_\alpha \xrightarrow{P} \mathscr{H}$, of it into a mixed Hodge diagram. We are interested in the cochain complex of the pair $(\mathscr{M}_\alpha, \mathscr{H})$. To fix the notation, let $\mathscr{H}$ be $\mathscr{E} \xleftarrow{\varphi} A_{\mathbf{C}} \xrightarrow{\bar{\varphi}} \bar{\mathscr{E}}$ and P be the diagram:

$$\begin{tikzcd}
 & & \mathscr{E} \\
\mathscr{M}_\alpha \arrow[urr, "\psi"] \arrow[rr, "\rho"] \arrow[drr, "\psi'"'] \arrow[urr, phantom, "H"', shift right=3ex] \arrow[drr, phantom, "H'", shift left=3ex] & & A_{\mathbf{C}} \arrow[u, "\varphi"'] \arrow[d, "\bar{\varphi}"] \\
 & & \bar{\mathscr{E}}
\end{tikzcd}$$

For the maps ψ, ρ, and ψ' we have relative cochain complexes. The maps φ and $\bar{\varphi}$ together with the homotopies H and H′ induce maps between these complexes. Define:

$$C^*(\mathscr{M}_\alpha, \mathscr{E})_\psi \xleftarrow{\ j\ } C^*(\mathscr{M}_\alpha, A_{\mathbf{C}})_\rho \xrightarrow{\ j'\ } C^*(\mathscr{M}_\alpha, \bar{\mathscr{E}})_{\psi'}$$

by:
$$\left(m, \varphi(a) - \int_0^1 H(m)\right) \leftarrow\!\shortmid (m, a) \mapsto \left(m, \bar{\varphi}(a) - \int_0^1 H'(m)\right).$$

Lemma **(6.2)**. — a) *The maps j and j' induce maps of cochain complexes. The induced maps on relative cohomology groups are isomorphisms and fit into a commutative diagram:*

$$\begin{tikzcd}
{} \arrow[d, dashed] & {} \arrow[d, dashed] & {} \arrow[d, dashed] \\
H^\ell(\mathscr{M}_\alpha) \arrow[r, equal] \arrow[d, "\psi^*"] & H^\ell(\mathscr{M}_\alpha) \arrow[r, equal] \arrow[d, "\rho^*"] & H^\ell(\mathscr{M}_\alpha) \arrow[d, "\bar{\psi}^*"] \\
H^\ell(\mathscr{E}) \arrow[d, "i_2^*"] & H^\ell(A_{\mathbf{C}}) \arrow[l, "\varphi^*"'] \arrow[r, "\bar{\varphi}^*"] \arrow[d, "i_2^*"] & H^\ell(\bar{\mathscr{E}}) \arrow[d, "i_2^*"] \\
H^{\ell+1}(\mathscr{M}_\alpha, \mathscr{E}) \arrow[d, "-\pi_1^*"] & H^{\ell+1}(\mathscr{M}_\alpha, A_{\mathbf{C}}) \arrow[l, "j^*"'] \arrow[r, leftrightarrow, "j'^*"] \arrow[d, "-\pi_1^*"] & H^{\ell+1}(\mathscr{M}_\alpha, \bar{\mathscr{E}}) \arrow[d, "-\pi_1^*"] \\
H^{\ell+1}(\mathscr{M}_\alpha) \arrow[r, equal] & H^{\ell+1}(\mathscr{M}_\alpha) \arrow[r, equal] & H^{\ell+1}(\mathscr{M}_\alpha)
\end{tikzcd}$$

b) *If we use the mixed Hodge structure on* $\mathrm{H}(\mathrm{A})$ *to define a bigrading for*

$$\mathrm{H}(\mathrm{A}_{\mathbf{C}})=\bigoplus_{r,s}\mathrm{A}^{r,s}(\mathrm{H}(\mathrm{A}_{\mathbf{C}})),$$

then ρ^* *sends* $\mathrm{H}(\mathscr{M}_\alpha^{r,s})$ *to* $\mathrm{A}^{r,s}(\mathrm{H}(\mathrm{A}_{\mathbf{C}}))$.

Proof. — *a)* By symmetry it suffices to consider only j. Let m be an element of $\mathscr{M}^n$ and a be in $(\mathrm{A}_{\mathbf{C}})^{n-1}$. Then:

$$dj(m,a)=d\Big(m,\varphi(a)-\int_0^1\mathrm{H}(m)\Big)=\Big(-dm,\varphi(da)-d\int_0^1\mathrm{H}(m)+\psi(m)\Big);$$

$$jd(m,a)=j(-dm,da+\rho(m))=\Big(-dm,\varphi(da)+\varphi\rho(m)+\int_0^1\mathrm{H}(dm)\Big).$$

By (5.5) these are equal. Thus j is a map of cochain complexes. It follows immediately that the diagram commutes. The five lemma then implies that j^* is an isomorphism.

b) The cohomology of $\mathscr{M}_\alpha$ has a bigrading since d maps $\mathscr{M}_\alpha^{r,s}$ to $\mathscr{M}_\alpha^{r,s}$. The existence of the homotopies H and H′ shows that $\varphi^*\rho^*=\psi^*$ and that $\bar\varphi^*\rho^*=(\psi')^*$. By (6.0) $\rho^*:\mathrm{H}(\mathscr{M}_\alpha^{r,s})\to(\varphi^*)^{-1}(\mathrm{R}^{r,s}(\mathrm{H}(\mathscr{E})))\cap(\bar\varphi^*)^{-1}(\mathrm{L}^{r,s}(\mathrm{H}(\bar{\mathscr{E}})))$. By the definition of the mixed Hodge structure on $\mathrm{H}(\mathrm{A})$, this latter space is $\mathrm{A}^{r,s}(\mathrm{H}(\mathrm{A}_{\mathbf{C}}))$.

We define subspaces of the relative cochain complexes similar to the spaces in section 4. These will induce a bigrading on the relative cohomology $\mathrm{H}(\mathscr{M}_\alpha,\mathrm{A}_{\mathbf{C}})$.

Definition **(6.3)**. — *a)* $\mathrm{R}^{r,s}(\mathscr{M}_\alpha,\mathscr{E})\subset\mathrm{C}(\mathscr{M}_\alpha,\mathscr{E})$ is the subspace:

$$\bigoplus_{\substack{p+q\leq r+s\\ p\geq r}}\mathscr{M}_\alpha^{p,q}\oplus\mathrm{R}^{r,s}(\mathscr{E}).$$

b) $\mathrm{L}^{r,s}(\mathscr{M}_\alpha,\bar{\mathscr{E}})\subset\mathrm{C}(\mathscr{M}_\alpha,\bar{\mathscr{E}})$ is the subspace $\displaystyle\bigoplus_{\substack{p+q\leq r+s\\ q\geq s}}\mathscr{M}_\alpha^{p,q}\oplus\mathrm{L}^{r,s}(\bar{\mathscr{E}})$.

c) $\mathrm{Dec}\,\mathrm{W}_{r+s}(\mathscr{M}_\alpha,\mathrm{A}_{\mathbf{C}})$ is the subspace $\displaystyle\bigoplus_{p+q\leq r+s}\mathscr{M}_\alpha^{p,q}\oplus\mathrm{Dec}\,\mathrm{W}_{r+s}(\mathrm{A}_{\mathbf{C}})$.

These subspaces are preserved by d. They give rise to subspaces $\mathrm{R}^{r,s}(\mathrm{H}(\mathscr{M}_\alpha,\mathscr{E}))$ and $\mathrm{L}^{r,s}(\mathrm{H}(\mathscr{M}_\alpha,\bar{\mathscr{E}}))$ and a filtration $\mathrm{Dec}\,\mathrm{W}(\mathrm{H}(\mathscr{M}_\alpha,\mathrm{A}_{\mathbf{C}}))$.

d) $\mathrm{A}^{r,s}(\mathrm{H}(\mathscr{M}_\alpha,\mathrm{A}_{\mathbf{C}}))$ is the intersection of $\mathrm{R}^{r,s}(\mathrm{H}(\mathscr{M}_\alpha,\mathscr{E}))$ with $\mathrm{L}^{r,s}(\mathrm{H}(\mathscr{M}_\alpha,\bar{\mathscr{E}}))$ when we identify the various relative cohomology groups by j^* and j'^*.

Lemma **(6.4)**. — a) *Let x be in $\mathrm{A}^{r,s}(\mathrm{H}(\mathscr{M}_\alpha,\mathrm{A}_{\mathbf{C}}))$. Then x has cocycle representatives $(m_x,e_x)\in\mathrm{R}^{r,s}(\mathscr{M}_\alpha,\mathscr{E})$, $(m_x,e'_x)\in\mathrm{L}^{r,s}(\mathscr{M}_\alpha,\bar{\mathscr{E}})$, and $(m_x,a_x)\in\mathrm{Dec}\,\mathrm{W}_{r+s}(\mathscr{M}_\alpha,\mathrm{A})$. We can choose these so that the cocycles $\varphi(a_x)-e_x-\int_0^1\mathrm{H}(m_x)$ in $\mathscr{E}$ and $\bar\varphi(a_x)-e'_x-\int_0^1\mathrm{H}'(m_x)$ in $\bar{\mathscr{E}}$ are exact.*

b) *Suppose given cocycles (m_x,e_x), (m_x,e'_x), and (m_x,a_x) as in part* a) *and cochains $h_x\in\mathrm{Dec}\,\mathrm{W}_{r+s}(\mathscr{E})$ and $h'_x\in\mathrm{Dec}\,\mathrm{W}_{r+s}(\bar{\mathscr{E}})$ such that $dh_x=\varphi(a_x)-e_x-\int_0^1\mathrm{H}(m_x)$ and $dh'_x=\bar\varphi(a_x)-e'_x-\int_0^1\mathrm{H}'(m_x)$. If the class of $[m_x,a_x]$ in cohomology is zero, then there are* *cochains* $(n_x, f_x)\in \mathrm{R}^{r,s}(\mathscr{M}_\alpha, \mathscr{E})$, $(n_x, b_x)\in \mathrm{Dec}\ \mathrm{W}_{r+s}(\mathscr{M}_\alpha, \mathrm{A}_{\mathbf{C}})$, *and* $(n_x, f'_x)\in \mathrm{L}^{r,s}(\mathscr{M}_\alpha, \bar{\mathscr{E}})$ *such that*:

1) $d(n_x, f_x)=(m_x, e_x)$, $d(n_x, b_x)=(m_x, a_x)$, $d(n_x, f'_x)=(m_x, e'_x)$, *and*

2) $\left(\varphi(b_x)-f_x-\int_0^1 \mathrm{H}(n_x)-h_x\right)$ *and* $\left(\bar{\varphi}(b_x)-f'_x-\int_0^1 \mathrm{H}'(n_x)-h'_x\right)$

are exact.

Proof. — *a)* By definition x has cocycle representatives (μ_x, ε_x) in $\mathrm{R}^{r,s}(\mathscr{M}_\alpha, \mathscr{E})$ and (μ'_x, ε'_x) in $\mathrm{L}^{r,s}(\mathscr{M}_\alpha, \bar{\mathscr{E}})$. The cocycle $-\rho(\mu_x)$ is exact in $\mathrm{A}_{\mathbf{C}}$ and is in $\mathrm{Dec}\ \mathrm{W}_{r+s}(\mathrm{A}_{\mathbf{C}})$. By (4.8), it is equal to $d(\gamma)$ for some $\gamma\in \mathrm{Dec}\ \mathrm{W}_{r+s}(\mathrm{A}_{\mathbf{C}})$. Then (μ_x, γ) is a cocycle in $\mathrm{Dec}\ \mathrm{W}_{r+s}(\mathscr{M}_\alpha, \mathrm{A}_{\mathbf{C}})$. The cocycle $j(\mu_x, \gamma)-(\mu_x, \varepsilon_x)$ is equal to $(0, \varphi(\gamma)-\int_0^1 \mathrm{H}(\mu_x)-\varepsilon_x)$. Consequently, $\varphi(\gamma)-\int_0^1 \mathrm{H}(\mu_x)-\varepsilon_x$ is a cocycle in $\mathscr{E}$. Clearly, it is in $\mathrm{Dec}\ \mathrm{W}_{r+s}(\mathscr{E})$. Since φ^* is strictly compatible with the weight filtrations on cohomology, $\left(\varphi(\gamma)-\int_0^1 \mathrm{H}(\mu_x)-\varepsilon_x\right)$ is cohomologous to $\varphi(\delta)$ for some $\delta\in \mathrm{Dec}\ \mathrm{W}_{r+s}(\mathrm{A}_{\mathbf{C}})$. The relative cocycle $(\mu_x, \gamma-\delta)$ in $\mathrm{Dec}\ \mathrm{W}_{r+s}(\mathscr{M}_\alpha, \mathrm{A}_{\mathbf{C}})$ represents x.

Consider the cocycle $(\mu'_x-\mu_x)$ in $\mathscr{M}_\alpha$. It is exact and is in $\bigoplus_{p+q\le r+s} \mathscr{M}_\alpha^{p,q}$. Thus we can write $(\mu'_x-\mu_x)=d(c)$ for some $c\in \bigoplus_{p+q\le r+s} \mathscr{M}_\alpha^{p,q}$. Let $c=b+b'$ with $b\in \bigoplus_{p\ge r} \mathscr{M}_\alpha^{p,q}$ and $b'\in \bigoplus_{q>s} \mathscr{M}_\alpha^{p,q}$. Define:

$$(m_x, e_x)=(u_x, \varepsilon_x)-d(b, 0), \qquad (m_x, e'_x)=(\mu'_x, \varepsilon'_x)+d(b', 0),$$

and:

$$(m_x, a_x)=(\mu_x, \gamma-\delta)-d(b, 0).$$

These three classes are cocycle representatives for x which lie in the appropriate subspaces. However, the second condition in part a) may not be fulfilled. We have $j^*(x)=[m_x, e_x]=[j(m_x, a_x)]$ in $\mathrm{H}(\mathscr{M}_\alpha, \mathrm{A}_{\mathbf{C}})$. Since $x\in \mathrm{A}^{r,s}(\mathrm{H}(\mathscr{M}_\alpha, \mathrm{A}_{\mathbf{C}}))$, there is $(n, \xi)\in \mathrm{R}^{r,s}(\mathscr{M}_\alpha, \mathscr{E})$ such that $j(m_x, a_x)=(m_x, e_x)+d(n, \xi)$. Thus:

$$\left[m_x, \varphi(a_x)-\int_0^1 \mathrm{H}(m_x)\right]=(m_x, e_x)+(-dn, d\xi+\psi(n)).$$

From this we see that $dn=0$ and that:

$$\varphi(a_x)-e_x-\psi(n)-\int_0^1 \mathrm{H}(m_x)=d\xi.$$

Similarly there is a cochain $(n', \xi')\in \mathrm{L}^{r,s}(\mathscr{M}_\alpha, \bar{\mathscr{E}})$ such that $dn'=0$ and:

$$\varphi'(a_x)-e'_x-\psi'(n')-\int_0^1 \mathrm{H}'(m_x)=d\xi'.$$

The three cocycles $(m_x, e_x+\psi(n))$, $(m_x, e'_x+\psi'(n'))$, and (m_x, a_x) satisfy part a) of the lemma.

b) We begin with cocycles (m, e) in $\mathrm{R}^{r,s}(\mathscr{M}_\alpha, \mathscr{E})$, (m, a) in $\mathrm{Dec}\ \mathrm{W}_{r+s}(\mathscr{M}_\alpha, \mathrm{A}_{\mathbf{C}})$, and (m, e') in $\mathrm{L}^{r,s}(\mathscr{M}_\alpha, \bar{\mathscr{E}})$ and cochains h in $\mathrm{Dec}\ \mathrm{W}_{r+s}(\mathscr{E})$ and h' in $\mathrm{Dec}\ \mathrm{W}_{r+s}(\bar{\mathscr{E}})$ such that:

$$dh=\varphi(a)-e-\int_0^1 \mathrm{H}(m)$$

and

$$dh'=\bar{\varphi}(a)-e'-\int_0^1 \mathrm{H}'(m).$$

We suppose that the relative cohomology class of (m, a) is zero. There are then cochains $n \in \mathcal{M}_\alpha^{r,s}$ and $b \in \operatorname{Dec} W_{r+s}(A_{\mathbf{C}})$ such that:

$$(m, a) = d(n, b) = (-dn, db + \rho(n)),$$

or $m = -dn$ and $a - \rho(n) = db$. Since (m, e) and (m, e') are cocycles, $e - \psi(n)$ and $e' - \psi'(n)$ must also be cocycles. In fact one sees that:

$$\varphi(a - \rho(n)) = e - \psi(n) + d\Big[h + \int_0^1 H(n)\Big].$$

Since $\varphi(a - \rho(n)) = \varphi(db) = d\varphi(b)$, both the cocycles $e - \psi(n)$ in $R^{r,s}(\mathcal{E})$, and $a - \rho(n)$ in $\operatorname{Dec} W_{r+s}(A_{\mathbf{C}})$, are exact. Likewise one shows that $e' - \psi'(n)$ in $L^{r,s}(\bar{\mathcal{E}})$ is exact. Choose cochains $f \in R^{r,s}(\mathcal{E})$, $b \in \operatorname{Dec} W_{r+s}(A_{\mathbf{C}})$, and $f' \in L^{r,s}(\bar{\mathcal{E}})$ such that $df = e - \psi(n)$, $db = a - \rho(n)$, and $df' = e' - \psi'(n)$. The cochains (n, f), (n, b), and (n, f') are as required by (6.4) *b)*, except that $\Big(\varphi(b) - f - \int_0^1 H(n) - h\Big)$ and $\Big(\bar{\varphi}(b) - f' - \int_0^1 H'(n) - h'\Big)$ may not be exact.

Note:

$$\begin{aligned} d\Big(\varphi(b) - f - \int_0^1 H(n) - h\Big) &= d\varphi(b) - df - d\int_0^1 H(n) - dh \\ &= \psi(n) - \varphi \circ \rho(n) - \Big(d\int_0^1 H(n) + \int_0^1 dH(n)\Big). \end{aligned}$$

Since H is a homotopy from $\varphi \circ \rho$ to ψ this last expression is 0. This shows that $\Big(\bar{\varphi}(b) - f - \int_0^1 H(n) - h\Big)$ is a cocycle. A similar computation shows that:

$$\Big(\bar{\varphi}(b) - f' - \int_0^1 H'(n) - h'\Big)$$

is a cocycle.

Using the fact that our cocycles are in $\operatorname{Dec} W_{r+s}(\mathcal{E})$ and $\operatorname{Dec} W_{r+s}(\bar{\mathcal{E}})$ respectively, and that $R^{r,s}(H(\mathcal{E})) + L^{r,s}(H(\bar{\mathcal{E}}))$ contains $\operatorname{Dec} W_{r+s}(H(A_{\mathbf{C}}))$ (4.9), we can change f, f', and b by cocycles (by an argument similar to the one in part a)) so that the "difference cocycles" become exact. This completes the proof of lemma (6.4).

Now we are ready to show that these subspaces of the relative cochains have all the properties suggested by the notation.

Proposition **(6.5)**. — a) $H(\mathcal{M}_\alpha, A_{\mathbf{C}}) = \bigoplus_{p,q \geq 0} A^{p,q}(H(\mathcal{M}_\alpha, A_{\mathbf{C}}))$.

b) $\operatorname{Dec} W_{r+s}(H(\mathcal{M}_\alpha, A_{\mathbf{C}})) = \bigoplus_{p+q \leq r+s} A^{p,q}(H(\mathcal{M}_\alpha, A_{\mathbf{C}}))$.

c) $\bigoplus_{\substack{p+q \leq r+s \\ p \geq r}} j^*(A^{p,q}(H(\mathcal{M}_\alpha, A_{\mathbf{C}}))) = R^{r,s}(H(\mathcal{M}_\alpha, \mathcal{E}))$.

d) $\bigoplus_{\substack{p+q \leq r+s \\ q \geq s}} j'^*(A^{p,q}(H(\mathcal{M}_\alpha, A_{\mathbf{C}}))) = L^{r,s}(H(\mathcal{M}_\alpha, \mathcal{E}))$.

e) *The long exact sequence of the pair* $(\mathcal{M}_\alpha, A)$:

$$(\dagger) \ \ldots \longrightarrow H^\ell(\mathcal{M}_\alpha) \xrightarrow{\rho^*} H^\ell(A_{\mathbf{C}}) \xrightarrow{\delta} H^{\ell+1}(\mathcal{M}_\alpha, A_{\mathbf{C}}) \xrightarrow{-\pi_1^*} \ldots$$

breaks up into a direct sum of long exact sequences:

$$(\dagger)_{r,s} \dots \longrightarrow \mathrm{H}^{\ell}(\mathcal{M}_{\alpha}^{r,s}) \xrightarrow{\rho^*} \mathrm{A}^{r,s}(\mathrm{H}^{\ell}(\mathrm{A}_{\mathbf{C}})) \xrightarrow{\delta} \mathrm{A}^{r,s}(\mathrm{H}^{\ell+1}(\mathcal{M}_{\alpha}, \mathrm{A}_{\mathbf{C}})) \longrightarrow \dots$$

Proof. — We begin by showing that $(\dagger)_{r,s}$ exists and is exact without assuming the bigrading in part a). By lemma (6.2) b), $\rho^* : \mathrm{H}(\mathcal{M}_{\alpha}^{r,s}) \to \mathrm{A}^{r,s}(\mathrm{H}(\mathrm{A}_{\mathbf{C}}))$. If $x \in \mathrm{A}^{r,s}(\mathrm{H}^{\ell}(\mathrm{A}_{\mathbf{C}}))$, then x has cocycle representatives $a_x \in \mathrm{R}^{r,s}(\mathscr{E})$ and $a'_x \in \mathrm{L}^{r,s}(\bar{\mathscr{E}})$ by propositions (4.6) a) and (4.6) b). Both $(0, a_x) \in \mathrm{R}^{r,s}(\mathcal{M}_{\alpha}, \mathscr{E})$ and $(0, a'_x) \in \mathrm{L}^{r,s}(\mathcal{M}_{\alpha}, \bar{\mathscr{E}})$ are representatives for the image of x. Thus, $\delta(x)$ is contained in $\mathrm{A}^{r,s}(\mathcal{M}_{\alpha}, \mathrm{A}_{\mathbf{C}})$. If $y \in \mathrm{A}^{r,s}(\mathrm{H}(\mathcal{M}_{\alpha}, \mathrm{A}_{\mathbf{C}}))$, then, by (6.4), y has representatives $(m_y, e_y) \in \mathrm{R}^{r,s}(\mathcal{M}_{\alpha}, \mathscr{E})$ and $(m_y, e'_y) \in \mathrm{L}^{r,s}(\mathcal{M}_{\alpha}, \bar{\mathscr{E}})$. The cocycle $(-m_y)$ represents $-\pi_1^*(y)$ and is in $\mathcal{M}_{\alpha}^{r,s}$. This shows that $(\dagger)_{r,s}$ exists. It is of order 2 at every point since it is embedded in the exact sequence $(\dagger)$.

Ker $\delta \subset$ Im ρ^* : This is immediate from (6.2).

Ker$(-\pi_1^*) \subset$ Im δ : Let $x \in \mathrm{A}^{r,s}(\mathrm{H}(\mathcal{M}_{\alpha}, \mathrm{A}_{\mathbf{C}}))$ be in Ker$(-\pi_1^*)$. Pick cocycle representatives as in (6.4) *a)*, (m_x, e_x), (m_x, a_x), and (m_x, e'_x). The condition that $\pi_1^*(x) = 0$ means that m_x is exact in $\mathcal{M}_{\alpha}$. Since $m_x \in \mathcal{M}_{\alpha}^{r,s}$, there is an element $n \in \mathcal{M}_{\alpha}^{r,s}$ with $dn = m_x$. Form new relative cocycles by adding $d(n, 0)$ to each of the three relative cocycles above. This allows us to assume that in addition $m_x = 0$. This means e_x, a_x, and e'_x are cocycles. Since $e_x - \varphi(a_x)$ and $e'_x - \bar{\varphi}(a_x)$ are exact (6.4), e_x and e'_x represent the same cohomology class. This class is in $(\varphi^*)^{-1}(\mathrm{R}^{r,s}(\mathrm{H}(\mathscr{E}))) \cap (\bar{\varphi}^*)^{-1}(\mathrm{L}^{r,s}(\mathrm{H}(\bar{\mathscr{E}})))$ which is $\mathrm{A}^{r,s}(\mathrm{H}(\mathrm{A}_{\mathbf{C}}))$. Its image under δ is x.

Ker $\rho^* \subset$ Im$(-\pi_1^*)$: Let $c \in \mathrm{H}^{\ell+1}(\mathcal{M}_{\alpha}^{r,s})$ be in Ker ρ^*. Then c is represented by a cocycle γ in $\mathcal{M}_{\alpha}^{r,s}$, and $\psi(\gamma)$, $\rho(\gamma)$, and $\psi'(\gamma)$ are all exact. By (4.6) *c)*, (4.7) *b)*, and (4.8) there are forms $e \in \mathrm{R}^{r,s}(\mathscr{E})$, $a \in \mathrm{W}_{r+s}(\mathrm{A}_{\mathbf{C}})$, and $e' \in \mathrm{L}^{r,s}(\bar{\mathscr{E}})$ such that $\psi(\gamma) = de$, $\rho(\gamma) = da$, and $\psi'(\gamma) = de'$. Form the relative cocycles $(-\gamma, e)$, $(-\gamma, a)$, and $(-\gamma, e')$ in $\mathrm{R}^{r,s}(\mathcal{M}_{\alpha}, \mathscr{E})$, Dec $\mathrm{W}_{r+s}(\mathcal{M}_{\alpha}, \mathrm{A}_{\mathbf{C}})$, and $\mathrm{L}^{r,s}(\mathcal{M}_{\alpha}, \bar{\mathscr{E}})$ respectively. We must choose e, a, and e' so that the three relative classes are cohomologous. This is done exactly as in the proof of the second part of (6.4) *a)*. Having achieved this, we find that the class they represent is in $\mathrm{A}^{r,s}(\mathrm{H}(\mathcal{M}_{\alpha}, \mathrm{A}_{\mathbf{C}}))$ and its image under $(-\pi_1^*)$ is $[\gamma] = c$. This completes the proof that $(\dagger)_{r,s}$ is exact.

Part a), b), c), and d) of (6.5) now follow easily by a " diagram chase " argument.

Now we return to the question of building a bigraded minimal model for a mixed Hodge diagram. Let $\mathcal{M}_{\alpha}$ be a bigraded algebra mapping to the mixed Hodge diagram, where $\mathcal{M}_{\alpha}$ is some stage of the minimal model. Then $\mathcal{M}_{\alpha+1} = \mathcal{M}_{\alpha} \otimes_d \Lambda(\mathrm{V})$, where V is the first non zero cohomology group of $(\mathcal{M}_{\alpha}, \mathrm{A}_{\mathbf{C}})$. By (6.5) V has a bigrading. We let $\mathcal{M}_{\alpha+1}$ have the bigrading which is the multiplicative extension of this one on V and the one already given on $\mathcal{M}_{\alpha}$. Now, we define d, ρ, ψ, ψ', H and H' on $\mathcal{M}_{\alpha+1}$ so as to extend what we have on $\mathcal{M}_{\alpha}$ and to satisfy (6.0).

First pick one-sided inverses s, p, and s' in the diagram:

$$
\begin{tikzcd}
\text{Cocycles}\ (\mathcal{M}_\alpha, \mathscr{E}) \arrow[dr, shift left] & \text{Cocycles}\ (\mathcal{M}_\alpha, \mathrm{A}_{\mathbf{C}}) \arrow[l, "j"'] \arrow[r, "j"] \arrow[d, shift left] & \text{Cocycles}\ (\mathcal{M}_\alpha, \bar{\mathscr{E}}) \arrow[dl, shift right] \\
& \mathrm{V} \arrow[ul, dashed, shift left, "s"] \arrow[u, dashed, shift left, "p"] \arrow[ur, dashed, shift right, "s'"'] &
\end{tikzcd}
$$

We can do this such that, for any $v \in \mathrm{V}^{r,s}$:

1) $s(v) \in \mathrm{R}^{r,s}(\mathcal{M}_\alpha, \mathscr{E})$, $p(v) \in \operatorname{Dec} \mathrm{W}_{r+s}(\mathcal{M}_\alpha, \mathrm{A}_{\mathbf{C}})$, and $s'(v) \in \mathrm{L}^{r,s}(\mathcal{M}_\alpha, \bar{\mathscr{E}})$,
2) the first coordinates all agree, *i.e.* $s_1(v) = p_1(v) = s'_1(v) \in \mathcal{M}_\alpha^{r,s}$, and
3) if we denote the second coordinates by $s_2(v)$, $p_2(v)$, and $s'_2(v)$, then:

$$\varphi p_2(v) - s_2(v) - \int_0^1 \mathrm{H}(p_1(v)) \quad \text{in } \mathscr{E}$$

and
$$\bar{\varphi} p_2(v) - s'_2(v) - \int_0^1 \mathrm{H}'(p_1(v)) \quad \text{in } \bar{\mathscr{E}}$$

are exact.

This follows from (6.4) *a)*. In 3) above the first cocycle is in $\operatorname{Dec} \mathrm{W}_{r+s}(\mathscr{E})$ and the second is in $\operatorname{Dec} \mathrm{W}_{r+s}(\bar{\mathscr{E}})$. Thus, in addition, it is possible to choose maps $h : \mathrm{V}^{r,s} \to \operatorname{Dec} \mathrm{W}_{r+s}(\mathscr{E})$ and $h' : \mathrm{V}^{r,s} \to \operatorname{Dec} \mathrm{W}_{r+s}(\bar{\mathscr{E}})$ such that:

$$dh(v) = \varphi p_2(v) - s_2(v) - \int_0^1 \mathrm{H}(p_1(v)),$$

and
$$dh'(v) = \bar{\varphi} p_2(v) - s'_2(v) - \int_0^1 \mathrm{H}'(p_1(v)).$$

Now we are ready to define $d : \mathrm{V} \to \mathcal{M}_\alpha$, and ρ, ψ, ψ', H, and H$'$ on $\mathcal{M}_{\alpha+1}$ extending the given maps on $\mathcal{M}_\alpha$. The map $d : \mathrm{V} \to \mathcal{M}_\alpha$ is given by $d(v) = -p_1(v)$. If $v \in \mathrm{V}^{r,s}$, then $d(v) = -s_1(v) = -s_2(v)$ and hence is in $\mathcal{M}_\alpha^{r,s}$ (6.4). We define:

$$\rho(v) = p_2(v), \qquad \psi(v) = s_2(v), \qquad \text{and} \qquad \psi'(v) = s'_2(v).$$

One checks easily that these extensions define maps of differential algebras satisfying (6.0) 1). Lastly, we define:

$$\mathrm{H}(v) = \varphi\rho(v) - d(h(v) \otimes t) - \int_0^t \mathrm{H}(p_1(v)),$$

and
$$\mathrm{H}'(v) = \bar{\varphi}\rho(v) - d(h'(v) \otimes t) - \int_0^t \mathrm{H}'(p_1(v)).$$

These extensions are homotopies from $\varphi\rho$ to ψ and $\bar{\varphi}\rho$ to ψ' respectively which satisfy (6.0). This completes the inductive step in the construction of the minimal model and proves the following theorem:

Theorem **(6.6).** — *Any mixed Hodge diagram has a bigraded minimal model. The bigrading induced on the cohomology of the diagram by its mixed Hodge structure agrees with the bigrading induced on the cohomology of the minimal model.*

The minimal model we constructed has the property that its bigrading induces one on each term $\mathcal{M}_\alpha$ in the canonical series. Furthermore the bigrading on $\mathcal{M}_{\alpha+1}=\mathcal{M}_\alpha \otimes \Lambda(V)$ is the multiplicative extension of the one on $\mathcal{M}_\alpha$ and one on V. Such a bigrading is called *compatible with the canonical series.*

Uniqueness and naturality will be consequences of the following result. Suppose that $\mathcal{H}$ is a mixed Hodge diagram and that:

$$\begin{tikzcd} \mathcal{N} \arrow[dr, "R"'] & & \mathcal{M} \arrow[dl, "P"] \\ & \mathcal{H} & \end{tikzcd}$$

are morphisms from bigraded minimal algebras, so that P induces an isomorphism on cohomology, and so that the bigrading on $\mathcal{N}$ is compatible with the canonical series.

Theorem **(6.7)**. — *There is a map* $\mu : \mathcal{N} \to \mathcal{M}$ *of bigraded differential algebras and a homotopy from* $P \circ \mu$ *to* R. *The map* μ *is well defined up to a homotopy preserving the bigradings.*

Corollary **(6.8)** (Naturality). — *Let* $\mathcal{H}_i$ *be mixed Hodge diagrams and* $P_i : \mathcal{M}_i \to \mathcal{H}$ *bigraded minimal models for* $i=1$ *and* 2. *Let the bigrading on* $\mathcal{M}_1$ *be compatible with the canonical series. Any morphism* $F : \mathcal{H}_1 \to \mathcal{H}_2$ *induces a map of bigraded minimal models* $f : \mathcal{M}_1 \to \mathcal{M}_2$. *The map* f *is well defined, up to a homotopy compatible with the bigradings, by requiring that* $P_2 \circ f$ *be homotopic to* $F \circ P_1$.

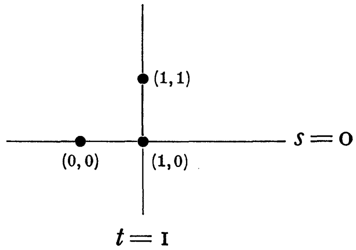
Proof **(6.8)**. — We have:

$$\begin{tikzcd} \mathcal{M}_1 \arrow[d, "P_1"] & \mathcal{M}_2 \arrow[d, "P_2"] \\ \mathcal{H}_1 \arrow[r, "F"] & \mathcal{H}_2 \end{tikzcd}$$

If $F \circ P_1$ is a morphism from $\mathcal{M}_1 \to \mathcal{H}_2$, then we can apply (6.7) to construct $\mu : \mathcal{M}_1 \to \mathcal{M}_2$. It remains to prove that $F \circ P_1$ is a morphism. We have:

$$\begin{tikzcd} & \mathcal{E}_1 \arrow[r, "g"] \arrow[d, phantom, "H_1" near start] & \mathcal{E}_2 \\ \mathcal{M}_1 \arrow[ur, "\psi"] \arrow[r, "\rho"] \arrow[dr, "\psi'"'] & (A_1)_C \arrow[u, "\varphi_1"] \arrow[r, "f_C"] \arrow[d, "\bar\varphi_1"'] \arrow[ur, phantom, "J"] \arrow[dr, phantom, "\bar J"] & (A_2)_C \arrow[u, "\varphi_2"'] \arrow[d, "\bar\varphi_2"] \\ & \bar{\mathcal{E}}_1 \arrow[r, "\bar g"'] & \bar{\mathcal{E}}_2 \end{tikzcd}$$

The reason that this diagram is not a morphism from $\mathcal{M}_1$ to $\mathcal{H}_2$ is that $J \circ \rho$ is a homotopy from $\varphi_2 \circ f_C \circ \rho$ to $g \circ \varphi_1 \circ \rho$ and $g \circ H_1$ is a homotopy from $g \circ \varphi_1 \circ \rho$ to $g \circ \psi$ but we need a homotopy from $\varphi_2 \circ f_{\mathbf{C}} \circ \rho$ to $g \circ \psi$. There is however a general proceedure for adjoining homotopies. Let $\mathrm{H} : \mathscr{M} \to \mathrm{A} \otimes (t, dt)$ and $\mathrm{H}' : \mathscr{M} \to \mathrm{A} \otimes (s, ds)$ be homotopies with $\mathrm{H}\big|_{t=1} = \mathrm{H}'\big|_{s=0}$. Let $\mathscr{C}$ represent the following subvariety in the plane:



Thus $\mathscr{C}$ is the differential algebra $\{(s, t, ds, dt)/(s(t-1), sdt, tds)\}$. H and H' together define a map $\widetilde{\mathrm{H}} : \mathscr{M} \to \mathrm{A} \otimes \mathscr{C}$ as follows. If

$$\mathrm{H}(m) = \sum_{i \geqq 0} (\alpha_i t^i + \beta_i t^i dt) \quad \text{and} \quad \mathrm{H}'(m) = \sum_{i \geqq 0} (\gamma_i s^i + \delta_i s^i ds),$$

then $\sum_{i \geqq 0} \alpha_i = \gamma_0$. Let:

$$\widetilde{\mathrm{H}}(m) = \sum_{i \geqq 0} (\alpha_i t^i + \beta_i t^i dt) + \sum_{i > 0} \gamma_i s^i + \sum_{i \geqq 0} \delta_i s^i ds.$$

One checks easily that in the presence of the above relations $\widetilde{\mathrm{H}}$ is a map of differential algebras. Since $\{(s, t, ds, dt)\} \to \mathscr{C}$ is an onto map which is an isomorphism on cohomology, we can lift $\widetilde{\mathrm{H}}$ to a map $\overline{\mathrm{H}} : \mathscr{M} \to \mathrm{A} \otimes (s, t, ds, dt)$. Necessarily:

$$\overline{\mathrm{H}}\Big|_{\substack{s=0\\t=0}} = \widetilde{\mathrm{H}}\Big|_{\substack{s=0\\t=0}} = \mathrm{H}\Big|_{t=0} \quad \text{and} \quad \overline{\mathrm{H}}\Big|_{\substack{s=1\\t=1}} = \widetilde{\mathrm{H}}\Big|_{\substack{s=1\\t=1}} = \mathrm{H}'\Big|_{s=1}.$$

Now restrict to the diagonal $\{s = t\}$, *i.e.* compose $\overline{\mathrm{H}}$ with the projection:

$$\mathrm{A} \otimes \{(s, t, ds, dt)\} \to \mathrm{A} \otimes \{(s, t, ds, dt)/(s = t, ds = dt)\}.$$

This composition is the required homotopy from $\mathrm{H}\big|_{t=0}$ to $\mathrm{H}'\big|_{s=1}$.

It remains only to remark that in our case, since $\mathrm{J} \circ \rho$ and $g \circ \mathrm{H}_1$ send $\mathscr{M}^{r,s}$ into $\mathrm{Dec}\, \mathrm{W}_{r+s}(\mathscr{E}_2) \otimes (t, dt)$, we can choose $\overline{\mathrm{H}}$ (and hence the resulting homotopy from $\rho \circ f_{\mathbf{C}} \circ \varphi_2$ to $g \circ \psi$) to have this property.

Now that we have made $\mathrm{F} \circ \mathrm{P}_1$ into a morphism we can apply (6.7) to prove (6.8).

Corollary **(6.9)**. — *If $\mathscr{H}_1$ and $\mathscr{H}_2$ are equivalent mixed Hodge diagrams and $\mathrm{P}_1 : \mathscr{M}_1 \to \mathscr{H}_1$ and $\mathrm{P}_2 : \mathscr{M}_2 \to \mathscr{H}_2$ are bigraded minimal models, then the equivalence of $\mathscr{H}_1$ with $\mathscr{H}_2$ defines an isomorphism of bigraded minimal models $\mu : \mathscr{M}_1 \xrightarrow{\cong} \mathscr{M}_2$ unique up to a homotopy compatible with the bigradings. In particular for any bigraded minimal model the bigrading is compatible with the canonical series.*

Proof of (6.9). — Using (6.6) and (6.7) we see that any two bigraded minimal models for a mixed Hodge diagram are isomorphic. Thus, since the construction of the bigraded minimal model in (6.6) is compatible with the canonical series, it follows that the bigrading on any minimal model is compatible with the canonical series. Given an elementary equivalence between mixed Hodge diagrams, (6.8) implies that there is induced a map between the bigraded minimal models, unique up to homotopy compatible with the bigradings. Since this map induces an isomorphism on cohomology it is an isomorphism of bigraded minimal models.

Proof of (6.7). — The bigraded map $\mu : \mathcal{N} \to \mathcal{M}$ and the homotopy from $R \circ \mu$ to P are constructed by induction over the canonical series for $\mathcal{N}$. Suppose we have:

$$\begin{tikzcd}
\mathcal{N}_\alpha \arrow[r, "\mu"] \arrow[d] & \mathcal{M} \arrow[d, "H'"', "H"] \arrow[dr, bend left, "\psi"] \arrow[dd, bend right=40, "\psi'"'] & \\
\mathcal{N} \arrow[rr, bend left, "\zeta"] \arrow[r, "J", "\lambda"'] \arrow[dr, "\zeta'"', "J'"] & A_{\mathbf{C}} \arrow[r, "\varphi"'] \arrow[d, "\bar{\varphi}"] & \mathcal{E} \\
& \bar{\mathcal{E}} &
\end{tikzcd}$$

and

$$\begin{tikzcd}
& \mathcal{E} \otimes (s, ds) \\
\mathcal{N}_\alpha \arrow[ur, "\bar{\Psi}", "K"'] \arrow[r, "\Lambda"] \arrow[dr, "\bar{\Psi}'"', "K'"] & A_{\mathbf{C}} \otimes (s, ds) \arrow[u, "\varphi"'] \arrow[d, "\bar{\varphi}"] \\
& \bar{\mathcal{E}} \otimes (s, ds)
\end{tikzcd}$$

a homotopy from $P \circ \mu$ to R. Let $\mathcal{N}_{\alpha+1} \cong \mathcal{N}_\alpha \otimes S(V)_k$. V has a bigrading $V = \bigoplus_{r,s} V^{r,s}$. We consider the three separate obstructions:

1) to extending μ and the homotopy $\bar{\Psi}$ into $\mathcal{E} \otimes (s, ds)$,
2) to extending μ and the homotopy Λ into $(A_{\mathbf{C}}) \otimes (s, ds)$, and
3) to extending μ and the homotopy $\bar{\Psi}'$ into $\bar{\mathcal{E}} \otimes (s, ds)$.

Call these obstructions α, β, and $\bar{\alpha}$. From (5.7) we see that, if $v \in V^{r,s}$, then
1) $\alpha(v) \in R^{r,s}(\mathcal{M}, \mathcal{E})$, $\beta(v) \in \operatorname{Dec} W_{r+s}(\mathcal{M}, A_{\mathbf{C}})$, and $\bar{\alpha}(v) \in L^{r,s}(\mathcal{M}, \bar{\mathcal{E}})$, and
2) the first coordinates of $\alpha(v)$, $\beta(v)$, and $\bar{\alpha}(v)$ all agree.

Claim:

$$j(\beta(v)) - \alpha(v) = d\left[0, \int_{t=0}^{t=1} Jv + \int_{s=0}^{s=1}\left(\int_{t=0}^{t=1} K(dv)\right)\right]$$

$$j'(\beta(v)) - \bar{\alpha}(v) = d\left[0, \int_{t=0}^{t=1} J'v + \int_{s=0}^{s=1}\left(\int_{t=0}^{t=1} K'(dv)\right)\right].$$

Explanation. — s is the variable for the homotopies $\overline{\Psi}$, Λ, and $\overline{\Psi}'$ whereas t is the variable for the homotopies H, H$'$, J, and J$'$.

Proof. — We consider the first equation:

$$\begin{aligned} j(\beta(v))-\alpha(v) &= \Big[0, \ -\varphi\lambda v+\zeta(v)+\int_{s=0}^{s=1}\varphi\Lambda dv-\int_{s=0}^{s=1}\overline{\Psi}(dv)-\int_{t=0}^{t=1}\mathrm{H}\mu(dv)\Big] \\ &= \Big[0, \ d\int_{t=0}^{t=1}\mathrm{J}v+\int_{t=0}^{t=1}d\mathrm{J}v-\int_{s=0}^{s=1}(\overline{\Psi}-\varphi\Lambda)\,(dv)-\int_{t=0}^{t=1}\mathrm{H}\mu(dv)\Big] \\ &= \Big[0, \ d\int_{t=0}^{t=1}\mathrm{J}v+\int_{t=0}^{t=1}(\mathrm{J}-\mathrm{H}\mu)\,(dv)-\int_{s=0}^{s=1}d\Big(\int_{t=0}^{t=1}\mathrm{K}dv+0\Big)\Big] \\ &= \Big[0, \ d\int_{t=0}^{t=1}\mathrm{J}v+\int_{t=0}^{t=1}\Big(d\int_{s=0}^{s=1}\mathrm{K}(dv)+\int_{s=0}^{s=1}d\mathrm{K}(dv)\Big)-\int_{s=0}^{s=1}d\Big(\int_{t=0}^{t=1}\mathrm{K}(dv)\Big)\Big] \\ &= \Big[0, \ d\int_{t=0}^{t=1}\mathrm{J}v+\int_{t=0}^{t=1}d\Big(\int_{s=0}^{s=1}\mathrm{K}(dv)\Big)-\int_{s=0}^{s=1}d\Big(\int_{t=0}^{t=1}\mathrm{K}(dv)\Big)\Big]. \end{aligned}$$

But:

$$\begin{aligned} d\int_{s=0}^{s=1}\Big(\int_{t=0}^{t=1}\mathrm{K}dv\Big) &= -\int_{s=0}^{s=1}d\Big(\int_{t=0}^{t=1}\mathrm{K}dv\Big)+\Big(\int_{t=0}^{t=1}\mathrm{K}dv\Big)_{s=0}^{s=1} \\ &= -\int_{s=0}^{s=1}d\Big(\int_{t=0}^{t=1}\mathrm{K}dv\Big)+\int_{t=0}^{t=1}\Big(\mathrm{K}dv\Big|_{s=0}^{s=1}\Big) \\ &= -\int_{s=0}^{s=1}d\Big(\int_{t=0}^{t=1}\mathrm{K}dv\Big)+\int_{t=0}^{t=1}d\Big(\int_{s=0}^{s=1}\mathrm{K}dv+0\Big). \end{aligned}$$

This proves the claim.

Since $\mathcal{M}$ is a bigraded minimal model for $\mathcal{H}$, all the relative cohomology groups vanish. Thus, applying (6.4) *b)*, we find cochains:

$$a(v)\in\mathrm{R}^{r,s}(\mathcal{M}_\alpha, \mathcal{E}), \qquad b(v)\in\mathrm{Dec}\ \mathrm{W}_{r+s}(\mathcal{M}_\alpha, \mathrm{A_C}), \qquad \bar{a}(v)\in\mathrm{L}^{r,s}(\mathcal{M}_\alpha, \mathcal{E}),$$
$$h(v)\in\mathrm{Dec}\ \mathrm{W}_{r+s}(\mathcal{E}), \qquad \text{and} \qquad h'(v)\in\mathrm{Dec}\ \mathrm{W}_{r+s}(\overline{\mathcal{E}})$$

such that:

1) the first coordinates $a_1(v)$, $b_1(v)$, and $\bar{a}_1(v)$ all agree,

2) $da(v)=\alpha(v)$, $db(v)=\beta(v)$, and $d\bar{a}(v)=\bar{\alpha}(v)$,

3) $dh(v)=\varphi b_2(v)-a_2(v)-\int_{t=0}^{t=1}\mathrm{H}(b_1(v))-\int_{t=0}^{t=1}\mathrm{J}v-\int_{s=0}^{s=1}\Big(\int_{t=0}^{t=1}\mathrm{K}(dv)\Big)$, and

4) $dh'(v)=\bar{\varphi}b_2(v)-\bar{a}_2(v)-\int_{t=0}^{t=1}\mathrm{H}'(b_1(v))-\int_{t=0}^{t=1}\mathrm{J}'v-\int_{s=0}^{s=1}\Big(\int_{t=0}^{t=1}\mathrm{K}'(dv)\Big)$.

We define $\mu(v)=-b_1(v)\in\mathcal{M}^{r,s}$:

(6.7) *a)*
$$\begin{cases} \overline{\Psi}(v) = -\psi(a_1(v))+\int_0^s\overline{\Psi}(dv)-d(a_2(v)\otimes s), \\ \Lambda(v) = -\rho(b_1(v))+\int_0^s\Lambda(dv)-d(b_2(v)\otimes s), \ \text{and} \\ \overline{\Psi}'(v) = -\psi'(\bar{a}_1(v))+\int_0^s\overline{\Psi}'(dv)-d(\bar{a}_2(v)\otimes s). \end{cases}$$

By the choice of $a(v)$, $b(v)$, and $\bar{a}(v)$, these homotopies have images in the required subspaces. They are indeed homotopies from $\psi\circ\mu$ to ζ, and from $\psi'\circ\mu$ to ζ' respectively (see (5.8)). The formulae giving the extensions of K and K′ over V are similar but more complicated:

$$\begin{aligned} \mathrm{K}(v) = &-\mathrm{H}(a_1(v)) + \int_0^s \mathrm{K}(dv) \\ &-d\Big(\Big\{\int_0^t \Big[-\mathrm{H}(a_1(v)) + \int_{s=0}^{s=1}\mathrm{K}(dv) - \mathrm{J}(v)\Big] + \varphi(b_2(v)) - d(h(v)\otimes t)\Big\}\otimes s\Big); \\ \mathrm{K}'(v) = &-\mathrm{H}'(\bar{a}_1(v)) + \int_0^s \mathrm{K}'(dv) \\ &-d\Big(\Big\{\int_0^t \Big[-\mathrm{H}'(\bar{a}_1(v)) + \int_{s=0}^{s=1}\mathrm{K}'(dv) - \mathrm{J}'(v)\Big] + \bar{\varphi}(b_2(v)) - d(h'(v)\otimes t)\Big\}\otimes s\Big). \end{aligned}$$

We check that K is the required extended homotopy:

$$\begin{aligned} \mathrm{K}(v)\Big|_{s=0} &= -\mathrm{H}(a_1(v)) = \mathrm{H}\mu(v). \\ \mathrm{K}(v)\Big|_{s=1} &= -\mathrm{H}(a_1(v)) + \int_{s=0}^{s=1}\mathrm{K}(dv) \\ &\qquad -d\Big(\int_0^t \Big[-\mathrm{H}(a_1(v)) + \int_{s=0}^{s=1}\mathrm{K}(dv) - \mathrm{J}(v)\Big] - \varphi(b_2(v))\Big) \\ &= -\mathrm{H}(a_1(v))\Big|_{t=0} + \Big(\int_{s=0}^{s=1}\mathrm{K}(dv)\Big)\Big|_{t=0} + \mathrm{J}(v) - \mathrm{J}(v)\Big|_{t=0} - \varphi(d(b_2(v))) \\ &= \varphi\rho\mu(v) + \varphi\Big(\int_{s=0}^{s=1}\Lambda(dv)\Big) + \mathrm{J}(v) - \varphi\lambda(v) - \varphi(d(b_2(v))). \end{aligned}$$

In going from line 1 to line 2 above we use the formula $d\int_0^t \omega = \omega - \omega\Big|_{t=0} - \int_0^t d\omega$. By the expression (6.7) *a)* for $\Lambda(v)$ we see that:

$$\begin{aligned} db_2 &= \int_{s=0}^{s=1}(\Lambda(dv)) + \rho\mu(v) - \Lambda(v)\Big|_{s=1} \\ &= \int_{s=0}^{s=1}(\Lambda(dv)) + \rho\mu(v) - \lambda(v). \end{aligned}$$

Thus $\mathrm{K}(v)\Big|_{s=1} = \varphi\mathrm{J}(v)$. Also:

$$\begin{aligned} \mathrm{K}(v)\Big|_{t=0} &= -\mathrm{H}(a_1(v))\Big|_{t=0} + \Big(\int_0^s \mathrm{K}(dv)\Big)\Big|_{t=0} - d(\varphi(b_2(v))\otimes s) \\ &= \varphi\rho\mu(v) + \varphi\Big(\int_0^s \Lambda(dv)\Big) - \varphi d(b_2(v)\otimes s) \\ &= \varphi\Lambda(v);\ \text{and:} \end{aligned}$$

$$\begin{aligned} \mathrm{K}(v)\Big|_{t=1} &= -\mathrm{H}(a_1(v))\Big|_{t=1} + \int_0^s \mathrm{K}(dv)\Big|_{t=1} \\ &\qquad -d\Big(\Big\{\int_{t=0}^{t=1}\Big[-\mathrm{H}(a_1(v)) + \int_{s=0}^{s=1}(\mathrm{K}dv) - \mathrm{J}(v)\Big] + \varphi(b_2(v)) - dh(v)\Big\}\otimes s\Big) \\ &= \psi\mu(v) + \int_0^s \overline{\Psi}(dv) - d(\{a_2(v)\}\otimes s). \end{aligned}$$

This last equality uses the formula for $dh(v)$ and the fact that:

$$\int_{s=0}^{s=1}\left(\int_{t=0}^{t=1}\mathrm{K}(dv)\right)=-\int_{t=0}^{t=1}\left(\int_{s=0}^{s=1}\mathrm{K}(dv)\right).$$

Thus we have $\mathrm{K}(v)\Big|_{t=1}=\overline{\Psi}(v)$.

Lastly:

$$d\mathrm{K}(v)=-d\mathrm{H}(a_1(v))+\mathrm{K}(dv)-\mathrm{K}(dv)\Big|_{s=0}=\mathrm{K}(dv).$$

The calculations for K' are similar.

This finishes the inductive construction of μ and the homotopy from $\mathrm{P}\circ\mu$ to R.

The proof that μ is unique up to homotopy is a relative version of the above argument. One inductively (over the canonical series for $\mathscr{N}$) constructs the homotopy. The details are left to the reader.

Let X be a smooth variety and $\mathscr{N}_\mathrm{X}$ the minimal model for the complex forms on X, $\mathscr{E}_{\mathrm{C}^\infty}(\mathrm{X};\mathbf{C})$. If $\mathrm{E}_{\mathrm{C}^\infty}(\mathrm{X})_\mathbf{C}\to\mathscr{E}(\log \mathrm{D})$ is a mixed Hodge diagram associated to some completion of X, then the minimal model $\mathscr{N}$ for $\mathscr{E}(\log\mathrm{D})$ receives a bigrading unique up to automorphism homotopic to the identity. The restriction map:

$$\mathscr{E}(\log\mathrm{D})\to\mathscr{E}_{\mathrm{C}^\infty}(\mathrm{X};\mathbf{C})$$

induces an isomorphism $\mathscr{N}\to\mathscr{N}_\mathrm{X}$ well-defined up to homotopy. This induces a bigrading on $\mathscr{N}_\mathrm{X}$ unique up to automorphism homotopic to the identity. If we change the compactification and/or the choices necessary to define the mixed Hodge diagram, we replace one mixed Hodge diagram with an equivalent one. The string of elementary equivalences connecting the two mixed Hodge diagrams all commute up to homotopy with the maps $\mathrm{E}_{\mathrm{C}^\infty}(\mathrm{X})_\mathbf{C}\to\mathscr{E}_{\mathrm{C}^\infty}(\mathrm{X},\mathbf{C})$. Thus the resulting bigrading on $\mathscr{N}_\mathrm{X}$ is unique up to automorphism homotopic to the identity.

Theorem **(6.10)**. — *Let* X *be a smooth variety and* $\mathscr{N}_\mathrm{X}$ *be the minimal model for the complex forms on* X, $\mathscr{E}_{\mathrm{C}^\infty}(\mathrm{X},\mathbf{C})$. $\mathscr{N}_\mathrm{X}$ *has a bigrading unique up to automorphism homotopic to the identity. If* $f:\mathrm{X}\to\mathrm{Y}$ *is an algebraic map, then* f *induces* $\hat{f}:\mathscr{N}_\mathrm{Y}\to\mathscr{N}_\mathrm{X}$, *a bigraded map which is well defined up to bigraded homotopy.*

Proof. — In light of the discussion preceeding the theorem we need only consider naturality. Let $f:\mathrm{X}\to\mathrm{Y}$ extend to $\bar{f}:\mathrm{V}_\mathrm{X}\to\mathrm{V}_\mathrm{Y}$, where V_X and V_Y are smooth compactifications of X and Y. Then, by (3.9), $\bar{f}$ induces a morphism of mixed Hodge diagrams unique up to homotopy commuting with the map $\bar{f}^*:\mathscr{E}_{\mathrm{C}^\infty}(\mathrm{Y},\mathbf{C})\to\mathscr{E}_{\mathrm{C}^\infty}(\mathrm{X},\mathbf{C})$ up to homotopy. By (6.9) $\bar{f}$ lifts to a morphism of bigraded minimal models $\hat{f}:\mathscr{N}_\mathrm{Y}\to\mathscr{N}_\mathrm{X}$ unique up to bigraded homotopy. We must show that the bigraded homotopy class of $\hat{f}:\mathscr{N}_\mathrm{Y}\to\mathscr{N}_\mathrm{X}$ does not depend on the choice of compactifications. If we are given two such $\bar{f}:\mathrm{V}_\mathrm{X}\to\mathrm{V}_\mathrm{Y}$, and $\bar{f}':\mathrm{V}'_\mathrm{X}\to\mathrm{V}'_\mathrm{Y}$, there is a third $\bar{f}'':\mathrm{V}''_\mathrm{X}\to\mathrm{V}''_\mathrm{Y}$ dominating both. From this and (6.9) the independence of f under change of compactification follows easily.

7. Filtrations in Homotopy Theory.

This section deals with the question of when certain filtrations on a differential algebra pass to the minimal model. Our discussion is very limited, and the application we have in mind is the weight filtration on the forms on a mixed Hodge diagram.

Definition **(7.1)**. — If $\mathcal{M}$ is a minimal algebra, then a filtration $W(\mathcal{M})$ is a *minimal filtration* if both d and $\wedge$ are strictly compatible with W.

As we shall see (7.3), a minimal filtration is determined up to isomorphism by its effect on cohomology. We, as always, want to work by induction over the canonical series for a minimal model. Our first result makes this possible when dealing with minimal filtrations.

Lemma **(7.2)**. — *Let $\mathcal{M}$ be a minimal differential algebra and $W(\mathcal{M})$ a filtration. Let $\{\mathcal{M}_\alpha\}$ be the canonical series for $\mathcal{M}$. Then $W(\mathcal{M})$ is a minimal filtration if and only if the following four conditions hold*:

a) W *restricted to each $\mathcal{M}_\alpha$ is a minimal filtration.*
b) *Let $V_{\alpha+1}$ be $I(\mathcal{M}_{\alpha+1})/I(\mathcal{M}_\alpha)$. As a subquotient of $\mathcal{M}_{\alpha+1}$ it receives a filtration, $W(V_{\alpha+1})$. The map $d: V_{\alpha+1} \to H(\mathcal{M}_\alpha)$ is strictly compatible with the filtrations.*
c) *The map $i^*: H(\mathcal{M}_{\alpha+1}) \to V_{\alpha+1}$ is strictly compatible with the filtrations.*
d) *The filtration on $\mathcal{M}_{\alpha+1}$ is isomorphic to the multiplicative extension of the one on $\mathcal{M}_\alpha$ and the one on $V_{\alpha+1}$.*

Proof. — Let us prove the necessity of the four conditions, leaving the sufficiency to the reader. Suppose that $W(\mathcal{M})$ is minimal and that inductively we have shown that $W(\mathcal{M}_\alpha)$ is also minimal. We have an exact sequence:

$$\dots \to H^k(\mathcal{M}_\alpha) \to H^k(\mathcal{M}_{\alpha+1}) \xrightarrow{i^*} V_{\alpha+1} \to H^{k+1}(\mathcal{M}_\alpha) \to H^{k+1}(\mathcal{M}_{\alpha+1}) \to \dots$$

(We assume for definiteness that the degree of the extension is k.) Define the relative cochain complex as before: $C^n = (\mathcal{M}^n \oplus \mathcal{M}^{n-1})$ with $d: C^n \to C^{n+1}$ sending (a, b) to $(-da, db+a)$. The vector space $V_{\alpha+1}$ is identified with $H^{k+1}(C)$. Giving C the direct sum filtration induces a filtration on $V_{\alpha+1}$. It follows easily that both i^* and d in the above sequence are strictly compatible with the filtrations, when one uses this filtration on $V_{\alpha+1}$. However, this filtration and the one defined in part b) are the same. Condition d) for $\mathcal{M}_{\alpha+1}$ is a consequence of b) and c) and lemma (1.3) *b)*. Condition a) is an immediate consequence of b), c), and d).

We also will need a technical lemma about the finiteness of minimal filtrations.

Lemma **(7.2)** a). — *Let $(\mathcal{M}, W)$ be a positive minimal filtration (positive means that $W_0(\mathcal{M})$ is only the constants). Suppose that $W_i(H(\mathcal{M}))$ is finite dimensional for all i. Then $W_i(\mathcal{M})$ is finite for all i.*

Proof. — We prove by induction on k that $W_k(\mathcal{M})$ is finite dimensional. Since the filtration $W(\mathcal{M})$ is the multiplicative extension of the filtration on the indecomposables it suffices to show that W_k(indecomposables) is finite dimensional. If W_{k-1} is finite dimensional, then W_k, when restricted to the decomposables, is finite dimensional. Since the image of d is contained in the decomposables, $d(W_k)$ must be finite dimensional. On the other hand the kernel of d, when restricted to the indecomposables, injects into the cohomology. Thus kernel($d\,|\,W_k$(indecomposables)) must also be finite dimensional. Consequently, W_k(indecomposables) is finite dimensional, and hence W_k itself is finite dimensional.

Proposition **(7.3)**. — *Let $\mathcal{M}$ and $\mathcal{N}$ be i-minimal algebras with minimal filtrations, $i \leqq \infty$. Suppose $f:\mathcal{M}\to\mathcal{N}$ is compatible with the filtrations. If f is an isomorphism of differential algebras and induces isomorphism $f^*: H^\ell(\mathcal{M}) \to H^\ell(\mathcal{N})$ of filtered cohomology groups for $\ell \leqq i$, then f is an isomorphism of filtered algebras.*

Proof. — Since the canonical series $\{\mathcal{M}_\alpha\}$ and $\{\mathcal{N}_\alpha\}$ are naturally defined, $f:\mathcal{M}_\alpha\to\mathcal{N}_\alpha$ is an isomorphism for all α. We will show by induction on α that $f:\mathcal{M}_\alpha\to\mathcal{N}_\alpha$ is an isomorphism of filtered algebras. Suppose we know this for $f|\mathcal{M}_\alpha$. By condition (7.2) d) and proposition (1.3) c), if the map induced by f on the quotient of the indecomposables $f: I(\mathcal{M}_{\alpha+1})/I(\mathcal{M}_\alpha) \to I(\mathcal{N}_{\alpha+1})/I(\mathcal{N}_\alpha)$ is an isomorphism of filtered vector spaces, then $f:\mathcal{M}_{\alpha+1}\to\mathcal{N}_{\alpha+1}$ is an isomorphism of filtered algebras.

Let $V_{\alpha+1} = I(\mathcal{M}_{\alpha+1})/I(\mathcal{M}_\alpha)$ and $U_{\alpha+1} = I(\mathcal{N}_{\alpha+1})/I(\mathcal{N}_\alpha)$. We have a commutative diagram:

$$\begin{tikzcd}
{} \arrow[r] & H^k(\mathcal{M}_{\alpha+1}) \arrow[r] \arrow[d, "\cong"', "f^*"] & V_{\alpha+1} \arrow[r] \arrow[d, "\cong"', "f^*"] & H^{k+1}(\mathcal{M}_\alpha) \arrow[r] \arrow[d, "\cong"', "f^*"] & H^{k+1}(\mathcal{M}_{\alpha+1}) \arrow[r] \arrow[d, "\cong"', "f^*"] & {} \\
{} \arrow[r] & H^k(\mathcal{N}_{\alpha+1}) \arrow[r] & U_{\alpha+1} \arrow[r] & H^{k+1}(\mathcal{N}_\alpha) \arrow[r] & H^{k+1}(\mathcal{N}_{\alpha+1}) \arrow[r] & {}
\end{tikzcd}$$

where k is the degree of the Hirsch extension for $\mathcal{M}_\alpha \subset \mathcal{M}_{\alpha+1}$, $k \leqq i$. By induction the third vertical map is an isomorphism of filtered vector spaces. Since the closed forms of $\mathcal{M}_{\alpha+1}$ of degree k equal the closed forms of $\mathcal{M}$ of degree k, $H^k(\mathcal{M}_{\alpha+1}) = H^k(\mathcal{M})$ and $H^k(\mathcal{N}_{\alpha+1}) = H^k(\mathcal{N})$ as filtered vector spaces. By hypothesis $f^*: H^k(\mathcal{M}) \to H^k(\mathcal{N})$ is an isomorphism of filtered vector spaces, since $k \leqq i$. Thus the first vertical map in our diagram is an isomorphism of filtered vector spaces. By conditions (7.2) *b)* and *c)*, it follows that $f^*: V_{\alpha+1} \to U_{\alpha+1}$ is also.

Definition **(7.4)**. — Let A be a differential algebra with a filtration W(A). The filtration *passes to the minimal model* if there is a minimal model for A, $\rho:\mathcal{M}\to A$, and a minimal filtration $W(\mathcal{M})$, such that ρ is compatible with the filtrations and is a quasi-isomorphism. Such a minimal model together with its filtration $(\mathcal{M}, W)$ is called a *filtered minimal model* for (A, W).

Proposition **(7.5)**. — a) *Given two i-minima lmodels* $\rho_j : (\mathscr{M}_j, \mathrm{W}) \to (\mathrm{A}, \mathrm{W})$ $(j=1,2)$, *where each* $(\mathscr{M}_j, \mathrm{W})$ *is a minimal filtration as in* (7.4), *there is an isomorphism of filtered minimal algebras* $\mathrm{I} : \mathscr{M}_1 \to \mathscr{M}_2$ *and a homotopy* H *from* ρ_1 *to* $\rho_2 \circ \mathrm{I}$ *which is compatible with the filtrations.*

b) *Given two filtered algebras and a map between them which is compatible with the filtrations and which is a quasi-isomorphism, then one filtration passes to the minimal model if and only if the other does.*

c) *Let* $f : (\mathrm{A}, \mathrm{W}) \to (\mathrm{B}, \mathrm{W}')$. *Suppose that both* W *and* W′ *pass to the minimal model. Then* f *induces a map* $\hat{f}$ *between the filtered minimal models. The map* $\hat{f}$ *is compatible with the filtrations and is well-defined up to a homotopy compatible with the filtrations.*

Proof. — Suppose that we have:

$$\begin{tikzcd} & (\mathrm{A}, \mathrm{W}) \arrow[d, "f"] \\ (\mathscr{M}, \mathrm{W}) \arrow[r, "\rho"] & (\mathrm{B}, \mathrm{W}) \end{tikzcd}$$

with f a quasi-isomorphism and $(\mathscr{M}, \mathrm{W})$ a minimal filtration. We will construct $\tilde{\rho} : (\mathscr{M}, \mathrm{W}) \to (\mathrm{A}, \mathrm{W})$ and a homotopy $\mathrm{H} : \mathscr{M} \to \mathrm{B} \otimes (t, dt)$ from ρ to $f \circ \tilde{\rho}$ which is compatible with the filtrations. We construct $\tilde{\rho}$ and H inductively over the canonical series $\{\mathscr{M}_\alpha\}$ for $\mathscr{M}$. Suppose that we have already defined them over $\mathscr{M}_\alpha$ and that $\mathscr{M}_{\alpha+1} = \mathscr{M}_\alpha \otimes \Lambda(\mathrm{V})_k$. We can choose this decomposition of $\mathscr{M}_{\alpha+1}$ so that the filtration on $\mathscr{M}_{\alpha+1}$ is the multiplicative extension of the one on $\mathscr{M}_\alpha$ and the one on V.

The existence of the homotopy H from $f \circ \tilde{\rho}$ to $\rho | \mathscr{M}_\alpha$ implies that on the level of spectral sequences we have a commutative diagram:

$$\begin{tikzcd} & \mathrm{E}(\mathrm{A}) \arrow[d, "\mathrm{E}(f)"] \\ \mathrm{E}(\mathscr{M}_\alpha) \arrow[ur, "\mathrm{E}(\tilde{\rho})"] \arrow[r, "\mathrm{E}(\rho|)"'] & \mathrm{E}(\mathrm{B}) \end{tikzcd}$$

The condition that f is a quasi-isomorphism is just the condition that $\mathrm{E}_i(f)$ is an isomorphism for all $i \geqq 1$.

If $v \in \mathrm{W}_\ell(\mathrm{V})$, then $\tilde{\rho}(dv) \in \mathrm{W}_\ell(\mathrm{A})$. Consider its class $[\tilde{\rho}(dv)]$ in $\mathrm{E}_1^\ell(\mathrm{A})$. The image of this class in $\mathrm{E}_1^\ell(\mathrm{B})$ is $\mathrm{E}(\rho|)([dv])$. Since $[dv]$ in $\mathrm{E}_1^\ell(\mathscr{M})$ is trivial it follows that $[\tilde{\rho}(dv)]$ is zero in $\mathrm{E}_1^\ell(\mathrm{A})$. Thus $\tilde{\rho}(dv)$ is da for some $a \in \mathrm{W}_\ell(\mathrm{A})$. Consequently, we can define a map $\tilde{\rho} : \mathscr{M}_{\alpha+1} \to \mathrm{A}$ which extends the given map $\tilde{\rho} : \mathscr{M}_\alpha \to \mathrm{A}$ and which is compatible with the filtrations. The obstruction to extending the homotopy H over $\mathscr{M}_{\alpha+1}$ assigns to each $v \in \mathrm{W}_\ell(\mathrm{V})$ the cohomology class $\left[\rho(v) - f \circ \tilde{\rho}(v) + \int_0^1 \mathrm{H}(dv)\right]$. Clearly this class is in $\mathrm{W}_\ell(\mathrm{H}(\mathrm{B}))$. Since f^* is a quasi-isomorphism, this class is the image of a class in $\mathrm{W}_\ell(\mathrm{H}(\mathrm{A}))$. This allows us to change our extension, $\tilde{\rho}$, over $\mathscr{M}_{\alpha+1}$, keeping it compatible with the filtrations, so that $\left[\rho(v) - f \circ \tilde{\rho}(v) + \int_0^1 \mathrm{H}(dv)\right]$ is zero in $\mathrm{E}_1^\ell(\mathrm{B})$.

Once we have made such a modification of $\widetilde{\rho}$ the extension of H is determined by choosing, linearly in V, $b_v \in \mathrm{B}$ such that:

1) $db_v = \rho(v) - f \circ \widetilde{\rho}(v) + \int_0^1 \mathrm{H}(dv)$, and

2) $b_v \in \mathrm{W}_\ell(\mathrm{B})$ whenever $v \in \mathrm{W}_\ell(\mathrm{V})$.

Then $\mathrm{H}(v) = \rho(v) + \int_0^t \mathrm{H}(dv) - db_v \otimes t - (-1)^{\deg(b_v)} b_v \otimes dt$. This completes the induction.

Applying this result to:

$$\begin{tikzcd} & & (\mathcal{M}', \mathrm{W}) \arrow[d, "\rho'"] \\ (\mathcal{M}, \mathrm{W}) \arrow[r, "\rho"] \arrow[urr, dashed, "\hat{f}"] & (\mathrm{A}, \mathrm{W}) \arrow[r, "f"] & (\mathrm{B}, \mathrm{W}) \end{tikzcd}$$

where $(\mathcal{M}, \mathrm{W})$ is a filtered minimal model for (A, W) and $(\mathcal{M}', \mathrm{W})$ is a filtered minimal model for (B, W), proves part c) of (7.5).

Applying the result to:

$$\begin{tikzcd} & (\mathrm{A}, \mathrm{W}) \arrow[d, "f"] \\ (\mathcal{M}, \mathrm{W}) \arrow[r, "\rho"] \arrow[ur, dashed, "\widetilde{\rho}"] & (\mathrm{B}, \mathrm{W}) \end{tikzcd}$$

where $(\mathcal{M}, \mathrm{W})$ is a filtered minimal model for (B, W) and f is a quasi-isomorphism, proves part b).

Applying it to:

$$\begin{tikzcd} & (\mathcal{M}', \mathrm{W}) \arrow[d, "\rho'"] \\ (\mathcal{M}, \mathrm{W}) \arrow[r, "\rho"] \arrow[ur, dashed] & (\mathrm{A}, \mathrm{W}) \end{tikzcd}$$

where both $\mathcal{M}$ and $\mathcal{M}'$ are filtered minimal models for (A, W), produces an isomorphism $\mathrm{I} : \mathcal{M} \to \mathcal{M}'$ compatible with the filtrations. I induces an isomorphism of filtered cohomology. By proposition (7.3), I is an isomorphism of filtered differential algebras.

Corollary **(7.6)**. — *Suppose that* (A, W) *passes to the minimal model. Then a filtered minimal model for* (A, W), $(\mathcal{M}, \mathrm{W})$, *is well-defined up to isomorphism. Furthermore, the isomorphism between any two such is itself well-defined up to homotopy compatible with the filtrations.*

Proof. — The uniqueness of $(\mathcal{M}, \mathrm{W})$ up to isomorphism follows immediately from (7.5) a). The uniqueness of the isomorphism up to homotopy compatible with the filtrations follows from (7.5) c).

Theorem **(7.7)**. — 1) *Let* (A, W) *be a filtered differential algebra defined over* $k \subset \mathbf{C}$. *Suppose* $(\mathrm{A}_{\mathbf{C}}, \mathrm{W})$ *passes to the minimal model. Then* (A, W) *does also.*

2) *Given two minimal filtrations on the same minimal model,* $(\mathscr{M}, \mathrm{W})$ *and* $(\mathscr{M}, \mathrm{W}')$, *such that over* $\mathbf{C}$ *the identity is homotopic to a filtered isomorphism, then the same is true over the field of definition.*

Proof. — 1) We construct $\{\mathscr{M}_\alpha \xrightarrow{\rho_\alpha} \mathrm{A}\}$, the minimal model for A. By induction we assume that at each stage we have a minimal filtration on $\mathscr{M}_\alpha$ so that ρ_α is compatible with the filtrations and so that ρ_α^* is strictly compatible with the induced filtrations on cohomology. Suppose we have $(\mathscr{M}_\alpha, \mathrm{W})$ as hypothesized. Let $\mathrm{V}_{\alpha+1}$ be the first non-zero cohomology group of $(\mathscr{M}_\alpha, \mathrm{A})$. The relative cochains $\mathrm{C}(\mathscr{M}_\alpha, \mathrm{A})$ have the direct sum filtration. This induces a filtration on $\mathrm{V}_{\alpha+1}$. Let $\mathscr{M}_{\alpha+1}$ be the filtered algebra $\mathscr{M}_\alpha \otimes \Lambda(\mathrm{V}_{\alpha+1})$. By choosing a splitting for the quotient map: $\mathrm{V}_{\alpha+1} \xrightarrow{s} \mathscr{Z}(\mathscr{M}_\alpha, \mathrm{A})$ we define $d : \mathrm{V}_{\alpha+1} \to \mathscr{M}_\alpha$ and a map of differential algebras $\rho_{\alpha+1} : \mathscr{M}_{\alpha+1} \to \mathrm{A}$. If we choose s to be compatible with the filtrations (and hence strictly compatible), then the filtration on $\mathscr{M}_{\alpha+1}$ will be minimal and $\rho_{\alpha+1}$ will be compatible with the filtrations. $\mathscr{M}_{\alpha+1} \xrightarrow{\rho_{\alpha+1}} \mathrm{A}$ is the next stage in the construction of the minimal model for A. Lastly, to prove 1) we must show that $\rho_{\alpha+1}^*$ is strictly compatible with the filtrations. We know that there is a minimal filtration on some minimal model for $(\mathrm{A}_{\mathbf{C}}, \mathrm{W})$, say $(\mathscr{N}, \mathrm{W}) \xrightarrow{\gamma} (\mathrm{A}_{\mathbf{C}}, \mathrm{W})$. For this minimal model $\gamma_{\alpha+1}^*$ is strictly compatible with the filtrations. By (7.5) and (7.3) there is a filtered isomorphism $\mathrm{I} : (\mathscr{M}_{\alpha+1}, \mathrm{W})_{\mathbf{C}} \to (\mathscr{N}_{\alpha+1}, \mathrm{W})$ so that $\gamma_{\alpha+1} \circ \mathrm{I}$ is homotopic to $(\rho_{\alpha+1})_{\mathbf{C}}$. Thus $(\rho_{\alpha+1})_{\mathbf{C}}^* = \gamma_{\alpha+1}^* \circ \mathrm{I}$ is strictly compatible with the filtrations. Thus so is $\rho_{\alpha+1}^*$.

2) Suppose inductively that we have $\varphi : \mathscr{M}_\alpha \to \mathscr{M}_\alpha$, homotopic to the identity, which is compatible with W and W′. We can extend φ to a map $\varphi : \mathscr{M} \to \mathscr{M}$ homotopic to the identity. By (1.3) *b)* we can find isomorphisms $\lambda : \mathscr{M}_\alpha \otimes \Lambda(\mathrm{V}_{\alpha+1}) \to \mathscr{M}_{\alpha+1}$ and $\lambda' : \mathscr{M}_\alpha \otimes \Lambda(\mathrm{V}_{\alpha+1}) \to \mathscr{M}_{\alpha+1}$ so that under the first W is the product filtration, and under the second W′ is. Since they are homotopic over $\mathbf{C}$, W and W′ induce the same filtration on $\mathrm{V}_{\alpha+1}$. Using the isomorphisms λ and λ', the map $\varphi | \mathscr{M}_{\alpha+1}$ sends $v \in \mathrm{V}$ to $v + x_v$. If whenever $v \in \mathrm{W}_i(\mathrm{V})$, $x_v \in \mathrm{W}'_i$, then φ would be compatible with the filtrations. Suppose that for some $v \in \mathrm{W}_i(\mathrm{V})$, $x_v \notin \mathrm{W}'_i$, say $[x_v] \in \mathrm{Gr}_n^{\mathrm{W}}$ is non-zero for some $n > i$. Since $dx_v \in \mathrm{W}'_i$, $[x_v]$ persists to ${}_{\mathrm{W}}\mathrm{E}_1^{-n} = {}_{\mathrm{W}}\mathrm{E}_\infty^{-n}$. If we can vary $\varphi | \mathscr{M}_{\alpha+1}$ by a homotopy which, when restricted to $\mathscr{M}_\alpha$, is compatible with the filtrations and which deforms $\varphi(v)$ into W'_{n-1}, then $[x_v] \in {}_{\mathrm{W}}\mathrm{E}_\infty^{-n}$ is zero. To see this, let $\mathrm{H} : \mathscr{M}_{\alpha+1} \to \mathscr{M}_{\alpha+1} \otimes (t, dt)$ be such a homotopy. Since $dv \in \mathrm{W}_i(\mathscr{M}_\alpha)$, $\mathrm{H}(dv) \in \mathrm{W}'_i$. Thus, if $\mathrm{H}(v) = \sum_j (\alpha_j t^j + \beta_j t^j \, dt)$, then we have:

1) $v + x_v = \alpha_0$,

2) $\sum_{j \geqq 0} \alpha_j \in \mathrm{W}_{n-1}$, and

3) $j\alpha_j \pm d\beta_{j-1} \in \mathrm{W}_i$.

Thus in $\mathrm{Gr}_n^{\mathrm{W}}$, $[x_v] = [\alpha_0] = -[\sum_{j \geqq 1} \alpha_j]$. Since $\alpha_j = \pm d\left(\frac{\beta_{j-1}}{j}\right)$ modulo W_i it follows that $[-\sum_{j \geqq 1} \alpha_j] = 0$ in ${}_{\mathrm{W}}\mathrm{E}_\infty^{-n}$.

Conversely, if $[x_v]=0$ in ${}_{\mathrm{W}}\mathrm{E}_{\infty}^{-n}$, we can vary φ by a homotopy relative to $\mathscr{M}_\alpha$ to make $\varphi(v)$ lie in W'_{n-1}. For if $[x_v]=0$, then $x_v=dy_v$ modulo W_{n-1}. Define a homotopy H to be the identity on $\mathscr{M}_\alpha$ and to send v to $v+x_v-d(y_v\otimes t)$.

By hypothesis we have $\psi:(\mathscr{M}_{\alpha+1})_{\mathbf{C}}\to(\mathscr{M}_{\alpha+1})_{\mathbf{C}}$ which sends W to W′ and which is homotopic to $\varphi_{\mathbf{C}}$. By (7.5) we can assume that this homotopy, when restricted to $\mathscr{M}_\alpha$, is compatible with the filtrations. This implies that $[x_v]\in{}_{\mathrm{W}}\mathrm{E}_{\infty}^{-n}(\mathscr{M}_\alpha)_{\mathbf{C}}$ is zero, and consequently so is $[x_v]\in{}_{\mathrm{W}}\mathrm{E}_{\infty}^{-n}(\mathscr{M}_\alpha)$. This allows us to deform φ so that:

$$\varphi(\mathrm{W}_i(\mathrm{V}_{\alpha+1}))\subset\mathrm{W}'_{n-1}.$$

Continuing by induction on n, we can finally deform φ until $\varphi(\mathrm{W}_i(\mathrm{V}_{\alpha+1}))\subset\mathrm{W}'_i$. This means that φ is compatible with the filtrations W and W′ on all of $\mathscr{M}_{\alpha+1}$.

Theorem **(7.8)**. — a) *Let* $(\mathrm{A},\mathrm{W})_{\mathbf{C}}\to(\mathscr{E},\mathrm{W},\mathrm{F})$ *be a mixed Hodge diagram defined over* $\mathbf{R}$, *and let* $(\mathrm{A}_0,\mathrm{W})$ *be a filtered algebra defined over* $\mathbf{Q}$. *Suppose that we have a quasi-isomorphism* $(\mathrm{A}_0,\mathrm{W})_{\mathbf{R}}\to(\mathrm{A},\mathrm{W})$. *Then* (A, Dec W) *and* $(\mathrm{A}_0, \mathrm{Dec}\,\mathrm{W})$ *pass to the minimal model.*

b) *Let* X *be an algebraic variety. Make choices needed to define* (E(X), W). *Let* $(\mathscr{M},\mathrm{W})$ *be its filtered minimal model. The map* $\mathrm{E}(\mathrm{X})\to\mathscr{E}(\mathrm{X})$ *identifies* $\mathscr{M}$ *with the minimal model of* $\mathscr{E}(\mathrm{X}),\mathscr{M}_{\mathrm{X}}$. *The induced filtration on* $\mathscr{M}_{\mathrm{X}}$ *is well-defined up to automorphism of* $\mathscr{M}_{\mathrm{X}}$ *homotopic to the identity, independent of the choices.*

Proof. — By (6.6) the minimal model $\mathscr{N}$ for $\mathscr{E}$ has a bigrading so that the induced total filtration is quasi-isomorphic to $(\mathscr{E},\mathrm{Dec}\,\mathrm{W})$. By (7.6) and (7.7), (A, Dec W) and $(\mathrm{A}_0,\mathrm{Dec}\,\mathrm{W})$ also pass to the minimal model. In the case of mixed Hodge diagrams for an algebraic variety X, we have various choices for (E(X), W) and we do not yet know that they are quasi-isomorphic *via* a quasi-isomorphism compatible with the restriction maps into $\mathscr{E}(\mathrm{X})$. However, by (3.5), this is true for their complex versions. Thus, given two such (E(X), W) and (E(X)′, W), their filtrations pass to the minimal models $(\mathscr{M},\mathrm{W})$ and $(\mathscr{M}',\mathrm{W})$. The identifications with $\mathscr{M}_{\mathrm{X}}$ induce an isomorphism $\varphi:\mathscr{M}\to\mathscr{M}'$ which over $\mathbf{C}$ is homotopic to a filtered isomorphism. Result (7.7) implies that φ itself is homotopic to a filtered isomorphism.

8. Mixed Hodge Structures for Minimal Models.

The first half of this section deals with generalities on extensions of mixed Hodge structures. The second half of the section is devoted to proving there is a family of mixed Hodge structures on the minimal model of a mixed Hodge diagram.

Let A and C be k-vector spaces ($k\subset\mathbf{R}$) with given mixed Hodge structures (A, W, F) and (C, W, F). We wish to classify all short exact sequences (extensions):

$$(*)\qquad 0\to\mathrm{A}\xrightarrow{\varphi}\mathrm{B}\xrightarrow{\psi}\mathrm{C}\to 0$$

where B has a mixed Hodge structure and φ and ψ are morphisms of mixed Hodge structures. The extension $0 \to A \xrightarrow{\varphi'} B' \xrightarrow{\psi'} C \to 0$ is equivalent to $(*)$ if and only if there is a commutative diagram of mixed Hodge structures:

$$\begin{tikzcd} & & B \arrow[dd, "\mathrm{I}", "\cong"'] \arrow[dr, "\psi"] & & \\ 0 \arrow[r] & A \arrow[ur, "\varphi"] \arrow[dr, "\varphi'"'] & & C \arrow[r] & 0 \\ & & B' \arrow[ur, "\psi'"'] & & \end{tikzcd}$$

We will classify extensions up to equivalence.

If X and Y are vector spaces with filtrations, W(X), F(X), W(Y), and F(Y), denote by $\mathrm{Hom}^{W}(X, Y)$ and $\mathrm{Hom}^{W}_{F}(X, Y)$ the subspace of homomorphisms compatible with W, and those compatible with both W and F.

Proposition **(8.1)**. — *There is a natural one-to-one correspondence between equivalence classes of extensions of* C *by* A *and*:

$$\mathrm{Hom}^{W}(C_{\mathbf{C}}, A_{\mathbf{C}})/\{\mathrm{Hom}^{W}_{F}(C_{\mathbf{C}}, A_{\mathbf{C}}) + (\mathrm{Hom}^{W}(C, A))_{\mathbf{C}}\}.$$

Proof. — Let $0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$ be an extension of mixed Hodge structures. Then φ and ψ are strictly compatible with W. We choose (unnaturally) a splitting:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i_1"] \arrow[dr, "\varphi"'] & A\oplus C \arrow[r, "\pi_2"] \arrow[d, "\mathrm{I}_1", "\cong"'] & C \arrow[r] & 0 \\ & & B \arrow[ur, "\psi"'] & & \end{tikzcd}$$

such that I_1 sends the direct sum filtration $W(A\oplus C)$ isomorphically onto W(B). In particular $C \overset{i_2}{\hookrightarrow} A\oplus C \xrightarrow{\mathrm{I}_1} B$ is strictly compatible with the filtrations W.

Over the complex numbers we choose any splitting:

$$\begin{tikzcd} 0 \arrow[r] & A_{\mathbf{C}} \arrow[r, "i_1"] \arrow[dr, "\varphi"'] & A_{\mathbf{C}}\oplus C_{\mathbf{C}} \arrow[r, "\pi_2"] \arrow[d, "\mathrm{I}_2", "\cong"'] & C_{\mathbf{C}} \arrow[r] & 0 \\ & & B_{\mathbf{C}} \arrow[ur, "\psi"'] & & \end{tikzcd}$$

such that:

(†) $\qquad \mathrm{I}_2 : (A_{\mathbf{C}}\oplus C_{\mathbf{C}}, W, F) \to (B_{\mathbf{C}}, W, F)$

is an isomorphism of bifiltered vector spaces. To show that such an I_2 exists consider the decomposition:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_C & \xrightarrow{\varphi} & B_C & \xrightarrow{\psi} & C_C & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ 0 & \longrightarrow & \bigoplus_{p,q} A^{p,q} & \xrightarrow{\varphi} & \bigoplus_{p,q} B^{p,q} & \xrightarrow{\psi} & \bigoplus_{p,q} C^{p,q} & \longrightarrow & 0 \end{array}$$

Since φ and ψ preserve the direct sum structures, we can choose a splitting for $\psi : B_C \to C_C$ which sends $C^{p,q}$ into $B^{p,q}$. Let $I_2 : A \oplus C \to B$ be the isomorphism induced by this splitting. It induces $I_2 : A^{p,q} \oplus C^{p,q} \xrightarrow{\cong} B^{p,q}$ for all p and q. The composition:

$$C \to A_C \oplus C_C \xrightarrow{I_2} B_C$$

is strictly compatible with both W and F. It is not, however, necessarily a map of mixed Hodge structures since it is not defined over k. This proves that a map as required by (†) exists.

The difference of the two splittings $(I_1)_C$ and I_2 is a homomorphism $d : C_C \to A_C$. Since both I_1 and I_2 are strictly compatible with W and since W(A) is the restriction of W(B) to A, it follows that d is compatible with W. The difference, d, is in general only a complex linear map since I_2 is only defined over $\mathbf{C}$. We are free to vary I_1 exactly by any element α in $\mathrm{Hom}^W(C, A)$, and to vary I_2 exactly by any element β in $\mathrm{Hom}_F^W(C_C, A_C)$. Changing I_1 and I_2 in this manner changes d by $\alpha+\beta$. Thus:

(8.2) $$[d] \in \mathrm{Hom}^W(C_C, A_C)/\{(\mathrm{Hom}_F^W(C_C, A_C)) + (\mathrm{Hom}^W(C, A))_C\}$$

is a well-defined invariant of the extension. Clearly, it remains unchanged if we replace the extension by an equivalent one.

Conversely, given two extensions $0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$ and $0 \to A \xrightarrow{\varphi'} B' \xrightarrow{\psi'} C \to 0$ whose difference invariants (8.2) are the same, we can choose splittings for ψ and ψ' $A \oplus C \underset{\cong}{\xrightarrow{I_1}} B$, $A \oplus C \underset{\cong}{\xrightarrow{I_1'}} B'$, $A_C \oplus C_C \underset{\cong}{\xrightarrow{I_2}} B_C$ and $A_C \oplus C_C \underset{\cong}{\xrightarrow{I_2'}} B_C'$ as before, such that the difference homomorphisms $d : C_C \to A_C$ and $d' : C_C \to A_C$ are equal. Let $I : B \to B'$ be given by $B \xrightarrow{I_1^{-1}} A \oplus C \xrightarrow{I_1'} B'$. This composition is a k-linear isomorphism of filtered vector spaces $(B, W) \to (B', W)$. Since the difference element for $(I_1)_C - I_2$ equals that for $(I_1')_C - I_2'$, $I : B_C \to B_C'$ is also the composition $B_C \xrightarrow{I_2^{-1}} A_C \oplus C_C \xrightarrow{I_2'} B_C'$. Thus I_C is an isomorphism which is compatible with F. Consequently $I : B \to B'$ is an isomorphism and a morphism of mixed Hodge structures, *i.e.* it is an isomorphism of mixed Hodge structures. Clearly $\psi' I = \psi$ and $I\varphi = \varphi'$.

Lastly, we show all classes $[d]$ in (8.2) arise as the invariants of extensions of mixed Hodge structures. We will make use of the next lemma.

Lemma **(8.3).** — *Suppose given* $0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$ *an exact sequence over* k, *and filtrations* $W(A)$, $W(B)$, $W(C)$, $F(A_{\mathbf{C}})$, $F(B_{\mathbf{C}})$, *and* $F(C_{\mathbf{C}})$ *so that* φ *and* ψ *are compatible with all filtrations. Suppose also that* (A, W, F) *and* (C, W, F) *define mixed Hodge structures. Then for* (B, W, F) *to be a mixed Hodge structure it is necessary and sufficient that* φ *and* ψ *strictly preserve* W *and that on the associated graded objects* $\mathrm{Gr}^W(\varphi)$ *and* $\mathrm{Gr}^W(\psi)$ *be strictly compatible with the filtrations induced by* F.

Proof. — Necessity follows immediately from theorem (1.12). We consider sufficiency. If φ and ψ are strictly compatible with W, then:

$$0 \longrightarrow \mathrm{Gr}_n^W(A) \xrightarrow{\mathrm{Gr}^W(\varphi)} \mathrm{Gr}_n^W(B) \xrightarrow{\mathrm{Gr}^W(\psi)} \mathrm{Gr}_n^W(C) \longrightarrow 0$$

is a short exact sequence. Thus to prove sufficiency we need only show that if $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is a short exact sequence strictly compatible with filtrations $F(X_{\mathbf{C}})$, $F(Y_{\mathbf{C}})$, and $F(Z_{\mathbf{C}})$, and if (X, F) and (Z, F) are Hodge structures of weight n, then (Y, F) is a Hodge structure of weight n.

(Y, F) is a Hodge structure of weight n if and only if for any $p+q=n+1$ $(F^p(Y_{\mathbf{C}}) \oplus \bar{F}^q(Y_{\mathbf{C}})) = Y_{\mathbf{C}}$ (see [4], (1.2.5)). We show first that for $p+q=n+1$:

$$F^p(Y_{\mathbf{C}}) \cap \bar{F}^q(Y_{\mathbf{C}}) = 0.$$

Since g preserves F and $\bar{F}$, we have that:

$$g(F^p(Y_{\mathbf{C}}) \cap \bar{F}^q(Y_{\mathbf{C}})) \subset F^p(Z_{\mathbf{C}}) \cap \bar{F}^q(Z_{\mathbf{C}}) = 0.$$

Thus $F^p(Y_{\mathbf{C}}) \cap \bar{F}^q(Y_{\mathbf{C}}) \subset \mathrm{Im}(f)$. Since f is injective and strictly compatible with F and $\bar{F}$, we have:

$$F^p(Y_{\mathbf{C}}) \cap \bar{F}^q(Y_{\mathbf{C}}) \subset f(F^p(X_{\mathbf{C}})) \cap f(\bar{F}^q(X_{\mathbf{C}})) = f(F^p(X_{\mathbf{C}}) \cap \bar{F}^q(X_{\mathbf{C}})) = f(0) = 0.$$

Now we show that if $p+q=n+1$, then $F^p(Y_{\mathbf{C}}) + \bar{F}^q(Y_{\mathbf{C}}) = Y_{\mathbf{C}}$. Let $y \in Y_{\mathbf{C}}$. Then $g(y) \in F^p(Z_{\mathbf{C}}) + \bar{F}^q(Z_{\mathbf{C}}) = g(F^p(Y_{\mathbf{C}})) + g(\bar{F}^q(Y_{\mathbf{C}}))$. This allows us to assume that $g(y)=0$. Then $y \in \mathrm{Im}(f) = f(F^p(X)) + f(\bar{F}^q(X_{\mathbf{C}}))$. Consequently:

$$y \in F^p(Y_{\mathbf{C}}) + \bar{F}^q(Y_{\mathbf{C}}).$$

This completes the proof of the lemma.

To complete the proof of (8.1), let $d : C_{\mathbf{C}} \to A_{\mathbf{C}}$ be any complex linear homomorphism compatible with W. Form:

$$A_{\mathbf{C}} \oplus C_{\mathbf{C}} \xrightarrow{\left(\begin{array}{c|c} \mathrm{Id}_A & d \\ \hline 0 & \mathrm{Id}_C \end{array}\right) = \alpha_d} (A \oplus C)_{\mathbf{C}}$$

Endow $A \oplus C$ with the direct sum filtration W. Push the direct sum filtration F^* on $A_{\mathbf{C}} \oplus C_{\mathbf{C}}$ forward *via* α_d to one $F_d((A \oplus C)_{\mathbf{C}})$.

Claim. — $(A \oplus C, W, F_d)$ is a mixed Hodge structure defined over k. Its difference-element is $[d]$.

Proof. — $A \oplus C$ is a k-vector space with a filtration W defined over k. Since d is compatible with W, α_d is strictly compatible with W. We have a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & A_{\mathbf{C}} \arrow[r, "i_1"] & (A \oplus C)_{\mathbf{C}} \arrow[r, "\pi_2"] & C_{\mathbf{C}} \arrow[r] & 0 \\
0 \arrow[r] & A_{\mathbf{C}} \arrow[r, "i_1"] \arrow[u, equal] & A_{\mathbf{C}} \oplus C_{\mathbf{C}} \arrow[r, "\pi_2"] \arrow[u, "\alpha_d"] & C_{\mathbf{C}} \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

If we use the direct sum filtrations W and F in the lower sequence then i_1 and π_2 are strictly compatible with W and the filtration induced by F on Gr^W. Since α_d sends W to W and F to F_d, the same is true for W and F_d in the upper sequence. Applying lemma (8.3) proves that $(A_{\mathbf{C}}, W, F_d)$ is a mixed Hodge structure. Clearly, its difference element is d.

Definition **(8.4)**. — If A is a k-vector space with a filtration W(A), and X is a complex vector space with a bi-grading $X = \bigoplus_{p,q} X^{p,q}$, then an isomorphism $I : X \to A_{\mathbf{C}}$ *defines a mixed Hodge structure on* A if:

a) $I(\bigoplus_{p+q \leq \ell} X^{p,q}) = (W_\ell(A))_{\mathbf{C}}$, and

b) if we define $F^p(A_{\mathbf{C}}) = I(\bigoplus_{r \geq p} X^{r,s})$, then (A, W, F) is a mixed Hodge structure.

Corollary **(8.5)**. — *Let* $0 \to A_1 \to A_2 \to A_3 \to 0$ *be a short exact sequence of k-vector spaces with each* A_i *having a weight filtration* $W(A_i)$, $i = 1, 2, 3$. *Let* $0 \to X_1 \xrightarrow{f} X_2 \xrightarrow{g} X_3 \to 0$ *be a short exact sequence of complex vector spaces with each* X_i *having a bi-grading* $X_i = \bigoplus_{p,q} X_i^{p,q}$. *Let*:

$$\begin{tikzcd}
0 \arrow[r] & (A_1)_{\mathbf{C}} \arrow[r] & (A_2)_{\mathbf{C}} \arrow[r] & (A_3)_{\mathbf{C}} \arrow[r] & 0 \\
0 \arrow[r] & X_1 \arrow[r] \arrow[u, "\cong"', "I_1"] & X_2 \arrow[r] \arrow[u, "\cong"', "I_2"] & X_3 \arrow[r] \arrow[u, "\cong"', "I_3"] & 0
\end{tikzcd}$$

be a commutative diagram. If I_1 *and* I_3 *induce mixed Hodge structures and*

$$I_2(\bigoplus_{p+q \leq \ell} X_2^{p,q}) = (W_\ell(A_2))_{\mathbf{C}},$$

then I_2 *also induces a mixed Hodge structure.*

Proof. — This is an immediate consequence of lemma (8.3).

At this point let us summarize our results to date. Let $\varphi : (E, W)_{\mathbf{C}} \to (\mathscr{E}, W, F)$ be a mixed Hodge diagram, and let $\mathscr{M} \xrightarrow{\rho} E$ be a minimal model for E, and $\mathscr{N} \xrightarrow{\psi} \mathscr{E}$ a complex minimal model for $\mathscr{E}$.

1) $\mathcal{N}$ has a bigrading $\mathcal{N} = \bigoplus_{p,q\geq 0} \mathcal{N}^{p,q}$ so that, if we define $W_k(\mathcal{N})$ to be $\bigoplus_{p+q\leq k} \mathcal{N}^{p,q}$ and $F^\ell(\mathcal{N})$ to be $\bigoplus_{p\geq \ell} \mathcal{N}^{p,q}$, then $(\mathcal{N}, W, F) \xrightarrow{\psi} (\mathcal{E}, \operatorname{Dec} W, F)$ is a quasi-isomorphism of bifiltered algebras. Given another such bigraded minimal model for $\mathcal{E}$, it is isomorphic to $\mathcal{N}$ by an isomorphism well-defined up to bigraded homotopy (6.6).

2) $\mathcal{M}$ has a minimal filtration $W(\mathcal{M})$ so that $\rho : (\mathcal{M}, W) \to (E, \operatorname{Dec} W)$ is a quasi-isomorphism. Any other such filtered minimal model $(\mathcal{M}', W)$ is isomorphic to $(\mathcal{M}, W)$ by an isomorphism unique up to homotopy compatible with the filtration (7.6).

Now we wish to meld these two results together. The fact that $\varphi : E_{\mathbf{C}} \to \mathcal{E}$ induces an isomorphism on cohomology implies that there is an isomorphism $I : \mathcal{N} \to \mathcal{M}_{\mathbf{C}}$ well-defined up to homotopy. Since $(E, \operatorname{Dec} W)_{\mathbf{C}} \xrightarrow{\varphi} (\mathcal{E}, \operatorname{Dec} W)$ is a quasi-isomorphism, the map I can be taken to be an isomorphism of filtered minimal models (7.5). Such a filtered isomorphism is well-defined up to homotopy compatible with the filtrations.

Theorem **(8.6)**. — *Any such isomorphism* I *as above defines a mixed Hodge structure on* $\mathcal{M}$. *The induced mixed Hodge structure on* $H(\mathcal{M})$ *agrees* via ρ^* *with the mixed Hodge structure that the mixed Hodge diagram defines on* $H(E)$.

Proof. — We prove that any such I induces a mixed Hodge structure by induction on the canonical series $\{\mathcal{M}_\alpha\}$ for $\mathcal{M}$ and $\{\mathcal{N}_\alpha\}$ for $\mathcal{N}$. Since the series are canonical, I restricted to $\mathcal{N}_\alpha$ induces an isomorphism $I : \mathcal{N}_\alpha \to (\mathcal{M}_\alpha)_{\mathbf{C}}$ for every α. We assume that for some fixed α, $I : \mathcal{N}_\alpha \to (\mathcal{M}_\alpha)_{\mathbf{C}}$ induces a mixed Hodge structure. We can write $\mathcal{M}_{\alpha+1} = \mathcal{M}_\alpha \otimes_d \Lambda(V)_k$ and $\mathcal{N}_{\alpha+1} = \mathcal{N}_\alpha \otimes_d \Lambda(V')_k$. We can suppose that the bigrading of $\mathcal{N}_{\alpha+1}$ induces one on V' and that the bigrading of $\mathcal{N}_{\alpha+1}$ induces the bigrading on $\mathcal{N}_\alpha$ (6.9). Likewise, we can assume that the filtration on $\mathcal{M}_{\alpha+1}$ is the multiplicative extension of the one on $\mathcal{M}_\alpha$ and the one induced on V (7.2).

We know already that the map induced by I on cohomology, $I : H(\mathcal{N}) \to H(\mathcal{M})_{\mathbf{C}}$ (which is the map φ^* when we make the natural identifications $H(\mathcal{M})_{\mathbf{C}} = H(E; \mathbf{C})$ and $H(\mathcal{N}) = H(\mathcal{E}; \mathbf{C})$) induces a mixed Hodge structure on cohomology. The reason is that the filtration $W(\mathcal{M})$ becomes the filtration Dec W on $H(E; \mathbf{Q})$ and the bigrading of $\mathcal{N}$ becomes the bigrading associated with the mixed Hodge structure on $H(\mathcal{E}; \mathbf{C})$. We have a commutative ladder of exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & H^k(\mathcal{M}_\alpha)_{\mathbf{C}} \arrow[r] & H^k(\mathcal{M})_{\mathbf{C}} \arrow[r] & V_{\mathbf{C}} \arrow[r] & H^{k+1}(\mathcal{M}_\alpha)_{\mathbf{C}} \arrow[r] & H^{k+1}(\mathcal{M})_{\mathbf{C}} \arrow[r, dashed] & {} \\
0 \arrow[r] & H^k(\mathcal{N}_\alpha) \arrow[r] \arrow[u, "\cong"', "I^*"] & H^k(\mathcal{N}) \arrow[r] \arrow[u, "\cong"', "I^*"] & V' \arrow[r] \arrow[u, "\cong"', "I^*"] & H^{k+1}(\mathcal{N}_\alpha) \arrow[r] \arrow[u, "\cong"', "I^*"] & H^{k+1}(\mathcal{N}) \arrow[r] \arrow[u, "\cong"', "I^*"] & {}
\end{tikzcd}$$

By the above discussion the second and fifth vertical arrows induce mixed Hodge structures. By the inductive hypothesis $I : \mathcal{N}_\alpha \to (\mathcal{M}_\alpha)_{\mathbf{C}}$ induces a mixed Hodge structure on $\mathcal{M}_\alpha$. The map $d : \mathcal{M}_\alpha \to \mathcal{M}_\alpha$ is compatible with both filtrations and hence is a morphism of mixed Hodge structures; by (1.12), 4) it follows that there is induced on $H(\mathcal{M}_\alpha)$ a mixed Hodge structure. This means that $I^* : H(\mathcal{N}_\alpha) \to H(\mathcal{M}_\alpha)_{\mathbf{C}}$ induces a mixed Hodge structure. Thus the first and fourth vertical arrows also induce mixed Hodge structures. Using (1.12), 4) and (8.5) we see that $I : V' \to V_{\mathbf{C}}$ induces a mixed Hodge structure. Now to finish the proof that $\mathcal{N}_{\alpha+1} \to (\mathcal{M}_{\alpha+1})_{\mathbf{C}}$ induces a mixed Hodge structure we filter both algebras as follows:

$$S_i(\mathcal{M}_{\alpha+1}) = \{\Sigma \omega \wedge v_{j_1} \wedge \ldots \wedge v_{j_t} \mid t \leq i, \ \omega \in \mathcal{M}_\alpha, \text{ and } v_{j_k} \in V\}$$

$$S_i(\mathcal{N}_{\alpha+1}) = \{\Sigma \omega \wedge v'_{j_1} \wedge \ldots \wedge v'_{j_t} \mid t \leq i, \ \omega \in \mathcal{N}_\alpha, \text{ and } v'_{j_k} \in V'\}.$$

Then I induces isomorphisms $I : S_i(\mathcal{N}_{\alpha+1}) \to S_i(\mathcal{M}_{\alpha+1})_{\mathbf{C}}$. One proves by induction on i that I induces a mixed Hodge structure, using (8.5).

Note. — 1) For forms of degree n in $\mathcal{M}$ the first possible non zero weight in the mixed Hodge structure is $\widetilde{W}_n$.

2) The mixed Hodge structure on $\mathcal{M}$ will depend in general on the choice of the homomorphism I.

Corollary **(8.7)**. — *Let* $(E, W)_{\mathbf{C}} \to (\mathcal{E}, W, F)$ *be a mixed Hodge diagram, and suppose that* (E', W) *is a filtered algebra defined over* $\mathbf{Q}$. *If* $(E', W)_{\mathbf{R}}$ *is quasi-isomorphic to* (E, W) *then any filtered isomorphism between the minimal model for* $\mathcal{E}$ *and the complex minimal model for* E' *defines a mixed Hodge structure over* $\mathbf{Q}$ *on the minimal model for* E'.

Proof. — This follows immediately from (8.6) and (7.7).

9. The Complex Homotopy Theory of Nonsingular Varieties and Mixed Hodge Structures.

In this section we deduce homotopy theoretic consequences of the bigrading (6.10) and the mixed Hodge structures (8.6) on the minimal model of a nonsingular variety. Throughout this section X is a nonsingular complex variety.

Theorem **(9.1)**. — *Suppose that* $\pi_1(X) = 0$.

a) *The homotopy groups* $\pi_n(X) \otimes \mathbf{Q}$ *have natural finite mixed Hodge structures*:

$$W_{-n}(\pi_n(X) \otimes \mathbf{Q}) = \pi_n(X) \otimes \mathbf{Q}.$$

The Whitehead product $\pi_n(X) \otimes \pi_m(X) \to \pi_{n+m-1}(X)$ *is a morphism of mixed Hodge structures.*

b) *The cohomology rings of the various stages of the rational Postnikov tower for* X, $H(X_n; \mathbf{Q})$, *have natural mixed Hodge structures with* $W_{\ell-1}(H^\ell(X_n; \mathbf{Q})) = 0$. *The maps* $X_n \xrightarrow{\pi} X_{n-1}$ *as well as* $X \to X_n$ *induce morphisms of mixed Hodge structures.*

c) *The rational k-invariants,* $k^{n+1} : [\pi_n(X) \otimes \mathbf{Q}]^* \to H^{n+1}(X; \mathbf{Q})$ *are morphisms of mixed Hodge structures.*

Proof. — Embed X as the complement of a divisor with normal crossings in a compact, non-singular variety V, $X = V - D$. Make the choices as required in sections 2 and 3 to define a filtered differential algebra $(E(X), W)$. This filtration passes to the minimal model by (7.7). Let $\mathcal{M}_X$ be the minimal model for $\mathcal{E}(X)$. By (7.8) the map $E(X) \to \mathcal{E}(X)$ induces a minimal filtration on $\mathcal{M}_X$ which, up to automorphism homotopic to the identity, is an invariant of X. Choosing an isomorphism from the bigraded minimal model for $\mathcal{E}(\log D)$ to $(\mathcal{M}_X, W)_{\mathbf{C}}$, $J : (\mathcal{N}, W) \xrightarrow{\cong} (\mathcal{M}_X, W)_{\mathbf{C}}$, induces a mixed Hodge structure on $\mathcal{M}_X$. The map J is defined only up to homotopy.

If we consider a rigid invariant of $\mathcal{M}_X$ such as the indecomposables, $I(\mathcal{M}_X)$, the cohomology $H(\mathcal{M}_X)$, or the cohomology of the various stages of the minimal model $H((\mathcal{M}_X)_\alpha)$, then it receives a canonical mixed Hodge structure from J. (A rigid invariant is one on which homotopic maps of minimal algebras induce the same map.) By (5.10), if $\pi_1(X) = 0$, then $I(\mathcal{M}_X)$ is the dual graded vector space to $\pi_*(X) \otimes \mathbf{Q}$, and $H((\mathcal{M}_X)_n)$ is the cohomology of the nth-stage of the Postnikov tower. Thus both these carry mixed Hodge structures. The map $d : I(\mathcal{M}_X)^{n+1} \to H^{n+2}((\mathcal{M}_X)_n)$, which is a morphism of mixed Hodge structures, becomes identified with the $(n+2)^{\text{nd}}$ k-invariant. The map $d : I(\mathcal{M}_X) \to I(\mathcal{M}_X) \wedge I(\mathcal{M}_X)$ is also a morphism of mixed Hodge structures and is dual to the Whitehead product:

$$(\pi_i(X) \otimes \mathbf{Q}) \otimes (\pi_j(X) \otimes \mathbf{Q}) \to \pi_{i+j-1}(X) \otimes \mathbf{Q}.$$

If we have an algebraic map $f : X \to X'$, then we can extend it over some completions to $\bar{f} : V \to V'$, where $V - X$ and $V' - X'$ are divisors with normal crossings. By (3.9), if we pick appropriate mixed Hodge diagrams for X and X′, then $\bar{f}$ will induce a morphism of them. By (6.10), (7.8), and (8.7) this will induce a morphism of mixed Hodge structures on the algebraic topological invariants mentioned above.

Theorem **(9.2)**. — *The tower of nilpotent rational Lie algebras associated with the rational nilpotent completion of* $\pi_1(X)$ *has the structure of a tower of Lie algebras in the category of mixed Hodge structures. This enrichment of structure is functorial with respect to algebraic maps.*

Proof. — The subalgebra of a minimal model generated in degree 1 is a rigid invariant. Hence, just as in (9.1), this subalgebra of the minimal model of $\mathcal{E}(X)$ has a canonical, functorial, mixed Hodge structure. Dualizing this gives the dual tower of rational Lie algebras a functorial mixed Hodge structure. By (5.11) this tower is the one associated to $\pi_1(X)$.

If X is a smooth variety, then the complex minimal model for $\mathcal{E}(X)$ has a bigrading $\mathcal{N} = \bigoplus_{r,s \geq 0} \mathcal{N}^{r,s}$. This bigrading induces one on the complex homotopy groups of X (provided that $\pi_1(X) = 0$) or in the tower of nilpotent complex Lie algebras associated to $\pi_1(X)$. The existence of the bigrading on the minimal model puts nontrivial homogeneity conditions on the differential. In the next lemma we shall produce a minimal differential algebra which does not posses a bigrading as in (6.6). In fact we show that it does not even have a grading with the properties which the associated grading to a bigrading as in (6.6) would have.

Lemma. — *Let $\mathscr{M}$ be a rational minimal differential algebra which through dimension* 10 *has the following structure*:

Degree	2	3	4	5	6	7	8	9	10
Generators	a	b	c	e		f	μ		
			η_{ab}		η_{bc-ae}	$\eta_{c^2-be-a^4}$		η_{bf-a^5}	

with $d(a)=d(b)=d(c)=d(e)=d(f)=0$, $d(\eta_\alpha)=\alpha$, and $d\mu=b\eta_{bc-ae}+e\eta_{ab}+fa$. The complex form of $\mathscr{M}$, $\mathscr{M}_{\mathbf{C}}$, does not have a splitting $\mathscr{M}_{\mathbf{C}}=\bigoplus_{i\geq 0}\mathscr{M}[i]$ with:

1) $\mathscr{M}[0]=$ground field,

2) $\mathscr{M}[i]=\bigoplus_{k\geq 0}(\mathscr{M}[i]\cap\mathscr{M}_{\mathbf{C}}^k)$,

3) $d:\mathscr{M}[i]\to\mathscr{M}[i]$, and

4) $\mathscr{M}[i]\wedge\mathscr{M}[j]\xrightarrow{\wedge}\mathscr{M}[i+j]$.

Proof. — We assume to the contrary that such a splitting exists and derive a contradiction. Call elements and subspaces of $\mathscr{M}[i]$ homogeneous of weight i. First note that a and b are homogeneous of some weights w_a and w_b. Likewise $c+\alpha a^2$ and $e+\beta ab$ are homogeneous of weights w_c and w_e for some appropriate elements α, $\beta\in\mathbf{C}$.

Claim 1. — $w_b+w_c=w_a+w_e$.

Proof. — $\operatorname{Im} d\cap\mathscr{M}_{\mathbf{C}}^7$ is homogeneous and generated by $\{bc-ae, a^2b\}$. The second element is homogeneous of weight $2w_a+w_b$. Hence we have a decomposition:

$$bc-ae=[b(c+\alpha a^2)]-[a(e+\alpha b)]-[(\alpha-\beta)a^2b]$$

with the first term in $\mathscr{M}[w_b+w_c]$, the second in $\mathscr{M}[w_a+w_e]$, and the third in $\mathscr{M}[2w_a+w_b]$. Clearly both of the first two terms are independent of the third. For $\operatorname{Im} d\cap\mathscr{M}_{\mathbf{C}}^7$ to be homogeneous of dimension two the first two terms must be of the same weight.

Claim 2. — $2w_c=w_b+w_e=4w_a$.

Proof. — $\operatorname{Im} d\cap\mathscr{M}_{\mathbf{C}}^8$ is homogeneous and generated by $\{c^2-be-a^4\}$. We decompose c^2 into its homogeneous components assuming that $w_c\neq 2w_a$.

Weight	$2w_c$	w_c+2w_a	$4w_a$
$c^2=$	$c^2+2\alpha a^2c+\alpha^2a^4$	$-2\alpha a^2(c+\alpha a^2)$	$+\alpha^2a^4$

Of course eb is homogeneous of weight w_b+w_e; and a^4 is homogeneous of weight $4w_a$. Thus c^2-be-a^4 can never be homogeneous if $w_c \neq 2w_a$. Hence $w_c=2w_a$, and c^2 itself is homogeneous. For the whole term to be homogeneous c^2, be, and a^4 must all be of the same weight, *i.e.* $w_b+w_e=4w_a=2w_c$.

The two equations in claim 1 and 2 imply that there is $T>0$ such that $w_a=2T$, $w_b=3T$, $w_c=4T$, and $w_e=5T$. Consequently, all a, b, c, e are homogeneous.

Claim 3:

$$\eta_{ab}+\alpha c+\beta a^2 \in \mathcal{M}[5T] \quad \text{for some} \quad \alpha, \beta \in \mathbf{C}$$
$$\eta_{bc-ae}+\gamma a^3+\delta ac \in \mathcal{M}[7T] \quad \text{for some} \quad \gamma, \delta \in \mathbf{C}.$$

Proof. — Since $\operatorname{Im} d \cap \mathcal{M}^5[5T]$ contains ab, there is some element in $\mathcal{M}^4[5T]$ whose differential is ab. All possible elements of this type are of the form $\eta_{ab}+\alpha c+\beta a^2$. Part two of the claim is proved similarly.

Claim 4. — $f+\varepsilon b\eta_{ab} \in \mathcal{M}[7T]$ for some $\varepsilon \in \mathbf{C}$.

Proof. — $\operatorname{Ker} d \cap \mathcal{M}^7_{\mathbf{C}}$ is homogeneous and generated by $\{f, ae, b\eta_{ab}, bc\}$. The subspace generated by $\{ae, bc, b\eta_{ab}\}$ is the sum of homogeneous subspaces. Thus $f+\varepsilon b\eta_{ab}+\varrho ae+\nu bc$ is homogeneous for some ϱ, ν, $\varepsilon \in \mathbf{C}$. Consider now the homogeneous space $\operatorname{Im} d \cap \mathcal{M}^{10}_{\mathbf{C}}$. It is generated by $\{bf-e^2, ac^2-abe-a^4, abe\}$. The last two elements are homogeneous of weight $10T$. Let us suppose that the first is not of weight $10T$ and let us decompose it into its homogeneous components.

Weight	$\neq 10T$		$=10T$
$bf-a^5=$	$(bf+\varrho abe)$	$+$	$(-\varrho abe-a^5)$

Since $\varrho abe+a^5$ is not in the subspace generated by abe and ac^2-a^4, $\operatorname{Im} d \cap \mathcal{M}[10T]$ is three-dimensional and hence all of $\operatorname{Im} d$. This implies that $(bf+\varrho abe) \in \mathcal{M}[10T]$. Consequently $bf \in \mathcal{M}[10T]$. Thus $f+\varepsilon b\eta_{ab} \in \mathcal{M}[7T]$.

Now at last we are in a position to derive the contradiction. Consider $\operatorname{Im} d \cap \mathcal{M}^9_{\mathbf{C}}$. It is generated by $\{b\eta_{bc-ae}+e\eta_{ab}+fa, abe\}$. The second term is homogeneous of weight $9T$. The first term has homogeneous coordinates as follows:

Weight	$9T$	$10T$
	$(fa+\varepsilon ab\eta_{ab}+\varepsilon \alpha abc)+\varepsilon\alpha a^3 b$	$-\varepsilon[ab\eta_{ab}+\alpha abc+\beta a^3 b]+(e\eta_{ab}+\alpha ec+\beta a^2 e)$
	$-\alpha ec-\beta a^2 e-\gamma a^3 b+\delta abc$	$+b\eta_{bc-ae}+\gamma a^3 b+\delta abc$

For any choices of α, β, γ, δ, and ε the term in $9T$ is independent of abe; whereas the term in $10T$ is nonzero. This is impossible since the subspace $\operatorname{Im} d \cap \mathcal{M}^9_{\mathbf{C}}$ is homogeneous.

Corollary. — *Not all finite, simply connected* CW-*complexes are homotopy equivalent to smooth complex varieties.*

Proof. — First let us build a finite CW-complex which has the above minimal algebra as its minimal model through dimension 11. Use this minimal algebra to build a Postnikov tower with nonzero homotopy groups in dimensions 2 through 9. Take the total space of the finite tower and truncate it at dimension 11. This will be a space with the above algebra as its 10-minimal model. It is also homotopy equivalent to a finite complex. Were this space homotopy equivalent to a smooth variety, then its complex minimal model would have a bigrading as in (6.6). The associated total grading to this bigrading is exactly what is ruled out by the previous calculation.

If A is a bigraded (or graded) vector space, then we denote by $\mathscr{F}(A)$ the free bigraded (or graded) Lie algebra generated by A. (In a bigraded (or graded) Lie algebra the bracket operation is homogeneous of type (0, 0) (or of type 0).) If $\mathfrak{J} \subset \mathscr{F}(A)$ is a homogeneous ideal, then we can form the quotient bigraded (or graded) Lie algebra $\mathscr{F}(A)/\mathfrak{J}$.

Let $X = V - D$, where V is a smooth, compact variety and D is a divisor with normal crossings. Define A to be $H_1(V; \mathbf{C}) \oplus \mathrm{Coker}(H_2(V; \mathbf{C}) \to H_0(\tilde{D}^1; \mathbf{C}))$. We give $H_1(V; \mathbf{C})$ the bigrading dual to the Hodge structure on $H^1(V; \mathbf{C})$. This means that $H_1(V; \mathbf{C}) = H_{-1,0}(V) \oplus H_{0,-1}(V)$ with $H_{-r,-s}(V) = (H^{r,s}(V))^*$. The cokernel summand is defined to be of bigradation $(-1, -1)$. Together these define a bigrading on A, $A = A_{-1,0} \oplus A_{0,-1} \oplus A_{-1,-1}$.

Theorem (**9.4**). — *In* $\mathscr{F}(A)$ *there is a homogeneous ideal* $\mathfrak{J}$ *such that the tower of bigraded, nilpotent, complex Lie algebras associated to* $\pi_1(X)$ *and the tower of bigraded nilpotent quotients of* $\mathscr{F}(A)/\mathfrak{J}$ *are isomorphic. The ideal* $\mathfrak{J}$ *has generators of types* $(-1, -1)$, $(-2, -1)$, $(-1, -2)$, *and* $(-2, -2)$ *only.*

Proof. — The complex Lie algebras associated to $\pi_1(X)$ are dual to the 1-minimal model for X and thus are a tower of bigraded Lie algebras:

$$0 \leftarrow \mathscr{L}_1 \leftarrow \mathscr{L}_2 \leftarrow \dots$$

$\mathscr{L}_1$ is the trivial Lie bracket on the bigraded vector space A. Choose a map of bigraded vector spaces $A \to \mathscr{L}_2$ lifting the identity $A = \mathscr{L}_1$. This determines a map $A \to \mathscr{L}_n$ for all n since the components of $\mathscr{L}_n$ of weight $\leqslant -2$ are equal to those of $\mathscr{L}_2$. As a result we have a map from the free Lie algebra on A to our tower:

$$\begin{tikzcd}
 & & \mathscr{F}(A) \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & & \\
0 & \mathscr{L}_1 \arrow[l] & \mathscr{L}_2 \arrow[l] & \mathscr{L}_3 \arrow[l] & \dots \arrow[l]
\end{tikzcd}$$

Each one of these maps is onto. Let $\mathfrak{J}_n$ be the kernel of $\mathscr{F}(\mathrm{A})\to\mathscr{L}_n$. We wish to show that there is a fixed homogeneous ideal $\mathfrak{J}\subset\mathscr{F}(\mathrm{A})$ generated in bidegrees $(-1,-1)$, $(-2,-1)$, $(-1,-2)$, and $(-2,-2)$, so that $\mathfrak{J}_n=\mathfrak{J}+\Gamma_{n+1}(\mathscr{F}(\mathrm{A}))$. That is what the theorem claims.

The way to get $\mathfrak{J}\subset\mathscr{F}(\mathrm{A})$ is to consider $\mathfrak{J}_n$, $n\geqq 4$, and then to take the subspace of $\mathfrak{J}_n$ of bidegrees $(-1,-1)$, $(-2,-1)$, $(-1,-2)$, and $(-2,-2)$. This vector space generates an ideal $\mathfrak{J}\subset\mathscr{F}(\mathrm{A})$.

Claim. — *a)* $\mathfrak{J}_n=\mathfrak{J}+\Gamma_{n+1}(\mathscr{F}(\mathrm{A}))$.

b) The resulting ideal $\mathfrak{J}$ is independent of n (provided $n\geqq 4$).

c) $\mathfrak{J}$ is generated in bidegrees $(-1,-1)$, $(-2,-1)$, $(-1,-2)$, and $(-2,-2)$.

Proof. — Part *c)* is clear. To prove *a)* and *b)* we observe that in general (in the case of negative weights) $\mathrm{H}^2(\mathscr{F}(\mathrm{A})/\mathfrak{J})$ is dual to the space of ideal generators for $\mathfrak{J}$, $(\mathfrak{J}/[\mathfrak{J},\mathscr{F}(\mathrm{A})])^*$. In our case $\mathrm{H}^2(\mathscr{F}(\mathrm{A})/\mathfrak{J}_n)=\mathrm{H}^2(\mathscr{L}_n)=\mathrm{H}^2(\mathscr{M}_n)$. $\mathrm{H}^2(\mathscr{M}_n)$, for $n\geqq 4$, has components of bidegrees $(1,1)$, $(2,1)$, $(1,2)$, and $(2,2)$ which are independent of n. The other homogeneous components of $\mathrm{H}^2(\mathscr{M}_n)$ are of total degree $\geqq n+1\geqq 5$, and this part of $\mathrm{H}^2(\mathscr{M}_n)$ is the kernel of $\mathrm{H}^2(\mathscr{M}_n)\to\mathrm{H}^2(\mathscr{M}_{n+1})$. Dualizing this we see that $\mathfrak{J}\subset\mathfrak{J}_n$ is independent of n provided that $n\geqq 4$, and that $\mathfrak{J}_n$ is generated by $\mathfrak{J}+\mathrm{V}_n$, where $\mathrm{V}_n\subset\mathfrak{J}_n$ is any vector space that projects onto $\mathfrak{J}_n/\mathfrak{J}_{n+1}$. Since:

$$\mathfrak{J}_n/\mathfrak{J}_{n+1}=\Gamma_{n+1}(\mathscr{F}(\mathrm{A})/\mathfrak{J}_{n+1}),$$

it follows that $\Gamma_{n+1}(\mathscr{F}(\mathrm{A}))$ projects onto $\mathfrak{J}_n/\mathfrak{J}_{n+1}$. Thus $\mathfrak{J}_n=\mathfrak{J}+\Gamma_{n+1}(\mathrm{A})$.

Corollary **(9.5)**. — *The complex nilpotent completion of* $\pi_1(\mathrm{X})$ *is determined by the bigraded nilpotent Lie algebra of* $(\pi_1(\mathrm{X})/\Gamma_5)\otimes\mathbf{C}$.

Theorem **(9.6)**. — *Let* X *be the complement of a divisor with normal crossings,* $\mathrm{X}=\mathrm{V}-\mathrm{D}$. *There is a natural isomorphism between the minimal model for* $(\mathrm{E}_{\mathrm{C}^\infty}(\mathrm{X})_{\mathbf{C}})$ *and the minimal model for* $\{{}_{\mathrm{W}}\mathrm{E}_1(\mathrm{E}_{\mathrm{C}^\infty}(\mathrm{X})_{\mathbf{C}}),d_1\}$.

Proof. — For this spectral sequence $\mathrm{E}_2=\mathrm{E}_\infty$. Since the weight filtration has a natural splitting over $\mathbf{C}$, we also have a natural isomorphism $\mathrm{E}_\infty\cong\mathrm{H}(\mathrm{X};\mathbf{C})$ of rings. Thus the cohomology ring of the differential algebra $\{{}_{\mathrm{W}}\mathrm{E}_1(\mathrm{E}_{\mathrm{C}^\infty}(\mathrm{X})_{\mathbf{C}}),d_1\}$ is identified with $\mathrm{H}(\mathrm{X};\mathbf{C})$. We have the bigrading of $\mathscr{N}_{\mathrm{X}}$, $\mathscr{N}_{\mathrm{X}}=\bigoplus_{r,s\geq 0}\mathscr{N}^{r,s}$ with d and wedge product of type $(0,0)$. The associated total grading gives a splitting of the filtration $\mathrm{Dec}\,\mathrm{W}(\mathscr{N}_{\mathrm{X}})$. Define $\mathrm{W}_k(\mathscr{N}_{\mathrm{X}}^n)=\mathrm{Dec}\,\mathrm{W}_{k-n}(\mathscr{N}_{\mathrm{X}}^n)$ for all k and n. The splitting of $\mathrm{Dec}\,\mathrm{W}$ yields one of $\mathrm{W}(\mathscr{N}_{\mathrm{X}})$. Such a splitting of $\mathrm{W}(\mathscr{N}_{\mathrm{X}})$ gives an isomorphism of filtered algebras $(\mathscr{N}_{\mathrm{X}},\mathrm{W})\cong{}_{\mathrm{W}}\mathrm{E}_0(\mathscr{N}_{\mathrm{X}})$. Since in the splitting of $\mathrm{Dec}\,\mathrm{W}(\mathscr{N}_{\mathrm{X}})$, d is homogeneous of degree zero, in the splitting of $\mathrm{W}(\mathscr{N}_{\mathrm{X}})$, d is homogeneous of degree -1. Thus $d_0:{}_{\mathrm{W}}\mathrm{E}_0(\mathscr{N}_{\mathrm{X}})\to{}_{\mathrm{W}}\mathrm{E}_0(\mathscr{N}_{\mathrm{X}})$ is zero, and $d_1:{}_{\mathrm{W}}\mathrm{E}_1(\mathscr{N}_{\mathrm{X}})\to{}_{\mathrm{W}}\mathrm{E}_1(\mathscr{N}_{\mathrm{X}})$ equals d. That is to say we have an isomorphism of differential algebras $(\mathscr{N}_{\mathrm{X}},d)\cong({}_{\mathrm{W}}\mathrm{E}_1(\mathscr{N}_{\mathrm{X}}),d_1)$.

The map $\psi:\mathscr{N}_{\mathrm{X}}\to\mathrm{E}_{\mathrm{C}^\infty}(\mathrm{X})_{\mathbf{C}}$ induces a map on spectral sequences. In particular

$E_1(\psi):\{{}_WE_1(\mathcal{N}_X), d_1\} \to \{{}_WE_1(E_{C^\infty}(X)_{\mathbf{C}}), d_1\}$. At E_2 it gives an isomorphism. Thus $\mathcal{N}_X = {}_WE_1(\mathcal{N}_X)$ maps to ${}_WE_1(E_{C^\infty}(X)_{\mathbf{C}})$ and induces an isomorphism on cohomology. This proves (9.6).

Corollary **(9.7)**. — *If* $X = V - D$ *is the complement of a divisor with normal crossings in a complete variety, then the complex homotopy type of* X *is determined functorially by*:

1) $H(V; \mathbf{C})$ *and the* $H(\widetilde{D}^p; \varepsilon^p_{\mathbf{C}})$, *(we denote* V *by* $\widetilde{D}^0$*)*
2) *the multiplication maps* $H(\widetilde{D}^p; \varepsilon^p_{\mathbf{C}}) \otimes H(\widetilde{D}^q; \varepsilon^q_{\mathbf{C}}) \to H(\widetilde{D}^{p+q}; \varepsilon^{p+q}_{\mathbf{C}})$ *for all* $p, q \geqslant 0$, *and*
3) *the Gysin maps* $H(\widetilde{D}^p; \varepsilon^p_{\mathbf{C}}) \to H(\widetilde{D}^{p-1}; \varepsilon^{p-1}_{\mathbf{C}})$.

Proof. — From the cohomology groups in 1) we build ${}_WE_1$ as a bigraded vector space. From the maps in 2) we define the multiplication on ${}_WE_1$. From the Gysin maps we define d_1. The corollary now follows from (9.6).

The splitting in (9.4) and the identification of minimal models in (9.6) are natural with respect to algebraic maps. Result (9.6) should be viewed as a generalization of the complex version of the main theorem in [5]. If X is a compact variety, then $X = V$ and there are no divisors. Thus ${}_WE_1^{p,q} = 0$ for $p \neq 0$ and ${}_WE_1^{0,q} = H^q(X)$. Thus (9.6) gives an equivalence between the minimal model for the complex forms on X and the minimal model for $H(X; \mathbf{C})$ in case X is compact.

10. The Rational Homotopy Theory of Nonsingular Varieties

In this section we deduce the rational analogues of (9.4), (9.6), and (9.7). The complex homotopy type of X is replaced by its rational type. The bigrading in (9.4) is replaced by its total grading. In passing to the rationals we can prove that gradings exists but don't show that they are canonical or natural. We begin with the analogue of (9.6).

Theorem **(10.1)**. — *Let* $X = V - D$ *be the complement in a smooth, compact variety of a divisor with normal crossings. There is an (unnatural) isomorphism between the minimal model for* $\mathcal{E}(X)$ (i.e. *the* $\mathbf{Q}$ *homotopy type of* X), *and the minimal model for* $\{{}_WE_1(E(X)), d_1\}$.

This result is a descent from $\mathbf{C}$ to $\mathbf{Q}$ and is a special case of an argument worked out jointly with Dennis Sullivan.

Proof. — The minimal model for $\mathcal{E}(X)$ has a minimal filtration $(\mathcal{M}, W)$. The proof consists of finding a splitting for this filtration, $\mathcal{M} = \bigoplus_r \mathcal{M}[r]$, such that:

1) $d : \mathcal{M}[r] \to \mathcal{M}[r]$,
2) $\mathcal{M}[r] \otimes \mathcal{M}[s] \xrightarrow{\wedge} \mathcal{M}[r+s]$, and
3) $W_l(\mathcal{M}) = \bigoplus_{r \leq l} \mathcal{M}[r]$.

Once we have such a splitting, it gives an isomorphism of differential algebras, $(\mathcal{M}, d) = ({}_{\mathrm{W}}\mathrm{E}_1(\mathcal{M}), d_1)$. Arguing as in (9.6), we see that $(\mathcal{M}, d)$ is a minimal model for $\{{}_{\mathrm{W}}\mathrm{E}_1(\mathrm{E}(\mathrm{X})), d_1\}$. The construction of the splitting is based on the following lemma.

Lemma **(10.2)**. — 1) *Let* A *be a finitely generated differential algebra defined over* k. *Suppose that* A *is free as a graded commutative algebra, and let* W *be a positive multiplicative filtration on* A. *Then* $\mathrm{Auto}^{\mathrm{W}}(\mathrm{A})$ *is a linear algebraic group.*

2) *If* $(\mathcal{M}, \mathrm{W})$ *is a minimal differential algebra with a positive minimal filtration, and if* $\mathrm{W}_\ell(\mathrm{H}(\mathcal{M}))$ *is finite dimensional for all* ℓ, *then* $\mathrm{Auto}^{\mathrm{W}}(\mathcal{M})$ *is the inverse limit of a sequence of algebraic groups and homomorphisms.*

Proof. — Let A be generated in degrees $\leqq \mathrm{N}-1$. Then Auto(A) is a subgroup of $\bigoplus_{i \leqq \mathrm{N}} \mathrm{GL}(\mathrm{A}^i)$. For any element $\alpha \in \bigoplus_{i \leqq \mathrm{N}} \mathrm{GL}(\mathrm{A}^i)$ to define a differential algebra homomorphism it is necessary that α commute with multiplication (when defined) and the differential of $\bigoplus_{i \leqq \mathrm{N}-1} \mathrm{A}^i$. These conditions are sufficient as well, by (1.3). Any such algebra homomorphism is automatically an isomorphism. Its inverse is defined by extending $\alpha^{-1} \in \bigoplus_{i \leqq \mathrm{N}} \mathrm{GL}(\mathrm{A}^i)$ multiplicatively to all of A. Lastly, $\alpha \in \bigoplus_{i \leqq \mathrm{N}} \mathrm{GL}(\mathrm{A}^i)$ extends to an isomorphism compatible with the filtration if and only if α itself is compatible with the filtration restricted to $\bigoplus_{i \leqq \mathrm{N}} \mathrm{A}^i$. This also follows from (1.3). These three conditions are quadratic, linear, and linear respectively. Thus $\mathrm{Auto}^{\mathrm{W}}(\mathrm{A})$ is a subalgebraic group of $\bigoplus_{i \leqq \mathrm{N}} \mathrm{GL}(\mathrm{A}^i)$.

By (7.2) *a)* $\mathrm{W}_\ell(\mathcal{M})$ is finite dimensional for all $\ell \geqq 0$. Let $\mathcal{M}_\ell$ be the subalgebra generated by $\mathrm{W}_\ell(\mathcal{M})$. One sees easily that $(\mathcal{M}_\ell, \mathrm{W})$ satisfies all the conditions of part one. Thus $\mathrm{Auto}^{\mathrm{W}}(\mathcal{M}_\ell)$ is a linear algebraic group. $\mathrm{Auto}^{\mathrm{W}}(\mathcal{M})$ is the inverse limit of the sequence:

$$\dots \longleftarrow \mathrm{Auto}^{\mathrm{W}}(\mathcal{M}_\ell) \longleftarrow \mathrm{Auto}^{\mathrm{W}}(\mathcal{M}_{\ell+1}) \longleftarrow \dots$$

We return now to the question of splitting the filtration $\mathrm{W}(\mathcal{M})$. First we consider the question of splitting $\mathrm{W}(\mathcal{M}_\ell)$. There is a diagram of algebraic groups and homomorphisms:

$$\begin{tikzcd} & & & \mathrm{G}_m \arrow[dl, dashed, "s"'] \arrow[d, hook, "i"] \\ 1 \arrow[r] & \mathrm{N} \arrow[r] & \mathrm{Auto}^{\mathrm{W}}(\mathcal{M}_\ell) \arrow[r, "\mathrm{gr}"] & \mathrm{Auto}(\mathrm{Gr}^{\mathrm{W}}(\mathcal{M}_\ell)). \end{tikzcd}$$

The kernel N of gr is a nilpotent group. G_m denotes the multiplicative group of the field. The inclusion $\mathrm{G}_m \overset{i}{\hookrightarrow} \mathrm{Auto}(\mathrm{Gr}^{\mathrm{W}}(\mathcal{M}_\ell))$ is the one defined by the grading, *i.e.* $a \in k$ acts by a^r on the r^{th} component of the grading. Splitting $\mathrm{W}(\mathcal{M}_\ell)$ is equivalent to lifting the map i to a map $s : \mathrm{G}_m \to \mathrm{Auto}^{\mathrm{W}}(\mathcal{M}_\ell)$.

On the level of complex points $\mathrm{Auto}^{\mathrm{W}}(\mathscr{M}_\ell) \xrightarrow{gr} \mathrm{Auto}(\mathrm{Gr}^{\mathrm{W}}(\mathscr{M}_\ell))$ is onto the image of $i(\mathrm{G}_m)$. This follows from the existence of the splitting for $(\mathscr{M}, \mathrm{W})_{\mathbf{C}}$. Hence, if we form the pull-back of the diagram we get:

$$\begin{tikzcd} 1 \arrow[r] & \mathrm{N} \arrow[r] \arrow[d, equal] & \mathrm{A} \arrow[r] \arrow[d, hook, "i'"] & \mathrm{G}_m \arrow[r] \arrow[d, hook, "i"] \arrow[l, bend right, dashed, "\varphi"'] & 1 \\ 1 \arrow[r] & \mathrm{N} \arrow[r] & \mathrm{Auto}^{\mathrm{W}}(\mathscr{M}_\ell) \arrow[r] & \mathrm{Auto}(\mathrm{Gr}^{\mathrm{W}}(\mathscr{M}_\ell)) & \end{tikzcd}$$

Any extension of G_m by a nilpotent group can be split, see ([2], page 363). The composition of such a splitting with i' gives the required lifting. Thus $\mathrm{W}(\mathscr{M}_\ell)$ admits a splitting. In fact the set of all splittings is the set of all sections in the above diagram, and hence is an algebraic variety with at least one k-rational point. Call this variety S_ℓ. Clearly the various S_ℓ fit together in an inverse system of algebraic varieties and algebraic maps:

$$\dots \longleftarrow \mathrm{S}_\ell \xleftarrow{r_\ell} \mathrm{S}_{\ell+1} \xleftarrow{r_{\ell+1}} \dots$$

Since a decreasing sequence of varieties stabilizes after a finite number of steps, the intersection $\bigcap_{t\geq \ell} \mathrm{Im}(\mathrm{S}_t \to \mathrm{S}_\ell)$ is equal to $\mathrm{Im}(\mathrm{S}_\mathrm{N} \to \mathrm{S}_\ell)$ for some sufficiently large N. The same is true, of course, on the level of k-rational points. Thus $\bigcap_{t\geq \ell} \mathrm{Im}(\mathrm{S}_t \to \mathrm{S}_\ell)$ has a k-rational point, or equivalently there is an element in the inverse limit of the k-rational points $\{x_\ell\}$. Such an element is a compatible sequence of splittings for the $\mathrm{W}(\mathscr{M}_\ell)$. Such a sequence of compatible splittings defines a splitting of $\mathrm{W}(\mathscr{M})$.

Corollary (**10.3**). — *Let* X *be the complement of a divisor with normal crossings* $\mathrm{X}=\mathrm{V}-\mathrm{D}$. *Let* A *be the graded vector space* $\mathrm{H}_1(\mathrm{V};\mathbf{Q}) \oplus \mathrm{Coker}(\mathrm{H}_2(\mathrm{V};\mathbf{Q}) \to \mathrm{H}_0(\widetilde{\mathrm{D}}^1;\mathbf{Q}))$, *where the first summand has type* -1 *and the second has type* -2. *There is an isomorphism between the rational nilpotent completion of* $\pi_1(\mathrm{X})$ *and the nilpotent quotients of* $\mathscr{F}(\mathrm{A})/\mathfrak{J}$ *for an appropriate homogeneous ideal* $\mathfrak{J}$. *This ideal has generators of type* -2, -3, *and* -4 *only.*

Proof. — Once we have the splitting as in (10.1) we apply the argument in (9.6). The ideal is determined by X and V. Only the identification of the rational nilpotent completion of $\pi_1(\mathrm{X})$ with the tower can be changed.

Corollary (**10.4**). — a) *The rational nilpotent completion of* X *is determined up to isomorphism by the graded Lie algebra associated to* $(\pi_1(\mathrm{X})/\Gamma_5)\otimes\mathbf{Q}$.

b) *If* $(\pi_1(\mathrm{X})/\Gamma_5)\otimes\mathbf{Q}$ *is isomorphic to* $(\mathrm{F}/\Gamma_5(\mathrm{F}))\otimes\mathbf{Q}$ *for some free group* F, *then* $(\pi_1(\mathrm{X})/\Gamma_n)\otimes\mathbf{Q}$ *is isomorphic to* $(\mathrm{F}/\Gamma_n(\mathrm{F}))\otimes\mathbf{Q}$ *for all n.*

In the case of a compact variety X the results of [3] show that $(\pi_1(\mathrm{X})/\Gamma_n)\otimes\mathbf{Q}$ is determined by $(\pi_1(\mathrm{X})/\Gamma_3)\otimes\mathbf{Q}$.

Example. — Let:

$$\begin{tikzcd} & E \arrow[d] \\ \mathbf{C}^* \arrow[ur] & T^2 \end{tikzcd}$$

be the $\mathbf{C}^*$ bundle over the torus with Chern class $[T]\in H^2(T^2)$. Then E is a nonsingular algebraic variety and $\pi_1(E)=\pi_1(E)/\Gamma_3=F(x,y)/\Gamma_3(F)$. This shows that to determine the rational nilpotent completion of the fundamental group of an open variety one needs information at least about $(\pi_1(X)/\Gamma_4)\otimes\mathbf{Q}$. I know of no example where no needs the full information about $(\pi_1/\Gamma_5)\otimes\mathbf{Q}$.

The fact that this example could be made an algebraic variety was shown to me by Pierre Deligne.

The fact that the rational homotopy type of X can be read off from the cohomological structure of V and the various subvarieties $\widetilde{D}^p$ tells us that the cohomological complexity of the divisor and its intersections puts an upper bound on the homotopy theoretic complexity of X. As typical of this we offer the following

Corollary **(10.5)**. — a) *If* X *is an affine variety which, when completed at* ∞, *has a smooth hyperplane section there, then the minimal models of* $\mathscr{E}(X)$ *and* $H(X;\mathbf{Q})$ *are isomorphic.*

b) *If* $X=V-\bigcup_i D_i$, *where* $D_i\cap D_j=\emptyset$ *for all* $i\neq j$, *then the tower of rational Lie algebras associated to* $\pi_1(X)$ *is determined by the graded Lie algebra associated to* $(\pi_1(X)/\Gamma_4)\otimes\mathbf{Q}$.

Proof. — *a)* $X=V-D$ where D is a smooth hyperplane section. The E_1-term of the Gysin spectral sequence is:

$$\begin{array}{c|c} H^{2n-2}(D) \longrightarrow & H^{2n}(V) \\ \vdots & \vdots \\ H^0(D) \longrightarrow & H^2(V) \\ & H^1(V) \\ & H^0(V) \\ \hline \end{array}$$

The Lefschetz theory [12] tells us that:

1) $\varphi: H^i(D)\to H^{i+2}(V)$ is injective for $i\neq n-1$,
2) $\varphi: H^i(D)\to H^{i+2}(V)$ is an isomorphism for $i>n-1$,
3) $\operatorname{Ker}\varphi\subset H^{n-1}(D)$ has a complement $I^{n-1}\subset H^{n-1}(D)$,
4) this complement is the image $H^{n-1}(V)\to H^{n-1}(D)$, and
5) $H^i(V)\to H^i(D)$ is an isomorphism for $i<n-1$.

Thus we define a projection from the E_1 term to:

$$
\begin{array}{c|l}
0 & 0 \\
\mathrm{Ker}\ \varphi & H^{n+1}(V)/\mathrm{Im}\ \varphi \\
0 & \vdots \\
\vdots & \\
 & H^2(V)/\mathrm{Im}\ \varphi \\
0 & H^1(V) \\
 & H^0(V) \\
\hline
\end{array}
$$

Since $\varphi(x)\smile y=\varphi(x\smile y|_D)$, the cup product on $H^*(V)$ defines a ring structure on $H^*(V)/\mathrm{Im}\ \varphi$. Thus this graded vector space is a differential graded algebra with $d=0$. The projection map from the E_1 term to this is a map of algebras because:

$$H^i(D)\otimes H^j(V)\to H^i(D)\otimes H^j(D)\to H^{n-1}(D)\to \mathrm{Ker}\ \varphi$$

is the zero map for $i+j=n-1$ and $i<n-1$. Clearly, the projection map commutes with the differentials and induces an isomorphism on cohomology. Thus the minimal model of $\mathscr{E}(X)$, the E_1-term and the above algebra with 0 differential are all isomorphic. This proves *a)*.

b) If $D_i\cap D_j=\emptyset$ for all $i\neq j$, then the only non-zero components in the grading of $H^2(X)$ are 2 and 3. From (9.6) we see that the tower of nilpotent Lie algebras are isomorphic to the nilpotent quotients of $\mathscr{F}(A)/\mathfrak{I}$, where $\mathfrak{I}$ is a homogeneous ideal with generators of types -2, and -3 only. Thus once we know $\mathscr{F}(A)/(\mathfrak{I}+\Gamma_4)$ as a graded Lie algebra we can determine $\mathfrak{I}$ as a homogeneous ideal.

REFERENCES

[1] A. Bousfield and D. Kan, Homotopy limits, completions and localizations, *Lecture Notes in Mathematics*, **304**, Berlin-Heidelberg-New York, Springer, 1972.

[2] A. Borel, *Linear algebraic groups*, New York, Benjamin, 1969.

[3] P. Deligne, Théorie de Hodge, I, *Actes du Congrès international des Mathématiciens*, I, Nice, 1970, 425-430.

[4] P. Deligne, Théorie de Hodge, II, *Publ. math. I.H.E.S.*, **40** (1971), 5-58.

[5] P. Deligne, P. Griffiths, J. Morgan and D. Sullivan, Real Homotopy theory of Kähler manifolds, *Invent. math.*, **29** (1975), 245-274.

[6] H. Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic 0, *Ann. of Math.*, **79** (1964), 109-326.

[7] A. Malcev, Nilpotent groups without torsion, *Izv. Akad. Nauk. SSSR, Math.*, **13** (1949), 201-212.

[8] J. Milnor, Morse Theory, *Ann. of Math. Studies*, **51**, Princeton, New Jersey, Princeton University Press, 1963.

[9] M. Nagata, Imbedding of an abstract variety in a complete variety, *J. Math. Kyoto*, **2** (1962), 1-10.

[10] J.-P. Serre, Sur la topologie des variétés algébriques en caractéristique *p*, *Symposium internacional de topologiá algebrica*, pp. 24-53, Mexico City, 1958.

[11] D. Sullivan, Infinitesimal Calculations in Topology, *Publ. math. I.H.E.S.*, **47** (1977), 269-331.

[12] A. Weil, *Introduction à l'étude des variétés kählériennes*, Paris, Hermann, 1958.

[13] H. Whitney, *Geometric Integration Theory*, Princeton, Princeton University Press, 1957.

[14] P. Deligne, Théorie de Hodge, III, *Publ. I.H.E.S.*, **44** (1974), 5-77.

Manuscrit reçu le 5 décembre 1976, révisé le 1er juin 1977.

Imprimé en France, à Vendôme
Imprimerie des Presses Universitaires de France
1978 — N° 26 256