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Algebraic K-theory and crystalline cohomology

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ALGEBRAIC K-THEORY AND CRYSTALLINE COHOMOLOGY

by Spencer BLOCH [1]

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INTRODUCTION

Let X be a smooth, projective variety defined over a perfect field k of characteristic $p \neq 0$. Let $W(k)$ be the ring of Witt vectors on k, $\bar{k}$ an algebraic closure of k, $X_{\bar{k}}$ the pull-back of X to $\bar{k}$. Among the interesting " cohomological invariants " associated to X on has:

a) The ℓ-adic (étale) cohomology $H^*_{\text{ét}}(X_{\bar{k}}, \mathbf{Z}/\ell^\nu\mathbf{Z})$ for $\ell \neq p$.

b) The p-adic (crystalline) cohomology $H^*_{\text{cris}}(X/W(k))$.

c) The Chow ring $CH^*(X)$ of algebraic cycles modulo rational equivalence.

These objects are not unrelated. For example, one has:

$$H^1_{\text{ét}}(X_{\bar{k}}, \mathbf{Z}/\ell^\nu\mathbf{Z}) \cong \{\ell^\nu\text{-torsion points in } \mathrm{Pic}(X_{\bar{k}}/\bar{k})\}(-1)$$

$$H^1_{\text{cris}}(X/W(\bar{k})) \cong \text{(Covariant) Dieudonné module of the } p\text{-divisible group associated to } \mathrm{Pic}(X_{\bar{k}}/\bar{k})$$

$$CH^1(X) \cong \{\bar{k}\text{-points in } \mathrm{Pic}(X_{\bar{k}}/\bar{k})\},$$

where $\mathrm{Pic}(X_{\bar{k}}/\bar{k})$ denotes the Picard scheme of X pulled back to the algebraic closure $\bar{k}$ of k. Various results make it unlikely that group schemes exist in general playing the role of the Picard scheme for degree >1. Barring that, one could ask whether there exist abelian group-valued functors defined on the category $(\mathrm{sch}/\bar{k})$ (or perhaps (Artinian, pointed $\mathrm{sch}/\bar{k}$)) whose closed points (resp. ℓ-torsion points, resp. Dieudonné module) compute $CH^*(X)$ (resp. $H^*_{\text{ét}}(X, \mathbf{Z}/\ell^\nu\mathbf{Z})$, $H^*_{\text{cris}}(X/W)$).

My hope is that the algebraic K-functors described by Quillen [24] will do the job. More precisely, let $\mathscr{K}_j$ denote the Zariski sheaf on X associated to the presheaf:

$$U \mapsto K_j(\Gamma(U, \mathcal{O}_X)).$$

One knows (theorem of Quillen [24], cf. also [14], [5]) that $H^j(X, \mathscr{K}_j) \cong CH^j(X)$. The purpose of this paper is to establish a similar relation between the " Dieudonné modules " of the functor $H^i(X, \mathscr{K}_j)$ and the piece of $H^{i+j-1}_{\text{cris}}(X/W)$ with " slopes " s satisfying $j-1 \leq s < j$. For reasons related to the present, far from complete, understanding of the K groups, we are forced to assume for much of the discussion that $\dim X < p = \mathrm{char}.k$ and $p \neq 2$. Also we will have nothing to say about relations between K-theory and ℓ-adic cohomology.

Given a functor:

$$F : (R\text{-algebras}) \to (\text{ab.grps.})$$

for a commutative ring R, one defines the *curves of length n on* F, C_nF, by:

$$C_nF = \mathrm{Ker}(F(R[T]/(T^{n+1})) \to F(R)).$$

When the functor F comes equipped with a suitable transfer map structure, C_nF is in a natural way a module over the ring Big $W_n(R)$ of all Witt vectors of length n over R. Moreover, the pro-system $\{C_nF\}_{n\geq 1}$ has endomorphisms f_m, v_m for all $m\geq 1$ satisfying various familiar identities [8]. If, in addition, R is a $\mathbf{Z}_{(p)}$-algebra for some p (*i.e.* all numbers ℓ with $(\ell, p)=1$ are invertible in R), then Big W(R) is isomorphic to a product of copies of the p-Witt vectors W(R). The corresponding decomposition on $C_{p^n}F$ yields the *typical curves* of length n on F, TC_nF, together with endomorphisms $f=f_p$ and $v=v_p$. If R is a perfect field and F is pro-represented by a p-divisible group over R, Cartier has shown that $TCF^{\wedge}=\varprojlim_n TC_nF$ is the covariant Dieudonné module of F.

We want to study the typical curves of the functors F_j^i defined on the category of k-algebras by:

$$F_j^i(A)=H^i(X_k\times \mathrm{Spec}(A), \mathscr{K}_j).$$

A crucial observation, due to Katz, is that the process of taking typical curves commutes with passage to cohomology, *i.e.*:

$$TC_nF_j^i=H^i(X, TC_n\mathscr{K}_j),$$

where $TC_n\mathscr{K}_j$ denotes the *sheaf* of typical curves on K_j (actually our $TC_n\mathscr{K}_j$ will denote the part generated by "symbols". It seems likely that this is the whole thing—one doesn't know). The interest is that the sheaves $TC_n\mathscr{K}_j$ form a complex ($d=\dim_k X$):

$$TC_n\mathscr{K}_1\xrightarrow{\delta} TC_n\mathscr{K}_2\to\dots\to TC_n\mathscr{K}_d\to TC_n\mathscr{K}_{d+1}.$$

To simplify notation, write:

$$C_n^q=TC_n\mathscr{K}_{q+1},\qquad C^q=\text{the pro-system of sheaves } \{TC_n\mathscr{K}_{q+1}\}_{n\geq 1}.$$

The principal results are:

Theorem (**0.1**). — *Let* X *be as above. Assume characteristic* $k=p\neq 0,2$ *and* $\dim_k X<p$. *Then the complex* $C_n^\bullet$ *mentioned above has the following properties:*

(i) $C_n^0=W_n$, *the sheaf of Witt vectors of length n studied by Serre* [25]. $C_n^q=(0)$ *for* $q>\dim_k X$ *and* $q<0$.

(ii) *Each* C_n^q *is a module over* W_n, *and the differentials* $\delta^q: C_n^q\to C_n^{q+1}$ *are* $W_n(k)$*-linear,* δ^0 *is a* W_n*-derivation.*

(iii) *Each pro-system* $C^q=\{C_n^q\}_{n\geq 1}$ *has endomorphisms* F, V *with* FV=VF=p. F *and* V *have the expected linearity properties with regard to the* W*-module structure. The diagrams:*

$$\begin{array}{ccc} C^q & \xrightarrow{\delta} & C^{q+1}\\ \downarrow{\scriptstyle pV} & & \downarrow{\scriptstyle V}\\ C^q & \xrightarrow{\delta} & C^{q+1}\end{array}\qquad\qquad \begin{array}{ccc} C^q & \xrightarrow{\delta} & C^{q+1}\\ \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle pF}\\ C^q & \xrightarrow{\delta} & C^{q+1}\end{array}$$

are commutative. In particular, there are endomorphisms $\mathscr{F}$ *and* $\mathscr{V}$ *of* $C^{\bullet}$ *given, respectively, by* $p^q F$ *and* $p^{\dim X - q} V$ *on* C^q.

(iv) *Each* C_n^r *is built up by a finite number of successive extensions of coherent sheaves. In particular,* $H^*(X, C_n^q)$ *is a* $W(k)$*-module of finite length for all* q, n, *and* $H^*(U, C_n^q) = (0)$ *for* $U \subset X$ *affine and* $* > 0$.

(v) *For any* n, *there is a canonical quasi-isomorphism of complexes* $C_n^{\bullet}/pC_n^{\bullet} \to \Omega_X^{\bullet}$, *where* $\Omega_X^{\bullet}$ *denotes the deRham complex.*

(vi) *Let* $H^*_{\mathrm{cris}}(X/W) = \varprojlim H^*_{\mathrm{cris}}(X/W_n)$ *be the* crystalline cohomology ([2], [3]) *of* $X/W(k)$, *and write* $H^*(X, C^{\bullet}) = \varprojlim_n H^*(X, C_n^{\bullet})$.

There exists a canonical isomorphism:

$$H^*_{\mathrm{cris}}(X/W) \overset{\rightarrow}{\approx} H^*(X, C^{\bullet}).$$

Under this isomorphism, the action of frobenius on $H^*_{\mathrm{cris}}(X/W)$ *is carried over to the action of the endomorphism* $\mathscr{F}$ *described in* (iii) *on* $H^*(X, C^{\bullet})$.

Using (vi) above and a standard " Mittag-Loeffler " argument, one gets a spectral sequence:

$$E_1^{s,t} = H^t(X, C^s) \Rightarrow H^{s+t}_{\mathrm{cris}}(X/W)$$

which I call the *slope spectral sequence.*

Theorem **(0.2)**. — *Let* X *be as in* (0.1), *and let* $\mathrm{Slope}^{\bullet} H^*_{\mathrm{cris}}$ *denote the filtration on* $H^*_{\mathrm{cris}}(X/W)$ *induced by the slope spectral sequence. Let* f *denote the frobenius endomorphism. The filtration* $\mathrm{Slope}^{\bullet} H^*_{\mathrm{cris}}$ *is stable under* f, *and we have*:

(i) *The action of* f *on* $\mathrm{Slope}^q H^*_{\mathrm{cris}}/p$-torsion *is divisible by* p^q.

(ii) $\mathrm{Slope}^q H^*_{\mathrm{cris}} \otimes \mathbf{Q}$ *is the greatest* f*-stable subspace of* $H^*_{\mathrm{cris}} \otimes \mathbf{Q}$ *on which the slopes are* $\geq q$.

Theorem **(0.3)**. — *Let* X *be smooth and proper over a perfect field* k *of characteristic* $p \neq 0, 2$, *and let* $s, t \geq 0$ *be integers with* $t < p$. *Then the group* $H^s(X, C^t)/p$-torsion *is a finitely generated* $W(k)$*-module. The endomorphisms* F, V *described in* (0.1) (iii) *give this group the structure of a Dieudonné module.*

Remark **(0.4)**. — One would like for $H^s(X, C^t)$ to be finitely generated over $W(k)[[V]]$. This is the case when $t = 0$, but *not* in general for $t > 0$.

Theorem **(0.5)**. — *Let* X *be as in* (0.1). *Then the slope spectral sequence degenerates up to torsion at* E_1.

The following application of these results was suggested by work of J. Milne.

Theorem **(0.6)**. — *Let* X *be as in* (0.1). *Write*:

$$H^*_{\mathrm{flat}}(X, \mathbf{Q}_p(1)) \underset{\text{déf.}}{=} (\varprojlim_n H^r_{\mathrm{flat}}(X, \mu_{p^n})) \otimes \mathbf{Q},$$

where H^r_{flat} *denotes cohomology in the flat topology, and* $\mu_{p^n} = \mathrm{Ker}(G_m \xrightarrow{p^n} G_m)$. *Then*:

$$H^r_{flat}(X, \mathbf{Q}_p(1)) \cong H^r_{cris}(X/W)^{(f=p)}_{\mathbf{Q}} = \{\alpha \in H^r_{cris} \otimes \mathbf{Q} \mid f\alpha = p\alpha\}.$$

All the above results are contained in section III. Section II exposes the rather elaborate algebraic structure on the $TC\mathscr{K}_j$ (differentials, transfer maps, W-module structure, filtrations, etc.). The key result here is the following (II.8.2.4) (this notation means II, § 8 (2.4)):

Theorem **(0.7)**. — *Let* R *be a local ring which is smooth (essentiellement lisse in the sense of* EGA) *over a perfect field* k *of characteristic* $p \neq 0,2$. *Let* $n, q \geq 1$ *be integers, and suppose* $q \leq p$. *Define* $T\Phi_n K_q(R)$ *by the exact sequence*:

$$0 \to T\Phi_n K_q(R) \to TC_n K_q(R) \to TC_{n-1} K_q(R) \to 0.$$

Then there is an exact sequence:

$$0 \to \Omega^{q-1}_R / D^{q-1}_n \to T\Phi_n K_q(R) \to \Omega^{q-2}_R / E^{q-2}_n \to 0,$$

where $\Omega^{\bullet}_R$ *denotes the exterior algebra on the Kähler differentials* Ω^1_R, *and* D^q_n, $E^q_n \subset \Omega^q_R$ *are defined by the* inverse Cartier sequence:

$$0 \to \Omega^q_R \xrightarrow{C^{-n}} \Omega^q_R / D^q_n \to \Omega^q_R / E^q_n \to 0.$$

Section I is devoted to necessary preliminaries concerning K-theory, Witt vectors, Chern classes of group representations, and crystalline cohomology. In particular, in (I.3.2.3) we correct a mistake in [6]. For expository accounts of a number of these topics, the reader is refered to the exposés of Berthelot and Illusie [3 *bis*], [21 *bis*].

I am indebted to B. Mazur for many helpful discussions on these and related subjects. Certainly the idea of applying typical curves to K-theory was his. I want to thank M. Stein and K. Dennis for their assistance in calculating the symbols. The key ideas involving crystalline cohomology arose either in a course taught by P. Berthelot at Princeton in 1973-74 or in private conversations with him. It is a pleasure to acknowledge his aid and inspiration. Finally, I want to thank the referee for an extremely careful and thorough job. Author (and reader) are in his debt.

I. — PRELIMINARIES

1. PRELIMINARIES ON WITT VECTORS

1. References for this section are [8], [12], [31]. Let R be a commutative ring with 1. The group W(R) of Witt vectors is defined by:

$$W(R) = (1 + TR[[T]])^{\times}.$$

Given $P(T) \in 1 + TR[[T]]$, we write $\omega(P)$ to denote the corresponding element of W(R). The group structure on W(R) will be written additively:

$$\omega(P.Q) = \omega(P) + \omega(Q).$$

Any $P \in 1 + TR[[T]]$ can be written uniquely as a product:

$$P(T) = \prod_{n \geq 1} (1 - a_n T^n)^{-1}, \qquad a_n \in R$$

and the elements $(a_1, a_2, \ldots)$ are called the Witt coordinates of $\omega(P)$.

W(R) has a canonical descending filtration:

$$\mathrm{Filt}^n W(R) = (1 + T^{n+1} R[[T]])^{\times}.$$

We write:

$$W_n(R) = W(R)/\mathrm{Filt}^n W(R) \cong (1 + TR_n)^{\times}.$$

Elements in $W_n(R)$ correspond *via* the Witt coordinate map to n-tuples $(a_1, \ldots, a_n)$ of elements of R, and:

$$W(R) \cong \varprojlim W_n(R),$$

i.e. W(R) is separated and complete in the topology induced by the Filt^n.

Proposition (**1.1**). — *There exists a unique structure of commutative ring on* W(R) *such that*:

$$\omega((1 - aT^m)^{-1}) . \omega((1 - bT^n)^{-1}) = \omega((1 - a^{n/r} b^{m/r} T^{mn/r})^{-r})$$

where $r = \mathrm{g.c.d.}(m, n)$. $1 \in W(R)$ *is represented by the power series* $(1 - T)^{-1}$.

Proof. — If $\omega \in W(R)$ has Witt coordinates $(a_1, a_2, \ldots)$ we have:

$$\omega = \sum_{n=1}^{\infty} \omega((1 - a_n T^n)^{-1}).$$

Note the infinite sum makes sense because of the topology on W(R), and it is canonical. Since:

$$\omega((1 - aT^m)^{-1}) . \omega((1 - bT^n)^{-1}) \in \mathrm{Filt}^{\max(m, n)} W(R)$$

we can extend the product to all pairs $\omega, \omega' \in W(R)$ by bilinearity.

If $R \to R'$ is an injection (resp. surjection) of rings, the induced map $W(R) \to W(R')$ is also injective (resp. surjective). This is useful, for example, in checking the above product satisfies the distributive laws. Using surjectivity one reduces to verifying the identity when R=polynomial ring/$\mathbf{Z}$. Then, using injectivity one replaces R by the algebraic closure of its quotient field. Now we can write $1 - aT^n = \prod_{\zeta^n=1} (1-\alpha\zeta T)$, and the problem reduces to showing:

$$\omega((1-aT)^{-1})(\omega((1-bT)^{-1})+\omega((1-cT)^{-1})) \\ = \omega((1-abT)^{-1})+\omega((1-acT)^{-1}).$$

This last identity follows easily from the observation that:

$$\omega((1-aT)^{-1}).\,\omega(P(T)) = \omega(P(aT)). \qquad \text{Q.E.D}$$

Note the filtration $\operatorname{Filt}^{\bullet}W(R)$ is not a ring filtration, *i.e.* $\operatorname{Filt}^n.\operatorname{Filt}^m \not\subset \operatorname{Filt}^{n+m}$ in general.

2. *Definition* **(2.1).**

For any $n \geq 1$, let $V_n : W(R) \to W(R)$ be the map $V_n\omega(P) = \omega(P(T^n))$.

Notice the map $\varphi_n : R[[T]] \to R[[T]]$, $T \mapsto T^n$ makes $R[[T]]$ a free module of rank n over itself. Thus there is defined a norm map $\varphi_{n_*} : R[[T]]^\times \to R[[T]]^\times$.

Definition **(2.2).** — Let $F_n : W(R) \to W(R)$ be the map induced by φ_{n_*}. We have:

$$F_n\omega(P(T)) = \sum_{\zeta^n=1} \omega(P(\zeta T^{1/n})).$$

Proposition **(2.3).** — (i) $V_n\omega((1-aT^m)^{-1}) = \omega((1-aT^{mn})^{-1})$.
(ii) $F_n\omega((1-aT^m)^{-1}) = \omega((1-a^{n/r}T^{m/r})^{-r})$, $r = \text{g.c.d.}(m, n)$.
(iii) $F_n \circ V_n$ = multiplication by n.
(iv) *If* $(m, n) = 1$, $V_n \circ F_m = F_m \circ V_n$.
(v) *If* $n \in \mathbf{Z}$ *is a unit in* R, *then* n *is a unit in* W(R).
(vi) *If* R *is a* $\mathbf{Z}/p\mathbf{Z}$*-algebra, then* $V_p \circ F_p$ = multiplication by p.
(vii) $F_n \circ F_m = F_{mn}$, $V_n \circ V_m = V_{nm}$.

Proof. — (i) and (ii) follow from the definitions. (iii) is a standard property of the norm map. (iv) and (vi) follow from (i) and (ii), and (v) follows from the binomial theorem, *i.e.* $(1-T)^{1/N} \in W(\mathbf{Z}[1/N])$. Q.E.D.

Proposition **(2.4).** — (i) $F_n : W(R) \to W(R)$ *is a ring homomorphism.*
(ii) V_n *satisfies the identity* $\omega.V_n(\omega') = V_n(F_n(\omega).\omega')$.
(iii) $V_n(\operatorname{Filt}^m W(R)) \subseteq \operatorname{Filt}^{mn+n-1} W(R)$.
(iv) $F_n(\operatorname{Filt}^{mn} W(R)) \subseteq \operatorname{Filt}^m W(R)$.

Proof. — These assertions follow easily from (2.3).

3. Recall the Möbius function μ is defined by

$$\mu(n)=\begin{cases} 0 & n \text{ contains a square factor} \\ (-1)^r & n=p_1 p_2 \dots p_r, \ \ p_i \text{ distinct primes} \end{cases}$$

Proposition (**3.1**). — *Let* R *be a* $\mathbf{Z}_{(p)}=\mathbf{Z}\left[\frac{1}{n}, n \text{ prime to } p\right]$*-algebra and let* $\mathrm{I}(p)$ *denote the set of integers* ≥ 1 *not divisible by* p. *Let* $\pi=\sum_{n\in \mathrm{I}(p)} \frac{\mu(n)}{n}\mathrm{V}_n\mathrm{F}_n$. *Then* π *is well-defined, and is a projection operator on* W(R). $\pi(\mathrm{W}(\mathrm{R}))=\bigcap_{\substack{n\in \mathrm{I}(p)\\ n>1}} \mathrm{Ker}\, \mathrm{F}_n$.

Proof. — One sees from (2.3) (v) and (2.4) (iii) that π is well-defined. Recall if k, m are relatively prime integers, one has:

$$\sum_{r\mid m} \mu(kr)=0.$$

Given $m\in \mathrm{I}(p)$, $m>1$, we compute:

$$\begin{aligned}
\mathrm{F}_m\circ\pi &= \sum_{n\in \mathrm{I}(p)} \frac{\mu(n)}{n}\mathrm{F}_m\mathrm{V}_n\mathrm{F}_n \\
&= \sum_{r\in \mathrm{I}(p)} \sum_{\substack{n\in \mathrm{I}(p)\\ (m,n)=r}} \frac{\mu(n)}{n}\mathrm{F}_{m/r}\mathrm{F}_r\mathrm{V}_r\mathrm{V}_{n/r}\mathrm{F}_{n/r}\mathrm{F}_r \\
&= \sum_{r\in \mathrm{I}(p)} \sum_{\substack{n\in \mathrm{I}(p)\\ (m,n)=r}} \frac{r\mu(n)}{n}\mathrm{V}_{n/r}\mathrm{F}_{nm/r} \\
&= \sum_{r\mid m} \sum_{\substack{k\\ (k,m)=1}} \frac{\mu(kr)}{k}\mathrm{V}_k\mathrm{F}_{mk} \\
&= \sum_{\substack{k\\ (k,m)=1}} \frac{1}{k}\mathrm{V}_k\mathrm{F}_{mk}\sum_{r\mid m}\mu(kr)=0.
\end{aligned}$$

The assertions of the proposition follow easily from this. Q.E.D.

Definition (**3.2**). — When R is a $\mathbf{Z}_{(p)}$-algebra, we define $\mathrm{W}^{(p)}(\mathrm{R})=\pi(\mathrm{W}(\mathrm{R}))\subset \mathrm{W}(\mathrm{R})$, where π is the projection operator defined above.

In fact $\mathrm{W}^{(p)}(\mathrm{R})$ is a ring, and $\pi:\mathrm{W}(\mathrm{R})\to \mathrm{W}^{(p)}(\mathrm{R})$ is a ring homomorphism. To see this, it is convenient to introduce the notion of ghost coordinates on W(R).

Definition (**3.3**). — The ghost map $\mathrm{W}(\mathrm{R}) \xrightarrow{\mathrm{gh}} \prod_{\infty} \mathrm{R}$ is defined to be the composite:

$$W(R) \cong (1+TR[[T]])^{\times} \xrightarrow{T\frac{d}{dT}\log} TR[[T]]^{+} \cong \prod_{\infty} R$$

$$P(T) \mapsto \frac{T\,dP/dT}{P}; \qquad a_n T^n \mapsto [0, \ldots, 0, na_n, 0, \ldots]$$

gh is clearly a homomorphism of abelian groups.

Proposition **(3.4)**. — (i) gh *is a ring homomorphism* (*for the product ring structure on* $\prod_{\infty} R$).

(ii) *If* R *has no* **Z**-*torsion*, gh *is injective. If* R *is a* **Q**-*algebra*, gh *is an isomorphism.*

(iii) *Assume* R *is a* $\mathbf{Z}_{(p)}$-*algebra, and let* $\rho : \prod_{\infty} R \to \prod_{\infty} R$ *be the projection operator*:

$$\rho(a_1, a_2, \ldots) = (a_1, 0, \ldots, 0, a_p, 0, \ldots, a_{p^2}, \ldots).$$

Then the ghost map induces a morphism of projection operators gh : $(W(R), \pi) \to (\prod_{\infty} R, \rho)$. *In other words, the diagram*:

$$\begin{array}{ccc} W(R) & \xrightarrow{gh} & \prod_{\infty} R \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \rho} \\ W(R) & \xrightarrow{gh} & \prod_{\infty} R \end{array}$$

commutes.

Proof. — (i) is the sort of universal assertion which it suffices to check for R = algebraically closed field. The polynomials $1+aT^n$ factor into linear factors so it suffices to check:

$$gh(\omega((1-aT)^{-1}).\omega((1-bT)^{-1})) = gh\ \omega((1-aT)^{-1}).gh\ \omega((1-bT)^{-1}).$$

Note $gh(\omega((1-aT)^{-1})) = (a, a^2, a^3, \ldots)$, so the assertion amounts to:

$$(a, a^2, a^3, \ldots)(b, b^2, b^3, \ldots) = (ab, a^2 b^2, \ldots)$$

which is clear. (ii) is also pretty clear.

For (iii), we define operators $\mathscr{V}_n$, $\mathscr{F}_n$ on $\prod_{\infty} R$, $n \in \mathbf{N}$, as follows:

$$\mathscr{V}_n(a_1, a_2, \ldots) = (0, \ldots, 0, na_1, 0, \ldots, 0, na_2, \ldots)$$

na_i appears as in-th coordinate.

$$\mathscr{F}_n(a_1, a_2, \ldots) = (a_n, a_{2n}, \ldots).$$

Using (2.3) (i), (ii), one checks that the diagrams:

$$\begin{array}{ccc} W(R) & \longrightarrow & \prod_{\infty} R \\ \downarrow{\scriptstyle V_n} & & \downarrow{\scriptstyle \mathscr{V}_n} \\ W(R) & \xrightarrow[gh]{} & \prod_{\infty} R \end{array} \qquad \begin{array}{ccc} W(R) & \longrightarrow & \prod_{\infty} R \\ \downarrow{\scriptstyle F_n} & & \downarrow{\scriptstyle \mathscr{F}_n} \\ W(R) & \xrightarrow[gh]{} & \prod_{\infty} R \end{array}$$

commute. Thus one has a commutative square for R a $\mathbf{Z}_{(p)}$-algebra:

$$\begin{array}{ccc} W(R) & \longrightarrow & \prod_\infty R \\ \pi \downarrow & & \downarrow \sum_{n\in I(p)} \frac{\mu(n)}{n} \mathscr{V}_n \circ \mathscr{F}_n \\ W(R) & \longrightarrow & \prod_\infty R \end{array}$$

Let $e_m \in \prod_\infty R$ be the element with m-th coordinate 1 and zeros elsewhere. We have:

$$\mathscr{F}_n(e_m) = \begin{cases} e_{m/n} & n \text{ divides } m \\ 0 & \text{otherwise} \end{cases}$$

from which it follows easily (using $\sum_{d|m} \mu(d) = 0,\ m > 1$) that:

$$\sum_{n \in I(p)} \frac{\mu(n)}{n} \mathscr{V}_n \circ \mathscr{F}_n(e_m) = \begin{cases} e_m & m = \text{power of } p \\ 0 & \text{otherwise} \end{cases}$$

Thus $\sum_{n \in I(p)} \frac{\mu(n)}{n} \mathscr{V}_n \circ \mathscr{F}_n = \rho$, proving (iii).

Proposition **(3.5)**. — *Let* R *be a* $\mathbf{Z}_{(p)}$*-algebra. Then* $W^{(p)}(R)$ *is a subring of* W(R) (*the inclusion* $W^{(p)}(R) \to W(R)$ *is not unital*), *and the projection* $\pi : W(R) \to W^{(p)}(R)$ *is a (unital) ring homomorphism.*

Proof. — By universality, we may assume R has no **Z**-torsion, so the ghost map is injective. We have:

$$\begin{array}{ccccccc} 0 & \longrightarrow & W^{(p)}(R) & \longrightarrow & W(R) & \xrightarrow{1-\pi} & W(R) \\ & & \cap \downarrow \text{gh} & & \cap \downarrow \text{gh} & & \cap \downarrow \text{gh} \\ 0 & \longrightarrow & \operatorname{Ker}(1-\rho) & \longrightarrow & \prod_\infty R & \xrightarrow{1-\rho} & \prod_\infty R \end{array}$$

If $\omega, \omega' \in W^{(p)}(R)$ but $\omega\omega' \notin W^{(p)}(R)$ we would have:

$$0 \neq gh(1-\pi)(\omega\omega') = (1-\rho) gh(\omega) . gh(\omega').$$

One checks easily that $\operatorname{Ker}(1-\rho)$ is a non-unital subring of $\prod_\infty R$, so:

$$(1-\rho) gh(\omega) gh(\omega') = 0,$$

a contradiction. The assertion that π is a ring homomorphism is proved similarly.

Notice that $\pi(1)$ is the unit element in $W^{(p)}(R)$.

Proposition **(3.6)**. — *Let* R *be a* $\mathbf{Z}_{(p)}$*-algebra. There is an isomorphism of rings*:

$$\prod_{n \in I(p)} W^{(p)}(R) \xrightarrow{\sim} W(R)$$

$$(\ldots, \omega_n, \ldots)_{n \in I(p)} \mapsto \sum_{n \in I(p)} \frac{1}{n} V_n(\omega_n)$$

with inverse:

$$\prod_{n \in I(p)} \pi \circ F_n : W(R) \to \prod_{n \in I(p)} W^{(p)}(R).$$

Proof. — Let's check, for example, that for $\omega \in W^{(p)}(R)$:

$$\prod_m \pi \circ F_m\left(\frac{1}{n} V_n(\omega)\right) = (0, \ldots, 0, \omega, 0, \ldots)$$

where ω appears in the n-th place. This is true because $\pi \circ V_m = 0$ for $m \in I(p)$ (use universality and (3.4) (iii)) so $\pi F_m V_n = 0$ unless $m = n$. Q.E.D.

4. Recall the Artin-Hasse exponential

$$\textbf{(4.1)} \qquad E(T) = \exp\left(\sum_{n=0}^{\infty} T^{p^n}/p^n\right) \in \mathbf{Z}_{(p)}[[T]].$$

The fact that E(T) has p-adically integral coefficients follows from the power series expansion:

$$E(T) = \prod_{n \in I(p)} (1 - T^n)^{-\mu(n)/n}.$$

(For more details, see [12], chapter III, § 1.)

Proposition **(4.2)**. — *Let* R *be a* $\mathbf{Z}_{(p)}$*-algebra, and view*:

$$W^{(p)}(R) = \pi(W(R)) \subset W(R) \cong (1 + TR[[T]])^\times.$$

The correspondence:

$$(a_0, a_1, a_2, \ldots) \to \prod_{n=0}^{\infty} E(a_n T^{p^n})$$

maps $\prod_\infty R$ *bijectively onto* $W^{(p)}(R)$ *(this is a bijection of sets, but it is not a homomorphism of groups)*.

The coordinates $(a_0, a_1, \ldots)$ associated to an element $\omega \in W^{(p)}(R)$ are called the *Witt coordinates* of ω. The map $gh : W^{(p)}(R) \to \prod_\infty R$ deduced from the ghost map on W(R) is given in terms of Witt coordinates by:

$$gh(a_0, a_1, \ldots) = (gh_0, gh_1, \ldots)$$

where:

$$gh_n = a_0^{p^n} + p a_1^{p^{n-1}} + \ldots + p^n a_n.$$

Proof. — Let's compute $\pi(\omega((1-aT^m)^{-1}))=\pi\omega$. If $m \neq$ power of p, then $\omega=V_n\omega'$ for some n prime to p so $\pi\omega=\pi V_n\omega'=0$. If $m=p^r$, $\omega=V_p^r\omega((1-aT)^{-1})$ and:

$$\pi\omega=V_p^r\pi(\omega((1-aT)^{-1}))=V_p^r E(aT)=E(aT^{p^r}).$$

The rest of the argument is straightforward and is left for the reader.

Remarks **(4.3).** — (i) When R is a $\mathbf{Z}_{(p)}$-algebra, $V=V_p$ and $F=F_p$ commute with the projection operator π and hence induce endomorphisms V and F of $W^{(p)}(R)$.

(ii) $W^{(p)}(R)$ has a descending filtration $\text{filt}^{\bullet}W^{(p)}(R)$ defined by:

$$\text{filt}^n W^{(p)}(R)=V^n(W^{(p)}(R)).$$

In terms of Witt coordinates:

$$\text{filt}^n W^{(p)}(R)=\{(a_0, a_1, \ldots) \mid a_0=a_1=\ldots=a_{n-1}=0\}.$$

The relation with the filtration $\text{filt}^{\bullet}W(R)$ on " big Witt " is given by:

$$\text{filt}^n W^{(p)}(R)=\pi(\text{filt}^m W(R))$$

for any m, $p^{n-1} \leq m < p^n$. $W_n^{(p)}(R)$ will denote $W^{(p)}(R)/\text{filt}^n W^{(p)}(R)$.

2. PRELIMINARIES ON ALGEBRAIC K-THEORY

1. Throughout this section, R will be a commutative ring with 1, and $q \geq 0$ an integer. Let $K_q(R)$ denote the q-th algebraic K-functor as defined by Quillen ([24], [14]). Actually, the more " naive " description:

(1.1) $$K_q(R)=\pi_q(K_0(R)\times B^+_{GL(R)})$$

will be adequate. Recall that $GL(R)=\varinjlim_n GL_n(R)$ is the infinite general linear group, $B_{GL(R)}$ is its classifying space as in [14], and $B^+_{GL(R)}$ is a certain H-space (in fact an infinite loop space) with the following property:

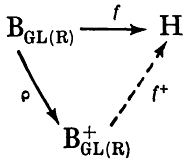
(1.2) There is given a map $\rho : B_{GL(R)} \to B^+_{GL(R)}$ which induces an isomorphism on homology:

$$H_*(B_{GL(R)}, \rho^*\mathscr{F}) \overset{\approx}{\to} H_*(B^+_{GL(R)}, \mathscr{F})$$

for any local system $\mathscr{F}$ on $B^+_{GL(R)}$.

Property (1.2) determines $B^+_{GL(R)}$ (up to homotopy) and it implies (via an obstruction theory argument) the following universal mapping property:

(1.3) Given a map $B_{GL(R)} \xrightarrow{f} H$ where H is an H-space, there exists a map f^+, unique up to homotopy, making the diagram:



commute.

In the cases of interest to us $H = K(A, n)$ will be the Eilenberg-Maclane space associated to some abelian group A (or, more generally, a complex $A^\bullet$ of abelian groups) and an integer $n \geq 1$. In this case, we get:

(1.4) $$\pi_n(f^+) : K_n(R) \to A = \pi_n(K(A, n)).$$

The notation $K_0(R, G)$ for a ring R and a group G will mean the Grothendieck group of representations of G on finitely generated projective R-modules, with relations given by short exact sequences of such. We will need a somewhat stronger universal mapping property for B^+_{GL} than (1.3). The following result is due to Quillen, the statement here being from [14].

Theorem **(1.5)**. — *Let* R *be a ring, and consider the two functors from spaces to abelian groups*:

$$X \mapsto K_0(R, \pi_1(X))$$
$$X \mapsto [X, B^+_{GL(R)}].$$

There is a morphism of functors:

$$\eta : K_0(R, \pi_1(\cdot)) \to [\cdot, B^+_{GL(R)}]$$

which has the following universal property: *given any morphism of functors*:

$$\xi : K_0(R, \pi_1(\cdot)) \to [\cdot, H]$$

where H *is an* H-*space*, *there exists a unique* ψ_ξ *making the diagram below commute*:

$$\begin{array}{ccc} K_0(R, \pi_1(\cdot)) & \xrightarrow{\ \xi\ } & [\cdot, H] \\ & {}_{\eta}\searrow \qquad \nearrow^{\psi_\xi} & \\ & [\cdot, B^+_{GL(R)}] & \end{array}$$

Corollary **(1.6)**. — *There is a canonical ring homomorphism*:

$$u : \mathrm{End}_{\mathrm{functorial}}(K_0(R, \cdot)) \to [B^+_{GL(R)}, B^+_{GL(R)}].$$

Proof of **(1.6)**. — Given $\alpha \in \mathrm{End}_{\mathrm{functorial}}(K_0(R, \pi_1(\cdot)))$, we get:

$$\begin{array}{ccc} K_0(R, \pi_1(\cdot)) & \xrightarrow{\ \alpha\ } & K_0(R, \pi_1(\cdot)) \\ \Big\downarrow{\scriptstyle \eta} & & \Big\downarrow{\scriptstyle \eta} \\ [\cdot, B^+_{GL(R)}] & \xrightarrow{\ \psi_{\eta\circ\alpha}\ } & [\cdot, B^+_{GL(R)}] \end{array}$$

Taking $\cdot = B^+_{GL(R)}$, define:

$$u(\alpha) = \psi_{\eta\circ\alpha}(\mathrm{Identity}) \in [B^+_{GL(R)}, B^+_{GL(R)}].$$

The fact that u is a ring homomorphism is straightforward using the uniqueness and functoriality of ψ. Q.E.D.

2. We will use repeatedly the fact that $\bigoplus_{q\geq 0} K_q(R)$ has a natural, associative, graded-commutative, ring structure induced by tensor product (R commutative with 1). To my knowledge, two constructions of the multiplication have appeared in print (Gersten [15] and Loday [32]) but no one has published a verification that the two structures coincide! Loday checks that his ring structure coincides up to sign with the pairing:

$$K_1(R)\times K_1(R)\to K_2(R)$$

defined by Milnor [30]. As this is a point of considerable import for us, we will adopt his definition, sketched below:

Step 1. — Tensor product defines a map:

$$\psi : B_{GL(R)}\times B_{GL(R)}\to B^+_{GL(R)}.$$

Indeed, a map $X\to B_{GL_n(R)}$ is given up to homotopy by a representation $\pi_1(X)\to GL_n(R)$. Let:

$$\rho_{n,m} : GL_n(R)\times GL_m(R)\to GL_{mn}(R)$$

denote the tensor product of the projection representations. Let:

$$\rho_{n,(m)} \text{ (resp. } \rho_{(n),m}) : GL_n(R)\times GL_m(R)\to GL_{mn}(R)$$

denote the tensor product of projection on the first factor (resp. second factor) with the trivial representation on $R^{\oplus m}$ (resp. $R^{\oplus n}$). Let $\rho_{(n),(m)}$ denote the tensor product of the trivial representations on $R^{\oplus n}$ and $R^{\oplus m}$. Let $\varphi_{n,m}$, $\varphi_{n,(m)}$, $\varphi_{(n),m}$, $\varphi_{(n),(m)}$, denote the composite maps:

$$B_{GL_n(R)}\times B_{GL_m(R)} \xrightarrow[\begin{pmatrix}\rho_{n,m}\\ \rho_{n,(m)}\\ \rho_{(n),m}\\ \rho_{(n),(m)}\end{pmatrix}]{} B_{GL_{mn}(R)}\longrightarrow B_{GL(R)}\longrightarrow B^+_{GL(R)}.$$

Since $B^+_{GL(R)}$ is an H-space, we can subtract homotopy classes of maps and define:

$$\psi_{n,m}=\varphi_{n,m}-\varphi_{(n),m}-\varphi_{n,(m)}+\varphi_{(n),(m)} : B_{GL_n(R)}\times B_{GL_m(R)}\to B^+_{GL(R)}.$$

The diagram:

$$\begin{array}{ccc} B_{GL_n(R)}\times B_{GL_m(R)} & \xrightarrow{\psi_{n,m}} & B^+_{GL(R)} \\ \downarrow & \nearrow_{\psi_{n+1,m+1}} & \\ B_{GL_{n+1}(R)}\times B_{GL_{m+1}(R)} & & \end{array}$$

commutes up to homotopy so we can stabilize to get:

$$\psi : B_{GL(R)}\times B_{GL(R)}\to B^+_{GL(R)}.$$

Step 2. Lemma **(2.1)**. — $(B_{GL(R)}\times B_{GL(R)})^+\simeq B^+_{GL(R)}\times B^+_{GL(R)}$.

Proof. — $B^+_{GL(R)}\times B^+_{GL(R)}$ is an H-space, and the map:

$$B_{GL(R)}\times B_{GL(R)}\to B^+_{GL(R)}\times B^+_{GL(R)}$$

is an isomorphism on homology. Q.E.D.

Step 3. — The map ψ above induces a map:

$$B^+_{GL(R)} \wedge B^+_{GL(R)} \to B^+_{GL(R)}.$$

Indeed, by (2.1) we can extend ψ to a map:

$$\mu : B^+_{GL(R)} \times B^+_{GL(R)} \to B^+_{GL(R)}.$$

The map $\psi_{n,m}$ is homotopically trivial on $B_{GL_n(R)} \times \{pt.\}$ and $\{pt.\} \times B_{GL_m(R)}$, so ψ is trivial on $B_{GL(R)} \times \{pt.\}$ and $\{pt.\} \times B_{GL(R)}$, whence μ factors:

(2.2) $$\mu : B^+_{GL(R)} \wedge B^+_{GL(R)} \to B^+_{GL(R)}.$$

This gives the ring structure on the higher K's. To get K_0, we use the fact that $K_0(R)$ acts on $B^+_{GL(R)}$ by tensor product (theorem of Quillen). Thus if we identify:

$$B^+_{GL(R)} = \{1\} \times B^+_{GL(R)} \subset K_0(R) \times B^+_{GL(R)},$$

μ extends to:

$$\mu : (K_0(R) \times B^+_{GL(R)}) \times (K_0(R) \times B^+_{GL(R)}) \to K_0(R) \times B^+_{GL(R)}.$$

3. Recall that $K_1(R) \supset R^*$, the group of units in R. The ring structure on K_* gives a map:

(3.1) $$\underbrace{R^* \otimes_{\mathbf{Z}} \ldots \otimes_{\mathbf{Z}} R^*}_{n \text{ times}} \to K_n(R).$$

When $n=2$, a map (the *symbol* map):

$$R^* \otimes_{\mathbf{Z}} R^* \to K_2(R)$$
$$r \otimes r' \mapsto \{r, r'\}$$

has been defined by Milnor [30]. In fact, Milnor defines a map $K_1(R) \otimes K_1(R) \to K_2(R)$.

Lemma **(3.2)**. — *The Milnor map* $K_1(R) \otimes K_1(R) \to K_2(R)$ *coincides with the multiplication defined by Loday up to a factor of* -1.

Proof. — Loday [32], p. 43, proposition (2.2.3).

We will frequently ignore the sign problem and use symbol notation for the multiplication $R^* \otimes \ldots \otimes R^* \to K_n(R)$, $r_1 \otimes \ldots \otimes r_n \mapsto \{r_1, \ldots, r_n\}$.

The following result is due to M. Stein [26].

Theorem **(3.3)**. — (i) *Assume* R *is semi-local and is additively generated by units* (e.g., R *local*). *Then* $K_2(R)$ *is generated by symbols.*

(ii) *Assume* R *local and let* $I \subset \operatorname{Rad} R$ *be an ideal. Then*:

$$K_2(R, I) \underset{\text{def.}}{=} \operatorname{Ker}(K_2(R) \to K_2(R/I))$$

is generated by symbols $\{1+i, u\}$, $i \in I$, $u \in R^*$.

4. The K functors are, of course, covariant for homomorphisms of rings; $f: \mathrm{R} \to \mathrm{S}$ induces $f^*: \mathrm{K}_*(\mathrm{R}) \to \mathrm{K}_*(\mathrm{S})$. If S admits a finite resolution by finitely generated projective R-modules, there is a transfer map:

$$f_*: \mathrm{K}_*(\mathrm{S}) \to \mathrm{K}_*(\mathrm{R})$$

([24], p. 111) which preserves degrees and satisfies a projection formula (at least for the Gersten product structure, (Gersten [15], (4.19))).

Theorem **(4.1)**. — *Let* $f: \mathrm{R} \to \mathrm{S}$ *be a ring homomorphism and assume* S *admits a finite resolution by finitely generated projective* R*-modules. Let* $r \in \mathrm{K}_i(\mathrm{R})$, $s \in \mathrm{K}_j(\mathrm{S})$. *Then*:

$$f_*(s.f^*(r)) = f_*(s).r.$$

Since we have distinguished between the Gersten and Loday product structures, we are morally obliged to at least sketch a proof valid in the Loday context. Happily, we will only need the theorem for S a projective R-module of finite type. With this restriction, one can proceed as follows: there are two maps which must be shown to be equal:

$$\alpha_1: \mathrm{B}^+_{\mathrm{GL(R)}} \wedge \mathrm{B}^+_{\mathrm{GL(S)}} \xrightarrow{f^* \times 1} \mathrm{B}^+_{\mathrm{GL(S)}} \times \mathrm{B}^+_{\mathrm{GL(S)}} \xrightarrow{\mu} \mathrm{B}^+_{\mathrm{GL(S)}} \xrightarrow{f_*} \mathrm{B}^+_{\mathrm{GL(R)}}$$

$$\alpha_2: \mathrm{B}^+_{\mathrm{GL(R)}} \wedge \mathrm{B}^+_{\mathrm{GL(S)}} \xrightarrow{1 \times f_*} \mathrm{B}^+_{\mathrm{GL(R)}} \times \mathrm{B}^+_{\mathrm{GL(R)}} \xrightarrow{\quad\quad \mu \quad\quad} \mathrm{B}^+_{\mathrm{GL(R)}}.$$

Let V_n, $\mathrm{R}^{\oplus n}$ (resp. W_m, $\mathrm{S}^{\oplus m}$) denote the standard and trivial representations of $\mathrm{GL}_n(\mathrm{R})$ (resp. $\mathrm{GL}_m(\mathrm{S})$). We have a morphism of functors:

$$f_*: \mathrm{K}_0(\mathrm{S}, \pi_1(\cdot)) \to \mathrm{K}_0(\mathrm{R}, \pi_1(\cdot)),$$

and hence can define:

$$x_1 = f_*(([\mathrm{V}_n \otimes_{\mathrm{R}} \mathrm{S}] - [\mathrm{R}^{\oplus n} \otimes_{\mathrm{R}} \mathrm{S}]) \otimes_{\mathrm{S}} ([\mathrm{W}] - [\mathrm{S}^{\oplus m}]))$$
$$x_2 = ([\mathrm{V}] - [\mathrm{R}^{\oplus n}]) \otimes_{\mathrm{R}} f_*([\mathrm{W}] - [\mathrm{S}^{\oplus m}])$$

in $\mathrm{K}_0(\mathrm{R}, \pi_1(\mathrm{B}_{\mathrm{GL}_n(\mathrm{R})} \times \mathrm{B}_{\mathrm{GL}_m(\mathrm{S})}))$.

As in (1.5), x_i gives rise to a map:

$$\beta_i: \mathrm{B}_{\mathrm{GL}_n(\mathrm{R})} \times \mathrm{B}_{\mathrm{GL}_m(\mathrm{S})} \to \mathrm{B}^+_{\mathrm{GL(R)}}.$$

One checks easily that this construction stabilizes. Passing to the plus construction, β_i gives rise to the map α_i. Note however that $x_1 = x_2$, so $\alpha_1 = \alpha_2$. Q.E.D.

3. CHERN CLASSES FOR GROUP REPRESENTATIONS

1. Let $\mathrm{M}_* = \mathrm{M}_0 \oplus \mathrm{M}_1 \oplus \ldots$ be a graded-commutative ring (for the application we shall have to consider a situation where each of the M_i is a complex of abelian groups, but let's keep things simple to begin). A *theory of chern classes* for group representations on projective R-modules will be a rule assigning to a representation $\rho: \mathrm{G} \to \mathrm{Aut}_{\mathrm{R}}(\mathrm{P})$

of an abstract group G on a finitely generated projective R-module P chern classes $c_i(\rho) \in H^i(G, M_i)$ (galois cohomology with G acting trivially on M_i), $i > 0$. The *total chern class* $c_t(\rho)$ will be the power series:

$$c_t(\rho) = 1 + c_1(\rho)t + c_2(\rho)t^2 + \dots$$

Note $\bigoplus H^i(G, M_i)$ is a *commutative* ring under cup product, so it makes sense to multiply $c_t(\rho) . c_t(\rho')$. We assume the following axioms:

(1.1) (Functoriality) Given $G' \xrightarrow{f} G \xrightarrow{\rho} \mathrm{Aut}(P)$ we have $c_t(\rho \circ f) = f^* c_t(\rho)$.

(1.2) (Triviality) $c_t(\text{triv.}) = 1$, where triv. : $G \to \mathrm{Aut}(R)$ is the trivial representation.

(1.3) (Additivity) Given $G \xrightarrow{\rho} \mathrm{Aut}(P)$, $G \xrightarrow{\rho'} \mathrm{Aut}(P')$, $c_t(\rho \oplus \rho') = c_t(\rho) . c_t(\rho')$.

(1.4) (Multiplicativity) Given $\rho : G \to \mathrm{Aut}(P)$, $\rho' : G \to \mathrm{Aut}(P')$, we have:

$$c_t(\rho \otimes \rho') = c_t(\rho) \times c_t(\rho'),$$

where $\times$ denotes multiplication in the sense of the λ-ring structure on $\mathbf{Z} \times \bigoplus_i H^i(G, M_i)$ ([17], § 3).

2. Suppose given a theory of chern classes as above. Let $\rho_{n,m} : GL_n(R) \hookrightarrow GL_m(R)$, $m \geq n$ denote the natural maps:

$$(*) \to \begin{pmatrix} * & 0 \\ 0 & I \end{pmatrix}.$$

It follows from the axioms that $c_t(\rho_{n,m})$ is independent of m, so we can write:

$$c_t(\rho_n) = c_t(\rho_{n,m}), \quad \text{where} \quad \rho_n : GL_n(R) \to GL(R).$$

Also $\rho^*_{n,m}(c_t(\rho_m)) = \rho_n$ so we have a class:

$$c_t(\rho) \underset{\text{def.}}{=} \varprojlim c_t(\rho_n) \in \varprojlim_n \prod_i H^i(GL_n(R), M_i)t^i.$$

In sum, there are universal chern classes, denoted $c_i(\mathrm{Id})$, in $\varprojlim H^i(GL_n(R), M_i)$ such that for any $\psi : G \to GL(R)$ factoring through some $GL_n(R)$, we have:

$$c_i(\psi) = \psi^*(c_i(\mathrm{Id})).$$

Consider the product of Eilenberg-Maclane spaces:

$$X = \prod_{n \geq 1} K(M_n, n).$$

The $c_i(\mathrm{Id})$ induce a map $B_{GL(R)} \xrightarrow{c} X$ which can be factored as in (§ 2, (1.3)) to give:

(2.1)
$$C^+ \times \mathrm{rk} : B^+_{GL(R)} \times K_0(R) \to X \times M_0$$
$$\pi_n(C^+) : K_n(R) \to M_n.$$

One might reasonably expect that the map:

$$\pi_*(C^+) : K_*(R) \to M_*$$

is a homomorphism of rings (this expectation led the author to grief in [6]). The truth of the matter is as follows:

Theorem **(2.3).** — *Let* $\varphi = \pi_*(C^+) : K_*(R) \to M_*$ *and let* $a_m \in K_m(R)$. *Then*:

$$\varphi(a_n . a_m) = \frac{-(m+n-1)!}{(m-1)!\,(n-1)!}\varphi(a_n).\varphi(a_m).$$

Corollary **(2.4).** — *Let* $N \geq 2$ *be an integer and suppose* $\frac{1}{n} \in M_0$ *for* $1 \leq n \leq N-1$. *Let*:

$$\varphi'_n = \frac{(-1)^{n-1}}{(n-1)!}\varphi_n : K_n(R) \to M_n, \quad n \leq N.$$

Then $\varphi' = \bigoplus_n \varphi'_n$ *is a ring homomorphism on the truncated rings* $\varphi' : \bigoplus_{n\leq N} K_n(R) \to \bigoplus_{n \leq N} M_n$.

The corollary follows easily from (2.3).

Proof of **(2.3).** — Let $S = \prod_{i \geq 1} H^i(GL(R), M_i)$, $T = \prod_{i\geq 1} H^i(GL(R)\times GL(R), M_i)$. Identify:

$$S = \{0\}\times S \subset \mathbf{Z}\times S$$
$$T = \{0\}\times T \subset \mathbf{Z}\times T.$$

Recall $\mathbf{Z}\times S$ and $\mathbf{Z}\times T$ are λ-rings [17].

The map $C^+ : B^+_{GL(R)} \to X$ corresponds to an element $c \in S$ which we can think of (roughly) as the total chern class of the identity map $GL(R) = GL(R)$. Similarly $C^+ \circ \mu : B^+_{GL(R)} \times B^+_{GL(R)} \to X$ corresponds to an element $b \in T$. Our assumption that the theory of chern classes satisfies the formula for tensor products implies that:

$$b = p_1^*(c).p_2^*(c),$$

multiplication taking place in the ideal T of the λ-ring $\mathbf{Z}\times T$.

Let $S' = \prod_{i\geq 1} H^i(X, M_i)$, $T' = \prod_{i \geq 1} H^i(X\times X, M_i)$. Again $\mathbf{Z}\times S'$, $\mathbf{Z}\times T'$ are λ-rings and the identity map $X = X$ gives a class $C' \in S'$. Let $p_1'^*, p_2'^* : S' \to T'$ be induced from the two projections. The class:

$$b' = p_1'^*(c').p_2'^*(c') \in T' \subset \mathbf{Z}\times T'$$

corresponds to a map $\mu' : X\times X \to X$, and the diagram:

$$\begin{array}{ccc} B^+_{GL(R)} \times B^+_{GL(R)} & \xrightarrow{\mu} & B^+_{GL(R)} \\ \downarrow{\scriptstyle C^+\times C^+} & & \downarrow{\scriptstyle C^+} \\ X\times X & \xrightarrow{\mu'} & X \end{array}$$

commutes (up to homotopy).

Noticet he class b' dies on $X\times\{pt.\}$ and $\{pt.\}\times X$, so μ' factors through a map $\mu' : X\wedge X \to X$. Thus we get a homomorphism:

$$\mu'_* : \pi_n(X)\otimes\pi_m(X) = M_n\otimes M_m \to \pi_{n+m}(X) = M_{n+m}.$$

Everything now follows from:

Lemma **(2.5)**. — *Let* $\alpha_n \in M_n$, $\beta_m \in M_m$. *Then*:

$$\mu'_*(\alpha_n\otimes\beta_m) = \frac{-(n+m-1)!}{(m-1)!\,(n-1)!}\alpha_n\beta_m.$$

Proof. — Think of α_n, β_m as maps:

$$\alpha_n : S^n \to K(M_n, n) \hookrightarrow X$$
$$\beta_m : S^m \to K(M_m, m) \hookrightarrow X.$$

We want to understand the composition:

$$S^{n+m} = S^n\wedge S^m \to K(M_n, n)\wedge K(M_m, m) \to X\wedge X \xrightarrow{\mu'} X.$$

Since the projection $X \to K(M_{n+m}, m+n)$ induces an isomorphism on π_{n+m}, it suffices to understand:

(2.6) $$S^{n+m} \to K(M_n, n)\wedge K(M_m, m) \to X\wedge X \xrightarrow{\mu'} X \to K(M_{n+m}, m+n).$$

Define $\mu'_{m,n}$ by the diagram:

$$\begin{array}{ccc} K(M_n, n)\times K(M_m, m) & \xrightarrow{\mu'_{m,n}} & K(M_{n+m}, m+n) \\ \downarrow & \nearrow_{(2.6)} & \\ K(M_n, n)\wedge K(M_m, m) & & \end{array}$$

Note $\mu'_{m,n}$ determines the arrow in (2.6) because:

$$H^*(K(M_n, n)\wedge K(M_m, m)) \subset H^*(K(M_n, n)\times K(M_m, m)).$$

On the other hand, $\mu'_{m,n}$ is determined by a cohomology class:

$$b'_{m,n} \in H^{n+m}(K(M_n, n)\times K(M_m, m), M_{n+m}).$$

Thinking of $K(M_n, n)\times K(M_m, m) \subset X\times X$, $b'_{n,m}$ is the piece of degree $n+m$ in $b'|_{K(M_n,n)\times K(M_m,m)}$.

Let $\gamma_i^{(1)}, \gamma_i^{(2)} \in H^i(X\times X, M_i)$ correspond to the maps:

$$X\times X \xrightarrow{p_1} X \xrightarrow{\text{projection}} K(M_i, i)$$
$$X\times X \xrightarrow{p_2} X \xrightarrow{\text{projection}} K(M_i, i)$$

and think of $b' = (b'_1, b'_2, \ldots)$ with $b'_n \in H^n(X\times X, M_n)$. Then:

$$b'_n = P_n(\gamma_1^{(1)}, \ldots, \gamma_n^{(1)}; \gamma_1^{(2)}, \ldots, \gamma_n^{(2)})$$

where the P_n are certain universal polynomials associated to the λ-ring structure [3 *ter*]. Since:

$$\gamma_i^{(1)}|_{K(M_n,\, n)\times K(M_m,\, m)} = 0 \qquad i \neq n$$
$$\gamma_j^{(2)}|_{K(M_n,\, n)\times K(M_m,\, m)} = 0 \qquad j \neq m,$$

we get:

$$b'_{n+m}|_{K(M_n,n)\times K(M_m,\, m)} = P_{n+m}(0,\, \ldots,\, \gamma_n^{(1)},\, 0,\, \ldots;\, 0,\, \ldots,\, \gamma_m^{(2)},\, 0,\, \ldots)$$

restricted to $K(M_n, n)\times K(M_m, m)$. By the computation in ([17], (1.18)):

$$P_{n+m}(0,\, \ldots,\, \gamma_n^{(1)},\, \ldots;\, 0,\, \ldots,\, \gamma_m^{(2)},\, \ldots) = \frac{-(n+m-1)!}{(n-1)!\,(m-1)!}\gamma_n^{(1)}\cdot\gamma_m^{(2)}.$$

Standard homotopy theory implies that:

$$\gamma_n^{(1)}\cdot\gamma_m^{(2)}|_{K(M_n,n)\times K(M_m,m)} \in H^{n+m}(K(M_n, n)\times K(M_m, m), M_{n+m})$$

induces a map $K(M_n, n)\wedge K(M_m, m) \xrightarrow{\mu''} K(M_{n+m}, n+m)$ which is ring multiplication $M_n\otimes M_m \to M_{n+m}$ on homotopy. Thus the diagram:

$$\begin{array}{ccc} K(M_n, n)\wedge K(M_m, m) & \xrightarrow{\mu''} & K(M_{n+m}, n+m) \\ & \underset{\mu'}{\searrow} & \Big\downarrow \text{mult. by } \frac{-(m+n-1)!}{(m-1)!\,(n-1)!} \\ & & K(M_{n+m}, n+m) \end{array}$$

commutes. This completes the proof of (2.5). Q.E.D.

3. For our purposes, the principal example of a theory of chern classes for group representations is the Hodge theory ([18], § 6). Given $\rho : G \to GL_n(R)$ we get:

(3.1) $$c_i(\rho) \in H^i(G, \Omega^i_{R/\mathbf{Z}})$$

where $\Omega^*_{R/\mathbf{Z}}$ is the exterior algebra on the R-module of absolute Kähler differentials $\Omega^1_{R/\mathbf{Z}}$. We get corresponding maps:

(3.2) $$d\log :\ K_n(R) \to \Omega^n_{R/\mathbf{Z}}$$

with $d\log(u) = \dfrac{du}{u}$ for $u \in R^* \subset K_1(R)$. From (2.3) above we get:

(3.3) $$d\log\{u_1,\, \ldots,\, u_n\} = (-1)^{n-1}\,(n-1)!\,\frac{du_1}{u_1}\wedge\ldots\wedge\frac{du_n}{u_n}.$$

A key tool in the K-theoretic computations in II will be the theory of crystalline chern classes constructed by Berthelot and Illusie [4]. Some details of this construction are given in § 4 below. At this point let me just outline how the crystalline classes differ from the Hodge classes. We suppose given a $\mathbf{Z}/p\mathbf{Z}$-algebra A and a flat lifting of A to an algebra B over $\mathbf{Z}/p^N\mathbf{Z}$ for some fixed N. Let $\Omega^{\bullet}_B$ denote the de Rham complex of B and let (for given $r \geq 0$) $\mathrm{Fil}^r_{(p)}\Omega^{\bullet}_B$ denote the subcomplex with:

$$(\mathrm{Fil}^r_{(p)}\Omega_B)^\ell = p^{[r-\ell]}\Omega^\ell_B$$

where $[r-\ell] = \min_{n \geq r-\ell}(n - \mathrm{ord}_p\, n!)$ for $r \geq \ell$, and $[r-\ell] = 0$ for $\ell \geq r$. In particular, $\mathrm{Fil}^r_{(p)}\Omega^{\bullet}_B$ contains the " segment ":

$$\ldots \to p\Omega^{r-1}_B \to \Omega^r_B \to \Omega^{r+1}_B \to \ldots$$

Given $\rho : G \to GL_n(A)$, the crystalline chern classes lie in the group hypercohomology of G acting (trivially) on $\mathrm{Fil}^{\bullet}_{(p)}\Omega^{\bullet}_B$:

$$c_r(\rho) \in H^{2r}(G, \mathrm{Fil}^r_{(p)}\Omega^{\bullet}_B).$$

In order to get maps on K-theory we proceed as follows (the author learned the argument below from Illusie, who attributes it to Quillen [21]): Let $K(2r, \mathrm{Fil}^r_{(p)}\Omega^{\bullet}_B)$ denote the Dold-Puppe construction applied to the complex $t_{\leq 0}(\mathrm{Fil}^r_{(p)}\Omega^{\bullet}_B[2r])$. (If $C^{\bullet}$ is a complex, $C^{\bullet}[n]$ denotes the complex $C[n]^r = C^{n+r}$. $t_{\leq 0}C^{\bullet}$ denotes the complex:

$$\to \ldots \to C^{-1} \to Z^0 \to 0 \to \ldots$$

The Dold-Puppe construction is given in [13].) The chern classes give maps:

$$\begin{aligned}
&B^+_{GL(A)} \to K(2r, \mathrm{Fil}^r_{(p)}\Omega^{\bullet}_B) \\
&K_r(A) \xrightarrow{\text{crys-}d\log} \pi_r(K(2r, \mathrm{Fil}^r_{(p)}\Omega^{\bullet}_B)) = H_r(\mathrm{Fil}^r_{(p)}\Omega^{\bullet}_B[2r]) \\
&\qquad = H^{-r}(\mathrm{Fil}^r_{(p)}\Omega^{\bullet}_B[2r]) \\
&\qquad = H^{+r}(\mathrm{Fil}^r_{(p)}\Omega^{\bullet}_B) \\
&\qquad = \Omega^r_{B,\,\mathrm{closed}}/pd\,\Omega^{r-1}_B.
\end{aligned} \tag{3.4}$$

The map:

$$\text{crys-}d\log : A^* \subset K_1(A) \to \Omega^1_{B,\,\mathrm{closed}}/pd\,B$$

is given by:

$$u \mapsto \frac{d\tilde{u}}{\tilde{u}}$$

where $\tilde{u} \in B^*$ is any lifting of u. Note that if $\tilde{u}'$ is another lifting, $\tilde{u} = \tilde{u}'(1+p\delta)$ so:

$$\frac{d\tilde{u}}{\tilde{u}} = \frac{d\tilde{u}'}{\tilde{u}'} + d\log(1+p\delta) \equiv \frac{d\tilde{u}'}{\tilde{u}'} \pmod{pd\,B}.$$

The graded group:

$$\mathbf{Z} \oplus \bigoplus_r \Omega^r_{B,\,\mathrm{closed}}/pd\,\Omega^{r-1}_B$$

forms a ring under wedge product of forms, and we have:

$$\text{crys-}d\log\{u_1, \ldots, u_n\} = (-1)^{-1}(n-1)!\,\frac{d\tilde{u}_1}{\tilde{u}_1} \wedge \ldots \wedge \frac{d\tilde{u}_n}{\tilde{u}_n}. \tag{3.5}$$

4. PRELIMINARIES ON CRYSTALLINE COHOMOLOGY

1. Let X be a smooth, projective variety over a perfect field k of characteristic $p \neq 0$. Let $n \geq 1$ be an integer, $S_n = \mathrm{Spec}(W_n(k))$. In addition let $W_n(\mathcal{O}_X)$ be the Zariski sheaf of rings obtained by taking the p-Witt vectors of length n on $\mathcal{O}_X$. We will see below that $X_n = (X, W_n(\mathcal{O}_X))$ is a scheme of finite type over S_n.

The purpose of this section is to recall (in the briefest possible way) the ideas involved in the crystalline topos $(X/S_n)_{\mathrm{cris}}$ and the crystalline cohomology $H^*_{\mathrm{cris}}(X/S_n)$. In particular, I want to:

a) Construct a map $j_n^* : H^*_{\mathrm{cris}}(X/S_n) \to H^*(X, \Omega^{\bullet}_{X_n/S_n, \gamma})$ which will be the key link between crystalline cohomology and the typical curves on K-theory. (The notation $\Omega^{\bullet}_{X_n/S_n, \gamma}$ means the de Rham complex of X_n/S_n with a certain compatibility with divided powers imposed. See below.)

b) Sketch the construction, due to Berthelot and Illusie, of crystalline chern classes. Indicate, in particular, the slight modification of their argument necessary to get chern classes for group representations on projective modules as discussed in § 3.3.

The reader who does not swing with the crystalline topos might do well to accept *a)* and *b)* above as established and move on to the next section.

2. Recall a *site* is a category with a topology, and a topos is the category of sheaves (of sets) on a site [19]. If we work in a fixed universe $\mathcal{U}$ (*universe* is a notion from set theory the meaning of which the author has never quite mastered, perhaps because he doesn't plan to ever leave the universe he is in now). The notion of $\mathcal{U}$-topos can be given a "coordinate free" definition as follows (*op. cit.*, IV, 1); it is a category E with the properties:

a) E is closed under finite projective limits.

b) Direct sums indexed by an element of $\mathcal{U}$ are representable, disjoint and universal in E.

c) Equivalence relations in E are effective and universal (*op. cit.*, I, (10.10)).

d) E admits a family of generators indexed by an element of $\mathcal{U}$ (*op. cit.*, I, (7.1)).

Here are some examples of topoi:

(2.1) The category of sheaves on a topological space X. That one was just to fix ideas. Of more interest to us will be the following two examples:

(2.2) Let E be a topos, and G a group in E, *i.e.* G is an object of E equipped with a map $\mu : G \times G \to G$ satisfying the usual group laws. A left action of G on an object F of E is a map $\tau : G \times F \to F$ satisfying the usual laws... The category whose objects are objects of E together with a left action of G is a topos, the classifying topos for G, and is denoted B_G. For example, if E is the category of sets (=sheaves on the one point space) and G is an abstract group, then B_G is the category of G-sets.

To carry things one step further, let G be an abstract group and let E be any old topos. A key property of topoi is that they have, themselves, a topology (the canonical topology) and every sheaf for this topology is representable (*i.e.* is of the form $X \mapsto \mathrm{Hom}_E(X, F)$ for some $F \in \mathrm{Ob}E$). Thus one can take the constant presheaf on E with value G, and sheafify it to get a group object $\widetilde{G} \in \mathrm{Ob}E$. The corresponding classifying topos $B_{\widetilde{G}}$ will be denoted B_G/E.

(2.3) The crystalline topos $(X/S_n)_{\mathrm{cris}}$ [3]. This will be the category of sheaves on a site $\mathrm{Cris}(X/S_n)$. An object of $\mathrm{Cris}(X/S_n)$ will be a triple (U, T, δ) with $U \subset X$ a Zariski open set, given as a closed subscheme of an S_n-scheme T:

$$\begin{array}{ccc} U & \overset{\mathscr{I}}{\hookrightarrow} & T \\ & \searrow \quad \swarrow & \\ & S_n & \end{array}$$

The Ideal $\mathscr{I} \subset \mathscr{O}_T$ defining U is equipped with a structure δ of *divided powers.* Roughly speaking, this means we are given maps of sets $\delta^{(m)} : \mathscr{I} \to \mathscr{I}$ for all $m \geq 0$ such that "$\delta^{(m)}(i) = i^m/m!$". In other words the $\delta^{(m)}(i)$ satisfy all the identities one would expect from $i^m/m!$, even though $\frac{1}{m!}$ may not be defined. It follows, for example, that $\mathscr{I}$ is a Nilideal:

$$i^{p^n} = p^n!\, \delta^{(p^n)}(i) = 0 \quad (p^n . \mathscr{O}_T = (0)).$$

Morphisms between objects in $\mathrm{Cris}(X/S_n)$ are commutative squares:

$$\begin{array}{ccc} U & \overset{\mathscr{I}}{\hookrightarrow} & T \\ \downarrow & & \downarrow \\ U' & \overset{\mathscr{I}'}{\hookrightarrow} & T' \end{array}$$

where the morphism $T \to T'$ is assumed compatible with the divided powers on $\mathscr{I}$ and $\mathscr{I}'$.

The most important example of a sheaf on $\mathrm{Cris}(X/S_n)$ (*i.e.* an object in $(X/S_n)_{\mathrm{cris}}$) is the structure sheaf $\mathscr{O}_{X/S_n}$ defined by:

$$\mathscr{O}_{X/S_n}(U, T, \delta) = \Gamma(T, \mathscr{O}_T).$$

$\mathscr{O}_{X/S_n}$ contains an Ideal $\mathscr{I}_{X/S_n}$ defined by:

$$\mathscr{I}_{X/S_n}(U, T, \delta) = \Gamma(T, \mathrm{Ker}(\mathscr{O}_T \to \mathscr{O}_U)).$$

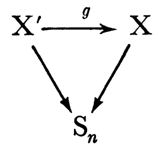
For $r \geq 1$, $\mathscr{I}^{[r]}_{X/S_n} \subset \mathscr{I}_{X/S_n}$ is defined to be the Subideal generated locally by all products of sections $\delta^{(r_1)}(i_1) . \delta^{(r_2)}(i_2) \ldots \delta^{(r_k)}(i_k)$ with $r_1 + \ldots + r_k \geq r$.

3. A morphism of topoi $f: E \to E'$ is a triple (f_*, f^*, φ) where $f_*: E \to E'$ and $f^*: E' \to E$ are functors, and:

$$\varphi : \mathrm{Hom}_E(f^*(X'), Y) \xrightarrow{\sim} \mathrm{Hom}_{E'}(X', f_*(Y))$$

is an isomorphism of bifunctors (in X′ and Y). In other words, the pair (f^*, f_*) are adjoint functors. Finally, we require that the functor f^* commute with finite projective limits. Here are some examples we will need.

(3.1) Given a morphism of schemes:

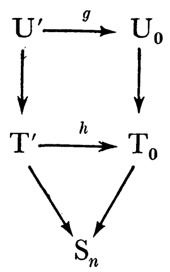


there is a morphism of topoi $g_{\mathrm{cris}} : (X'/S_n)_{\mathrm{cris}} \to (X/S_n)_{\mathrm{cris}}$. The construction of g_{cris} is conveniently described in several steps.

Step 1. — A representable object in $(X/S_n)_{\mathrm{cris}}$ is by definition a sheaf of the form:

$$(U, T, \delta) \mapsto \mathrm{Hom}_{\mathrm{cris}(X/S_n)}((U, T, \delta), (U_0, T_0, \delta_0))$$

for some fixed (U_0, T_0, δ_0). As is customary, we identify such a sheaf with the corresponding object (U_0, T_0, δ_0). The sections of the sheaf $g^*_{\mathrm{cris}}(U_0, T_0, \delta_0)$ on an object $(U', T', \delta') \in \mathrm{Cris}(X'/S_n)$ are by definition the morphisms $h: T' \to T_0$ such that the diagram:



commutes, and such that h is compatible with the divided power structures on $\mathrm{Ker}(\mathcal{O}_{T_0} \to \mathcal{O}_{U_0})$ and $\mathrm{Ker}(\mathcal{O}_{T'} \to \mathcal{O}_{U'})$. If $U' \not\subseteq g^{-1}(U_0)$, the space of sections over (U', T', δ') is empty.

Step 2. — For $F' \in (X'/S_n)_{\mathrm{cris}}$ any sheaf, $g_{\mathrm{cris}*}(F')$ is defined on an object (U, T, δ) by:

$$g_{\mathrm{cris}*}(F')(U, T, \delta) = \mathrm{Hom}_{(X/S_n)\mathrm{cris}}((U, T, \delta), g_{\mathrm{cris}*}(F'))$$
$$= \mathrm{Hom}_{(X'/S_n)\mathrm{cris}}(g^*_{\mathrm{cris}}(U, T, \delta), F').$$

Step 3. — For $F \in (X/S_n)_{\mathrm{cris}}$ any sheaf, one can write

$$F = \varinjlim \text{ (representable sheaves)}$$

and then define

$$g_{\mathrm{cris}*}(\mathrm{F}) = \varinjlim\, (g^*_{\mathrm{cris}}(\text{representable sheaves})).$$

(3.2) Let $(\mathrm{X}/\mathrm{S}_n)_{\mathrm{zar}}$ denote the topos of sheaves for the Zariski topology on X. There is a morphism of topoi $u_{\mathrm{X}/\mathrm{S}_n} : (\mathrm{X}/\mathrm{S}_n)_{\mathrm{cris}} \to (\mathrm{X}/\mathrm{S}_n)_{\mathrm{zar}}$ defined by:

$$u_{\mathrm{X}/\mathrm{S}_n^*}(\mathrm{F})(\mathrm{U}) = \varinjlim_{(\mathrm{U},\mathrm{T},\delta)} \mathrm{F}(\mathrm{U}, \mathrm{T}, \delta).$$

In other words, a section of $u_{\mathrm{X}/\mathrm{S}_n^*}(\mathrm{F})$ over U is a compatible family of sections of F, one for each divided power thickening $\mathrm{U} \to \mathrm{T}$. The reader familiar with the dictionary between crystalline sheaves and connections may want to think of $u_{\mathrm{X}/\mathrm{S}_n^*}$ as the *horizontal sections* functor.

(3.3) We will need a number of morphisms of topoi related to the classifying topos B_G of a group G in a topos E (2.2). The *forgetful morphism*:

$$\pi : \mathrm{B}_\mathrm{G} \to \mathrm{E}$$

is given by $\pi_*(\mathrm{F}) = \mathrm{F}$ (forget the G-structure), and $\pi^*(\mathrm{F}) = \mathrm{G} \times \mathrm{F}$ (with G acting by multiplication on the left). The *invariants* morphism:

$$\Gamma_\mathrm{G} : \mathrm{B}_\mathrm{G} \to \mathrm{E}$$

is defined by $\Gamma_{\mathrm{G}*}(\mathrm{F}) =$ subsheaf of G-invariant sections of F, and $\Gamma^*_\mathrm{G}(\mathrm{F}) = \mathrm{F}$, viewed as a G-object with trivial action.

Finally, the terminology " classifying topos " can be explained as follows: Let F be a G-object in E, with G acting (say) on the right. F is said to be a (right) *torseur* under G if there exists a covering $\coprod_i \mathrm{X}_i$ of the final object e in E such that the pullback $\mathrm{F} \times_e \mathrm{X}_i$ is isomorphic to $\mathrm{G} \times_e \mathrm{X}_i$ with G acting by right multiplication (all i).

Proposition **(3.4)**. — *Let* $\tau : \mathrm{E}' \to \mathrm{E}$ *be a morphism of topoi. There exists a* 1—1 *correspondence between (isomorphism classes of) torseurs under* $\tau^*(\mathrm{G})$ *in* E′ *and (isomorphism classes of) morphisms* $\rho : \mathrm{E}' \to \mathrm{B}_\mathrm{G}$ *such that the diagram*:

$$\begin{array}{ccc} \mathrm{E}' & \xrightarrow{\rho} & \mathrm{B}_\mathrm{G} \\ & \searrow \quad \swarrow {\scriptstyle \Gamma_\mathrm{G}} & \\ & \mathrm{E} & \end{array}$$

commutes (up to given isomorphism of functors).

Proof (See SGA 4, exposé IV, exercise 5.9). — Note $\Gamma^*_\mathrm{G}(\mathrm{G}) = \mathrm{G}$ acting trivially on itself. Let E_G denote the object in B_G given by G acting by left multiplication on itself. $\Gamma^*_\mathrm{G}(\mathrm{G})$ acts on E_G (by multiplication on the right), and I claim E_G is a $\Gamma^*_\mathrm{G}(\mathrm{G})$-torseur. In fact $E_G \to e$ is a covering of the final object in B_G (because of the identity section e, this map is covering in E), and we have:

$$\Gamma_G^*(G) \times E_G \overset{\approx}{\to} E_G \times E_G$$
$$(g, h) \mapsto (h, hg)$$

where G acts on the second factor in $\Gamma_G^*(G) \times E_G$ and G acts diagonally on $E_G \times E_G$.

Given $\tau : E' \to E$, it remains to show that the assignment $f \mapsto f^* E_G$ defines an equivalence of category between the category of morphisms of topoi $E' \to B_G$ over E, and the category of (right $\tau^*(G)$-)torseurs on E'. Given a torseur T' on E', we define a functor:

$$f_{T'}^* : (\text{left G-objects in E}) \to E'$$
$$f_{T'}^*(H) = T' \overset{\tau^*(G)}{\times} \tau^*(H)$$
$$= \text{quotient}(\tau^*(G) \times T' \times \tau^*(H) \underset{p_2}{\overset{p_1}{\rightrightarrows}} T' \times \tau^*(H))$$

where $p_1(g, t', h) = (t', h)$ and $p_2(g, t', h) = (t'g, gh)$. Notice this completely determines $f_{T'*}$. Indeed, for $F' \in E'$, the sheaf $f_{T'*}(F')$ on B_G is given by:

$$f_{T'*}(F')(H) = \text{Hom}_{B_G}(H, f_{T'*}(F')) = \text{Hom}_{E'}(f_{T'}^*(H), F').$$

The verification that $(f_{T'}^*, f_{T'*})$ give a morphism of topoi $E' \to B_G$ with the desired properties is left for the reader. Q.E.D.

4. Let E be a topos, $A \in E$ an abelian group (*i.e.* an abelian group object in E). By taking the right derived functors of the global sections functor $\text{Hom}_E(e, A)$ one can define cohomology groups $H^*(E, A)$. Here are two examples.

(4.1) Let X/S_n be as above, and let $\mathcal{O}_{X/S_n}$ be the crystalline structure sheaf discussed in (2.3). Define:

$$H^*_{\text{cris}}(X/W(k)) = \varprojlim_n H^*((X/S_n)_{\text{cris}}, \mathcal{O}_{X/S_n}).$$

The main result of Berthelot's thesis [3] is that $H^*_{\text{cris}}(X/W(k))$ is a Weil cohomology (*i.e.* satisfies Poincaré duality, Künneth formula, and the various other axioms necessary to prove rationality of the Zeta function). Concisely said, $H^*_{\text{cris}}(X/W)$ is *the* p-adic cohomology theory for a smooth proper variety X.

(4.2) (Crystalline chern classes). — Let S be a scheme with $p \in \mathcal{O}_S$ nilpotent. The *big* crystalline site CRIS(S) is the category whose objects are triples (X, Y, δ) where X and Y are S-schemes with $X \to Y$ defined by an Ideal $\mathcal{I}$ with divided powers δ. A morphism $(X', Y', \delta') \to (X, Y, \delta)$ is a commutative diagram compatible with the divided powers:

$$\begin{array}{ccccc} X' & \hookrightarrow & Y' & \searrow & \\ \downarrow & & \downarrow & & S \\ X & \hookrightarrow & Y & \nearrow & \end{array}$$

A *covering* $\coprod_i (X_i, Y_i, \delta_i) \to (X, Y, \delta)$ is a collection of maps with each X_i Zariski open in X and $X = \bigcup_i X_i$. The associated topos S_{CRIS} has a sheaf of rings $\mathcal{O}_{S_{CRIS}}$ defined by $\mathcal{O}_{S_{CRIS}}(X, Y, \delta) = \Gamma(Y, O_Y)$, and we have $\mathcal{I}_{S_{CRIS}} = \mathcal{O}_{S_{CRIS}}$, an Ideal with divided powers.

In similar fashion one defines S_{ZAR}, the big topos of Zariski sheaves, and there is a morphism (horizontal sections (3.2)) $u_S : S_{CRIS} \to S_{ZAR}$. Let G be a group in S_{ZAR}, given as acting on a locally free $\mathcal{O}_S$-Module of finite type $\mathcal{E}$. (We have in mind $G = GL(r)$ acting on $\mathcal{O}_S^{\oplus r}$.) Berthelot and Illusie [4] construct chern classes for this representation with:

$$c_i(\mathcal{E}) \in H^{2i}(B_G/S_{CRIS}, \mathcal{I}^{[i]}_{B_G/S_{CRIS}}).$$

Here B_G/S_{CRIS} is viewed as a ringed topos with sheaf of rings $\mathcal{O}_{S_{CRIS}}$ with trivial G-action $= \mathcal{O}_{B_G/S_{CRIS}}$. $\mathcal{I}_{B_G/S_{CRIS}} \subset \mathcal{O}_{B_G/S_{CRIS}}$ corresponds to $\mathcal{I}_{S_{CRIS}} \subset \mathcal{O}_{S_{CRIS}}$.

(4.3) (Crystalline chern classes for group representations). — Let $G = GL(r)$ acting on $\mathcal{O}_S^{\oplus r}$ in (4.2), and let X be an S-scheme, H an abstract group. Suppose given a locally free $\mathcal{O}_X$-Module $\mathcal{F}$ of rank r, together with a representation $\rho : H \to \mathrm{Aut}(\mathcal{F})$. These data define a $GL(r)$-torseur on B_H/X_{ZAR} and hence a morphism:

$$B_H/X_{ZAR} \to B_{GL(r)}/X_{ZAR}.$$

Let $u_{B_H/(X/S)_{CRIS}} : B_H/(X/S)_{CRIS} \to B_H/X_{ZAR}$ be the horizontal sections morphism. Pulling back the above torseur, we get a commutative diagram of topoi:

$$\begin{array}{ccc} B_H/(X/S)_{CRIS} & \longrightarrow & B_{GL(r)}/S_{CRIS} \\ {\scriptstyle u_{B_H/(X/S)_{CRIS}}} \Big\downarrow & & \Big\downarrow {\scriptstyle u_{B_{GL(r)}/S_{CRIS}}} \\ B_H/X_{ZAR} & \longrightarrow & B_{GL(r)}/S_{ZAR} \end{array}$$

In particular, the chern classes discussed in (4.2) can be pulled back to give classes:

$$c_i(\rho) \in H^{2i}(B_H/(X/S)_{CRIS}, \mathcal{I}^{[i]}_{B_H/(X/S)_{CRIS}})$$

where again $B_H/(X/S)_{CRIS}$ is viewed as a ringed topos with sheaf of rings $\mathcal{O}_{(X/S)_{CRIS}}$ with trivial H-action.

(4.4) Let me show how, given the chern classes $c_i(\rho)$ described in (4.3), one deduces the existence of crystalline chern classes as discussed in § 3.3. The first point is that cohomology is the same whether computed in the big or little crystalline topos, so we can think of:

$$c_i(\rho) \in H^{2i}(B_H/(X/S)_{cris}, \mathcal{I}^{[i]}_{B_H/(X/S)_{cris}}).$$

Now take $S = \mathrm{Spec}(\mathbf{Z}/p^N\mathbf{Z})$ for some fixed N. Let $X = \mathrm{Spec}(A_0)$ where A_0 is a $\mathbf{Z}/p\mathbf{Z}$-algebra, and let $Y = \mathrm{Spec}(A)$ where A is a $\mathbf{Z}/p^N\mathbf{Z}$-algebra with $A/pA \cong A_0$, so $X \overset{i}{\hookrightarrow} Y$ is a closed immersion. We get morphisms of topoi:

$$B_H/(X/S)_{\mathrm{cris}} \xrightarrow{i_{\mathrm{cris}}} B_H/(Y/S)_{\mathrm{cris}} \xrightarrow{u_{B_H/Y/S}} B_H/Y_{\mathrm{zar}} \xrightarrow{\Gamma} (\mathrm{sets})$$

where Γ denotes the composite:

$$B_H/Y_{\mathrm{zar}} \xrightarrow{\Gamma_H} Y_{\mathrm{zar}} \xrightarrow{\Gamma} (\mathrm{sets}).$$

In the notation of derived categories:

$$c_i(\rho) \in H^{2i}(R\Gamma \circ Ru_{B_H/Y/S*} \circ Ri_{\mathrm{cris}*}(\mathscr{I}^{[i]}_{B_H/X/S})).$$

To analyse the right hand side, we recall two further results of Berthelot-Illusie ([4], formulas (1.3) and (1.2.1)):

(i) For any $r \geq 0$, we have:

$$Ri_{\mathrm{cris}*}(\mathscr{I}^{[r]}_{B_H/X/S}) = K^{[r]}_{B_H/Y/S}$$

where:

$$K_{B_H/Y/S} = \mathscr{I}_{B_H/Y/S} + p\mathscr{O}_{B_H/Y/S}$$

and the superscript $[r]$ denotes the r-th divided power (2.3) of the ideal.

(ii) There is a natural map compatible with products:

$$Ru_{B_H/Y/S*}(K^{[r]}_{B_H/Y/S}) \to \mathrm{Fil}^r_{(p)}(\Omega^{\bullet}_{B_H/Y/S})$$

where $\mathrm{Fil}^r_{(p)}(\Omega^{\bullet}_{Y/S})$ denotes the complex of Zariski sheaves:

$$\mathrm{Fil}^r_{(p)}(\Omega^{\bullet}_{Y/S})^{\ell} = p^{[r-\ell]}\Omega^{\ell}_{Y/S}$$

discussed in § 3.3, and $\mathrm{Fil}^r_{(p)}(\Omega^{\bullet}_{B_H/Y/S})$ denotes the same complex viewed as a complex in $B_H/(Y/S)_{\mathrm{zar}}$ with trivial H-action. We have now:

$$\begin{aligned}
&\bigoplus_r H^{2r}(R\Gamma \circ Ru_{B_H/Y/S*} \circ Ri_{\mathrm{cris}*}(\mathscr{I}^{[r]}_{B_H/X/S})) \\
&\overset{(i)}{\cong} \bigoplus_r H^{2r}(R\Gamma \circ Ru_{B_H/Y/S*}(K^{[r]}_{B_H/X/S})) \\
&\overset{(ii)}{\to} \bigoplus_r H^{2r}(R\Gamma(\mathrm{Fil}^r_{(p)}(\Omega^{\bullet}_{B_H/Y/S}))) \\
&\cong \bigoplus_r H^{2r}(H, \mathrm{Fil}^r_{(p)}\Omega^{\bullet}_A).
\end{aligned}$$

(The final isomorphism holds because $Y = \mathrm{Spec}\, A$ is affine, and $\Omega^{\bullet}_{Y/S}$ is a complex of quasi-coherent sheaves. $H^{2r}(H, \mathrm{Fil}^r_{(p)}(\Omega^{\bullet}_A))$ denotes the $2r$-th hypercohomology of the group H acting trivially on the de Rham complex $\Omega^{\bullet}_A$.) The corresponding chern classes $c_r(\rho) \in H^{2r}(H, \mathrm{Fil}^r_{(p)}\Omega^{\bullet}_A)$ are the ones discussed in § 3.3.

5. Let X, S_n be as in § 4.1. It remains to construct the map:

$$j^*_n : H^*_{\mathrm{cris}}(X/S_n) \to H^*(X, \Omega^{\bullet}_{X_n/S_n, Y})$$

discussed in that section.

Lemma (5.1). — *Let* R *be a* $\mathbf{Z}_{(p)}$-*algebra and let* $W_n(R)$ *be the ring of p-Witt vectors of length n over* R. *Let* $f \in R$ *and let* $g \in W_n(R)$ *be any lifting of* f (i.e. *g maps to f under the natural map* $W_n(R) \twoheadrightarrow R$). *Write* R_f, $W_n(R)_g$ *for the localizations with respect to the multiplicative systems defined by powers of f and g. Then*:

$$W_n(R_f) \cong W_n(R)_g.$$

Proof. — If g' is another lifting of f, we have $g' = g + \omega$ where $\omega^n = 0$, so g' is invertible in $W_n(R)_g$. It follows that $W_n(R)_g$ is independent of the choice of g lifting f. Taking $g = (f, 0, \ldots, 0)$ (Witt coordinates) and using the formula (valid in $W_n(R_f)$):

$$g^{-1}.(a_1, \ldots, a_n) = (a_1 f^{-1}, a_2 f^{-p}, \ldots, a_n f^{-p^{n-1}}),$$

the lemma follows easily.

Lemma (5.2). — *Let* R *be a k-algebra of finite type, where k is a perfect field of characteristic p. Then* $W_n(R)$ *is a* $W_n(k)$-*algebra of finite type.*

Proof. — Let $x_1, \ldots, x_N$ generate R as a k-algebra and also as an $R^{p^{n-1}}$-module ($R^{p^r} = \{x^{p^r} \mid x \in R\}$). Let $S \subset W_n(R)$ be the $W_n(k)$-subalgebra generated by elements of the form:

$$(\underbrace{0, \ldots, 0}_{j}, x_i, 0, \ldots, 0) = V^j(x_i)$$

for all i, j. The formula:

$$(x, 0, \ldots, 0).V^j(y) = V^j(x^{p^j}.y)$$

implies $S \supset (0, \ldots, 0, R, *, \ldots, *)$. But:

$$(a_1, \ldots, a_n) = \sum_i V^{i-1}(a_i)$$

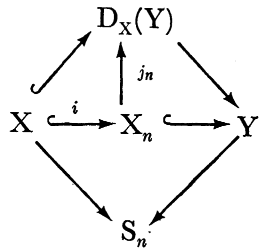
so $S = W_n(R)$. Q.E.D.

Corollary (5.3). — *Let* X *be a scheme of finite type over a perfect field k. Then the ringed space* $(X, W_n(\mathcal{O}_X))$ *is a scheme of finite type over* $W_n(k)$.

Lemma (5.4). — *Let* X, k, S_n, X_n *be as in* § 4.1. *Then* X_n *is projective over* S_n.

Proof. — By assumption X is projective. Let $\mathcal{L}$ on X be an ample line bundle. The map $f \mapsto (f, 0, \ldots, 0)$ defines a homomorphism $\mathcal{O}_X^* \to \mathcal{O}_{X_n}^*$, so $\mathcal{L}$ lifts to an $\mathcal{L}_n$ on X_n. To show $\mathcal{L}_n$ is ample, it suffices by ([20], III, (2.6.1)) to verify that for any coherent sheaf $\mathcal{F}$ on X_n, we have $H^1(X_n, \mathcal{F} \otimes \mathcal{L}_n^{\otimes m}) = (0)$ for $m \gg 0$. (To apply this criterion, we need to know that $X_n \to S_n$ is universally closed, and X_n is separated. Closedness is clear. For separation, note that $\Gamma(U, W_n(\mathcal{O}_X)) = W_n(\Gamma(U, \mathcal{O}_X))$, so the intersection of two affines is affine.) Since $X \to X_n$ is defined by a nilpotent Ideal, $\mathcal{F}$ has a finite filtration such that the successive quotients $\mathcal{F}_i/\mathcal{F}_{i-1}$ are $\mathcal{O}_X$-Modules. Thus $H^1(X_n, (\mathcal{F}_i/\mathcal{F}_{i-1}) \otimes_{\mathcal{O}_{X_n}} \mathcal{L}_n^{\otimes m}) = H^1(X_n, (\mathcal{F}_i/\mathcal{F}_{i-1}) \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes m}) = (0)$, assuming $m \gg 0$. It follows that $H^1(X_n, \mathcal{F} \otimes \mathcal{L}_n^{\otimes m}) = (0)$ so $\mathcal{L}_n$ is ample as claimed. Q.E.D.

Now choose an embedding $X_n \hookrightarrow Y = \mathbf{P}^N_{S_n}$ for some N, so we have $X \hookrightarrow X_n \hookrightarrow Y$. Let $D_X(Y)$ denote the *divided power envelope* of X in Y ([3], I.2). This is a Y-scheme obtained by putting divided powers on the Ideal of X in Y in a universal way. We obtain a diagram:



The morphism $j_n : X_n \to D_X(Y)$ arises from the universal nature of the divided powers on $D_X(Y)$, together with the fact that the Ideal $V(\mathcal{O}_{X_n})$ defining $X \hookrightarrow X_n$ has divided powers γ. Namely:

$$\gamma^{(m)}(V(x)) = \frac{p^{m-1}}{m!} V(x^m).$$

Following Berthelot, we define a complex:

$$\mathcal{O}_{D_X(Y)} \otimes_{\mathcal{O}_Y} \Omega^{\bullet}_{Y/S_n} \quad (\Omega^{\bullet}_{Y/S_n} = \text{de Rham complex})$$

to be essentially as written, except that relations of the sort:

$$d(x^{[r]} \otimes dy) = x^{[r-1]} \otimes dx\, dy$$

are imposed ($dy \in \Omega^q_Y$; $x \in \operatorname{Ker}(\mathcal{O}_Y \to \mathcal{O}_X)$; $x^{[r]} = r$-th divided power of x, viewed as an element in $\operatorname{Ker}(\mathcal{O}_{D_X(Y)} \to \mathcal{O}_X)$). A basic result of Berthelot ([3], V, (2.3.2)) is:

$$H^{\bullet}_{\text{cris}}(X/W_n) \cong H^{\bullet}_{\text{zar}}(X, \mathcal{O}_{D_X(Y)} \otimes_{\mathcal{O}_Y} \Omega^{\bullet}_Y).$$

We now easily prove:

Proposition (5.5). — *There is a canonical map:*

$$j_n^* : H^{\bullet}_{\text{cris}}(X/W_n) \to H^{\bullet}(X, \Omega^{\bullet}_{X_n/S_n, \gamma}),$$

where $\Omega^{\bullet}_{X_n/S_n, \gamma}$ *denotes the de Rham complex with compatibilities like:*

$$d(\gamma^{(m)}(x)\,dy) = \gamma^{(m-1)}(x)\; dx\, dy$$

imposed. $j_0^* : H^{\bullet}_{\text{cris}}(X/k) \to H^{\bullet}(X, \Omega^{\bullet}_X)$ *is the standard identification, and for* $m \geq n$ *the diagram below commutes:*

$$\begin{array}{ccc} H^{\bullet}_{\text{cris}}(X/W_m) & \xrightarrow{j_m^*} & H^{\bullet}(X, \Omega^{\bullet}_{X_m/S_m, \gamma}) \\ \downarrow & & \downarrow \\ H^{\bullet}_{\text{cris}}(X/W_n) & \xrightarrow{j_n^*} & H^{\bullet}(X, \Omega^{\bullet}_{X_n/S_n, \gamma}) \end{array}$$

Proof. — We simply compose:

$$H^*_{\text{cris}}(X/W_n) \cong H^*(X, \mathcal{O}_{D_X(Y)} \otimes \Omega^{\bullet}_{Y/S_n}) \xrightarrow{j_n^*} H^*(X, \Omega^{\bullet}_{X_n/S_n,\, \gamma}).$$

To see that this map is canonical, suppose we have another embedding $X_n \to Y'$. Replacing Y' by $Y' \underset{S_n}{\times} Y$ we may assume given a diagram:

$$\begin{array}{ccc} X_n & \hookrightarrow & Y' \\ & \searrow & \downarrow \\ & & Y \end{array}$$

and hence:

$$\begin{array}{ccc} & \overset{j'_n}{\nearrow} & D_X(Y') \\ X_n & & \downarrow \\ & \underset{j_n}{\searrow} & D_X(Y) \end{array}$$

The desired independence from the choice of Y follows from the commutative diagram:

$$\begin{array}{ccccc} & & H^*(\mathcal{O}_{D_X(Y')} \otimes \Omega^{\bullet}_{Y'}) & & \\ & \nearrow\!\!\wr & \uparrow & \overset{j_n'^*}{\searrow} & \\ H^*_{\text{cris}}(X/W_n) & & & & H^*(X, \Omega^{\bullet}_{X_n/S_n,\, \gamma}) \\ & \searrow\!\!\wr & & \underset{j_n^*}{\nearrow} & \\ & & H^*(\mathcal{O}_{D_X(Y)} \otimes \Omega^{\bullet}_{Y}) & & \end{array}$$

Q.E.D.

II. — LOCAL STRUCTURE OF TYPICAL CURVES

1. CURVES ON K_*. GENERALITIES

1. Let R be a commutative ring with 1, T a variable, and m, q integers ≥ 0. We write $R_m = R[T]/(T^{m+1})$ and $R_\infty = R[[T]]$. Define the *curves of length m on* K_q:

(1.1) $$C_m K_q(R) = \mathrm{Ker}(K_q(R_m) \xrightarrow[T\mapsto 0]{} K_q(R)) \qquad 1 \leq m \leq \infty.$$

Define a decreasing filtration $\mathrm{filt}^{\bullet}\, C_m K_q(R)$ by the exact sequences:

(1.2) $$0 \to \mathrm{filt}^n\, C_m K_q(R) \to C_m K_q(R) \to C_n K_q(R).$$

Thus $\mathrm{filt}^0 = C_m K_q$ and $\mathrm{filt}^m = (0)$.

We will be particularly interested in the subgroup:

$$SC_m K_q(R) \subset C_m K_q(R)$$

generated by symbols. More precisely, define:

(1.3) $$SC_m K_q(R) = C_m K_q(R) \cap \{R_m^* \otimes \ldots \otimes R_m^*\} \subset K_q(R_m)$$

where $\{\ \}$ denotes the multiplication in K-theory. Alternately:

(1.4) $$SC_m K_q(R) = \{(1+TR_m)^* \otimes \ldots \otimes R_m^*\} \subset K_q(R_m).$$

The reader can check that the two descriptions (1.3) and (1.4) coincide. (Use the fact that $K_q(R_m) \to K_q(R)$ is split.) There is an induced filtration $\mathrm{filt}^{\bullet} SC_m K_q(R)$ and exact sequences for $n \leq m$:

$$0 \to \mathrm{filt}^n\, SC_m K_q(R) \to SC_m K_q(R) \to SC_n K_q \to 0.$$

Finally, define $\hat{C}K_q(R) = \varprojlim C_n K_q(R)$, $S\hat{C}K_q(R) = \varprojlim SC_n K_q(R)$. An immediate consequence of (I.2.3.3) is:

Proposition **(1.5)**. — *Let* R *be a local ring. Then* $SC_m K_q(R) = C_m K_q(R)$ *for* $q \leq 2$. *Moreover* $\mathrm{filt}^n\, C_m K_2(R)$ *is generated by symbols*:

$$\{1 + T^{n+1} a, b\}, \qquad a \in R_m, \qquad b \in R_m^*.$$

I would conjecture that an analogous result should be true also for $q > 2$. (*)

(*) *Added in proof*: Recent results of Keune suggest that the conjecture is false, even for $q = 3$.

2. Let R be a commutative ring with 1, and let R_m be as above. Given an integer $n \geq 1$, define:

$$\varphi_n : R_m \to R_{mn+n-1} \qquad (\text{resp. } \varphi_n : R_\infty \to R_\infty)$$
$$\varphi_n | R = 1_R, \qquad \varphi_n(T) = T^n.$$

Note that φ_n makes R_{mn+n-1} a free R_m-module of rank n, so we have maps (I, § 2, 4):

$$K_q(R_m) \underset{\varphi_{n*}}{\overset{\varphi_n^*}{\rightleftarrows}} K_q(R_{mn+n-1}).$$

The diagrams:

$$\begin{array}{ccc} K_q(R_m) & \xrightarrow{\varphi_n^*} & K_q(R_{mn+n-1}) \\ & \searrow \quad \swarrow & \\ & K_q(R) & \end{array}$$

$$\begin{array}{ccc} K_q(R_{mn+n-1}) & \xrightarrow{\varphi_{n*}} & K_q(R_m) \\ \downarrow & & \downarrow \\ K_q(R) & \xrightarrow{\times n} & K_q(R) \end{array}$$

are commutative. Indeed, the upper one is clear and the lower follows easily from the projection formula (I, § 2 (4.1)). We denote by V_n, F_n the induced maps:

(2.1)
$$V_n : C_m K_q(R) \to C_{mn+n-1} K_q(R) \qquad (\text{resp. } V_n : K_q(R)\,{}_\infty C \hookleftarrow)$$
$$F_n : C_{mn+n-1} K_q(R) \to C_m K_q(R) \qquad (\text{resp. } F_n : C_\infty K_q(R) \hookleftarrow)$$

In terms of symbols when $q=2$, for example, we have:

(2.2)
$$V_n\{P(T), Q(T)\} = \{P(T^n), Q(T^n)\}, \qquad P, Q \in R_m^*$$
$$F_n\{P(T), Q(T^n)\} = \{\prod_{\zeta^n=1} P(\zeta T^{1/n}), Q(T)\}.$$

The second formula follows from (I.2.4.1) and the fact that $\varphi_{n*}(P(T)) = \prod_{\zeta^n=1} P(\zeta T^{1/n})$ where $P(T) \in K_1(R_m)$. The action of F_n on a general symbol $\{P(T), Q(T)\}$ is more difficult to describe.

Proposition **(2.3)**. — *For integers* $m \leq r \leq \infty$, *the diagrams:*

$$\begin{array}{ccc} C_{nr+n-1} K_q(R) & \longrightarrow & C_{nm+n-1} K_q(R) \\ \downarrow{\scriptstyle F_n} & & \downarrow{\scriptstyle F_n} \\ C_r K_q(R) & \longrightarrow & C_m K_q(R) \end{array}$$

$$\begin{array}{ccc} C_{nr+n-1}K_q(R) & \longrightarrow & C_{nm+n-1}K_q(R) \\ \uparrow{\scriptstyle V_n} & & \uparrow{\scriptstyle V_n} \\ C_rK_q(R) & \longrightarrow & C_mK_q(R) \end{array}$$

commute.

Proof. — The lower square commutes by functoriality. For the upper square, we can apply the lemma below to the cocartesian diagram of rings:

$$\begin{array}{ccc} R_{nr+n-1} & \longrightarrow & R_{nm+n-1} \cong R_{nr+n-1} \otimes_{R_r} R_m \\ \uparrow{\scriptstyle \varphi_n} & & \uparrow{\scriptstyle \varphi_n} \\ R_r & \longrightarrow & R_m \end{array}$$

Lemma (2.4). — *Suppose given rings* A, B, C, *and ring homomorphisms* $\varphi : A \to B$, $\psi : A \to C$. *Assume* B (*viewed as an* A-*module* via φ) *is finitely generated and projective. Then the diagram*:

$$\begin{array}{ccc} K_q(B) & \xrightarrow{\psi_B^*} & K_q(B \otimes_A C) \\ \downarrow{\scriptstyle \varphi_*} & & \downarrow{\scriptstyle \varphi_{C*}} \\ K_q(A) & \xrightarrow{\psi^*} & K_q(C) \end{array}$$

is commutative.

Proof. — Note $B \otimes_A C$ is a finitely generated projective C-module so φ_{C*} is defined. With notation as in (I, § 2, 1), it follows from the discussion there that a morphism of functors:

$$K_0(B, \cdot) \to K_0(C, \cdot)$$

gives rise to a map $K_q(B) \to K_q(C)$ for all q. We thus reduce to verifying that the diagram:

$$\begin{array}{ccc} K_0(B, \cdot) & \xrightarrow{\psi_B^*} & K_0(B \otimes_A C, \cdot) \\ \downarrow{\scriptstyle \varphi_*} & & \downarrow{\scriptstyle \varphi_{B*}} \\ K_0(A, \cdot) & \xrightarrow{\psi^*} & K_0(C, \cdot) \end{array}$$

commutes. This in turn is a consequence of the canonical isomorphism of C-modules, valid for any B-module M:

$$M \otimes_A C \cong M \otimes_B (B \otimes_A C). \qquad \text{Q.E.D.}$$

Corollary **(2.5)**. — V_n, F_n *induce maps*:

$$C_m K_q(R) \underset{F_n}{\overset{V_n}{\rightleftarrows}} C_{nm+n-1} K_q(R).$$

Proposition **(2.6)**. — *The maps* V_n, F_n *satisfy the following relations*:

(i) $F_n \circ V_n =$ *multiplication by* n.

(ii) $V_n \circ V_m = V_{nm}$; $F_n \circ F_m = F_{nm}$.

It seems quite likely that the following relations hold as well. We will verify them explicitly later in the cases where we need them:

(iii) If $(m, n) = 1$ then $V_m \circ F_n = F_n \circ V_m$.

(iv) If characteristic $R = p$, then $V_p \circ F_p =$ multiplication by p.

2. THE MODULE STRUCTURE ON $C_n K_q(R)$

1. Let R be commutative with 1 and let $R_n = R[t]/(t^{n+1})$. For $m \geq 0$ an integer, define a category $\mathrm{Fil}_m \mathrm{Nil}(R)$ by taking as objects pairs (P, g) with P a finitely generated projective R-module, and $g: P \to P$ an R-linear endomorphism satisfying $g^{m+1} = 0$. Morphisms $\varphi: (P, g) \to (Q, h)$ are R-linear maps $\varphi: P \to Q$ satisfying $\varphi \circ g = h \circ \varphi$. $\mathrm{Fil}_m \mathrm{Nil}(R)$ is an exact category in the sense of Quillen [24], as is:

$$\mathrm{Nil}(R) = \varinjlim \mathrm{Fil}_m \mathrm{Nil}(R).$$

Let $\mathscr{P}(R_n)$ denote the category of finitely generated projective R_n-modules. For an R-algebra A, we have a " bilinear functor ":

$$\textbf{(1.1)} \qquad F: \mathscr{P}(R_n) \times \mathrm{Fil}_n \mathrm{Nil}(A) \to \mathscr{P}(A)$$
$$(M, (P, g)) \mapsto M \otimes_{R_n} (P, g)$$

where $\otimes_{R_n}(P, g)$ means $\otimes_{R_n} P$ with $t \in R_n$ acting via g on P. For $m \geq n$ there are inclusions $\mathrm{Fil}_n \mathrm{Nil}\, A \underset{i}{\hookrightarrow} \mathrm{Fil}_m \mathrm{Nil}\, A$ and restrictions $\mathscr{P}(R_m) \xrightarrow{r} \mathscr{P}(R_n)$ giving commutative diagrams:

$$\textbf{(1.2)} \qquad \begin{array}{ccccc} & & \mathscr{P}(R_n) \times \mathrm{Fil}_n \mathrm{Nil}\, A & & \\ & \overset{r \times 1}{\nearrow} & & \overset{F}{\searrow} & \\ \mathscr{P}(R_m) \times \mathrm{Fil}_n \mathrm{Nil}\, A & & & & \mathscr{P}(A) \\ & \underset{1 \times i}{\searrow} & & \underset{F}{\nearrow} & \\ & & \mathscr{P}(R_m) \times \mathrm{Fil}_m \mathrm{Nil}\, A & & \end{array}$$

A " bilinear functor " on exact categories leads to a pairing on the K-groups [15]. The full force of this pairing applied to (1.1) should enable one to define a graded ring structure on $\bigoplus_{q\geq 1} C_n K_q(R)$, but we will consider only the pairings:

(1.3) $$K_p(R_n) \times K_0(\mathrm{Fil}_n \,\mathrm{Nil}\, A) \to K_p(A)$$

and in the limit:

(1.4) $$CK_p(R) \times \left(\frac{K_0(\mathrm{Nil}\, A)}{K_0(\mathrm{Fil}_0 \,\mathrm{Nil}\, A)}\right) \to K_p(A).$$

2. Classically, one constructs a homomorphism:

(2.1) $$\partial : K_1(A[T, T^{-1}]) \to K_0(\mathrm{Nil}\, A) / K_0(\mathrm{Fil}_0 \,\mathrm{Nil}\, A)$$

as follows ([33], p. 656): an element $z \in K_1$ can be interpreted as a class of automorphisms over the ring from which we choose a representative:

$$\zeta : \bigoplus_N A[T, T^{-1}] \xrightarrow{\cong} \bigoplus_N A[T, T^{-1}].$$

Multiplying if necessary by a power of T, we can assume the matrix of ζ does not involve T^{-1}. Then ζ lifts to an endomorphism $\bar{\zeta}$ of $\bigoplus_N A[T]$. One checks that the quotient $(\bigoplus_N A[T]) / \mathrm{Image}\, \bar{\zeta} = P$ is a finitely generated projective A-module with a nilpotent endomorphism g (g = multiplication by T), and that the class of (P, g) in $K_0(\mathrm{Nil}\, A) / K_0(\mathrm{Fil}_0 \,\mathrm{Nil}\, A)$ is independent of the various choices.

Now consider the case $A = R_n$. We have an R-algebra homomorphism:

$$f : R_n \to R_n[T, T^{-1}], \quad f(t) = tT^{-1}$$

and hence a map on K-groups:

$$\partial \circ f^* : K_1(R_n) \to K_0(\mathrm{Nil}\, R_n) / K_0(R_n).$$

It is clear that $K_1(R) \subset K_1(R_n)$ lies in the kernel of $\partial \circ f^*$, so we get finally:

(2.2) $$\partial \circ f^* : C_n K_1(R) = W_n(R) \to K_0(\mathrm{Nil}\, R_n) / K_0(R_n).$$

Together with (1.4), this yields a pairing:

(2.3) $$\hat{C}K_p(R) \times W_n(R) \to C_n K_p(R)$$

and in the limit (using the fact that (1.4) is functorial in A):

(2.4) $$\hat{C}K_p(R) \times W(R) \xrightarrow{\langle\ \rangle} \hat{C}K_p(R).$$

We will show that the action of $W(R)$ on $\hat{C}K_p(R)$ defined by $-\langle\ \ \rangle$ gives $\hat{C}K_p(R)$ the structure of a $W(R)$-module.

3. *Lemma* **(3.1)**. — *Let* $r \in R$, *and let* $\omega = (1 - rt) \in W(R)$. *The map*:

$$\langle\ , \omega\rangle : \hat{C}K_p(R) \to \hat{C}K_p(R)$$

is induced by the map of R-*algebras* $R_n \to R_n$, $t \mapsto rt$.

Proof. — The element $f^*\omega \in K_1(R_n[T, T^{-1}])$ is represented by:

$$R_n[T, T^{-1}] \xrightarrow[\cdot(1-rtT^{-1})]{\cong} R_n[T, T^{-1}]$$

so the class of $\partial f^*\omega$ in $K_0(\mathrm{Nil}\, R_n)/K_0(R)$ is given by the pair:

$$(R_n[T]/(T-rt), \text{mult. by } T) = (P, g).$$

Thus the endomorphism $\langle\ , \omega\rangle$ is induced by the endomorphism on the level of categories:

$$\mathscr{P}(R_n) \to \mathscr{P}(R_n), \qquad M \mapsto M \otimes_{R_n} (P, g) = M \otimes_{R_n} (R_n, \text{mult. by } rt).$$

The assertion is now clear. Q.E.D.

Recall we have maps $V_n, F_n : \hat{C}K_p(R) \to \hat{C}K_p(R)$ induced respectively by the maps $\varphi_n : R_m \to R_{nm+n-1}$, $\varphi_n(t) = t^n$, and the transfer on K-theory associated to the φ_n. On $W(R) = \hat{C}K_1(R)$, $V_n(1-rt) = (1-rt^n)$.

Lemma (**3.2**). — *For* $\alpha \in \hat{C}K_p(R)$ *we have*:

$$\langle \alpha, (1-rt^n)\rangle = V_n(\langle F_n\alpha, (1-rt)\rangle).$$

Proof. — Let $M \in \mathrm{Ob}\,\mathscr{P}(R_{mn+n-1})$ for some m. The lemma follows from the chain of isomorphisms:

$$M \overset{(1)}{\otimes}_{R_{mn+n-1}} (R_{mn+n-1}[T]/(T^n - rt^n), T)$$
$$\cong \varphi_{n*}(M) \overset{(2)}{\otimes}_{R_m} (R_{mn+n-1}[T]/(T-rt^n), T)$$
$$\cong \varphi_{n*}(M) \overset{(3)}{\otimes}_{R_m} (R_m[T]/(T-rt), T) \overset{(4)}{\otimes}_{R_m} R_{nm+n-1}.$$

The tensor products labeled (1), (2), and (3) are taken with the module structure on the right obtained by letting t act on $R_?[T]/(\dots)$ by multiplication by T. Tensor product (4) is taken with respect to the standard R_m-module structure on $R_m[T]/(T-rt)$. The left and right hand sides of the above correspond respectively to the maps:

$$\langle\ , (1-rt^n)\rangle : \hat{C}K_p(R) \to \hat{C}K_p(R)$$
$$V_n \circ \langle\ , (1-rt)\rangle \circ F_n : \hat{C}K_p(R) \to \hat{C}K_p(R).$$

This completes the proof. Q.E.D.

Lemma (**3.3**). — *For* $\alpha \in \hat{C}K_p(R)$, *we have*:

$$\langle V_n\alpha, (1-rt)\rangle = V_n\langle \alpha, (1-r^n t)\rangle.$$

Proof. — Let $M \in \mathscr{P}(R_m)$. The map $\langle\ , (1-rt)\rangle \circ V_n$ is given on the level of modules by:

$$M \mapsto M \otimes_{R_m} R_{mn+n-1} \otimes_{R_{mn+n-1}} (R_{mn+n-1}[T]/(T-rt), T)$$
$$\wr\|$$
$$M \otimes_{R_m} (R_{mn+n-1}[T]/(T-rt), T^n).$$

The map $V_n \circ \langle\ , (1-r^n t)\rangle$ is given by:

$$M \overset{(1)}{\otimes}_{R_m} (R_m[T]/(T-r^n t), T) \overset{(2)}{\otimes}_{R_m} R_{mn+n-1}$$
$$\cong M \otimes_{R_m} (R_{mn+n-1}[T]/(T-r^n t^n), T)$$

where $\overset{(1)}{\otimes}$ is with respect to the R_m-module structure on $R_m[T]/(T-r^n t)$ given by $\cdot t =$ multiplication by T, and $\overset{(2)}{\otimes}$ is with respect to the natural R_m-module structure on $R_m[T]/(T-r^n t)$. The lemma follows. Q.E.D.

Lemma (3.4). — *Let* $\varphi : R \to R'$ *be a ring homomorphism making* R′ *a finitely generated projective* R-*module. The transfer in* K-*theory induces maps* $\varphi_* : \hat{C}K_p(R') \to \hat{C}K_p(R)$. *We have*:

(i) $\varphi_* \circ V_n = V_n \circ \varphi_*$.

(ii) *For* $\omega \in W(R')$, $\alpha \in \hat{C}K_p(R)$, $\varphi_* \langle \omega, \varphi^* \alpha \rangle = \langle \varphi_* \omega, \alpha \rangle$.

(iii) *For* $\omega \in W(R)$, $\alpha \in \hat{C}K_p(R')$, $\varphi_* \langle \varphi^* \omega, \alpha \rangle = \langle \omega, \varphi_* \alpha \rangle$.

Proof. — (i) follows from (§ 1 (2.4)). For (ii), we have first a commutative diagram (notation as in (2.2)):

$$\begin{array}{ccccc}
W_n(R') & \xrightarrow{f^*} & K_1(R'_n[T, T^{-1}]) & \xrightarrow{\partial} & K_0(\text{Nil } R'_n)/K_0(R'_n) \\
\downarrow \varphi_* & & \downarrow \varphi_* & & \downarrow \varphi_* \\
W_n(R) & \xrightarrow{f^*} & K_1(R_n[T, T^{-1}]) & \xrightarrow{\partial} & K_0(\text{Nil } R_n)/K_0(R_n)
\end{array}$$

Using this, one reduces (ii) to the projection formula for tensor products of modules. Namely, for $M \in \mathscr{P}(R_n)$ and $(P, g) \in \text{Nil}(R'_n)$, there exists a canonical isomorphism of R_n-modules:

$$M \otimes_{R_n} (P, g) \cong M \otimes_{R_n} R'_n \otimes_{R'_n} (P, g).$$

The proof of (iii) is similar, and is omitted. Q.E.D.

Theorem (3.5). — *The pairing*:

$$W(R) \times \hat{C}K_p(R) \to \hat{C}K_p(R)$$
$$(\omega, \alpha) \mapsto -\langle \alpha, \omega \rangle$$

induces a W(R)-*module structure on* $\hat{C}K_p(R)$. *We have the "projection formula"*:

$$V_n \omega . \alpha = V_n(\omega . F_n \alpha)$$
$$\omega . V_n \alpha = V_n(F_n \omega . \alpha).$$

For $p = 0$, *the pairing coincides with the ring structure on* W(R).

Proof. — The pairing has been defined, and the first projection formula for $\omega=(1-rt)$ follows from (3.2). If $\omega=(1-rt^m)=V_m(1-rt)=V_m\omega_0$ for $\omega_0=(1-rt)$, we get:

$$V_n\omega.\alpha=V_{nm}\omega_0.\alpha\underset{(3.2)}{=}V_{nm}(\omega_0.F_{nm}\alpha)=V_nV_m(\omega_0.F_mF_n\alpha)$$
$$\underset{(3.2)}{=}V_n(V_m\omega_0.F_n\alpha)=V_n(\omega.F_n\alpha).$$

Notice in particular that $(1-rt^m).\alpha\in\operatorname{Ker}(\hat{C}K_p(R)\to C_{m-1}K_p(R))$. Since any $\omega\in W(R)$ can be written as an infinite product:

$$\omega=\prod_{m\geq 1}(1-r_mt^m),$$

the first projection formula follows by linearity.

The second projection formula for $\omega=(1-rt)$ is the content of (3.3). Suppose $\omega=(1-rt^m)$. Let $R'=R[X]/(X^m-r)$, let $x\in R'$ be the image of X, and let $\varphi:R\to R'$ be the structure map. Writing $\omega_0=(1-xt)\in W(R')$, we have $\omega=\varphi_*\omega_0$.

Now using (3.3) and (3.4) we get:

$$\omega.V_n\alpha=\varphi_*\omega_0.V_n\alpha=\varphi_*(\omega_0.V_n\varphi^*\alpha)=\varphi_*V_n(F_n\omega_0.\varphi^*\alpha)$$
$$=V_n\varphi_*(F_n\omega_0.\varphi^*\alpha)=V_n(\varphi_*F_n\omega_0.\alpha).$$

One checks from the formulas that $\varphi_*F_n\omega_0=\varphi_*(1-x^nt)=F_n\omega$. The second projection formula for a general ω follows by linearity and continuity.

It remains to show the pairing $(\omega,\alpha)\to-\langle\omega,\alpha\rangle$ gives a module structure, *i.e.* that the map:

$$W(R)\to\operatorname{End}(\hat{C}K_p(R))$$

is a ring homomorphism. For elements $\omega_0=(1-rt)^{-1}$ and $\omega_1=(1-st)^{-1}$, the product in W(R) is $\omega_0.\omega_1=(1-rst)^{-1}$ and multiplicativity follows from (3.1). Consider next $\omega_0.V_m\omega_1.\alpha$ and $V_m\omega_0.\omega_1.\alpha$:

$$\omega_0.(V_m\omega_1.\alpha)=\omega_0.V_m(\omega_1.F_m\alpha)=V_m(F_m\omega_0.(\omega_1.F_m\alpha))$$
$$\underset{(3.1)}{=}V_m((F_m\omega_0.\omega_1).F_m\alpha)=V_m(F_m\omega_0.\omega_1).\alpha=(\omega_0.V_m\omega_1).\alpha$$
$$V_m\omega_0.(\omega_1.\alpha)=V_m(\omega_0.F_m(\omega_1.\alpha))\underset{(\text{by }(3.7)\text{ below})}{=}V_m(\omega_0.(F_m\omega_1.F_m\alpha))$$
$$=V_m((\omega_0.F_m\omega_1).F_m\alpha)=(V_m\omega_0.\omega_1).\alpha.$$

The case $V_m\omega_0.V_n\omega_1.\alpha$ is similar, and is left for the reader. Q.E.D.

Remark **(3.6)**. — We will be most interested in the subgroup $S\hat{C}K_p(R)$ generated by symbols. This subgroup is clearly stable under the operations V_n and under multiplication by elements $(1-rt)^{-1}\in W(R)$. To verify for a particular R that $S\hat{C}K_p(R)$ is a W(R)-module, it suffices, by (§ 3 (5.1)), to show stability under F_n. This will be checked for R local in § 5 (1.1).

Proposition (**3.7**). — *Let* $\omega \in \mathrm{W(R)}$, $\alpha \in \hat{\mathrm{C}}\mathrm{K}_p(\mathrm{R})$, *and let* $n \geq 1$ *be an integer. Then* $\mathrm{F}_n(\omega . \alpha) = \mathrm{F}_n(\omega) . \mathrm{F}_n(\alpha)$.

Proof. — By specialization and linearity we reduce to the case $\mathrm{R} = \mathrm{S}[r]$, r an independent variable, and $\omega = (1 - rt^m)^{-1}$. Using the fact that the K-theory of a polynomial ring injects in the K-theory of the Laurent ring, we reduce to the case $r \in \mathrm{R}^\times$.

Let $\mathrm{R}' = \mathrm{R}[\mathrm{X}]/(\mathrm{X}^m - r)$, write $x \in \mathrm{R}'$ for the image of X, and let $i : \mathrm{R} \to \mathrm{R}'$ be the natural map. Fix $q \geq 1$ and let $\varphi_n : \mathrm{R}_{q-1} \to \mathrm{R}_{nq-1}$ be the map $t \mapsto t^n$. The squares in the diagram below are cocartesian:

$$\begin{array}{ccccc}
\mathrm{R}_{qn-1} & \xrightarrow{i} & \mathrm{R}'_{qn-1} & \xrightarrow[t \mapsto xt]{\cong} & \mathrm{R}'_{qn-1} \\
\uparrow{\scriptstyle \varphi_n} & & \uparrow{\scriptstyle \varphi_n} & & \uparrow{\scriptstyle \varphi_n} \\
\mathrm{R}_{q-1} & \xrightarrow{i} & \mathrm{R}'_{q-1} & \xrightarrow[t \mapsto x^n t]{\cong} & \mathrm{R}'_{q-1}
\end{array}$$

so by (§ 1 (2.4)) we get a commutative diagram of K-groups:

$$\begin{array}{ccccccc}
\mathrm{K}_p(\mathrm{R}_{qn-1}) & \xrightarrow{i^*} & \mathrm{K}_p(\mathrm{R}'_{qn-1}) & \xrightarrow[t \mapsto xt]{} & \mathrm{K}_p(\mathrm{R}'_{qn-1}) & \xrightarrow{i_*} & \mathrm{K}_p(\mathrm{R}_{qn-1}) \\
\downarrow{\scriptstyle \mathrm{F}_n} & & \downarrow{\scriptstyle \mathrm{F}_n} & & \downarrow{\scriptstyle \mathrm{F}_n} & & \downarrow{\scriptstyle \mathrm{F}_n} \\
\mathrm{K}_p(\mathrm{R}_{q-1}) & \xrightarrow{i^*} & \mathrm{K}_p(\mathrm{R}'_{q-1}) & \xrightarrow[t \mapsto x^n t]{} & \mathrm{K}_p(\mathrm{R}'_{q-1}) & \xrightarrow{i_*} & \mathrm{K}_p(\mathrm{R}_{q-1})
\end{array}$$

Using the projection formula (3.4) (ii), the top and bottom horizontal arrows are multiplication by $\omega = i_*(1 - xt)^{-1}$ and $\mathrm{F}_n\omega = i_*(1 - x^n t)^{-1}$ respectively. Q.E.D.

3. COMPUTATION OF $\mathrm{SC}_n\mathrm{K}_q(\mathrm{R})$

1. As before, we fix a ring R and an integer $q \geq 1$. Define $\Phi_n = \Phi_n\mathrm{K}_q(\mathrm{R})$ by the exact sequence:

$$0 \to \Phi_n \to \mathrm{SC}_n\mathrm{K}_q(\mathrm{R}) \to \mathrm{SC}_{n-1}\mathrm{K}_q(\mathrm{R}) \to 0.$$

The purpose of this section is to begin the computation of Φ_n for R a smooth, local k-algebra, k a perfect field of characteristic $p \neq 0,2$.

Proposition (**1.1**). — *Viewing* T *as an element of* $\mathrm{K}_1(\mathrm{R}_\infty[\mathrm{T}^{-1}])$, *the product structure on* K_* *gives a map*:

$$\cdot\mathrm{T} : \mathrm{K}_q(\mathrm{R}_\infty[\mathrm{T}^{-1}]) \to \mathrm{K}_{q+1}(\mathrm{R}_\infty[\mathrm{T}^{-1}]).$$

Assuming R *is regular and local, there is an induced map*:

$$SC_\infty K_q(R) \xrightarrow{\cdot T} SC_\infty K_{q+1}(R).$$

Proof. — Let $i : R \to R_\infty$ be the natural map. The localization sequence ([24], (3.2)):

$$\ldots \to K_q(R_\infty) \to K_q(R_\infty[T^{-1}]) \to K_{q-1}(R) \to K_{q-1}(R_\infty) \to \ldots$$

breaks into split exact 3-term sequences:

$$(1.1.1) \qquad 0 \to K_{q+1}(R_\infty) \to K_{q+1}(R_\infty[T^{-1}]) \underset{s}{\rightleftarrows} K_q(R) \to 0,$$

the splitting being given by $s(x) = i^*(x).T$. In fact, for any ring A, let H_A = category of A-modules which admit a finite resolution by finitely generated projectives. For $S \subset A$ a multiplicative monoid of central non-zero divisors, let $H_{A,S} \subset H_A$ be the sub-category of modules M such that $M \otimes_A A[S^{-1}] = (0)$. The sequence of spaces:

$$BQH_{A,S} \to BQH_A \to BQH_{A[S^{-1}]}$$

is a homotopy fibration [15 *bis*]. In our case, tensor product induces a diagram of fibrations [15]:

$$\begin{array}{ccccc}
\mathrm{Nil}(\mathbf{Z}) \wedge BQ\mathscr{P}(R) & \longrightarrow & BQH_{\mathbf{Z}[T]} \wedge BQ\mathscr{P}(R) & \longrightarrow & BQH_{\mathbf{Z}[T,T^{-1}]} \wedge BQ\mathscr{P}(R) \\
\downarrow & & \downarrow & & \downarrow \\
BQH_{R_\infty, T} & \longrightarrow & BQH_{R_\infty} & \longrightarrow & BQH_{R_\infty[T^{-1}]}
\end{array}$$

whence a diagram of homotopy groups:

$$\begin{array}{ccc}
x.(T) & \longmapsto & 1.x \\
\pi_{q+1}(BQH_{\mathbf{Z}[T,T^{-1}]} \wedge BQ\mathscr{P}(R)) & \longrightarrow & \pi_q(\mathrm{Nil}(\mathbf{Z}) \wedge BQ\mathscr{P}(R)) \\
\downarrow & & \downarrow \\
\pi_{q+1}(BQH_{R_\infty[T^{-1}]}) & & \pi_q(BQH_{R_\infty, T}) \\
\wr\Vert & & \wr\Vert \\
i^*(x).T \in K_q(R_\infty[T^{-1}]) & \longrightarrow & K_{q-1}(R) \ni x
\end{array}$$

$$x.(T) \mapsto i^*(x).T, \qquad 1.x \mapsto x, \qquad i^*(x).T \mapsto x$$

Returning to (1.1.1), we see in particular that $SC_\infty K_{q+1} \subset K_{q+1}(R_\infty[T^{-1}])$. To prove (1.1), it suffices to show $T.SC_\infty K_q(R) \subset SC_\infty K_{q+1}(R)$. $SC_\infty K_q(R)$ is generated by symbols $\{x_1, \ldots, x_q\}$ with $x_i \in R^*_\infty$ and $x_1 \in 1+TR_\infty$ (§ 1, 1). Using the fact that R is local, one checks easily that at least one of the elements x_1, $(1-T)x_1$ has the form $1-Tu$ for $u \in R^\times_\infty$. Using the *Steinberg identity* $\{T, 1-T\}=1$, we have, computing in $K_{q+1}(R_\infty[T^{-1}])$:

$$\{T, x_1, \ldots, x_q\}=\{T, (1-T)\, x_1, x_2, \ldots, x_q\}=\{T, 1-Tu, x_2, \ldots, x_q\}$$
$$=\{u^{-1}, 1-Tu, x_2, \ldots, x_q\} \in SC_\infty\, K_{q+1}(R). \qquad \text{Q.E.D.}$$

Remark (**1.2**). — It will be convenient to view $\{T, x_1, \ldots, x_q\}$ as a element in $SC_n K_{q+1}(R)$ for $n<\infty$. This is really an abuse of notation in that $x \mapsto 1$ in $R^\times_n$ does not for example imply $\{T, x\}=1$ in $C_n K_2(R)$.

Proposition (**1.3**). — *Assume* R *regular local,* $1/2 \in R$, *and* $n<\infty$. *Then* $SC_n K_q(R)$ *is generated by symbols*:

$$\{x, r_1, \ldots, r_{q-1}\} \quad \text{and} \quad \{x, T, r_1, \ldots, r_{q-2}\}$$

with $x \in 1+TR_\infty$, $r_i \in R^\times$.

Proof. — We have $\{T, T\}=\{T, -1\}$, so if suffices to prove the assertion for $q=2$. We proceed by induction on n, the case $n=0$ being trivial. Using (I.2.3.3), we see that $\Phi_n K_2(R)$ is generated by symbols $\{1+rT^n, x\}$, $r \in R$, $x \in R^\times_n$. Similarly:

$$\text{Tangent } K_2(R_n) \underset{(\text{dfn.})}{=} \operatorname{Ker}(K_2(R_n[\varepsilon]/(\varepsilon^2)) \underset{\varepsilon \mapsto 0}{\longrightarrow} K_2(R_n))$$

is generated by symbols $\{1+b\varepsilon, a\}$, $b \in R_n$, $a \in R^\times_n$. Therefore we get a surjection:

$$\text{(1.3.1)} \qquad \text{Tangent } K_2(R_n) \underset{\varepsilon \mapsto T^n}{\twoheadrightarrow} \Phi_n K_2(R).$$

The following is due to Van der Kallen [27].

Lemma (**1.4**). — *Let* A *be a commutative ring with* $1/2 \in A$. *Then* Tangent $K_2(A) \simeq \Omega^1_A$, *the module of absolute Kähler differentials of* A. *The isomorphism is given explicitly for* A *generated additively by units, by*:

$$b\frac{da}{a} \mapsto \{1+b\varepsilon, a\} \qquad a \in A^\times,\ b \in A.$$

For our application, notice:

$$\Omega^1_{R_n} \simeq (\Omega^1_R \otimes \mathbf{Z}[T]/(T^{n+1})) \oplus (R \otimes \Omega^1_{\mathbf{Z}[T]/(T^{n+1})}).$$

It follows that Tangent $K_2(R_n)$ is generated by symbols:

$$\{1+b\varepsilon, r\}, \qquad b \in R_n, \qquad r \in R^\times$$

and:
$$\{1+b\varepsilon, 1-T\}.$$

The map $R_n[\varepsilon]\to R_n$, $\varepsilon\mapsto T^n$ sends $T\varepsilon\mapsto 0$, so the surjection (1.3.1) shows that $\Phi_n K_2(R)$ is generated by symbols:

$$\{1+sT^n, r\} \text{ and } \{1+sT^n, 1-T\} \quad s\in R,\ r\in R^\times.$$

Computing in $C_\infty K_2(R)$ we have:

$$\begin{aligned}\{1-sT^{n+1}, T\} &= \{(1-T)(1-sT^{n+1}), T\}\\ &= \{1-T(1+sT^n-sT^{n+1}), T\}\\ &= \{(1-T)(1-sT^{n+1}), 1+sT^n-sT^{n+1}\}^{-1}\\ &\equiv \{1+sT^n, 1-T\} \pmod{T^{n+1}}.\end{aligned}$$

Thus $\Phi_n K_2(R)$ is generated by symbols:

$$\{1+sT^n, r\} \text{ and } \{1-sT^{n+1}, T\} \quad s\in R,\ r\in R^\times.$$

Finally $C_n K_2(R)$ is built up by successive extensions of the $\Phi_m K_2(R)$, so the proof of (1.3) is complete. Q.E.D.

In the course of the proof we have seen:

Corollary **(1.5)**. — *In* $C_n K_q(R)$ *we have*:

$$\{1-sT^{n+1}, T, \ldots\} = \{1+sT^n, 1-T, \ldots\}$$

$$\{1-sT^m, T, \ldots\} = 1, \quad m\geq n+2.$$

Define a filtration $\mathrm{filt}'\, SC_\infty K_q(R)$ by taking filt'^n to be generated by symbols:

$$\{x, r_1, \ldots, r_{q-1}\} \text{ and } \{y, T, r_1, \ldots, r_{q-2}\},$$

$$x\in 1+T^{n+1}R_\infty,\quad y\in 1+T^{n+2}R_\infty,\quad r_i\in R^\times.$$

Define $\Phi'_n = \Phi'_n K_q(R) = \mathrm{filt}'^{n-1}/\mathrm{filt}'^n$. Notice that, by (1.3), filt'^0 is dense in $SC_\infty K_q(R)$ in the sense that it maps surjectively to $SC_n K_q(R)$ for any $n<\infty$. There are natural maps $\mathrm{filt}'^n\to\mathrm{filt}^n$, and hence $\Phi'^n\to\Phi^n$. It follows from the proof of (1.4) that $\mathrm{filt}'^n\, SC_\infty K_2(R) = \mathrm{filt}^n\, SC_\infty K_2(R)$ so $\Phi'_n K_2(R)\cong\Phi_n K_2(R)$.

2. *Proposition* **(2.1)**. — *We continue to assume* R *regular local, and* $1/2\in R$. *Then there is a well-defined homomorphism of abelian groups*:

$$\rho'_1 : \Omega^{q-1}_R \to \Phi'_n K_q(R)$$

$$\rho'_1(r\,ds_1\wedge\ldots\wedge ds_{q-1}) = \{1+rs_1s_2\ldots s_{q-1}T^n, s_1, \ldots, s_{q-1}\},\quad s_i\in R^\times.$$

Proof. — Let M be the free R-module on generators ds, $s\in R^\times$, and define N by the exact sequence:

$$0\to N\to M\to\Omega^1_R\to 0.$$

Then $\Omega^{q-1}_R\cong\Lambda^{q-1}_R M/P$ where P is the submodule of $\Lambda^{q-1}_R M$ generated as an abelian group by elements $n\wedge ds_2\wedge\ldots\wedge ds_{q-1}$, $n\in N$. Since ρ'_1 clearly induces a map of abelian groups:

$$\Lambda^{q-1}_R M\to\Phi'_n$$

it suffices to show $\rho_1'(\tau \wedge ds_2 \wedge \ldots \wedge ds_{q-1}) = 0$ when $\tau = \sum_i r^{(i)} ds^{(i)} = 0$ in Ω^1_{R}. Since $s_2 s_3 \ldots s_{q-1}\tau = 0$ we have $\prod_i \{1 + s_2 \ldots s_{q-1} r^{(i)} s^{(i)} \mathrm{T}^n, s^{(i)}\} = 1$ in $\Phi'_n \mathrm{K}_2(\mathrm{R})$ by (1.6) and (1.4). Multiplying by $\{s_2, \ldots, s_{q-1}\} \in \mathrm{K}_{q-2}(\mathrm{R})$ gives:

$$\rho_1'(\tau \wedge ds_2 \wedge \ldots \wedge ds_{q-1}) = \prod_i \{1 + s_2 \ldots s_{q-1} r^{(i)} s^{(i)} \mathrm{T}^n, s^{(i)}, s_2, \ldots, s_{q-1}\} = 1.$$

Q.E.D.

Proposition **(2.2)**. — *Let* R *be regular local with* $1/2 \in \mathrm{R}$. *Then there is a well-defined homomorphism*:

$$\rho_2' : \Omega^{q-2}_{\mathrm{R}} / d\Omega^{q-3}_{\mathrm{R}} \to \Phi'_n \mathrm{K}_q(\mathrm{R})$$
$$\rho_2'(r ds_1 \wedge \ldots \wedge ds_{q-2}) = \{1 + r s_1 \ldots s_{q-2} \mathrm{T}^{n+1}, s_1, \ldots, s_{q-2}, \mathrm{T}\}.$$

Proof. — It follows from the description of filt′ $\mathrm{SC}_\infty \mathrm{K}^*$ that multiplication by T induces a map:

$$\Phi'_{n+1} \mathrm{K}_{q-1}(\mathrm{R}) \xrightarrow{\cdot \mathrm{T}} \Phi'_n \mathrm{K}_q(\mathrm{R}).$$

This can be composed with $\rho_1' : \Omega^{q-2} \to \Phi'_{n+1} \mathrm{K}_{q-1}$ to give $\rho_2' : \Omega^{q-2}_{\mathrm{R}} \to \Phi'_n \mathrm{K}_q(\mathrm{R})$. We must show $\rho_2'(d\Omega^{q-3}_{\mathrm{R}}) = (1)$. But using $\{s, -s\} = 1$ we see:

$$\begin{aligned}\{1 + s_1 \ldots s_{q-2} \mathrm{T}^{n+1}, s_1, \ldots, s_{q-2}, \mathrm{T}\} \\ &= \{1 + s_1 \ldots s_{q-2} \mathrm{T}^{n+1}, \pm s_1 s_2 \ldots s_{q-2}, s_2, \ldots, s_{q-2}, \mathrm{T}\} \\ &= \{1 + s_1 \ldots s_{q-2} \mathrm{T}^{n+1}, \pm \mathrm{T}^{n+1}, s_2, \ldots, s_{q-2}, \mathrm{T}\}^{-1} \\ &= \{1 + s_1 \ldots s_{q-2} \mathrm{T}^{n+1}, \pm 1, s_2, \ldots, s_{q-2}, \mathrm{T}\}^{-1}.\end{aligned}$$

By assumption $1/2 \in \mathrm{R}$ so the bottom symbol is trivial. Q.E.D.

Remarks **(2.3)**. — (i) We can compose the ρ_i' with $\Phi'_n \to \Phi_n$ to get maps:

$$\rho_1 : \Omega^{q-1}_{\mathrm{R}} \to \Phi_n \mathrm{K}_q(\mathrm{R}); \qquad \rho_2 : \Omega^{q-2}_{\mathrm{R}} / d\Omega^{q-3}_{\mathrm{R}} \to \Phi_n \mathrm{K}_q(\mathrm{R}).$$

(ii) Looking at the generators for filt'^{n-1}, we see:

$$\rho' = \rho_1' \oplus \rho_2' : \Omega^{q-1}_{\mathrm{R}} \oplus \Omega^{q-2}_{\mathrm{R}} / d\Omega^{q-3}_{\mathrm{R}} \to \Phi'_n \mathrm{K}_q(\mathrm{R})$$

is surjective.

4. COMPUTATION OF $\mathrm{SC}_n \mathrm{K}_q(\mathrm{R})$ (Continued)

1. In this section R will be a regular local $\mathbf{F}_p$-algebra, $p \neq 2$. Eventually we will assume R smooth over a perfect field k and q an integer $\leq p$. We keep the notations of § 3.

Proposition **(1.1).** — *The maps* $\rho_2', \rho_2 : \Omega_R^{q-2}/d\Omega_R^{q-3} \to \Phi_n', \Phi_n$ *are zero unless* $p \mid n+1$.

Proof. — It suffices to show $\rho_2'((n+1)\, r ds_1 \wedge \ldots \wedge ds_{q-2}) = 0$ where $r, s_i \in R^\times$. Indeed, if $r \notin R^\times$ then $r+1 \in R^\times$ and we simply add $ds_1 \wedge \ldots \wedge ds_{q-2}$ to the form. We have:

$$\begin{aligned} &\{1 + rs_1 \ldots s_{q-2} T^{n+1}, s_1, \ldots, s_{q-2}, T\}^{n+1} \\ &\quad = \{1 + rs_1 \ldots s_{q-2} T^{n+1}, s_1, \ldots, s_{q-2}, -rs \ldots s_{q-2}\}^{-1} \in \mathrm{filt}'^n \, SC\, K_q(R). \end{aligned}$$

Q.E.D.

Recall (I.3.3.3) that for any commutative ring A with $\frac{1}{(q-1)!} \in A$ there is a map:

$$\text{(1.2)} \qquad \frac{(-1)^{q-1}}{(q-1)!} d \log : K_q(A) \to \Omega^q_{A/\mathbf{Z}}$$

given on symbols by:

$$\frac{(-1)^{q-1}}{(q-1)!} d \log \{a_1, \ldots, a_q\} = \frac{da_1}{a_1} \wedge \ldots \wedge \frac{da_q}{a_q}.$$

Proposition **(1.3).** — *If* $p \nmid n$ *and* $q \leq p$, *the maps* ρ_1, ρ_1' *are injective.*

Proof. — The assumption implies $\frac{1}{(q-1)!} \in R$. We have:

$$\frac{(-1)^{q-1}}{(q-1)!} d \log : K_q(R_n) \to (\Omega_R^q \otimes_{\mathbf{F}_p} \mathbf{F}_p[t]/(T^{n+1})) \oplus (\Omega_R^{q-1} \otimes_{\mathbf{F}_p} \Omega^1_{\mathbf{F}_p[t]/(T^{n+1})})$$

and:

$$\frac{(-1)^{q-1}}{(q-1)!} d \log(\rho_1(\tau)) = (\text{something}) \oplus (\tau \otimes (n T^{n-1} dT)).$$

It follows that ρ_1 (and hence also ρ_1') is injective if $p \nmid n$. Q.E.D.

Proposition **(1.4).** — *Let* $n = mp^r$, $p \nmid m$, $r \geq 1$. *Then* $\operatorname{Ker} \rho_1 \supseteq \operatorname{Ker} \rho_1' \supseteq D_r$, *where* $D_r \subseteq \Omega_R^{q-1}$ *is the subgroup generated by elements*:

$$a^{p^\ell - 1} da \wedge \frac{da_1}{a_1} \wedge \ldots \wedge \frac{da_{q-2}}{a_{q-2}}; \qquad a \in R, \qquad a_1, \ldots, a_{q-2} \in R^\times, \qquad 0 \leq \ell \leq r-1$$

$(D_0 = (0)$ *and* $D_1 = d\Omega^{q-2})$.

Proof. — It will suffice to show:

$$\rho_1'(a^{p^\ell - 1} da) = 0 \text{ in } \Phi_n' K_2(R)$$

(the desired element is obtained from this one by multiplication by:

$$\{a_1, \ldots, a_{q-2}\} \in K_{q-2}(R).)$$

Suppose first $a \in R^{\times}$. Then:

$$\begin{aligned}\rho_1'(a^{p^\ell - 1}da) &= \{1 + a^{p^\ell}T^{mp^r}, a\} \\ &= \{1 + aT^{mp^{r-\ell}}, a\}^{p^\ell} \\ &= \{1 + a^{p^\ell}T^{mp^r}, \pm T\}^{-mp^{r-\ell}} = 0 \text{ in } \Phi_n' K_2(R)\end{aligned}$$

since $r > \ell$.

To finish the proof, it will suffice to show $D_r \subset \Omega^1_R$ is generated by elements $a^{p^\ell - 1}da$ for $a \in R^{\times}$. For this, recall Serre [25] defines a map $d : W_r^{(p)}(R) \to \Omega^1_R$ which is defined on a vector $(a_0, \ldots, a_{r-1})$ by:

$$d(a_0, \ldots, a_{r-1}) = da_{r-1} + a_{r-2}^{p-1} da_{r-2} + \ldots + a_0^{p^{r-1}} da_0.$$

Clearly $D_r = d(W_r^{(p)}(R))$ so it suffices to show:

Lemma **(1.5)**. — $W_r^{(p)}(R)$ *is additively generated by vectors all of whose Witt components are zero or invertible.*

Proof of **(1.5)**. — By induction on r, $W_1^{(p)}(R) = R$ local, so we may assume $r > 1$. Now use the exact sequence $0 \to W_1^{(p)}(R) \to W_r^{(p)}(R) \to W_{r-1}^{(p)}(R) \to 0$. Q.E.D.

2. The following theorem was proven in (I, § 3, 3).

Theorem **(2.1)** — *Let* A *be an* $\mathbf{F}_p$*-algebra, and let* A_N *be a flat* $\mathbf{Z}/p^{N+1}\mathbf{Z}$*-algebra (some* $N \geq 1$*) such that* $A_N/pA_N \cong A$. *Wedge product of differentials makes:*

$$\bigoplus_{q=0}^{p} \Omega^q_{A_N, \text{closed}}/pd\Omega^{q-1}_{A_N}$$

a graded ring and there is a natural ring homomorphism:

$$\bigoplus_q \frac{(-1)^{q-1}}{(q-1)!} \text{crys-}d\log : \bigoplus_{q=0}^{p} K_q(A) \to \bigoplus_{q=0}^{p} \Omega^q_{A_N, \text{closed}}/pd\Omega^{q-1}_{A_N}$$

with property that $\text{crys-}d\log(a) = d\tilde{a}/\tilde{a}$ *for* $a \in A^{\times}$ *and* $\tilde{a} \in A_N$ *any lifting.*

Proposition **(2.2)**. — *Now assume in addition to the above hypotheses on* R *that* R *is smooth over a perfect field* k. *Let* $n = mp^r$, $r \geq 1$, $p \nmid m$, *and assume* $q \leq p$. *Then:*

$$\operatorname{Ker} \rho_1 = \operatorname{Ker} \rho_1' = D_r.$$

In particular, $\rho_1' : \Omega^{q-1}_R/D_r \xrightarrow{\approx} \Phi_n' K_q(R)$.

Proof. — Using (1.1), it suffices to show $\operatorname{Ker} \rho_1 \subseteq D_r$. Let $\tilde{R}$ be a p-adically complete and separated local ring, flat over $\mathbf{Z}_p$, such that $R = \tilde{R}/p\tilde{R}$. Let:

$$\sum_I a_I db_I \in \Omega^{q-1}_R \qquad (db_I = db_{i_1} \wedge \ldots \wedge db_{i_{q-1}})$$

and assume $\rho_1(\sum_I a_I db_I) = 1$. Choose liftings $\tilde{a}_I, \tilde{b}_i \in \tilde{R}$, and let $\tilde{A} = \tilde{R}[T]/(T^{n+1})$.

Applying (2.1) to the lifting $\widetilde{A}/p^{N+1}\widetilde{A}$ of R_n for large N:

$$(2.3)\qquad \frac{(-1)^{q-1}}{(q-1)!}\text{crys-}d\log\circ\rho_1(\sum_I a_I db_I)\equiv nT^{n-1}dT\wedge\sum_I \widetilde{a}_I d\widetilde{b}_I + T^n\sum_I d\widetilde{a}_I\wedge d\widetilde{b}_I \pmod{p^N\Omega^q_{\widetilde{A}}+pd\Omega^{q-1}_{\widetilde{A}}}.$$

Let α denote the right side of (2.3) (viewed as an element in $\Omega^q_{\widetilde{A}}$ for the particular liftings $\widetilde{a}_I$, $\widetilde{b}_i$). By assumption:

$$\alpha\equiv pd\omega_N \pmod{p^N}$$

for some $\omega_N=T^n\xi_N+\eta_N T^{n-1}dT+$terms of lower order in T, $\xi_N\in\Omega^{q-1}_{\widetilde{R}}$, $\eta_N\in\Omega^{q-2}_{\widetilde{R}}$ (we write $\Omega^*_{\widetilde{R}}$ for the p-adically separated differentials).

We may now differentiate and divide by n ($\Omega^*_{\widetilde{A}}$ is torsion free) to get:

$$\sum_I \widetilde{a}_I d\widetilde{b}_I\equiv p\xi_N-pn^{-1}d\eta_N \pmod{p^{N-r}\Omega^{q-1}_{\widetilde{R}}}.$$

In particular, for $N\gg 0$ we have (replacing η_N by $m^{-1}\eta_N$):

$$d\eta_N\in p^{r-1}\Omega^{q-1}_{\widetilde{R}}$$

$$\sum_I a_I db_I\equiv p^{1-r}d\eta_N \pmod{p\Omega^{q-1}_{\widetilde{R}}}.$$

Everything now follows from:

Lemma **(2.4)**. — *Let* $\gamma\in\Omega^{q-1}_R$. *Then* $\gamma\in D_r$ *if and only if there exists an* $\eta\in\Omega^{q-2}_{\widetilde{R}}$ *such that* $d\eta\in p^{r-1}\Omega^{q-1}_{\widetilde{R}}$ *and* $\gamma\equiv p^{1-r}d\eta \pmod{p\Omega^{q-1}_{\widetilde{R}}}$.

Proof. — For $I=(i_1,\dots,i_{q-2})$, $b_I=(b_{i_1},\dots,b_{i_{q-2}})$ a $(q-2)$-tuple of units in R or $\widetilde{R}$, I will write:

$$\frac{db_I}{b_I}=\frac{db_{i_1}}{b_{i_1}}\wedge\dots\wedge\frac{db_{i_{q-2}}}{b_{i_{q-2}}},\qquad db_I=db_{i_1}\wedge\dots\wedge db_{i_{q-2}}.$$

Notice $d(a)\wedge db_I=d(ab_{i_1}\dots b_{i_{q-2}})\wedge\dfrac{db_I}{b_I}$ so $D_1=\Omega^{q-1}_{R,\text{exact}}$. (2.4) clearly holds in this case. Also if $\gamma=a^{p^\ell-1}da\wedge\dfrac{db_I}{b_I}$ we can take:

$$\eta=p^{r-\ell-1}\widetilde{a}^{p^\ell}\frac{d\widetilde{b}_I}{\widetilde{b}_I}$$

for a lifting.

Suppose now we have η with $d\eta\in p^{r-1}\Omega^{q-1}_{\widetilde{R}}$, $r>1$. I claim η can be written (up to alteration by a closed form which doesn't affect $d\eta$):

$$\eta=\sum_I\gamma_{0,I}^{p^{r-1}}\frac{d\delta_{0,I}}{\delta_{0,I}}+\dots+p^{r-1}\sum_I\gamma_{r-1,I}\frac{d\delta_{r-1,I}}{\delta_{r-1,I}};\qquad \delta_{j,I}\in\widetilde{R}^{\times q-2}.$$

Assume inductively we can write:

$$\eta=\sum_{I}\alpha_{0,I}^{p^{r-2}}\frac{d\beta_{0,I}}{\beta_{0,I}}+\ldots+p^{r-2}\sum_{I}\alpha_{r-2,I}\frac{d\beta_{r-2,I}}{\beta_{r-2,I}}$$

so:

$$(2.5)\qquad 0\equiv p^{2-r}\,d\eta=\sum_{I}\alpha_{0,I}^{p^{r-2}-1}\,d\alpha_{0,I}\wedge\frac{d\beta_{0,I}}{\beta_{0,I}}+\ldots+\sum_{I}d\alpha_{r-2,I}\frac{d\beta_{r-2,I}}{\beta_{r-2,I}}\pmod{p}.$$

(Note $\alpha_{0,I}\in\widetilde{R}$, $\beta_{0,I}\in\widetilde{R}^{q-2}$.)

Recall the *Cartier operator* $C:\Omega^{q-1}_{R,\,\text{closed}}\to\Omega^{q-1}_{R}$ is defined, and has the following properties:

$$C(d\omega)=0,\qquad C\left(\frac{db_I}{b_I}\right)=\frac{db_I}{b_I},\qquad C(f^p.\omega)=fC(\omega).$$

The inverse Cartier operator C^{-1} induces a p-linear isomorphism:

$$C^{-1}:\ \Omega^{q-1}_{R}\xrightarrow{\sim}\Omega^{q-1}_{R,\,\text{closed}}/D_1.$$

We have $C^{-1}(D_r)=D_{r+1}/D_1$, so there is a map:

$$C^{-r}:\ \Omega^{q-1}_{R}\hookrightarrow\Omega^{q-1}_{R}/D_r.$$

Applying C^{r-2} to (2.5) kills all but the first term, and we get:

$$\sum_{I}d\alpha_{0,I}\wedge\frac{d\beta_{0,I}}{\beta_{0,I}}\equiv 0\pmod{p},$$

i.e. $\sum_{I}\alpha_{0,I}\frac{d\beta_{0,I}}{\beta_{0,I}}$ is closed $(\mathrm{mod}\,p)$. This implies for some $\gamma_{0,I}$, $\delta_{0,I}$ (since:

$$C^{-1}:\ \Omega^{q-1}_{R}\xrightarrow{\sim}\Omega^{q-1}_{R,\,\text{closed}}/D_1)$$

that
$$\sum_{I}\alpha_{0,I}\frac{d\beta_{0,I}}{\beta_{0,I}}\equiv\sum_{I}\gamma_{0,I}^{p}\frac{d\delta_{0,I}}{\delta_{0,I}}\pmod{p,\,D_1}.$$

Apply $(C^{-1})^{r-2}$ to get:

$$\sum_{I}\alpha_{0,I}^{p^{r-2}}\frac{d\beta_{0,I}}{\beta_{0,I}}\equiv\sum_{I}\gamma_{0,I}^{p^{r-1}}\frac{d\delta_{0,I}}{\delta_{0,I}}\pmod{p,\,D_r}.$$

Notice elements in D_r can be lifted to closed forms over $\widetilde{R}$ and changing η by a closed form doesn't change $d\eta$. Thus we may assume:

$$\eta=\sum_{I}\gamma_{0,I}^{p^{r-1}}\frac{d\delta_{0,I}}{\delta_{0,I}}+p\eta'$$

for some $\eta'\in\Omega^{q-2}_{\widetilde{R}}$. Now:

$$\eta'=\frac{\eta-\sum_{I}\gamma_{0,I}^{p^{r-1}}\frac{d\delta_{0,I}}{\delta_{0,I}}}{p}$$

satisfies $d\eta' \in p^{r-2}\Omega_{\widetilde{R}}^{q-1}$. By induction, we can change η' by a closed form to get:

$$\eta' = \sum_I \gamma_{1,I}^{p^{r-2}} \frac{d\delta_{1,I}}{\delta_{1,I}} + \ldots + p^{r-2} \sum_I \gamma_{r-1,I} \frac{d\delta_{r-1,I}}{\delta_{r-1,I}}$$

whence:

$$\eta = \sum_I \gamma_{0,I}^{p^{r-1}} \frac{d\delta_{0,I}}{\delta_{0,I}} + \ldots + p^{r-1} \sum_I \gamma_{r-1,I} \frac{d\delta_{r-1,I}}{\delta_{r-1,I}}. \tag{2.6}$$

It follows easily from this that $p^{1-r} d\eta \in D_r \pmod p$. Q.E.D.

Remark **(2.7)**. — Notice that to represent η as in (2.6), it was only necessary to modify the original η by a closed form whose reduction mod p lay in D_r.

3. *Proposition* **(3.1)**. — *Let* R *be as above, and suppose* $n+1 = mp^r$, $p \nmid m$, $r \geq 1$. *Assume also* $q \leq p$. *We have a map* ρ *defined by the composition*:

$$\Omega_{R}^{q-1} \oplus (\Omega_{R}^{q-2}/d\Omega_{R}^{q-3}) \xrightarrow[\rho_1' \oplus \rho_2']{} \Phi_n' \to \Phi_n.$$

Let $E_r \subseteq \Omega_{R,\text{closed}}^{q-2}$ *be the subgroup generated by* D_{r+1} *together with elements of the form* $a^{p^r} \frac{db_I}{b_I}$. *Then* $\operatorname{Ker} \rho \subseteq \{0\} \oplus (E_r/D_1)$.

Proof. — We have:

$$\frac{(-1)^{q-1}}{(q-1)!} d\log \circ \rho(\omega_{q-1}, \omega_{q-2}) = T^{n-1}\omega_{q-1} \wedge dT + T^n d\omega_{q-1} + T^n d\omega_{q-2} \wedge dT$$

whence $\operatorname{Ker} \rho \subset \{0\} \oplus (\Omega_{R,\text{closed}}^{q-2}/d\Omega_{R}^{q-3}) = \{0\} \oplus (E_1/D_1)$. Let $\widetilde{A} = \widetilde{R}[T]/(T^{n+1})$ be a lifting of R_n as before and consider the composition:

$$\frac{(-1)^{q-1}}{(q-1)!} \text{crys-}d\log \circ \rho_2 : \Omega_{R}^{q-2} \to \Omega_{\widetilde{A}}^{q}/p\, d\Omega_{\widetilde{A}}^{q-1} + p^N \Omega_{\widetilde{A}}^{q}, \qquad N \gg 0.$$

Let $\omega \in \Omega_{R}^{q-2}$ be such that $\rho_2(\omega) = 0$, and let $\widetilde{\omega} \in \Omega_{\widetilde{R}}^{q-2}$ be a lifting. We have:

$$\frac{(-1)^{q-1}}{(q-1)!} \text{crys-}d\log \rho_2(\omega) = T^n d\widetilde{\omega} \wedge dT \in p\, d\Omega_{\widetilde{A}}^{q-2} + p^N \Omega_{\widetilde{A}}^{q-1}.$$

Note in this case $\Omega_{\widetilde{A}}^{*}$ has torsion, *viz.*, $p^r T^n dT = 0$ but $p^{r-1} T^n dT \neq 0$. Suppose:

$$T^n d\widetilde{\omega} \wedge dT \equiv pd(T^n \zeta + \eta T^n dT + \text{terms of lower order in T}) \pmod{p^N}.$$

(Here $\zeta \in \Omega_{\widetilde{R}}^{q-1}$, $\eta \in \Omega_{\widetilde{R}}^{q-2}$.) Equating powers of T we get:

$$T^n d\widetilde{\omega} \wedge dT \equiv pT^n d\eta \wedge dT \pmod{p^N}$$

so $d(\widetilde{\omega} - p\eta) \in p^r \Omega_{\widetilde{R}}^{q-1}$. As in the proof of (2.4) (cf. also (2.7)) we can write:

$$\widetilde{\omega} = \sum_I \gamma_I^{p^r} \frac{d\beta_I}{\beta_I} + \mu \pmod p$$

where $\mu \equiv$ element in $D_{r+1} \pmod p$. It follows that $\omega \in E_r + D_{r+1}$. Q.E.D.

Proposition **(3.2)**. — *Assume as above that* $n+1=mp^r$, $r\geq 1$, $p\nmid m$, *and* $q\leq p$. *Then* $\operatorname{Ker}\rho_2'=\operatorname{Ker}\rho_2=E_r/D_1$.

Proof. — We have from (3.1):

$$\operatorname{Ker}\rho_2'\subset\operatorname{Ker}\rho_2\subset E_r/D_1$$

so it suffices to show $E_r/D_1\subset\operatorname{Ker}\rho_2'$. D_{r+1} leads to symbols:

$$\alpha=\{1+a^{p^\ell}T^{mp^r}, a, a_1, \ldots, a_{q-3}, T\} \quad \ell\leq r.$$

(Note if $q\leq 2$, $(0)=D_{r+1}\subset\Omega^{q-2}$.) We have:

$$\begin{aligned}\alpha&=\{1+aT^{(n+1)p^{-\ell}}, a, \ldots, T\}^{p^\ell}\\ &=\{1+a^{p^\ell}T^{n+1}, -T^{(n+1)p^{-\ell}}, \ldots, T\}=1.\end{aligned}$$

Other elements in E_r lead to symbols:

$$\begin{aligned}\{1+a^{p^r}T^{mp^r}, a_1, \ldots, a_{q-2}, T\}&=\{1+aT^m, \ldots, T\}^{p^r}\\ &=\{1+aT^m, a_1, \ldots, a_{q-2}, -a\}^{-p^r/m}\\ &=\{1+a^{p^r}T^{n+1}, a_1, \ldots, a_{q-2}, -a\}^{1/m}=1 \text{ in } \Phi_n'.\end{aligned}$$

(Note Φ_n' is p-torsion, so $1/m$ makes sense.) Q.E.D.

4. We can now prove:

Theorem **(4.1)**. — *Let* R *be a smooth local k-algebra, where k is a perfect field of characteristic* $p\neq 0,2$. *Fix* $q\leqslant p$. *Then the two filtrations* $\operatorname{filt}^\bullet SC_\infty K_q(R)$ *and* $\operatorname{filt}'^\bullet SC_\infty K_q(R)$ *coincide, so* $\Phi_nK_q(R)=\Phi_n'K_q(R)$. *Moreover*:

(i) *If* $n\not\equiv 0, -1 \pmod p$ $\rho_1:\Omega_R^{q-1}\cong\Phi_nK_q(R)$

(ii) *If* $n=mp^r$, $r\geq 1$, $p\nmid m$, $\rho_1:\Omega_R^{q-1}/D_r\overset{\cong}{\to}\Phi_nK_q(R)$

(iii) *If* $n+1=mp^r$, $r\geq 1$, $p\nmid m$:

$$\rho=\rho_1\oplus\rho_2:\ \Omega_R^{q-1}\oplus(\Omega_R^{q-2}/E_r)\cong\Phi_n.$$

Proof. — We have by (1.1), (1.3) that for $n\not\equiv 0, -1 \pmod p$:

$$\Omega_R^{q-1}\underset{\rho_1'}{\cong}\Phi_n'K_q(R)\hookrightarrow\Phi_nK_q(R). \tag{4.2}$$

When $n=mp^r$, $r\geq 1$, $p\nmid m$ we get by (1.1), (1.3), (2.2):

$$\Omega_R^{q-1}/D_r\underset{\rho_1'}{\cong}\Phi_n'K_q(R)\hookrightarrow\Phi_nK_q(R). \tag{4.3}$$

When $n+1=mp^r$, $r\geq 1$, $p\nmid m$ we get by (§ 4 (2.3) (ii), (3.1), (3.2)):

$$\Omega_R^{q-1}\oplus(\Omega_R^{q-2}/E_r)\underset{\rho_1'\oplus\rho_2'}{\cong}\Phi_n'K_q(R)\hookrightarrow\Phi_nK_q(R). \tag{4.4}$$

In particular, $\Phi_n'\hookrightarrow\Phi_n$ for all n. Writing filt, filt′ for the respective filtrations on $SC_\infty K_q(R)$ we see $\operatorname{filt}'^{n-1}\cap\operatorname{filt}^n=\operatorname{filt}'^n$. Thus, for any $m\leq n$:

$$\operatorname{filt}'^{n-m}\cap\operatorname{filt}^n=\operatorname{filt}'^{n-m}\cap\operatorname{filt}^{n-m+1}\cap\operatorname{filt}^{n-m+2}\cap\ldots\cap\operatorname{filt}^n=\operatorname{filt}'^n.$$

Taking $m=n$, we get:

$$\mathrm{filt}^n=\mathrm{filt}'^0\cap\mathrm{filt}^n=\mathrm{filt}'^n.$$

Thus $\mathrm{filt}^\bullet=\mathrm{filt}'^\bullet$ so $\Phi_n\cong\Phi'_n$ and the theorem follows from (4.2)-(4.4). Q.E.D.

Corollary **(4.5).** — *${}^\bullet$T induces a map:*

$$\mathrm{SC}_n\mathrm{K}_{q-1}(\mathrm{R})\to\mathrm{SC}_{n-1}\mathrm{K}_q(\mathrm{R}).$$

Proof. — We have:

$$\mathrm{SC}_n\mathrm{K}_{q-1}(\mathrm{R})=\mathrm{SC}_\infty\mathrm{K}_{q-1}(\mathrm{R})/\mathrm{filt}^n=\mathrm{SC}_\infty\mathrm{K}_{q-1}(\mathrm{R})/\mathrm{filt}'^n$$
$$\mathrm{SC}_{n-1}\mathrm{K}_q(\mathrm{R})=\mathrm{SC}_\infty\mathrm{K}_q(\mathrm{R})/\mathrm{filt}'^{n-1}.$$

It follows easily from the definitions that the map ${}^\bullet\mathrm{T}:\mathrm{SC}_\infty\mathrm{K}_{q-1}(\mathrm{R})\to\mathrm{SC}_\infty\mathrm{K}_q(\mathrm{R})$ carries filt'^n to filt'^{n-1}. Q.E.D.

5. MODULE STRUCTURE ON $\mathrm{S\hat{C}K}_q$

1. Let R be a local ring. Recall we have defined:

$$\hat{\mathrm{C}}\mathrm{K}_q(\mathrm{R})=\varprojlim\mathrm{C}_n\mathrm{K}_q(\mathrm{R})\qquad\mathrm{S}\hat{\mathrm{C}}\mathrm{K}_q(\mathrm{R})=\varprojlim\mathrm{SC}_n\mathrm{K}_q(\mathrm{R}).$$

We have $\mathrm{S}\hat{\mathrm{C}}\mathrm{K}_q(\mathrm{R})\subset\hat{\mathrm{C}}\mathrm{K}_q(\mathrm{R})$. In this section we check that $\mathrm{S}\hat{\mathrm{C}}\mathrm{K}_q(\mathrm{R})$ is a sub-W(R)-module of $\hat{\mathrm{C}}\mathrm{K}_q(\mathrm{R})$. As noted in (§ 2 (3.6)), it suffices to show:

Proposition **(1.1).** — $\mathrm{S}\hat{\mathrm{C}}\mathrm{K}_q(\mathrm{R})$ *is stable under the operators* F_n.

Proof. — $\mathrm{S}\hat{\mathrm{C}}\mathrm{K}_q(\mathrm{R})$ is generated topologically by symbols:

$$\{\omega,r_1,\ldots,r_{q-1}\},\quad\{\omega,r_1,\ldots,r_{q-2},\mathrm{T}\},\quad r_i\in\mathrm{R}^\times,\quad\omega\in\mathrm{W}(\mathrm{R}).$$

By an easy application of the projection formula:

$$\mathrm{F}_n\{\omega,r_1,\ldots,r_{q-1}\}=\{\mathrm{F}_n\omega,r_1,\ldots,r_{q-1}\}.$$

Again by the projection formula:

$$\mathrm{F}_n\{\omega,\mathrm{T},r_1,\ldots,r_{q-2}\}=\mathrm{F}_n\{\omega,\mathrm{T}\}.\{r_1,\ldots,r_{q-2}\}$$

(multiplication in $\mathrm{K}_*(\mathrm{R}_\infty)$)

so we reduce to the case $q=2$. But $\mathrm{S}\hat{\mathrm{C}}\mathrm{K}_2(\mathrm{R})=\hat{\mathrm{C}}\mathrm{K}_2(\mathrm{R})$. Q.E.D.

Proposition **(1.2).** — *Let m, n be positive integers with* $(m,n)=1$. *Then* $\mathrm{F}_n.\mathrm{V}_m=\mathrm{V}_m.\mathrm{F}_n$ *on* $\mathrm{S}\hat{\mathrm{C}}\mathrm{K}_q(\mathrm{R})$.

Proof. — For symbols $\{\omega,r_1,\ldots,r_{q-1}\}$ this follows because $\mathrm{F}_n\mathrm{V}_m=\mathrm{V}_m\mathrm{F}_n$ on W(R).

For symbols $\{\omega,\mathrm{T},r_1,\ldots,r_{q-2}\}$ we reduce (cf. the proof of (§ 2 (3.7))) to the case $\omega=i_*\omega_0$ where $i:\mathrm{R}\to\mathrm{R}'$ is a ring homomorphism, R' is a finitely generated free

R-module, and $\omega_0 = (1 - rt)^{-1}$ for $r \in R'^{\times}$. The transfer map commutes with F_n and V_m (§ 2 (3.4)), so we get:

$$F_n V_m\{\omega, T, \dots\} = F_n V_m\{i_*\omega_0, T, \dots\} = F_n V_m i_*\{\omega_0, T, \dots\}$$
$$= i_* F_n V_m\{\omega_0, T, \dots\} \underset{(*)}{=} i_* V_m F_n\{\omega_0, T, \dots\} = V_m F_n\{\omega, T, \dots\}.$$

(The interchange of V_m and F_n at $(*)$ is justified because $\{\omega_0, T, \dots\} = \{\omega_0, r, \dots\}^{-1}$, one of the first sort of symbols.) Q.E.D.

Proposition (**1.3**). — *Suppose* R *is a* $\mathbf{Z}/p\mathbf{Z}$*-algebra,* $p \neq 2$. *Then* $V_p \circ F_p =$ multiplication by p *on* $\widehat{SCK}_q(R)$.

Proof. — It suffices to consider symbols $\{1 - rt^m, t, r_1, \dots, r_{q-2}\}$ with $r, r_i \in R^{\times}$. If $(p, m) = 1$, $1/m \in W(R)$ by (I.1.2.3 (v)) so it suffices to note:

$$V_p F_p\{1 - rt^m, t, \dots\}^m = V_p F_p\{1 - rt^m, r, \dots\}^{-1}$$
$$= \{V_p F_p(1 - rt^m), r, \dots\}^{-1} = \{1 - rt^m, r, \dots\}^{-p} = \{\omega, t, \dots\}^p.$$

If $m = pn$ we can use the projection formula:

$$V_p F_p\{1 - rt^{np}, t, \dots\} = V_p\{1 - rt^n, t, \dots\} = \{1 - rt^{np}, t, \dots\}^p. \qquad \text{Q.E.D.}$$

The following result is straightforward, and is left to the reader.

Proposition (**1.4**). — *Let* $r_1, \dots, r_{q-1} \in R^*$, *and let* $\omega, \omega' \in W(R)$. *Then*:

$$\omega.\{\omega', r_1, \dots, r_{q-1}\} = \{\omega\omega', r_1, \dots, r_{q-1}\},$$

where $\omega\omega' \in W(R)$ *denotes the product in the sense of Witt vectors.*

6. THE DERIVATION

1. In this section R will be local and smooth over a perfect field k of characteristic $p \neq 0, 2$. Let $\delta^q : \widehat{SCK}_q(R) \to \widehat{SCK}_{q+1}(R)$ be induced by the maps:

$$\cdot T : SC_n K_q(R) \to SC_{n-1} K_{q+1}(R) \quad (\S\ 4\ (4.5)).$$

Proposition (**1.1**). — (i) $\delta^{q+1} \circ \delta^q = 0$.

(ii) *We have* $V_n \circ \delta^q = n\,\delta^q \circ V_n$; $nF_n \circ \delta^q = \delta^q \circ F_n$; *and* $V_n F_n \delta^q = \delta^q V_n F_n$.

(iii) δ^q *is a* $W(k)$*-homomorphism.*

Proof. — (i) We have $\{T, T\} = \{T, -1\}$. Since the $\widehat{SCK}_q(R)$ are $\mathbf{Z}_q$-modules, they have no 2-torsion, so any symbol $\{T, -1, \dots\}$ is trivial.

(ii) It suffices to check on generators (we write $n\{\ \}$ instead of $\{\ \}^n$):

$$n\,\delta^q V_n\{1-aT^m, r_1, \ldots, r_{q-1}\}=\{1-aT^{mn}, r_1, \ldots, r_{q-1}, T^n\}$$
$$=V_n\,\delta^q\{1-aT^m, r_1, \ldots, r_{q-1}\}.$$
$$n\,\delta^q V_n\{1-aT^m, r_1, \ldots, r_{q-2}, T\}=n\{\ldots, T^n, T\}=0$$
$$=V_n\,\delta^q\{1-aT^m, \ldots, T\}$$
$$nF_n\,\delta^q\{1-aT^m, r_1, \ldots, r_{q-1}\}=F_n\{1-aT^m, r_1, \ldots, r_{q-1}, T^n\}$$
$$=\{F_n(\omega(1-aT^m)), r_1, \ldots, r_{q-1}, T\}$$
$$=\delta^q F_n\{1-aT^m, r_1, \ldots, r_{q-1}\}.$$

The other identities are more complicated. We may assume n prime. If $(n, p)=1$:

$$\delta^q F_n\{1-aT^m, T, r_1, \ldots, r_{q-2}\}=\delta^q F_n\,\delta^{q-1}\{1-aT^m, r_1, \ldots, r_{q-2}\}$$
$$=\frac{1}{n}\delta^q\,\delta^{q-1}F_n\{\ldots\}=0$$
$$=nF_n\,\delta^q\{1-aT^m, T, \ldots\}.$$

If $(m, p)=1$ we have:

$$m\,\delta^q F_n\{1-aT^m, T, r_1, \ldots, r_{q-2}\}=\delta^q F_n\{1-aT^m, a^{-1}, r_1, \ldots, r_{q-2}\}$$
$$=nF_n\,\delta^q\{1-aT^m, a^{-1}, r_1, \ldots\}$$
$$=mnF_n\,\delta^q\{1-aT^m, T, \ldots\}$$

and we can cancel the m. Finally if $n=p$, $m=rp$, we have:

$$\delta^q F_n\{1-aT^{rp}, T, r_1, \ldots, r_{q-2}\}=\{1-aT^r, T, r_1, \ldots, r_{q-2}, T\}$$
$$=0$$
$$=nF\delta^q\{1-aT^{rp}, T, \ldots\}.$$

It remains to show $V_nF_n\,\delta^q=\delta^q V_nF_n$. This is an easy consequence of the above when $(n, p)=1$. When $n=p$, we have:

$$\delta^q V_pF_p\{1-aT^m, r_1, \ldots, r_{q-1}\}=p\{1-aT^m, r_1, \ldots, r_{q-1}, T\}.$$

Assuming first $(m, p)=1$, we get:

$$V_pF_p\,\delta^q\{1-aT^m, r_1, \ldots, r_{q-1}\}=V_pF_p\{1-aT^m, r_1, \ldots, r_{q-1}, T\}$$
$$=\frac{1}{m}V_pF_p\{1-aT^m, r_1, \ldots, r_{q-1}, a^{-1}\}$$
$$=\frac{p}{m}\{1-aT^m, r_1, \ldots, r_{q-1}, a^{-1}\}$$
$$=p\{1-aT^m, r_1, \ldots, r_{q-1}, T\}.$$

If $m=pr$ we have:

$$V_pF_p\,\delta^q\{1-aT^{pr}, r_1, \ldots, r_{q-1}\}=V_pF_p\{1-aT^{pr}, r_1, \ldots, r_{q-1}, T\}$$
$$=V_p\{1-aT^r, r_1, \ldots, r_{q-1}, T\}$$
$$=p\{1-aT^{pr}, r_1, \ldots, r_{q-1}, T\}.$$

The verification that:

$$\delta^q V_pF_p\{1-aT^m, r_1, \ldots, r_{q-2}, T\}=V_pF_p\,\delta^q\{\ldots, T\}=0$$

is similar, and is omitted, as is the straightforward verification of (iii). Q.E.D.

We continue to assume R local and smooth over a perfect field k of characteristic $p \neq 0, 2$. We have $\delta_n^{q-1} : SC_nK_{q-1}(R) \to SC_{n-1}K_q(R)$. Let $\theta \in SC_nK_{q-1}(R)$ be a symbol such that $\delta_n^q\theta = 0$. At this point it is convenient to start writing the group law on $C_nK_q(R)$ *additively*.

Lemma **(1.2)**. — *Let* $\omega, \omega' \in W_n(R)$. *Then:*

$$\omega . \delta_n^{q-1}(\omega'\,\theta) + \omega'\delta_n^{q-1}(\omega\theta) = \delta_n^{q-1}(\omega\omega'\theta).$$

Proof. — We may assume as usual:

$$\omega = \omega(1 - aT^r)^{-1}, \qquad \omega' = \omega(1 - bT^s)^{-1}, \qquad a, b \in R^\times.$$

Suppose first $r = s = 1$. We have $\{\theta, T\} = 0$, so by (§ 2 (3.1)), $0 = \{\omega\omega'\theta, abT\}$ and:

$$\omega\,\delta^{q-1}(\omega'\theta) = \{\omega\omega'\theta, aT\}$$
$$\omega'\delta^{q-1}(\omega\theta) = \{\omega\omega'\theta, bT\} = \{\omega\omega'\theta, a^{-1}\}$$
$$\omega\,\delta^{q-1}(\omega'\theta) + \omega'\delta^{q-1}(\omega\theta) = \{\omega\omega'\theta, T\} = \delta^{q-1}(\omega\omega'\theta).$$

Now let r, s be arbitrary. Write:

$$Q(Z) = Z^r - a, \qquad Q'(Z') = Z'^s - b$$
$$R[z, z'] = R[Z, Z']/(Q, Q'),$$

and let $k : R \hookrightarrow R[z, z']$ be the natural map. Write ω_1, ω_1' for $\omega((1 - zT)^{-1})$, $\omega((1 - z'T)^{-1})$, viewed as elements of $R[z]$, $R[z']$, respectively. We know from the above that:

$$\omega_1\delta(\omega_1'k^*\theta) + \omega_1'\delta(\omega_1k^*\theta) = \delta(\omega_1\omega_1'k^*\theta),$$

so (1.2) will follow from:

Lemma **(1.3)**:

$$k_*(\omega_1\,\delta(\omega_1'k^*\theta)) = \omega\,\delta(\omega'\theta); \qquad k_*(\omega_1'\,\delta(\omega_1k^*\theta)) = \omega'\,\delta(\omega\theta);$$
$$k_*\delta(\omega_1\omega_1'k^*\theta) = \delta(\omega\omega'\theta).$$

Proof. — I'll use k_* to denote any of the various transfer maps. The reader can check (via the usual universal argument) $k_*(\omega_1) = \omega$, $k_*(\omega_1') = \omega'$, $k_*(\omega_1\omega_1') = \omega\omega'$. Thus:

$$\begin{aligned} k_*\,\delta(\omega_1\omega_1'k^*\theta) &= k_*\{\omega_1\omega_1'k^*\theta, T\} \\ &= \{k_*(\omega_1\omega_1'k^*\theta), T\} \\ &\underset{(\S 2, (3.4\,(ii)))}{=} \{k_*(\omega_1\omega_1')\theta, T\} \\ &= \{\omega\omega'\theta, T\} = \delta(\omega\omega'\theta). \end{aligned}$$

It remains to show, for example:

$$k_*(\omega_1\,\delta(\omega_1'k^*\theta)) = \omega\,\delta(\omega'\theta).$$

Write $R \xrightarrow{k'} R[z'] \xrightarrow{k''} R[z, z']$ so $k = k'' \circ k'$. We have:

$$\begin{aligned}
k_*(\omega_1 \delta(\omega_1' k^* \theta)) &= k'_* k''_*(\omega_1 . k''^*(\delta(\omega_1' k'^* \theta))) \\
&= k'_*(k'^*(\omega) . \delta(\omega_1' k'^* \theta)) \\
&\underset{(\S 2, (3.4 (ii)))}{=} \omega . k'_*\{\omega_1' k'^* \theta, T\} = \omega\{k'_*(\omega_1' k'^* \theta), T\} \\
&\underset{(\S 2, (3.4 (iii)))}{=} \omega\{\omega' \theta, T\} = \omega\, \delta(\omega' \theta).
\end{aligned}$$

Q.E.D.

Corollary **(1.4)**. — *The map*:

$$\delta_n^1 : W_n(R) = C_n K_1(R) \to C_{n-1} K_2(R)$$

is a $W_n(R)$*-derivation,* $\delta_n^1(\omega\omega') = \omega\, \delta_n^1(\omega') + \omega'\, \delta_n^1(\omega)$.

Proof. — Take $\theta = 1 \in W_n(R)$ and apply (1.2).

2. *Theorem* **(2.1)**. — *Let* R *be local, smooth over k perfect of characteristic* $\neq 0, 2$. *Let* $\mathscr{SC}_{n-\bullet} K_{1+\bullet}(R)$ *denote the complex*:

$$W_n(R) \xrightarrow{\delta_n^1} SC_{n-1} K_2(R) \xrightarrow{\delta_{n-1}^2} SC_{n-2} K_3(R) \longrightarrow \dots$$

Let $\Omega^\bullet_{W_n(R)}$ *denote the (absolute) deRham complex of* $W_n(R)$. *Then there is a morphism of complexes*:

$$\varphi_n : \Omega^\bullet_{W_n(R)} \to \mathscr{SC}_{n-\bullet} K_{1+\bullet}(R)$$

with the property that $\varphi_n^0 : W_n(R) \to W_n(R)$ *is the identity and* $\varphi_n^1 : \Omega^1_{W_n(R)} \to SC_{n-1} K_2(R)$ *is induced by the derivation* $\delta_n^1 : W_n(R) \to SC_{n-1} K_2(R)$. *The* φ_n^q *are homomorphisms of* $W_n(R)$*-modules. These conditions characterize the* φ_n^q.

Proof. — Assume inductively $\varphi_n^{q-1} : \Omega^{q-1}_{W_n(R)} \to SC_{n-q+1} K_q(R)$ is defined. The idea is to define:

$$\begin{aligned}
\varphi_n^q(\omega\, d\omega_1 \wedge \dots \wedge d\omega_q) &= \omega\, \delta^q \varphi_n^{q-1}(\omega_1 d\omega_2 \wedge \dots \wedge d\omega_q) \\
&= \omega\, \delta^q \omega_1 \varphi^{q-1}(d\omega_2 \wedge \dots \wedge d\omega_q).
\end{aligned}$$

Claim. — $\varphi^q(\omega\, d\omega_1 \wedge \dots \wedge d\omega_q) + \varphi^q(\omega_1 d\omega \wedge \dots \wedge d\omega_q) = \varphi^q(d(\omega_1 \omega) \wedge \dots \wedge d\omega_q)$.

Indeed, this follows from (1.2), taking $\theta = \varphi^{q-1}(d\omega_2 \wedge \dots \wedge d\omega_q)$.

Claim. — If $\omega_i = \omega_j$, we have $\varphi^q(\omega\, d\omega_1 \wedge \dots \wedge d\omega_q) = 0$. Indeed, if $i, j > 1$ this will hold by induction. We thus can reduce to the case $i = 1$, $j = 2$, and it suffices to show $\varphi^2(d\omega \wedge d\omega) = 0$, *i.e.* $\delta^2 \omega \varphi^1(d\omega) = 0$. But

$$\begin{aligned}
\delta^2 \omega \varphi^1(d\omega) &= \delta^2 \varphi^1(\omega\, d\omega) = \delta^2 \varphi^1\left(\frac{1}{2} d(\omega^2)\right) \\
&= \frac{1}{2} \delta^2 \delta^1(\omega^2) = 0.
\end{aligned}$$

Claim. — If $\omega_i = \omega_i' + \omega_i''$, we have:

$$\varphi^q(d\omega_1 \wedge \ldots \wedge d\omega_q) = \varphi^q(d\omega_1 \wedge \ldots \wedge d\omega_i' \wedge \ldots \wedge d\omega_q) + \varphi^q(d\omega_1 \wedge \ldots \wedge d\omega_i'' \wedge \ldots \wedge d\omega_q).$$

Indeed if $i > 1$ this is true inductively. For $i = 1$ we have:

$$\begin{aligned}\varphi^q(d(\omega_1' + \omega_1'') \wedge \ldots \wedge d\omega_q) &= \delta^q(\omega_1' + \omega_1'')\, \varphi^{q-1}(d\omega_2 \wedge \ldots \wedge d\omega_q) \\ &= \varphi^q(d\omega_1' \wedge \ldots \wedge d\omega_q) + \varphi^q(d\omega_1'' \wedge \ldots \wedge d\omega_q).\end{aligned}$$

These three claims suffice to prove the existence of a $W_n(R)$-linear map:

$$\varphi_n^q : \Omega^q_{W_n(R)} \to SC_{n-q}K_{q+1}(R).$$

The fact that $\varphi_n^q \circ d = \delta^q \circ \varphi_n^{q-1}$ is easily checked. Q.E.D.

3. Let $\Omega^q_{W(R)} = \varprojlim_n \Omega^q_{W_n(R)}$. Note the above constructions give maps:

$$\varphi^q : \Omega^q_{W(R)} \to S\hat{C}K_{q+1}(R).$$

Proposition **(3.1).** — *Let* $f_p : W(R) \to W(R)$ *denote the p-frobenius* ($p = \text{char. } R$). *Let* $f_p^{(q)} : \Omega^q_{W(R)} \to \Omega^q_{W(R)}$ *be the induced map. Finally, let* $F_p^{(q)} : S\hat{C}K_q(R) \to S\hat{C}K_q(R)$ *be its frobenius. Then:*

$$\varphi^q \circ f_p^{(q)} = p^q F_p^{(q+1)} \circ \varphi^q.$$

Proof. — We have $f_p^{(0)} = f_p = F_p^{(1)}$, $\varphi^0 = \text{identity}$, so the assertion holds when $q = 0$. Also for $\omega \in W(R)$, $\alpha \in \Omega^q_{W(R)}$, $\beta \in S\hat{C}K_{q+1}(R)$ we have:

$$f_p^{(q)}(\omega\alpha) = f_p(\omega)\, f_p^{(q)}(\alpha); \qquad F_p^{(q+1)}(\omega\beta) = f_p(\omega)\, F_p^{(q+1)}(\beta),$$

the right-hand identity being (§ 2 (3.7)). It therefore suffices to verify the identity on exact q-forms:

$$\varphi^q f_p^{(q)}(d\tau) = p^q F_p^{(q+1)} \varphi^q.$$

But:

$$\begin{aligned}\varphi^q f_p^{(q)}(d\tau) &= \varphi^q\, d(f_p^{(q-1)}(\tau)) = \delta^q \varphi^{q-1} f_p^{(q-1)}(\tau) \\ &\underset{\text{(by induction)}}{=} p^{q-1}\, \delta^q F_p^{(q)} \varphi^{q-1}(\tau) \\ &\underset{(1.1)\,(\text{ii})}{=} p^q F_p^{(q+1)}\, \delta^q \varphi^{q-1}(\tau) = p^q F_p^{(q+1)} \varphi^q(d\tau).\end{aligned}$$

Q.E.D.

Remark **(3.2).** — It follows from (1.1) that the map $\mathscr{F}_p : S\hat{C}K_\bullet(R) \to S\hat{C}K_\bullet(R)$ given by $\mathscr{F}_p = p^{q-1} F_p^{(q)}$ on $S\hat{C}K_q(R)$ is a map of complexes. If we let $f_p : \Omega^\bullet_{W(R)} \to \Omega^\bullet_{W(R)}$ be the map induced by frobenius, we have by (3.1):

$$\mathscr{F}_p \circ \varphi = \varphi \circ f_p.$$

One final property of $S\hat{C}K_\bullet(R)$ will be important when we deal with divided powers. For $\omega = V_p(u) \in V_p W(R)$ and $n \in \mathbf{N}$ define:

$$\gamma^{(n)}\omega = \frac{p^{n-1}}{n!} V_p(u^n).$$

The assignment $\omega \mapsto \gamma^{(n)}\omega$, $n \in \mathbf{N}$, $\omega \in V_p W(R)$ defines a *divided powers* structure on the ideal $V_p W(R)$ (cf. I.4.5). $\gamma^{(n)}$ is well defined because V_p is injective on $W(R)$, and $p^{n-1}/n!$ is an integer (see proof of (3.3) below).

Proposition **(3.3)**. — *Given* $\eta \in S\hat{C}K_q(R)$ *with* $\delta^q \eta = 0$, *and* $\omega \in V_p W(R)$, *we have* $\delta^q(\gamma^{(n)}\omega\eta) = \gamma^{(n-1)}\omega\, \delta^q(\omega\eta)$. *In particular* $\delta^1(\gamma^{(n)}\omega) = \gamma^{(n-1)}\omega\, \delta^1(\omega)$.

Proof. — $\gamma^{(0)}\omega = 1$, $\gamma^{(1)}\omega = \omega$, $\gamma^{(2)}\omega = \frac{p}{2} V_p(u^2) = \frac{1}{2}\omega^2$, where $\omega = V(u)$. Since $p > 2$, the derivation property of δ^q implies the assertion in these cases. Assuming $n, p > 2$ we have:

$$\operatorname{ord}_p \frac{p^{n-3}}{(n-1)!} \geq 0.$$

Indeed, writing $n = \sum_i a_i p^i$ with $0 \leq a_i < p$, we find $\operatorname{ord}_p n! = (n - \sum_i a_i)/(p-1)$. For $n \geq 2$, $p \geq 3$, we find $n - \operatorname{ord}_p n! \geq 2$. Now we compute:

$$\begin{aligned}
\delta^q(\gamma^{(n)}\omega\eta) &= \frac{p^{n-1}}{n!}\delta^q(V_p(u^n)\ \eta) = \frac{p^{n-1}}{n!}\delta^q(V_p(u^n F_p \eta)) \\
&= \frac{p^{n-2}}{n!} V_p \delta^q(u^n F_p \eta) \underset{(1.2)}{=} \frac{p^{n-2}}{(n-1)!} V_p(u^{n-1}\delta^q(uF_p\eta)) \\
&\underset{(\S 5,\, 1.3)\text{ and }(\S 2,\, 3.5)}{=} \frac{p^{n-3}}{(n-1)!} V_p(u^{n-1})\ V_p \delta^q(uF_p\eta) = \frac{p^{n-2}}{(n-1)!} V_p(u^{n-1})\ \delta^q V_p(uF_p\eta) \\
&= \gamma^{(n-1)}\omega\, \delta^q(\omega\eta).
\end{aligned}$$

Q.E.D.

Corollary **(3.4)**. — *The map* $\varphi_n : \Omega^{\bullet}_{W_n(R)} \to \mathcal{S}\mathcal{C}_{n-\bullet} K_{1+\bullet}(R)$ *factors through a map* $\varphi_n : \Omega^{\bullet}_{W_n(R),\gamma} \to \mathcal{S}\mathcal{C}_{n-\bullet} K_{1+\bullet}(R)$, *where* $\Omega^{\bullet}_{W_n(R),\gamma}$ *denotes the complex of differentials compatible with the divided power structure on* $W_n(R)$ (I, § 4 (5.5)).

7. TYPICAL CURVES, FILTRATIONS, AND EXACT SEQUENCES

1. Throughout this section R will be a smooth local k-algebra, k a perfect field of characteristic $p \neq 0, 2$. Recall $W^{(p)}(R) \subset W(R)$ is a direct factor with projection operator:

$$\pi = \sum_{n \in I(p)} \frac{\mu(n)}{n} V_n F_n : W(R) \to W^{(p)}(R).$$

The corresponding idempotent $e = 1_{W^{(p)}(R)} \in W(R)$ is given by:

$$e = \sum_n \frac{\mu(n)}{n} V_n F_n(1) = \sum_n \frac{\mu_n}{n} V_n(1)$$

$$e = \omega(E(T)^{-1}), \qquad E(T) = \text{Artin-Hasse exponential (I, § 1 (4.1))}.$$

For any W(R)-module M, we get a $W^{(p)}(R)$-module $e.M$. In particular, we define the *typical curves* on $K^q(R)$ by:

(1.1) $$T\hat{C}K_q(R) = e.S\hat{C}K_q(R).$$

Viewing F_n, V_n as endomorphisms of $S\hat{C}K_q(R)$, the same prescription $\pi = \sum_{I(p)} \frac{\mu_n}{n} V_n F_n$ defines a projection operator on $S\hat{C}K_q(R)$ and we have:

$$T\hat{C}K_q(R) = \pi(S\hat{C}K_q(R)).$$

Indeed for $\alpha \in S\hat{C}K_q(R)$:

$$e\alpha = \sum_{n \in I(p)} \frac{\mu_n}{n} V_n(1)\alpha = \sum_n \frac{\mu_n}{n} V_n F_n \alpha = \pi\alpha.$$

The maps:

$$\delta^q : S\hat{C}K_q(R) \to S\hat{C}K_{q+1}(R);$$

$$\{\ \} : W(R) \otimes \underbrace{R^\times \otimes \ldots \otimes R^\times}_{q-1} \to S\hat{C}K_q(R),$$

$$\{\ \}(\omega \otimes r_1 \ldots \otimes r_{q-1}) = \{\omega, r_1, \ldots, r_{q-1}\}$$

both commute with the operation of π ((§ 6 (1.1)) and the projection formula), so we get:

$$\delta^q : T\hat{C}K_q(R) \to T\hat{C}K_{q+1}(R)$$

$$\{\ \} : W^{(p)}(R) \otimes R^\times \otimes \ldots \otimes R^\times \to T\hat{C}K_q(R).$$

In particular, it follows from (§ 3 (1.3)) and (I, § 1 (4.2)) that $T\hat{C}K_q(R)$ is topologically generated by symbols:

$$\{E(aT^{p^m}), r_1, \ldots, r_{q-1}\}, \quad a, r_1, \ldots, r_{q-1} \in R^\times$$

(1.2) $$\{E(aT^{p^m}), r_1, \ldots, r_{q-2}, T\}.$$

Define a descending filtration $\mathrm{filt}^\bullet\, T\hat{C}K_q(R)$ by taking $\mathrm{filt}^n\, T\hat{C}K_q(R)$ to be generated topologically by symbols as in (1.2) for $m \geq n$. Thus $\mathrm{filt}^0 = T\hat{C}K_q(R)$. Define:

(1.3) $$TC_nK_q(R) = T\hat{C}K_q(R)/\mathrm{filt}^n\, T\hat{C}K_q(R)$$

$$T\Phi_nK_q(R) = \mathrm{filt}^n/\mathrm{filt}^{n+1}.$$

Proposition **(1.4)**. — *Assume* $q \leq p$. *Then* $\mathrm{filt}^n\, W^{(p)}(R).T\hat{C}K_q(R) \subseteq \mathrm{filt}^n\, T\hat{C}K_q(R)$, *so* $TC_nK_q(R)$ *is a* $W_n^{(p)}(R)$*-module for all n. Moreover,* $\delta^q(\mathrm{filt}^n\, T\hat{C}K_q(R)) \subseteq \mathrm{filt}^n\, T\hat{C}K_{q+1}(R)$ *so there is an induced* $W(k)$*-linear map* $\delta^q : TC_nK_q(R) \to TC_nK_{q+1}(R)$.

Proof. — The last assertion is clear from the definition and (§ 6 (1.1) (iii)). For the first we have by (§ 5 (1.4)) and (§ 6 (1.2)):

$$\omega.\{\omega', r_1, \ldots, r_{q-1}\} = \{\omega\omega', r_1, \ldots, r_{q-1}\}$$

$$\omega, \omega' \in W^{(p)}(R) \qquad r_1, \ldots, r_{q-1} \in R^\times$$

$$\omega.\{\omega', r_1, \ldots, r_{q-2}, T\} = \{\omega\omega', r_1, \ldots, r_{q-2}, T\} - \omega'\{\omega, r_1, \ldots, r_{q-2}, T\}.$$

Taking $\omega \in \text{filt}^n W^{(p)}(R)$ we are reduced to showing that $\text{filt}^n T\hat{C}K_q(R)$ is a $W^{(p)}(R)$-submodule of $T\hat{C}K_q(R)$. This is a consequence of:

Lemma (**1.5**):

$$\text{filt}^n T\hat{C}K_q(R) = T\hat{C}K_q(R) \cap \text{filt}^{p^n-2} S\hat{C}K_q(R).$$

Proof. — The symbols:

$$\{E(aT^{p^m}), r_1, \ldots, r_{q-1}\}, \quad \{E(aT^{p^m}), r_1, \ldots, r_{q-2}, T\} \qquad r_i \in R^\times, \quad a \in R, \quad m \geq n$$

are trivial mod T^{p^n-1} (§ 3 (1.5)). These symbols generate $\text{filt}^n T\hat{C}K_q(R)$ topologically so, by definition of $\text{filt}^\bullet S\hat{C}K_q(R)$ (§ 1 (1.5)), we have:

$$\text{filt}^n T\hat{C}K_q(R) \subseteq T\hat{C}K_q(R) \cap \text{filt}^{p^n-2} \hat{C}K_q(R).$$

On the other hand, by (§ 4 (4.1)) we have $\text{filt}^{p^n-2} \hat{C}K_q(R)$ generated topologically by symbols:

$$\{1+aT^m, r_1, \ldots, r_{q-1}\}, \quad \{1+aT^{m+1}, r_1, \ldots, r_{q-2}, T\}, \quad m \geq p^n-1$$

so $\pi(\text{filt}^{p^n-2} \hat{C}K_q(R)) \subseteq \text{filt}^n T\hat{C}K_q(R)$. Thus if $\alpha \in T\hat{C}K_q(R) \cap \text{filt}^{p^n-2} \hat{C}K_q(R)$ we have:

$$\alpha = \pi(\alpha) \in \text{filt}^n T\hat{C}K_q(R). \qquad \text{Q.E.D.}$$

Remark (**1.6**). — (i) It follows from the above that $T\hat{C}K_q(R) \cong \varprojlim TC_n K_q(R)$.
(ii) For $a \in R^\times$ we have $-\{E(aT), T\} = \pi\{1-aT, T\} = -\pi\{1-aT, a\} = \{E(aT), a\}$.
(iii) $T\hat{C}K_q(R)$ has endomorphisms $F = F_p$ and $V = V_p$. We have:

$$V^n T\hat{C}K_q(R) \subset \text{filt}^n T\hat{C}K_q(R)$$

but the inclusion is *not* in general an equality, essentially because:

$$V\{E(aT), T\} = p\{E(aT^p), T\} \neq \{E(aT^p), T\}.$$

2. *Theorem* (**2.1**). — *Recall* (§ 4 (1.4) *and* (3.1)) *we have defined*: $D_n = D_n(\Omega^q_R) =$ *subgroup of* Ω^q_R *generated by differentials* $a^{p^\ell-1} da \wedge \frac{da_1}{a_1} \wedge \ldots \wedge \frac{da_q}{a_q}$; $a, a_i \in R^\times$, $\ell \leq n-1$; $E_n = E_n(\Omega^q_R) =$ *subgroup of* Ω^q_R *generated by differentials* $a^{p^n} \frac{da_1}{a_1} \wedge \ldots \wedge \frac{da_q}{a_q}$ *plus* D_{n+1}. *By convention,* $D_0 = 0$, $E_0 = \Omega^q_R$. *Note also* $D_1 = \text{Image}(d : \Omega^{q-1}_R \to \Omega^q_R)$ *and* $E_1 = \text{Ker}(\Omega^q_R \xrightarrow{d} \Omega^{q+1}_R)$. (*See* (2.4) *below.*)

Assume R *smooth and local over* k, *a perfect field of characteristic* $\neq 0, 2$. *Let* $q \leq p$. *For all* n, *there are exact sequences of* k-*vector spaces and* p^{-n}-k-*linear morphisms*:

$$0 \to \Omega^{q-1}_R / D_n \to T\Phi_n K_q(R) \to \Omega^{q-2}_R / E_n \to 0.$$

The left hand arrow sends the class of $a\frac{dr_1}{r_1}\wedge\ldots\wedge\frac{dr_{q-1}}{r_{q-1}}$ *to* $\{E(aT^{p^n}), r_1, \ldots, r_{q-1}\}$ *while the right hand maps* $\{E(aT^{p^n}), r_1, \ldots, r_{q-2}, T\}$ *to* $a\frac{dr_1}{r_1}\wedge\ldots\wedge\frac{dr_{q-2}}{r_{q-2}}$. *In particular:*

$$T\Phi_0 K_q(R) \cong \Omega_R^{q-1} = TC_1 K_q(R)$$
$$0 \to \Omega_R^1/D_n \to T\Phi_n K_2(R) \to R/R^{p^n} \to 0.$$

Proof. — One checks by looking at generators as above that:

$$\text{filt}^n\, T\hat{C}K_q(R) = \pi(\text{filt}^{p^n-2}\, S\hat{C}K_q(R))$$
$$\text{filt}^{n+1}\, T\hat{C}K_q(R) = \pi(\text{filt}^{p^n}\, S\hat{C}K_q(R)).$$

Writing $\text{filt}^i = \text{filt}^i\, S\hat{C}K_q(R)$, we have by (§ 4 (4.1)):

$$(2.2)\qquad \begin{array}{ccccccccc}
0 & \longrightarrow & \Omega_R^{q-1}/D_n & \longrightarrow & \text{filt}^{p^n-2}/\text{filt}^{p^n} & \longrightarrow & \Omega_R^{q-1}\oplus\Omega_R^{q-2}/E_n & \longrightarrow & 0 \\
 & & \wr\| & & \Big| & & \wr\| & & \\
 & & \Phi_{p^n}K_q(R) & & \Big|\,\pi & & \Phi_{p^n-1}K_q(R) & & \\
 & & {\scriptstyle(*)}\searrow & & \downarrow & & \downarrow & & \\
 & & & & T\Phi_n K_q(R) & \longrightarrow & \text{Coker}(*) & \longrightarrow & 0
\end{array}$$

Claim 1. — The map $(*)$ above is injective. Indeed, by (§ 4 (4.1)) $\Phi_{p^n}K_q(R)$ is generated by symbols $\{1+aT^{p^n}, r_1, \ldots, r_{q-1}\}$. Note $E(aT^{p^n}) \equiv 1+aT^{p^n} \pmod{T^{p^n+1}}$ so a relation:

$$\sum_i \{E(a_i T^{p^n}), r_1^{(i)}, \ldots, r_{q-1}^{(i)}\} \in \text{filt}^{n+1}\, T\hat{C}K_q(R) = T\hat{C}K_q(R) \cap \text{filt}^{p^{n+1}-2}$$

implies a relation:

$$\sum_i \{1+a_i T^{p^n}, \ldots\} \in \text{filt}^{p^n}, \quad (\textit{i.e.}\text{ trivial mod } T^{p^n+1}\ (\S\ 1\ (1))),$$

proving the claim.

Claim 2. — The right-hand vertical arrow in (2.2) identifies Coker$(*)$ with Ω_R^{q-2}/E_n. Indeed, the factor $\Omega_R^{q-1} \subset \Phi_{p^n-1}$ (§ 4 (4.1) (iii)) is generated by symbols $\{1+aT^{p^n-1}, r_1, \ldots, r_{q-1}\} = V_{p^n-1}\{1+aT, r_1, \ldots, r_{q-1}\}$. We have seen (I.1.3.6) that $\pi . V_m = 0$ for $(m, p) = 1$, so the above symbols die in $T\Phi_n K_q(R)$, hence also in Coker$(*)$.

We have now $\Omega_R^{q-2}/E_n \twoheadrightarrow \text{Coker}(*)$.

$$a\frac{dr_1}{r_1}\wedge\ldots\wedge\frac{dr_{q-2}}{r_{q-2}} \to \{E(aT^{p^n}), r_1, \ldots, r_{q-2}, T\} \quad (\text{mod image}(*)).$$

Suppose there is a relation:

$$\sum_i \{E(a_i T^{p^n}), r_1^{(i)}, \ldots, r_{q-2}^{(i)}, T\} \equiv \sum \{E(c_j T^{p^n}), s_1^{(j)}, \ldots, s_{q-1}^{(j)}\}$$
$$(\text{mod filt}^{n+1}\, T\hat{C}K_q(R)).$$

Working in $\hat{C}K_q(R)$ we certainly have the same congruence mod $\text{filt}^{p^n-1}\hat{C}K_q(R)$ (*i.e.* mod T^{p^n}). Thus in $\Phi_{p^n-1}K_q(R)$ we have:

$$0=\rho_2\left(\sum_i a_i \frac{dr_1^{(i)}}{r_1^{(i)}}\wedge\ldots\wedge\frac{dr_{q-2}^{(i)}}{r_{q-2}^{(i)}}\right)$$

with ρ_2 as in (§ 3 (2.3)) (i). From (§ 4 (4.1)) (iii) we get:

$$\sum_i a_i \frac{dr_1^{(i)}}{r_1^{(i)}}\wedge\ldots\wedge\frac{dr_{q-2}^{(i)}}{r_{q-2}^{(i)}}\in E_n. \qquad \text{Q.E.D.}$$

Corollary **(2.3)**. — *Let* R *be as above. Then* $TC_nK_q(R)=(0)$ *for* $q\geq \dim_k R+2$. (It is not necessary to assume $q\leq p$.)

Proof. — In fact $S\hat{C}K_q(R)=0$ for $q\geq\dim_k R+2$. Indeed, by (§ 3 (1.3)) and the definition of Φ'_n in § 3 (1), it suffices to show $\Phi'_nK_q(R)=(0)$ for all n. Using (§ 3 (3.4)) (ii) and the assertion $\text{Ker}\,\rho'_2\supset E_r/D_1$ proved in (§ 4 (3.2)) we get ($n+1=mp^r$, $p\nmid m$):

$$\rho':\ \Omega_R^{q-1}\oplus(\Omega_R^{q-2}/E_r)\twoheadrightarrow\Phi'_nK_q(R).$$

If $q>\dim_k R+2$, the left-hand side is zero. If $q=\dim_k R+2$, the map:

$$C^{-1}:\ \Omega_R^{q-2}\to\Omega_R^{q-2}/D_1$$

is an isomorphism, so $E_r=\Omega_R^{q-2}$ for all r and again the left side is zero. Q.E.D.

Remark **(2.4)**. — There is an exact sequence:

$$0\longrightarrow\Omega_R^{q-2}\xrightarrow{C^{-n}}\Omega_R^{q-2}/D_n\longrightarrow\Omega_R^{q-2}/E_n\longrightarrow 0$$

where C^{-n} is the n-th iterate of the inverse Cartier operator (§ 5 (2)). Also:

$$D_{\ell+1}/D_\ell \underset{C^\ell}{\cong} D_1=\text{Image}\,(d)\quad\text{for any}\quad \ell\geq 0.$$

Thus $T\Phi_nK_q(R)$ is built up from finitely many successive extensions of R-modules of finite type.

3. Suppose now that $\dim_k R<p$. The complex $TC_nK_\bullet(R)$ is defined (§ 4 (4.5)) (more generally for $\dim R\geq p$ we can define $TC_nK_\bullet(R)$ by truncating at $TC_nK_p(R)$):

$$0\to TC_nK_1(R)\xrightarrow{\delta^1}TC_nK_2(R)\xrightarrow{\delta^2}\ldots$$

with $TC_nK_q(R)$ placed in degree $q-1$. To simplify notation, let $C_n^q=TC_nK_{q+1}(R)$ so $C_n^\bullet$ is a complex:

$$0\to C_n^0\to\ldots\to C_n^{\dim R}\to 0.$$

For $q\geq 0$, define $\text{trunc}_q(C_n^\bullet/pC_n^\bullet)$ to be the complex:

$$0\to C_n^0/pC_n^0\to\ldots\to C_n^{q-1}/pC_n^{q-1}\to C_n^q/VC_n^q\to 0.$$

Proposition (**3.1**). — (i) *Let* R *be as above, and assume* $\dim R < p$. *The complex* $C_1^\bullet$ *is isomorphic to the deRham complex* $\Omega_R^\bullet$.

(ii) *Let* R *be smooth and local over a perfect field* k, char. $\neq 2$. *Let* $\ell < p$. *The natural maps* $C_n^q \to C_1^q \cong \Omega_R^q$ *induce a quasi-isomorphism* $\mathrm{trunc}_\ell(C_n^\bullet/pC_n^\bullet) \to t_{\leq \ell}\Omega_R^\bullet$, *where* $t_{\leq \ell}\Omega_R^\bullet$ *denotes the complex*:

$$0 \to R \to \Omega_R^1 \to \ldots \to \Omega_R^\ell \to 0 \ldots$$

(quasi-isomorphism = isomorphism on homology). In particular, taking $q = \dim R + 1$, *we obtain a quasi-isomorphism* $C_n^\bullet/pC_n^\bullet \to \Omega_R^\bullet$.

Proof. — We have a square:

$$\begin{array}{ccc} \Omega_R^{q-1} & \xrightarrow{d} & \Omega_R^q \\ \rho \downarrow \wr & & \rho \downarrow \wr \\ C_1^{q-1} & \xrightarrow{\delta^q} & C_1^q \end{array}$$

with vertical arrows ρ (isomorphisms by (2.1)) given by:

$$\rho\left(a\frac{dr_1}{r_1}\wedge \ldots \wedge \frac{dr_\ell}{r_\ell}\right) = \{1 + aT, r_1, \ldots, r_\ell\}, \qquad \ell = q-1, q.$$

The square commutes up to sign because:

$$\begin{aligned} \delta^q\{1+aT, r_1, \ldots, r_{q-1}\} &= \{1+aT, r_1, \ldots, r_{q-1}, T\} \\ &= (-1)^q\{1+aT, a, r_1, \ldots, r_{q-1}\} = (-1)^q \rho\, d\left(a\frac{dr_1}{r_1}\wedge \ldots \wedge \frac{dr_{q-1}}{r_{q-1}}\right). \end{aligned}$$

It remains to show the map on cohomology:

$$\psi^q : H^q(\mathrm{trunc}_\ell(C_n^\bullet/pC_n^\bullet)) \to H^q(t_{\leq \ell}\Omega_R^\bullet)$$

is an isomorphism for all q. One knows that $H^q(\Omega_R^\bullet)$ is generated by classes:

$$a^p\frac{dr_1}{r_1}\wedge \ldots \wedge \frac{dr_q}{r_q}.$$

Since the cycles $Z^q(\mathrm{trunc}_\ell\, C_n^\bullet/pC_n^\bullet)$ contain the images of symbols $\{E(a^pT), r_1, \ldots, r_q\}$ (use (1.6) (ii)) and since the natural map $C_n^\ell/VC_n^\ell \to \Omega_n^\ell$ is surjective, it follows that ψ^q above is surjective for all q.

The maps $C_n^q \to \Omega_R^q$ are surjective, so given $\alpha \in Z^q(\mathrm{trunc}_\ell\, C_n^\bullet/pC_n^\bullet)$ with $\psi^q(\alpha) = 0$, we can modify α by a boundary $\delta\beta$ and assume $\alpha \mapsto 0$ in Ω_R^q. Thus α is represented by a class in $\mathrm{filt}^1\, TC_nK_{q+1}(R)$, and we can assume:

$$\alpha = \sum_i \{E(a_iT^{p^{n_i}}), r_1^{(i)}, \ldots, r_q^{(i)}\} + \sum_j \{E(b_jT^{p^{m_j}}), s_1^{(j)}, \ldots, s_{q-1}^{(j)}, T\}$$

with m_j and $n_i \geq 1$. Obviously, we can drop the sum on the right without changing the class of α in $H^q(\mathrm{trunc}_\ell\, C^\bullet_n/pC^\bullet_n)$, so we can assume $\alpha \in VC^q_n$. When $q=\ell$, this shows the class of α is zero, so ψ^ℓ is injective. For $q<\ell$ we know:

$$\delta(\alpha)=\sum_i \{E(a_i T^{p^{n_i}}), r_1^{(i)}, \ldots, r_q^{(i)}, T\} \in VC_n^{q+1}$$

(note $pC_n^{q+1} \subset VC_n^{q+1}$).

Let $m=\min(n_i)$, and suppose $n_1=\ldots=n_r=m$. Note $m \geq 1$. $\delta(\alpha)$ maps to an element in $V_p C_{p^m} K_{q+2}(R)$. The composition:

$$C_{p^m}K_{q+2}(R) \xrightarrow{d\log} \Omega^{q+2}_{R[T]/(T^{p^m+1})} \longrightarrow \Omega^{q+1}_R \wedge dT$$

kills $V_p C_{p^m} K^{q+2}(R)$, so we get:

$$\sum_i da_i \wedge \frac{dr_1^{(i)}}{r_1^{(i)}} \wedge \ldots \wedge \frac{dr_q^{(i)}}{r_q^{(i)}} = 0.$$

(Note $q<\ell<p$ at this point, so the numerical factor $(-1)^{q+1}(q+1)!$ entering in the definition of the $d\log$ (I, § 3 (3.3)) is a unit.) Thus:

$$\sum_i a_i \frac{dr_1^{(i)}}{r_1^{(i)}} \wedge \ldots \wedge \frac{dr_q^{(i)}}{r_q^{(i)}} = \sum_j c_j^p \frac{dt_1^{(j)}}{t_1^{(j)}} \wedge \ldots \wedge \frac{dt_q^{(j)}}{t_q^{(j)}} + \text{exact}$$

for some c's and t's. It follows from (2.1) that:

$$\alpha \equiv \sum_j \{E(c_j^p T^{p^m}), t_1^{(j)}, \ldots, t_q^{(j)}\} \pmod{\mathrm{filt}^{m+1}}$$

$$\equiv 0 \pmod{\mathrm{filt}^{m+1}, p}.$$

An easy induction on m shows $\alpha \in pC^q_n$. Q.E.D.

4. Let $\mathscr{C}$ be a category, I a filtering index set. A pro-object in $\mathscr{C}$ indexed by **I** is a collection $\{C_i\}_{i\in I}$ together with maps $C_j \xrightarrow{f_{ij}} C_i$ for $j \geq i$, satisfying $f_{ij}\circ f_{jk}=f_{ik}$. The set of morphisms between two pro-objects in the pro-category Pro $\mathscr{C}$ is given by:

$$\mathrm{Hom}(\{C_i\}_{i\in I}, \{D_j\}_{j\in J}) = \varprojlim_J \varinjlim_I \mathrm{Hom}(C_i, D_j).$$

In our examples, $I=\mathbf{N}$=the natural numbers, $\mathscr{C}$ is an abelian category, and all maps $\{C_n\} \xrightarrow{\varphi} \{D_n\}$ have the property that there exists an $m \in \mathbf{Z}$ and maps $\varphi_n : C_n \to D_{n+m}$ for all n defining φ. A pro-object $\{C_n\}$ is "essentially 0" if for all n there exists an $N(n) \geq n$ such that $f_{n,N}=0$. Such an object is isomorphic to the zero pro-system $\{0\}$ in Pro φ. If φ is realized by maps $\varphi_n : C_n \to D_{n+m}$, we can define pro-objects Ker φ, Coker φ by:

$$(\mathrm{Ker}\,\varphi)_n = \mathrm{Ker}\,\varphi_n, \qquad (\mathrm{Coker}\,\varphi)_n = \mathrm{Coker}\,\varphi_n.$$

Ker φ and Coker φ are independent up to canonical isomorphism in Pro φ of the choice of m.

Example (4.1). — Define $TCK_q(R)$ to be the pro-object $\{TC_nK_q(R)\}$. We have endomorphisms p, $V=V_p$ and $F=F_p$ of $TCK_q(R)$, where p and V arise from maps $TC_nK_q(R) \to TC_nK_q(R)$, but F is best realized as a map:

$$TC_nK_q(R) \to TC_{n-1}K_q(R).$$

As endomorphisms of pro-objects we have $p=F\circ V=V\circ F$. The rationale for working with pro-objects rather than with $T\hat{C}K_q(R)$ at this point will become clear when we work with sheaves.

Proposition (4.2). — *Let* R *be as before, and assume* $q \leq p$. *The pro-objects* $\mathrm{Ker}(p)$, $\mathrm{Ker}(V)$, $\mathrm{Ker}(F)$ *are essentially* 0.

Proof. — It suffices to show $\mathrm{Ker}(p)$ is essentially 0. Recall $TC_nK_q(R)$ has a filtration with associated graded $= \bigoplus_{m=0}^{n-1} T\Phi_m K_q(R)$. We have:

$$p.\mathrm{filt}^m TC_nK_q(R) \subseteq \mathrm{filt}^{m+1} TC_nK_q(R),$$

so there are induced maps:

$$\text{“}p\text{”} : T\Phi_m K_q(R) \to T\Phi_{m+1} K_q(R).$$

If we show "p" injective, it will follow that:

$$\mathrm{Ker}(p)_n \subseteq \mathrm{filt}^{n-1} TC_nK_q(R)$$

so the transition maps $\mathrm{Ker}(p)_n \to \mathrm{Ker}(p)_{n-1}$ are zero.

Lemma (4.3). — "p" : $T\Phi_m K_q(R) \to T\Phi_{m+1} K_q(R)$ *is injective.*

Proof. — We have:

$$\text{“}p\text{”}\{E(aT^{p^m}), r_1, \ldots, r_{q-1}\} = \{E(a^p T^{p^{m+1}}), r_1, \ldots, r_{q-1}\}$$
$$\text{“}p\text{”}\{E(aT^{p^m}), r_1, \ldots, r_{q-2}, T\} = \{E(a^p T^{p^{m+1}}), r_1, \ldots, r_{q-2}, T\}.$$

From this one gets a map of exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega^{q-1}/D_m & \longrightarrow & T\Phi_m K_q(R) & \longrightarrow & \Omega^{q-2}/E_m & \longrightarrow & 0 \\
 & & \downarrow \alpha & & \downarrow \text{“}p\text{”} & & \downarrow \beta & & \\
0 & \longrightarrow & \Omega^{q-1}/D_{m+1} & \longrightarrow & T\Phi_{m+1} K_q(R) & \longrightarrow & \Omega^{q-2}/E_{m+1} & \longrightarrow & 0
\end{array}$$

so it suffices to show α and β are injective. Both α and β are induced by the inverse Cartier operator, and injectivity follows from diagrams like:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & D_m & \longrightarrow & \Omega^{q-1} & \longrightarrow & \Omega^{q-1}/D_m & \longrightarrow & 0 \\
 & & \wr\Vert\, C^{-1} & & \cap\big\downarrow C^{-1} & & \big\downarrow \alpha & & \\
0 & \longrightarrow & D_{m+1}/D_1 & \longrightarrow & \Omega^{q-1}/D_1 & \longrightarrow & \Omega^{q-1}/D_{m+1} & \longrightarrow & 0
\end{array}$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E_m & \longrightarrow & \Omega^{q-2} & \longrightarrow & \Omega^{q-2}/E_m & \longrightarrow & 0 \\
 & & \wr\Vert & & \cap\big\downarrow & & \big\downarrow \beta & & \\
0 & \longrightarrow & E_{m+1}/D_1 & \longrightarrow & \Omega^{q-2}/D_1 & \longrightarrow & \Omega^{q-2}/E_{m+1} & \longrightarrow & 0
\end{array}$$

Q.E.D.

5. Let $\{C_n\} \xrightarrow{f} \{D_n\} \xrightarrow{g} \{E_n\}$ be morphisms of pro-objects (in an abelian category), and assume f is realized by maps $f_n : C_n \to D_{n+m}$ and g is realized by maps:

$$g_n : D_n \to E_{n+r}, \qquad m, r \in \mathbf{Z}.$$

I will say the sequence is exact if the pro-object $\{H_n\}$ defined by $H_n = \operatorname{Ker} g_n / \operatorname{Image} f_{n-m}$ is essentially zero.

Proposition **(5.1)**. — *Define a map*:

$$e_n : R^\times / R^{\times p^n} \to TC_n K_2(R)$$
$$e_n(r) = \{E(T), r\}$$

and let $e : \{R^\times / R^{\times p^n}\}_{n \geq 0} \to TCK_2(R)$ *be the corresponding map of pro-systems. Then there is an exact sequence of pro-systems*:

$$0 \longrightarrow \{R^\times / R^{\times p^n}\}_{n\geq 0} \xrightarrow{e} TCK_2(R) \xrightarrow{1-F} TCK_2(R).$$

If, moreover, R *is strictly henselian, then* $1-F$ *is surjective.*

Proof. — I claim first that $e_n(r)=0$ implies $r=s^{p^n}$ in $R^\times$. Indeed, when $n=1$ we get by (2.1) that $dr/r=0$ in Ω^1_R, whence $r=s^p$. In general we may assume inductively that $r=u^{p^n}$, so:

$$0 = \{E(T), r\} = \{E(T^{p^n}), u\}.$$

Recall we have $\Omega^1_R/D_n \hookrightarrow T\Phi_n K_2(R)$ and $\{E(T^{p^n}), u\}$ is the class of du/u. Applying the Cartier operator n times we get $du/u=0$ in Ω^1_R so $u=s^p$, $r=s^{p^{n+1}}$.

The map F can be realized as a map $F : TC_n K_q(R) \to TC_{n-1} K_q(R)$ (any q). Indeed, by (§ 1 (2.2)):

$$F\{E(aT^{p^m}), r_1, \ldots, r_{q-2}, T\} = \begin{cases} \{E(aT^{p^{m-1}}), r_1, \ldots, r_{q-2}, T\} & \text{if } m \geq 1 \\ -\{E(a^pT), r_1, \ldots, r_{q-2}, -a\} & \text{if } m=0 \end{cases}$$

$$F\{E(aT^{p^m}), r_1, \ldots, r_{q-1}\} = \{E(a^pT^{p^m}), r_1, \ldots, r_{q-1}\}.$$

Abusing notation, I will write:

$$1-F : TC_nK_2(R) \to TC_{n-1}K_2(R)$$

for the obvious thing. Symbols $\{E(T), r\}$ certainly lie in the kernel of $1-F$.

Lemma (**5.2**). — *The kernel of* $1-F : TC_nK_2(R) \to TC_{n-1}K_2(R)$ *is generated by symbols* $\{E(T^{p^m}), r\}$, $m \leq n-1$, *together with certain elements in* $\mathrm{filt}^{n-1}TC_nK_2(R)$.

Proof. — Notice first that the lemma suffices to show the sequence:

$$\{R^\times/R^{\times p^n}\} \longrightarrow TCK_2(R) \xrightarrow{1-F} TCK_2(R)$$

is exact.

Suppose we have:

$$\alpha = \{\prod_{m<n} E(a_m T^{p^m}), T\} + \sum_{i, m_i < n} \{E(b_i T^{p^{m_i}}), c_i\}$$

such that $F(\alpha) = \alpha$. We proceed as follows.

Step 1. — We may assume all the $a_m \in R^\times \cup \{0\}$. Indeed, we can insure this by multiplying by suitable (trivial) symbols $\{E(T^{p^m}), T\}$.

Step 2. — We may assume $a_0 = 0$. Indeed, if $a_0 \in R^\times$ we have (2.1):

$$\{E(a_0T), T\} = -\{E(a_0T), -a_0\}.$$

Thus:

$$F\alpha = \{\prod_{m<n-1} E(a_m T^{p^{m+1}}), T\} + \sum_{i, m_i < n} \{E(b_i^p T^{p^{m_i}}), c_i\}.$$

Step 3. — If $n=1$ there is nothing to prove. Assume $n>1$, so it makes sense to speak of the images $\bar{\alpha} = \overline{F\alpha}$ in $T\Phi_0K_2(R) \cong \Omega^1_R$. We have:

$$\bar{\alpha} = \sum_{i, m_i = 0} b_i \frac{dc_i}{c_i} = \overline{F\alpha} = da_1 + \sum_{i, m_i = 0} b_i^p \frac{dc_i}{c_i}.$$

The exact sequence:

$$0 \longrightarrow R^\times/R^{\times p} \longrightarrow \Omega^1_R \xrightarrow{1-C^{-1}} \Omega^1_R/D_1$$

gives for some $s_0 \in R^\times$:

$$\sum_{i, m_i = 0} b_i \frac{dc_i}{c_i} = \frac{ds_0}{s_0}$$

and $\alpha \equiv \{E(T), s_0\} \bmod \mathrm{filt}^1 TC_nK_2(R)$.

Step 4. — Suppose inductively that for some r, $1 \leq r < n-1$:

$$\alpha = \{E(T), s_{r-1}\} + \alpha_r$$

$$\alpha_r = \{\prod_{m \geq r} E(a_m T^{p^m}), T\} + \sum_{i, m_i \geq r} \{E(b_i T^{p^{m_i}}), c_i\}.$$

Computing mod filt^r we get:

$$0 \equiv \alpha_r \equiv F\alpha_r \equiv \{E(a_r T^{p^{r-1}}), T\}.$$

Hence by (§ 7 (2.1)) $a_r = u^{p^r}$ for some $u \in R^\times$. Thus:

$$\{E(a_r T^{p^r}), T\} = p^r\{E(uT), T\} = p^r\{E(uT), -u\}$$

so we may assume $a_r = 0$.

Step 5. — Computing in $T\Phi_r K_2(R)$ we get:

$$\alpha \equiv \sum_{i, m_i = r} \{E(b_i T^{p^{m_i}}), c_i\} \equiv F\alpha \equiv \{E(a_{r+1} T^{p^r}), T\} + \sum_{i, m_i = r} \{E(b_i^p T^{p^{m_i}}), c_i\}.$$

Again by (§ 7 (2.1)) we have $a_{r+1} = a'^{p^r}_{r+1}$ and:

$$\sum_{i, m_i = r} b_i \frac{dc_i}{c_i} \equiv -a'^{p^r - 1}_{r+1}\, da'_{r+1} + \sum_{i, m_i = r} b_i^p \frac{dc_i}{c_i} \bmod D_r.$$

From the exact sequence:

$$0 \longrightarrow R^\times / R^{\times p} \longrightarrow \Omega^1_R / D_r \xrightarrow{C^{-1} - 1} \Omega^1_R / D_{r+1}$$

we conclude:

$$\sum_{i, m_i = r} b_i \frac{dc_i}{c_i} \equiv \frac{ds_r}{s_r} \bmod D_r,$$

for some $s_r \in R^\times$. Thus:

$$\alpha_r \equiv \{E(T^{p^r}), s_r\} = \{E(T), s_r^{p^r}\} \bmod \text{filt}^{r+1}$$

and $\alpha = \{E(T), s_{r-1} s_r^{p^r}\} + \alpha_{r+1}$. This process continues until $r = n - 2$ so we get for some s:

$$\alpha \equiv \{E(T), s\} \bmod \text{filt}^{n-1}\, TC_n K_2(R),$$

proving (5.2).

To complete the proof of (5.1), we must show $1 - F : TC_n K_2(R)$ is surjective when R is strictly Henselian. For symbols $\{E(aT^{p^m}), r\}$ to lie in the image, it suffices that the equation $X^p - X - a = 0$ have a solution in R. Such a solution exists since the equation is separable. Finally:

$$\{E(aT^{p^m}), T\} = (F - 1)\{\prod_{\ell = m+1}^{\infty} E(aT^{p^\ell}), T\}.$$

This completes the proof of (5.1). Q.E.D.

III. — GLOBAL RESULTS

1. Sheaves of typical curves

Let X be a smooth, projective variety over a perfect field k of characteristic $p \neq 0,2$. We will assume throughout that $\dim X < p$. I claim first that the constructions of the previous sections can be globalized to yield sheaves $\mathscr{T}\mathscr{C}_n\mathscr{K}_{q,X} = \mathscr{T}\mathscr{C}_n\mathscr{K}_q$ for the Zariski topology on X with stalks:

$$\mathscr{T}\mathscr{C}_n\mathscr{K}_{q,X,x} \cong \mathrm{TC}_n\mathrm{K}_q(\mathscr{O}_{X,x}).$$

Indeed, we may define $\mathscr{C}_n\mathscr{K}_{q,X}$ to be the Zariski sheaf associated to the presheaf $U \mapsto C_nK_q(\Gamma(U, \mathscr{O}_X))$. $\mathscr{S}\mathscr{C}_n\mathscr{K}_{q,X} \subset \mathscr{C}_n\mathscr{K}_{q,X}$ will be the subsheaf of sections which are represented at each stalk by symbols. By functoriality of the sheaf construction, $\mathscr{T}\mathscr{C}_n\mathscr{K}_{q,X}$ and $\mathscr{C}_n\mathscr{K}_{q,X}$ will be sheaves of big-$W(\mathscr{O}_X)$-modules. In characteristic $p \neq 0$, we can take typical components and define $\mathscr{T}\mathscr{C}_n\mathscr{K}_{q,X}$.

We will be particularly interested in the projective system of sheaves of typical curves on X:

$$\mathscr{T}\mathscr{C}\mathscr{K}_{q,X} = \{\mathscr{T}\mathscr{C}_n\mathscr{K}_{q,X}\}_{n \geq 1}$$

and in the projective system of complexes (II, 6.2.1), (II.7.1.4):

$$\{\mathscr{T}\mathscr{C}_n\mathscr{K}_{\bullet,X}, \delta_n^{(q)}\}_{n \geq 1}.$$

This complex (of pro-sheaves) will be called the *complex of typical curves* on X. To simplify notation, we will write:

$$C_n^{q-1} = \mathscr{T}\mathscr{C}_n\mathscr{K}_{q,X}$$
$$C^{q-1} = \mathscr{T}\mathscr{C}\mathscr{K}_{q,X}$$
$$\{C^\bullet, \delta^q\} = \{\mathscr{T}\mathscr{C}\mathscr{K}_{\bullet,X}, \delta\}.$$

Notice $C_n^0 = W_{n,X}$ is the sheaf of Witt vectors studied by Serre [25]. We will frequently write $W_{n,X}$ in place of C_n^0.

Proposition (**1.1**). — *Let* X *be as above. For any n, q the Zariski cohomology groups* $H^*(X, C_n^q)$ *are* $W(k)$*-modules of finite length. There is a Hodge type spectral sequence*:

$$E_1^{s,t} = H^t(X, C_n^s) \Rightarrow H^{s+t}(X, C_n^\bullet)$$

where $\mathbf{H}^*$ *denotes the hypercohomology of the complex. In particular* $\mathbf{H}^*(X, C_n^\bullet)$ *has finite length, and* $\mathbf{H}^\ell = (0)$ *for* $\ell > 2 \dim_k X$.

Proof. — The computations in II, § 7 (cf. II.7.2.4) show that C_n^q is built up by a finite number of extensions from coherent sheaves on X. Since X is proper, the cohomology groups of coherent sheaves are finite k-vector spaces, so the finiteness assertion follows. Also we have $C_n^q=(0)$ for $q>\dim X$ by (II.7.2.3). The other assertions are standard business, using these facts.

Recall ([20], III.0.13) a projective system $\{A_{n\in n}\}_{\mathbf{N}}$ of abelian groups is said to satisfy the Mittag-Loeffler condition if for all n, there exists an $N=N(n)$ such that for $m\geq N$, $\mathrm{Image}(A_m\to A_n)=\mathrm{Image}(A_N\to A_n)$.

Proposition (**1.2**). — (i) *If the* A_n *are modules of finite length over a ring* R *and if the transition maps are* R*-module maps, then* $\{A_n\}$ *satisfies Mittag-Loeffler.*

(ii) *If* $\{A_n\}$ *satisfies Mittag-Loeffler and if* $\{A_n\}\to\{B_n\}$ *is a map of projective systems induced by surjective maps* $A_n\twoheadrightarrow B_n$, *then* $\{B_n\}$ *satisfies Mittag-Loeffler.*

(iii) *Let* $\{A_n\}$, $\{B_n\}$, $\{C_n\}$ *be projective systems of abelian groups,* $A=\varprojlim A_n$, $B=\varprojlim B_n$, $C=\varprojlim C_n$. *Given exact sequences* $0\to A_n\to B_n\to C_n$ *compatible with the transition maps, we get exact sequences* $0\to A\to B\to C$.

(iv) *Suppose with notation as in* (iii) *we are given exact sequences* $A_n\to B_n\to C_n\to 0$, *and assume* $\{A_n\}$ *satisfies Mittag-Loeffler. Then the sequence* $A\to B\to C\to 0$ *is exact.*

Proof. — This is all standard. See, for example [*op. cit.*].

Corollary (**1.3**). — *Let* $\{\mathscr{A}_n\}$, $\{\mathscr{B}_n\}$, $\{\mathscr{C}_n\}$ *be projective systems of abelian sheaves on* X *and suppose given exact sequences* $0\to\mathscr{A}_n\to\mathscr{B}_n\to\mathscr{C}_n\to 0$ *compatible with transition maps. Assume that cohomology groups* $H^*(X,\mathscr{A}_n)$, $H^*(X,\mathscr{B}_n)$, $H^*(X,\mathscr{C}_n)$ *are modules of finite length over a ring* R, *and that the* R*-module structure is preserved by the transition maps. Define* $H^*(X,\mathscr{A})=\varprojlim_n H^*(X,\mathscr{A}_n)$, $H^*(X,\mathscr{B})=\varprojlim H^*(X,\mathscr{B}_n)$, $H^*(X,\mathscr{C})=\varprojlim H^*(X,\mathscr{C}_n)$. *Then one gets a long-exact sequence of cohomology*:

$$\ldots\to H^{n-1}(X,\mathscr{A})\to H^{n-1}(X,\mathscr{B})\to H^{n-1}(X,\mathscr{C})\to H^n(X,\mathscr{A})\to\ldots$$

Define:

$$H^t(X,C^q)=\varprojlim_n H^t(X,C_n^q)$$

$$H^t(X,C^\bullet)=\varprojlim_n H^t(X,C_n^\bullet).$$

From (1.1) and (1.2) one gets:

Proposition (**1.4**). — *By passing to the limit, there exists a spectral sequence (slope spectral sequence)*:

$$E_1^{s,t}=H^t(X,C^s)\Rightarrow H^{s+t}(X,C^\bullet).$$

Lemma (**1.5**). — *Let* $\{\mathscr{A}_n\}$, $\{\mathscr{B}_n\}$, $\{\mathscr{C}_n\}$ *be projective systems of sheaves of abelian groups on* X, *and suppose given exact sequences*:

$$0\to\mathscr{A}_n\to\mathscr{B}_n\to\mathscr{C}_n\to 0$$

compatible with the transition maps. Assume the projective system $\{\mathscr{A}_n\}$ *is essentially zero. Then* $\varprojlim H^*(X, \mathscr{B}_n) \cong \varprojlim H^*(X, \mathscr{C}_n)$.

Proof. — We have long exact sequences of cohomology:

$$\ldots \longrightarrow H^{q-1}(X, \mathscr{C}_n) \xrightarrow{\partial_n^{q-1}} H^q(X, \mathscr{A}_n) \xrightarrow{\alpha_n^q} H^q(X, \mathscr{B}_n) \xrightarrow{\beta_n^q} H^q(X, \mathscr{C}_n).$$

The systems $\{\mathrm{Ker}\ \alpha_n^q\}_{n\geq 1}$ and $\{\mathrm{Ker}\ \beta_n^q\}_{n\geq 1}$ are essentially zero, and hence satisfy Mittag-Loeffler. Writing:

$$H^{q-1}(X, \mathscr{C}) \xrightarrow{\partial^{q-1}} H^q(X, \mathscr{A}) \xrightarrow{\alpha} H^q(X, \mathscr{B}) \xrightarrow{\beta} H^q(X, \mathscr{C}) \xrightarrow{\partial^q}$$

for the inverse limit of these sequences, we infer from (1.2) that Image $\alpha = \mathrm{Ker}\ \beta$ and Image $\beta = \mathrm{Ker}\ \partial^q$. Since $H(X, \mathscr{A}) = 0$, this implies $H^q(X, \mathscr{B}) \cong H^q(X, \mathscr{C})$. Q.E.D.

Proposition **(1.6)**. — *There are long exact sequences*:

$$\ldots \to H^{n-1}(X, C^q) \to H^{n-1}(X, C^q/pC^q) \to H^n(X, C^q) \xrightarrow{p} H^n(X, C^q) \to \ldots$$
$$\ldots \to H^{n-1}(X, C^q) \to H^{n-1}(X, C^q/VC^q) \to H^n(X, C^q) \xrightarrow{V} H^n(X, C^q) \to \ldots$$
$$\ldots \to H^{n-1}(X, C^\bullet) \to H^{n-1}(X, C^\bullet/VC^\bullet) \to H^n(X, C^\bullet) \xrightarrow{V} H^n(X, C^\bullet) \to \ldots$$

Proof. — Fix q and define $\mathscr{K}er(p)_r$, $\mathscr{C}oker(p)_r$ by the exact sequence:

$$0 \to \mathscr{K}er(p)_r \to C_r^q \xrightarrow{p} C_r^q \to \mathscr{C}oker(p)_r \to 0.$$

I will first show by induction on r that the cohomology of the sheaves $\mathscr{K}er(p)_r$ and $\mathscr{C}oker(p)_r$ has finite length over $W(k)$. $C_1^q \cong \Omega_X^q$, so the assertion is clear for $r=1$. For $r>1$ we have:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T\Phi_{r-1}K_{q+1} & \longrightarrow & C_r^q & \longrightarrow & C_{r-1}^q & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle 0} & & \Big\downarrow{\scriptstyle p} & & \Big\downarrow{\scriptstyle p} & & \\ 0 & \longrightarrow & T\Phi_{r-1}K_{q+1} & \longrightarrow & C_r^q & \longrightarrow & C_{r-1}^q & \longrightarrow & 0 \end{array}$$

The natural map $\mathscr{K}er(p)_r \to \mathscr{K}er(p)_{r-1}$ is zero (cf. the proof of (II.7.4.2)) so the snake lemma gives:

$$\mathscr{K}er(p)_r \cong T\Phi_{r-1}K_{q+1}$$
$$0 \to \mathscr{K}er(p)_{r-1} \to T\Phi_{r-1}K_{q+1} \to \mathscr{C}oker(p)_r \to \mathscr{C}oker(p)_{r-1} \to 0.$$

The assertion follows from the fact that the cohomology of $T\Phi_{r-1}K_{q+1}$ is of finite length. (It also follows from this argument that $p.C_r^q \cong C_{r-1}^q$.) The first exact sequence in (1.6) is obtained from:

$$0 \to \mathscr{K}er(p)_r \to C_r^q \xrightarrow{p} C_r^q \to \mathscr{C}oker(p)_r \to 0$$

by splitting into 3-term sequences and applying (1.5). The other exact sequences are derived similarly. Q.E.D.

Corollary (**1.7**). — $H^n(X, C^{\bullet})$ *is a finitely generated* $W(k)$*-module for all* n.

Proof. — By (II.7.3.1):

$$H^n(X, C^{\bullet}/pC^{\bullet}) \simeq H^n(X, \Omega^{\bullet}_X).$$

Hence $H^n(X, C^{\bullet})/p.H^n(X, C^{\bullet})$ is a finite k-vector space. The action of p on $H^n(X, C^{\bullet}) = \varprojlim H^n(X, C^{\bullet}_r)$ is topologically nilpotent and this group is complete and separated, hence finitely generated. Q.E.D.

2. Comparison and finiteness theorems

Theorem (**2.1**). — *Let* X *be smooth and projective over a perfect field* k *of characteristic* $p \neq 0, 2$. *Assume* $\dim X < p$. *Let* $H^{*}_{\mathrm{cris}}(X/W) = \varprojlim_n H^{*}_{\mathrm{cris}}(X/W_n)$ *denote the crystalline cohomology. Then there is a canonical isomorphism*:

$$H^{*}_{\mathrm{cris}}(X/W) \cong H^{*}(X, C^{\bullet}).$$

Proof. — Recall we have defined the complex $\Omega^{\bullet}_{W_n, Y}$ (I.4.5.5) and we have a map:

$$\alpha : H^{*}_{\mathrm{cris}}(X/W_n) \to H^{*}(X, \Omega^{\bullet}_{W_n, Y}).$$

It follows from (II.6.3.4) that there is a map of complexes $\varphi_n : \Omega^{\bullet}_{W_n, Y} \to C^{\bullet}_n$. We compose to get $\beta^q_n : H^q_{\mathrm{cris}}(X/W_n) \to H^q(X, C^{\bullet}_n)$ and in the limit:

$$\beta^q : H^q_{\mathrm{cris}}(X/W) \to H^q(X, C^{\bullet}).$$

We get a diagram with exact rows:

$$\begin{array}{ccccccccc}
H^{q-1}(X, \Omega^{\bullet}_X) & \to & H^q_{\mathrm{cris}}(X/W) & \xrightarrow{p} & H^q_{\mathrm{cris}}(X/W) & \to & H^q(X, \Omega^{\bullet}_X) & \to & H^{q+1}_{\mathrm{cris}}(X/W) \\
\wr\| & & \downarrow{\scriptstyle \beta^q} & & \downarrow{\scriptstyle \beta^q} & & \wr\| & & \downarrow{\scriptstyle \beta^{q+1}} \\
H^{q-1}(X, \Omega^{\bullet}_X) & \longrightarrow & H^q(X, C^{\bullet}) & \xrightarrow{p} & H^q(X, C^{\bullet}) & \to & H^q(X, \Omega^{1}) & \to & H^{q+1}(X, C^{\bullet})
\end{array}$$

(Existence and exactness for the top row follows from the base change theorem ([3], V.3.5.8) plus universal coefficients ([3], VII.1.1.11). Commutativity of the diagram follows from compatibility of the construction of α (I, § 3, 5) with reduction modulo p.) Notice that everything in the above diagram is zero for $q > 2 \dim X$, so β^q is an isomorphism. Proceeding by downward induction on q assume β^{q+1} is an isomorphism. From the diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & H^q_{\mathrm{cris}}/pH^q_{\mathrm{cris}} & \longrightarrow & H^q(X, \Omega^{\bullet}) & \longrightarrow & H^{q+1}_{\mathrm{cris}} \\
 & & \downarrow & & \wr\| & & \wr\downarrow{\scriptstyle \beta^{q+1}} \\
0 & \to & H^q(C^{\bullet})/pH^q(C^{\bullet}) & \to & H^q(X, \Omega^{\bullet}) & \longrightarrow & H^{q+1}(C^{\bullet})
\end{array}$$

it follows that β^q induces an isomorphism mod p. Since everything is finitely generated over $W(k)$, β^q is surjective.

The Kernel ${}_pH^q_{\mathrm{cris}}$ of multiplication by p maps surjectively to the corresponding Kernel ${}_pH^q(C^\bullet)$. We now have acyclic complexes:

$$\begin{array}{ccccccccc} 0 \to & {}_pH^q_{\mathrm{cris}} & \to & H^q_{\mathrm{cris}} & \xrightarrow{p} & H^q_{\mathrm{cris}} & \longrightarrow & \dots \\ & \downarrow & & \downarrow{\scriptstyle\beta^q} & & \downarrow{\scriptstyle\beta^q} & & \downarrow{\scriptstyle\wr\|} \\ 0 \to & {}_pH^q(C^\bullet) & \to & H^q(C^\bullet) & \to & H^q(C^1) & \to & \dots \end{array}$$

with all vertical arrows surjective. Thus the complex of kernels is acyclic, so multiplication by p on Ker β^q is surjective and β^q is necessarily an isomorphism. Q.E.D.

Theorem (**2.2**). — *The groups*:

$$H^n(X, C^q)/p\text{-torsion}$$

are finitely generated $W(k)$*-modules. These modules, equipped with operators* F, V *induced from the corresponding operators on* C^q, *are thus the Dieudonné modules of certain formal groups associated to* X.

Proof. — Let $\sigma : W(k) \to W(k)$ denote the Frobenius automorphism, and let $A = W(k)[[V]]$ denote the "Hilbert ring" of power series in V with commutation relation $V.\sigma(x) = x.V$, $x \in W(k)$.

Lemma (**2.3**). — (i) A *is noetherian (on the left and the right)*.

(ii) *Any finitely generated* A*-module* M *is* V*-adically complete and separated*, i.e.:

$$M \cong \varprojlim M/V^nM.$$

Proof. — (i) Let $\mathfrak{J} \subset A$ be (say) a left ideal, and let $f \in \mathfrak{J}$, $f \neq 0$. An argument of Manin [23] shows that f can be written $f = u.f_0$ where $f_0 \in W(k)[V]$ is a monic polynomial and u is a unit. It follows that A/Af is finitely generated as a $W(k)$-module, whence $\mathfrak{J}/Af$ is finitely generated, as is $\mathfrak{J}$ as an A-module.

(ii) The assertion is clear if M is free. More generally write M as a quotient of a finitely generated free A-module:

$$0 \to R \to F \to M \to 0.$$

Let ${}_{V^n}M$ denote the Kernel of multiplication by V^n on M. We get for each $n \geq 1$ a long exact sequence:

$$0 \to {}_{V^n}M \to R/V^nR \to F/V^nF \to M/V^nM \to 0.$$

The sequence ${}_{V^n}M$ is an ascending sequence of submodules of M and is therefore stationary by (i). Thus the prosystem $\{{}_{V^n}M\}$ with transition maps given by multiplication by V is essentially zero. Applying (1.2) we get a diagram with exact rows:

$$\begin{array}{ccccccccc}
0 \to & R & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0 \\
 & \downarrow & & \downarrow \wr & & \downarrow & & \\
0 \to & \varprojlim R/V^nR & \to & \varprojlim F/V^nF & \to & \varprojlim M/V^nM & \to & 0
\end{array}$$

In particular M maps onto $\varprojlim M/V^nM$. Since R satisfies the same hypotheses as M, $R \twoheadrightarrow \varprojlim R/V^nR$ and (easy diagram chase) $M \cong \varprojlim M/V^nM$. Q.E.D.

Lemma **(2.4)**. — *Let* H *be a finitely generated* A*-module. Assume that* H/VH *has finite length as a* $W(k)$*-module, and let* $T \subset H$ *be the submodule generated by all* p*-torsion and all* V*-torsion elements. Then* H/T *is a finitely generated* $W(k)$*-module.*

Proof. — Replacing H by H/T, we may assume H has no p-torsion and no V-torsion. Let $B = A[V^{-1}] = W(k)((V))$ be the ring of "Hilbert Laurent series", and let $H_B = B \otimes_A H$. We have $H \hookrightarrow H_B$. I claim first that H_B is a torsion B-module (compare [23], proof of proposition (2.1) (2)). It suffices to show every $x \in H$ is killed by some $a \in A$.

Suppose the map $i_x : A \to H$, $i_x(a) = ax$, is injective for some $x \in H$. Since H/VH has finite length we have $p^r H \subset VH$ for some $r \gg 0$. Thus $p^r V^{-1} : H_B \to H_B$ stabilizes $H \subset H_B$. Let $H' = \{b \in B \mid bx \in H\}$. H' is an A-module and $i_x : H' \to H$, $i_x(b) = bx$. Note if $i_x(b) \in VH$, we have $bx = Vh$, $V^{-1}bx \in H$, so $b \in VH'$. Thus $H'/VH' \hookrightarrow H/VH$. In particular H'/VH' is of finite length, so H' is finitely generated as an A-module.

Let $\{h'_i\}$ be a finite set of generators of H' and choose $n \gg 0$ such that $V^n h'_i \in VA$ for all i. Since $p^r V^{-1} \in H'$ we have:

$$p^{rn} V^{-n} = \sum_i a_{i,n} h'_i, \qquad a_{i,n} \in A$$

whence $p^{rn} \in VA$. This is a contradiction, so H_B is torsion as claimed.

To finish the proof of (2.4), we apply ([23], corollary 2, p. 27 and lemma (2.1), p. 28) to get:

$$H_B = \bigoplus_{\text{finite}} \text{Modules of the form } B/Bq$$

for suitable q:

$$q = V^n + \sum_{i=1}^{n} a_i V^{n-i}, \qquad a_i \in pW(k).$$

Notice B/Bq is a finitely generated module over the *quotient field* K of $W(k)$. In fact, $V^{-1} = (V^{n-1} + a_1 V^{n-2} + \ldots + a_{n-1}) \cdot a_n^{-1}$ so B/Bq is generated by $1, V, \ldots, V^{n-1}$. Thus $H \subset$ finitely generated K-module. Since $p^r H \subset VH$ we have $\bigcap_n p^n H = (0)$. Now choose a finite set $\{h_i\}$ of elements in H which form a basis for $H \otimes K$ and let $H_0 \subset H$

be the finitely generated free $W(k)$-module generated by the h_i. Either $p^N H \subseteq H_0$ for some N, in which case H is finitely generated over $W(k)$ as desired, or there exist elements $x_\ell \in H$, $1 \leq \ell < \infty$ such that $p^\ell x_i \in H_0 - pH_0$. In the latter case, using the compacity of H_0, there exists a sequence of integers $\ell_i \to \infty$ such that $p^\ell x_{\ell_i} \to x \in H_0 - pH_0$. Thus $x \in \bigcap_n p^n H$, contradicting the fact that this intersection is zero. Q.E.D.

Given an integer $q \geq 0$, let $t_{\leq q} C^\bullet$ denote the pro-system of complexes of sheaves on X (X as in § 1):

$$C^0 \to C^1 \to \ldots \to C^q \to 0.$$

The complex $t_{\leq q} C^\bullet$ has an endomorphism $V_{(q)}$ which is $p^{q-r}V$ on C^r (II.6.1.1) and there is an exact sequence of complexes with compatible endomorphisms:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C[-q] & \longrightarrow & t_{\leq q}C^\bullet & \longrightarrow & t_{\leq q-1}C^\bullet & \longrightarrow & 0 \\ & & \downarrow {\scriptstyle V} & & \downarrow {\scriptstyle V_{(q)}} & & \downarrow {\scriptstyle pV_{(q-1)}} & & \\ 0 & \longrightarrow & C[-q] & \longrightarrow & t_{\leq q}C^\bullet & \longrightarrow & t_{\leq q-1}C^\bullet & \longrightarrow & 0 \end{array} \tag{2.5}$$

Lemma (**2.6**). — *The cohomology* $H^\bullet(X, t_{\leq q}C^\bullet)$ *is a finitely generated* A*-module with* $V \in A$ *acting by* $V_{(q)}$. *Moreover* $H^\bullet(X, t_{\leq q}C^\bullet)/V.H^\bullet(X, t_{\leq q}C^\bullet)$ *is a* $W(k)$*-module of finite length.*

Proof. — The Kernel of $V_{(q)}$ is essentially zero, so by (1.5) we get a long exact sequence:

$$\ldots H^n(X, t_{\leq q}C^\bullet) \xrightarrow{V_{(q)}} H^n(X, t_{\leq q}C^\bullet) \to H^n(X, t_{\leq q}C^\bullet/V_{(q)}t_{\leq q}C^\bullet) \\ \to H^{n+1}(X, t_{\leq q}C^\bullet) \to \ldots$$

Everything will follow if we show the cohomology of $t_{\leq q}C^\bullet/V_{(q)}t_{\leq q}C^\bullet$:

$$W/p^qVW \to C^1/p^{q-1}VC^1 \to \ldots \to C^q/VC^q$$

has finite length.

When $q=0$ we get $W/VW = \mathcal{O}_X$ and the assertion is true. In general, there is a sequence of complexes (vertical arrows exact up to essentially zero pro-objects):

$$\begin{array}{ccccccc} 0 & & & & 0 & & \\ \downarrow & & & & \downarrow & & \\ C^0/p^{q-1}VC^0 & \longrightarrow & \ldots & \longrightarrow & C^{q-1}/VC^{q-1} & \longrightarrow & 0 \\ \downarrow {\scriptstyle p} & & & & \downarrow {\scriptstyle p} & & \downarrow \\ C^0/p^qVC^0 & \longrightarrow & \ldots & \longrightarrow & C^{q-1}/pVC^{q-1} & \longrightarrow & C^q/VC^q \\ \downarrow & & & & \downarrow & & \downarrow \\ C^0/pC^0 & \longrightarrow & \ldots & \longrightarrow & C^{q-1}/pC^{q-1} & \longrightarrow & C^q/VC^q \\ \downarrow & & & & \downarrow & & \downarrow \\ 0 & & & & 0 & & 0 \end{array}$$

Using (II.7.3.1), we get a long exact cohomology sequence:

$$\ldots \to H^{n-1}(X, t_{\leq q}\Omega^{\bullet}_X) \to H^n(X, t_{\leq q-1}C^{\bullet}/V_{(q-1)}t_{\leq q-1}C^{\bullet}) \to H^n(X, t_{\leq q}C^{\bullet}/V_{(q)}t_{\leq q}C^{\bullet}) \to H^n(X, t_{\leq q}\Omega^{\bullet}_X) \to \ldots$$

The finiteness assertion for the cohomology of $t_{\leq q}C^{\bullet}/V_{(q)}t_{\leq q}C^{\bullet}$ follows by induction.
Q.E.D.

We can now complete the proof of (2.2). The sequence (2.5) gives a cohomology sequence:

$$H^{n+q-1}(X, t_{\leq q-1}C^{\bullet}) \to H^n(X, C^q) \to H^{n+q}(X, t_{\leq q}C^{\bullet}).$$

This is an exact sequence of A-modules if we let V act on the left-hand group by $pV_{(q-1)}$, on the middle by V itself, and on the right by $V_{(q)}$. Truncate this sequence to get:

$$0 \to M \to H^n(X, C^q) \to N \to 0.$$

We know by (2.4) and (2.6) M/(p-torsion) is finitely generated over W(k) (note V-torsion $\subset$ p-torsion because each of these modules has an endomorphism F with FV = some power of p. Also note that M itself is *not* in general a finitely generated A-module because the action of V is *via* $pV_{(q-1)}$, not $V_{(q-1)}$). Further N is finitely generated over A and $N/p\text{-torsion} \subset H^{n+q}(X, t_{\leq q}C^{\bullet})/p\text{-torsion}$ is finitely generated over W(k).

Since A is noetherian, we get $\text{tors}(N) = {}_{p^\ell}N$ for $\ell \gg 0$. The exact sequence $\text{Tor}_1^W(H^n(X, C^q), K/W) \to \text{Tor}_1^W(N, K/W) \to M \otimes K/W$ shows:

$$\text{Coker}(\text{tors}\, H^n(X, C^q) \to \text{tors}\, N) \subset (M/\text{tors}\, M) \otimes (W(k)/p^\ell W(k)).$$

In particular, this cokernel is finitely generated over W(k). It follows easily that $H^n(X, C^q)/p$-torsion is finitely generated over W(k), completing the proof of (2.2).
Q.E.D.

3. Slope spectral sequence.

Definition (**3.1**). — Assume $\dim X < p$. The spectral sequence:

$$E_1^{s,t} = H^t(X, C^s) \Rightarrow H^{s+t}_{\text{cris}}(X/W)$$

will be called the *slope spectral sequence*.

Proposition (**3.2**). — *The slope spectral sequence degenerates up to torsion at* E_1. (Note (3.2) implies (2.2).)

Proof. — Let $\mathscr{V}$ be the endomorphism of $C^{\bullet}$ given by $p^{\dim X - q}.V$ on C^q. The corresponding endomorphisms $\mathscr{V}_r^{s,t}$ on $E_r^{s,t}/p$-torsion will have slopes σ. (Recall the *slopes* of $\mathscr{V}_r^{s,t}$ are the p-adic ordinals of the eigenvalues, the point being that in fact the eigenvalues are not quite well defined because $\mathscr{V}_r^{s,t}$ is not W(k)-linear, whereas their ordinals are defined ([12], [3 *bis*], [23 *bis*])), satisfying:

$$\dim X - s < \sigma \leq \dim X - s + 1.$$

Indeed, on $E_1^{s,t}/p$-torsion we have $\mathscr{V}_1^{s,t}=p^{\dim X - s}.V$ and V is topologically nilpotent (II, § 7 (1.6) (iii)) so $\sigma>\dim X-s$. On the other hand $E_1^{s,t}$ admits an endomorphism F such that $VF=FV=p$, so $\sigma\leq\dim X-s+1$. Since $E_r^{s,t}/p$-torsion is a subquotient of $E_1^{s,t}/p$-torsion we get the same inequalities for the slopes of $\mathscr{V}_r^{s,t}$.

To show the differentials d_r are torsion, note we have commutative squares:

$$\begin{array}{ccc} E_r^{s,t}/p\text{-torsion} & \xrightarrow{d_r} & E_r^{s+r,t-r+1}/p\text{-torsion} \\ \downarrow{\scriptstyle \mathscr{V}^{s,t}} & & \downarrow{\scriptstyle \mathscr{V}_r^{s+r,t-r+1}} \\ E_r^{s,t}/p\text{-torsion} & \xrightarrow{d_r} & E_r^{s+r,t-r+1}/p\text{-torsion} \end{array}$$

Since $\mathscr{V}_r^{s,t}$ and $\mathscr{V}_r^{s+r,t-r+1}$ have no slopes in common, the d_r must be trivial. Q.E.D.

Let $\mathrm{Slope}^\bullet\, H^*_{\mathrm{cris}}(X/W)$ be the filtration defined by the slope spectral sequence (3.1).

Proposition **(3.3)**. — *The filtration* $\mathrm{Slope}^\bullet\, H^*_{\mathrm{cris}}(X/W)$ *is stable under the frobenius f. Moreover, the action of f on*:

$$(\mathrm{Slope}^\ell\, H^*_{\mathrm{cris}}(X/W))/p\text{-torsion}$$

is divisible by p^ℓ.

Proof. — $\mathrm{Slope}^\ell\, H^*_{\mathrm{cris}}$ is identified with the image of the map:

$$\rho^{(\ell)}:\ H^*(X, \mathrm{Slope}^\ell\, C^\bullet)\to H^*(X, C^\bullet)$$

where:
$$(\mathrm{Slope}^\ell\, C^\bullet)^r=\begin{cases} 0 & r<\ell \\ C^r & r\geq\ell. \end{cases}$$

$\mathrm{Slope}^\ell\, C^\bullet$ has an endomorphism $\mathscr{F}^{(\ell)}$ such that the square:

$$\begin{array}{ccc} H^*(X, \mathrm{Slope}^\ell\, C^\bullet) & \xrightarrow{\rho^{(\ell)}} & H^*_{\mathrm{cris}}(X/W) \\ \downarrow{\scriptstyle p^\ell\mathscr{F}^{(\ell)}} & & \downarrow{\scriptstyle f} \\ H^*(X, \mathrm{Slope}^\ell\, C^\bullet) & \xrightarrow{\rho^{(\ell)}} & H^*_{\mathrm{cris}}(X/W) \end{array}$$

commutes. By (3.2), $\rho^{(\ell)}$ is injective mod p-torsion, so f is divisible by p^ℓ on the image. Q.E.D.

Corollary **(3.4)**. — *Let* K *be the quotient field of* W(*k*). *The filtration*:

$$\mathrm{Slope}^\bullet\, H^*_{\mathrm{cris}}(X/W)\otimes K$$

is given by:

$$\text{Slope}^{\ell}\, H^*_{\text{cris}}\otimes K = \text{largest subspace of } H^*_{\text{cris}}\otimes K \text{ stable under frobenius on which the slopes } s \text{ satisfy } \ell \leq s < \infty.$$

Proof. — Using (3.3), it suffices to show that on:

$$(H^*_{\text{cris}}/\text{Slope}^{\ell}\, H^*_{\text{cris}})\otimes K \cong H^*(X, t_{\leq \ell-1}C^\bullet)\otimes K$$

the slopes all satisfy $0 \leq s < \ell$. This follows from the existence of a topologically nilpotent operator $\mathscr{V}_{(\ell)}$ on $H^*(X, t_{\leq \ell-1}C^\bullet)$ ($\mathscr{V}_{(\ell)}$ defined by taking $p^{\ell-1-q}.V$ on C^q) such that $f\circ \mathscr{V}_{(\ell)} =$ multiplication by p^ℓ. Q.E.D.

Example (**3.5**). — It follows from the above that the Witt vector cohomology $H^\bullet(X, W)\otimes K$ is the part of $H^*_{\text{cris}}\otimes K$ of slope <1 (compare [1]).

4. Relation with flat cohomology.

Let X be smooth and proper over a perfect field k of characteristic $\neq 0,2$. Let $H^*_{\text{fl}}(X, \mu_{p^\nu})$ denote the cohomology in the flat topology of the sheaf $\mu_{p^\nu} = \text{Ker}(G_m \xrightarrow{p^\nu} G_m)$. Let

$$H^n_{\text{fl}}(X, \mathbf{Z}_p(1)) = \varprojlim_{\nu} H^n_{\text{fl}}(X, \mu_{p^\nu}), \qquad H^n_{\text{fl}}(X, \mathbf{Q}_p(1)) = H^n_{\text{fl}}(X, \mathbf{Z}_p(1))_{\mathbf{Q}}.$$

Finally, let $H^*_{\text{cris}}(X/W)$ denote the crystalline cohomology and let f be the frobenius endomorphism of H^*_{cris}.

Theorem (**4.1**). — (i) *Assume* $k=\bar{k}$. *There exist short exact sequences for all* $n\geq 0$:

$$0 \to H^n_{\text{fl}}(X, \mathbf{Q}_p(1)) \to H^{n-1}(X, C^1)_{\mathbf{Q}} \xrightarrow{1-F} H^{n-1}(X, C^1)_{\mathbf{Q}} \to 0$$

where $F : C^1 = \mathscr{C}\mathscr{C}\mathscr{K}_2 \to \mathscr{C}\mathscr{C}\mathscr{K}_2$ *is the frobenius*.

(ii) *Assume* $k=\bar{k}$ *and* $\dim X < p$. *We get*:

$$H^n_{\text{fl}}(X, \mathbf{Q}_p(1)) \cong H^n_{\text{cris}}(X/W)^{(f=p)}_{\mathbf{Q}} = \{\alpha\in H^n_{\text{cris}}\otimes\mathbf{Q} \mid f\alpha = p\alpha\}.$$

(iii) *Without assumption on* k, *if the pro-system* $\{H^*_{\text{fl}}(X, \mu_{p^\nu})\}_{\nu\geq 1}$ *satisfies the Mittag-Loeffler condition for all values of* $*$, *then there is a long-exact sequence*:

$$\ldots \to H^n_{\text{fl}}(X, \mathbf{Z}_p(1)) \to H^{n-1}(X, C^1) \xrightarrow{1-F} H^{n-1}(X, C^1) \to H^{n+1}_{\text{fl}}(X, \mathbf{Z}_p(1)) \to \ldots$$

Proof. — The pro-system $\{C^1_n\} = C^1$ consists of sheaves which are extensions of coherent sheaves on X, so $H^*_{\text{Zariski}}(X, C^1) = H^*_{\text{étale}}(X, C^1)$. Let $\pi : X_{\text{fl}} \to X_{\text{ét}}$ be the morphism of topoi. We have ([20 *bis*] (11.7)):

$$R^n\pi_*\mu_{p^\nu} = \begin{cases} G_m/G_m^{p^\nu} & n=1 \\ 0 & n\neq 1 \end{cases}$$

so:

$$H^n_{\text{fl}}(X, \mu_{p^\nu}) \cong H^{n-1}_{\text{ét}}(X, G_m/G_m^{p^\nu}).$$

If these cohomology groups satisfy Mittag-Loeffler, we can use (II.7.5.1) to get exact sequences:

$$\ldots \to \varprojlim_{\nu} H^{n-1}_{\text{ét}}(X, G_m/G_m^{p^\nu}) \to H^{n-1}(X, C^1) \xrightarrow{1-F} H^{n-1}(X, C^1) \to \ldots,$$

proving (iii).

More generally, let $R_n = \operatorname{Ker}(1-F : C^1_{n,\text{ét}} \to C^1_{n-1,\text{ét}})$. We have:

$$\{G_{m,\text{ét}}/G^{p^\nu}_{m,\text{ét}}\} \hookrightarrow \{R_\nu\}$$

with cokernel essentially zero (II, § 7 (5.2)). By an argument similar to (1.5), we get:

$$\varprojlim_{\nu} H^n_{\text{ét}}(X, G_m/G_m^{p^\nu}) \cong \varprojlim_{\nu} H^n_{\text{ét}}(X, R_\nu).$$

Truncate the long exact sequences:

$$\ldots \xrightarrow{\partial^n_\nu} H^n_{\text{ét}}(X, R_\nu) \xrightarrow{\alpha^n_\nu} H^n_{\text{ét}}(X, C^1_\nu) \xrightarrow{\beta^n_\nu} H^n_{\text{ét}}(X, C^1_{\nu-1}) \xrightarrow{\partial^{n+1}_\nu} \ldots$$

to get:

$$0 \to \text{Image } \beta^n_\nu \to H^n_{\text{ét}}(X, C^1_{\nu-1}) \to \operatorname{Ker} \alpha^n_\nu \to 0$$

$$0 \to \text{Image } \partial^n_\nu \to H^n_{\text{ét}}(X, R_\nu) \to \operatorname{Ker} \beta^n_\nu \to 0.$$

Note $\{H^*(X, C^1_\nu)\}$ satisfies Mittag-Loeffler (the modules have finite length), as do the quotient pro-objects $\{\text{Image } \beta^n_\nu\}_\nu$ and $\{\text{Image } \partial^n_\nu\}_\nu$. Passing to the limit, we conclude that the complex:

$$\ldots \xrightarrow{\partial^n} \varprojlim_{\nu} H^n_{\text{ét}}(X, R_\nu) \xrightarrow{\alpha^n} H^n_{\text{ét}}(X, C^1) \xrightarrow{\beta^n} H^n_{\text{ét}}(X, C^1) \xrightarrow{\partial^{n+1}}$$

is exact except possibly on the right, where we only know $\text{Image } \beta^n \subset \operatorname{Ker} \partial^{n+1}$. The modules $H^n_{\text{ét}}(X, C^1)_{\mathbf{Q}}$ are finitely generated over $K =$ quotient field of $W(k)$, and $\beta^n = 1 - F = 1$-frobenius linear map. Assuming $k = \bar{k}$, one knows from the theory of frobenius linear maps that such endomorphisms are always surjective, so:

$$\operatorname{Ker} \partial^{n+1} \subset H^n(X, C^1)_{\mathbf{Q}} = \text{Image } \beta^n \otimes \mathbf{Q}.$$

Finally, if $\dim X < p$, it follows from (3.4) and (3.2) that:

$$H^n_{\text{cris}}(X/W)^{(f=p)}_{\mathbf{Q}} \cong H^{n-1}(X, C^1)^{(F=1)}_{\mathbf{Q}}. \qquad \text{Q.E.D.}$$

IV. — OPEN PROBLEMS

The paper raises a number of questions. For the convenience of the reader, I will describe three problems of immediate import and three others of a somewhat more vague and open-ended character.

Problem 1. — Let R be a commutative ring and let $C_n^i(R) = C_n K_{i+1}(R)$. One would like to give $C_n^*(R) = \bigoplus_{i \geq 0} C_n^i(R)$ the structure of graded ring (commutative in the graded sense), compatible with the module structure $C_n^i(R) \times C_n^0(R) \to C_n^i(R)$ defined in II, § 2. Using [15], one can define pairings $C_n^i(R) \times C_n^j(R) \to C_n^{i+j}(R)$ but associativity and commutativity are unclear. This product structure should give the complex of typical curves the structure of a differential graded algebra. It should be compatible with the product structure in crystalline cohomology and should induce a product structure on the slope spectral sequence (III, § 1 (1.4)).

Problem 2. — Eliminate the hypothesis $\dim X < \mathrm{char}.\, k$. This is really a problem about the topology of BGL^+ which can be thought of as follows: Let $M_* = \bigoplus_{i \geq 0} M_i$ be a graded ring and suppose we have a theory of Chern classes $C(\rho)$ for representations ρ of abstract groups G on finitely generated projective R-modules (R some fixed ring). We suppose $C^i(\rho) \in H^i(G, M_i)$ and that the $C(\rho)$ satisfy the usual identities for a theory of Chern classes. Fix an integer q and view C^q as a map $C^q : BGL^+(R) \to K(M_q, q)$. Let $BGL^+(R)(q-1)$ denote the space obtained from $BGL^+(R)$ by killing $\pi_1, \ldots, \pi_{q-1}$. One needs to " divide " C^q by constructing a map $c'^q : BGL^+(R)(q-1) \to K(M_q, q)$ in such a way that the diagram:

$$\begin{array}{ccc} BGL^+(R)(q-1) & \xrightarrow{c'^q} & K(M_q, q) \\ \big\downarrow & & \big\downarrow {\scriptstyle \text{multiplication by } (-1)^q (q-1)!} \\ BGL^+(R) & \xrightarrow{\quad c^q \quad} & K(M_q, q) \end{array}$$

commutes up to homotopy. Since $\pi_q(BGL^+(R)(q-1)) = \pi_q(BGL^+(R)) = K_q(R)$ this construction will yield $\dfrac{\text{“}(-1)^q\text{”}}{(q-1)!} c^q : K_q(R) \to M_q$. This is related to a theorem of Adams for BU. Adams showed that the q-th Chern character in $H^{2q}(BU, \mathbf{Q})$ pulled back to an integral cohomology class in $H^{2q}(BU(2q-1))$.

Problem 3. — Eliminate the hypothesis $p \neq 2$ which occurs throughout the computations.

The following problems are somewhat broader and less precise.

Problem 4. — Understand the kernel of $1-F : TCK_n(R) \to TCK_n(R)$. There is a map of pro-systems:

$$e : \{K_{n-1}(R)/p^m K_{n-1}(R)\}_{m \geq 1} \to \operatorname{Ker}(1-F)$$

analogous to the map defined in (II, § 8 (5.1)). Is e an isomorphism? This question is closely related to recent work of Milne and Parshin on duality and classfield theory for surfaces. An affirmative answer should enable one to compute galois groups of fields of the form $K = R((t))$, where $k = \mathbf{F}_q((s))$. Parshin has shown that for L/K galois of degree prime to p, there is an isomorphism $K_2^{top}(K)/NK_2^{top}(L) \cong \operatorname{Gal}(L/K)$, where K_2^{top} denotes the topological K_2-group and $N : K_2^{top}(L) \to K_2^{top}(K)$ is the norm. An affirmative answer in problem 4 for $n=3$ should enable one to remove the hypothesis $[L : K]$ prime to p. (It will be necessary to modify the definition of $TC_m K_3(K)$ to take into account the topology on K.)

Problem 5. — Suppose we are given a lifting of our variety X/k to a variety $\widetilde{X}/W(k)$. Find corresponding natural liftings of the formal groups associated to the Dieudonné modules $H^i(X, C^j)$. The Tate modules of these lifting groups should be related to the p-adic étale cohomology of the geometric generic fibre of $\widetilde{X}$, $H^*_{\text{étale}}(\widetilde{X}_{\overline{K}}, \mathbf{Z}_p)$ (cf. [1]).

Problem 6. — Understand geometrically the variation of slopes in the $H^i(X_t, C^j)$ for an interesting family of varieties $\{X_t\}$. I have in mind here some analogue of what Artin has done for K-3 surfaces using $H^2(X, C^0)$ [1 *bis*].

Recently, Deligne and Illusie, basing their approach on some older work of Lubkin, have constructed a complex which coincides with the complex of typical curves in degree $< p$, and which computes the crystalline cohomology in all degrees. In particular, one has a slope spectral sequence without restriction on dimension. The discussion of problem 2 above is thus out of date, though it remains of importance to link the Deligne-Illusie-Lubkin approach to the typical curves in dimensions $\geq p$.

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