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INFINITESIMAL COMPUTATIONS IN TOPOLOGY

by DENNIS SULLIVAN

This paper was written in the effort to understand the nature of the mathematical object presented by a diffeomorphism class of compact smooth manifolds. Under suitable restrictions on the fundamental group and the dimension we find a rather understandable and complete answer to the question posed with " finite ambiguity ".

Roughly speaking, our answer (§ 13) is that this mathematical object behaves up to " finite ambiguity " like a finite dimensional real vector space with additional structure provided by tensors, lattices, and canonical elements.

This algebraic model is derived directly from the differential forms on the manifold M by the following procedure (§ 5). One proceeds dimension by dimension to construct a smallest possible subdifferential algebra of forms with the same cohomology. Forms are chosen in each dimension to add cohomology not already achieved or to create necessary relations among cohomology classes in the subalgebra. Actually one imagines the forms chosen generically so that no relation other than graded commutativity holds in the subalgebra, or practically one replaces the inclusion by a homomorphism from the free graded commutative algebra generated by the choices. This process may be discontinued when the dimension of the manifold is reached whereafter it becomes formal. One arrives at the essential part of the *minimal model* $\mathcal{M}$ of M which is a graded vector space with the additional structure of a graded commutative multiplication and a differential (respectively trilinear and bilinear tensors).

This object is well defined up to isomorphism. To prove this isomorphism the algebraic notion of homotopy between maps of differential algebras is developed (§ 3). We note that the underlying set of the model is not well defined (as it is for the cohomology ring for example). The ambiguity is precisely described by the *inner automorphisms* of a minimal model—those of the form $\exp(di+id)$ where i is any derivation of degree -1 (§ 6).

When M has a nilpotent homotopy system (1) the minimal model has finite type with finitely many generators x_α in each dimension k. In fact the spaces of generators (or indecomposables) are naturally isomorphic to the dual spaces over $\mathbf{R}$ of the homotopy groups of M and there are canonical integrals for the homotopy periods $\left\{\int_{\text{sphere}}(x_\alpha-\Delta^{-1}sdx_\alpha)\right\}$ described in § 11.

(1) π_1 is nilpotent and π_n is a nilpotent π_1-module, $n>1$.

The indecomposable spaces of the model $\mathscr{M}$ then inherit natural lattices of integral periods—those functionals giving integral values to the integral homotopy.

Similarly the cohomology of the model inherits lattices from the usual integrals over cycles. Actually we also have more lattices coming from the fact that the inductive stages $\mathscr{M}^k$ of the model $\mathscr{M}$ exactly describe (§ 10) the real homotopy theory of the inductive stages of the Postnikov system of M, $\{M \ldots \to M^k \to \ldots \to *\}$. So each $\mathscr{M}^k$ has natural lattices in its cohomology [2].

To describe this relationship between models and Postnikov systems we need Whitney's differential forms on general spaces and a cellular de Rham theorem (§ 7). Actually we make use of a rational polynomial de Rham theory (§ 7) and a spatial realization construction (§ 8) which canonically yields the Postnikov system of the rational homotopy type from the stages of the $\mathbf{Q}$-minimal model.

A manifold M also possesses a natural sequence of real cohomology classes $p_i \in H^{4i}(M, \mathbf{R})$ (which are actually integral) with a differential form representative constructed as is well known from a choice of connection in the tangent bundle. A connection Θ assigns to each coordinate system α a matrix of 1-forms θ_α. One forms the expressions $d\theta_\alpha - \theta_\alpha \circ \theta_\alpha = \Omega_\alpha$ to arrive at the curvature tensor Ω. One then forms the differential forms $\{\text{trace}\, \Omega \circ \Omega, \text{trace}\, \Omega \circ \Omega \circ \Omega \circ \Omega, \ldots\}$ representing the Pontryagin characteristic classes.

Note that in this construction essential use is made of the graded commutativity of differential forms. For the curvature form is not closed $(d\Omega_\alpha = \Omega_\alpha \circ \theta_\alpha - \theta_\alpha \circ \Omega_\alpha)$ and closed forms result by applying trace, which satisfies $\text{trace}(AB - BA) = 0$ in a commutative context. Similarly in the construction of the model the commutativity of forms plays the important role of allowing simple algebraic models to be used. Finally this commutativity plays a crucial part in the direct calculation of transgressive fibrations by the Chevalley-Hirsch-Koszul formula (§ 7).

Let us now state the main theorems of § 13. So far one has constructed from the differential forms an algebraic invariant consisting of the minimal model essentially determined by two tensors on a finite dimensional vector space, some lattices on certain canonical subquotients (which are fixed by the inner automorphisms) and certain canonical elements in cohomology, the Pontryagin classes. Then we have $(\pi_1 = e$, dimension $\geq 5)$.

Theorem **(13.1)**. — *For each positive integer b, there are only finitely many diffeomorphism types of closed manifolds whose homology torsion is bounded by b and whose algebraic invariant —model, lattices, and Pontryagin class—is isomorphic to a given one* [3].

Using work of Wall and others allows a generalization to the nilpotent case which we don't state here.

(2) The relevant lattice can be described geometrically in terms of homologies with spherical boundaries.
(3) That is an isomorphism between *real* models preserving lattices and Pontryagin classes.

Theorem **(13.2)**. — *Any rational cohomology ring satisfying Poincaré duality and* $H^1=0$ *may be realized by a manifold with possibly one singular point in dimension* $4k$. *Furthermore the lower Pontryagin classes and rational homotopy type may be prescribed arbitrarily with this ring.*

Necessary and sufficient arithmetic conditions for the removal of the singular point in dimension $4k$ may be stated (§ 13).

Theorem **(13.5)** describes an algebraic group whose arithmetic subgroups are commensurable with the component group of diffeomorphisms. In particular this component group modulo a normal nilpotent group is commensurable to those automorphisms of the integral cohomology ring which preserve the Pontryagin classes and extend to the model.

Again any commensurability class of arithmetic groups (perhaps extended by an Abelian group) can be realized as the component group of diffeomorphisms in simple examples (§ 13).

Along the way to these theorems about manifolds one obtains a general picture of algebraic topology (after tensoring with $\mathbf{Q}$, $\mathbf{R}$, or $\mathbf{C}$) in terms of differential forms and minimal differential algebras (§ 2, § 8, § 10).

For spaces with nilpotent homotopy systems there is a perfect replica ([4]) in terms of nilpotent minimal d.g.a.'s (§ 2) of the rational (real or complex) homotopy theory (§ 10). Furthermore the integral homotopy theory is determined up to finite ambiguity by $\mathbf{Q}$-minimal models, lattices, and torsion constraints (§ 10).

The additional point beyond this $\mathbf{Q}$-story is the role played by algebraic groups (§ 6) and the automorphisms groups of integral homotopy types which are *arithmetic groups*. Again all commensurability classes of arithmetic groups occur in this way even for simply connected spaces of " quadratic complexity " (§ 10).

Part of this general picture of algebraic topology is the " calculability " of minimal models. We make the general claim that any reasonable geometric construction on spaces can be mirrored by a finite algebraic one with minimal models. Paragraph 11 contains four examples of this. The papers of Wu [Wu] contain more examples and develop this philosophical viewpoint further. Such algebraic constructions (following the geometric) can be justified in the nilpotent case by (§ 10) and the functorial nature of the construction of § 8.

We have written the paper with the more general non-nilpotent problems in mind. The reason is that besides the graded commutative product of forms there is one more important infinitesimal computation relevant to algebraic topology.

Namely certain local systems can be described infinitesimally by matrices θ of 1-forms satisfying $d\theta-\theta\circ\theta=0$. Twisted cohomology can then be described with ordinary forms (§ 1) and the non-nilpotent minimal models (§ 2) enter as canonical computations (§ 7, § 8).

([4]) The possibility of such a picture had already been demonstrated by Quillen [Q] in a more abstract setting.

In effect general non abelian calculations in discrete groups are replaced by the quadratic equations on forms (above) together with linear equations associated to twisted cohomology (§ 2).

The domain of validity of this method is quite general (§ 2, [DS], and § 9) but its actual boundary and range of application is an open problem (§ 14).

For example the nilpotent theory can be applied very well to Kaehler manifolds whose models are a formal consequence of the cohomology ring. In § 12 we give a number of corollaries of this for Kaehler manifolds going beyond the " real formality " of [DGMS] which in turn is based on the earlier developments of this paper. New theorems are the $\mathbf{Q}$-formality of Kaehler manifolds (Theorem (12.1)) and the corollary:

Theorem **(12.4)**. — *The diffeomorphism type of a simply connected Kaehler manifold* ($\dim_{\mathbf{C}}>2$) *is finitely determined by its integral cohomology ring and the Pontryagin classes. The component group of diffeomorphisms modulo a unipotent subgroup is commensurable to the automorphisms of the cohomology ring fixing the Pontryagin classes.*

The analytic facts about forms (§ 12, [DGMS]), underlying this theorem are valid on the twisted level for unitary representations and the (non-nilpotent) consequences are waiting to be adduced.

This paper represents a synthesis and reformulation of old ideas rather than the invention of new mathematics. The only originality pertains to the naturality of the classification problem posed with finite ambiguity and the preference for calculations or coordinates (a choice of model) over the invariant or functorial emphasis.

One could be led to these ideas by the following train of thought. Starting from the basic problem of classifying manifolds one assumes compactness to achieve countability. One then must hold the fundamental group somewhat at bay because all finitely presented groups occur. The modern methods of surgery (constructive cobordisms) and Smale's homotopy cobordism in the hands of Browder and Novikov ([Br] and [N]) reduced the problem of classification ($\pi_1=e$, dimension>4) to that of the underlying homotopy type and a (finite) refinement of the tangent bundle.

This refinement of the tangent bundle can be described exactly [Su_3 and Su_4] in terms of real K-theory at odd primes and ordinary cohomology at the prime 2 in the homeomorphism context (continuous or piecewise linear after Kirby and Siebenmann [KS]). However the finite refinement for the smooth classification consists of a known part (again describable in terms of K-theory) and a part constructed from the stable cohomotopy of the manifold [Su_4]. Unfortunately the latter finite group is not readily calculable and we reach a definite knowledge barrier. Another obstruction to a complete classification is the homotopy type which had been included as an invariant up to now. Also the automorphism group of the homotopy type (a group more or less entirely unknown before 1970) acts on cohomology, K-theory, etc., and the true invariants are really its orbits.

After some struggling with these ideas and difficulties one might decide that a reformulation is in order. One can be guided in this reformulation by the deep work [K] of Dan Kan in homotopy theory. One can really see " to the bottom of the well " in integral homotopy theory of simply connected (or even nilpotent) spaces using Kan's semi-simplicial groups (in minimal form and made nilpotent) to arrive at the idea

$$\text{finite homotopy type} \sim \text{finite diagram of finitely generated nilpotent groups.}$$

There are tedious technicalities but one can achieve (5) the theorem that the automorphism group is arithmetic (even after passing to homotopy classes) and the objects behave like nilpotent groups. For example, Quillen's work of 1968 [Q] on rational homotopy theory can be viewed in the following steps — tensor the above diagram of nilpotent groups with $\mathbf{Q}$, pass to the Lie algebras, shuffle these together to get a differential Lie algebra. In Kan's terms this is Quillen's equivalence of homotopy categories:

$$\begin{matrix}\text{rational homotopy theory} & & \text{differential Lie Algebras (connected)} \\ \text{of simply connected spaces} & \sim & \text{(or 1-connected differential coalgebras).}\end{matrix}$$

These ideas inspire one to consider the integral classification problems with *finite ambiguity* with the gap between that and the rational classification to be filled in by arithmetic group theory.

But this is only the abstract picture. To apply these images to manifolds one would want to compute in terms of geometrical data. Thus differential forms come to mind.

Of course simultaneously one also has in mind the extensive calculations of algebraic topology in terms of the Steenrod algebra — an algebraic complexity in the torsion that arises directly from the non-commutativity of cochain multiplication. This complexity is due to lack of symmetry in the finite simplicial approximations while the simplicity of the calculus for forms must come from the infinitesimal nature of wedge product (one computes at each point).

Thus one might try to connect the general insight of Kan and Quillen's work to manifolds, starting from the de Rham algebra of forms. However, there is a curious algebraic obstruction coming from the fact that while the dual of a co-algebra is an algebra, the dual of an algebra is not a co-algebra unless the algebra is controlled in some sense.

One is led to replace the large de Rham algebra by a connected d.g.a. which is smaller and has the same cohomology. This may be dualized to a differential co-algebra which may then be compared to Quillen's objects.

This comparison turns out to be unnecessary however, because one discovers

(5) This was done in 1971, but it seemed to me that the proof of § 10 was preferable. Recently Wilkerson has independently applied Kan's method to achieve similar results (*Topology*, 1976).

instead the minimal model describing in a very transparent way the homotopy theory of the space. One has achieved instead a more or less equivalent theory to Quillen's for simply connected spaces starting from the geometrical objects, differential forms, which are natural and convenient for calculations in manifolds. This theory extends as it is to nilpotent spaces and somehow beyond using the second infinitesimal computation $d\theta - \theta \circ \theta = 0$.

Although most of this work was done in 1971-72 there were many versions and troublesome points to be worked out, especially the notion of algebraic homotopy. I am indebted to John Morgan for much help and inspiration in all these matters. More recently Steve Halperin's critical remarks and general ideas have been helpful and he has worked out many technical points independently. Similar remarks apply to Roy Douglas.

Finally, in reading the literature backwards in time: Quillen, 1969 [Q], then Kan, 1958 [K], Whitney, 1957 [W], Thom, 1955 [TC], Henri Cartan, 1950 [C], de Rham, 1929 [D], Elie Cartan, 1928 [C], and Poincaré, 1895 [P], the relevance and utility of differential forms for algebraic topology became more and more evident while the ideas presented here seemed be more and more well known to these authors with a maximum being reached in the period around 1950. The spectacular success of the spectral sequence in the hands of Serre, 1951 [Se_1] and Borel, 1953 [B] for the integral homology of $K(\pi, n)$'s and homogeneous spaces perhaps accounted for the shift of emphasis. This progress in algebraic topology has not continued at the same rate however and we have suggested here that one might therefore recall the older methods of differential forms, which are evidently quite powerful.

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Summary

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§ 2. *On the structure of differential algebras.*

The definition of minimality — $\mathscr{A}$ is free connected and (image d) is decomposable. The relation between Lie algebras and minimal algebras generated in degree one. The structure of general minimal algebras (Theorem (2.1)) is a multilinear solvable extension of a dual Lie algebra. The analogy with the homotopy theory of a space. Any free d.g.a. is the tensor product of a contractible d.g.a. and a minimal d.g.a. (Theorem (2.1)). The notion of a nilpotent, solvable, etc., minimal d.g.a.

§ 3. *Algebraic maps, deformations, and obstructions.*

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A definition (3.1) of homotopy between maps (which has finite type *a priori*) and its justification, Proposition (3.2). Obstruction for homotopy (Propositions (3.3) and (3.5)). Homotopy is an equivalence relation, (3.4). Remark on homotopies of homotopies, etc.

§ 4. *Algebraic Fibrations.*

The notion of algebraic fibration. The *infinitesimal condition* for the π_1-action. Various results about two isomorphisms implying a third, *e.g.* a map between minimal algebras inducing an isomorphism in degree one and an isomorphism of sufficiently many twisted cohomologies is an isomorphism (Corollary (4.2)).

§ 5. *Building homological models.*

The direct method of building a minimal d.g.a. mapping to an arbitrary d.g.a. by an isomorphism of (various) cohomologies. The uniqueness up to isomorphism of such " minimal models " (Theorem (5.1)). Examples of minimal models of familiar spaces, function spaces, complements of links.

§ 6. *The automorphism group of a differential algebra.*

The automorphisms of a nilpotent d.g.a. homotopic to the identity are precisely the " inner automorphisms " — those of the form $\exp(di+id)$ (Propositions (6.3) and (6.4)). Thus all the groups associated to one d.g.a. are algebraic matrix groups, having the same reductive parts, which is represented faithfully on the homotopy or homology (even the spherical homology) (Theorem (6.1)).

Appendix: algebraic and arithmetic groups.

The basic relevant facts are enunciated. For example in § 13 we will use the statement that an isomorphism over the reals between two nilpotent d.g.a.'s which is rational on cohomology can be changed to a rational isomorphism of d.g.a.'s. This uses A.2 of the appendix and Theorem (6.1).

§ 7. *Differential forms and de Rham's theorem on general spaces.*

Theorem (7.1) is the " cellular " de Rham theorem (proved directly by the 5-lemma). There are various constructions of forms on general spaces: simplicial, polyhedral, semi-simplicial, stratifiable. The main algebraic consequence is a simple d.g.a. formula for the total space of a transgressive fibration in terms of a d.g.a. for the base (Theorem (7.2)). This formula may be iterated for any fibration and one perceives the relation between minimal models and homotopy theory.

§ 8. *The spatial realization of a d.g.a., its homotopy and cohomology.*

The spatial realization, $\langle\mathscr{A}\rangle$, is the semi-simplicial complex whose simplices are maps of $\mathscr{A}$ into forms on standard simplices. In the nilpotent case over $\mathbf{Q}$ the homotopy of $\langle\mathscr{A}\rangle$ is dual to the generators of $\mathscr{A}$ and the cohomology of $\langle\mathscr{A}\rangle$ is isomorphic to that of $\mathscr{A}$ via the natural map $\mathscr{A}\to$ forms on $\langle\mathscr{A}\rangle$ (Theorem (8.1)). In the non-nilpotent case over $\mathbf{R}$ (C^∞ forms) the homotopy and the continuous cohomology $\langle\mathscr{A}\rangle$ can be computed (Theorem (8.1)′).

§ 9. *Poincaré's analytic definition of the fundamental group.*

Poincaré defined π_1 as a universal object for global (multivalued) solutions of locally integrable differential systems. Theorem (9.1) justifies this idea explicitly. The non-nilpotent theory depends on this infinitesimal definition of π_1.

§ 10. *Integral homotopy theory and minimal algebras.*

Combining § 5, § 7, § 8 we derive a computation of the rational homotopy of a nilpotent space by the minimal model of its de Rham algebra of forms (Theorem (10.1)). Theorem (10.2) explains how space $\rightarrow$ minimal model is finite to one on maps and onto for objects. Theorem (10.3) characterizes the image of automorphisms as lying in the commensurability class of the arithmetic group determined by the automorphism group of the model. Theorem (10.4) extends the model to an algebraic invariant determining the integral homotopy type and its automorphisms up to finite ambiguity.

§ 11. *Algebraic constructions that mirror topological ones.*

The first paragraph gives the algebraic formula for the *space of closed curves* in a given space. The second describes (via the path space) explicit differential forms integrands which give a complete set of *homotopy periods.* The third describes the higher homotopy of the *automorphism group of a space* algebraically (it is in fact the algebraic analogue of a complex considered by Nijenhuis in differential geometry and pseudo-group theory). The universal fibration with given nilpotent fibre is sketched. The fourth part *begins* the discussion of what one must do for general *spaces of maps* or *spaces of sections.* One corollary is an explicit algorithm for computing Gelfand-Fuks cohomology of a manifold. (The algorithm is given, its validity is not proved.)

§ 12. *Formal computations and Kaehler manifolds.*

A *formal* minimal model is one determined (formally) by its cohomology ring. There is a direct proof of the *formality over* $\mathbf{C}$ for a Kaehler manifold. Formality over $\mathbf{Q}$ is equivalent to formality over any larger field (Theorem (12.1)). The explicit algorithm for constructing the formal model (passing via differential Lie algebras) is given. Other examples of formal spaces. Integral homotopy types which are formal over $\mathbf{Q}$ have many continuous *endomorphisms* (Theorem (12.2)). Thus so do Kaehler manifolds (Theorem (12.4)). Thus $\mathbf{Q}$-maps can be essentially lifted to integral maps (when the range is formal) (Corollary (12.3)). *Finite determination* of the diffeomorphism type of a simply connected Kaehler manifold (Theorem (10.5)). Theorem (10.6) describes the $\mathbf{Q}$-homotopy theory of *holomorphic maps.* Theorem (10.7) characterizes formality in terms of grading automorphisms. The vast unstable $\mathbf{Q}$-homotopy of Thom spaces is quickly computed by formality.

§ 13. *Algebraic invariants for the classification and construction of manifolds and diffeomorphisms.*

Manifolds are closed simply connected of dimension at least five. Theorem (13.1) describes an algebraic invariant constructed from the forms on the manifold which determines the diffeomorphism type up to finitely many possibilities. Theorem (13.2) gives necessary and sufficient conditions in terms of the homology intersection for realizing the rational part of the invariant by a manifold. Theorem (13.5) describes the $\mathbf{Q}$-algebraic group which determines the commensurability class of the component group of diffeomorphisms (as its " arithmetic subgroup "). Corollary (13.4) asserts that two isometries which agree on homology are algebraically isotopic.

§ 14. *Questions, problems, and further remarks.*

Theorem (14.1) states that a Riemannian metric determines a canonical embedding of the model into the forms (generalizing the harmonic forms of Hodge Theory). Problems and questions about the above theory and geometric properties are formulated. Areas for future study might include infinitesimal isometries, complex structures, symplectic structures, and sign of curvature. Also the non simply connected (non-nilpotent) theory adumbrated above might be developed further and applied. There are also interesting questions in commutative algebra related to minimal models.

1. Twisted cohomology of differential algebras.

We will define the cohomology of a differential graded algebra (d.g.a.) $\mathscr{A}$ ([6]) with twisted coefficients. The coefficient system consists of a vector space V (in degree zero) and a *twisting matrix* $V \xrightarrow{\Theta} \mathscr{A}_1 \otimes V$ satisfying the *integrability condition*. One expression of this condition is that the natural $\mathscr{A}$-derivation d_Θ on $\mathscr{A} \otimes V$ defined by Θ on V and d in $\mathscr{A}$ $(d_\Theta(a \otimes v) = da \otimes v + \bar{a} \circ \Theta v)$ is a differential, $d_\Theta \circ d_\Theta = 0$. In terms of a basis $\{v_\alpha\}$ of V write $\Theta v_\alpha = \sum_\beta \theta_{\alpha\beta} v_\beta$, $\theta_{\alpha\beta} \in \mathscr{A}_1$, and the *integrability condition* becomes the familiar equation:

$$d\theta_{\alpha\gamma} = \sum_\beta \theta_{\alpha\beta} \wedge \theta_{\beta\gamma} \quad \text{or} \quad d\Theta - \Theta \circ \Theta = 0.$$

We can define the twisted cohomology of $\mathscr{A}$ with coefficients in V *(tensor version)* to be that of the complex $(\mathscr{A} \otimes V, d_\Theta)$.

We can also consider the complex $\mathrm{Hom}(V, \mathscr{A})$ which has two degree $+1$ maps obtained using d in $\mathscr{A}$ and the composition of Θ and multiplication in $\mathscr{A}$. The difference of these (called d and Θ) is the differential d_Θ. In terms of the basis $\{v_\alpha\}$ of V an element of $\mathrm{Hom}(V, \mathscr{A})$ is a set $\{a_\alpha\}$ taken from $\mathscr{A}$. Then $d_\Theta = d - \Theta$, that is:

$$d\{a_\alpha\} = \{da_\alpha - \sum_\beta \theta_{\alpha\beta} a_\beta\},$$

and we have the *hom version* of twisted cohomology.

The definitions of twisted cohomology extend immediately to differential modules $\mathscr{M}$ over $\mathscr{A}$ ($\mathscr{M}$ is a module over $\mathscr{A}$ with a differential satisfying $d(a.m) = da.m + \bar{a}dm$, $a \in \mathscr{A}$, $m \in \mathscr{M}$).

Now we mention a method for showing the twisted cohomology vanishes. Let $\mathscr{A} \xrightarrow{s} \mathscr{A}$ (or $\mathscr{M} \xrightarrow{s} \mathscr{M}$) be a contracting homotopy, *i.e.* $ds + sd = 1$. Let $\mathcal{O}_s$ denote the subspace of $\mathscr{A}_1$ whose left multiplication operators anti-commute with s.

Proposition **(1.1)**. — *The twisted cohomology of $\mathscr{A}$ (or $\mathscr{M}$) vanishes if the twisting matrix has entries in $\mathcal{O}_s$.*

Proof. — Consider for example the complex $\mathrm{Hom}(V, \mathscr{A})$. Let $\tilde{s}$ denote the map of degree -1 induced by s. Then we have $\Theta\tilde{s} + \tilde{s}\Theta = 0$ because the entries of Θ lie in $\mathcal{O}_s$. Then $d_\Theta \tilde{s} + \tilde{s} d_\Theta = (d - \Theta)\tilde{s} + \tilde{s}(d - \Theta) = (d\tilde{s} + \tilde{s}d) - (\Theta\tilde{s} + \tilde{s}\Theta) = d\tilde{s} + \tilde{s}d = 1$. The other cases are similar.

([6]) Throughout the paper (except § 10) the ground ring is a field of characteristic zero — usually **Q**, **R** or **C**; $\bar{a} = (-1)^{\dim a} a$.

Topological Example. — Suppose $\mathscr{A}$ is the deRham algebra of forms on some manifold M (or space, § 7). Suppose (V, Θ) is a finite dimensional coefficient system for $\mathscr{A}$. Then:

Theorem (**1.2**). — Θ *determines a local system over* M *in the topological sense. The local differential forms for this system with the corresponding differential are isomorphic to* Hom(V, $\mathscr{A}$) *with the differential* d_Θ. *Thus the algebraic and the topological definitions of twisted cohomology agree for finite dimensional coefficients over the reals or complexes.*

Proof. — The movement of V along paths in M is described by the differential system $dv=\Theta(v)$. We can allow each v_0 in V to develop along any path in M by this rule. Going around small closed paths brings v_0 back to itself because $d\Theta=\Theta\circ\Theta$ (see § 9).

We obtain in this way a foliation of $M\times V$ transverse to the vertical factor V so that each leaf covers M evenly. Note the movement of V is linear because the rule is. So we also have a linear action of $\pi_1 M$ on V.

A " form " for topological twisted cohomology is an equivariant linear map of V into the forms on the universal covering space $\widetilde{M}$ of M (relative to the action of $\pi_1 M$ on each).

We can view such a " form " as a form on $\widetilde{M}\times V$ only defined in the $\widetilde{M}$ directions—depending linearly on V and invariant under the π_1 symmetry on $\widetilde{M}\times V$.

Passing to the quotient we obtain forms on $M\times V$ (by the above) only defined in the leaf-directions. The twisted d is the geometric one along the leaves.

Now such forms on $M\times V$ defined in the leaf directions are the same as forms on $M\times V$ defined in the M directions since the tangent plane to a leaf is canonically identified by projection to the tangent plane below in M. M-direction forms in $M\times V$ depending linearly on V are just Hom(V, M) and under this identification the d along the leaves goes over to $d_\Theta=(d-\Theta)$ (see Deligne's and Katz's Springer notes for more details).

Remark. — A finite dimensional local system over M can be given by a global matrix of 1-forms Θ iff the corresponding vector bundle is continuously trivial. A trivialization precisely determines such a Θ and the above calculation downstairs on M is valid.

It is a theorem [DS] that any *finite dimensional local system* over **C** on a compact polyhedron is continuously trivial on some finite covering space. Thus the above explicit d.g.a. formulation is valid after passing to a *finite* cover.

2. On the structure of differential algebras.

We will study the structure of differential algebras $\mathscr{A}$ which are *connected* in degree zero [7] and *free* of algebraic relations besides graded commutativity. From the theorem

[7] That is, degree zero consists of the ground field k and $\mathscr{A}=k\oplus\mathscr{A}^+$ where $\mathscr{A}^+$ is the part in positive degrees.

below we can reduce our study to those satisfying the further *condition of minimality*:

$$d\mathscr{A} \subseteq \mathscr{A}^+ . \mathscr{A}^+ \quad \text{(minimal condition).}$$

These minimal algebras constitute a natural categorical closure of classical Lie algebras relative to the concepts of cohomology and representations. They also provide efficient algebraic models for topological spaces.

Degree one case:

By a *dual Lie algebra* we mean a free-connected differential algebra which is generated in *degree one*. The minimal condition is automatically satisfied. The differential $\mathscr{A}_1 \xrightarrow{d} \wedge^2 \mathscr{A}_1$ in terms of a basis $\{x_k\}$ of $\mathscr{A}_1$ has the form:

$$(2.1) \qquad dx_k = \sum_{i,j} a_{ij}^k x_i \wedge x_j.$$

Dualizing d yields a Lie algebra with structure constants a_{ij}^k, $[y_i, y_j] = \sum_k a_{ij}^k y_k$. The Jacobi identity comes from the condition that $d \circ d = 0$.

Going from a Lie algebra to a dual Lie algebra is the construction of the complex for Lie algebra cohomology. One uses the structure constants (as above) to define the differential in the exterior algebra on the space dual to the underlying vector space of the Lie algebra.

The classical structure theory for finite dimensional Lie algebras when read on the dual side implies in particular that $\mathscr{A}$ is obtained in two stages:

(i) One first forms a tensor product of simple algebras (those having no proper differential subalgebras generated in degree one, and such that d is injective in degree one). This is the semi-simple part.

(ii) One then forms an iterated sequence of extensions of the form $\mathscr{A}(x_1, x_2, \ldots, x_n)$ where $dx_i = \sum_j a_{ij} x_j + b_i$, the a_{ij} and b_i are taken from $\mathscr{A}$. The (a_{ij}) describe an action and the (b_i) a twisted cohomology class for $\mathscr{A}$ in degree 2. This is the solvable part.

Note that step (i) is several uncoupled systems of quadratic equations (2.1) and step (ii) is a triangular system of linear equations.

We will see that the further structure of a minimal algebra is described by linear equations generalizing step (ii).

The general minimal case:

Suppose now $\mathscr{A}$ is a minimal differential algebra whose structure has been described for the differential subalgebra $\mathscr{A}^{k-1}$ generated by elements of degree $< k$ where $k-1$ is at least one. Let $\{x_\alpha\}$ be a complete set of algebraic generators of $\mathscr{A}$ in degree k. Since $d\mathscr{A} \subset \mathscr{A}^+ . \mathscr{A}^+$ we can write equations:

$$dx_\alpha = \sum_\beta \theta_{\alpha\beta}^k x_\beta + a_\alpha$$

where the $\Theta^k = (\theta_{\alpha\beta}^k)$ are taken from degree one and the a_α lie in $\mathscr{A}^{k-1}$.

We claim that $(\theta^k_{\alpha\beta})$ defines an integrable local system Θ^k and that $\{a_\alpha\}$ is a (hom) cocycle of $\mathscr{A}^{k-1}$ for these coefficients (§ 1). To see this one differentiates the equations and uses $d^2=0$ with the freeness of $\mathscr{A}$.

Writing the structure equations in vector-matrix form we have $(\Theta=\Theta^k)$:

$$dx=\Theta(x)+a.$$

Differentiating:

$$\begin{aligned} d^2x=0 &= (d\Theta)(x)-\Theta(dx)+da \\ &= (d\Theta)(x)-\Theta(\Theta(x)+a)+da \\ &= ((d\Theta)-\Theta\circ\Theta)(x)+(d-\Theta)(a). \end{aligned}$$

Since $\mathscr{A}$ is free of relations besides graded commutativity, we obtain two equations $d\Theta-\Theta\circ\Theta=0$ and $(d-\Theta)(a)=0$. Since:

$$(d-\Theta)(d-\Theta)(v)=(d^2-\Theta\circ d-d\circ\Theta+\Theta\circ\Theta)(v)=-(d(\Theta)-\Theta\circ\Theta)(v),$$

we define $d_\Theta=d-\Theta$ and conclude:

(i) $d^2=0$ implies $d_\Theta\circ d_\Theta=0$ and $d_\Theta a=0$;

(ii) $dx=\Theta(x)+a$ may be rewritten $d_\Theta x=a$.

Also note if we change (in all ways) the generators x_α then $\{a_\alpha\}$ changes by (every possible) d_Θ-coboundary in $\mathscr{A}^{k-1}$.

Now the differential algebra $\mathscr{A}$ is the union of the canonical subalgebras:

$$\mathscr{A}^1\subset\mathscr{A}^2\subset\ldots\subset\mathscr{A}^k\subset\mathscr{A}.$$

Thus we have proved the

Theorem **(2.1)**. — *A minimal differential algebra* $\mathscr{A}=\bigcup_k\mathscr{A}^k$ *is determined up to isomorphism by*:

(i) *the dual Lie algebra* $\mathscr{A}^1\subset\mathscr{A}$ *defined by the* quadratic *equations*:

$$dx_k=\sum_{i,j}a^k_{ij}x_i\wedge x_j$$

giving d *on the one-dimensional generators*;

(ii) *an inductive sequence of twisted cohomology classes*:

$$a^{k+1}=[a_\alpha] \text{ in } \mathrm{H}^{k+1}(\mathscr{A}^{k-1},\Theta^k) \qquad k=2,3,\ldots$$

determining the differential algebra extension $\mathscr{A}^k=\mathscr{A}^{k-1}(x_\alpha)$. *The cohomology class and twisting matrix in* $\mathscr{A}^{k-1}$ *are determined by (and determine) the* linear *structure equations*:

$$dx_\alpha=\sum_\beta\theta^k_{\alpha\beta}x_\beta+a_\alpha \quad \text{or} \quad d_\Theta x=a$$

giving d *on the new generators of* $\mathscr{A}^k$ *over* $\mathscr{A}^{k-1}$.

Remark. — The sequence of twisting matrices Θ^2, Θ^3, ... define representations of the Lie algebra in dimension one on the spaces dual to the indecomposable spaces.

Analogy to topology:

The analogy between a minimal d.g.a. and the homotopy theory of a space is striking:

fundamental group	: dual Lie algebra $\mathscr{A}^1$
k-th homotopy group	: space of k-dimensional generators of $\mathscr{A}$
action of fundamental group on higher homotopy	: twisting matrices $\Theta^2, \Theta^3, \ldots$
cohomological k-invariants of the Postnikov system	: cohomology classes of structure $a^3, a^4, \ldots$

We can pursue the analogy further:

Massey products	: Massey products
Whitehead products on homotopy	: dual graded Lie algebra defined by all the quadratic terms in d.

Note a Massey product $\langle a, b, c\rangle$ can be computed (modulo ideal (a, c)) by solving $a\wedge b = d\eta$ and $b\wedge c = d\nu$. Then $a\eta \pm \nu c$ represents $\langle a, b, c\rangle$.

Relation to Lie algebras:

(i) We can define *nilpotent* minimal algebras. We suppose the dual Lie algebra $\mathscr{A}^1$ and the twisting matrices Θ^k are nilpotent. This property is equivalent to the existence of a refined sequence of differential subalgebras $\mathscr{A}_1 \subset \mathscr{A}_2 \subset \ldots$ so that

a) each dimension is exhausted by some $\mathscr{A}_k$,

b) $\mathscr{A}_k$ is a linear extension of $\mathscr{A}_{k-1}$ without twisting terms in the d-formulae. (Thus $dx = \Theta x + a$ becomes $dx = a$.)

Note nilpotent algebras are structured by ordinary cohomology. If $\mathscr{A}$ has no generators in degree one, then $\mathscr{A}$ is nilpotent. This gives many examples.

Dropping the finiteness condition *a)* leads to the notion of generalized nilpotent algebras.

We will see that these nilpotent algebras give canonical models for the algebraic topology of spaces whose homotopy system $(\pi_1, \pi_2, \ldots)$ is nilpotent (§ 10).

(ii) We can define *solvable* minimal algebras. This means $\mathscr{A}$ can be built up to any dimension by a *finite* number of linear steps. It is clearly enough to assume this for dimension one. For then each dimension can be achieved in one linear step.

If $\mathscr{A}^1$ can be built in a longer sequence of linear steps, we have the notion of *generalized solvable* algebras.

(iii) In the general case, the part of $\mathscr{A}$ above dimension one behaves like an enlargement of the solvable part of $\mathscr{A}^1$, though this enlargement has the new ingredient of symmetric algebra.

Reduction to the minimal case:

Define a contractible algebra to be one isomorphic to a tensor product of those of the form $\Lambda(x, dx)$. Then we claim the

Theorem **(2.2)**. — *Every free connected differential algebra is isomorphic (with differentials) to the tensor product of a unique minimal algebra and a unique contractible algebra* (8).

Proof. — The differential in $\mathscr{A}$ induces one d' on the graded spaces of generators or indecomposables. $\mathscr{A}$ is minimal if and only if d' is zero. If d' is not zero, write this complex as a direct sum of its homology and an acyclic subcomplex (a linear version of the theorem).

It is easy to lift the acyclic part back to $\mathscr{A}$ so that it generates a contractible differential subalgebra, $\mathscr{C} \subset \mathscr{A}$.

Then $\mathscr{A}/\text{ideal}\,\mathscr{C}^+$ is our desired minimal algebra $\mathscr{M}$. Since $\mathscr{A}$ is free we can write $\mathscr{A} \simeq \mathscr{C} \otimes \mathscr{M}$ as algebras, but $\mathscr{M}$ need not be closed under d. However, it works at the beginning. Let $\mathscr{M}_1$ be $d^{-1}(\wedge^2 \mathscr{A}_1)$. Then $\mathscr{M}^1$ is the algebra generated by $\mathscr{M}_1$. Using $d^2 = 0$ and the freeness of $\mathscr{A}$ one sees that $\mathscr{M}^1$ is actually closed under d.

Suppose inductively we have chosen a splitting $\mathscr{M}^{k-1} \subset \mathscr{A}$ closed under d. If:

$$dx_\alpha = \sum_\beta \theta_{\alpha\beta} x_\beta + a_\alpha$$

are the structure equations for the extension $\mathscr{M}^{k-1} \subset \mathscr{M}^k$, $k > 1$, then $\{a_\alpha\}$ is a cocycle in $\mathscr{A}$ which is exact in $\mathscr{M}^k$ (and therefore $\mathscr{M}$) and our problem is to show it is exact in $\mathscr{A}$.

For this we have

Proposition **(2.3)**. — *Suppose* $\mathscr{A} = \mathscr{C} \otimes \mathscr{M}$ *as algebras where* $\mathscr{C}$ *is a differential subalgebra which is connected and contractible. Then* $\mathscr{A} \to \mathscr{A}/\text{ideal}\,\mathscr{C}^+$ *is an isomorphism on cohomology for all coefficients.* (Proof below.)

Granting this we can solve the desired equations in $\mathscr{A}$, namely, $\{a_\alpha\}$ is a coboundary, and we have our lifting on $\mathscr{M}^k$. Thus $\mathscr{M}$ lifts to $\mathscr{A}$ and we have a tensor product decomposition of differential algebras $\mathscr{A} \simeq \mathscr{M} \otimes \mathscr{C}$.

Uniqueness will follow because the generators of $\mathscr{M}$ and $\mathscr{C}$ correspond to the homology and acyclic part respectively of the complex of indecomposables for $\mathscr{A}$. Let us call this " indecomposable homology " the *dual homotopy spaces of* $\mathscr{A}$ (Definition (2.3)). See [Q] for similar definition.

If we choose the splitting $\mathscr{C} \subset \mathscr{A}$ differently resulting in $\mathscr{M}' = \mathscr{A}/\text{ideal}\,\mathscr{C}^+$, the composition $\mathscr{M} \xrightarrow{\text{splitting}} \mathscr{A} \to \mathscr{M}'$ induces an isomorphism on the *dual homotopy spaces* and is therefore an isomorphism of differential algebras.

We now prove the proposition (2.3). For simplicity assume $\mathscr{C} = \wedge(x, dx)$ has only two generators. Write $\mathscr{A} = \mathscr{M} \otimes \mathscr{C}$ as $\mathscr{C}$-modules and consider the sequence of

(8) The generality of this theorem is due to a conversation with Richard Body.

powers of the ideal of $\mathscr{C}^+$, $\mathscr{A} \supset \mathfrak{J} \supset \mathfrak{J}^2 \supset \dots$. The successive quotients $\mathfrak{J}^k/\mathfrak{J}^{k+1}$ are isomorphic to $\mathscr{M} \otimes (x^k, x^{k-1}dx)$ with the product differential. So it is easy to define a contracting homotopy for $\mathfrak{J}^k/\mathfrak{J}^{k+1}$ commuting (up to appropriate sign) with the operations of left multiplication by $\mathscr{M}$. By Proposition (1.1), $\mathfrak{J}^k/\mathfrak{J}^{k+1}$ is acyclic for all twisted coefficients. Since degree $x>0$ we're done by applying the 5-lemma finitely many times.

3. Algebraic maps, deformations, and obstructions.

We study maps from $\mathscr{A}$ which are fixed on some subdifferential algebra $\mathscr{B}$. The receiving algebra $\mathscr{C}$ as well as $\mathscr{B}$ are arbitrary; but we assume $\mathscr{A}$ is built up from $\mathscr{B}$ by successive extensions ("adding new generators"):

$$dx_\alpha = \sum_\beta \theta_{\alpha\beta} x_\beta + a_\alpha, \qquad \text{degree } x_\alpha > 0,$$

where $\{\theta_{\alpha\beta}\}$ and $\{a_\alpha\}$ belong to earlier stages. The x_α are free generators.

After proposition (3.1) we assume the twisting coefficients $\{\theta_{\alpha\beta}\}$ lie in $\mathscr{B}$. Note that $a_\alpha = \mathrm{P}_\alpha(x_\beta)$ is a polynomial in earlier x_β with coefficients in $\mathscr{B}$. We call the linear spaces of x_α the *dual homotopy* of the pair $(\mathscr{A}, \mathscr{B})$. (Definition 3.1)

Proposition **(3.1)**. — *The inductive construction of a map $\mathscr{A} \to \mathscr{C}$ extending $\mathscr{B} \xrightarrow{\vee} \mathscr{C}$ is obstructed by a sequence of classes in the cohomology of $\mathscr{C}$ with coefficients in the dual homotopy of $(\mathscr{A}, \mathscr{B})$ (twisted by the inductive map applied to $\{\theta_{\alpha\beta}\}$).*

Proof. — To build a map f we have to successively solve the structure equations in $\mathscr{C}$. The obstructions are then $\{f(a_\alpha)\}$ which defines a $\hom(d - f(\theta_{\alpha\beta}))$ cocycle (§ 1).

We will define a notion of deformation of map or homotopy having some advantages over that described in earlier reports ([Su$_1$], [DGMS], and [S-V]). There a homotopy was a d.g.a. map $\mathscr{A} \to \mathscr{C}(t, dt)$ in analogy with the topological definition $\mathrm{X} \times \mathrm{I} \to \mathrm{Y}$. Now we describe a notion analogous to the topological one expressed by $\mathrm{X} \to \mathrm{Y}^{\mathrm{I}}$.

Let us write $\mathscr{A} = \mathscr{B}(x_\alpha)$, $\mathscr{A} \otimes_{\mathscr{B}} \mathscr{A} = \mathscr{B}(x_\alpha, y_\alpha)$, and $\mathscr{A}^{\mathrm{I}} = \mathscr{B}(x_\alpha, y_\alpha, \bar{\delta}_\alpha)$. Here $\bar{\delta}_\alpha$ is a new generator in degree one less than $\delta_\alpha = \delta(x_\alpha, y_\alpha) = x_\alpha - y_\alpha$. Note that $d\delta_\alpha = \sum_\beta \theta_{\alpha\beta}\delta_\beta + \mathrm{P}_\alpha(x_\beta) - \mathrm{P}_\alpha(y_\beta)$.

Proposition **(3.2)**. — *We can define a differential in $\mathscr{B}(x_\alpha, y_\alpha, \bar{\delta}_\alpha)$ extending that on $\mathscr{B}(x_\alpha, y_\alpha)$ of the form:*

$$d\bar{\delta}_\alpha = \delta_\alpha + \sum_\beta \theta_{\alpha\beta}\bar{\delta}_\beta + \text{earlier terms}.$$

Moreover, $\mathrm{ideal}(\bar{\delta}_\alpha, \delta_\alpha) = \mathrm{ideal}(\bar{\delta}_\alpha, d\bar{\delta}_\alpha)$ *has vanishing cohomology for all coefficients in* $\mathscr{B}$.

Proof. — The proof goes by direct induction on the new generators using the vanishing of cohomology for the ideal. Note for the induction this vanishing follows from the equality of ideals and the isomorphisms:

$$\mathcal{A}^{\mathrm{I}} \simeq \mathcal{B}(x_\alpha) \otimes \text{algebra}(\bar{\delta}_\alpha, d\bar{\delta}_\alpha) \quad \text{and} \quad \mathcal{A}^{\mathrm{I}}/\text{ideal}(\bar{\delta}_\alpha, \delta_\alpha) \simeq \mathcal{A}.$$

Also $P_\alpha(x_\beta) - P_\alpha(y_\beta)$ belongs to the ideals, we can solve in the ideal:

$$d_\Theta\{\eta_\alpha\} = \{P_\alpha(x_\beta) - P_\alpha(y_\beta)\}$$

and let η_α be the " earlier terms ".

Remark. — One also sees inductively two such differentials differ by changing bases $\bar{\delta}_\alpha \mapsto \bar{\delta}_\alpha +$decomposable from ideal. We give explicit formulae in § 6, proposition (6.3).

Definition **(3.1)**. — Two maps of $\mathcal{A}$ into $\mathcal{C}$ extending $\mathcal{B} \xrightarrow{\nu} \mathcal{C}$ are *homotopic* (rel ν) if the combined map of $\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A}$ into $\mathcal{C}$ extends over $\mathcal{A}^{\mathrm{I}}$.

By the remark following proposition (3.2) the homotopy relation is independent of the choice of d in $\mathcal{A}^{\mathrm{I}}$ (mod decomposables).

Proposition **(3.3)**. — *The inductive construction of a homotopy between two maps of $\mathcal{A}$ into $\mathcal{C}$ is obstructed by a sequence of classes in the cohomology of $\mathcal{C}$ with coefficients in the dual homotopy of $(\mathcal{A}, \mathcal{B})$.*

The proof is the same as proposition (3.1).

Corollary **(3.4)**. — *Homotopy is an equivalence relation, and we can add homotopies.*

Proof. — With obvious notation, we use the obstruction theory to extend the inclusion:

$$\mathcal{B}(x_\alpha, z_\alpha) \to \mathcal{B}(x_\alpha, y_\alpha, z_\alpha, \bar{\delta}(x_\alpha, y_\alpha), \bar{\delta}(y_\alpha, z_\alpha))$$

over $\mathcal{B}(x_\alpha, z_\alpha, \bar{\delta}(x_\alpha, z_\alpha))$ using the vanishing of the cohomology of:

$$\text{ideal } (\bar{\delta}(x_\alpha, y_\alpha), \bar{\delta}(y_\alpha, z_\alpha), \delta(x_\alpha, y_\alpha), \delta(y_\alpha, z_\alpha)).$$

This proves transitivity by adding homotopies.

Symmetry follows by transposing the d in $\mathcal{B}(x_\alpha, y_\alpha, \bar{\delta}_\alpha)$ by the involution:

$$x_\alpha \mapsto y_\alpha, \quad y_\alpha \mapsto x_\alpha, \quad \bar{\delta}_\alpha \mapsto -\bar{\delta}_\alpha$$

and using the uniqueness remark about d in the proof of proposition (3.2).

Now consider a map $\mathcal{C}' \xrightarrow{\pi} \mathcal{C}$ and a lifting ν' of $\mathcal{B} \xrightarrow{\nu} \mathcal{C}$ up to $\mathcal{C}'$, $\mathcal{B} \xrightarrow{\nu'} \mathcal{C}'$.

Proposition **(3.5)**. — *The construction of a lifting of $\mathcal{A} \xrightarrow{f} \mathcal{C}$ into $\mathcal{C}'$ up to homotopy is obstructed by a sequence of classes in the relative cohomology of $(\mathcal{C}' \to \mathcal{C})$ with coefficients in the dual homotopy of $(\mathcal{A}, \mathcal{B})$.*

Proof. — Now $x_\alpha \to \mathscr{C}$ is determined and $y_\alpha \to \mathscr{C}'$ and $\bar{\delta}_\alpha \to \mathscr{C}$ are to be found. The obstructions are obtained as usual by examining the structure equation:

$$d\bar{\delta}_\alpha = (x_\alpha - y_\alpha) + \sum_\beta \theta_{\alpha\beta}(x_\beta - y_\beta) + Q_\alpha(x_\beta, y_\beta).$$

As an application consider the equivalence classes of maps up to homotopy $[\mathscr{A}, \mathscr{C}; \nu]$. Assume $\mathscr{C}' \xrightarrow{\pi} \mathscr{C}$ induces an isomorphism of cohomology with coefficients in the dual homotopy of $(\mathscr{A}, \mathscr{B})$.

Corollary **(3.6)**. — $[\mathscr{A}, \mathscr{C}'; \nu'] \xrightarrow{\pi} [\mathscr{A}, \mathscr{C}; \nu]$ *is a bijection.*

Proof. — Proposition (3.5) applied to maps of $\mathscr{A}$ fixed on $\mathscr{B}$ ($\mathscr{A}^{I}$ fixed on $\mathscr{A} \otimes_{\mathscr{B}} \mathscr{A}$) implies π is surjective (injective) since the relative cohomology vanishes.

Remark. — This proof prompts the observation that homotopies of homotopies, homotopies of homotopies of homotopies, etc. are special cases of ordinary homotopies with varying $\mathscr{B}$ and $\mathscr{A}$.

Now assume that the number of x_α's is finite and that $\mathscr{C}$ is finite dimensional in each degree of an x_α and one less. Fix $\mathscr{B} \xrightarrow{\nu} \mathscr{C}$.

Proposition **(3.7)**. — *The set of deformation classes of maps* $[\mathscr{A}, \mathscr{C}; \nu]$ *has the structure of an affine algebraic variety modulo an equivalence relation which is the image of an affine algebraic variety.*

Proof. — Consider the array:

$$\mathscr{M}ap(\mathscr{B}(x_\alpha, y_\alpha, \bar{\delta}_\alpha), \mathscr{C}) \xrightarrow{R} \mathscr{M}ap(\mathscr{B}(x_\alpha), \mathscr{C}) \times \mathscr{M}ap(\mathscr{B}(y_\alpha), \mathscr{C}) \xrightarrow{E} [\mathscr{A}, \mathscr{C}; \nu]$$

where $\mathscr{M}ap(\cdot, \mathscr{C})$ means the set of all d.g.a. maps. Now E is onto and each of the $\mathscr{M}ap$ sets are subsets of linear spaces $\mathrm{Hom}((x_\alpha), \mathscr{C})$, etc. described by the algebraic conditions commutation with d.

Remark. — The actual structure of $[\mathscr{A}, \mathscr{C}; \nu]$ is somewhat better because the homotopies between two maps f and g are parametrized by a space of cocycles of $\mathscr{C}$. Thus the image of R is obtained by dividing an affine algebraic variety by a linear foliation.

4. Algebraic Fibrations.

Let $\mathscr{E}$ be a differential algebra and $\mathscr{B} \subset \mathscr{E}$ a differential subalgebra. We assume $\mathscr{E}$ is a free module over $\mathscr{B}$ so we can write $\mathscr{E} \sim \mathscr{B} \otimes \mathscr{F}$. Then if $\mathscr{B}$ is connected (or provided with an augmentation $\mathscr{B} \to$ ground field) $\mathscr{F}$ inherits the structure of a differential algebra via the identification $\mathscr{F} \sim \mathscr{E}/\text{ideal } \mathscr{B}^{+}$.

This situation is the algebraic analogue of a fibration and we refer to $\mathscr{E}$, $\mathscr{B}$, $\mathscr{F}$ respectively as the total space, the base, and the fiber.

If $\mathscr{B}$ is not connected we will have to assume later that the differential in $\mathscr{E}$ has the form on $\mathscr{F}$ in degree i:

$$d\mathscr{F}_i \subset \mathscr{F}_{i+1} + \mathscr{F}_i \otimes \mathscr{B}_1 + \mathscr{F}_{i-1} \otimes \mathscr{B}_2 + \ldots$$

We call this property the *infinitesimal condition* (namely $\mathscr{F}_{i+1}$ instead of $\mathscr{F}_{i+1} \otimes \mathscr{B}_0$). With this condition the $\mathscr{F} \otimes \mathscr{B}_1$ part of $d\mathscr{F}$ defines twisting matrices in $\mathscr{B}$ for $\mathrm{H}^*\mathscr{F}$. These are called the *action coefficients* of the fibration.

We study the cohomological relationships between $\mathscr{B}$, $\mathscr{E}$, and $\mathscr{F}$. Note that proposition (2.3) is an example and may be paraphrased here as: if the base is connected and contractible then $\mathscr{E}$ and $\mathscr{F}$ have isomorphic cohomology for all coefficients.

We begin with an application of the algebraic analogue of *induced fibration.*

A map between fibrations is a map of pairs $(\mathscr{E}, \mathscr{B}) \to (\mathscr{E}', \mathscr{B}')$ and one sees easily that maps can be factored so that the base is the same for one factor and the fiber is the same for the other. Namely:

$$\mathscr{E} = \mathscr{B} \otimes \mathscr{F} \xrightarrow{f} \mathscr{B}' \otimes \mathscr{F}' = \mathscr{E}'$$

equals the composition of (" equal bases " and then " equal fibres "):

$$\mathscr{B} \otimes \mathscr{F} \xrightarrow{g} \mathscr{B}' \otimes \mathscr{F} \quad \text{and} \quad \mathscr{B}' \otimes \mathscr{F} \xrightarrow{h} \mathscr{B}' \otimes \mathscr{F}';$$

g is essentially $(f \mid \mathscr{B}) \otimes \text{Identity}$ and h is the product of $(f \mid \mathscr{F})$ and the identity on $\mathscr{B}'$. The differential algebra structure on $\mathscr{B}' \otimes \mathscr{F}$ is obtained by pushing forward that on $\mathscr{B} \otimes \mathscr{F}$ by $f \mid \mathscr{B}$ (" the induced fibration ").

We study these two types of maps in turn. We prove four propositions and state in the proof of each which type of twisted cohomology (tensor or hom) is meant in the statement.

First the case of equal bases $\mathscr{B} \otimes \mathscr{F} \xrightarrow{f} \mathscr{B} \otimes \mathscr{F}'$.

Proposition **(4.1)**. — *If* $\mathrm{H}^*\mathscr{F} \overset{f}{\sim} \mathrm{H}^*\mathscr{F}$ *(ordinary coefficients) then* $\mathrm{H}^*\mathscr{E} \overset{f}{\sim} \mathrm{H}^*\mathscr{E}'$ *for all coefficients coming from* $\mathscr{B}$.

Proof. — The subcomplexes degree($\mathscr{B}$-coefficients) $\geq k$ are closed under d. The quotient complexes are just the $\mathscr{B}_k$ with zero differential tensor the fiber complexes. The conclusion for *either* hom or tensor twisted cohomology follows easily from the five lemma.

From now on we assume the *infinitesimal condition* concerning $d\mathscr{F}$. We can then speak of the action coefficients. Again we assume equal bases, $\mathscr{B} \otimes \mathscr{F} \xrightarrow{f} \mathscr{B} \otimes \mathscr{F}'$.

Proposition **(4.2)**. — *Suppose the fibers* $\mathscr{F}$ *and* $\mathscr{F}'$ *are generalized solvable* (§ 2); *and* $\mathrm{H}\mathscr{E} \overset{f}{\sim} \mathrm{H}\mathscr{E}'$ *for all* inductive *action coefficients (see proof). Then the fibers* $\mathscr{F}$ *and* $\mathscr{F}'$ *are actually isomorphic as differential algebras. Thus the total spaces* $\mathscr{E}$ *and* $\mathscr{E}'$ *are isomorphic as differential algebras.*

Proof. — In the first non-zero degree of the fiber $\mathscr{F}'$, the $\mathscr{F}'$ cocycles $\mathscr{C}_i' \subset \mathscr{F}_i'$ satisfy $d\mathscr{C}_i' \subset \mathscr{C}_i' \otimes \mathscr{B}_1 + \mathscr{B}_{i+1}$. Thus $\mathscr{C}_i' \subset \mathscr{E}$ determines a twisted (hom) cocycle in the relative complex $(\mathscr{E}, \mathscr{B})$, the coefficients being part of the action coefficients.

Using $\mathrm{H}^i \mathscr{E} \xrightarrow{f} \mathrm{H}^i \mathscr{E}'$ is onto for $\mathscr{C}_i'$ cohomology (hom version) implies $\mathscr{C}_i \xrightarrow{f} \mathscr{C}_i'$ is onto.

Similarly, if j is the first non-zero degree of $\mathscr{F}$, $\mathrm{K}_j =$ kernel of $(\mathscr{C}_j \to \mathscr{C}_j')$ defines a (hom) cocycle going to zero under f. Applying injectivity of $\mathrm{H}^j \mathscr{E} \to \mathrm{H}^j \mathscr{E}'$ with K_j coefficients (hom version) forces K_j to be zero.

Being generalized solvable the fibers have cocycles in this first non-zero degree. We conclude $j = i$ and $\mathscr{C}_i \overset{f}{\sim} \mathscr{C}_i'$.

We amalgamate the subalgebra generated by $\mathscr{C}_i \sim \mathscr{C}_i'$ into the base and repeat the argument as often as necessary to reach our conclusion that $\mathscr{F} \overset{f}{\sim} \mathscr{F}'$ and $\mathscr{E} \overset{f}{\sim} \mathscr{E}'$.

Note we keep meeting new action coefficients in the aggrandizing base (these are the *inductive* action coefficients).

Corollary **(4.1)**. — *A map between two generalized solvable algebras which gives isomorphisms on all cohomology with homotopy action coefficients* (§ 3) *is an isomorphism of differential algebras.*

Corollary **(4.2)**. — *A map between two minimal algebras* (§ 2) *which induces an isomorphism in degree* 1 *and an isomorphism on cohomology with coefficients in the higher homotopy action is an isomorphism of differential algebras.*

Now we treat the case of " equal fibers ", $\mathscr{B} \otimes \mathscr{F} \xrightarrow{f} \mathscr{B}' \otimes \mathscr{F}$.

Let Σ be any collection of coefficients which is closed under tensor products and contains the appropriate action coefficients (for both fibrations).

Proposition **(4.3)**. — *If* $\mathrm{H}\mathscr{B} \overset{f}{\sim} \mathrm{H}\mathscr{B}'$ *(coefficients in* Σ*) and* $d = 0$ *in the common fiber, then* $\mathrm{H}\mathscr{E} \overset{f}{\sim} \mathrm{H}\mathscr{E}'$ *(coefficients in* Σ*).*

Proof. — When d is zero in the fiber $\mathscr{F}$, degree($\mathscr{F}$-coefficient)$\leq k$ defines a filtration by d-subcomplexes. The successive quotients are the complexes for computing the (tensor) cohomology of the base with coefficients in the action. We tensor this picture with the various coefficients from Σ, apply the 5-lemma, and we're done.

Remark. — Proposition (4.3) is valid for generalized nilpotent fibers if Σ contains the homotopy action (which is then defined in $\mathscr{B}$).

Proposition **(4.4)**. — *Suppose the common fiber is solvable. Then* $\mathrm{H}\mathscr{E} \overset{f}{\sim} \mathrm{H}\mathscr{E}'$ *for* Σ*-coefficients implies* $\mathrm{H}\mathscr{B} \sim \mathrm{H}\mathscr{B}'$ *for coefficients in* $\Sigma_{\mathscr{B}}$ *(those of* Σ *coming from* $\mathscr{B}$*).*

Proof. — Consider the case when $d = 0$ on $\mathscr{F}$ and the subsequent filtration by degree $\mathscr{F}$ as in Proposition (4.3). Let $(\mathscr{E})_n$ denote the hypothesis up to n and $(\mathscr{B})_n$ the conclusion up to n. Now assuming $(\mathscr{E})_n$ we prove $(\mathscr{B})_k$ for $k \leq n$ by induction.

If $(\mathscr{B})_{k-1}$ is true, since $\mathscr{F}$ is connected in degree zero, we have isomorphisms of tensor cohomology with $\Sigma_{\mathscr{B}}$ coefficients on the successive quotients (other than $\mathscr{B}$ itself) up to k. The five lemma a number of times and $(\mathscr{E})_n$ then give $(\mathscr{B})_k$ for $k \leq n$.

Now we turn to the solvable fiber. We have the natural interpolating differential algebras $\mathscr{B} = \mathscr{E}_0 \subset \mathscr{E}_1 \subset \mathscr{E}_2 \subset \ldots \mathscr{E}$ determined by the solvable series. Assume Σ contains the inductive action coefficients. Since the series is finite in each degree, $\mathrm{H}\mathscr{E}_{\mathrm{N}} \sim \mathrm{H}\mathscr{E}$ in degree $\leq n$ for all coefficients and N sufficiently large. Now assuming Σ contains the inductive action coefficients we apply the above to deduce:

$$(\mathscr{E}_{\mathrm{N}})_n \Rightarrow (\mathscr{E}_{\mathrm{N}-1})_n \Rightarrow \ldots \Rightarrow (\mathscr{E}_0)_n$$

which is $(\mathscr{B})_n$. This completes the proof for the tensor version of twisted cohomology.

We can combine these statements in various ways. For example let:

$$(\mathscr{E}, \mathscr{B}) \xrightarrow{f} (\mathscr{E}', \mathscr{B}')$$

be a map of algebraic fibrations with finite type nilpotent fibers and Σ be a collection of coefficients containing the homotopy action and closed under tensor product.

Consider the statements

(i) $\mathrm{H}\mathscr{B} \overset{f}{\sim} \mathrm{H}\mathscr{B}'$ for Σ-coefficients

(ii) $\mathrm{H}\mathscr{E} \overset{f}{\sim} \mathrm{H}\mathscr{E}'$ for Σ-coefficients

(iii) $\mathscr{F} \overset{f}{\sim} \mathscr{F}'$ as differential algebras.

Theorem **(4.5)**. — *Any two of the above assertions implies the third.*

Remark. — The proofs and statements of propositions (2.3), (4.1), and (4.3) can be expanded into a spectral sequence theory $\mathrm{H}^*(\mathscr{B}, \mathrm{H}^*\mathscr{F}) \Rightarrow \mathrm{H}^*\mathscr{E}$. We have instead chosen the direct computational path because we *already* have *commutative* differential algebras.

5. Building Homological Models.

First Case (ordinary coefficients):

To each differential algebra $\mathscr{A}$ we will associate a homological model $\mathscr{M}(\mathscr{A})$. $\mathscr{M}(\mathscr{A})$ maps to $\mathscr{A}$ inducing an isomorphism on cohomology. $\mathscr{M}(\mathscr{A})$ as a differential algebra is free, connected, minimal, and generalized nilpotent. $\mathscr{M}(\mathscr{A})$ is characterized up to isomorphism by all these properties. The map $\mathscr{M}(\mathscr{A}) \to \mathscr{A}$ is well defined up to homotopy.

The construction of $\mathscr{M}(\mathscr{A}) \xrightarrow{\rho} \mathscr{A}$ is straightforward. For its description we refer to the algebraic operation $\mathscr{M} \mapsto \mathscr{M}(y)$ where dy is a cocycle c in $\mathscr{M}$ as " killing " the cohomology class of c. ($\mathscr{M}(y)$ means $\mathscr{M}$ with the new generator y added.) If $c = 0$, we say we have " added " the cohomology class of y.

Suppose now that $\mathcal{M}_k \xrightarrow{\rho_k} \mathcal{A}$ has been constructed so that ρ_k induces an isomorphism on cohomology in degree $\leq k$ and an injection in degree $k+1$. Then we "add" cohomology to $\mathcal{M}_k$ (if necessary) and map accordingly to "make ρ_k" onto in degree $k+1$. Then we add variables to "kill" the kernel on cohomology in degree $k+2$ and map accordingly. If there are terms in degree one we may have to again kill the kernel in degree $k+2$, and again, and again... The union of these constructions in the desired $\mathcal{M}_{k+1} \xrightarrow{\rho_{k+1}} \mathcal{A}$.

To start the induction we assume $H^0\mathcal{A}$ is the ground field and take:

$$\mathcal{M}_0 = H^0\mathcal{A} = \text{" constants of } \mathcal{A}\text{ "}.$$

$\mathcal{M} = \bigcup_k \mathcal{M}^k$ is the desired model, $\mathcal{M}(\mathcal{A})$.

If $\mathcal{M}' \to \mathcal{A}$ is another homological model, we can lift $\mathcal{M} \to \mathcal{A}$ (up to homotopy) to $\mathcal{M}'$ by applying the obstruction theory for homotopy, Proposition (3.5). The map $\mathcal{M}' \to \mathcal{M}$ induces an isomorphism on cohomology and is therefore an isomorphism of differential algebras (which is well defined up to homotopy) (Propositions (4.2) and (3.6)).

If in the construction of $\mathcal{M}(\mathcal{A}) \to \mathcal{A}$ there are only finitely many steps in each degree, then $\mathcal{M}(\mathcal{A})$ is nilpotent. Otherwise $\mathcal{M}(\mathcal{A})$ is generalized nilpotent, being formed by an infinite number of "nilpotent steps".

Example (nilpotent spaces).

If $\mathcal{A}$ is the de Rham complex (over $\mathbf{Q}$ or $\mathbf{R}$) of a nilpotent homotopy type X, $\mathcal{M}(\mathcal{A})$ is nilpotent and the realization of $\mathcal{M}(\mathcal{A})$ (§ 8) is the rational or real form of the homotopy type X. In particular, the dual homotopy spaces of $\mathcal{M}(\mathcal{A})$ are the dual homotopy groups of X. If the Betti numbers of X are not finite, one has to take into account the natural topology on dual spaces and cohomology. (See Remark (ii), § 8.) Also for an arbitrary space we can associate a generalized nilpotent model well defined up to isomorphism.

Example (Computations).

If $\mathcal{A}$ is the de Rham algebra of X, then $\mathcal{M}(X) = \mathcal{M}(\mathcal{A})$ in the following cases is: ($|x|$ means degree x, $\wedge(x, y, \ldots)$ means free graded commutative algebra on $x, y, \ldots$) for:

$X = S^1$	$\wedge(y)$, $\lvert y\rvert = 1$, $dy = 0$;
$X = S^2$	$\wedge(x, y)$, $\lvert x\rvert = 2$, $dy = x^2$;
$X = \mathbf{C}P^2$	$\wedge(x, y)$, $\lvert x\rvert = 2$, $dy = x^3$;
$X =$ Lie group	$\wedge(x_1, x_2, \ldots, x_n)$, $\lvert x_i\rvert$ odd, $dx_i = 0$;
$X =$ Grassmannian of 2-planes in $\mathbf{C}^4$	$\wedge(c_1, c_2, u, v)$, $\lvert c_1\rvert = 2$, $\lvert c_2\rvert = 4$, $du = c_1^3 - 2c_1c_2$, $dv = c_2c_1^2 - c_2^2$;

X = Riemann surface of genus $g>1$	$\wedge(x_1, y_1, \ldots, x_g, y_g; \ldots; \ldots; \ldots)$, $\lvert x_i\rvert = \lvert y_i\rvert = 1$, $dx_i = dy_i = 0$, and the number of 1-dimensional generators between the k and $(k+1)$-st semi-colon is $\sim g^k$ (actually $\frac{1}{2}g(g-1)-1$ for $k=1$); (See § 12 for the computation.)
X = Maps (S^1 into S^2)	$\wedge(x, \bar{x}, y, \bar{y})$, $\lvert x\rvert = 2$, $\lvert \bar{x}\rvert = 1$, $dx = 0$, $d\bar{x} = 0$, $dy = x^2$, $d\bar{y} = 2x\bar{x}$; (See § 11 for the reasoning);
X = complement of Borromean rings in S^3	$\wedge(x, y, z; u, v, w)$, $\lvert x\rvert = \lvert y\rvert = \lvert z\rvert = 1$, $dx = dy = dz = 0$, $du = xy$, $dv = xz$, $dw = yz$;
X = $K(\pi, n)$	$\wedge(\mathrm{Hom}(\pi, \mathbf{R}))$, $d \equiv 0$, $\lvert \mathrm{Hom}(\pi, \mathbf{R})\rvert = n$. (See § 8 for the proof.)

Second Case (Twisted Coefficients and Relative):

We assume we have differential algebras $\mathscr{A}$ and $\mathscr{B}$, a map $\mathscr{B} \to \mathscr{A}$ and a collection Σ of coefficients in $\mathscr{B}$ for which $H^1\mathscr{B} \to H^1\mathscr{A}$ is injective.

Theorem (**5.1**). — *We can minimally* (9) *extend* $\mathscr{B} \to \mathscr{A}$ *to* $\mathscr{B}(x_\alpha) \xrightarrow{f} \mathscr{A}$ *where:*

degree $x_\alpha > 0$

so that f induces an isomorphism of Σ-cohomology. Any two such extensions $\mathscr{B}(x_\alpha)$ and $\mathscr{B}(x'_\alpha)$ are isomorphic as differential algebras. The isomorphisms $\mathscr{B}(x'_\alpha) \xrightarrow{i} \mathscr{B}(x_\alpha)$ are well defined up to homotopy by the condition that $f \circ i$ is homotopic to f'.

Proof. — The construction of $\mathscr{B}(x_\alpha)$ and the map i are the same as the nilpotent case using the propositions of § 3. A useful device is to form the direct sum of all the coefficients in Σ and work with this one (hom) cohomology theory.

That i is an isomorphism of differential algebras is proposition (4.2).

Definition (**5.1**). — We call $\mathscr{B}(x_\alpha)$ the *model* of $\mathscr{A}$ over $\mathscr{B}$ relative to the coefficients Σ.

Example 1 (semi-simple case). — If $\mathscr{B}$ is a dual Lie algebra (generated in degree 1 and free of relations) which is finite dimensional and semi-simple (namely $\mathscr{B}_1 \xrightarrow{d} \wedge^2\mathscr{B}_1$ is injective) then $H^1(\mathscr{B}) = H^2\mathscr{B} = 0$ for all finite dimensional coefficients Σ [Wh$_1$]. So any $\mathscr{B} \to \mathscr{A}$ can be extended to a model $\mathscr{B}(x_\alpha) \to \mathscr{A}$ for any collection of finite dimensional coefficients Σ in $\mathscr{B}$.

Example 2 (solvable case). — Suppose $\mathscr{B}$ is just the exterior algebra on one generator θ in degree one and $\mathscr{B} \to \mathscr{A}$. Suppose θ is not exact and let Σ be the set of non-zero

(9) With twisting coefficients taken from Σ.

elements of the ground field so that $\theta \neq df - \sigma\theta f$ for $\sigma \in \Sigma$ and $f \in \mathscr{A}_0$. Writing $g = 1 + \sigma f$ this is equivalent to $\sigma\theta \neq dg/g$, g in $\mathscr{A}_0$.

Then for $\{\sigma\theta\}$ coefficients $\mathscr{B} \to \mathscr{A}$ is injective on H^1 and we can form a (solvable) model $\mathscr{B}(x_\alpha) \to \mathscr{A}$ inducing an isomorphism for these coefficients.

Example 3 (spaces). — We can apply examples 1 and 2 to the de Rham complexes $\mathscr{E}$ of spaces. We need the local systems embodied in $\mathscr{B} \to \mathscr{E}$ and model-building begins. The non-nilpotent examples require real coefficients, and C^∞ forms.

6. The automorphism group of a differential algebra.

One knows those self-mappings of a chain complex homotopic to the identity are just those of the form $1 + di + id$ where i is any degree -1 mapping of the complex. We show here that those self-mappings of a finite type nilpotent differential algebra $\mathscr{A}$ homotopic to the identity are just those of the form $1 + (di + id) + \frac{1}{2}(di + id)^2 + \ldots$, namely, $\exp(di + id)$ where i is any degree -1 *derivation* of the algebra. This is the basic result, proposition (6.4). Now we discuss some consequences.

We term these $\exp(di + id)$ automorphisms (see Remarks) *inner* automorphisms and the homotopy classes of automorphisms *outer* automorphisms, because of the basic result. We refer to the group of automorphisms acting on the cohomology, the dual homotopy (the spaces of indecomposables) (§ 3) and the spherical cohomology (cocycles mod. decomposables) as the *homology automorphisms*, the *homotopy automorphisms*, and the *spherical homology automorphisms*.

Say that an automorphism σ of a graded vector space is *unipotent* if $\sigma - I$ is nilpotent in each degree, $(\sigma - I)^N = 0$ for some N. A group of automorphisms is *unipotent* if each of its elements is unipotent. (On a finite dimensional vector space a (conjugate of a) unipotent subgroup is contained in upper triangular matrices with one's on the diagonal [BH].)

The *reductive* part of a group of automorphisms is by definition the quotient by the maximal normal unipotent subgroup. (See Appendix on algebraic groups.)

Theorem **(6.1)**. — a) *The symmetry groups associated to a nilpotent differential algebra,* $\mathscr{A}$:

(i) *all automorphisms,*
(ii) *outer automorphisms (homotopy classes),*
(iii) *homology automorphisms,*
(iv) *homotopy automorphisms,*
(v) *spherical homology automorphisms,*

differ from one another only by normal unipotent subgroups.

b) *If the differential algebra* $\mathscr{A}$ *is finitely generated, then each group above is naturally an algebraic matrix group and each has the same reductive part. Therefore, each has the form* $G(\mathscr{A}) \underset{\sim}{\times} U'$

where $G(\mathscr{A})$ *is reductive and the (normal) unipotent factor* U' *varies as we go down the list* [10]. *Case* (ii) *is the non obvious case.*

c) *Ignoring products with multiplicative groups, every connected algebraic group occurs as* p *varies in the example* $\mathscr{A}=\wedge(x_1, \ldots, x_n; dy=p(x_1, \ldots, x_n))$ *where the* x_i *are closed, have degree* 2 *and* $p(x_1, \ldots, x_n)$ *is a homogeneous polynomial of some (large) degree. Note the groups* (i), (ii), (iii), (iv), *and* (v) *are equal in this case.*

In the appendix we discuss some facts about algebraic groups relative to our theory and applications to topology.

Now let G be any group of outer automorphisms (perhaps coming from topology —like the fundamental group acting on the universal cover of a compact manifold). Let Π, H and (ΠH) denote the irreducible pieces (irreducible subquotients) of the representation of G on homotopy, homology, and spherical homology, respectively.

Corollary **(6.2)**. — *For the abstract group* G *of outer automorphisms of the nilpotent d.g.a.* $\mathscr{A}$, *the representations* Π *and* H *are constructed by finitely many algebraic operations (tensor product, subspace, etc.) from the common representation* (ΠH) *on spherical homology—the specific algorithm being provided by* $\mathscr{A}$.

The proof of the corollary can be safely left to the reader of the ensuing propositions (especially (6.4)).

Remark. — These results are true when $\mathscr{A}$ is obtained by finitely many linear extensions from $\mathscr{B}$ which contains the twisting coefficients, if we consider only automorphisms which fix $\mathscr{B}$. In the propositions below we assume the d.g.a. is minimal nilpotent.

Remark. — Note that in the case of a not necessarily nilpotent dual Lie algebra, i may be regarded as an element in the Lie algebra, $di+id$ reduces to id which is $\mathrm{ad}(i)$, and $\exp(di+id)$ is induced by conjugation in the Lie group. Also $di+id$ is nilpotent for all i iff the Lie algebra is nilpotent (Engel's Theorem) and $\exp(di+id)$ makes sense algebraically (without consideration of topologies) only in this case. Since inner automorphisms are very much (cohomologically) the identity, the first result rings true even in the non nilpotent cases where it is unformulated.

Remark. — In case $\mathscr{A}$ consists of forms on a manifold, think of i as given by a vector field; $di+id$ is then the Lie derivative on forms, and $\exp(di+id)$ generates the induced flow on forms. This provides the geometric motivation for algebraic homotopy. In the propositions below, however, we assume that $\mathscr{A}$ is a nilpotent minimal d.g.a.

Proposition **(6.1)**. — *The inner derivations, those of the form* $di+id$ *where* i *is a derivation of degree* -1, *commute with* d *and form a Lie algebra under bracket* $[X, Y]=X\circ Y-Y\circ X$.

---

[10] $G \underset{\sim}{\times} H$ means the semi-direct product of the two groups G and H; H is normal and G acts on H.

Proof. — One calculates $[d, dj+jd]=0$ since $d^2=0$. Then:

$$[id+di, jd+dj]=d[i, jd+dj]+[i, jd+dj]d$$

using $[d, dj+jd]=0$.

Proposition **(6.2)**. — *The exponentials of inner derivations (called inner automorphisms) make sense algebraically and form a subgroup of automorphisms of $\mathscr{A}$ commuting with d.*

Proof. — In a nilpotent minimal algebra, d increases the monomial weight and i preserves it. Thus $di+id$ is nilpotent in each degree and:

$$\exp(di+id)=1+(di+id)+\tfrac{1}{2}(di+id)^2+\ldots$$

makes sense algebraically. It is not hard to see we have algebra automorphisms commuting with d. Composition is the classical Baker-Campbell-Hausdorff formula and Proposition (6.1):

$$\exp \mathrm{X} \circ \exp \mathrm{Y}=\exp(\mathrm{X}+\mathrm{Y}-\tfrac{1}{2}[\mathrm{X}, \mathrm{Y}]+\tfrac{1}{12}[\mathrm{X}[\mathrm{X}, \mathrm{Y}]]+\ldots).$$

Proposition **(6.3)**. — $\exp(di+id)$ *is homotopic to the identity with a canonical homotopy provided by i.*

Proof. — Write $\mathscr{A}=\mathscr{B}(x)$, $\mathscr{A}^{\mathrm{I}}=\mathscr{B}(x, u, du)$ (see § 3) and let ∂ be the derivation of $\mathscr{A}^{\mathrm{I}}$ defined by $\partial(x)=u$, $\partial(u)=i.i(x)$ and $\partial(du)=idi(x)$. Calculating modulo decomposables $(d\partial+\partial d)x\equiv du$, $(d\partial+\partial d)\, du\equiv 0$, $(d\partial+\partial d)\, u\equiv 0$.

Thus $e=\exp(d\partial+\partial d)$ is defined, and $\mathscr{A}^{\mathrm{I}}\simeq(\mathscr{B}(x)\otimes e\mathscr{B}(x))(u)$. So if $y=ex$ and $\mathscr{A}^{\mathrm{I}}\xrightarrow{\pi}\mathscr{A}$ is defined by $x\mapsto x$, $u\mapsto ix$, then $i.\pi=\pi.\partial$, $y\mapsto\exp(di+id)x$ and π is a homotopy of $\exp(di+id)$ to the identity.

The case $i=0$ provides $\mathscr{A}^{\mathrm{I}}$ with a canonical differential, as promised after Proposition (3.2).

Proposition **(6.4)**. — *An automorphism of a nilpotent minimal algebra is unipotent if and only if the corresponding automorphism of spherical homology is unipotent.*

Proof. — By induction, σ(new generator x) is defined modulo cocycles by $\sigma(dx)$ which is already defined.

Remark. — A unipotent automorphism σ ($\sigma=\mathrm{I}+\mathrm{X}$, X nilpotent) can be written uniquely as exp D where D is a derivation of degree zero commuting with d. Namely, use $\mathrm{D}=\log\sigma=\log(\mathrm{I}+\mathrm{X})=\mathrm{X}-\tfrac{1}{2}\mathrm{X}^2+\ldots$

Proposition **(6.5)**. — *If an automorphism σ is homotopic to the identity, we can write* $\sigma=\exp(di+id)$.

Proof. — The homotopy to the identity H provides us with the preliminary information that σ is the identity on cohomology and so is unipotent by Proposition (6.4). Thus we can write $\sigma=\exp \mathrm{D}$ and our problem is to write $\mathrm{D}=di+id$.

Solving for i inductively meets cohomology obstructions like those for constructing a homotopy. (For a new generator we have to find ix in $d(ix) = \mathrm{D}x - idx$. Since $d(\mathrm{D}x - idx) = d\mathrm{D}x - didx = \mathrm{D}(dx) - di(dx) = id(dx) = 0$ we have a cohomology obstruction.)

To insure these inductive obstructions vanish we combine the given homotopy H of (identity, σ) with the homotopy $\pi(i)$ coming from i (Proposition (6.3)) via the transitivity argument (Corollary (3.4)) to obtain $\mathrm{H} + \pi(i)$ a homotopy of the identity. Inductively we solve for i so that this self-homotopy of the identity is homotopically trivial (see Remark after Corollary (3.6)). This is possible because the cohomological ambiguity in solving $di + id = \mathrm{D}$ is the same as that for constructing a homotopy (Proposition (3.3)). Since H exists, the inductive cohomological obstructions to constructing i subject to « $\mathrm{H} + \pi(i)$ is homotopic to the trivial self homotopy » will vanish.

Proof of Theorem (6.1). — The first assertion follows from proposition (6.4).

For the second part use the Levi-decomposition (see the Appendix) for the group of all automorphisms $\mathrm{Aut}(\mathscr{A}) = \mathrm{G}(\mathscr{A}) \underset{\sim}{\times} \mathrm{U}(\mathscr{A})$ where $\mathrm{U}(\mathscr{A})$ is the maximal normal unipotent subgroup of $\mathrm{Aut}(\mathscr{A})$, $\mathrm{G}(\mathscr{A}) = \mathrm{Aut}(\mathscr{A})/\mathrm{U}(\mathscr{A})$ and $\underset{\sim}{\times}$ denotes semi-direct product. Clearly Aut $\mathscr{A}$ is an algebraic group of matrices and [B-HC] applies. Then since the surjections (i) $\to$ ((ii), (iii), (iv), or (v)) have *full* unipotent kernels we can easily form the quotients $\mathrm{G}(a) \underset{\sim}{\times} \mathrm{U}/\mathrm{U}'$ by looking at the action of $\mathrm{G}(a)$ on the nilpotent Lie algebra of U. (The word full means exp of a Lie subalgebra. This is obvious for (i) $\to$ ((iii), (iv), or (v)) and is a consequence of Propositions (6.3) and (6.5) for (i) $\to$ (ii).)

We leave the third part and Corollary (6.2) for the reader.

Appendix (Algebraic groups and arithmetic groups)

The labeled statements below are used in what follows. The others yield corollaries about the structure of automorphism groups of spaces and manifolds (after sections 10 and 13 are completed) which we don't reiterate.

A useful reference for most of the remarks below is Borel-Harish-Chandra [B-HC].

An *algebraic group* is a group in the context of algebraic varieties. Each one is an extension of an Abelian variety which is a projective variety by an *algebraic matrix group* which is an affine variety. Only the latter concern us and we define them now.

Let k be a subfield of $\mathbf{C}$ (for us usually $k = \mathbf{Q}, \mathbf{R}$, or $\mathbf{C}$) and let V be a finite dimensional vector space over k. GL(V) denotes the linear automorphisms of V and one refers to the group of matrices with entries in any field K containing k as the K-points of GL(V).

An *algebraic matrix group* G *over* k is determined by any subgroup of the $\mathbf{C}$-points of GL(V) defined by polynomial equations in the entries where the coefficients of the polynomials lie in k. We can speak of the K-points of G for any field K between k and $\mathbf{C}$.

A *unipotent group* U is an algebraic matrix group which for some representation above lies in the subgroup of complex matrices with ones on the diagonal and zeroes below (in some basis). This condition is independent of the representation. Moreover, there is essentially one $\mathbf{Q}$-algebraic group structure on the underlying $\mathbf{Q}$-points of a $\mathbf{Q}$-unipotent group.

Every algebraic matrix group G has a unique maximal normal unipotent subgroup U which is also defined over k.

The quotient G/U has the structure (over k) of an algebraic matrix group G_r called the *reductive group* associated to G.

There is a semi-direct product isomorphism $G \simeq G_r \ltimes U$ of algebraic groups called the Levi decomposition. Two Levi decompositions are related by inner automorphisms from U. All this is defined over k.

Unipotent groups have (nilpotent) Lie algebras from which they are constructed algebraically by the Campbell-Hausdorff formula—which has only finitely many terms in this case.

Thus if $U' \subset U$ is normal and invariant under the action of the reductive group G_r the semi-direct product $G_r \ltimes U/U'$ is also an algebraic group (even an algebraic matrix group).

Suppose $G \xrightarrow{\rho} G'$ is a homomorphism of algebraic groups defined over $\mathbf{Q}$ with unipotent kernel. Then $\mathbf{Q}$-points of G' which come from $\mathbf{C}$-points of G already come from $\mathbf{Q}$-points of G.

Here one speaks of a " principal homogeneous space " $\rho^{-1}(x)$ of the unipotent group $\rho^{-1}(e)$ where x is a rational point at G'. If ρ sends a $\mathbf{C}$-point of G to x then $\rho^{-1}(x)$ is an algebraic variety defined over $\mathbf{Q}$ which is one free orbit of the unipotent group $\rho^{-1}(e)$ (in the sense that this is true for the $\mathbf{C}$-points). Then the theorem is

(A.I) " A principal homogeneous space of a unipotent group (all) defined over $\mathbf{Q}$ has a rational point " [Se 2].

Arithmetic groups are defined in $\mathbf{Q}$-algebraic matrix groups G by choosing a lattice (a finitely generated subgroup of biggest rank) in the rational vector space V. Then let $G_{\mathbf{Z}}$ denote the automorphisms of G which yield isomorphisms of the lattice (not only preserve it).

A different choice of V or lattice in V leads to a commensurable subgroup $G'_{\mathbf{Z}}$. Namely, $G_{\mathbf{Z}} \cap G'_{\mathbf{Z}}$ is a subgroup of finite index in each.

Any subgroup of the $\mathbf{Q}$-algebraic matrix group G commensurable to a $G_{\mathbf{Z}}$ is called an *arithmetic group*.

Any finitely generated torsion free nilpotent group is an arithmetic group in a unipotent group over $\mathbf{Q}$ (and this structure is essentially unique).

The Levi decomposition induces on some subgroup of finite index Γ in an arithmetic group a semi-direct product decomposition $\Gamma \simeq \Gamma_r \ltimes N$ where N is a finitely generated nilpotent group and Γ_r is arithmetic for the reductive part.

Arithmetic groups are finitely presented. Any isotropy subgroup of a vector in an algebraic representation is also arithmetic and thus also finitely presented. (Compare the free group example (§ 10).)

(A.2) " If $G \to G'$ is a homomorphism of $\mathbf{Q}$-algebraic matrix groups defined over $\mathbf{Q}$ then the image of any arithmetic group in G is commensurable to the image of the $\mathbf{Q}$-points of G intersected with any arithmetic subgroup of G′ [B-HC]. "

7. Differential forms and deRham's theorem on general spaces.

There are several variants of what we now consider. To fix ideas we may think of forms ([11]) on simplices and compatible collections on these simplicial complexes. (Example (i) below.) Thus in two triangles with a common edge a one form is a pair of one forms (one on each triangle) agreeing on vectors in the common edge. (See Figure 7.1.) Any such partially continuous one form can be integrated along any reasonable path on this complex.

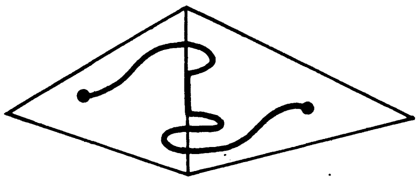

FIG. 7.1

More generally we can integrate $n \times n$ matrices of 1-forms to move $\mathbf{R}^n$ along paths and construct local systems (Theorem (1.2)) and integrate single k-forms over k-chains to obtain k-cochains. This integration is the reason that the construction is valid topologically. The advantage for topology comes from the good algebraic properties of infinitesimal calculation—notably the (graded) commutative multiplication of forms.

We will define the general situation in familiar but (as yet) undefined terms (the italicized words). We treat *spaces* made up inductively of *cells* of increasing dimension. Cells have *boundaries* of lower complexity which are attached by admissible maps to the inductive space.

We must have the notion of *form on a cell.* We build up the notion of *form on a space* inductively by extending over a new cell an inductive form pulled back to the boundary of that new cell. For simplicity let us further assume we have an " integration map " to ordinary cochains on the space giving a chain map and an isomorphism between form cohomology and ordinary cohomology for cells. Only one more property is needed for a deRham theorem.

([11]) This idea is due to Whitney (1956) and Thom (1959).

This is the extension property — any form in the boundary of a cell extends over the entire cell. Then we can state and prove the " cellular deRham theorem ".

Theorem (**7.1**). — *For any such notion of cell, space, form, and integration map, form cohomology is isomorphic to ordinary cohomology — the isomorphism being given by the integration map.*

Proof. — We are tacitly assuming ordinary cochains satisfy the extension property. Then for both forms and cochains we have short exact sequences (and natural maps between them defined by integration) corresponding to $X_{k-1} \to X_k \to (X_k, X_{k-1})$, where X_k is the union of all cells up to k. The calculation of (X_k, X_{k-1}), namely, forms or cochains on X_k which vanish on X_{k-1}, breaks into a product over the k-cells (whose interiors are disjoint). Then we have the sequences corresponding to:

$$\text{boundary} \to \text{cell} \to (\text{cell, boundary})$$

and now the middle is good by hypothesis while the boundary term is good by induction. We apply the 5-lemma to the long sequence of cohomology, take a direct product, put this in the first sequences on the right, apply the 5-lemma again, and we're done.

Remark. — Since this discussion is equivariant with deck transformations we also have implicitly obtained twisted cohomology in terms of forms on covering spaces. (Compare Theorem (1.2).)

Example (simplicial complexes).

Take simplices and simplicial complexes (of arbitrary cardinality) for the notions of *cell* and *space*. A *form* on a cell is a form on a neighborhood of the simplex in the affine space it generates. Compatibility is defined by restriction to the affine spaces of various faces.

(i) (polynomial forms). — One example is to take forms on a k-simplex which are polynomials in the affine coordinates $x_1, \ldots, x_k$ times the various constant forms $dx_{i_1} \wedge \ldots \wedge dx_{i_r}$. We let the coefficients of the polynomials lie in any field containing the rationals, $\mathbf{Q}$. Integration over simplices is a rational operation defining simplicial cochains with values in the same field.

The extension property is a non-trivial fact [12]. The idea is to fix attention on one face of the boundary and let π be the (singular) map which projects from the opposite vertex. Let x be the linear coordinate which is 1 on the face and zero on the vertex, and ω the form to be extended from the boundary. By subtracting off $x^N \pi^* \omega$ (which is a polynomial form for large N since ω is) we obtain a new extension problem which is zero on that face. We move to an adjacent face and repeat the process. Since π^* preserves zero we are eventually reduced by this process to a trivial extension problem.

The acyclicity of the simplex can be done by pure algebra — for the forms on a

(12) An elegant proof was given by a talented young mathematician in an M.I.T. seminar of 1972. He was Bruce Renshaw (1946-1974) whose primary interests were number theory, geometry, and music.

k-simplex are just (the contractible algebra) polynomials in $x_1, \ldots, x_k$ (in degree zero) tensor the exterior algebra on $dx_1, \ldots, dx_k$ (in degree one).

(ii) (C^∞ forms). — Another example (needed over the reals for non-nilpotent considerations and for relating to classical deRham) is to take smooth forms (of class C^∞) on neighborhoods of a simplex in its own affine space. The extension goes through as in example (i) although x^N is replaced by a C^∞ function which is one *near* the face and zero *near* the vertex (Griffiths). The acyclicity of the forms on a simplex is the classical Poincaré lemma ([13]).

(iii) (polyhedral forms). — We can take forms on a simplex to be the direct limit (under rectilinear subdivision) of either of the previous examples. The extension lemma is proved by coning the subdivision on the boundary into an interior point. In the polynomial case we can let the subdivision (for example, rational subdivision) respect the field and continue to work over that field. The remainder of the verification is formal.

(iv) (flat forms). — These comprise a maximal geometric class of forms relevant for topology. This discussion is the body of Whitney's book " Geometric Integration ".

Example (semisimplicial complexes).

We may repeat the discussion of example (i) for semisimplicial sets. We take collections of forms on the (abstract simplices) compatible with the face and degeneracy structure. The role of cell is played by *non-degenerate* simplices, and we can interpret this discussion literally on the Milnor geometric realization $[M_1]$ which has one cell for each non-degenerate simplex.

The discussion can be enhanced with notions of continuity or measurability on the collection of k-simplices and the corresponding collections of forms on k-simplices. The basic operations, for example extension above, preserve this structure.

Example (stratified spaces).

In the above cell-space abstraction we didn't require that cells be contractible. Thus these notions can be extended to stratified sets — thought of inductively as obtained by attaching manifolds with boundary with a careful statement about the geometry of the attaching map.

It would be interesting to carry this out in detail — the basic idea being that a form should have values only on multivectors tangent to the strata.

Remark. — The local nature of the multiplication of forms means that it agrees (cohomologically) with any other locally defined notion of cup product (or Massey

([13]) According to deRham — this lemma is due to Elie Cartan who referred to it as the " converse of Poincaré's lemma " that $d \circ d = 0$.

product) for cochains. (See Whitney's book, p. 286, Theorem 18A—although we don't use this identification but rather define cup product by wedge products of forms.)

Now we illustrate the advantages of commutative multiplication in a fibration formula. This is the (twisted) $K(\pi, n)$-analogue of the Chevalley-Hirsch-Koszul formula for principal Lie group bundles which was current in 1950 and ignored later in topology (see [B] and [C]). The evident power and simplicity of the CHK formula helped prompt me to the present theory after Armand Borel kindly explained it to me in 1970.

We explain our hypothesis in concrete (but lengthy) terms which makes the proof self-evident and obviates [14] the need to choose a technical definition of topological fibration. We concentrate on the twisted case over $\mathbf{R}$. The untwisted case over $\mathbf{Q}$ is also valid and simpler.

Assume we have two spaces E and B, for which forms are defined, and an admissible map $E \xrightarrow{\pi} B$ satisfying a certain cohomological fibration property. Since we want to treat the twisted case, assume we have an $n \times n$ matrix of one-forms Θ on B satisfying the integrability condition $d\Theta - \Theta \circ \Theta = 0$. We then assume we have a $d_\Theta = (d - \Theta)$-cocycle in the forms on B, $c = (c_1, \ldots, c_n)$ so that $\pi^* c = d_\Theta \xi$ can be solved in E. We further assume that the solution $\xi = (\xi_1, \ldots, \xi_n)$ has been chosen so that over each cell σ of B the multiplication map (with π^*):

$$\mathscr{B}(\sigma, \partial\sigma)(\xi_1, \ldots, \xi_n) \xrightarrow{m} \mathscr{E}(\sigma, \partial\sigma), \qquad \sigma . \xi \mapsto \pi^* \sigma . \xi,$$

induces an isomorphism on cohomology. Here $\mathscr{B}(\sigma, \partial\sigma)$ and $\mathscr{E}(\sigma, \partial\sigma)$ denote respectively the forms on σ vanishing on $\partial\sigma$ and the forms on $\pi^{-1}\sigma$ vanishing on $\pi^{-1}\partial\sigma$. The differential on the left is defined by:

$$d(b . \xi) = db . \xi + \bar{b} . \Theta_\sigma(\xi) + \bar{b} . c_\sigma$$

where b belongs to $\mathscr{B}(\sigma, \partial\sigma)$, c_σ (resp. Θ_σ) denotes c (resp. Θ) restricted to σ, and $\mathscr{B}(\sigma, \partial\sigma)(\xi_1, \ldots, \xi_n)$ denotes $\mathscr{B}(\sigma, \partial\sigma)$ (the ring without unit) tensor the free commutative algebra (with unit) on $\xi_1, \ldots, \xi_n$. Note that $c_\sigma = d_\Theta \eta$ on σ so in the new basis $(\xi - \eta) = (\xi_1 - \eta_1, \ldots) = (\xi_1', \ldots)$ we have $d(b . \xi') = db . \xi' - \Theta(b\xi')$. So our condition is that $\mathscr{E}(\sigma, \partial\sigma)$ (in a coherent way) looks homologically like $(\text{fibre}) \times (\sigma, \partial\sigma)$ where the fibre has cohomology the free commutative algebra on $\xi_1, \ldots, \xi_n$. (Note the twisted cohomology of $(\sigma, \partial\sigma)$ agrees with the untwisted, see Theorem (1.2).)

Then let $\mathscr{B}$ denote either the forms on B or any d.g.a. mapping to the forms on B, so Θ comes from $\mathscr{B}$ and the map is an isomorphism for all finite dimensional coefficients defined in $\mathscr{B}$.

Theorem **(7.2)**. — *The natural map* $\mathscr{B}(\xi_1, \ldots, \xi_n) \xrightarrow{m}$ forms on E *defined by multiplication induces an isomorphism on cohomology for all finite dimensional coefficients induced by twisting matrices in* $\mathscr{B}$. *The* d *on the left is defined as usual by* $d\xi = \Theta\xi + c$.

[14] For our track.

Proof. — First let $\mathscr{B}$ be the forms on B and go by direct induction over the cells or skeleta. Then replace the forms by any other $\mathscr{B}$ using Proposition (4.3).

Example. — Suppose E is a fibration over B with fibre $K(\pi, n)$. Suppose $\pi \otimes \mathbf{R} = V$ is a finite dimensional real vector space and the representation of $\pi_1 B$ on V is *infinitesimally given* by a twisting matrix Θ of one forms on B. (This is often true, see Remark after Theorem (1.2).) Then the primary obstruction to a cross section gives a cohomology class represented (tensor $\mathbf{R}$) by a d_Θ-closed form c so that the above holds. Thus the cohomological formulae for E in Theorem (7.2) can be written. This is an exemple of the generalized CHK formula.

We conclude with some geometric remarks about the simplicial or polyhedral forms.

(i) There is a natural chain map inverse to integration due to Whitney. Essentially a vertex becomes the name of the barycentric coordinate on its star and we extend inductively using d. An explicit formula is the following — the basic cochain $\langle x_0, \ldots . x_n \rangle$ goes to $\sum_i (-1)^i x_i dx_0 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n$. For the geometric interpretation see [Su$_1$] or [W].

Two applications are: Since the cocycles are carried precisely onto the locally constant forms we have on the simplicial *cocycles* a canonical *commutative associative ring* structure. Since a form can be integrated over any tiny chain we have a *canonical extension* of cochains on one triangulation to all finer subdivisions (commuting with d and only defined over $\mathbf{Q}$ or $\mathbf{R}$).

(ii) It seems the polynomial forms make up the smallest natural, commutative, associative d.g.a. containing the simplicial cochains and giving the correct homology. One can argue on the unit interval that the variable t satisfying no polynomial relation must be there in degree zero to make H^0 correct. In particular, there is no commutative associative multiplication on the cochains themselves or on any other finite dimensional model (compare (i) above).

(iii) The polyhedral forms are exactly multiplicative for compact polyhedra $\mathscr{E}_{K \times L} = \mathscr{E}_K \otimes \mathscr{E}_L$, as one readily sees by considering the product of two affine spaces.

8. The spatial realization of a differential algebra, its homotopy and cohomology.

If $\mathscr{A}$ is a differential algebra, define an $\mathscr{A}$-differential system on a space X to be a d.g.a. map of $\mathscr{A}$ into the forms on X (§ 7). We can define a space $\langle \mathscr{A} \rangle$ carrying the universal $\mathscr{A}$-differential system. $\langle \mathscr{A} \rangle$ is defined by the sets of all $\mathscr{A}$-differential systems on standard simplices $\Delta^0, \Delta^1, \ldots$ which form a semi-simplicial set with face operators, degeneracy operators, and a topology induced by one on differential forms. Note we have various definitions of $\langle \mathscr{A} \rangle$ depending on type of forms used on standard simplices.

There is a natural mapping:

$$\mathscr{A} \xrightarrow{\pi} \text{forms on } \langle \mathscr{A} \rangle \qquad (\text{see § 7})$$

which we use to compute the cohomology of $\langle \mathscr{A} \rangle$ when $\mathscr{A}$ is a minimal d.g.a. (§ 2). For the nilpotent case we work over $\mathbf{Q}$ or $\mathbf{R}$ and define $\langle \mathscr{A} \rangle$ using polynomial forms on the standard simplices. In the non-nilpotent case we work over $\mathbf{R}$ and define $\langle \mathscr{A} \rangle$ with smooth forms (of class C^∞) on standard simplices.

The rough statement about $\langle \mathscr{A} \rangle$ that we want is that the homotopy corresponds to the generators of $\mathscr{A}$ and the cohomology is computed by $\mathscr{A}$ (via π).

We will first give the statement for $\mathscr{A}$ nilpotent and minimal over $\mathbf{Q}$ of finite type.

Theorem **(8.1)**. — (i) *$\mathscr{A} \xrightarrow{\pi} \mathbf{Q}$-polynomial forms on $\langle \mathscr{A} \rangle$ induces an isomorphism of cohomology over $\mathbf{Q}$.*

(ii) *The fundamental group of $\langle \mathscr{A} \rangle$ is the $\mathbf{Q}$-nilpotent group defined by the $\mathbf{Q}$-Lie algebra dual to $\mathscr{A}^1$.*

(iii) *The higher homotopy groups of $\langle \mathscr{A} \rangle$ are naturally the dual spaces of the indecomposable spaces of $\mathscr{A}$ in degrees > 1.*

Proof of theorem (8.1). — First the homotopy of $\langle \mathscr{A} \rangle$.

First case. — Let $\mathscr{A}^1$ be a nilpotent minimal algebra generated in degree one. Then $\mathscr{A}^1$ determines a dual Lie algebra (over $\mathbf{Q}$) which determines a nilpotent Lie group by the Campbell-Hausdorff formula. The $\mathbf{Q}$-points in the group and the Lie algebra correspond bijectively under the log and exponential maps (which are polynomial). Integration of a $\mathbf{Q}$-polynomial form $\mathscr{A}^1$-simplex (defining a simplex of $\langle \mathscr{A}^1 \rangle$) determines a $\mathbf{Q}$-polynomial map of a simplex into the Lie group. In fact $\langle \mathscr{A}^1 \rangle$ is just the $\mathbf{Q}$-polynomial singular complex of the group modulo the action of left translation. Since the $\mathbf{Q}$-polynomial singular complex is the same as that of the $\mathbf{Q}$-vector space of the Lie algebra (using log) and the latter is clearly contractible it follows that $\pi_1 \langle \mathscr{A}^1 \rangle$ is as described in (i) of Theorem (8.1).

Second case. — Let $\mathscr{A}$ be a general minimal nilpotent differential algebra. Then using the fact that the higher cohomology of the circle vanishes we obtain $\pi_1 \langle \mathscr{A} \rangle = \pi_1 \langle \mathscr{A}^1 \rangle$. Similarly, for higher spheres we can reduce the question of maps $S^k \to \langle \mathscr{A} \rangle$ to ordinary cohomology and the k-dimensional generators of $\mathscr{A}$ using remark (i) below.

This completes the homotopy computation — which is rather close to the definition of $\langle \mathscr{A} \rangle$. We note one further case, namely, $\langle \mathscr{A} \rangle$ is contractible if $\mathscr{A}$ is contractible (§ 2). This follows by the same argument.

Now we turn to cohomology which is essentially more non-trivial making use as it does of the commutativity of differential form multiplication.

Suppose (in general) that $\mathscr{A}$ is obtained by a linear extension of $\mathscr{B}$:

$$\mathscr{A} = \mathscr{B}(x_1, \ldots, x_n)$$

where $(x_1, \ldots, x_n)$ is a basis of V in degree k. Then we claim the natural map $\langle\mathscr{A}\rangle \xrightarrow{\pi} \langle\mathscr{B}\rangle$ is in a very precise sense a fibration with fibre $\langle V, k\rangle$.

Actually, we say the following: if Δ is a non-degenerate simplex of $\mathscr{B}$, (Δ) denotes the semisimplicial set generated by the standard simplex of dimension dim (Δ), $(\partial\Delta)$ is the semisimplicial subset carried by the boundary, then the quotient of the pair $(\pi^{-1}(\Delta), \pi^{-1}(\partial\Delta))$ is *isomorphic* as semisimplicial set to the quotient of the pair $\langle V, k\rangle \times ((\Delta), (\partial\Delta))$ [15].

This follows by looking at the structure equation of the extension, $dx = b$, see § 2. An $\mathscr{A}$-simplex is a $\mathscr{B}$-simplex Δ together with a solution of the structure equation on Δ. Two solutions differ by a solution of $dy=0$. If we add the important remark that the degeneracies of a non-degenerate simplex in $\mathscr{B}$ are all *distinct* our isomorphism is established. In this sense $\langle\mathscr{A}\rangle \to \langle\mathscr{B}\rangle$ is a fibration with fibre $\langle V, k\rangle$. Now when one computes cohomology of a semisimplicial set (as it is or realized) one can ignore the subcomplexes generated by the degenerate simplices (in the Milnor realization [M_1] for example, there is one cell for each non-degenerate simplex).

So we filter $\langle\mathscr{B}\rangle$ by the skeleton

$$\langle\mathscr{B}\rangle_k = \{\text{non-degenerate simplices of dimension} \leq k \text{ plus all their degeneracies}\}.$$

Similarly we filter $\langle\mathscr{A}\rangle$ by $\langle\mathscr{A}\rangle_k = \pi^{-1}\langle\mathscr{B}\rangle_k$. Then we study the map:

$$\text{forms on } \langle\mathscr{B}\rangle(x_1, \ldots, x_n) \xrightarrow{f} \text{forms on } \langle\mathscr{A}\rangle$$

by induction over k using the five lemma. We obtain (assuming for the moment the computation is valid for $\langle V, k\rangle$):

Proposition **(8.2)**. — *f is an isomorphism of rational cohomology.*
Compare Theorem (7.2).

Note Proposition (8.2) is the analogue of Proposition (4.1).
The next step is to use Proposition (4.3) to replace forms on $\langle\mathscr{B}\rangle$ by $\mathscr{B}$, namely:

Proposition **(8.3)**. — *$\mathscr{B}(x_1, \ldots, x_n) \to$ forms on $\langle\mathscr{B}\rangle(x_1, \ldots, x_n)$ is an isomorphism on rational cohomology if $\mathscr{B} \to$ forms on $\langle\mathscr{B}\rangle$ is.*

The argument will be completed in the nilpotent case over **Q** by using Proposition (8.2) and (8.3) over and over. This uses the knowledge of rational cohomology of the Eilenberg-Maclane spaces $\langle V, k\rangle$, which is well known since Serre—being one of the main applications of the spectral sequence of a fibration.

We have in hand a direct argument. Namely, the description of $\langle V, 1\rangle$ above, knowledge of the circle, direct products, and direct limits takes care of $\langle V, 1\rangle$ over **Q**.

Considering the contractible algebra $\mathscr{A}$ so that $\langle\mathscr{A}\rangle$ fibres over $\langle V, k+1\rangle$ with

[15] This is the key step in the proof.

fibre $\langle V, k\rangle$, the remark above that $\langle\mathscr{A}\rangle$ is also contractible, and finally Proposition (8.2) to put us in a position to use Proposition (4.4) to give a converse of Proposition (8.3) in this case, provides an induction $k \to k+1$. This computes $\langle V, k\rangle$ as desired.

Note that a key point in Proposition (8.2) is that forms on $\langle\mathscr{B}\rangle_k \to$ forms on $\langle\mathscr{B}\rangle_{k-1}$ (similarly for $\langle\mathscr{A}\rangle_k \to \langle\mathscr{A}\rangle_{k-1}$) is onto. This is the extension remark (§ 7).

Remarks. — (i) If we consider maps of $\mathscr{A}$ into (forms on X) $=\mathscr{E}$X, divided into deformation classes — a deformation being a map of $\mathscr{A}$ into the forms on $X \times I$ — then one can prove by basic subdivision and cellular approximation arguments that:

homotopy classes of maps $(X \to \langle\mathscr{A}\rangle) \sim$ deformation classes of maps $(\mathscr{A} \to \mathscr{E}X)$.

(Note there is a natural map from right to left.) This is one sense in which $\langle\mathscr{A}\rangle$ carries the universal $\mathscr{A}$-differential system.

(ii) In the infinite nilpotent case over $\mathbf{Q}$, topologies in another form must be remembered in the study of homologically infinite spaces. Then $\mathscr{A}$ will have an inverse limit topology and $\langle\mathscr{A}\rangle$ should be defined by continuous maps of $\mathscr{A}$ into forms. Then Theorem (8.1) is true using continuous duals.

(iii) Note that $\mathscr{A}^1 \to \pi_1\langle\mathscr{A}^1\rangle$ provides a direct construction or definition of the simply connected Lie group associated to the dual Lie algebra $\mathscr{A}^1$. Also since $\pi_1\langle\mathscr{A}^1\rangle$ always exists and is a group, we have a *group construction* for ANY (dual) Lie algebra generalizing the classical one. $\mathrm{Diff}^\infty(M)$ arises in this way — in the continuous version of $\langle\mathscr{A}^1\rangle$ where $\mathscr{A}^1$ is the (continuous) dual of the Lie algebra of C^∞ vector fields on M, assumed compact.

(iv) It is rare to have spaces where both the homotopy and the cohomology are known explicitly. Other cochain contexts lead to constructions of spaces like $\langle\mathscr{A}\rangle$ where the homotopy groups are known almost from the definition, but the cohomology is elusive. We only succeed here because of the commutative product on the level of forms (essentially Theorem (7.2)).

(v) We have analyzed in the course of the proof the characteristic zero cohomology of Eilenberg-Maclane spaces — a key particular case of the statement.

(vi) Extending the remark (v) further, one sees in the construction $\langle\mathscr{A}\rangle$ natural " Postnikov systems " over $\mathbf{Q}$ or $\mathbf{R}$ for spaces — defined canonically once algebraic models $\mathscr{A}$ are built from the deRham complex.

Now we give a statement about $\langle\mathscr{A}\rangle$ in the non-nilpotent case over $\mathbf{R}$ (which we don't use in the sequel and thus we only sketch the proof). For simplicity assume $\mathscr{A}$ has finite type over $\mathbf{R}$ and let $G_{\mathscr{A}}$ denote the real simply connected Lie group determined by the Lie algebra dual to $\mathscr{A}^1$.

" *Theorem* " **(8.1)′**. — (i) *The fundamental group of* $\langle\mathscr{A}^1\rangle$ *is the real Lie group* $G_{\mathscr{A}}$ (*whose topology is determined by the topology on the simplices of* $\langle\mathscr{A}^1\rangle$).

(ii) *The higher homotopy groups of* $\langle \mathscr{A}^1 \rangle$ *are isomorphic to the homotopy groups of* $G_{\mathscr{A}}$ *(which of course are finitely generated Abelian groups).*

(iii) *The map* $\mathscr{A}^1 \xrightarrow{\pi}$ *forms on* $\langle \mathscr{A}^1 \rangle$ *induces an isomorphism on the cohomology of* $\mathscr{A}^1$ *(which is by definition the Lie algebra cohomology of* $G_{\mathscr{A}}$*) and the continuous cohomology of* $\langle \mathscr{A}^1 \rangle$ *(for all finite coefficients coming from* $\mathscr{A}^1$*).*

(iv) *The natural map* $\langle \mathscr{A} \rangle \to \langle \mathscr{A}^1 \rangle$ *has a simply connected homotopy fibre* $\mathscr{F}$. *The homotopy groups of* $\mathscr{F}$ *are real vector spaces dual to the indecomposable spaces of* $\mathscr{A}$ *in degrees* >1.

(v) *The map* $\mathscr{A} \xrightarrow{\pi}$ *forms on* $\langle \mathscr{A} \rangle$ *induces an isomorphism between the cohomology of* $\mathscr{A}$ *and the continuous cohomology of* $\langle \mathscr{A} \rangle$ *(for all finite dimensional coefficients coming from* $\mathscr{A}^1$*).*

Sketch proof of theorem (8.1)′:

We begin with the remark that most of the theorem is harder to state than to prove. For by integration an $\mathscr{A}^1$-simplex is just a C^∞ map of the simplex into $G_{\mathscr{A}}$ defined up to left translation. Thus $\langle \mathscr{A}^1 \rangle$ is just the C^∞ singular complex of $G_{\mathscr{A}}$ mod the (discrete) action of $G_{\mathscr{A}}$ on the left. Thus (i) and (ii) are immediate. (iii) is a reformulation of the theorem of Van Est [VE]. (iv) follows as in the proof of (iii), Theorem (8.1) above. (v) follows from the inductive fibration argument used in the nilpotent case (Theorem (8.1)) based on continuous cohomology. We have not developed the details of continuous cohomology required to finish this outline so the proof of (v) should be regarded as incomplete. Note however that (v) is a plausible generalization of Van Est's theorem from Lie algebras to minimal d.g.a.'s.

9. Poincaré's analytical definition of the fundamental group.

In his first remarks on topology, the Comptes Rendus note of 1892 " Sur l'analysis situs ", Poincaré asks to what extent the Betti numbers determine a closed manifold up to continuous deformation. He then introduces the " fundamental group " by an analytic construction and constructs infinitely many 3-manifolds with distinct fundamental groups and equal Betti numbers.

" ... Soient maintenant $F_1, F_2, \ldots, F_p$ p fonctions quelconques... "

" Je ne suppose pas que les fonctions F soient uniformes, mais je suppose que si le point $(x_1, x_2, \ldots, x_{n+1})$ décrit sur la surface un contour fermé *infiniment petit*, chacune des fonctions F revient à sa valeur primitive. Cela posé, supposons que notre point décrive sur la surface un contour fermé *fini*, il pourra se faire que nos p fonctions ne reviennent pas à leurs valeurs initiales, mais deviennent :

$$F'_1, F'_2, \ldots, F'_p$$

ou, en d'autres termes, qu'elles subissent la substitution :

$$(F_1, F_2, \ldots, F_p; F'_1, F'_2, \ldots, F'_p)."$$

" Toutes les substitutions correspondant aux divers contours fermés que l'on peut tracer sur la surface forment un groupe qui est discontinu (au moins en ce qui concerne sa forme). "

" Ce groupe dépend évidemment du choix des fonctions F_j; supposons d'abord que ces fonctions soient les plus générales que l'on puisse imaginer en ne s'imposant pas d'autre condition que celle que nous avons énoncée plus haut; et soit G le groupe correspondant... "

" Le groupe G peut donc servir à définir la forme de la surface et s'appeler le groupe de la surface. " And then in " Analysis situs " (1895):

" Mais pour mieux fixer les idées et bien que cela n'ait rien d'essentiel, supposons que les fonctions F soient définies de la manière suivante. Elles devront satisfaire à des équations différentielles de la forme :

$$(1) \qquad dF_i = X_{i,1}dx_1 + X_{i,2}dx_2 + \ldots + X_{i,n}dx_n,$$

où les coefficients $X_{i,k}$ seront des fonctions données des x_k et des F_i. Ces fonctions devront être uniformes, finies et continues... "

" Je suppose également que..., les équations (1) satisfassent aux conditions d'intégrabilité, qui peuvent s'écrire :

$$\frac{dX_{i,k}}{dx_q} + \frac{dX_{i,k}}{dF_1}X_{1,q} + \frac{dX_{i,k}}{dF_2}X_{2,q} + \ldots + \frac{dX_{i,k}}{dF_\lambda}X_{\lambda,k}$$
$$= \frac{dX_{i,q}}{dx_k} + \frac{dX_{i,q}}{dF_1}X_{1,k} + \frac{dX_{i,q}}{dF_2}X_{2,k} + \ldots + \frac{dX_{t,q}}{dF_\lambda}X_{\lambda,k}.$$

" Si alors le point M décrit sur la variété V un contour infiniment petit, les fonctions F reviendront à leurs valeurs primitives. "

From this analytical situation Poincaré then deduces the considerations about curves which have come down to the present to define the fundamental group.

In the current language we could say that Poincaré was considering representations of the fundamental group of V which were given infinitesimally by a rule:

$$(2) \qquad dF = \omega(x, F).$$

F is a quantity which evolves by this rule and $\omega(x, F)$ is a 1-form depending on the point x in V and the quantity F, with values in the space where dF resides.

As Poincaré described it the rule of development for F was to be globally given ([16]) and globally integrable. The first condition means the bundle with fibre $\mathscr{F} = \{F\}$ is actually the product bundle. While the second means that $V \times \mathscr{F}$ is foliated by leaves (of evolution of F) which cover V evenly. The resulting diffeomorphisms of $\mathscr{F}$ — the holonomy in the foliation — are the substitutions (mentioned by Poincaré) of the representation infinitesimally given by (2).

([16]) The x_i in (1) are global coordinates for a neighborhood of V in some Euclidean space.

If we begin with a differential rule like $dF=\omega(x, F)$, satisfying the integrability condition, the global integrability is a subtle question unless F moves in a compact space or one has some estimate like $|\omega(x, F)| \leq k|F|$ which insures the growth of F along paths is at most exponential.

For example if F belonged to a finite dimensional linear space and $\omega(x, F)$ depended linearly on F, then $dF=\omega(x, F)$ would be globally integrable and $\omega(x, F)$ agrees with the twisting matrix discussed in § 1 which defines a flat connection on the product bundle.

One knows that finite dimensional matrix representations do not detect every fundamental group (*e.g.* a finitely generated group with no subgroups of finite index) but of course bounded linear (even unitary) representations in Hilbert space detect any group (*e.g.* the regular representation on the square summable functions in the group). We claim

Theorem **(9.1)**. — *Any representation of the fundamental group of* V *into the bounded linear operators on Hilbert space* $\mathscr{F}$ *is given infinitesimally by a Poincaré one form* $\omega(x, F)$ *via the rule*:

$$dF=\omega(x, F), \quad F \in \mathscr{F}.$$

Moreover $\omega(x, F)$ *is determined uniquely by the representation up to deformation* (*any two are related by one on* $V \times I$).

Proof. — The representation determines a bundle with discrete structure group and fibre $\mathscr{F}$. This bundle admits a differentiable trivialization (which is unique up to deformation) inside the structure group of all bounded operators with the norm topology (we smooth Kuiper's trivialization [Ku] by approximation).

Writing down the inclination of the leaves of the foliation given by the discrete structure group relative to the Kuiper product structure yields $\omega(x, F)$.

Remark. — Conversely, any rule on V:

$$dF=\omega(x, F)$$

satisfying the local integrability condition when $\omega(x, F)$ is a continuous (in x) family of bounded operators on Hilbert space (or Banach space) can be integrated to yield representations of the fundamental group of V into the bounded linear operators on the space of the F.

10. Integral homotopy theory and minimal algebras.

If the homotopy system $(\pi_1; \pi_2, \pi_3, \ldots)$ of a space X is nilpotent there is a clear relation between its homotopy theory and the algebraic theory of nilpotent differential algebras over $\mathbf{Q}$.

We will think of the process, for any space:

$$\text{space} \to \begin{matrix}\text{rational}\\ \text{forms}\end{matrix} \to \begin{matrix}\text{homological}\\ \text{model}\end{matrix} \to \begin{matrix}\text{spatial}\\ \text{realization}\end{matrix} \quad (\S\,7,\ \S\,5,\ \S\,8),$$

as the passage from integral homotopy theory to (nilpotent) rational homotopy theory. In notation, we write:

$$X \to (\mathscr{E}X = \text{forms on } X) \to (\mathscr{X} = \text{model of } \mathscr{E}X) \to \langle \mathscr{X} \rangle.$$

Now assume X is a nilpotent homotopy type having finite type over the integers.

Theorem **(10.1)**. — *If $\mathscr{X}$ is the minimal model of the* $\mathbf{Q}$*-polynomial forms of some complex representing* X, *then*

(i) $\pi_i X \otimes \mathbf{Q}$ *is dual to the indecomposable spaces of* $\mathscr{X}$.

(ii) $H^*(X, \mathbf{Q})$ *is the cohomology of* $\mathscr{X}$.

(iii) *The homotopy classes of maps of a space* Y *into the rational homotopy type of* X *are computed algebraically* (§ 3) *by mapping* $\mathscr{X}$ *into the forms on* Y.

Proof of Theorem (10.1)

The natural map $X \xrightarrow{\ell} \langle \mathscr{X} \rangle$ has certain properties derived in § 8, Theorem (8.1), namely: ℓ induces an isomorphism of $\mathbf{Q}$-cohomology and the homotopy groups of $\mathscr{X}$ are dual to the indecomposables of $\mathscr{X}$ (assuming $\mathscr{X}$ is nilpotent). Also, the homotopy classes of maps of a space Y into $\langle \mathscr{X} \rangle$ are just the d.g.a. maps $\{\mathscr{X} \to \text{forms on } Y\}$ up to homotopy (Remark (i), § 8).

It follows easily by an argument we give below [17] (or Theorem (2.1) [Su_2]) that $X \to \langle \mathscr{X} \rangle$ tensors homotopy groups with $\mathbf{Q}$, $\pi_i \langle \mathscr{X} \rangle = \pi_i X \otimes \mathbf{Q}$, and this is also true for π_1 which is a nilpotent group. In summary $\langle \mathscr{X} \rangle$ is properly the rational homotopy type of X and those properties of X reflected in its rational homotopy type are expressed clearly in terms of the minimal model $\mathscr{X}$.

Now we relate maps into X and maps into its rational homotopy type, the automorphisms of X and the automorphisms of $\mathscr{X}$, and finally the classification of X in terms of $\mathscr{X}$. The proofs are at the end of the section.

Theorem **(10.2)**. — (i) *If* X *is nilpotent of finite type over* $\mathbf{Z}$ *and* Y *is a finite complex, then on homotopy sets the induced map*:

$$[Y, X] \to [Y, \text{rational homotopy type of } X]$$

is finite to one.

(ii) *Any nilpotent differential algebra* $\mathscr{A}$ *of finite type over* $\mathbf{Q}$ *has a* $\mathbf{Z}$*-form. Namely, there is a space* X *whose homotopy system* $(\pi_1; \pi_2, \pi_3, \ldots)$ *is nilpotent and of finite type over* $\mathbf{Z}$ *and a map* $\mathscr{A} \to$ (rational forms on X) *inducing an isomorphism on cohomology.*

Now assume in addition to the above that either X is a finite complex or that the homotopy system vanishes after some point. Recall a commensurability class of groups is one generated by the operations of : (i) passing to a subgroup of finite index,

[17] " Uniqueness of the model " argument, see note before the proofs of Theorems (10.1)-(10.4).

and (ii) passing to a quotient with finite kernel. Also see the Appendix to § 6 on algebraic and arithmetic groups.

Theorem **(10.3)** (Automorphism groups are arithmetic). — *Consider the discrete group of homotopy classes of self-equivalences,* Aut X. *Then* Aut X *and the naturally associated* **Q**-*algebraic matrix group* $\mathrm{Aut}_{\mathbf{Q}}\,X = \mathrm{Aut}\,\mathscr{X}/$(inner automorphisms) *have the following properties*:

(i) Aut X *is commensurable with a full arithmetic subgroup of* $\mathrm{Aut}_{\mathbf{Q}}\,X$.

(ii) *The natural action of* Aut X *on the integral homology is compatible with an algebraic matrix representation of* $\mathrm{Aut}_{\mathbf{Q}}\,X$ *on the vector spaces of rational homology.*

(iii) *The reductive part of* $\mathrm{Aut}_{\mathbf{Q}}\,X$ *is faithfully represented on the natural subspace of homology generated by maps of spheres into* X.

(iv) *As we vary* X *through finite complexes,* Aut X *runs through every commensurability class of groups containing arithmetic groups. In fact, it suffices to take* X *a skeleton of spaces with two non-zero homotopy groups to realize all classes.*

Consequences of Theorem (10.3)

a) Because of (i) we have that Aut X is finitely presented.

b) Because of (i) we have up to commensurability a semi-direct product decomposition of Aut X as $N \rtimes \Gamma$ where N is a normal finitely generated nilpotent subgroup of Aut X and Γ is the arithmetic subgroup of some semisimple **Q**-algebraic group — like $SL(n, \mathbf{Z})$ or the automorphisms of some quadratic form. Much is known about these latter groups — the " semisimple part " of Aut X.

c) Because of (ii) we know that the subgroup of Aut X which fixes any cohomology class is again arithmetic and thus finitely presented. This rules out such a situation as Aut X = free group on $k \geq 2$ generators and $\mathrm{Aut}\,X \to H_*X$ is some representation such as $\{x, y\} \subset \mathrm{Aut}\,X$ which acts on $H^*X = \mathbf{Z}\oplus\mathbf{Z}\oplus\mathbf{Z}$ by $x \mapsto \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $y \mapsto \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Here the isotropy group of the vector $(0, 0, 1)$ is the commutator subgroup of $\{x, y\}$ which is not finitely generated ([18]).

d) Because of (iii) we know that Γ, the " semisimple " part of Aut X, is faithfully represented on the spherical homology (again up to commensurability). Thus we can " see " the difficult part of the group theory of Aut X acting on the homology (even the spherical homology and thus we also " see " the semisimple part of Γ acting on the homotopy). For example, all eigenvalue considerations for elements of Aut X are generated in the action of Γ on the spherical homology.

Now we can study classification. Note that a **Z**-form $(X, \mathscr{A} \to \text{forms on } X)$ of a finitely generated nilpotent d.g.a. $\mathscr{A}$ determines natural graded lattices (a finitely

([18]) I owe this example to P. Deligne.

generated subgroup of maximal rank in each degree) in the indecomposable spaces of $\mathscr{A}$, $\pi_k\mathscr{A}^k$ and in the structure space of $\mathscr{A}$, $\mathscr{S}(\mathscr{A}) = \bigoplus_k \mathrm{H}^{k+1}(\mathscr{A}^{k-1})$. We call these lattices the structure lattice of the **Z**-form X of $\mathscr{A}$.

Subject to a torsion constraint, and assuming $\mathscr{A}$ is finitely generated nilpotent, we have

Theorem **(10.4)**. — *There are only finitely many integral homotopy types with given model $\mathscr{A}$ and given structure lattice in $\mathscr{S}(\mathscr{A}) \bigoplus_k \pi_k\mathscr{A}$.*

Note. — A *torsion constraint* is the obvious proviso arising from the fact that there are infinitely many (torsion) homotopy types with the rational homotopy type of a point. Thus a *torsion constraint* is any condition on the homotopy types in question to restrict the torsion in the integral homotopy (or the integral homology) to a finite number of possibilities [19].

The idea of the proofs of all these Theorems (10.2), (10.3), (10.4) is that the Postnikov system of X is precisely mirrored in the algebraic structure of $\mathscr{A}$ (or $\mathscr{X}$).

Recall a Postnikov system is (first for *any* space X) a sequence of spaces $\ldots \to \mathrm{X}_k \to \ldots \to \mathrm{X}_1 \to *$ receiving a map from X which identifies the homotopy system of X_k with $(\pi_1; \pi_2, \ldots, \pi_k, 0, 0, \ldots)$, and (second for nilpotent spaces) a refined sequence:

$$\mathrm{X}_k = \mathrm{Y}_n^{k-1} \to \mathrm{Y}_{n-1}^{k-1} \to \ldots \to \mathrm{Y}_1^{k-1} = \mathrm{X}_{k-1}$$

so that the induced map of homotopy systems $\mathrm{Y}_j \to \mathrm{Y}_{j-1}$ is onto, the kernel K_j is only non-zero in degree k, and the corresponding fibration is principal, *i.e.* $\pi_1\mathrm{Y}_{j-1}$ acts trivially on the kernel K_j.

We will have the exactly analogous picture for the d.g.a. $\mathscr{X}$: (first) the sequence of d.g.a.'s $\mathscr{X}^1 \subset \mathscr{X}^2 \subset \ldots$ (generated in degree $\leq k$) which exists because of the minimal condition (§ 2) and then the refined sequence $\mathscr{X}^{k-1} = \mathscr{Y}_1^{k-1} \subset \ldots \subset \mathscr{Y}_n^{k-1} = \mathscr{X}^k$ which exists when $\mathscr{X}$ is nilpotent (§ 2).

Just as the fibrations $\mathrm{Y}_j \to \mathrm{Y}_{j-1}$ are structured by ordinary cohomology classes (of Y_{j-1}), the extensions $\mathscr{Y}_{j-1} \subset \mathscr{Y}_j$ are structured by ordinary classes (of $\mathscr{Y}_{j-1}$). In the passage from $\mathrm{X} \to \mathscr{X}$ (and $\mathrm{Y}_j \to \mathscr{Y}_j$) the only change is that the coefficients of the structure classes have been tensored with **Q** $(\mathrm{K}_j \to \mathscr{K}_j)$. This follows because $\mathrm{X} \xrightarrow{\ell} \langle \mathscr{X} \rangle$ tensors homotopy with **Q**.

Note. — Actually, it is easy to prove this last fact about ℓ directly (without the spectral sequences of Theorem (2.1) of [Su$_2$] which also worked for prime localizations). One merely applies Theorem (7.2) inductively to the Postnikov system of X and then appeals to the uniqueness of the model (§ 5) to relate the homotopy of X and $\mathscr{X}$.

[19] If $\mathscr{A}$ is fixed a torsion constraint in homology is equivalent to one in homotopy — up to some fixed dimension.

This " uniqueness of the model " argument can work in non-nilpotent situations over $\mathbf{R}$ (where [Su_2] definitely doesn't apply) to compute (homotopy groups)$\otimes\mathbf{R}$.

Note. — The minimal $\mathbf{Z}$-refinement of $X_k \to X_{k-1}$, such that ordinary cohomology structures the $Y_j \to Y_{j-1}$, may refine the rational picture. This causes no real problem. Actually, there is a possible extension of the discussion because it is enough to assume the action of $\pi_1 Y_{j-1}$ is trivial on $K_j \otimes \mathbf{Q}$ and the above point is obviated.

The proofs. — To prove Theorem (10.2) (ii) we merely walk along the algebra $\mathscr{A}$ and replicate a space inductively choosing $\mathbf{Z}$-forms of the structure classes of § 2. Inductively, given Y_{j-1} we pull back the structure class c of $\mathscr{Y}_{j-1} \subset \mathscr{Y}_j$ to Y_{j-1} and let K_j be any finitely generated abelian group (for example, without torsion) mapping with finite kernel onto a lattice of $\mathscr{K}_j$ containing the image of $H_{k+1}(Y_{j-1}, \mathbf{Z}) \xrightarrow{c} \mathscr{K}_j$. Then the pulled back c lifts (in finitely many ways) to a class on Y_{j-1} with coefficients in K_j to structure $Y_j \to Y_{j-1}$. $\mathscr{Y}_j$ serves as a model of Y_j using Theorem (7.2) (in the untwisted form over $\mathbf{Q}$). This proves Theorem (10.2) (ii).

Now we prove Theorem (10.4) assuming for the moment Theorem (10.3) (i). Consider the $\mathbf{Z}$-forms of $\mathscr{A}$ $\{(X, \mathscr{A} \xrightarrow{f} \text{forms on } X)\}$ compatible with the given structure lattice subject to a torsion constraint.

The difficult point in the finiteness result is to control the arrows f. In particular for two such arrows the induced maps of structure lattices differ by an element in the group Γ consisting of those lattice isomorphisms coming from an outer automorphism of $\mathscr{A}$ (uniqueness of the model, § 5).

The group of homotopy equivalences of X, Aut X, acts on Γ dividing it into finitely many classes. (By Theorem (6.1) and Theorem (10.3) (i) we can apply A.2 of appendix, § 6, to the homomorphism of $\mathbf{Q}$-algebraic groups:

$$(\text{outer Aut } \mathscr{A}) \to (\text{aut } \mathscr{S}(\mathscr{A}) \bigoplus_k \pi_k \mathscr{A})$$

to conclude the image of Aut X has finite index in Γ.)

Now we can finish by Postnikov induction. To prove Theorem (10.3) (i) we note that Aut Y_j is constructed from Aut Y_{j-1} by three operations:

(i) product with the " arithmetic " group Aut K_j;

(ii) pass to the subgroup of the product preserving the " structure " $\sigma^* c = \tau_* c$, σ in Aut Y_{j-1}, τ in Aut K_j, c in $H^{j+1}(Y_{j-1}, K_j)$;

(iii) extend by an abelian kernel a quotient of $H^j(Y_{j-1}, K_j)$.

This picture passes smoothly to Aut $\mathscr{Y}_j$ and Aut $\mathscr{Y}_{j-1}$ over $\mathbf{Q}$ showing the condition (ii) and the extension of (iii) are of an *algebraic nature* and straightforward induction is possible. This proves Theorem (10.3) (i).

Theorem (10.3) (ii) follows from the definitions.

Theorem (10.3) (iii) follows from Proposition (6.4).

Theorem (10.3) (iv) follows from Theorem (6.1), part (iii), since the arithmetic points of product of multiplicative group $\mathbf{Q}^*$ is finite, and we can realize that model by an odd sphere fibration over a product of $\mathbf{C}P^\infty$'s and then take a large skeleton.

We prove Theorem (10.2) (i) by a similar but more elaborate induction. One proves the desired assertion together with others by induction. The others are that (up to homotopy) homotopies between two maps (an orbit of a free nilpotent group action) and homotopies of homotopies (an abelian group), etc., are just "tensored by $\mathbf{Q}$" in the passage to $\mathbf{Q}$-homotopy theory. An extended 5-lemma involving groups, orbits, and sets carries the induction. This scheme is used twice in [Su_2]; the first part is in fact part of Lemma (2.8) there. This scheme proves other things (analogous to the Hasse principle for maps in [Su_2]) such as going from $\mathbf{Q}$ to any larger field is injective on maps (up to homotopy). For this last point and for a direct proof of the above without [Su_2] we can use the work of § 3 (see the remark after Proposition (3.6)) to build the rational (or larger field) form of the required long exact sequence (corresponding to the homotopy sequence of $\mathrm{Y}_j^{\mathrm{Y}} \to \mathrm{Y}_{j-1}^{\mathrm{Y}}$) by algebra.

11. Algebraic constructions that mirror topological ones.

We give algebraic constructions for the space of all closed curves on a given space, the path fibration of a given space, the universal fibration with a given fibre, the space of all cross sections of a given fibration and topological applications of each. The third is sketchy and the fourth needs more work.

Assume $\mathscr{A}$ is a minimal nilpotent differential algebra. We leave to the reader the relative case ($\mathscr{A}$ is a linear extension of $\mathscr{B}$ which contains the twisting coefficients) and the details of duality and (algebraic) continuity in infinite dimensional vector spaces.

Space of closed curves. — Write $\mathscr{A} = \wedge(x_\alpha) = \wedge(x)$ and form $\wedge\mathscr{A} = \wedge(x, y)$ where $|y| = |x| - 1$. Let s be the derivation of degree -1 defined on generators by $x \mapsto y$, $y \mapsto 0$. The differential in $\wedge\mathscr{A}$ is defined inductively by the condition $ds + sd = 0$. Namely, dx is as in $\mathscr{A}$ and $dy = -sdx$. Note inductively that $d^2y = -dsdx = sd(dx) = 0$.

Recall the ordinary loop space is $\Omega\mathscr{A} = \wedge(y)$, $dy = 0$. Note we have the algebraic fibration $\mathscr{A} \hookrightarrow \wedge\mathscr{A}$ whose fibre is $\wedge\mathscr{A}/\text{ideal } \mathscr{A}^+ = \Omega\mathscr{A}$.

The correctness of this formula is verified by the universal map $\mathscr{A} \xrightarrow{\psi} \wedge\mathscr{A}(\xi)$, $|\xi| = 1$, $d\xi = 0$, defined by $x \mapsto x + \xi y$. ψ is universal for maps $\mathscr{A} \to \mathscr{C}(\xi)$, $|\xi| = 1$, $d\xi = 0$. See [S-V].

Topological application. — Computing directly with this formula gives the following.

Theorem [S-V]. — *If* M *is a closed simply connected manifold whose cohomology ring is not singly generated, then the Betti numbers of the space of all closed curves on* M *are not bounded. Consequently, by Gromoll-Mayer there are infinitely many geometrically distinct periodic geodesics for any riemannian metric on* M.

The calculation begins in [Su_1] and is given completely in [S-V] with some elementary and interesting commutative algebra for such computations provided by Mme Vigué.

The theorem leaves out the interesting case $M=S^2$. To illustrate the formula we compute this case. Then $\mathscr{A}=\wedge(x_1, x_2)$ where $dx_2=x_1^2$, so $\wedge\mathscr{A}=\wedge(x_1, x_2, y_1, y_2)$ where $dy_1=0$, $dy_2=-2x_2y_1$. One easily calculates each Betti number is one and the cohomology ring structure is trivial. Massey products abound.

The case $\pi_1 M=\mathbf{Z}$ is an interesting case to consider now. For most other (known) fundamental groups the geodesic question falls to a direct geometric argument. One should be able to use solvable models for this. For any space with fundamental group $\mathbf{Z}$ has a solvable model mirroring the homotopy structure exactly (and thus good for the computation of the space of closed curves) whenever the action of $\mathbf{Z}$ on homotopy (tensor the reals) is part of an action of $\mathbf{R}$. Thus we can surely treat the case of finite dimensional higher homotopy after passing to a finite cover—an acceptable operation in this problem.

Similarly, geodesics invariant under an isometry should be treated with solvable models. (See recent work of K. Grove and S. Halperin.)

The path fibration. — Let $P(\mathscr{A})$ be $\wedge(x,y)$ with the derivation s of degree -1 as in $\wedge\mathscr{A}$ above. Now however a canonical differential d is defined inductively so that $ds+sd=\Delta$ is an isomorphism in positive degrees.

We define dy inductively by $dy=x-\Delta^{-1}dsx$. Note inductively that:

$$d^2y=dx-d\Delta^{-1}sdx=dx-\Delta^{-1}(ds(dx))=dx-\Delta^{-1}((ds+sd)(dx))=dx-dx=0.$$

Also note that Δ is a derivation, $\Delta x=x$ and $\Delta y=y$ mod decomposables, so Δ^{-1} is still defined and we can continue.

Of course we have the algebraic fibration $\mathscr{A}\hookrightarrow P\mathscr{A}$. Again the fibre is:

$$P\mathscr{A}/\text{ideal }\mathscr{A}^+=\Omega\mathscr{A}.$$

Clearly $P\mathscr{A}$ is contractible, s is a contracting homotopy (or $dy\equiv x$), so we have the algebraic path fibration.

Topological application. — The formulae defining d in the path space tell us how to construct the linear functionals on the homotopy groups by integrating differential forms. Let $\mathscr{A}=\wedge(x)\to$ forms on M be a model and $S^n\to M$ be a smooth map. We pull back the generators x $(=x_1, x_2, \dots)$ to forms on the sphere. For the generators y of dimension $<n-1$ we can solve the equations $dy=x-\Delta^{-1}sdx$ inductively. (In fact, we may do this canonically once some geometrical apparatus is chosen once and for all in the sphere—namely, a direction for the Poincaré lemma, a metric, etc.)

For $|y|=n-1$, we may not be able to solve and in fact we have

Homotopy periods. — $\left\{\int_{S^n} x_\alpha-\Delta^{-1}sdx_\alpha\right\}$, $|x_\alpha|=n$.

Note that these forms can be constructed and are closed in any cylinder connecting two maps. Thus they are homotopy invariants.

Also this discussion works with homology spheres and homology cylinders which suggests what nilpotent models of non-nilpotent spaces are measuring. (This remark is verified by recent work by Hausmann on homology spheres.)

Finally, Gromov [Gr] has made the simple but important observation that the existence of these formulae shows for a compact nilpotent space M the set of homotopy classes of maps of spheres (even compact manifolds) into M with dilatation less than λ grows at most polynomially in λ. A specific estimate is given by the homotopy periods above.

In the case of S^2 we recover the classical integral formula for the Hopf invariant of $S^3 \to S^2$ [Wh$_2$].

The universal fibration with given fibre (sketch). — Given $\mathscr{A}$, define a differential Lie algebra $\mathscr{L}\mathscr{A}$ as follows: in degree $k>0$ put the derivations of $\mathscr{A}$ decreasing dimension by k. In degree zero put the derivations of degree zero commuting with the differential d in $\mathscr{A}$. The differential δ of degree -1 is defined in $\mathscr{L}\mathscr{A}$ by $\varphi \to d\varphi - \bar{\varphi}d$.

We say that $\mathscr{L}\mathscr{A}$ determines the base of the universal algebraic fibration with fibre $\mathscr{A}$ (satisfying the infinitesimal condition (§ 4)). The obvious map:

$$(\text{generators of } \mathscr{A}) \to \operatorname{Hom}(\mathscr{L}(\mathscr{A}), \mathscr{A})$$

yields the higher terms in the formula:

$$d\mathscr{A} \subset \mathscr{A}_{i+1} + \mathscr{B}_1 \otimes \mathscr{A}_i + \mathscr{B}_2 \otimes \mathscr{A}_{i-1} + \dots$$

where the universal base is the differential algebra $\wedge \mathscr{L}\mathscr{A}$ defined by $\mathscr{L}\mathscr{A}$ (see § 12), its elements being the multilinear functions on $\mathscr{L}(\mathscr{A})$.

Note in any fibration with fibre $\mathscr{A}$ the d-formula leads to a map:

$$\text{dual } \mathscr{B}_j \to \operatorname{Hom}(\text{generators of } \mathscr{A}_i, \mathscr{A}_{i+1-j})$$

which is contained in degree $j-1$ of $\mathscr{L}(\mathscr{A})$. Thus the j-dimensional generators of $\wedge \mathscr{L}\mathscr{A}$ map to $\mathscr{B}_j$ and this map will induce the given fibration from the universal one.

Now the homology of $\mathscr{L}\mathscr{A}$ is the homotopy of $\wedge \mathscr{L}\mathscr{A}$, the universal base (see § 2, § 12). Thus in degree one we find the Lie algebra — all derivations of $\mathscr{A}$ commuting with d modulo those of the form $di + id$ — namely, the outer derivations (compare § 3). This fits with the calculation of the homotopy classes of automorphisms of $\mathscr{A}$ (§ 3). (The algebraic analogue of $\pi_1 \mathrm{B}\,\mathrm{Aut}\,\mathrm{X}$.)

In degree $k>1$, the homology of $\mathscr{L}\mathscr{A}$ gives the derivations of $\mathscr{A}$, decreasing dimension by $(k-1)$ and commuting with d, modulo those of the form $d\varphi \pm \varphi d$. One can see these are just the homotopy classes of maps $\mathscr{A} \to \mathscr{A}(\xi)$, $|\xi| = k-1$, $d\xi = 0$ which are the identity $\operatorname{mod}(\xi)$. (This is algebraic analogue of $\{X \times S^{k-1} \to X\}$ computing $\pi_{k-1}(X^X, \text{identity}) = \pi_k \mathrm{B}\,\mathrm{Aut}\,\mathrm{X}$.)

Topological application. — We can put this algebra into motion as follows: let $\eta\mathscr{L}\mathscr{A}$ be the nilpotent part of $\mathscr{L}(\mathscr{A})$—namely, in degree zero we only take the maximal nilpotent ideal of $\mathscr{L}\mathscr{A}$ and in positive degrees all of $\mathscr{L}\mathscr{A}$. Then the corresponding d.g.a. (§ 2) $\wedge(\eta\mathscr{L}\mathscr{A})$ is nilpotent and can be realized by a space $\widetilde{\mathrm{B}}$ over $\mathbf{Q}$ (§ 8).

The reductive part $\mathrm{G}(\mathscr{A})$ of $\mathrm{Aut}\,\mathscr{A}$ (§ 6) acts on $\eta\mathscr{L}\mathscr{A}$ by isomorphisms, thus also on $\wedge\eta\mathscr{L}\mathscr{A}$ and thus also on the realization $\langle\wedge\eta\mathscr{L}\mathscr{A}\rangle=\widetilde{\mathrm{B}}$ by free simplicial maps. Then $\widetilde{\mathrm{B}}/\mathrm{G}(\mathscr{A})$ is defined and will be B Aut X where X is the rational homotopy type realizing $\mathscr{A}$. The spatial realization of the universal algebraic fibration with fibre $\mathscr{A}$ will be the universal topological one with fibre X. (This can be justified by the above calculations of homotopy groups.) Note that $\widetilde{\mathrm{B}}$ is the classifying space for fibrations with unipotent action (also called nilpotent action).

For a sample calculation let $\mathrm{X}=\mathbf{C}\mathrm{P}^n$ (over $\mathbf{Q}$) so that $\mathscr{A}=\wedge(x,y)$, $dy=x^{n+1}$. Let (a, b) denote the derivation of $\mathscr{A}$ taking a to b and annihilating the other generator. Then one calculates directly:

$$\begin{array}{ccccccccc} & 1 & 2 & 3 & 4 & 5 & \dots & 2n+1 & \\ \mathscr{L}(\mathscr{A})= & \{0\}\oplus & \{0\}\oplus & \{(y, x^n)\}\oplus & \{(x, 1)\}\oplus & \{(y, x^{n-1})\}\oplus & 0\dots 0\oplus & \{(y, 1)\}\oplus & 0\dots \end{array}$$

and δ is zero except $\delta(x, 1)=(n+1)(y, x^n)$ since:

$$\delta(x, 1)(y)=(x, 1)\,dy=(x, 1)\,x^{n+1}=(n+1)\,x^n.$$

Thus shifting up one, B Aut X has homotopy groups of rank 1 in dimensions 4, 6, 8, ..., $2n+2$ and thus has minimal model $\wedge(x_4, x_6, \dots, x_{2n+2})$, $d\equiv 0$. (This fits with the idea that the rational homotopy theory of $\mathbf{C}\mathrm{P}^n$-bundles contains all the rational characteristic classes but the first—of a possibly associated $\mathbf{C}^{n+1}$-bundle.)

The space of cross sections of a fibration. — Suppose $\mathscr{B}\subset\mathscr{E}$ is an algebraic fibration with fibre $\mathscr{A}=\mathscr{E}/\text{ideal}\,\mathscr{B}^+$. We write $\mathscr{E}=\mathscr{A}\otimes\mathscr{B}$ as algebras and fix a cross section $\mathscr{E}\xrightarrow{\varphi}\mathscr{B}$ ($\mathscr{B}\subset\mathscr{E}\xrightarrow{\varphi}\mathscr{B}$ is the identity). Define a differential algebra Γ for the space of all cross sections in the connected component of φ as follows:

The generators of Γ are pairs (a, b^*) where a is a generator of $\mathscr{A}$ (we include 1 for the unbased theory and not for the based theory) and b^* is a (dual) additive generator of $\mathscr{B}$. The degree of (a, b^*) is $|a|-|b^*|$. We set the (a, b^*) of negative degrees equal to zero and we convert the (a, b^*) of degree zero into scalars by $(a, b^*)=\langle\varphi(a), b^*\rangle$.

The map $a\overset{e}{\mapsto}\sum\limits_b(a, b^*)b$ leads to the universal evaluation map $\mathscr{E}\xrightarrow{e}\Gamma\otimes\mathscr{B}$ which forces the definition of $d(a, b^*)=(da, b^*)\pm(a, \partial b^*)$. The second term is computed bilinearly with ∂ the dual of d on $\mathscr{B}$. The first term is computed using the diagonal (dual to the multiplication of $\mathscr{B}$) to evaluate b^* on da, the differential of a in $\mathscr{E}$. For example, if $da=a_1.a_2$ and $\Delta b^*=\sum\limits_i b_i'\otimes b_i''$ then $(da, b^*)=\sum\limits_i(b_i', a_1).(b_i'', a_2)$ and we're reduced to the basic symbols (a, b^*).

The evaluation map e is universal for a connected family of cross sections extending φ, namely, the composition $\mathscr{E} \xrightarrow{a} \mathscr{B} \otimes \mathscr{C} \xrightarrow{j} \mathscr{B}$ is φ where a is the family and $j\mathscr{C}^+ = 0$. ($\mathscr{C}$ is an arbitrary connected d.g.a.)

Topological application. — The universal property of e and the fact that this algebra is based on differential forms help one to *first* construct the map between the two sides of the Bott-Haefliger conjecture about Gelfand-Fuks cohomology of vector fields on a manifold [H] and *second* to compute the space of cross sections occurring there.

The algorithm above applied to the associated fibration over M^n with fibre Y_n (the $U(n)$ bundle over $2n$-skeleta BU_n on which $O(n)$ acts) yields a differential algebra determining the nilpotent homotopy type of the dual Lie algebra of vector fields on M — in particular, its cohomology is the cohomology of vector fields. We can also imagine non-nilpotent formulae and twisted cohomology.

We note here a (relatively) convenient form of the fibre Y_n for doing this computation. The $2n$-skeleton of BU_n is formal (§ 12) and so a model $\wedge_n$ is constructed directly from the homology coalgebra (§ 12). Then $\wedge_n(\xi_1, \ldots, \xi_n)$ with $|\xi_i| = 2i - 1$ and $d\xi_i = c_i$ the i-th Chern class is a model of Y_n that displays the $O(n)$ symmetry. (Note the algorithm for Γ didn't require that $\mathscr{A}$ be minimal, but it does have to be free of relations — and thus large in this case since Y_n is a bouquet of spheres.)

12. Formal computation and Kaehler manifolds.

Some minimal algebras can be computed formally from their cohomology rings. For example, if M is a complex manifold and $\mathscr{E}$M denotes the complex valued differential forms with the two differentials ∂ and $\bar\partial$ we have the canonical diagram of differential algebras:

$$\begin{array}{ccc} \mathscr{E}^{\bar\partial}(M) = \{\omega \in \mathscr{E}M : \bar\partial\omega = 0\} \xrightarrow{i} \mathscr{E}M & \qquad & \bar\partial \text{ closed forms} \to \text{all forms} \\ \downarrow{\scriptstyle p} & & \downarrow \\ \mathscr{H} = \{\omega \in \mathscr{E}M : \bar\partial\omega = 0\}/\{\omega \in \mathscr{E}M : \omega = \bar\partial\eta\} & & \text{cohomology} \end{array}$$

The differentials are induced by $d = \partial + \bar\partial$, i is the inclusion and p is the projection.

For a compact Kaehler manifold the induced differential on $\mathscr{H}$ is trivial and p and i induce isomorphisms of cohomology ([20]). Thus from § 5 and § 3 we can build the homological model (over **C**) of the forms on a compact Kaehler manifold directly from the cohomology ring of M (over **C**).

Thus for a Kaehler manifold M there is a d.g.a. map:

$$(\text{model of } M) \xrightarrow{\pi} (\text{cohomology of } M)$$

inducing an isomorphism of cohomology (over **C**).

We say that a nilpotent differential algebra is *formal* if such a map π exists.

([20]) This follows directly from the Hodge theory, see [We] and [DGMS] where other proofs of formality over **C** and **R** are also given. The results of this paper provide the background for and certain extensions of those of [DGMS].

Theorem (**12.1**). — *The notion of formality for a nilpotent minimal algebra is independent of the ground field. Therefore the rational model of a compact Kaehler manifold is formal over* $\mathbf{Q}$. *In particular, one can deduce the model from the cohomology ring.*

This theorem is a consequence of the characterization of formality given in Theorem (12.7), and extends the results of [DGMS] to the ground field $\mathbf{Q}$.

Now we describe the computation of a formal model $\mathscr{M}$ from its own cohomology ring $\mathscr{H}$. Let $\{x_i\}$ denote an additive basis of the dual of $\mathscr{H}^+$ and let $\Delta x_k = \sum_{i,j} a_{ij}^k x_i \otimes x_j$ be the diagonal map (dual to multiplication in $\mathscr{H}^+$). There are two steps to the formal model associated to $\mathscr{H}$.

Step (i). — Form the free graded Lie algebra $\mathscr{L}(\mathscr{H})$ on the x_i shifted down one in dimension and define a differential in $\mathscr{L}(\mathscr{H})$ by:

$$\partial x_k = \sum_{i,j} a_{ij}^k [x_i, x_j].$$

The homology of $\mathscr{L}(\mathscr{H})$ as a Lie algebra will be the homotopy of the model with its Whitehead products ([21]). Note for a nilpotent (*e.g.* simply connected) compact Kaehler manifold this yields direct computation of the (homotopy)$\otimes \mathbf{Q}$ from the cohomology ring (by Theorem (12.1) and Theorem (10.1)).

Even for a Riemann surface of genus g the algorithm comes very close. For there we have $x_1, y_1, \ldots, x_g, y_g, \omega$ as an additive basis (of H_1 and H_2), the diagonal map $\Delta x_i = \Delta y_i = 0$, $\Delta \omega = \sum_i x_i \otimes y_i$, the differential Lie algebra defined by $\partial x_i = \partial y_i = 0$, $\partial \omega = \sum_i [x_i, y_i]$ in the free Lie algebra on $x_1, y_1, x_2, y_2, \ldots, x_g, y_g$ in degree zero and ω in degree 1, and finally the homology of this Lie algebra — the free Lie algebra on $x_1, y_1, \ldots, x_g, y_g$ modulo the relation:

$$[x_1, y_1] + [x_2, y_2] + \ldots + [x_g, y_g] = 0.$$

Step (ii). — From the differential Lie algebra $\mathscr{L}(\mathscr{H})$ we form a differential algebra $\wedge \mathscr{L}(\mathscr{H})$ whose generators are the dual additive generators of $\mathscr{L}(\mathscr{H})$ (shifted back up by one) with a differential defined by:

$$d(\text{generator}) = \text{linear term} + \text{quadratic term}$$

where the linear term is the dual of ∂ and the quadratic term is the dual of the bracket [,] in $\mathscr{L}(\mathscr{H})$ see [Q].

This differential algebra is minimal precisely when $\Delta = 0$ *i.e.* products in $\mathscr{H}^+$ are zero. Then $\mathscr{L}(\mathscr{H})$ (and its homology) is the free Lie algebra on the homology and the model $\wedge \mathscr{L}(\mathscr{H})$ is the minimal model of a one point union of spheres with the same homology.

The presence of the linear term in d explains why the homotopy of $\wedge \mathscr{L}\mathscr{H}$ (in

([21]) The format of differential Lie algebras for homotopy computations in simply connected spaces was introduced by Quillen [Q], and his paper is a good reference for this discussion.

the sense of § 2) is the homology of $\mathscr{L}\mathscr{H}$. Such a d.g.a. description starting from $\mathscr{H}$ is justified by Theorem (12.5) below by computing the cohomology of $\wedge\mathscr{L}\mathscr{H}$ to be $\mathscr{H}$.

Besides Kaehler manifolds many other spaces arising in previous computations of algebraic topology are formal. Examples are:

(i) Lie groups and classifying spaces.

(ii) Some homogeneous spaces G/H—one formality condition is that the ideal of H^*B_H in H^*B_G has a regular sequence (see (v) and [GVH]).

(iii) Canonical skeleta of formal spaces (namely, using rational cells to take off exactly the homology up to some point), for example, the $2n$-skeleton of BU_n which figures in characteristic classes of foliations (§ 11). That canonical skeleta of formal spaces are formal follows by considering the sub-differential Lie algebra of $\mathscr{L}(\mathscr{H})$ generated by $|x_i| \leq k$, and this fact and proof generalize to any sub-coalgebra of homology.

(iv) The unstable Thom spaces MU_n (and MSO_n) maps of V into which describe the classification of submanifolds of V up to cobordism in $V \times I$. These have a rich homotopy structure—see the computation at the end of this section.

(v) (Generalizing (ii)) the model:

$$\wedge(x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_r)$$

with $dy_i = \mathscr{P}_i(x_1, \ldots, x_n)$ where $|x_i|$ is even, $dx_i = 0$, and $\mathscr{P}_i$ is not a zero divisor in $\wedge(x_1, \ldots, x_n)/(\mathscr{P}_1, \ldots, \mathscr{P}_{i-1})$ (a regular sequence). The criterion of Theorem (12.5) is easy to fulfill. The requisite computation of cohomology goes by induction over the $\mathscr{P}_i$.

If $r = n$ these cohomology rings satisfy Poincaré duality and Theorem (13.2) applies to build manifolds with these cohomology rings, if the signature is sufficiently divisible.

These differential algebras are actually intrinsically formal, namely, any other model with the same cohomology ring is formal (there is a map) and thus isomorphic to this one ([22]). With this remark the finite determination results of [BD] become a particular case of Theorem (10.4).

The integral forms of formal homotopy types enjoy certain symmetry properties useful in obstruction theory. With the appropriate nilpotent and finiteness condition on X we can say:

Theorem **(12.2)**. — *Any integral form* X *of a formal homotopy type has sufficiently many endomorphisms* $X \xrightarrow{f} X$ *(which are rational isomorphisms) to localize the homology*:

$$\varinjlim_{f} H_*(X, \mathbf{Z}) = H_*(X, \mathbf{Q}).$$

([22]) A remark of Steve Halperin.

Proof. — This is proved by a direct Postnikov induction lifting enough of the grading automorphisms (see below). We omit the details here because a generalization was obtained with R. Body which received a separate treatment [BS].

Corollary **(12.3)**. — *For all finite complexes* Y *the homotopy set*:

$$[\mathrm{Y}, \text{rational homotopy type of X}] = [\text{model of X, forms on Y}]$$

is just the direct limit $\varinjlim_f [\mathrm{Y}, \mathrm{X}]$.

Proof. — Straightforward obstruction theory.

Thus we have

Theorem **(12.4)**. — (i) *A nilpotent* (e.g., *simply connected*) *compact Kaehler manifold admits continuous self-mappings of any sufficiently divisible degree.* (*Compare* $\mathbf{C}\mathrm{P}^n$.)

(ii) *Two nilpotent* (e.g., *simply connected*) *compact Kaehler manifolds with isomorphic rational cohomology rings have continuous maps between them inducing these isomorphisms (up to composition with a grading automorphism—see below).*

Proof. — This follows from Theorems (12.1), (12.2), and Corollary (12.3).

Note. — Recall any self-mapping of a Riemann surface of genus $g>1$ either has degree 0 or degree ± 1 (23). The theorem applies to the sphere and torus, though, which of course have self-maps of all degrees.

Actually, for the Kaehler manifold we can take the endomorphisms so that H^i is multiplied by q^i, q sufficiently divisible.

Applying § 10 and § 13 we also have (real dimension $\neq 4$)

Theorem **(12.5)**. — a) *The diffeomorphism type of a simply connected Kaehler manifold is determined up to a finite number of the possibilities by*:

(i) *the integral cohomology ring*,

(ii) *the (rational) Pontryagin classes.*

b) *Up to commensurability and normal nilpotent subgroups the arithmetic group of self-homotopy equivalences (up to homotopy) is just the group of automorphisms of* (i) *and the arithmetic group of self-diffeomorphisms (up to deformation) is just the group of automorphisms of* (i) *fixing* (ii).

Proof. — This follows from Theorems (13.1) and (13.3).

Finally, applying the " injectivity " of maps in going from $\mathbf{Z}\to\mathbf{Q}$ and $\mathbf{Q}\to\mathbf{C}$ (§ 10) we have, because the diagram $\mathscr{H} \leftarrow \mathscr{E}^{\partial}\mathrm{M} \to \mathscr{E}\mathrm{M}$ is natural for holomorphic maps:

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(23) True for a closed $\mathrm{K}(\pi, 1)$ manifold of non-zero Euler characteristic.

Theorem **(12.6)**. — *The homotopy theory of holomorphic maps between (nilpotent) compact Kaehler manifolds is determined up to a finite number of possibilities over* $\mathbf{Z}$ *or exactly over* $\mathbf{Q}$ *(where it is the formal map) by the induced map on cohomology (over* $\mathbf{Q}$ *or* $\mathbf{C}$*).*

Proof. — This follows from Theorem (10.1).

To justify the descent of formality from $\mathbf{C}$ to $\mathbf{Q}$ we give the following characterization of formal homotopy types. If k is the ground field we have the action of k^*, the multiplicative group of k, on cohomology, $x \mapsto \alpha^i x$, α in k^*, $|x| = i$. These are the *grading automorphisms.*

Theorem **(12.7)**. — *A nilpotent algebra* $\mathscr{A}$ *is formal iff the grading automorphisms of* $\mathrm{H}\mathscr{A}$ *lift to* $\mathscr{A}$, *or iff all automorphisms of* $\mathrm{H}\mathscr{A}$ *lift to* $\mathscr{A}$.

Remark (Proof of Theorem (12.1)). — By the description of the automorphisms of $\mathscr{A}$ and cohomology automorphisms (§ 6) the second criterion is clearly independent of the ground field. Thus Theorem (12.1) (over $\mathbf{Q}$) follows from the existence of the $(\partial, \bar{\partial})$ diagram (over $\mathbf{C}$), the results on algebraic groups (§ 6), and theorem (12.7).

Proof of Theorem (12.7). — If $\mathscr{A}$ is formal, we have a d.g.a. map $\mathscr{A} \to \mathrm{H}\mathscr{A}$ inducing isomorphisms on cohomology. The obstruction theory for maps (§ 3, Proposition (3.6)) implies all automorphisms of $\mathrm{H}\mathscr{A}$ lift to $\mathscr{A}$.

Suppose the grading automorphisms of $\mathrm{H}\mathscr{A}$ lift to $\mathscr{A}$. Using one of these, say α (or its diagonizable part—note Corollary (6.2)), we obtain a direct sum decomposition $\mathscr{A} = \bigoplus_i \mathscr{A}_i$ where α acts on $\mathscr{A}_i$ as multiplication by α^i. Let $\mathrm{W}_k = \bigoplus_{i \geq k} \mathscr{A}_i$; then one sees by induction that $\mathrm{W}_k \cap (\text{dimension} > k) = 0$. Thus $(\text{dimension } k) \cap \mathrm{W}_k = \mathrm{S}$ is contained in the cycles, $(\text{dimension} < k) \cap \mathrm{W}_k = \mathfrak{I}$ is an ideal and the quotient map:

$$\mathscr{A} \to \mathscr{A}/(\mathfrak{I}, d\mathfrak{I}) = \mathrm{S}/(\mathrm{S} \cap d\mathfrak{I})$$

of d.g.a.'s induces an isomorphism on cohomology because the non-bounding cycles of degree i are all represented in W_i.

Homotopy groups of Thom spaces

For a sample formal calculation consider the problem of computing the rational homotopy groups of the (unstable) Thom spaces MU_d—the union of the one point compactifications of the canonical $\mathbf{C}^d$-bundle over the complex grassmannians. These spaces can be defined by cofibration sequences:

$$\mathrm{BU}_{d-1} \to \mathrm{BU}_d \to \mathrm{MU}_d$$

which shows the cohomology of MU_d is the ideal $\mathfrak{I}$ of the d-th Chern class c_d in the polynomial algebra on $c_1, \ldots, c_d$. It is easy to see that MU_d is formal, since BU_{d-1} and BU_d are, the induced map between them is formal, and this map is onto for cohomology.

If we form $\mathfrak{I}(y)$ with $dy \doteq c_d$, we have killed the bottom homotopy group of MU_d in a total space. It is easy to see the map $\mathfrak{I}(y) \to \mathfrak{I}/c_d\mathfrak{I}$ obtained by setting y and c_d equal to zero induces an isomorphism on cohomology. But $\mathfrak{I}/c_d\mathfrak{I} = \mathfrak{I}/\mathfrak{I}^2$ is the zero ring on monomials in $c_1, \ldots, c_d$ with weight exactly one in c_d (besides c_d itself). The formal space with this ring is the bouquet on those even dimensional spheres. Thus the rational homotopy of MU_d (shifted down one) in dimensions $> 2d$ is the free Lie algebra on these (dual) monomials (shifted down one).

For example, $\pi_4 MU_2 \otimes \mathbf{Q} = \mathbf{Q}$ and in dimensions greater than four we have the free Lie algebra on:

$$(c_2 c_1), (c_2 c_1^2), (c_2 c_1^3), \ldots$$

in degrees 5, 7, 9, ... for $\pi_{*-1} MU_2$.

We stop here, but one may compute the bracket structure with the generator of degree $2d$ using the differential Lie algebra formula above. The rational homotopy of MU_d was first computed by Burlet [Bu].

There are interesting connections with isolated complex singularities going back to the rigid cone in $\mathbf{C}^6$ of 2×3 matrices of rank ≤ 1.

The fact that this isolated singularity cannot be deformed away follows because the link of the singularity in S^{11} represents a nontrivial multiple of the generator $[c_2 c_1, c_2 c_1]$ in $\pi_{11} MU_2 \otimes \mathbf{Q} = \mathbf{Q}$.

Conjecture. — Links of complex isolated singularities only involve single brackets in $\pi_* MU_d \otimes \mathbf{Q}$.

13. Algebraic invariants for the classification and construction of manifolds and diffeomorphisms.

We will describe algebraic pictures of manifolds and diffeomorphisms that make sense for the general manifold and can be converted back into geometry with only finite ambiguity in the case of simply connected manifolds of dimension at least five.

Let us associate to each manifold M its underlying homotopy type X together with the orbit of the rational Pontryagin class in H^*X under the action of the group of self homotopy equivalences Aut X. We obtain a map:

$$\{\text{diffeomorphism types}\} \xrightarrow{N} \{\text{homotopy types, cohomology orbits}\}.$$

Novikov [N] proved by surgery and Smale's h-cobordism theorem that in the class of closed manifolds which are simply connected and have dimension at least five that N is finite to one. (Later Browder [Br] studied the image and the author ([Su_3] and [Su_4]) determined the kernel precisely in the homeomorphism context.)

We will now apply differential forms to obtain a completely algebraic invariant for the diffeomorphism type up to finite ambiguity.

Recall that by successively solving the equation $dx = y$ a finite number of times where y is made up algebraically (by addition and wedge product) from previously constructed forms we can build a homological model $\mathscr{M}=\wedge(x_1, x_2, \ldots, x_r; d)$ *over* $\mathbf{R}$ (§ 5) of the manifold M up to its dimension. The manifold M also determines integral lattices in the structure space $\mathscr{S}(\mathscr{M})=\bigoplus_k (\mathrm{H}^{k+1}(\mathscr{M}^{k-1}) \oplus \pi_k\mathscr{M})$ (§ 10).

Finally, recall the differential form construction of the Pontryagin classes from a connection in the tangent bundle. If the connection Θ is described in the presence of partial framings α by matrices of 1-forms Θ_α, one defines the curvature tensor by $\Omega_\alpha = d\Theta_\alpha - \Theta_\alpha \circ \Theta_\alpha$ and then the (independent of α) closed forms $\{\operatorname{trace} \Omega_\alpha \circ \ldots \circ \Omega_\alpha\}$ represent the Pontryagin classes (so applying to Θ, exterior differentiation, addition, and wedge product of forms).

For the integral part of our invariant we take the integral lattice of the structure space (Theorem (10.4)), the integral lattices in $\mathrm{H}^{4k}(\mathrm{M}, \mathbf{R})$ and finally the torsion coefficients of homology.

Theorem **(13.1)**. — *The collection of simply connected closed manifolds of dimension greater than four having isomorphic algebraic invariants*:

(i) *the homological model* $\mathscr{M}$ *(over the reals) of the smooth forms up to the dimension,*

(ii) *the (real) Pontryagin classes,*

(iii) *the integral structure mentioned above (lattices and torsion coefficients),*

falls into finitely many *diffeomorphism classes.*

Proof. — See the end of this section.

Remark. — The members of these finite families of diffeomorphism types have identical properties from many points of view, but not all.

Geometric realization of invariants

Let us consider rational model and the Pontryagin class together as the rational part of the invariant explicit in Theorem (13.1). For the rational part of the invariant let us note that when the top dimension is $4k$, a signature is defined up to sign, and the k-th Pontryagin class is determined by the lower ones when the signature is zero, and by the signature and a choice of " fundamental class " $\mathrm{H}^n(\mathscr{M}) \sim \mathbf{Q}$ when the signature is nonzero. (Using the Thom-Hirzebruch formula.)

The first theorem concerns the realization of the rational model and the (lower) Pontryagin classes by a manifold subject to this convention. Notice that realization only depends on the position of the Pontryagin classes in the cohomology ring.

Theorem **(13.2)**. — *Given any rational model and Pontryagin class (excluding the top component in dimension $4k$) whose cohomology satisfies* $\beta_1=0$ *and Poincaré duality over* $\mathbf{Q}$, *we can say the following*:

(i) *There is a simply connected manifold realizing this algebraic data if the dimension is not $4k$ or when the dimension is $4k$ if we allow one singular point — the cone on a $(4k-1)$-$\mathbf{Q}$-homology sphere.*

(ii) *To remove the singularity in a realization of dimension $4k$ it is necessary and sufficient that:*

a) *when the signature is zero the quadratic form on* H^{2k} *(unambiguously defined over* $\mathbf{Q}$*) is a sum of squares* $\sum_i \pm x_i^2$ *(over* $\mathbf{Q}$*);*

b) *when the signature is nonzero there is a choice of fundamental class* $H^{4k} \underset{\mu}{\sim} \mathbf{Q}$ *so that the quadratic form on* H^{2k}, $\langle y^2, \mu\rangle$, *is a sum of squares* $\sum_i \pm x_i^2$ *and the numbers* $\langle p_{i_1} \dots p_{i_r}, \mu\rangle$ *are integers satisfying the congruences of a cobordism class* [St].

Proof. — See the end of this section.

Example. — Let Φ be any homogeneous cubic form (symmetric) on a rational vector space H. Then there is a simply connected closed six-manifold so that $H^2 = H$, $H^4 = H^*$, p_1 contained in H^4 is arbitrary, and the cup product structure is given by Φ. Compare [Su_5].

Remark. — We leave to the reader the formulation and proof of the analogous theorem on almost complex manifolds involving the signature and the Euler characteristic.

Path Components of diffeomorphisms

Our classification is completed by describing the automorphisms — namely the path components of Diff M, denoted $\{\mathrm{Diff}\, M\}$.

The homotopy smoothings sequence (see below) leads to the rough description of the homomorphism:

$$\{\text{components of diffeomorphisms}\} \xrightarrow{\psi} \{\text{components of homotopy equivalences}\}:$$

(i) the image of ψ is commensurable with the isotropy group of the Pontryagin class;

(ii) the kernel of ψ is commensurable to a quotient of $H^3 \oplus H^7 \oplus \dots \oplus H^{4i-1} \oplus \dots$ where $4i-1 <$ dimension M and cohomology is taken with integer coefficients ([24]).

We will describe this extension in terms of $\mathbf{Q}$-algebraic groups. $\{\mathrm{Diff}\, M\}$ will lie in the corresponding arithmetic commensurability class.

A quick description is to take automorphisms of the algebraic model of the principal tangent bundle over M up to homotopy.

The detailed description goes as follows.

Define an *algebraic diffeomorphism* to be a pair (σ, ξ) where σ is an automorphism of the model $\mathscr{M}$ and ξ satisfies $d\xi = \sigma p - p$. $\xi = (\xi_1, \xi_2, \dots)$ is called the *distortion of the Pontryagin class*, $p = (p_1, p_2, \dots)$. The composition rule is:

$$(\sigma, \xi) \circ (\sigma', \xi') = (\sigma \circ \sigma', \xi + \sigma\xi').$$

([24]) Actually, surgery only constructs a concordance and then we apply the theorem of Cerf that in this case ($\dim \geq 5$, $\pi_1 = e$) concordance implies isotopy.

An *algebraic isotopy* of (σ, ξ) to the identity is a derivation i of degree -1 of $\mathscr{M}$ so that $\sigma = \exp(di + id)$ (§ 3) and a homology between ξ and the homology $\sigma p \sim p$ induced by i (see § 3).

If we add the boundary condition that we take no component of p or ξ in the top dimension of M, then the algebraic group we want for $\{\mathrm{Diff}\,\mathrm{M}\}$ is the quotient of *algebraic diffeomorphisms* by the unipotent subgroup of those *algebraically isotopic* to the identity.

Theorem **(13.3)**. — *If* $\pi_1 \mathrm{M} = e$ *and dimension* $\mathrm{M} > 5$, *then the component group of* Diff M *is commensurable to the arithmetic groups in the algebraic group just described. Thus these component groups are finitely presented.*

As a sample calculation using Theorem (13.3) let us describe the computation of the kernel of (diffeomorphisms $\rightarrow$ homotopy equivalences).

Let D denote the distortion cohomology groups $\mathrm{H}^3 \oplus \mathrm{H}^7 \oplus \ldots \oplus \mathrm{H}^{4i-1} \oplus \ldots$ where $(4i-1) < \dim \mathrm{M}$.

Let $\mathrm{I} \subset \mathrm{D}$ be the indeterminacy subgroup defined by $\mathrm{I} = \{ip\}$ where p is the Pontryagin class and i is a derivation of the model $\wedge(x_1, \ldots, x_r, d)$ of degree -1 satisfying $di + id = 0$.

Corollary **(13.3)**. — *Then the kernel of "diffeomorphisms to homotopy equivalences" is commensurable to a lattice of* D/I.

Proofs. — Theorem (13.3) and Corollary (13.3) are proved at the end of the section.

Remark. — Thus a nonzero Pontryagin class cuts down the diffeomorphism group in two steps, by the orbit of that cohomology class and by creating an indeterminacy in the distortion of the cocycle within that class.

Geometric interpretation. — In a Riemannian manifold we have the following analogy to this algebraic picture. First a diffeomorphism induces an automorphism σ of all the forms. Also there are canonical closed forms $p = (p_1, p_2, \ldots)$ defined by the curvature tensor Ω, $\{\mathrm{tr}\,\Omega \circ \ldots \circ \Omega\}$, for the Pontryagin classes. There are canonical distortion elements $\xi = (\xi_1, \xi_2, \ldots)$ solving $d\xi = \sigma p - p$. ξ is determined by connecting the original metric and the new metric (distorted by the diffeomorphism from the old) by their linear interpolation.

These elements are valid for computation in the above abstract theory.

Corollary **(13.4)**. — *Two isometries of* M *which induce the same map on real homology are algebraically isotopic.*

Proof. — They induce the same map on the canonical minimal model of a Riemannian manifold ((§ 14) and [Su1]) and they have zero distortions.

Realization of arithmetic groups as $\{\mathrm{Diff}\,\mathrm{M}\}$.

Theorem **(13.5).** — *If* G *is any* **Q**-*algebraic group we can construct a* closed manifold M *so that* $\{\mathrm{Diff}\,\mathrm{M}\}$ *is commensurable to the arithmetic subgroups in a vector space extension of* G.

Construction. — Let M be the double along the boundary ∂ of a neighborhood in $\mathbf{R}^{2N}$ (N large) of the finite complex X constructed in Theorem (10.3).

Then $H^*(M, \mathbf{Q})$ is even dimensional and the Pontryagin class is zero. So $\{\mathrm{Diff}\,\mathrm{M}\}$ is commensurable to Aut M by the above. But Aut M $\to$ Aut X (restricting to a skeleton) is easily seen to be onto and the lift of an element in Aut X is well defined up to a self homotopy of the identity of ∂. These self homotopies compose in an Abelian fashion and this is seen by a disjoint support argument.

Proof of Theorem (13.1). — We consider a sequence of equivalence relations (i), (ii), (iii), (iv), (v) on any set of closed simply connected n-manifolds having the same torsion numbers, $n>4$. These are generated by the notions:

(i) there is an isomorphism between real models up to the dimension n preserving the structure lattices, the lattice in $H^{4*}(\mathbf{R})$ and the Pontryagin classes (real);

(ii) there is an isomorphism of rational models doing the same (as (i));

(iii) there is an integral homotopy equivalence between two of the manifolds;

(iv) there is an integral homotopy equivalence between two of the manifolds preserving the rational Pontryagin classes;

(v) there is a diffeomorphism between two of the manifolds.

Say one relation has finite index in a coarser one if each class of the latter only contains finitely many classes of the former.

Step 1. — The relations (i) and (ii) are equal.

Proof. — The words " up to the dimension n " can be dropped in (i) or put in (ii) without changing the relations. One sees this by the straightforward inductive argument over the stages above the dimension n.

Now look at the induced isomorphism σ of the rational structure spaces for a chosen isomorphism realizing (i). The isomorphisms between the two models (up to the dimension n) agreeing with σ form a **Q**-principal homogeneous space (Appendix § 6) of the unipotent group (Proposition (6.4)) fixing the structure lattice of one. It thus has a rational point (A.1, appendix, § 6) and the manifolds are in relation (ii).

Step 2. — (ii)$\cap$(iii) has finite index in (ii).

Proof. — Theorem (10.4).

Step 3. — (iv) has finite index in (ii)$\cap$(iii).

Proof. — Let Aut M denote the group of integral homotopy equivalences of a particular manifold M of one (ii)$\cap$(iii) equivalence class.

For all the other manifolds of this (ii)$\cap$(iii) class push their Pontryagin classes to $H^{4*}(M, \mathbf{Q})$ by all possible (iii) equivalences. By the definitions these classes lie in one orbit of the Pontryagin class of M under the group Γ of automorphisms of $H^{4*}(M, \mathbf{Q})$ which give isomorphisms of the lattice and come from an automorphism of the rational model of M.

By theorem (10.2) (i) we may apply (A.2, appendix, § 6) to the homomorphism of $\mathbf{Q}$-algebraic groups (outer automorphisms of $\mathbf{Q}$-model) $\to$ (automorphisms $H^{4*}(M, \mathbf{Q})$) and deduce the image of Aut M has finite index in Γ.

So the classes above are covered by finitely many orbits of the action of Aut M on $H^{4*}(M, \mathbf{Q})$, and (iv) has finite index in (ii)$\cap$(iii).

Step 4. — (v) has finite index in (iv).

Proof. — This follows from the homotopy-smoothings sequence. In fact this statement was first proved by Novikov. (See Theorem (10.8) and p. 112 of [Wa].)

Step 5. — (v) has finite index in (i).

Proof. — This follows from steps 1-4 and completes the proof of the theorem.

Proof of Theorem (13.2). — There are several steps which we first state and then prove:

(i) We construct a space X representing the model $\mathscr{M}$.

(ii) Then we find a map of nonzero degree of a closed n-manifold $M \xrightarrow{f} X$ so that:

a) $f^* p_i$ is the Pontryagin class of M; and

b) if $n = 4k$ the quadratic form on $\langle y^2, f_* M\rangle$ is equivalent to a sum of squares $\sum_i \pm x_i^2$.

(iii) We do rational surgery on f to preserve (ii) and reduce the kernel on $H_*(\ , \mathbf{Q})$ to zero. Since a map of nonzero degree is always onto (X satisfies Poincaré duality) the desired realization is constructed.

Now (i) is accomplished using the spatial realization of $\mathscr{M}$ (§ 8).

For (ii) the solutions of *a)* up to cobordism correspond by the classical argument of Thom to $\pi_n M$ where M is the Thom space of the induced bundle over the fibre product of $X \xrightarrow{p} \Pi K(\mathbf{Q}, 4i)$ and $BO \xrightarrow{\overline{P}} \Pi K(\mathbf{Q}, 4i)$ where $\overline{P}$ is the universal dual Pontryagin class.

Now $\pi_n M \otimes \mathbf{Q} = H_n M \otimes \mathbf{Q} = H_n X = \mathbf{Q}$ so maps of nonzero degree satisfying (ii) *a)* exist. If the signature of X is zero, then condition (ii) *b)* is automatic.

If the signature is nonzero and the conditions of part (ii) of the theorem are satisfied one observes the image of $\pi_n M$ in $\pi_n MSO$ is characterized by rational conditions and pulls back to $\pi_n M$ an element in $\pi_n MSO$ with those numbers. A little thought shows (ii) *b)* is satisfied.

Now for (iii). The less technical aspects of surgery are sufficient for us. Below the middle dimension one uses the rational Hurewicz theorem to represent the rational kernel by embedded spheres. Condition (ii) *a)* implies a multiple embedding has trivial normal bundle and surgery can be done to kill this element preserving (ii) *a)*. (See [M2] and [Br] for details.) Condition (ii) *b)* is preserved by Witt's theorem about cancelling quadratic forms over a field.

The middle dimensional surgery arguments greatly simplify here. The odd dimensional cases are easy (no torsion and linking considerations arise), the $(4k+2)$ case is treated by killing twice any embedded $(2k+1)$-sphere which might have a nontrivial normal bundle, and the $4k$ case rests on the quadratic form which is arranged by (ii) *b)*, using Witt's theorem, and signature $\mathrm{M} =$ signature X.

If (ii) *a)* above were satisfied and not (ii) *b)* we could isolate the middle dimensional kernel by writing M as a connected sum along a rational homology sphere, then discard the kernel side and cone to achieve the realization with one singular point.

Proof of (13.3). — The surgery theory along with Cerf's theorem shows path components of Diff M (except for finite obstructions) correspond to deformation classes of pairs consisting of a self homotopy equivalence plus a covering bundle map on the complement of a point in the principal tangent bundle (see [N] for exactly this viewpoint or [Su_3] for the relative form of the above surgery theorem).

We have described this picture rationally by the algebraic diffeomorphisms modulo algebraic isotopy. The actual picture is an integral form of the rational one (§ 10).

14. Questions, Problems, and further remarks.

Geometry. — The reflection of the analytical or geometrical properties of the manifold in the differential forms and then in the algebraic structure of the models provides a method of discovering relationships between the analysis or geometry of the manifold and its topology.

Given a riemannian metric one can use the Hodge decomposition [Su_1] when building the model and arrive at the

Theorem (**14.1**). — *A compact riemannian manifold* M *has a canonical model*:

$$\mathscr{M} \to \textit{forms on } \mathrm{M}$$

on which the group of isometries acts in algebraic consequence of the action on harmonic forms.

In this canonical construction of the model of a riemannian manifold the deviation from formality (§ 12) is related to the incompatibility of wedge products and harmonicity of forms. Thus for a symmetric space the construction of the model is formal.

Problem 1. — Systematize the relationship between product structure of harmonic forms and the Massey product structure of the model and its deviation from formality for a Riemannian manifold.

In another direction one can think of the covariant differentiation of the metric as a map $\Omega^p \xrightarrow{\Delta} \Omega^p \otimes \Omega^1$ so that the composition with skew-symmetrization $\Omega^p \otimes \Omega^1 \to \Omega^{p+1}$ is the exterior d. The curvature is the deviation of Δ from satisfying the integrability condition of § 1.

Question 1. — Can one express the condition of signed curvature (positive or negative) in algebraic terms via this factoring and derive topological consequences from the form of the models?

We have seen in § 12 that the models of Kaehler manifolds are formal — so the form of the model is dictated by the cohomology ring. The map of the model into the forms is still interesting — in fact the canonical model given by a metric has a certain holomorphic sense. Namely, the part generated in degree one (— which is very large for a Riemann surface) can be constructed canonically from the complex structure.

Question 2. — Does the canonical model in degree one for compact complex manifolds (admitting Kaehler metrics) provide a useful holomorphic invariant?

The diagram of differential algebras yielding formality of a Kaehler manifold:

$$\{\bar\partial \text{ cohomology}\} \leftarrow \{\bar\partial \text{ closed forms}\} \rightarrow \{\text{all smooth forms}\}$$

makes sense for any complex manifold and should yield information in the compact case.

Problem 2. — Does the finite dimensional differential algebra $\{\bar\partial\text{-cohomology}, \partial\}$ determine the model of a *compact complex manifold*?

There are definite connections with the topology here because Morgan has observed that formality results if the Frölicher ∂, $\bar\partial$ spectral sequence collapses at E^1 and a natural complex conjugation property holds. (See Morgan's work on the models of open complements of analytic subvarieties of Kaehler manifolds — to appear (*Publications math. I.H.E.S.*, volume 48).)

It would be interesting to have strong necessary conditions on the topology that a compact manifold M admits infinitely many isometries for some metric. This is equivalent to having a nontrivial action of the circle (by isometries then if desired). One can work with the invariant forms I defined by $(di+id)\,\omega=0$ where i is contraction by the vector field of the action.

Using I and i one can form an algebraic model of the Borel fibration:

$$M \to M_S \to \mathbf{C}P^\infty$$

where $M_S = (M \times S^\infty)/S^1$. Namely form $I \otimes \wedge(x)$ where $|x| = 2$, $dx = 0$, and:

$$d(a \otimes 1) = da \otimes 1 + i(a) \otimes x.$$

$d^2 = 0$ here is equivalent to $di + id = 0$ and $i^2 = 0$ on I.

Problem 3. — Derive necessary topological conditions for M to admit an infinitesimal isometry for some metric, perhaps combining the geometrical homological picture of the fixed point set and the algebraic picture of the Borel fibration expressed above in the differential forms.

Besides compact complex manifolds and infinitesimal isometries the subject o compact symplectic manifolds begs attention in this discussion.

Here we have a closed 2-form ω on a $2n$-manifold that is locally equivalent to $dx_1 \wedge dx_2 + \ldots + dx_{2n-1} \wedge dx_{2n}$. Thus ω^n is a volume form and the cohomology of M contains $1, \omega, \omega^2, \ldots, \omega^n$.

Question 3. — Is there any further topological condition for the existence of a nondegenerate closed 2-form besides this cohomological one? In particular does the ω somehow structure or influence the formation of a model in the differential forms?

Finally, all these problems can be enhanced (where successful) [25] by the discussion of twisted models which encourage their own questions and problems.

We have seen that local systems can be treated by the above differential algebras when they are infinitesimally given by a twisting matrix Θ satisfying $d\Theta - \Theta \circ \Theta = 0$. Actually if Θ is an infinite dimensional operator one has to be able to solve the differential system $dv = \Theta(v)$ (Theorem (1.2)). If the vector space has a norm and Θ is a bounded operator the classical picture prevails (§ 9).

Question 4. — Outside the case of bounded operators on normed spaces, when does a twisting matrix Θ determine a local system and which local systems are *infinitesimally given* in this way?

Answering this question gives the boundary of the topological situations describable by the infinitesimal computations of this paper in terms of positive degree differential algebras.

Problem 4. — Extend the computations of homotopy and cohomology of the spatial realization (§ 8) to the case where infinite dimensional (infinitesimally given) representations occur.

Problem 4 leads one to infinite dimensional Lie algebras and pseudo-groups and the significance of the realization. The first example is the Lie algebra $\mathscr{V}_q$ of formal vector fields on $\mathbf{R}^q$. The realization $\langle v^q \rangle$ of the (continuous) dual Lie algebra v^q receives

[25] For example, Kaehler manifolds (§ 12).

a map from the classifying space of codimension q-foliations with trivial normal bundle $B\overline{\Gamma}(q) \xrightarrow{j_\infty} \langle v^q \rangle$.

Question 4. — Is j_∞ a homotopy equivalence?

Problem 5. — Study the homotopy type of the realization of other (geometric) Lie algebras (infinite dimensional).

Knots and Links. — Models should be applicable to the study of invariants of classical knots and links.

For links the nilpotent structure of the complement is already very interesting (as seen in the work of Fox, Chen, and Milnor) and nilpotent models are useful here — since they follow the structure of the lower central series.

For knots the nilpotent model is trivial (since the complement is a homology circle) but the solvable models can be used to study the derived series.

These examples are interesting in a reverse direction because they allow one to " see " geometrically in a three dimensional space rather arbitrary topological complexity expressed by elaborate models.

General π_1-theory

If we were willing to allow terms in degree zero in the algebraic picture, then a differential form description of completely general spaces is possible. One begins with some algebra $\mathscr{A}(\pi_1)$ for the $K(\pi_1, 1)$ (intuitively the forms on the covering space) on which π_1 acts and which is regular in an appropriate sense. Then one forms linear extensions to follow any Postnikov system with this π_1. The discussion is essentially the equivariant form of the simply connected one and is more general but less powerful than the infinitesimal models described above.

Commutative algebra and the Hsiang-Allday problem

Finally, for the deeper calculations and applications we might ask what is the nature of the mathematical structure presented by a minimal differential algebra from the point of view of algebraic geometry.

For example let $y_1, y_2, \ldots$ denote the exterior generators of a nilpotent differential algebra $\mathscr{A}$ and write $dy_k = P_k(x_1, x_2, \ldots)$ mod. ideal $(y_1, y_2, \ldots)$ where the $x_1, x_2, \ldots$ are the polynomial generators. By introducing the P_k as relations in the polynomial algebra (dimension by dimension) one obtains a sequence of commutative rings $A_1 \to A_2 \to \ldots$ which is an invariant of $\mathscr{A}$.

Similarly, one has a sequence of quotient d.g.a's obtained by setting (y_i, dy_i) equal to zero (dimension by dimension) $\mathscr{A} \to \mathscr{A}_1 \to \mathscr{A}_2 \to \ldots$ The limiting cohomology (which is stable by dimension) is the limit of the algebras A_i.

One interesting remark is that if the P_k (for one dimension) are not zero divisors then $H^*\mathscr{A}_k \sim H^*\mathscr{A}_{k+1}$ (Proposition 2, [S-V]).

These considerations are related to various properties relating growth of homotopy and cohomology. For example (see [H-V]).

Question 5. — If the nilpotent algebra $\mathscr{A}$ had k exterior generators and more than k polynomial generators, is $H^*\mathscr{A}$ infinite dimensional?

Given the homotopy theory above this question contains the topological one expressed by W. Y. Hsiang: If the total rank of the homotopy of a simply connected finite complex is finite, is the total odd rank at least as great as the total even rank? In other words is the homotopy Euler characteristic X_π always nonpositive when it is defined. (Hsiang's corollary would be that a compact Lie group G of rank r cannot act with only finite isotropy on a space satisfying $|X_\pi|<r$. One sees this by looking at the rational fibration $G \to X \to X/G$.)

Added remark. — A paper by Steve Halperin has recently appeared which solves Question 5 and other related ones in a most satisfactory way. Also see the work of Allday.

Problem 6. — Study the general relationship between the homotopy and the cohomology of a nilpotent differential algebra. For example, what is the significance of the algebraic properties of the canonical algebra of cycles — a curious homotopy invariant?

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