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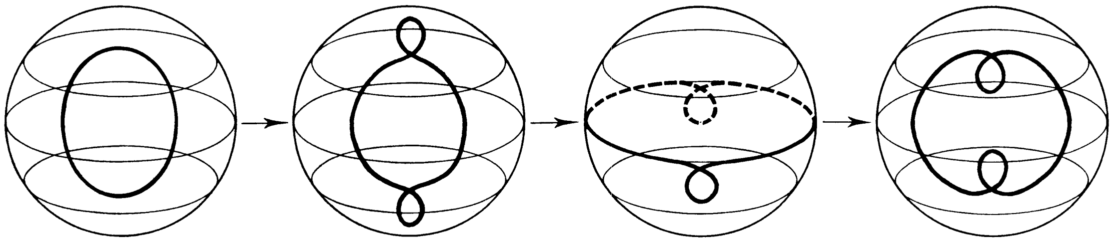
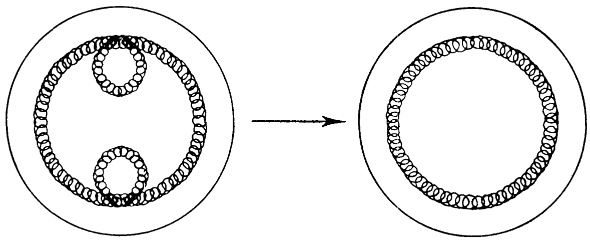
A COUNTEREXAMPLE TO THE PERIODIC ORBIT CONJECTURE

by DENNIS SULLIVAN

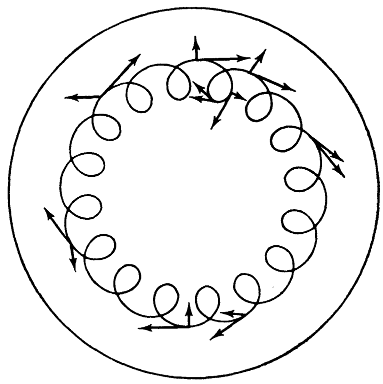
We will construct *a flow on a compact five-manifold so that every orbit is periodic but the length of orbits is unbounded.* The construction is based on the well known deformation (through immersed curves) on the two-dimensional sphere S^2 which introduces two twists or kinks:



By repeating this operation over and over we can produce a moving curve γ_t on the two-sphere S^2 whose geodesic curvature goes uniformly to infinity:



For fixed time t we consider all the congruent versions γ_t^α of γ_t obtained by rotating the sphere by elements α of SO_3. We add to the curve γ_t a vector field of constant length $1/t$ which uniformly turns exactly once around the tangent vector as the point traces the curve with a good parameter. We also consider the unit tangent vector field and we have after rotating a four-dimensional family of " curves of clocks " on S^2:



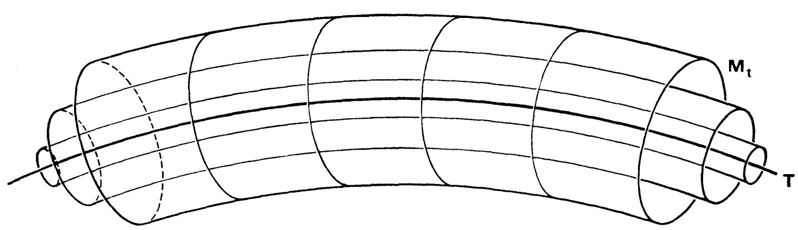
The family being parametrized by t and $\alpha \in SO_3$.

Now consider the 1-dimensional family of four-manifolds

$$M_t = \{(x, v_1, v_2) : x \in S^2, v_1, v_2 \in \text{tangent space of } x, |v_1| = 1, |v_2| = 1/t\}.$$

For $1 \leq t < \infty$, the M_t fill up a deleted tubular neighborhood of the unit tangent bundle of S^2

$$T = \{(x, v_1) : x \in S^2, v_1 \in \text{tangent space of } x, |v_1| = 1\}$$

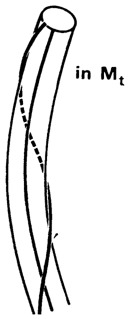


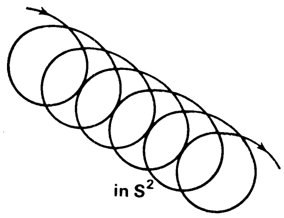
and $W = T \cup \bigcup_{1 \leq t < \infty} M_t$ is in a natural way a compact five-manifold with boundary, with M at a uniform distance of $1/t$ from T.

The clock structure on γ_t and its rotates by SO_3 give a three-dimensional family of curves γ_t^α in M_t ($1 \leq t < \infty$, $\alpha \in SO_3$). The fibres of the natural projection $T \to S^2$ give a family of curves in T. *We claim all these curves define the promised 1-dimensional foliation of* W (which one can double to hide the boundary if that is desired (see Addendae 1 and 3)). There are two points to check.

(i) All the γ_t^α with t fixed and α varying in SO_3 exactly fill M_t by embedded circles because each clock (x, v_1, v_2) occurs exactly once among the family of curves γ_t^α.

(ii) As t approaches infinity the tangent directions of γ_t^α approach the tangent directions of the curves in T. For if we consider tracing a small portion of γ_t for t large we see the two hands of the clock turning very rapidly (essentially together), and that portion of γ_t^α courses rapidly around a small toroidal tube about a curve of T going uniformly once around the $1/t$ direction and uniformly once around the unit direction in each circuit. The S^2 component of the unit vector field defined by our curves is of



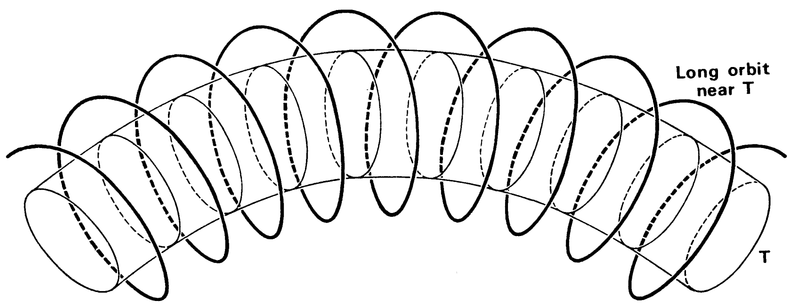


the order of the reciprocal curvature k^{-1}. With an appropriate time parameter we easily have then a Lipschitz vector field.

The Lipschitz continuity implies we can choose foliation coordinates which are smooth in the leaf direction and Lipschitz continuous in the transverse direction. I see no reason why the example cannot be made C^∞ (see Addendum 1).

Of course the lengths of the curves γ_t^α are unbounded near T. In fact as $t \to \infty$ each curve γ_t^α accumulates at every point of the three-dimensional submanifold T of W.

Put differently (see Addendum 4) the example can be viewed as a one parameter family of S^1 fibrations which passes (continuously in the space of foliations with all leaves compact) to an inequivalent fibration.



Historical Note and Further Problems:

The structure of this example—namely a flow on a five-dimensional neighborhood of RP^3 ($=T=SO_3$) fibred by circles in which nearby orbits are longer and longer as we come closer and closer to RP^3 (the " bad set ") was almost forced in an interesting way by the thoughts, ideas, and theorems of various people.

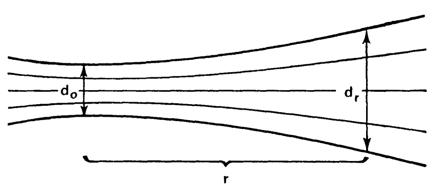
First of all the surprisingly deep argument in David Epstein's proof [E1] that no such example exists in dimension three is based at the core on the fact that the " bad set " is at most two dimensional and can be treated like a trivial fibration in circles. If the " bad set " were S^3 (or RP^3) fibred by Hopf circles Epstein's argument would break down very clearly because in this case there would be no cross-section (even homologically).

Secondly, Bob Edwards made a very suggestive step towards a possible counter-example by recalling there is an isotopy of a curve in S^3 which makes it arbitrarily long on the one hand and arbitrarily close tangentially speaking to the Hopf fibres on the other hand. Last year at IHES he posed the strategic question—does the existence of such an isotopy in a fibration imply that the fibre is homologous to zero (as in the Hopf fibration of S^3 or RP^3)?

It turned out this question could be dealt with very naturally by the picture of a geometric or foliation current developed in Ruelle-Sullivan [RS]. The idea is that a moving Edwards' submanifold, thought of as a real cycle or current (after normalization to total mass 1 by dividing by the length), represents a homology class moving to zero. On the other hand the limiting current at infinite time is a smear of fibres parametrized by a mass distribution in the base. This current is clearly homologous to a real multiple of one fibre which is then homologous to zero. This is the argument which evolved into the theorem in Edwards-Millet-Sullivan [EMS] stating that there is an upper bound on the volume of leaves if the homology classes of leaves (1) lie in an open half space of the real homology space (1). This homology condition negates the possibility of a " bad set " looking like a fibration whose fibre is homologous to zero.

The moving immersed curve on S^2 which gains more and more coils provides by lifting to the unit tangent bundle (which is closely related to the Hopf fibration of S^3) an intrinsic picture of an Edwards curve, which was modified by Thurston to find a real analytic example (Addendum 2).

The truth or falsity of the general compact leaf conjecture (2) was always in a precarious balance. There is a " proof " which fails if nearby leaves separate too quickly:



a strong inequality of the form $d_r \leq d_0 \log(a+br)$ (being true on the " average ") is needed in that argument, but it seemed tantalizingly close anyway. This argument evolving from a discussion with Grauert does lead to positive results under certain

(1) Of the bad set.
(2) Namely, if all leaves are compact is the volume of leaves locally bounded.

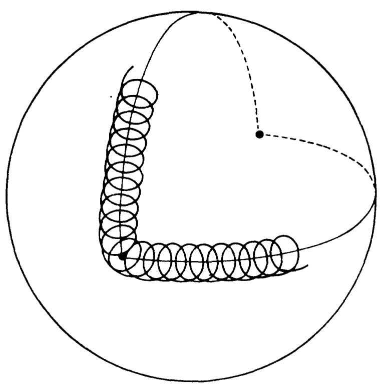
geometric assumptions. These were obtained already in the Warwick thesis of A. W. Wadsley [W].

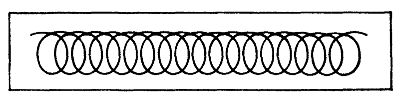
An interesting open question is the influence of analyticity, either real or complex, on the structure of foliations with compact leaves. If the ambient manifold is Kaehler and the foliation is complex analytic the theorem in [EMS] implies the result. The general complex analytic case is unknown and perhaps more likely than the real analytic case (see the real analytic example of Thurston in Addendum 2 and discussion of Problems (Addendum 5)).

The beautiful structure of the " bad set " as analysed by Epstein's three-dimensional argument and the interplay with the interesting theorems of Newman [D] and Montgomery [M] as described in [E2] and [Mi] and [NW] are no longer hypothetical in compact manifolds, and one can assume that this awesome geometry can really occur in foliations with compact leaves. Then Epstein's original work can be the beginning of a classification of the singularities or bad sets of these foliations. The question of how much of the " Epstein-Ehresmann-Reeb hierarchy " can occur as " bad set " is a new problem.

Addendum 1) (1)

Kuiper found a way to coordinatize the example so that it is clearly C^∞. One imagines two half-circles pinned at antipodal points, one stationary and the other rotating on the surface of the sphere. A small wheel moves tangent to the sphere with center traversing the two half-circles periodically according to a fixed C^∞ parametrization. A point on the rim of the wheel will trace out the immersed closed curve γ_t on the sphere if we choose the rotation of the moving half circle appropriately. If t is the angular velocity of the wheel of radius e^{-t} a C^∞ example will result. The calculations are facilitated by developing an equatorial strip onto a flat strip.





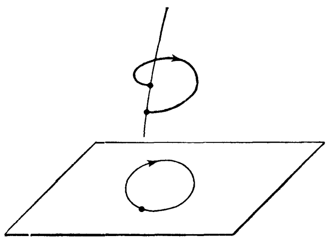
He also remarked we can put the example on $S^5 = (S^3 \times D^2) \cup (S^1 \times D^4)$.

(1) These are remarks added after an unannotated version of the first part was circulated in mimeographed form.

Addendum 2)

Thurston transformed the example in a beautiful way to make it real analytic. First he considers the constant curvature flow on T the bundle of unit tangent vectors on the torus. One can think of unit tangent vectors flowing around circles of radius r in the plane with velocity v (and then divide by the lattice). This flow is lifted to a certain four manifold M fibred by circles over T. From the two parameter family of flows on $M^4 = M$ we extract a periodic one parameter family to yield the example on $M \times S^1$. On M which is real analytic there is an analytic 1-form η so that η restricted to each fibre of $M \to T$ is the uniform 1-form on S^1 and $d\eta$ is the pulled up area form from the torus via $M \to T \to$ torus. The flow on T is lifted to M by specifying the η-component.

If a small curve in T would be lifted to M by keeping the η-component of its tangent vector zero, the lifted curve would have two points on the fibre a distance apart



equal to area of the projected curve on the torus. Since we want closed orbits we let the η-component be $\dfrac{\text{area}}{\text{period}} = \dfrac{\pi r^2}{2\pi r/v} = \dfrac{vr}{2}$, and then we have just the right vertical compensation. In coordinates let X denote the unit vector field on T corresponding to the geodesic flow, let Y denote the vertical vector field in the fibring $T \to$ torus, then the vector field $Z(v, r)$ on M given by the conditions:

a) the projection of Z in T is $vX + \dfrac{v}{r}Y$

b) η evaluated on Z is $\dfrac{vr}{2}$,

will cover the constant curvature flow on T and have closed orbits for v and r positive and finite. Thurston introduces a single analytic parameter u by $u = \text{arc cotg}\, r$ setting $(v, v/r, vr/2) = (\sin 2u, 2\sin^2 u, \cos^2 u)$. For each u one obtains a non-zero analytic vector field on M. The total system on $M \times \{u \in S^1\}$ is real analytic, all orbits are periodic (one checks directly the exceptional values of u), and the periods are unbounded near $u = n\pi$.

For definiteness Thurston's flow is on the 5-manifold $R^5 \bmod (\Gamma \times Z \times Z)$ where Γ

are the integral matrices among $\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$, x, y, z the first 3-variables of R^5. $Z \times Z$ is the group generated by unit translations on the fourth and fifth variables of R^5. On $N = R^3 \bmod \Gamma$ we have the 1-form η defined by the (invariant by Γ) formula $\eta = dz - x dy$. N fibres over the torus and $M = N \times S^1$ fibres over $T = (\text{torus}) \times S^1$ and this defines η on M^4 (whose extra twisted circle (over T) replaces the second hand of the clock in the first example).

Addendum 3)

Edwards pointed out a simple general construction—which clarifies the first example. Let G be a compact Lie group which is fibred by circles. Let $\gamma_t : S^1 \to G$ be a smooth family of immersions $0 < t \leq 1$ so that the tangent directions of γ_t approach those of the fibring of G as $t \to 0$.

Then we can fill up $D^2 \times G$ by circles using

a) the fibring of G for $0 \times G$

b) the translates of the graph of γ_t on (the circle of radius t)$\times G$, $0 < t \leq 1$.

If the approach of γ_t to the tangent direction of the fibring is fast enough we obtain a foliation with compact leaves. If the length of γ_t is unbounded we have unbounded leaves.

A variant of this is to fill up $(G \times S^1)_t$ $(-1 \leq t \leq 1)$, $t \neq 0$ by the translates of the graph of $\gamma_{|t|}$ as in *b)* but fill up $(G \times S^1)_0$ by the fibring of G product the point foliation of S^1. We can identify $(G \times S^1)_{-1}$ and $(G \times S^1)_1$ to obtain an example on $G \times S^1 \times S^1$ with a bad set at $(G \times S^1)_0$.

In this form the relation between the first example and the analytic example is more transparent. In Thurston's example on $N \times S^1 \times S^1$, G is replaced by the 3-dimensional real nilpotent Lie group divided by a discrete subgroup. The bad set is $(N \times S^1)_0$ and $(N \times S^1)_\pi$.

Addendum 4)

Another feature of examples in this circular form (see Addendum 3) is that we see an isotopy of the total space of a circle fibration so that all the fibres grow arbitrarily long and yet become tangent to a new circle fibration.

This question of one fibration moving continuously (through the space of foliations) to an inequivalent fibration has been studied recently by Rosenberg and Langevin (to appear). They rule out such a phenomenon (using other work of Thurston) if $H_1(\text{fibre}, R) = 0$. Of course in the example we just described the fibre is S^1. The theorem in [EMS] rules out this phenomenon if the fibre is not homologous to zero in the total space (for the value of the parameter where unboundedness occurs).

Addendum 5 (Problems)

These examples bring three problems into sharp focus:

a) codimension three case;
b) complex analytic case;
c) the structure of infinities of the volume function.

We discuss these in turn.

Note. — In codimension 2 case there are no such examples in general ([EMS] or [Vogt]) by generalyzing Epstein's argument.

Codimension 3. — The construction of a flow example in dimension 4 would require a new idea. One can rule out numerous candidates for bad sets B by special arguments:

a) $B = S^1 \times I/(x, 0) \sim (2x, 1)$ (1). This is the canonical example causing trouble in the "Weaver step" of the codimension two proof [EMS]. This bad set can be ruled out by the idea that the generic orbit γ is central in π_1(neighborhood of B).

Now γ must go to zero under the natural map $\pi_1 B \rightarrow H_1 B$. It then lifts to the corresponding cyclic cover $\widetilde{B}$. Here because γ is wrapping around geometrically we again find a contradiction to the center statement. Note $\pi_1 B = \{x, y : xyx^{-1} = y^2\}$.

b) The bad set B cannot be a manifold in dimension 4.

Case i) : if $\dim B = 2$, the homological argument of [EMS] mentioned above works to rule out this bad set.

Case ii) : if $\dim B = 3$, B is fibred by circles with base M^2. If $M^2 \neq S^2$, the above center type argument leads to a contradiction (due to Epstein and Hirsch). If $M = S^2$, we can assume $B = S^3$ with the Hopf fibration (2). Then we can get a contradiction using Seifert's stability [S] — which asserts a vector field near the Hopf vector field has a closed orbit near a Hopf fibre. This contradicts the infinite wrapping around of nearby generic orbits (idea of Hirsch and Epstein).

(There is a possible imprecision here in the $B = S^3$ case. If the boundaries of invariant neighborhoods wobble too much we may not achieve the Seifert hypothesis. However, an example based on this wobbling would be difficult to construct.)

Complex analytic case:

We have remarked above that real analytic examples exist and no complex analytic examples exist in a Kaehler manifold. Thurston's real analytic example has the further beauty of being locally like affine space filled with individually homogeneous helices.

(1) Namely, B is the mapping torus of the degree two map of S^1 to itself.
(2) Ken Millet informs me Seifert fibrations succumb to the same Seifert stability argument.

It seems reasonable to suppose there might be a complex analytic example even in this restricted affine realm. (The geometric-topological properties of these affine manifolds is very much unknown.)

It is possible to move an analytic submanifold $T=S^1\times S^1$ by a holomorphic vector field on $M^3=\begin{pmatrix}1 & z & z'' \\ 0 & 1 & z' \\ 0 & 0 & 1\end{pmatrix}\Big/$(discrete group) so that the volume of T goes to infinity (Deligne). Here we have an affine structure and the movement is geometrically like that of translating a complex line in $\mathbf{C}^3$.

We only obtain a " multi-valued foliation " by this example. The question of a complex analytic example is open. Holmann has a pretty theorem ruling out complex analytic examples on a compact manifold M whose leaves are given by a holomorphic vector field—namely an action $M\times\mathbf{C}\to M$. The proof is as follows:

a) Consider the pull back of the diagonal under the graph of the action

$$M\times\mathbf{C}\to M\times M.$$

b) This $\mathbf{C}$-analytic set projects properly to $\mathbf{C}$ and defines a $\mathbf{C}$-analytic subset of $\mathbf{C}$ (by the Remmert proper mapping theorem).

c) This subset misses a neighborhood of zero in $\mathbf{C}$ (because we have a foliation and periods can't be arbitrarily small) and is therefore discrete. Q.E.D.

See [H1] and [H2] for more information on these questions.

Hierarchy:

Many questions can be put about the structure of the bad set in these examples.

(i) Is the natural filtration always finite?

(ii) What sort of closed sets occur?

(iii) Can a " Thom stratification " of the leaf space be achieved after allowing for the non-Hausdorffness.

For a related construction (repairing the non-Hausdorffness of leaf spaces) see Williams branched manifolds in *I.H.E.S. Publications*, n° 43.

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