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THE PICARD GROUP OF A SCHEME OVER AN ARTIN RING

by JOSEPH LIPMAN [1]

CONTENTS

INTRODUCTION

In proposing a construction for the local Picard scheme of a complete local ring of mixed characteristic (with perfect residue field), Grothendieck has raised the following problem [2]:

Let R *be a local Artin ring with perfect residue field* k *of characteristic* $p>0$, *and let* $f: X \to \mathrm{Spec}(R)$ *be a proper map. Give a natural construction of a group-scheme* **P** *locally of finite type over* k, *together with an embedding*

$$\mathrm{Pic}(X) \hookrightarrow \mathbf{P}(k)$$

which is bijective if k *is algebraically closed.*

(1) Supported by National Science Foundation grant GP-29216 at Purdue University.
(2) Quoted from a letter to the author dated September 1969.

Under certain conditions the local Picard scheme should be obtained by factoring out a discrete group from an inverse limit of such **P**'s (cf. [**SGA** 2, pp. 189-191]) ([1]).

If the usual Picard functor $\mathbf{Pic}_{X/R}$ of X over R is an R-scheme, then we can take **P** to be Greenberg's *realization* of $\mathbf{Pic}_{X/R}$ over k (cf. [**Gr**, § 4]). In the application to local Picard schemes, however, the maps of the above type $f: X \to \mathrm{Spec}(R)$ which appear are very far from being flat, so we cannot expect $\mathbf{Pic}_{X/R}$ to be a scheme. Grothendieck's idea is that, even so, *the fppf sheaf associated to the realization of* $\mathbf{Pic}_{X/R}$ *might still be a k-scheme.* This turns out to be correct, and therein lies the theme of this paper.

Example. — Let T be a three-dimensional regular local ring with regular parameters u, v, w, and algebraically closed residue field k; let $S = T/(u^3+v^3+w^3)$; let Z be the desingularization of S obtained by blowing up the maximal ideal $\mathfrak{M}$ of S; let $R = S/\mathfrak{M}^2$ and let $X = Z \otimes_S R$. Then $Y = X_{red}$ is the projective plane cubic curve over k defined by the equation $U^3+V^3+W^3=0$, and it can be shown that *the fppf sheaf associated to the realization* $\mathbf{P}'$ *of* $\mathbf{Pic}_{X/R}$ *is* $\mathbf{Pic}_{Y/k}$. For any k-algebra A, we find that $\mathbf{P}'(A) = \mathbf{Pic}_{Y/k}(\mathbf{R}(A)/\mathfrak{M}\mathbf{R}(A))$, $\mathbf{R}$ being the Greenberg algebra associated to R (cf. Appendix A). $\mathbf{P}'$ is an *étale sheaf*, but $\mathbf{P}'$ is not equal to its associated *fppf* sheaf $\mathbf{Pic}_{Y/k}$ (there is a natural surjective map $\mathbf{R}(A)/\mathfrak{M}\mathbf{R}(A) \to A$ whose kernel is nilpotent, but not, in general, zero). Thus $\mathbf{P}'$ *is not a k-scheme, and consequently* $\mathbf{Pic}_{X/R}$ *is not an* R-*scheme* ([2]).

A detailed discussion of the main results is given in § 1. (At first sight it will seem that the functor **P** which we study is not the one just described; this apparent anomaly is set straight in Remark (1.8).) Our basic results and methods are motivated by the classical theory of the Picard functor of a scheme over a field. (In fact, in the special case when $p\mathcal{O}_X = (0)$, so that X is actually a k-scheme, our functor **P** becomes *identical* with $\mathbf{Pic}_{X/k}$.) However there are new difficulties to be dealt with. For example, as in the classical case, a number of questions about **P** are treated by " linearizing " them; but whereas in the former case the linearized questions are trivial, this is hardly so in the present situation. Indeed, the solutions of the linear problems, as exemplified by Theorems (2.4) and (8.1), constitute the main methodological novelty ([3]).

A weaker version of parts I and II was distributed as a preprint in late summer, 1971. Part I in its present form was worked out during a visit to Harvard University in the fall semester of 1971, and the results were presented there at a seminar held in January, 1972.

It remains to thank Professor Grothendieck for the generous communication, in the above-mentioned correspondence, of his ideas on the local Picard scheme, and for his subsequent encouragement.

(1) Boutot [**Bt**] gives a different type of construction (equicharacteristic case).

(2) To prove the assertions in this example, I need Corollary (0.2), Proposition (A.1), Corollary (C.6), and considerations of the type found in the first half of § 2.

(3) Our results are further developed in [**L**], where they are used in proving (for example) that *for a complete local ring* A *with algebraically closed residue field, if* A *is factorial then so is the formal power series ring* A[[T]].

0. Preliminaries (sheaves; Witt vectors).

All rings are understood to be commutative, with identity; "subring" means "subring containing the identity"; all homomorphisms of rings preserve identity elements.

Let k be a perfect field of characteristic $p>0$. There is a fully faithful embedding of the category of k-schemes into the category of covariant (set-)functors of k-algebras: to the scheme Z is associated the functor

$$h_Z(A) = \mathrm{Hom}_k(\mathrm{Spec}(A), Z).$$

We may therefore think of certain functors (viz. those which are, up to isomorphism, of the form h_Z) and their morphisms as schemes and morphisms of schemes. For example we shall say that a functor F *is an algebraic* (resp. *locally algebraic*) *k-scheme* if $F \cong h_Z$, where Z is a scheme of finite type (resp. locally of finite type) over k. If, in addition, F is a functor into the category of groups (equivalently: Z is a group-scheme) then we say that F *is an algebraic* (resp. *locally algebraic*) *k-group*.

We shall use freely the language of *topologies and sheaves* on the category of k-algebras, as presented in [**DG**, chap. III, § 1]. Here we review briefly a few pertinent points. For a k-algebra A, a finite family $(B_i)_{i\in I}$ of A-algebras is said to *cover* A for the *fpqc* (resp. *fppf*; resp. étale; resp. Zariski) topology if the B_i are flat over A (resp. flat and finitely presented; resp. étale; resp. rings of fractions of the form $A_f = A[1/f]$, $f \in A$) and if furthermore the union of the images of the $\mathrm{Spec}(B_i)$ in Spec(A) is all of Spec(A). A covariant set-valued (or group-valued, or ring-valued) functor F of k-algebras is said to be a *sheaf* (for a fixed one of the above topologies) if the following condition holds:

For any k-algebra A *and any covering family* $(B_i)_{i\in I}$ *the canonical diagram*

$$F(A) \to \prod_i F(B_i) \rightrightarrows \prod_{i,j} F(B_i \otimes_A B_j)$$

is exact.

For a fixed topology, there is associated to any functor F—into the category of sets, or of groups, or of rings—a sheaf $F^\sim$—into the same category—and a morphism of functors $F \mapsto F^\sim$ such that any morphism of functors $F \to G$ with G a *sheaf* factors uniquely through $F \mapsto F^\sim$. $F^\sim$ is obtained from F by "pasting together elements which agree locally". More precisely, for a k-algebra A we say that $\xi_1, \xi_2 \in F(A)$ *agree locally* if there exists a family $(B_i)_{i\in I}$ covering A such that the canonical images of ξ_1 and ξ_2 in $F(B_i)$ are the same for each $i \in I$; this defines a functorial equivalence relation R(A) on F(A), whence a quotient functor $F_0 = F/R$, and $F^\sim(A)$ is the direct limit of the kernels of the diagrams

$$\prod_i F_0(B_i) \rightrightarrows \prod_{i,j} F_0(B_i \otimes_A B_j)$$

as $(B_i)_{i\in I}$ runs through "all" covering families of A. $F_0(A)$ can be identified (functorially) with the image of the canonical map $F(A) \to F^\sim(A)$.

The *abelian functors* (i.e. functors into the category of abelian groups) and their morphisms form an abelian category, and the *abelian sheaves* are the objects of a full abelian subcategory [**DG**, p. 331, (3.5)]. The functor $F \mapsto F^{\sim}$ from the category of abelian functors to the category of abelian sheaves is *exact*; its right adjoint, the inclusion functor of the category of sheaves into that of presheaves, is left exact [*ibid.*, (3.6)].

The following observation will be very useful:

Lemma (**0.1**). — *Let* B *be a k-algebra. Then there exists a filtered inductive system* $(B_\alpha)_{\alpha \in J}$ *of* B-*algebras such that each* B_α *is a free finitely generated* B-*module, and such that if*

$$\overline{B} = \varinjlim_{\alpha} B_\alpha$$

then $\overline{B}^p = \overline{B}$ (i.e. *the Frobenius endomorphism* $x \mapsto x^p$ *of* $\overline{B}$ *is surjective*).

(Note that $\overline{B}$ is a flat B-algebra, and that $\mathrm{Spec}(\overline{B}) \to \mathrm{Spec}(B)$ is surjective, since $\overline{B}$ is integral over B; so $\overline{B}$ is *faithfully flat* over B.)

Proof. — If the system $(B_\alpha)_{\alpha \in J}$ has the required properties, relative to B, and if B′ is a homomorphic image of B, then clearly the system $(B_\alpha \otimes_B B')_{\alpha \in J}$ has the required properties relative to B′. Thus it suffices to treat the case $B = k[(X_\gamma)]$, where $(X_\gamma)_{\gamma \in G}$ is a family (not necessarily finite) of independent indeterminates. In this case, let Q be an algebraic closure of the field of fractions of B, and for each pair $\alpha = (L, n)$, where L is a finite subset of G and n is a positive integer, set

$$B_\alpha = B[(X_\gamma^{p^{-n}})_{\gamma \in L}] \subseteq Q.$$

The B_α form, in an obvious way, an inductive system (in which the maps are just inclusion maps), and one checks that this is as desired. Q.E.D.

Corollary (**0.2**). — *Let* $F \to G$ *be a morphism of functors of k-algebras, such that* $F(B) \to G(B)$ *is bijective whenever* B *is a k-algebra with* $B^p = B$. *Then the associated morphism of fpqc sheaves* $F^{\sim} \to G^{\sim}$ *is an* isomorphism. *The same is true for the associated morphism of fppf sheaves, provided that* F *and* G *commute with filtered direct limits.*

Proof. — Using Lemma (0.1) (and the " note " following it) we see that associated *fpqc* sheaves can be constructed (as before) entirely out of covering families $(B_i)_{i \in I}$ in which, for all i, $B_i^p = B_i$ (for if $B_i^p \neq B_i$, then by Lemma (0.1) we can replace B_i by a faithfully flat B_i-algebra $\overline{B}_i$ with $\overline{B}_i^p = \overline{B}_i$); the assertion of (0.2) for *fpqc* sheaves results.

From the method of construction of associated *fppf* sheaves, the second assertion is a straightforward consequence of the following two facts:

(i) For any k-algebra B, if $\xi_1, \xi_2 \in F(B)$ have the same image in G(B), then there exists a B-algebra B′ such that B′ is a free finitely generated B-module—and hence a flat finitely presented B-algebra [**EGA** 0I, p. 136, Cor. (6.3.7)]—and such that ξ_1 and ξ_2 have the same image in F(B′).

(*Proof:* Let $\overline{B}$ be as in Lemma (0.1); then $F(\overline{B}) \to G(\overline{B})$ is bijective, so ξ_1, ξ_2 have the same image in $F(\overline{B})$; but F commutes with direct limits...)

(ii) For any k-algebra B, if $\eta \in G(B)$, then there exists B′ as in (i), and $\xi \in F(B')$ whose image in $G(B')$ is the same as that of η. (*Proof:* similar to that of (i).)

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We shall make frequent use of the *Witt vectors*; all the facts we need concerning them are immediate consequences of those few which we now review. (For details cf. (for example) [**S**, pp. 45-53].)

Let k be, again, a perfect field of characteristic $p>0$. The k-ring-scheme **W** of Witt vectors has as its underlying scheme $\mathrm{Spec}(k[X_0, X_1, X_2, \ldots])$ (where $(X_i)_{i\geq 0}$ is a family of independent indeterminates). Addition and multiplication are defined by certain polynomials

$$S_i(Y, Z),\ P_i(Y, Z) \in k[Y_0, Y_1, \ldots, Y_i, Z_0, Z_1, \ldots, Z_i] \qquad (i \geq 0)$$

so that for any k-algebra A, addition in

$$\mathbf{W}(A) = \{(a_0, a_1, a_2, \ldots) \mid a_i \in A,\ i \geq 0\}$$

is given by

$$(a_0, a_1, a_2, \ldots) + (a'_0, a'_1, a'_2, \ldots) \\ = (S_0(a_0, a'_0), S_1(a_0, a_1, a'_0, a'_1), S_2(a_0, a_1, a_2, a'_0, a'_1, a'_2), \ldots)$$

and similarly for multiplication, with P_i in place of S_i. As for explicit formulae, we will need only the following three:

(i) $S_0(Y_0, Z_0) = Y_0 + Z_0$

(ii) $p(a_0, a_1, a_2, \ldots) = (0, a_0^p, a_1^p, a_2^p, \ldots)$

(iii) $(a, 0, 0, 0, \ldots)(a_0, a_1, a_2, \ldots) = (aa_0, a^p a_1, a^{p^2} a_2, \ldots)$

The ring $\mathbf{W}(k)$ is a complete discrete valuation ring whose maximal ideal is generated by p and whose residue field is k. For any complete local ring R, with perfect residue field K, and any homomorphism $k \to K$, there is a *unique* homomorphism $\mathbf{W}(k) \to R$ making the following diagram commute:

$$\begin{array}{ccc} \mathbf{W}(k) & \longrightarrow & R \\ \downarrow & & \downarrow \\ k & \longrightarrow & K \end{array}$$

If R is an Artin ring, with maximal ideal $\mathfrak{m}$, and if $[K:k]<\infty$, then the preceding map $\mathbf{W}(k) \to R$ makes R into a $\mathbf{W}(k)$-module of finite length. (Consider the filtration $R \supseteq \mathfrak{m} \supseteq \mathfrak{m}^2 \supseteq \mathfrak{m}^3 \supseteq \ldots$)

Let N be an integer >0. The k-ring-scheme $\mathbf{W}_N$ of *Witt vectors of length* N is such that for any k-algebra A

$$\mathbf{W}_N(A)=\{(a_0, a_1, \ldots, a_{N-1}) \mid a_i \in A, 0 \leq i < N\},$$

addition and multiplication being given by the above polynomials S_i, P_i $(0 \leq i < N)$. As a scheme, then, $\mathbf{W}_N$ is the affine space $\mathrm{Spec}(k[X_0, X_1, \ldots, X_{N-1}])$. $\mathbf{W}_1(A)$ is canonically isomorphic to A (this follows from (i) and (iii) above). We have a " truncation " homomorphism of ring schemes $\rho_N : \mathbf{W} \to \mathbf{W}_N$ given by

$$\rho_N(a_0, a_1, a_2, \ldots, a_n, \ldots)=(a_0, a_1, a_2, \ldots, a_{N-1}).$$

$\rho_N(A) : \mathbf{W}(A) \to \mathbf{W}_N(A)$ is *surjective* for all k-algebras A, and from (ii) above we see that

$$\ker(\rho_N(A)) \supseteq p^N \mathbf{W}(A),$$

with *equality* if $A^p = A$.

Similarly, if $M \geq N$, we have a truncation map $\rho_{NM} : \mathbf{W}_M \to \mathbf{W}_N$, and clearly

$$\rho_N = \rho_{NM} \circ \rho_M.$$

1. Discussion of results.

In this section we describe, and comment on, the main results of the paper.

Let us say that a scheme X (with structure sheaf $\mathcal{O}_X$) is *complete* if the following two (equivalent) conditions hold:

a) $H^0(X, \mathcal{O}_X)$ is an Artin ring, and the canonical map $X \to \mathrm{Spec}(H^0(X, \mathcal{O}_X))$ is proper.

b) There exists an Artin ring R and a proper map $X \to \mathrm{Spec}(R)$.

In what follows we consider a triple (X, k, ι) *with*

— X a complete scheme,

— k a perfect field of characteristic $p>0$,

— $\iota : k \to H^0(X, \mathcal{O}_X)_{\mathrm{red}}$ a ring homomorphism *via* which $H^0(X, \mathcal{O}_X)_{\mathrm{red}}$ is a finite k-algebra.

(For any ring S, we set $S_{\mathrm{red}} = S/(\text{nilradical of } S)$.)

X and k being as above, there is another way of looking at ι which is actually the point of view we will take throughout most of the paper. $H^0(X, \mathcal{O}_X)$ is a product of local Artin rings whose residue fields are finite over k, so ι lifts uniquely to a homomorphism from the Witt vectors $\mathbf{W}(k)$ to $H^0(X, \mathcal{O}_X)$, and by this lifting $H^0(X, \mathcal{O}_X)$ becomes a finite $\mathbf{W}(k)$-algebra (cf. § 0). Composing the finite map $\mathrm{Spec}(H^0(X, \mathcal{O}_X)) \to \mathrm{Spec}(\mathbf{W}(k))$ with the natural map $X \to \mathrm{Spec}(H^0(X, \mathcal{O}_X))$, we get a map

$$f : X \to \mathrm{Spec}(\mathbf{W}(k))$$

and clearly:

(i) f is proper;

(ii) $f(X)$ is supported in the closed point of $\mathrm{Spec}(\mathbf{W}(k))$; *i.e.* there exists an integer $N>0$ such that $p^N \mathcal{O}_X=(0)$ (so that X is proper, via f, over the ring $\mathbf{W}_N(k)$ of Witt vectors of length N).

It is easily checked that in this way we obtain a *one-one correspondence* between triples (X, k, ι) as above and triples (Y, k, g) with Y a scheme and $g: Y \rightarrow \mathrm{Spec}(\mathbf{W}(k))$ a proper map such that $g(Y)$ is supported in the closed point of $\mathrm{Spec}(\mathbf{W}(k))$.

For any scheme Z, Pic(Z) denotes, as usual, the group of isomorphism classes of invertible $\mathcal{O}_Z$-modules [1]. Using inverse images of invertible sheaves, one makes Pic(Z) into a contravariant functor of schemes, which can be identified via Čech cohomology with the functor $H^1(Z, \mathcal{O}_Z^*)$ ($\mathcal{O}_Z^*$ = sheaf of units of $\mathcal{O}_Z$) [**EGA** 0I, pp. 124-126].

Our basic goal is, roughly speaking, to endow Pic(X) *with some natural structure — depending on* ι *— of locally algebraic k-group.*

One way of doing this is given by Theorem (1.2) just below. There are other reasonable, and seemingly different, approaches *but they lead to the same result* (cf. remarks (1.7) and (1.8) at the end of this section).

For any k-algebra A, let

$$X_A = X \otimes_{\mathbf{W}(k)} \mathbf{W}(A)$$

(X being a $\mathbf{W}(k)$-scheme via the above f, and $\mathbf{W}(A)$ being a $\mathbf{W}(k)$-algebra in the obvious way). X_A — and hence $\mathrm{Pic}(X_A)$ — varies functorially with A.

Definition (**1.1**). — The (covariant) functor $\mathbf{P}=\mathbf{P}(X, k, \iota)$ from k-algebras A to abelian groups is the *fpqc* sheaf associated to the functor $\mathrm{Pic}(X_A)$.

Theorem (**1.2**). — **P** *is a locally algebraic k-group.*

The proof of (1.2) occupies §§ 2-4; briefly, it goes as follows. Let $\mathcal{N}$ be the Nilradical of $\mathcal{O}_X$, let X_n ($n>0$) be the subscheme of X defined by the $\mathcal{O}_X$-Ideal $\mathcal{N}^n$ (so that $X_n = X$ for large n), and let $\mathbf{P}_n$ be the *fpqc* sheaf associated to the functor $\mathrm{Pic}(X_{n,A}) = \mathrm{Pic}(X_n \otimes_{\mathbf{W}(k)} \mathbf{W}(A))$. We proceed by induction on n. To begin with, $\mathbf{P}_1$ turns out to be the usual Picard functor of the scheme $X_1 = X_{red}$ over the *field* k; so by a well-known theorem of Murre and Grothendieck, $\mathbf{P}_1$ is a locally algebraic k-group. Then, to pass from $\mathbf{P}_{n-1}$ to $\mathbf{P}_n$, we use the truncated exponential map $x \mapsto 1+x$ to reduce the problem to one of representing a functor defined in terms of cohomology of coherent sheaves: more specifically, to proving Theorem (2.4) (for details cf. § 2). Section 3 is devoted entirely to Proposition (3.1), which allows us in § 4 to apply some simple facts about Greenberg modules (Appendix A) to complete the proof.

[1] Pic(Z) is a *set* because, for example, every invertible $\mathcal{O}_Z$-Module is isomorphic to a subsheaf of the sheaf G given by $G(U) = \prod_{z \in U} \mathcal{O}_{Z,z}$ (U open in Z).

The foregoing inductive process also yields some information about the relation of $\mathbf{P}$ to $\mathbf{P}_1$: for example, the canonical homomorphism $\mathbf{P}\to\mathbf{P}_1$ is quasi-compact, with unipotent kernel and cokernel. (For this, and other related results, cf. (2.5), (2.7) and (2.11) in § 2.)

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For Theorem (1.2) to be useful, we need more information about the relation between $\mathrm{Pic}(X_A)$ and $\mathbf{P}(A)$, for k-algebras A. More precisely, we want to know something about the kernel and cokernel of the canonical map $\mathrm{Pic}(X_A)\to\mathbf{P}(A)$. In part II (§§ 6, 7) we obtain results in this direction *under the assumption that* $A^p=A$ (i.e. the Frobenius endomorphism $x\mapsto x^p$ of A is surjective). In particular—and probably most significantly—these results apply when A is a perfect field.

The main result of this sort is Theorem (7.5):

If $A^p=A$, *then, with* $k_1=H^0(X,\mathcal{O}_X)_{\mathrm{red}}$, *there is a natural exact sequence*

$$0\to\mathrm{Pic}(k_1\otimes_k A_{\mathrm{red}})\to\mathrm{Pic}(X_A)\to\mathbf{P}(A)\to\mathrm{Br}(k_1\otimes_k A_{\mathrm{red}})\to\mathrm{Br}(X_A)$$

(*Here* " Br " *denotes* " cohomological Brauer group ".)

Theorem (7.5) contains most of the results of § 6 as corollaries. These corollaries have to be proved independently, however, because they are used, to a large extent, in the proof of (7.5). Specifically, what is needed is Corollary (6.11): *if* $\mathbf{P}^{\mathrm{et}}$ *is the* étale sheaf *associated to the functor* $\mathrm{Pic}(X_A)$, *then the canonical map*

$$\mathbf{P}^{\mathrm{et}}(A)\to\mathbf{P}(A)$$

is bijective *whenever* $A^p=A$.

(We also mention here that the proof of (7.5) uses (via Corollary (C.6) of Appendix C) the following remarkable property of Witt vectors (Lemma (C.2)): *if* B *is an étale* A-*algebra* (A *being a k-algebra*) *then* $\mathbf{W}_m(B)$ *is an étale* $\mathbf{W}_m(A)$-*algebra* ($m\geq 1$).)

Here is another example (further indications about § 6 are given in the remarks at the beginning of part II):

If K *is a normal algebraic field extension of k such that every connected component of* X_{red} *has a* K-*rational point, and if* A *is a perfect field containing* K, *then* $\mathrm{Pic}(X_A)\to\mathbf{P}(A)$ *is bijective.*

(By (6.9), $X_{\mathrm{red}}\otimes_k K$ has a section over $k_1\otimes_k K$, so the assertion is a special case of (6.7). To derive it from (7.5) we need to see that the map $\mathrm{Br}(k_1\otimes_k A)\to\mathrm{Br}(X_A)$ is injective; this map is defined in § 7 to be $\alpha\circ\beta^{-1}$, where α, β are as in the canonical commutative diagram

$$\begin{array}{ccc}
\mathrm{Br}(H^0(X_A,\mathcal{O}_{X_A})) & \xrightarrow{\alpha} & \mathrm{Br}(X_A) \\
\beta\downarrow\wr & & \downarrow \\
\mathrm{Br}(k_1\otimes_k A) & \xrightarrow[\gamma]{} & \mathrm{Br}(X_{\mathrm{red}}\otimes_k A);
\end{array}$$

but because of the above mentioned section, γ is injective, whence so is $\alpha\circ\beta^{-1}$.)

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Part III (§§ 8, 9) deals with the Lie algebras of various k-groups. An upper bound for the dimension of the k-group $\mathbf{P}$ is given by the dimension of its Lie algebra, Lie($\mathbf{P}$); this latter dimension is shown in Theorem (9.1) to be

$$\textbf{(1.3)} \qquad \lambda(\mathrm{H}^1(\mathrm{X}, \mathcal{O}_\mathrm{X})) + \sum_{t\geq 0} \lambda(\mathrm{coker}(\nu_t))$$

where " λ " denotes the *length* of a $\mathbf{W}(k)$-module, and ν_t is the canonical map

$$\nu_t : \mathrm{H}^1(\mathrm{X}, p^t\mathcal{O}_\mathrm{X}) \to \mathrm{H}^1(\mathrm{X}, p^t\mathcal{O}_\mathrm{X}/p^{t+1}\mathcal{O}_\mathrm{X}).$$

What we actually show in § 9 is that Lie($\mathbf{P}$) has the same dimension as Lie($\mathbf{H}$), where $\mathbf{H}$ is the linear version of $\mathbf{P}$, *i.e.* the *fpqc* sheaf associated to the functor $\mathrm{H}^1(\mathrm{X}_\mathrm{A}, \mathcal{O}_{\mathrm{X}_\mathrm{A}})$ of k-algebras A. (By Theorem (2.4) $\mathbf{H}$ is an affine algebraic k-group.) The above given dimension (1.3) can then be read off from the complete description of Lie($\mathbf{H}$) contained in Theorem (8.1). This dimension is $\geq \lambda(\mathrm{H}^1(\mathrm{X}, \mathcal{O}_\mathrm{X}))$, with equality if and only if the k-scheme $\mathbf{H}$ is reduced.

In the classical case, when X is a scheme over k (*i.e.* $p\mathcal{O}_\mathrm{X} = (0)$), the well-known and easily proved fact is that Lie($\mathbf{P}$) is naturally isomorphic to $\mathrm{H}^1(\mathrm{X}, \mathcal{O}_\mathrm{X})$. In contrast, the proof of Theorem (9.1) is long and tedious, depending on many other results in the paper; and I could not find a *natural* isomorphism between Lie($\mathbf{P}$) and Lie($\mathbf{H}$). Hopefully this state of affairs can be improved upon.

As one consequence of Theorems (8.1) and (9.1) we have (cf. Proposition (8.5)):

If $\mathrm{H}^2(\mathrm{X}, p^t\mathcal{O}_\mathrm{X}/p^{t+1}\mathcal{O}_\mathrm{X}) = 0$ *for all* $t \geq 0$ (for example if $\dim \mathrm{X} = 1$) *then* $\mathbf{P}$ *is smooth, of dimension* $\lambda(\mathrm{H}^1(\mathrm{X}, \mathcal{O}_\mathrm{X}))$.

***

To further acquaintance with the functor $\mathbf{P}$, we add here some remarks concerning the dependence of $\mathbf{P}$ on (X, k, ι).

(1.4) Let $\mathrm{X}^1, \mathrm{X}^2, \ldots, \mathrm{X}^m$ be the connected components of X. For each $j = 1, 2, \ldots, m$, X^j is open and closed in X, so X^j is a complete scheme and $\mathrm{X} = \coprod_{j=1}^m \mathrm{X}_j$. Let

$$\mathbf{P}^i = \mathbf{P}(\mathrm{X}^i, k, \pi^i \circ \iota)$$

where π^i is the projection map

$$\pi^i : \mathrm{H}^0(\mathrm{X}, \mathcal{O}_\mathrm{X})_{\mathrm{red}} = \prod_{j=1}^m \mathrm{H}^0(\mathrm{X}^j, \mathcal{O}_{\mathrm{X}^j})_{\mathrm{red}} \to \mathrm{H}^0(\mathrm{X}^i, \mathcal{O}_{\mathrm{X}^i})_{\mathrm{red}}.$$

There is then a natural map

$$\mathbf{P} \to \prod_{i=1}^n \mathbf{P}^i$$

which is easily seen to be an *isomorphism.*

So we may, at our convenience, assume that X *is connected.* In this case $\mathrm{H}^0(\mathrm{X}, \mathcal{O}_\mathrm{X})$ is a local Artin ring, and $\mathrm{H}^0(\mathrm{X}, \mathcal{O}_\mathrm{X})_{\mathrm{red}}$ is a finite field extension of k (*via* ι).

(1.5) $\mathbf{P}=\mathbf{P}(X, k, \iota)$ varies *functorially* with (X, k, ι), in the following way. Let (X', k', ι') be another triple satisfying the same conditions as (X, k, ι), and let $\mathbf{P}'=\mathbf{P}(X', k', \iota')$. Suppose we are given (g, θ) such that $g: X' \to X$ is a morphism of schemes and $\theta: k \to k'$ is a homomorphism of fields for which the following diagram commutes:

$$\begin{array}{ccc} H^0(X, \mathcal{O}_X)_{\rm red} & \xrightarrow{g^0} & H^0(X', \mathcal{O}_{X'})_{\rm red} \\ \uparrow{\scriptstyle \iota} & & \uparrow{\scriptstyle \iota'} \\ k & \xrightarrow[\theta]{} & k' \end{array}$$

g^0 being the map induced by g. (It amounts to the same thing to require that the corresponding diagram

$$(\sharp)\qquad \begin{array}{ccc} X & \xleftarrow{g} & X' \\ \downarrow & & \downarrow \\ \operatorname{Spec}(\mathbf{W}(k)) & \xleftarrow[\operatorname{Spec}(\mathbf{W}(\theta))]{} & \operatorname{Spec}(\mathbf{W}(k')) \end{array}$$

commute.) Let $X^* = X \otimes_{\mathbf{W}(k)} \mathbf{W}(k')$, and let

$$\iota^*: k' \to H^0(X^*, \mathcal{O}_{X^*})_{\rm red}$$

correspond to the projection map $f^*: X^* \to \operatorname{Spec}(\mathbf{W}(k'))$. (Note that f^* is proper, and that $p^N \mathcal{O}_X = (0)$ implies $p^N \mathcal{O}_{X^*} = (0)$, whence $H^0(X^*, \mathcal{O}_{X^*})_{\rm red}$ is finite over k', *via* ι^* [1].) Let $\mathbf{P}^* = \mathbf{P}(X^*, k', \iota^*)$. Then:

(i) $\mathbf{P}^* = \theta^*(\mathbf{P}) = \mathbf{P} \otimes_k k'$

(*i.e. the functor* $\mathbf{P}^*$ *is the* restriction *of* $\mathbf{P}$ *to* k'*-algebras*);

(ii) *there is a natural homomorphism*

$$\mathbf{P}^* \to \mathbf{P}'.$$

To check (i), let A′ be any k'-algebra, and observe that

$$X^*_{A'} = X^* \otimes_{\mathbf{W}(k')} \mathbf{W}(A') = (X \otimes_{\mathbf{W}(k)} \mathbf{W}(k')) \otimes_{\mathbf{W}(k')} \mathbf{W}(A') = X \otimes_{\mathbf{W}(k)} \mathbf{W}(A') = X_{A'};$$

thus the functor $\operatorname{Pic}(X^*_{A'})$ of k'-algebras A′ is the restriction to k'-algebras of the functor $\operatorname{Pic}(X_A)$ of k-algebras A, and since restriction commutes with passage to associated *fpqc* sheaves, (i) follows. As for (ii), ($\sharp$) gives a natural $\mathbf{W}(k')$-morphism $X' \to X^*$, from which we obtain, in succession, the functorial maps (for k'-algebras A′):

$$X'_{A'} \to X^*_{A'}$$
$$\operatorname{Pic}(X^*_{A'}) \to \operatorname{Pic}(X'_{A'})$$
$$\mathbf{P}^* \to \mathbf{P}'.$$

---

[1] In fact, since $\mathbf{W}(k')$ is flat over $\mathbf{W}(k)$, it is easily seen that

$$H^0(X^*, \mathcal{O}_{X^*})_{\rm red} = H^0(X, \mathcal{O}_X)_{\rm red} \otimes_k k'$$

(1.6) Notation remains as in the preceding remark (1.5). For any functor F of k'-algebras, the functor $\theta_* F$ of k-algebras A is defined by

$$\theta_* F(A) = F(k' \otimes_k A).$$

To the above homomorphism $\mathbf{P}^* \to \mathbf{P}'$ there corresponds a homomorphism

$$\mathbf{P} \to \theta_* \mathbf{P}'$$

as follows: for any k-algebra A, the natural map $A \to k' \otimes_k A$ determines, by functoriality, a map

$$\mathbf{P}(A) \to \mathbf{P}(k' \otimes_k A) = \mathbf{P}^*(k' \otimes_k A)$$

(cf. (i) above) which can be composed with

$$\mathbf{P}^*(k' \otimes_k A) \to \mathbf{P}'(k' \otimes_k A) = \theta_* \mathbf{P}'(A)$$

to give the desired functorial map

$$\delta_A : \mathbf{P}(A) \to \theta_* \mathbf{P}'(A).$$

For each A, then, setting $A' = k' \otimes_k A$, we have the natural commutative diagram

$$\begin{array}{ccc} \mathrm{Pic}(X_A) & \longrightarrow & \mathrm{Pic}(X'_{A'}) \\ \downarrow & & \downarrow \\ \mathbf{P}(A) & \xrightarrow{\delta_A} & \mathbf{P}'(A') = \theta_* \mathbf{P}'(A) \end{array}$$

In Corollary (6.13) we show:

If $X = X'$ *and* g *is the identity map, then the above homomorphism* $\mathbf{P} \to \theta_* \mathbf{P}'$ *is an isomorphism.*

In particular, if $X = X'$ is connected, if $k' = H^0(X, \mathcal{O}_X)_{\mathrm{red}}$ and $\theta = \iota$, then we have

$$\mathbf{P}(X, k, \iota) = \iota_*(\mathbf{P}(X, k', 1_{k'}))$$

(1 = identity map). In view of remark (1.4), this should allow us, whenever it seems advantageous, to assume that X is connected, $k = H^0(X, \mathcal{O}_X)_{\mathrm{red}}$, and ι = identity.

* * *

Finally, we describe two other definitions of the functor $\mathbf{P}$.

(1.7) In practice, X may be presented to us as a scheme proper over some particular local Artin ring R whose residue field—call it k—is perfect of characteristic $p > 0$. ($\iota : k \to H^0(X, \mathcal{O}_X)_{\mathrm{red}}$ can then be taken to be the map induced by $R \to H^0(X, \mathcal{O}_X)$: note that $H^0(X, \mathcal{O}_X)$ is finite over R.) In this case, a reasonable candidate for the Picard functor (indeed the one suggested by Grothendieck, cf. Introduction) is the following:

R is naturally a $\mathbf{W}(k)$-algebra of finite length (§ 0) and so we have a corresponding Greenberg algebra $\mathbf{R}$ together with an *isomorphism* of $\mathbf{W}(k)$-algebras $\mathrm{R} \xrightarrow{\approx} \mathbf{R}(k)$ (Appendix A, Proposition (A.1)). For any k-algebra A let

$$X_A^\# = X \otimes_{\mathbf{R}(k)} \mathbf{R}(A)$$

and let $\mathbf{P}^\#$ be the *fppf sheaf* associated to the functor $\mathrm{Pic}(X_A^\#)$. The proof of Theorem (1.2), slightly modified (§ 5), shows that $\mathbf{P}^\#$ *is a locally algebraic k-group.*

How is this $\mathbf{P}^\#$ related to the previously defined $\mathbf{P}$? Well actually *they are isomorphic* (so that $\mathbf{P}^\#$ depends only on X, k and ι!). To see this note first that $\mathbf{P}^\#$ is also the *fpqc* sheaf associated to $\mathrm{Pic}(X_A^\#)$ (because $\mathrm{Pic}(X_A^\#)$ and $\mathbf{P}^\#$ clearly have the same associated *fpqc* sheaf; but $\mathbf{P}^\#$, being a scheme, *is* an *fpqc* sheaf). Now since $\mathbf{R}$ is (canonically) a $\mathbf{W}$-algebra, we have, for all k-algebras A, a functorial map

$$X_A^\# = X \otimes_{\mathbf{R}(k)} \mathbf{R}(A) \rightarrow X \otimes_{\mathbf{W}(k)} \mathbf{W}(A) = X_A$$

which is an *isomorphism* if $A^p = A$ (since then $\mathbf{R}(A) = \mathbf{R}(k) \otimes_{\mathbf{W}(k)} \mathbf{W}(A)$ (Proposition (A.1) (ii)); hence there is a functorial homomorphism

$$\mathrm{Pic}(X_A) \rightarrow \mathrm{Pic}(X_A^\#)$$

which is *bijective* if $A^p = A$; so by Corollary (0.2) there results an *isomorphism of associated fpqc sheaves*

$$\mathbf{P}(X, k, \iota) \xrightarrow{\approx} \mathbf{P}^\#.$$

It may be observed that the proof that $\mathbf{P}$ is a k-group is somewhat neater than the corresponding proof for $\mathbf{P}^\#$, because $\mathbf{P}$ does not take into account certain " finiteness " features. On the other hand, the *fppf* result is stronger (for example, as we have just seen, it implies that $\mathbf{P}$ is isomorphic to $\mathbf{P}^\#$). Another bonus for working with the *fppf* topology is that for any algebraically closed field extension K of k, we know without further ado that the canonical map

$$\mathrm{Pic}(X_K^\#) \rightarrow \mathbf{P}^\#(K)$$

is bijective [**DG**, p. 291, Remark 1.15].

(1.8) Let X, R, $\mathbf{R}$, k be as in (1.7) above. What about the usual Picard functor of X over R, namely the étale sheaf $\mathbf{Pic}_{X/R}$ associated to the functor $\mathrm{Pic}(X \otimes_R T)$ of R-algebras T? In general, of course, $\mathbf{Pic}_{X/R}$ is not an R-scheme. (Cf. [**FGA**, p. 232-06]; for the applications we have in mind (cf. Introduction) X will not even be flat over R.) Still, by a theorem of Greenberg [**Gr**, p. 643], *if* $\mathbf{Pic}_{X/R}$ *does happen to be an* R-*scheme, then the functor* $\mathbf{Pic}_{X/R}(\mathbf{R}(A))$ *of k-algebras* A *is a k-scheme*; and this suggests that we look more closely at the functor $\mathbf{Pic}_{X/R}(\mathbf{R}(A))$, even when $\mathbf{Pic}_{X/R}$ is not an R-scheme.

As it turns out, however, Corollary (C.6) implies that $\mathbf{Pic}_{X/R}(\mathbf{R}(A))$ *is the étale sheaf associated to the functor* $\mathrm{Pic}(X_A^\#)$ *of remark* (1.7). Hence (cf. (1.7)) *the fppf sheaf associated to the functor* $\mathbf{Pic}_{X/R}(\mathbf{R}(A))$ *is* $\mathbf{P}^\#$ $(=\mathbf{P})$.

Furthermore, when $A^p = A$ then every étale A-algebra B satisfies $B^p = B$ (Lemma (6.12)), so that $X_B^\sharp = X_B$ (cf. (1.7)); hence $\mathbf{Pic}_{X/R}(\mathbf{R}(A)) = \mathbf{P}^{et}(A)$, where $\mathbf{P}^{et}$ is the *étale sheaf* associated to the functor $\mathrm{Pic}(X_A)$, and so (6.11) says: *if* $A^p = A$, *then the canonical map*

$$\mathbf{Pic}_{X/R}(\mathbf{R}(A)) \to \mathbf{P}^\sharp(A) = \mathbf{P}(A)$$

is bijective.

I. — REPRESENTABILITY OF THE FUNCTOR P

2. Linearization.

In this section we " linearize " Theorem (1.2), i.e. we reduce the problem of representing **P** to one of representing functors defined in terms of cohomology of coherent sheaves. The technique is quite similar to that of Oort [**O**].

We begin with some preliminaries. Consider a triple $(Y, \mathcal{O}, \mathcal{J})$, where Y is any topological space, $\mathcal{O}$ is a sheaf of rings (commutative, with identity) on Y, and $\mathcal{J}$ is an $\mathcal{O}$-Ideal (sheaf) such that $\mathcal{J}^2 = (0)$. Setting $\overline{\mathcal{O}} = \mathcal{O}/\mathcal{J}$, we have the exact sequence

$$0 \to \mathcal{J} \to \mathcal{O} \to \overline{\mathcal{O}} \to 0.$$

Also if $\mathcal{O}^*$ (resp. $\overline{\mathcal{O}}^*$) is the sheaf of units of $\mathcal{O}$ (resp. $\overline{\mathcal{O}}$), there is an exact sequence of (multiplicative) abelian sheaves

$$1 \to 1 + \mathcal{J} \to \mathcal{O}^* \to \overline{\mathcal{O}}^* \to 1$$

($\mathcal{O}^* \to \overline{\mathcal{O}}^*$ is surjective because $\mathcal{J}^2 = (0)$.)

The two resulting long exact cohomology sequences *(contra-)vary functorially* with the triple $(Y, \mathcal{O}, \mathcal{J})$: given a second such triple $(Y', \mathcal{O}', \mathcal{J}')$ and a morphism

$$(Y', \mathcal{O}', \mathcal{J}') \to (Y, \mathcal{O}, \mathcal{J}),$$

i.e. a pair (ψ, φ) where $\psi : Y' \to Y$ is a continuous map and $\varphi : \psi^*(\mathcal{O}) \to \mathcal{O}'$ is a homomorphism of sheaves of rings such that $\varphi(\psi^*\mathcal{J}) \subseteq \mathcal{J}'$, we obtain canonical homomorphisms of the two cohomology sequences defined as above for $(Y, \mathcal{O}, \mathcal{J})$ into the corresponding sequences for $(Y', \mathcal{O}', \mathcal{J}')$ (cf. [**G**, Prop. (3.2.2)]); and if $(Y'', \mathcal{O}'', \mathcal{J}'') \to (Y', \mathcal{O}', \mathcal{J}')$ is another morphism of triples, then these cohomology homomorphisms satisfy obvious transitivity relations vis-à-vis

$$(Y'', \mathcal{O}'', \mathcal{J}'') \to (Y', \mathcal{O}', \mathcal{J}') \to (Y, \mathcal{O}, \mathcal{J}).$$

The " truncated exponential " map exp from the (additive) abelian sheaf $\mathcal{J}$ to the (multiplicative) abelian sheaf $1 + \mathcal{J}$ is defined by

$$\exp(a) = 1 + a \qquad (a \in \Gamma(U, \mathcal{J});\ U \text{ any open subset of } Y).$$

Since $\mathscr{J}^2=(0)$, it is immediate that exp is an *isomorphism of abelian sheaves*. One checks that exp *varies functorially* with $(Y, \mathcal{O}, \mathscr{J})$: given $(\psi, \varphi) : (Y', \mathcal{O}', \mathscr{J}') \to (Y, \mathcal{O}, \mathscr{J})$ as above, the resulting diagram

$$\begin{array}{ccc} \psi^*(\mathscr{J}) & \xrightarrow{\psi^*(\exp)} & \psi^*(1+\mathscr{J}) = 1+\psi^*(\mathscr{J}) \\ \Big\downarrow{\scriptstyle\varphi} & & \Big\downarrow{\scriptstyle\varphi} \\ \mathscr{J}' & \xrightarrow[\exp]{} & 1+\mathscr{J}' \end{array}$$

commutes.

* * *

Now let us return to the situation of Theorem (1.2), where we have a proper map $f : X \to \operatorname{Spec}(\mathbf{W}(k))$. Let $\mathcal{N}$ be the Nilradical of $\mathcal{O}_X$, and for $n \geq 1$ let X_n be the subscheme of X whose underlying topological space is the same as that of X and whose structure sheaf is $\mathcal{O}_n = \mathcal{O}_X/\mathcal{N}^n$. For $n \geq 2$ let $\mathscr{J}_n = \mathcal{N}^{n-1}/\mathcal{N}^n$, so that for any k-algebra A we have an exact sequence of sheaves on $X_{n,A} = X_n \otimes_{\mathbf{W}(k)} \mathbf{W}(A)$:

$$0 \to \mathscr{J}_n \mathcal{O}_{n,A} \to \mathcal{O}_{n,A} \to \mathcal{O}_{n-1,A} \to 0$$

($\mathcal{O}_{m,A}$ being the structure sheaf on $X_{m,A}$, $m \geq 1$). Since $\mathscr{J}_n^2 = (0)$, we can apply the preceding considerations to the triple $(X_{n,A}, \mathcal{O}_{n,A}, \mathscr{J}_n \mathcal{O}_{n,A})$ to obtain, for $n \geq 2$ and $i \geq 0$, *exact sequences (of abelian groups) which vary functorially with the k-algebra* A:

(2.1) $$H^i(\mathcal{O}_{n,A}) \to H^i(\mathcal{O}_{n-1,A}) \to H^{i+1}(\mathscr{J}_n \mathcal{O}_{n,A}) \to H^{i+1}(\mathcal{O}_{n,A}) \to H^{i+1}(\mathcal{O}_{n-1,A})$$

(2.2) $$H^0(\mathcal{O}^*_{n-1,A}) \to H^1(\mathscr{J}_n \mathcal{O}_{n,A}) \to \operatorname{Pic}(X_{n,A}) \to \operatorname{Pic}(X_{n-1,A}) \to H^2(\mathscr{J}_n \mathcal{O}_{n,A})$$

(The cohomology is taken on the topological space $|X_{n,A}| = |X_A|$. In (2.2) we have identified $1+\mathscr{J}_n \mathcal{O}_{n,A}$ with $\mathscr{J}_n \mathcal{O}_{n,A}$ (via exp), and $H^1(\mathcal{O}^*_{m,A})$ with $\operatorname{Pic}(X_{m,A})$, $m \geq 1$.)

The proof of Theorem (1.2) will be by induction on n. The inductive step from $n-1$ to n will be achieved by applying the next lemma to the exact sequences of *fpqc* sheaves associated to (2.1) and (2.2).

Lemma **(2.3)**. — *Let*

$$\mathbf{F}_1 \to \mathbf{F}_2 \to \mathbf{F} \to \mathbf{F}_3 \to \mathbf{F}_4$$

be an exact sequence of abelian sheaves on the category of k-algebras with the fpqc (resp. *fppf*) *topology. Assume that* $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, $\mathbf{F}_4$ *are schemes, with* $\mathbf{F}_1$ *and* $\mathbf{F}_2$ *affine and algebraic over k. Then* $\mathbf{F}$ *is a scheme, and the morphism* $\mathbf{F} \to \mathbf{F}_3$ *is affine. If, in addition,* $\mathbf{F}_3$ *is locally algebraic over k, then so is* $\mathbf{F}$.

Proof. — Let $\mathbf{C}$ be the *fppf* cokernel of $\mathbf{F}_1 \to \mathbf{F}_2$. Then $\mathbf{C}$ is an affine algebraic k-group [**DG**, p. 331, (3.5) and p. 342, (5.6)]. Consequently, $\mathbf{C}$ is its own associated *fpqc* sheaf, so that $\mathbf{C}$ is also the *fpqc* cokernel of $\mathbf{F}_1 \to \mathbf{F}_2$. So in either case (*fppf* or *fpqc*) if $\mathbf{K}$ is the kernel of $\mathbf{F}_3 \to \mathbf{F}_4$ (i.e. the inverse image in $\mathbf{F}_3$ of the *zero-section* $\mathrm{Spec}(k) \to \mathbf{F}_4$), then we have an exact sequence of sheaves

$$0 \to \mathbf{C} \to \mathbf{F} \to \mathbf{K} \to 0.$$

$\mathbf{K}$ is a closed subscheme of $\mathbf{F}_3$, and if $\mathbf{F}_3$ is locally algebraic over k then so is $\mathbf{K}$. The conclusion follows now from [**DG**, p. 363, (1.8) and (1.9) (*mutatis mutandis*, cf. p. 297 (3.3))]. Q.E.D.

To be able to use Lemma (2.3), we need to know that certain sheaves are schemes. In §§ 3, 4 we shall prove:

Theorem (**2.4**). — *With notation as in Theorem* (1.2), *let* $\mathscr{F}$ *be a coherent* $\mathcal{O}_X$*-Module. Then for all* $i \geq 0$, *the fpqc sheaf associated to the functor* $\mathrm{H}^i(\mathrm{X}_\mathrm{A}, \mathscr{F} \otimes_{\mathcal{O}_\mathrm{X}} \mathcal{O}_{\mathrm{X}_\mathrm{A}})$ *of k-algebras* A *is an affine algebraic k-scheme.*

This being granted, we can complete the proof of Theorem (1.2) in the following way.

First, the sequence (2.1) may be viewed as an exact sequence of functors of k-algebras. The corresponding sequence of associated *fpqc* sheaves is exact, and according to Theorem (2.4) (with $\mathrm{X} = \mathrm{X}_n$ or X_{n-1}, $\mathscr{F} = \mathcal{O}_n$ or $\mathcal{O}_{n-1}$) all the sheaves in this sequence, other than the middle one, are affine algebraic k-schemes. From Lemma (2.3), we conclude that *the middle term (i.e. the sheaf associated to the functor* $\mathrm{H}^{i+1}(\mathscr{J}_n \mathcal{O}_{n,\mathrm{A}})$ $(i \geq 0)$) *is also an affine algebraic k-scheme.*

Next, look at the sequence (2.2). Let $\mathbf{H}^*$ be the *fpqc* sheaf associated to the functor $\mathrm{H}^0(\mathcal{O}^*_{n-1,\mathrm{A}})$. We want to show that $\mathbf{H}^*$ *is affine and algebraic over k*. By Theorem (2.4) the *fpqc* sheaf—call it $\mathbf{H}$—associated to the functor $\mathrm{H}^0(\mathcal{O}_{n-1,\mathrm{A}})$ is an affine algebraic k-scheme, and clearly $\mathbf{H}$ is a *ring-scheme* over k. It is easily seen that for any k-algebra A

$$\mathbf{H}^*(\mathrm{A}) \cong \{\text{group of units in the ring } \mathbf{H}(\mathrm{A})\}.$$

It follows that $\mathbf{H}^*$ is isomorphic to the inverse image, under the *multiplication* map $\mathbf{H} \times_k \mathbf{H} \to \mathbf{H}$, of the 1-section of $\mathbf{H}$ (the isomorphism being induced by the first projection of $\mathbf{H} \times_k \mathbf{H}$ onto $\mathbf{H}$). Thus $\mathbf{H}^*$ may be identified with a closed subscheme of $\mathbf{H} \times_k \mathbf{H}$, and so $\mathbf{H}^*$ is indeed an affine algebraic k-scheme.

Now, let $\mathbf{P}_n$ be the *fpqc* sheaf associated to the functor $\mathrm{Pic}(\mathrm{X}_{n,\mathrm{A}})$. For large n, $\mathrm{X}_n = \mathrm{X}$, and so $\mathbf{P}_n = \mathbf{P}$. For any $n \geq 2$, we see, by the foregoing remarks, and by Lemma (2.3) applied to the exact sequence of *fpqc* sheaves associated to (2.2), that *if* $\mathbf{P}_{n-1}$ *is a locally algebraic k-scheme, then also* $\mathbf{P}_n$ *is a locally algebraic k-scheme.*

Thus we are reduced to studying $\mathbf{P}_1$. But $\mathbf{P}_1$ *is just the usual Picard functor* $\mathbf{Pic}_{\mathrm{X}_1/k}$ *of the reduced scheme* X_1 *over k, i.e.* the *fpqc* sheaf associated to the functor $\mathrm{Pic}(\mathrm{X}_1 \otimes_k \mathrm{A})$. Indeed, by definition, $\mathbf{P}_1$ is the sheaf associated to

$$\mathrm{Pic}(\mathrm{X}_1 \otimes_{\mathbf{W}(k)} \mathbf{W}(\mathrm{A})) = \mathrm{Pic}(\mathrm{X}_1 \otimes_k \mathbf{W}(\mathrm{A})/(p));$$

but there is a functorial map (induced by the truncation ρ_1 of § 0)

$$\mathbf{W}(A)/(p) \to A = \mathbf{W}_1(A)$$

which is *bijective* if $A^p = A$; hence there exists a functorial map

$$\mathrm{Pic}(X_1 \otimes_k \mathbf{W}(A)/(p)) \to \mathrm{Pic}(X_1 \otimes_k A)$$

which is *bijective* if $A^p = A$; by Corollary (0.2) we have therefore an *isomorphism* of associated sheaves:

$$\mathbf{P}_1 \overset{\approx}{\to} \mathbf{Pic}_{X_1/k}.$$

Since $\mathbf{Pic}_{X_1/k}$ is a locally algebraic k-group [**M**; **SGA** 6, Exposé XII; **A**, § 7], we are done (modulo Theorem (2.4)).

* * *

Theorem (2.4)—and hence Theorem (1.2)—being taken for granted, we add here some remarks on the "Néron-Severi" group $\pi_0(\mathbf{P})$ (cf. (2.7)), and on the relation of $\mathbf{P}$ to $\mathbf{P}_1 = \mathbf{P}(X_1, k, \iota)$, where now $X_1 \to X$ is an arbitrary nilpotent immersion (cf. (2.5) and (2.11)). These results will not be needed elsewhere in this paper, but will prove useful in future applications. We retain the preceding notation, except that $\mathcal{N}$ may now be *any* nilpotent coherent $\mathcal{O}_X$-Ideal.

Proposition **(2.5)** (cf. [**SGA** 6, Exposé XII, (3.5) and (3.6)]). — *The canonical map* $u : \mathbf{P} \to \mathbf{P}_1$ *is of the form* $v \circ w$, *where* v *is a closed immersion, and* w *is affine, faithfully flat and finitely presented. The kernel and (fppf) cokernel of* u *are unipotent algebraic* k*-groups.*

Proof. — For the first assertion, and the fact that the *fppf* cokernel of u is a locally algebraic k-group, it suffices that u be affine [**SGA** 3, p. 315]. Arguing as above we see that the canonical map $u_n : \mathbf{P}_n \to \mathbf{P}_{n-1}$ is affine ($n \geq 2$), whence so is $u = u_2 \circ u_3 \circ \ldots$

Next, consider the sequence

$$\mathbf{P}_{n+1} \xrightarrow{u_{n+1}} \mathbf{P}_n \xrightarrow{u_n} \mathbf{P}_{n-1} \qquad (n \geq 2).$$

We shall see in the following two paragraphs that if the kernel and (*fppf*) cokernel of both u_{n+1} and u_n are unipotent, then so are the kernel and cokernel of $u_n \circ u_{n+1}$. Note that all of these kernels and cokernels are in any case locally algebraic k-groups (since u_{n+1}, u_n, and $u_n \circ u_{n+1}$ are affine maps) so if they are unipotent (hence—by definition—affine, hence quasi-compact) then they are algebraic over k. To complete the proof of (2.5) by induction, it will then suffice to show that the kernel and cokernel of u_n are indeed unipotent.

So suppose that $\ker(u_{n+1})$ and $\ker(u_n)$ are unipotent. We have a natural exact sequence of *fppf* sheaves

$$0 \to \ker(u_{n+1}) \to \ker(u_n \circ u_{n+1}) \xrightarrow{u'_{n+1}} \ker(u_n).$$

From the commutative diagram

$$\begin{array}{ccc} \ker(u_n \circ u_{n+1}) & \xrightarrow{u'_{n+1}} & \ker(u_n) \\ \cap\downarrow & & \cap\downarrow \\ \mathbf{P}_{n+1} & \xrightarrow{u_{n+1}} & \mathbf{P}_n \end{array}$$

in which the vertical arrows are closed immersions, we see that the map u'_{n+1} is affine, so that $\ker(u_n \circ u_{n+1})$ is affine and there exists a commutative diagram

$$\begin{array}{ccc} \ker(u_n \circ u_{n+1}) & \xrightarrow{u'_{n+1}} & \ker(u_n) \\ & {\scriptstyle w'_{n+1}}\searrow \quad \nearrow{\scriptstyle v'_{n+1}} & \\ & \mathrm{C} & \end{array}$$

in which v'_{n+1} is a *closed immersion*, and w'_{n+1} is an *epimorphism* of abelian k-groups. Since $\ker(u_n)$ is unipotent, so is C [**DG**, p. 485 (2.3)], and by *loc. cit.* the exact sequence

$$0 \to \ker(u_{n+1}) \to \ker(u_n \circ u_{n+1}) \xrightarrow{w'_{n+1}} \mathrm{C} \to 0$$

shows then that $\ker(u_n \circ u_{n+1})$ is unipotent.

Suppose that $\operatorname{coker}(u_{n+1})$ and $\operatorname{coker}(u_n)$ are unipotent. From the natural exact sequence

$$\operatorname{coker}(u_{n+1}) \xrightarrow{u''_n} \operatorname{coker}(u_n \circ u_{n+1}) \to \operatorname{coker}(u_n) \to 0$$

we obtain an exact sequence

$$0 \to \mathrm{K} \to \operatorname{coker}(u_n \circ u_{n+1}) \to \operatorname{coker}(u_n) \to 0$$

where $\mathrm{K} = \operatorname{coker}(u_{n+1})/\ker(u''_n)$ is an affine algebraic k-group [**DG**, p. 342, (5.6)]. By (2.3), $\operatorname{coker}(u_n \circ u_{n+1})$ is affine. Since $\operatorname{coker}(u_{n+1})$ is unipotent, so is its quotient K; as above, it follows that $\operatorname{coker}(u_n \circ u_{n+1})$ is unipotent.

Let us show now that the kernel and cokernel of u_n are unipotent. We have previously established an exact sequence

$$\mathbf{H}^1 \to \mathbf{P}_n \xrightarrow{u_n} \mathbf{P}_{n-1} \to \mathbf{H}^2$$

where $\mathbf{H}^i$ $(i = 1, 2)$ is the *fpqc* sheaf associated to the functor $\mathrm{H}^i(\mathscr{J}_n \mathscr{O}_{n,\mathrm{A}})$. As before, Theorem (2.4) implies that $\mathbf{H}^i$ is an affine algebraic k-group; hence, clearly, $\mathbf{H}^i$ is a *Greenberg module* (cf. remarks immediately preceding Appendix A), and so $\mathbf{H}^i$ is unipotent and connected [**DG**, p. 601, (1.2)]. $\operatorname{Ker}(u_n)$ is a quotient of $\mathbf{H}^1$, so it is unipotent (cf. preceding argument about K). We will show in a moment that $\operatorname{coker}(u_n)$ is *quasi-compact* (hence *algebraic* over k); thus [**DG**, p. 249, (5.1) (b)] $\operatorname{coker}(u_n)$ is a *closed* subgroup of $\mathbf{H}^2$, and so is unipotent.

We show now that $\mathrm{coker}(u_n)$ is quasi-compact. For convenience, we write " u " instead of " u_n ". For any locally algebraic k-group $\mathbf{Q}$, let $\mathbf{Q}^0$ be the connected component of the identity, and let $\pi_0(\mathbf{Q})$ be the étale k-group $\mathbf{Q}/\mathbf{Q}^0$ (cf. [**DG**, chap. II, § 5, n° 1], from which we will take, tacitly, a number of facts).

Let

$$u^0 : \mathbf{P}_n^0 \to \mathbf{P}_{n-1}^0, \qquad u_0 : \pi_0(\mathbf{P}_n) \to \pi_0(\mathbf{P}_{n-1})$$

be the maps induced by u. We have an exact (" serpent ") sequence of *fppf* sheaves

(2.6) $$0 \to \ker(u^0) \to \ker(u) \xrightarrow{\alpha} \ker(u_0) \xrightarrow{\beta} \mathrm{coker}(u^0) \xrightarrow{\gamma} \mathrm{coker}(u) \xrightarrow{\delta} \mathrm{coker}(u_0) \to 0.$$

Since u is quasi-compact (in fact affine) it follows easily that u^0 and u_0 are quasi-compact, and hence all the sheaves in (2.6) are locally algebraic k-groups. $\mathrm{Coker}(u^0)$ is, moreover, connected and algebraic over k; hence $\ker(\delta) \cong \mathrm{coker}(u^0)/\ker(\gamma)$ is a *connected* subgroup of $\mathrm{coker}(u)$. Since $\mathrm{coker}(u_0)$ is an *étale* k-group, we have that $(\mathrm{coker}(u))^0$ is a subgroup of $\ker(\delta)$, and consequently $(\mathrm{coker}(u))^0 = \ker(\delta)$. It follows that

$$\pi_0(\mathrm{coker}(u)) \cong \mathrm{coker}(u_0).$$

So to prove that $\mathrm{coker}(u)$ is quasi-compact, it will suffice to show that the étale k-group $\mathrm{coker}(u_0)$ is quasi-compact, i.e. that $\mathrm{coker}(u_0)(\bar{k})$ *is a finite group* ($\bar{k}$=algebraic closure of k).

Note that $\mathrm{coker}(u)$ is annihilated by p^t for some $t>0$, since, as above, $\mathrm{coker}(u)$ is a subsheaf of $\mathbf{H}^2$, and $\mathbf{H}^2$ has a composition series consisting of subgroups of the additive group $\mathbf{W}_1$ [**DG**, p. 487, (2.5) (vii)]. So $p^t\,\mathrm{coker}(u_0) = 0$, and we need only show that $\mathrm{coker}(u_0)(\bar{k})$ is a *finitely generated* abelian group; since $\mathrm{coker}(u_0)(\bar{k})$ is a homomorphic image of $\pi_0(\mathbf{P}_{n-1})(\bar{k})$, the conclusion follows from the next proposition (with $X = X_{n-1}$):

Proposition **(2.7)**. — $\pi_0(\mathbf{P})(\bar{k})$ *is a finitely generated abelian group* ($\bar{k}$=*algebraic closure of* k).

Proof. — For this proof only, take $\mathcal{N}$ to be the Nilradical of $\mathcal{O}_X$, and let us show by induction on n that $\pi_0(\mathbf{P}_n)(\bar{k})$ is finitely generated for all $n \geq 1$. (For large n, we obtain the desired result.) When $n=1$, this is the theorem of Néron-Severi [**SGA** 6, Exposé XIII, Théorème (5.1)]. For the inductive step from $n-1$ to n, recall that (with notation as in (2.6)) u_0 is a quasi-compact map, so that $\ker(u_0)$ is quasi-compact; but $\ker(u_0)$ is étale over k, so $\ker(u_0)(\bar{k})$ is a finite group, and the desired conclusion follows from the exact sequence

$$0 \to \ker(u_0)(\bar{k}) \to \pi_0(\mathbf{P}_n)(\bar{k}) \to \pi_0(\mathbf{P}_{n-1})(\bar{k}).$$

This completes the proof of (2.7) and of (2.5).

We can extract more information from the preceding arguments. We first observe that (with the notation of (2.6)):

(2.8) u^0 *and* u_0 *are affine maps.*

(*Proof:* For u^0 this follows immediately from the fact that u is affine. We have just remarked that u_0 is quasi-compact and that $\ker(u_0)$ is finite over k; from this it follows easily that $u_0 = v_0 \circ w_0$, where v_0 is a closed immersion and u_0 is a *finite* faithfully flat map; thus u_0 is even a finite map.)

Furthermore:

(2.9) $\ker(u^0)$ *and* $\operatorname{coker}(u^0)$ *are unipotent.*

(*Proof:* As above, $\ker(u)$, being a quotient of the connected group $\mathbf{H}^1$, is itself connected, whence $\ker(u) = \ker(u^0)$, so that $\ker(u^0)$ is unipotent. Furthermore we deduce that, in (2.6), β maps $\ker(u_0)$ *isomorphically* onto $\ker(\gamma)$, so $\ker(\gamma)$ is finite over k, whence γ is a finite map; and since $\operatorname{coker}(u)$ is affine, so therefore is $\operatorname{coker}(u^0)$. By [**DG**, p. 501, (1.1)] $\operatorname{coker}(u^0)$ is the direct product of a multiplicative group $\mathbf{M}$ and a unipotent group $\mathbf{U}$. The composed map

$$\mathbf{M} \hookrightarrow \operatorname{coker}(u^0) \xrightarrow{\gamma} \operatorname{coker}(u) \hookrightarrow \mathbf{H}^2$$

must be trivial [**DG**, p. 486, (2.4)], so $\mathbf{M}$ is a subgroup of $\ker(\gamma) \cong \ker(u_0)$, and therefore $\mathbf{M}$ is *étale* over k. But $\mathbf{M}$ is a quotient of the connected group $\operatorname{coker}(u^0)$, so $\mathbf{M}$ is *connected*, and hence $\mathbf{M}=0$. Thus $\operatorname{coker}(u^0)=\mathbf{U}$.)

(2.10) $\ker(u_0)$ *and* $\operatorname{coker}(u_0)$ *are unipotent.*

(*Proof:* As above, $\ker(u_0)$ is isomorphic to a closed subgroup of $\operatorname{coker}(u^0)$, and $\operatorname{coker}(u^0)$ is unipotent; so $\ker(u_0)$ is unipotent. As for $\operatorname{coker}(u_0)$, we have already shown it to be a finite étale k-group annihilated by p^t for some $t>0$, so it is unipotent [**DG**, p. 485, (2.2) (b), and p. 488, (2.6)].)

In view of (2.8), (2.9), (2.10), the proof of (2.5) can be copied, *mutatis mutandis*, to give:

Corollary **(2.11)**. — *Let* $u : \mathbf{P} \to \mathbf{P}_1$ *be as in Proposition* (2.5), *and let* $u^0 : \mathbf{P}^0 \to \mathbf{P}_1^0$, $u_0 : \pi_0(\mathbf{P}) \to \pi_0(\mathbf{P}_1)$ *be the induced maps. Then the conclusions of* (2.5) *hold with either* u^0 *or* u_0 *in place of* u. *In particular, the kernel and cokernel of* u_0 *are finite étale* k*-groups which are annihilated by some power of* p.

3. On the homology of bounded complexes [1].

The key to the representability theorem (2.4) is the following elementary result on complexes, inspired by [**EGA**, 0_{III}, (11.9.2)].

As usual, for a ring A, a *bounded complex* $\mathrm{P}_\bullet = (\mathrm{P}_i, \delta_i)_{i \in \mathbf{Z}}$ of A-modules is a sequence of A-modules and A-homomorphisms

$$\ldots \to \mathrm{P}_{i+1} \xrightarrow{\delta_{i+1}} \mathrm{P}_i \xrightarrow{\delta_i} \mathrm{P}_{i-1} \to \ldots$$

[1] The notation in this section is completely independent of notation in other sections.

with $\delta_i \circ \delta_{i+1} = 0$ for all i, and $P_i = (0)$ for all but finitely many i. The kernel and image of δ_i are denoted respectively by $Z_i(P_\bullet)$ and $B_{i-1}(P_\bullet)$, and the homology modules $H_i(P_\bullet)$ $(i \in \mathbf{Z})$ are defined by

$$H_i(P_\bullet) = Z_i(P_\bullet)/B_i(P_\bullet).$$

Proposition **(3.1)**. — *Let* A *be a ring of the form* $D/\mathfrak{m}^n$, *where* D *is a discrete valuation ring with maximal ideal* $\mathfrak{m}$, *and* $n > 0$ *is a positive integer. Let* $t \in A$ *be a generator of the maximal ideal of* A, *and let* $P_\bullet = (P_i, \delta_i)_{i \in \mathbf{Z}}$ *be a bounded complex of* A-*modules. For each* $r = 0, 1, 2, \ldots, n$ *we have then a subcomplex* $t^r P_\bullet = (t^r P_i, \delta_i)_{i \in \mathbf{Z}}$ *of* $P_\bullet$; *assume that for each* i, r *as above, the homology module* $H_i(t^r P_\bullet)$ *is finitely generated over* A. *Then there exists a bounded complex of* finitely generated A-*modules* $Q_\bullet = (Q_i, d_i)_{i \in \mathbf{Z}}$ *and an* A-*homomorphism* $u : Q_\bullet \to P_\bullet$ *of complexes, such that for any* A-*module* M *the following condition holds*:

(*) *The homology maps*

$$H_i(Q_\bullet \otimes_A M) \to H_i(P_\bullet \otimes_A M) \qquad (i \in \mathbf{Z})$$

induced by $u \otimes 1_M : Q_\bullet \otimes_A M \to P_\bullet \otimes_A M$ *are all* isomorphisms.

(Here, of course, $P_\bullet \otimes_A M$ is the complex

$$\ldots \to P_i \otimes_A M \xrightarrow{\delta_i \otimes 1_M} P_{i-1} \otimes_A M \to \ldots,$$

and similarly for $Q_\bullet \otimes_A M$.)

Proof [1]. — We begin by noting that: if $Q_\bullet$ is any complex of A-modules and $u : Q_\bullet \to P_\bullet$ is an A-homomorphism of complexes inducing *isomorphisms* of homology

$$\textbf{(3.2)} \qquad H_i(t^r Q_\bullet) \xrightarrow{\approx} H_i(t^r P_\bullet), \quad i \in \mathbf{Z}; \quad 0 \le r \le n,$$

then (*) *holds for any* A-*module* M. Indeed, from the commutative diagram (with exact rows)

$$\begin{array}{ccccccccc} 0 \to & t^r Q_\bullet & \to & Q_\bullet & \to & Q_\bullet \otimes_A (A/t^r) & \to 0 \\ & \downarrow & & \downarrow u & & \downarrow & \\ 0 \to & t^r P_\bullet & \to & P_\bullet & \to & P_\bullet \otimes_A (A/t^r) & \to 0 \end{array}$$

we obtain the commutative diagram with exact rows

$$\begin{array}{ccccccccccc} \ldots \to & H_{i+1}(t^r Q_\bullet) & \to & H_{i+1}(Q_\bullet) & \to & H_{i+1}(Q_\bullet \otimes_A (A/t^r)) & \to & H_i(t^r Q_\bullet) & \to & H_i(Q_\bullet) & \to \ldots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \ldots \to & H_{i+1}(t^r P_\bullet) & \to & H_{i+1}(P_\bullet) & \to & H_{i+1}(P_\bullet \otimes_A (A/t^r)) & \to & H_i(t^r P_\bullet) & \to & H_i(P_\bullet) & \to \ldots \end{array}$$

from which we see, using the " five-lemma ", that (*) holds for $M = A/t^r$ $(0 \le r \le n)$; it follows that (*) holds whenever M is a finitely generated A-module, since such an M

[1] It is not necessary to read this proof to understand the rest of the paper.

is a direct sum of A-modules of the form A/t^r $(0 \leq r \leq n)$; finally, since any A-module is the inductive limit of its finitely generated submodules, and since " inductive limit " commutes with both tensor product and homology, we see easily that $(*)$ holds for all A-modules M.

Accordingly, *we shall proceed by induction to construct a bounded complex* $Q_\bullet$ *of finitely generated* A*-modules, and an* A*-homomorphism* $u: Q_\bullet \to P_\bullet$ *such that the resulting homology maps* (3.2) *are all isomorphisms.*

By assumption $P_\bullet$ is bounded, so there exists an integer b such that $P_i = (0)$ for all $i \leq b$ [1]. Let j be an integer $\geq b-1$, and suppose that we have found a complex $Q_\bullet = Q_\bullet^{(j)} = (Q_i, d_i)_{i \in \mathbf{Z}}$ of finitely generated A-modules, and an A-homomorphism $u = u^{(j)}: Q_\bullet^{(j)} \to P_\bullet$, such that $Q_i = (0)$ if $i \leq b-1$ or if $i > j$, and such that the following conditions hold:

(I_j) For $i < j$ and $0 \leq r \leq n$ the homomorphism

$$H_i(t^r Q_\bullet) \to H_i(t^r P_\bullet)$$

induced by u is an isomorphism.

(II_j) For $0 \leq r \leq n$ the composed homomorphism

$$v_{jr}: Z_j(t^r Q_\bullet) \xrightarrow{u_j} Z_j(t^r P_\bullet) \xrightarrow{\text{canonical}} H_j(t^r P_\bullet)$$

is surjective.

(III_j) Let N_j be the kernel of v_{j0}. Then for $0 \leq r \leq n$ the kernel of v_{jr} is $t^r N_j$.

(IV_j) (For $0 \leq r \leq n$ and any A-module E, set

$${}_rE = (0) : t^r \subseteq E$$

i.e. ${}_rE$ is the submodule of E consisting of all elements which are annihilated by t^r.)

Each member of $H_{j+1}(t^r P_\bullet)$ is the homology class of an element $t^r \mu \in Z_{j+1}(t^r P_\bullet)$ with

$$\delta_{j+1}(\mu) \in u_j({}_rN_j).$$

Remarks. — (i) There is one and only one complex $Q_\bullet^{(b-1)}$ as above, namely take $Q_i = (0)$ for all $i \in \mathbf{Z}$.

(ii) What we are going to do is to construct, under the above conditions, a finitely generated A-module Q_{j+1}^* and A-homomorphisms $d_{j+1}^*: Q_{j+1}^* \to N_j \subseteq Z_j(Q_\bullet)$ and $u_{j+1}^*: Q_{j+1}^* \to P_{j+1}$ such that the middle square in the following diagram commutes:

$$\text{(3.3)} \qquad \begin{array}{ccccccccccc} \cdots \longrightarrow & 0 & \longrightarrow & Q_{j+1}^* & \xrightarrow{d_{j+1}^*} & Q_j & \xrightarrow{d_j} & Q_{j-1} & \longrightarrow \cdots \\ & \downarrow{\scriptstyle u_{j+2}} & & \downarrow{\scriptstyle u_{j+1}^*} & & \downarrow{\scriptstyle u_j} & & \downarrow{\scriptstyle u_{j-1}} & \\ \cdots \longrightarrow & P_{j+2} & \xrightarrow{\delta_{j+2}} & P_{j+1} & \xrightarrow{\delta_{j+1}} & P_j & \xrightarrow{\delta_j} & P_{j-1} & \longrightarrow \cdots \end{array}$$

---

[1] Actually, all we need in what follows is that A is a noetherian ring, that $t \in A$ satisfies $t^n = 0$ and that $H_i(t^r P_\bullet) = (0)$ for all $i \leq b$ and $0 \leq r \leq n$.

Since d^*_{j+1} maps Q^*_{j+1} into $Z_j(Q_\bullet)$, we have $d_j \circ d^*_{j+1} = 0$. Thus we may replace $Q_{j+1} = (0)$, d_{j+1}, and u_{j+1} by Q^*_{j+1}, d^*_{j+1}, u^*_{j+1} respectively, to get a new complex $Q^{(j+1)}_\bullet$ and an A-homomorphism $u^{(j+1)} : Q^{(j+1)}_\bullet \to P_\bullet$, as depicted in (3.3). The construction will be such that for $Q^{(j+1)}_\bullet$ and $u^{(j+1)}$ the conditions (I_{j+1}), (II_{j+1}), (III_{j+1}) and (IV_{j+1}) are satisfied.

(iii) Starting with $j = b - 1$ (cf. remark (i)), and repeating the construction described in remark (ii), we eventually obtain $u^{(j)} : Q^{(j)}_\bullet \to P_\bullet$ where j is an integer with the property that $P_i = (0)$ for all $i > j$ (such a j exists because $P_\bullet$ is bounded). Then $Z_j(P_\bullet) \to H_j(P_\bullet)$ is an isomorphism; hence $u_j(N_j) = (0)$, and we have a commutative diagram

$$\begin{array}{ccc} N_j & \xrightarrow{\text{inclusion}} & Q_j \\ \big\downarrow & & \big\downarrow{\scriptstyle u_j} \\ (0) = P_{j+1} & \xrightarrow{\delta_{j+1}} & P_j \end{array}$$

Setting $Q^*_{j+1} = N_j$, $d^*_{j+1} =$ inclusion map of N_j into Q_j, and $u^*_{j+1} =$ zero-map, we can form the complex $Q^{(j+1)}_\bullet$ and the A-homomorphism $u^{(j+1)}$ as in remark (ii). From (I_j), (II_j) and (III_j) (for $Q^{(j)}_\bullet$, $u^{(j)}$) we obtain (I_{j+1}) (for $Q^{(j+1)}_\bullet$, $u^{(j+1)}$), and it follows at once that for $(Q_\bullet, u) = (Q^{(j+1)}_\bullet, u^{(j+1)})$ all the homology maps (3.2) are isomorphisms as desired [1].

It remains then to carry out the construction described in remark (ii). Q^*_{j+1} will be given as a submodule of the A-module

$$R_{j+1} = N_j \times_{P_j} P_{j+1} = \{(x, y) \in N_j \times P_{j+1} \mid u_j(x) = \delta_{j+1}(y)\}.$$

The mappings d^*_{j+1}, u^*_{j+1} are taken to be the projections, namely

$$d^*_{j+1}(x, y) = x, \qquad u^*_{j+1}(x, y) = y,$$

and then obviously (3.3) will be commutative, as required.

To build up Q^*_{j+1}, we begin by constructing three finite subsets of R_{j+1}. First, N_j, being an A-submodule of Q_j, is finitely generated; let $\{\lambda'_1, \lambda'_2, \ldots, \lambda'_p\}$ be a basis of N_j. N_j is the kernel of v_{j0}, so $u_j(N_j) \subseteq B_j(P_\bullet)$; hence we can choose $\lambda_1, \lambda_2, \ldots, \lambda_p$ in P_{j+1} such that

$$u_j(\lambda'_k) = \delta_{j+1}(\lambda_k) \qquad (k = 1, 2, \ldots, p)$$

i.e. $\qquad (\lambda'_k, \lambda_k) \in R_{j+1}$.

---

[1] Of course we are interested ultimately in condition (I_j), the other conditions being necessary only to carry out the inductive procedure. Since these conditions may appear rather involved, we should point out that they are *unavoidable* in the sense that we have the (easily checked) implications

$$(I_j) \Rightarrow (III_{j-1}) \quad \text{and} \quad (I_j) \Rightarrow (II_{j-1}) \Rightarrow (IV_{j-2}).$$

This is not to say that there could not exist a less complicated proof of Proposition (3.1)!

Second, for each r with $0 \leq r \leq n$ choose a finite set of elements

$$\{t^r \mu_{1r}, t^r \mu_{2r}, \ldots, t^r \mu_{q_r r}\}$$

in $Z_{j+1}(t^r P_\bullet)$ whose homology classes generate $H_{j+1}(t^r P_\bullet)$. By (IV_j) we may make the choice so that

$$\delta_{j+1}(\mu_{\ell r}) = u_j(\mu'_{\ell r}) \quad \text{with} \quad \mu'_{\ell r} \in {}_r N_j \quad (\ell = 1, 2, \ldots, q_r);$$

thus $(\mu'_{\ell r}, \mu_{\ell r}) \in R_{j+1}$.

Third, for each r with $0 \leq r \leq n$ choose a finite set of elements

$$\{t^r \nu_{1r}, t^r \nu_{2r}, \ldots, t^r \nu_{s_r r}\}$$

in $Z_{j+2}(t^r P_\bullet)$ whose homology classes generate $H_{j+2}(t^r P_\bullet)$. We have

$$u_j(0) = 0 = \delta_{j+1}(\delta_{j+2}(\nu_{mr})) \quad (m = 1, 2, \ldots, s_r)$$

i.e. $(0, \delta_{j+2}(\nu_{mr})) \in R_{j+1}$.

Next, let Q'_{j+1} be the A-submodule of R_{j+1} generated by all the elements

$$(\lambda'_k, \lambda_k) \qquad (k = 1, 2, \ldots, p)$$
$$(\mu'_{\ell r}, \mu_{\ell r}) \qquad (0 \leq r \leq n,\ \ell = 1, 2, \ldots, q_r)$$
$$(0, \delta_{j+2}(\nu_{mr})) \qquad (0 \leq r \leq n,\ m = 1, 2, \ldots, s_r).$$

Define Q^*_{j+1} to be $Q'_{j+1} + E$, where E is a finitely generated submodule of R_{j+1} which is chosen in accordance with Lemma (3.5) below in such a way that the following is true: *let*

$$B = (0) \times B_{j+1}(P_\bullet) \subseteq N_j \times_{P_j} P_{j+1} = R_{j+1};$$

then for all r with $0 \leq r \leq n$,

(3.4) $$Q^*_{j+1} \cap (B + {}_r R_{j+1}) = (Q^*_{j+1} \cap B) + {}_r Q^*_{j+1}.$$

We can now easily verify (I_{j+1}), (II_{j+1}), (III_{j+1}) and (IV_{j+1}) for $Q_\bullet^{(j+1)}$, $u^{(j+1)}$ (cf. remark (ii) above).

(I_{j+1}): $\lambda'_1, \ldots, \lambda'_p$ generate N_j, and $d^*_{j+1}(\lambda'_k, \lambda_k) = \lambda'_k$ $(k = 1, 2, \ldots, p)$; thus d^*_{j+1} maps Q^*_{j+1} *surjectively* onto N_j, and so for $0 \leq r \leq n$

$$d^*_{j+1}(t^r Q^*_{j+1}) = t^r N_j.$$

Now (I_{j+1}) follows at once from (II_j) and (III_j).

(II_{j+1}): We have $\mu'_{\ell r} \in {}_r N_j$, so

$$0 = t^r \mu'_{\ell r} = d^*_{j+1}(t^r(\mu'_{\ell r}, \mu_{\ell r}))$$

i.e. $t^r(\mu'_{\ell r}, \mu_{\ell r}) \in Z_{j+1}(t^r Q_\bullet^{(j+1)})$.

Since $u^*_{j+1}(t^r(\mu'_{\ell r}, \mu_{\ell r})) = t^r \mu_{\ell r}$, (II_{j+1}) results from the definition of $\mu_{\ell r}$.

(III_{j+1}): Chasing through the definitions, we come down to showing:
if $x=t^r(y,z)$, *with* $(y,z)\in Q^*_{j+1}$ *such that* $t^r y=0$ *and* $t^r z\in B_{j+1}(t^r P_\bullet)$, *then* $x=t^r(0,z')$, *with* $(0,z')\in Q^*_{j+1}$ *such that* $z'\in B_{j+1}(P_\bullet)$.

This can be rephrased as follows:
if $(y,z)\in Q^*_{j+1}$, *and* $t^r(y,z)\in t^r B$, *then* $t^r(y,z)\in t^r(Q^*_{j+1}\cap B)$.

But this last statement is just another form of (3.4).

(IV_{j+1}): Since $t^r\nu_{mr}\in Z_{j+2}(t^r P_\bullet)$, we have

$$t^r(0,\delta_{j+2}(\nu_{mr}))=(0,\delta_{j+2}(t^r\nu_{mr}))=(0,0).$$

It follows that $(0,\delta_{j+2}\nu_{mr})\in {}_rN_{j+1}$, and hence

$$\delta_{j+2}(\nu_{mr})\in u^*_{j+1}({}_rN_{j+1}).$$

From this—and the definition of ν_{mr}—(IV_{j+1}) is immediate.

The following lemma will complete the proof:

Lemma **(3.5)**. — *Let* R *be an* A-*module and let* Q *and* B *be two submodules of* R, *with* Q *finitely generated. Then there exists a finitely generated submodule* E *of* R *such that, if* $Q^*=Q+E$, *then, for all* r *with* $0\le r\le n$

$$Q^*\cap(B+{}_rR)=(Q^*\cap B)+{}_rQ^*. \tag{3.6}$$

Proof. — We proceed by descending induction. (3.6) is obvious for any Q^* if $r\ge n$, since then ${}_rR=R$ and ${}_rQ^*=Q^*$. It evidently suffices therefore to show: *for fixed* s, *if*

$$Q\cap(B+{}_rR)=(Q\cap B)+{}_rQ$$

for all $r>s$, *then there exists a finitely generated submodule* E *of* R *such that, if* $Q^*=Q+E$, *then* (3.6) *holds for all* $r\ge s$.

Now since Q is finitely generated, so also is its submodule $Q\cap(B+{}_sR)$, and hence we can find a finitely generated submodule E of ${}_sR$ such that

$$Q\cap(B+{}_sR)\subseteq B+E. \tag{3.7}$$

Let $Q^*=Q+E$ and let $y\in Q^*\cap(B+{}_rR)$ for some fixed $r\ge s$. Set $y=q+e$ $(q\in Q,\ e\in E)$. Since $e\in{}_sR\subseteq{}_rR$, we have

$$q=y-e\in Q\cap(B+{}_rR). \tag{3.8}$$

If $r>s$, then we conclude that

$$q\in(Q\cap B)+{}_rQ$$

and since $e\in{}_sE\subseteq{}_sQ^*\subseteq{}_rQ^*$, therefore

$$y=q+e\in(Q\cap B)+{}_rQ+{}_rQ^*\subseteq(Q^*\cap B)+{}_rQ^*.$$

Hence (3.6) holds if $r>s$.

If $r=s$, then we conclude from (3.7) and (3.8) that $q \in B+E$, i.e. $q=b+e'$ ($b \in B$, $e' \in E$); hence

$$b=q-e' \in Q_s^* \cap B$$

and
$$y=q+e=b+(e'+e) \in (Q_s^* \cap B)+{}_sQ_s^*.$$

Hence (3.6) holds when $r=s$. Q.E.D.

Remark **(3.9)**. — We can augment (3.1) as follows (this will be used only in Theorem (8.1) in order to prove that a certain map is natural):

Let $S_\bullet=(S_i, \partial_i)_{i\in \mathbf{Z}}$ *be a bounded complex of finitely generated* A*-modules, and let* $v: S_\bullet \to P_\bullet$ *be an* A*-homomorphism of complexes* (A, $P_\bullet$ *as before*). *Then* $Q_\bullet$ *and* u *can be chosen as in* (3.1), *and with the additional property that* $v=u\circ w$ *for some* A*-homomorphism* $w: S_\bullet \to Q_\bullet$.

Proof. — In the proof of (3.1), we introduce an additional inductive datum; let $S_\bullet^{(j)}$ be the complex obtained from $S_\bullet$ by annihilating S_i for all $i>j$, and let

$$w=w^{(j)}: S_\bullet^{(j)} \to Q_\bullet^{(j)}$$

be an A-homomorphism such that $u_i^{(j)}\circ w_i^{(j)}=v_i$ for all $i\leq j$.

In passing from j to $j+1$, what we will then need is an A-homomorphism

$$w_{j+1}: S_{j+1} \to Q_{j+1}^*$$

such that

$$u_{j+1}^*\circ w_{j+1}=v_{j+1}$$

and
$$d_{j+1}^*\circ w_{j+1}=w_j\circ \partial_{j+1}$$

(cf. remark (ii) in the proof of (3.1)). But it is immediate that

$$w_j(\partial_{j+1}S_{j+1}) \subseteq N_j$$

so that we have the map

$$w'_{j+1}: S_{j+1} \to N_j \times_{P_j} P_{j+1}=R_{j+1}$$

given by
$$w'_{j+1}(s)=(w_j(\partial_{j+1}s), v_{j+1}(s)).$$

Thus all we have to do is to modify the construction of Q_{j+1}^* so as to have

$$w'_{j+1}(S_{j+1}) \subseteq Q_{j+1}^*.$$

This is accomplished simply by replacing Q'_{j+1} by Q''_{j+1}, where

$$Q''_{j+1}=Q'_{j+1}+w'_{j+1}(S_{j+1}).$$

4. Representing the cohomology of a coherent sheaf.

Let $f: X \to \mathrm{Spec}(\mathbf{W}(k))$ be a proper morphism, as before, with

$$f(X) \subseteq \{\text{closed point of } \mathrm{Spec}(\mathbf{W}(k))\},$$

and let $\mathscr{F}$ be a coherent $\mathcal{O}_X$-Module. With the aid of Proposition (3.1), we will now prove Theorem (2.4) concerning the functor of k-algebras

$$H^i(X_A, \mathscr{F}_A) \qquad (i \geq 0,\ X_A = X \otimes_{\mathbf{W}(k)} \mathbf{W}(A), \quad \mathscr{F}_A = \mathscr{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_A}).$$

This functor can be computed by Čech cohomology, as follows. X is quasi-compact, so there exists a finite affine open covering of X, say $\mathfrak{U} = (U_j)_{1 \leq j \leq m}$. Let $C_\bullet = C_\bullet(\mathfrak{U}, \mathscr{F})$ be the corresponding Čech complex:

$$\ldots \to 0 \to C_0 \to C_{-1} \to C_{-2} \to \ldots \to C_{-m} \to 0 \to \ldots$$

(We put the Čech complex in this form to accord with § 3; for the usual form set $C^i = C_{-i}$.) The argument on p. 94 of [**EGA** III] gives an A-*functorial isomorphism of* $\mathbf{W}$(A)-*modules*

$$H_{-i}(C_\bullet \otimes_{\mathbf{W}(k)} \mathbf{W}(A)) \xrightarrow{\approx} H^i(X_A, \mathscr{F}_A) \ (^1).$$

To get the functor $H_{-i}(C_\bullet \otimes_{\mathbf{W}(k)} \mathbf{W}(A))$ into manageable form we use Proposition (3.1). Fix an integer $N > 0$ such that $p^N \mathcal{O}_X = 0$, so that X is proper over $\mathrm{Spec}(\mathbf{W}_N(k))$ $(\mathbf{W}_N(k) = \mathbf{W}(k)/(p^N))$. Then $C_\bullet$ is a bounded complex of $\mathbf{W}_N(k)$-modules, and for any integer $r \geq 0$ and any $i \in \mathbf{Z}$, the homology modules

$$H_{-i}(p^r C_\bullet) = H_{-i}(C_\bullet(\mathfrak{U}, p^r \mathscr{F})) = H^i(X, p^r \mathscr{F})$$

are *finitely generated* over $\mathbf{W}_N(k)$, since X is proper over $\mathbf{W}_N(k)$ and $p^r \mathscr{F}$ is a coherent $\mathcal{O}_X$-Module. Hence (Proposition (3.1)) there exists a complex $Q_\bullet$ of *finitely generated* $\mathbf{W}_N(k)$-modules, and a homomorphism of complexes $Q_\bullet \to C_\bullet$ which produces for each $\mathbf{W}_N(k)$-module M and each $i \in \mathbf{Z}$ an *isomorphism*

$$H_{-i}(Q_\bullet \otimes_{\mathbf{W}_N(k)} M) \xrightarrow{\approx} H_{-i}(C_\bullet \otimes_{\mathbf{W}_N(k)} M).$$

Taking $M = \mathbf{W}(A)/(p^N)$, A being a k-algebra, we get an isomorphism

$$H_{-i}(Q_\bullet \otimes_{\mathbf{W}(k)} \mathbf{W}(A)) \xrightarrow{\approx} H_{-i}(C_\bullet \otimes_{\mathbf{W}(k)} \mathbf{W}(A)),$$

and clearly this is an A-*functorial isomorphism of* $\mathbf{W}$(A)-*modules*.

Note that each Q_i $(i \in \mathbf{Z})$ is a $\mathbf{W}(k)$-module of finite length. Thus, to prove Theorem (2.4), it suffices now to show:

Proposition **(4.1)**. — *Let* E, F, G *be* $\mathbf{W}(k)$-*modules of finite length. Let* $\alpha : E \to F$, $\beta : F \to G$ *be* $\mathbf{W}(k)$-*module homomorphisms such that* $\beta \circ \alpha = 0$. *For any k-algebra* A *set*

$$E_A = E \otimes_{\mathbf{W}(k)} \mathbf{W}(A)$$
$$F_A = F \otimes_{\mathbf{W}(k)} \mathbf{W}(A)$$
$$G_A = G \otimes_{\mathbf{W}(k)} \mathbf{W}(A)$$

---

(1) [**EGA** III, p. 94] contains a reference to [**EGA** 0_{III}, (12.1.4.2)] whose proof (as given) seems incomplete: what is the map $\Gamma(\mathscr{C}^\bullet(\mathfrak{U}', \psi^* \mathscr{F})) \to \Gamma(\psi^* \mathscr{L}^\bullet)$? Never mind; we can use instead the homotopy-commutative diagram

$$\begin{array}{ccc} \Gamma(\psi^* \mathscr{C}^\bullet(\mathfrak{U}, \mathscr{F})) & \longrightarrow & \Gamma(\mathscr{C}^\bullet(\mathfrak{U}', \psi^* \mathscr{F})) \\ \downarrow & & \downarrow \\ \Gamma(\psi^* \mathscr{L}^\bullet) & \longrightarrow & \Gamma(\mathscr{L}'^\bullet). \end{array}$$

$$\begin{aligned}
&\alpha_A : E_A \to F_A = \alpha \otimes_{\mathbf{W}(k)} (1_{\mathbf{W}(A)}) \qquad (1_{\mathbf{W}(A)} = \textit{identity map of } \mathbf{W}(A)).\\
&\beta_A : F_A \to G_A = \beta \otimes_{\mathbf{W}(k)} (1_{\mathbf{W}(A)})\\
&I_A = \textit{image of } \alpha_A\\
&K_A = \textit{kernel of } \beta_A \qquad (\supseteq I_A (\textit{since } \beta_A \circ \alpha_A = 0))\\
&H_A = K_A / I_A \qquad (= \textit{homology of } \{(E \xrightarrow{\alpha} F \xrightarrow{\beta} G) \otimes_{\mathbf{W}(k)} \mathbf{W}(A)\}).
\end{aligned}$$

Let Φ *be any one of the functors*

$$A \to E_A \text{ (resp. } F_A, G_A, I_A, K_A, H_A)$$

and let $\Phi^{\sim}$ *be the associated fpqc sheaf. Then* $\Phi^{\sim}$ *is an affine algebraic k-scheme.*

Proof. — For $\Phi = (A \to E_A, F_A \text{ or } G_A)$, the result is given by Proposition (A.1) (i) (Appendix A).

Since K_A is the kernel of $F_A \to G_A$, therefore $(K_A)^{\sim}$ (the sheaf associated to $(A \to K_A)$) is the kernel of $(F_A)^{\sim} \to (G_A)^{\sim}$, so $(K_A)^{\sim}$ is a closed subscheme of $(F_A)^{\sim}$.

We have an exact sequence

$$\textbf{(4.2)} \qquad 0 \to I_A \to F_A \to (F/I)_A = (F/I) \otimes_{\mathbf{W}(k)} \mathbf{W}(A)$$

$(I = \alpha(E))$, whence $(I_A)^{\sim}$ is the kernel of $(F_A)^{\sim} \to ((F/I)_A)^{\sim}$; as above, $((F/I)_A)^{\sim}$ is an affine algebraic k-scheme, so $(I_A)^{\sim}$ is a closed subscheme of $(F_A)^{\sim}$.

Similarly, from the exact sequence

$$\textbf{(4.3)} \qquad 0 \to H_A \to F_A / I_A = (F/I)_A \to G_A$$

we conclude that $(H_A)^{\sim}$ is a closed subscheme of $((F/I)_A)^{\sim}$. Q.E.D.

This completes the proof of Theorem (2.4), and of Theorem (1.2).

* * *

Later on, we will make use of the following observation:

Corollary **(4.4)**. — *With the notation of* (4.1), *if* $A^p = A$, *then the canonical map*

$$\Phi(A) \to \Phi^{\sim}(A)$$

is bijective.

Proof. — For $\Phi(A) = E_A, F_A$ or G_A, this is contained in Proposition (A.1) (ii). For $\Phi(A) = K_A$, use the commutative diagram (with exact rows)

$$\begin{array}{ccccccc}
0 & \longrightarrow & K_A & \longrightarrow & F_A & \longrightarrow & G_A \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & (K_A)^{\sim}(A) & \longrightarrow & (F_A)^{\sim}(A) & \longrightarrow & (G_A)^{\sim}(A)
\end{array}$$

in which, as we have just remarked, the last two vertical arrows are bijective. The proofs for $\Phi(A)=I_A$ and $\Phi(A)=H_A$ are similar (cf. (4.2) and (4.3)).

5. Definition of P as an fppf sheaf.

As in remark (1.7), we consider a local Artin ring R, with residue field k, and a scheme X proper over R. Let **R** be the Greenberg algebra, over k, associated to R; for each k-algebra A set $X^{\#}_A = X \otimes_{R(k)} \mathbf{R}(A)$; and let $\mathbf{P}^{\#}$ be the *fppf sheaf* associated to the functor $\mathrm{Pic}(X^{\#}_A)$. Except for the changes indicated below, the proof of Theorem (1.2) given in §§ 2-4 applies more or less *verbatim* (with "*fpqc*" replaced by "*fppf*", and **W** by **R**, so that X_A becomes $X^{\#}_A$, etc.) to show that $\mathbf{P}^{\#}$ *is a locally algebraic k-group.*

The canonical map $R \to H^0(X, \mathcal{O}_X)$ gives rise to a homomorphism

$$\iota : k \to H^0(X, \mathcal{O}_X)_{\mathrm{red}}.$$

As in remark (1.7), we deduce an *isomorphism of functors*

$$\mathbf{P}(X, k, \iota) \overset{\approx}{\to} \mathbf{P}^{\#}.$$

* * *

In order to apply to $\mathbf{P}^{\#}$, the proof of Theorem (1.2) must be modified in two places.

First, to begin the inductive argument, we need to know (cf. middle of § 2): *if* $X = X_1$, i.e. *if* X *is reduced* (so that X is actually a k-scheme) *then there is an isomorphism*

$$(5.1) \qquad \mathbf{P}^{\#} \overset{\approx}{\to} \mathbf{Pic}_{X/k}.$$

To establish such an isomorphism, note that we have a ring homomorphism, functorial in A

$$\mathbf{R}(A) \to \mathbf{W}_1(A) = A$$

obtained by passage to associated *fpqc* sheaves from the functorial map

$$R \otimes_{\mathbf{W}(k)} \mathbf{W}(A) \to k \otimes_{\mathbf{W}(k)} \mathbf{W}(A),$$

cf. Proposition (A.1). From this we deduce a homomorphism of functors

$$(5.2) \qquad \mathrm{Pic}(X \otimes_{\mathbf{R}(k)} \mathbf{R}(A)) \to \mathrm{Pic}(X \otimes_k A)$$

which is *bijective* if $A^p = A$, since then $\mathbf{R}(A) = \mathbf{R}(k) \otimes_{\mathbf{W}(k)} \mathbf{W}(A)$ (Prop. (A.1) (ii)) and $A = \mathbf{W}(A)/(p)$, so that

$$X \otimes_{\mathbf{R}(k)} \mathbf{R}(A) = X \otimes_{\mathbf{W}(k)} \mathbf{W}(A) = X \otimes_k A.$$

Now the functor **R**, being isomorphic as a scheme to some affine space (Prop. (A.1) (i)), commutes with filtered direct limits. [**EGA** IV, (8.5.2) and (8.5.5)] shows then that the functor $\mathrm{Pic}(X \otimes_{\mathbf{R}(k)} \mathbf{R}(A))$ commutes with such limits, as does $\mathrm{Pic}(X \otimes_k A)$. From Corollary (0.2), it follows that (5.2) gives rise to an *isomorphism of associated fppf sheaves*, and this is the desired isomorphism (5.1).

Secondly, as in § 4, we want to show that the *fppf* sheaf associated to the functor $H_{-i}(C_{\bullet}\otimes_{\mathbf{R}(k)}\mathbf{R}(A))$ is an affine algebraic k-scheme. To apply the result of § 3, we note that if N is an integer with $p^N R=(0)$, then $\mathbf{R}$ is a $\mathbf{W}_N$-algebra, and there is a functorial homomorphism

$$\psi_A : H_{-i}(C_{\bullet}\otimes_{\mathbf{W}_N(k)}\mathbf{W}_N(A)) \to H_{-i}(C_{\bullet}\otimes_{\mathbf{R}(k)}\mathbf{R}(A))$$

which, as above, is *bijective* if $A^p=A$; since both homology and tensor products commute with direct limits, as do the functors $\mathbf{W}_N$ and $\mathbf{R}$, Corollary (0.2) shows that the homomorphism of *fppf* sheaves associated to ψ_A is an *isomorphism*; in other words, *we may replace* $\mathbf{R}$ *by* $\mathbf{W}_N$. We have, as in § 4, a functorial isomorphism

$$H_{-i}(Q_{\bullet}\otimes_{\mathbf{W}_N(k)}\mathbf{W}_N(A)) \to H_{-i}(C_{\bullet}\otimes_{\mathbf{W}_N(k)}\mathbf{W}_N(A))$$

and we are then reduced to proving Proposition (4.1), with " $\mathbf{W}$ " replaced by " $\mathbf{W}_N$ " and "*fpqc*" by "*fppf*". The proof is practically the same (use (A.2) in addition to (A.1) (i)).

Remark **(5.3)**. — Corollary (4.4) holds with *fpqc* replaced by *fppf*. The proof is the same, with (A.2) in place of (A.1) (ii).

II. — RELATION OF $\mathbf{P}(A)$ TO $\mathrm{Pic}(X_A)$ WHEN $A^p=A$

In this part **II** we obtain some information about the kernel and cokernel of the canonical map $\mathrm{Pic}(X_A)\to\mathbf{P}(A)$, under the assumption that the k-algebra A satisfies $A^p=A$ (Corollaries (6.7), (6.8); the most general result along these lines is Theorem (7.5), but we need some of the results of § 6 to prove it). As in § 2 (cf. also [**SGA** 6, Exposé XII, Corollaire (3.3)]) the underlying idea is to use " dévissage " to reduce to the case of Picard functors of schemes over fields (Proposition (6.2)). We find that when $A^p=A$, $\mathbf{P}(A)$ is related to $\mathrm{Pic}(X_A)$—in the classical way—by Galois descent (Corollary (6.10)) or via the *étale* topology (Corollary (6.11)). We also elucidate the dependence of $\mathbf{P}=\mathbf{P}(X, k, \iota)$ on k and ι (Corollary (6.13)).

Throughout $f: X\to \mathrm{Spec}(\mathbf{W}(k))$ *and* $\mathbf{P}$ *will be as in* § 1; *we also set* $X_1=X_{red}$, *and* $k_1=H^0(X_1, \mathcal{O}_{X_1})$.

6. Determination of P(A) ($A^p=A$).

It is convenient to begin with a simple observation:

Lemma **(6.1)**. — *Let*

$$\begin{array}{ccc} S & \xrightarrow{\alpha} & T \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\delta} \\ U & \xrightarrow{\gamma} & V \end{array}$$

be a commutative diagram of homomorphisms of abelian groups, and let the resulting map

$$\sigma : S \to T \times_V U = \{(t, u) \in T \times U \mid \delta(t) = \gamma(u)\}$$

be defined, as usual, by

$$\sigma(s) = (\alpha(s), \beta(s)) \qquad (s \in S).$$

Then the following conditions are equivalent:

(i) σ *is an isomorphism.*

(ii) *The map* $\ker(\alpha) \to \ker(\gamma)$ *induced by* β *is bijective, and the map* $\operatorname{coker}(\alpha) \to \operatorname{coker}(\gamma)$ *induced by* δ *is injective.*

(*Here* " ker " = *kernel and* " coker " = *cokernel.*)

(ii)′ *The map* $\ker(\beta) \to \ker(\delta)$ *induced by* α *is bijective, and the map* $\operatorname{coker}(\beta) \to \operatorname{coker}(\delta)$ *induced by* γ *is injective.*

Proof. — By symmetry it suffices to prove the equivalence of (i) and (ii). This is straightforward; the necessary verifications are left to the reader.

Proposition **(6.2)**. — *Let* $\mathbf{P}_1$ *be the fpqc sheaf associated to the functor*

$$\operatorname{Pic}(X_{1,A}) = \operatorname{Pic}(X_1 \otimes_{\mathbf{W}(k)} \mathbf{W}(A))$$

of k-algebras A. *If* $A^p = A$, *then the natural commutative diagram*

$$\begin{array}{ccc} \operatorname{Pic}(X_A) & \longrightarrow & \operatorname{Pic}(X_{1,A}) \\ \downarrow & & \downarrow \\ \mathbf{P}(A) & \longrightarrow & \mathbf{P}_1(A) \end{array}$$

gives rise (cf. Lemma (6.1)) *to an* isomorphism

$$\operatorname{Pic}(X_A) \xrightarrow{\sim} \operatorname{Pic}(X_{1,A}) \times_{\mathbf{P}_1(A)} \mathbf{P}(A)$$

Proof. — Let $\mathcal{N}_X$ be the Nilradical of $\mathcal{O}_X$, and for $n \geq 2$ let X_n be the closed subscheme of X defined by $\mathcal{N}_X^n$. As in § 2 (cf. (2.2)) we have an exact A-functorial sequence

$$0 \to E_n(A) \to \operatorname{Pic}(X_{n,A}) \to \operatorname{Pic}(X_{n-1,A}) \to F_n(A)$$

where: $E_n(A) = \text{cokernel of } H^0(\mathcal{O}^*_{n-1,A}) \to H^1(\mathcal{J}_n \mathcal{O}_{n,A})$

$$F_n(A) = H^2(\mathcal{J}_n \mathcal{O}_{n,A}).$$

Denoting associated *fpqc* sheaves with " ~ ", we shall show:

(6.3) $F_n(A) \to F_n^\sim(A)$ *is injective* $(A^p = A)$.

(6.4) *There is an isomorphism of functors*

$$\{\text{kernel of } (H^1(\mathcal{O}_{n,A}) \to H^1(\mathcal{O}_{n-1,A}))\} \xrightarrow{\sim} E_n(A).$$

Now we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_n(A) & \longrightarrow & \mathrm{Pic}(X_{n,A}) & \xrightarrow{\alpha} & \mathrm{Pic}(X_{n-1,A}) & \longrightarrow & F_n(A) \\ & & \downarrow & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\delta} & & \downarrow \\ 0 & \longrightarrow & E_n^{\sim}(A) & \longrightarrow & \mathbf{P}_n(A) & \xrightarrow{\gamma} & \mathbf{P}_{n-1}(A) & \xrightarrow{\tau} & F_n^{\sim}(A) \end{array}$$

in which the top row is, as above, exact, and the bottom row, obtained from the top row by passing to associated *fpqc* sheaves and " evaluating " at A, is exact except perhaps at $\mathbf{P}_{n-1}(A)$. Since anyway $\tau\circ\gamma=0$, we find from (6.3) above that δ induces an *injective* map $\mathrm{coker}(\alpha)\to\mathrm{coker}(\gamma)$. Furthermore, since $A^p=A$, (4.4) shows that if $\mathbf{H}^i_m$ is the *fpqc* sheaf associated to the functor $H^i(\mathcal{O}_{m,A})$ $(i, m\geq 1)$ then the canonical map $H^1(\mathcal{O}_{m,A})\to\mathbf{H}^1_m(A)$ is *bijective* (read through § 4 to see why (4.4) is applicable); it follows then from (6.4) above (cf. proof of (4.4)) that $E_n(A)\to E_n^{\sim}(A)$ *is bijective.* So Lemma (6.1) gives us an *isomorphism*

$$\mathrm{Pic}(X_{n,A}) \xrightarrow{\sim} \mathrm{Pic}(X_{n-1,A})\times_{\mathbf{P}_{n-1}(A)}\mathbf{P}_n(A)$$

Since $X_n=X$ for large n, Proposition (6.2) follows by induction on n.

Proof of **(6.3)**. — We have a commutative diagram

$$\begin{array}{ccccccc} H^1(\mathcal{O}_{n,A}) & \longrightarrow & H^1(\mathcal{O}_{n-1,A}) & \longrightarrow & F_n(A) & \longrightarrow & H^2(\mathcal{O}_{n,A}) \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle\varphi} & & \downarrow \\ \mathbf{H}^1_n(A) & \xrightarrow{\lambda} & \mathbf{H}^1_{n-1}(A) & \xrightarrow{\mu} & F_n^{\sim}(A) & \xrightarrow{\nu} & \mathbf{H}^2_n(A) \end{array}$$

in which the top row is exact and the bottom row is obtained from the top row by evaluating associated *fpqc* sheaves at A (so that $\nu\circ\mu=0$ and $\mu\circ\lambda=0$). As above, the vertical arrows other than φ are *bijective.* So by diagram chasing, we can conclude that φ is injective, as desired, ***provided that the kernel of μ equals the image of λ.*** But if $\mathbf{I}_n$ is the image, in the category of *fpqc* sheaves, of the canonical map $\boldsymbol{\lambda}:\mathbf{H}^1_n\to\mathbf{H}^1_{n-1}$, then $\mathbf{I}_n$ is the kernel of $\mathbf{H}^1_{n-1}\to F_n^{\sim}$ (since passage to associated sheaves is an *exact* functor), and so $\mathbf{I}_n(A)$ is the kernel of μ; thus we need to show that λ $(=\boldsymbol{\lambda}(A))$ maps $\mathbf{H}^1_n(A)$ *onto* $\mathbf{I}_n(A)$. Since, by § 4, $\boldsymbol{\lambda}$ is a homomorphism of Greenberg modules, therefore the kernel of $\boldsymbol{\lambda}$ is a Greenberg module and hence a connected unipotent algebraic k-group [**DG**, p. 601, (1.2)]. So the conclusion results from the following lemma:

Lemma **(6.5)**. — *Let*

$$0\to\mathbf{E}\to\mathbf{H}\to\mathbf{I}\to 0$$

be an exact sequence of abelian sheaves on the category of k-algebras with the fpqc topology. Suppose that $\mathbf{E}$ *is a connected unipotent algebraic k-group. Then for any k-algebra* A *such that* $A^p = A$, *the resulting sequence*

$$0 \to \mathbf{E}(A) \to \mathbf{H}(A) \to \mathbf{I}(A) \to 0$$

is exact.

Proof. — It suffices to show that $H^1(A, \mathbf{E}) = (0)$ (the cohomology being taken with respect to the *fpqc* topology). The long exact cohomology sequence associated to the exact sequence

$$0 \to \mathbf{E}_{\mathrm{red}} \to \mathbf{E} \to \mathbf{E}/\mathbf{E}_{\mathrm{red}} \to 0$$

shows that we may assume that $\mathbf{E}$ is either *smooth* or *infinitesimal.*

In the smooth case, $\mathbf{E}$ has a composition series whose quotients are isomorphic to the additive group $\mathbf{W}_1$ [**DG**, p. 495, (3.9)]. In the infinitesimal case, since any closed subgroup and any quotient group of an infinitesimal unipotent k-group is again infinitesimal and unipotent, $\mathbf{E}$ has a composition series whose quotients are isomorphic to ${}_p\mathbf{W}_1$, the kernel of the Frobenius endomorphism $\mathfrak{F}$ of $\mathbf{W}_1$ [**DG**, chap. IV, § 2, no. 2]. So we need only treat the cases $\mathbf{E} = \mathbf{W}_1$, $\mathbf{E} = {}_p\mathbf{W}_1$.

By descent theory

$$H^1(A, \mathbf{W}_1) = (0)$$

(cf. [**DG**, p. 383, (6.6)] or [**SGA** 4, Exposé VII, remarque (4.5)]). From the exact sequence

$$0 \to {}_p\mathbf{W}_1 \to \mathbf{W}_1 \xrightarrow{\mathfrak{F}} \mathbf{W}_1 \to 0$$

we deduce then that

$$H^1(A, {}_p\mathbf{W}_1) \cong A/A^p = (0).$$

This completes the proof of Lemma (6.5), and of (6.3).

Proof of **(6.4)**. — We use an argument due essentially to Oort [**O**, § 6].

With notation as in the beginning of § 2, we can associate to each triple $(Y, \mathcal{O}, \mathcal{J})$ a diagram with exact rows

$$\begin{array}{ccccccccc}
H^0(\mathcal{O}) & \xrightarrow{\gamma} & H^0(\bar{\mathcal{O}}) & \xrightarrow{\delta} & H^1(\mathcal{J}) & \longrightarrow & H^1(\mathcal{O}) & \longrightarrow & H^1(\bar{\mathcal{O}}) \\
 & & & & \Big\downarrow{\scriptstyle e = H^1(\exp)} & & & & \\
 & & H^0(\bar{\mathcal{O}}^*) & \xrightarrow{\delta^*} & H^1(1+\mathcal{J}) & \longrightarrow & H^1(\mathcal{O}^*) & \longrightarrow & H^1(\bar{\mathcal{O}}^*)
\end{array}$$

which varies *functorially* with the triple $(Y, \mathcal{O}, \mathcal{J})$. Suppose that $\mathcal{J} \subseteq \mathcal{N}$, where $\mathcal{N}$ is some $\mathcal{O}$-Ideal with the properties:

a) $\mathcal{N}$ is a Nil-ideal (*i.e.* for each $y \in Y$ the stalk $\mathcal{N}_y$ consists entirely of nilpotent elements of $\mathcal{O}_y$).

b) $\mathcal{N}\mathcal{J} = (0)$.

c) In the canonical diagram

$$\begin{array}{ccc} H^0(\mathcal{O}) & \xrightarrow{\gamma} & H^0(\bar{\mathcal{O}}) = H^0(\mathcal{O}/\mathcal{J}) \\ & {\searrow}^{u} \quad {\swarrow}_{\bar{u}} & \\ & H^0(\mathcal{O}/\mathcal{N}) & \end{array}$$

the maps u and $\bar{u}$ have the same image in $H^0(\mathcal{O}/\mathcal{N})$. (*c)* holds, for instance, if u is *surjective.*)

Since the kernel of $\bar{u}$ is $H^0(\mathcal{N}/\mathcal{J})$, *c)* is equivalent to:

c') $$H^0(\mathcal{O}/\mathcal{J}) = \gamma(H^0(\mathcal{O})) + H^0(\mathcal{N}/\mathcal{J}).$$

With these assumptions, we will show that:

$(*)$ $$e(\text{image of } \delta) = (\text{image of } \delta^*).$$

$(*)$ implies that *e induces an isomorphism of* $\operatorname{coker}(\delta)$ $(=\ker(H^1(\mathcal{O}) \to H^1(\bar{\mathcal{O}})))$ *onto* $\operatorname{coker} \delta^*$. Thus, granting $(*)$, to prove (6.4), we have only to check that *a)*, *b)* and *c)* hold for $Y = X_A$, $\mathcal{O} = \mathcal{O}_{n,A}$, $\mathcal{J} = \mathcal{N}_X^{n-1}\mathcal{O}_{n,A}$, $\mathcal{N} = \mathcal{N}_X\mathcal{O}_{n,A}$. Let us do this.

Since $\mathcal{N}_X$ is the Nilradical of $\mathcal{O}_X$, and $\mathcal{O}_{n,A} = \mathcal{O}_{X_A}/\mathcal{N}_X^n\mathcal{O}_{X_A}$, *a)* and *b)* are obvious.

As for *c)*, it is enough to show that u is surjective, and this results from the next lemma (with $X = X_n$).

Lemma **(6.6)**. — *For any k-algebra* A, *the canonical map*

$$u_A : H^0(X_A, \mathcal{O}_{X_A}) \to H^0(X_{1,A}, \mathcal{O}_{X_{1,A}})$$

is surjective.

Proof. — We have a natural commutative diagram

$$\begin{array}{ccc} H^0(X, \mathcal{O}_X) \otimes_{W(k)} \mathbf{W}(A) & \longrightarrow & H^0(X_A, \mathcal{O}_{X_A}) \\ \downarrow{\scriptstyle u_k \otimes 1} & & \downarrow{\scriptstyle u_A} \\ H^0(X_1, \mathcal{O}_{X_1}) \otimes_{W(k)} \mathbf{W}(A) & \xrightarrow[\omega]{} & H^0(X_{1,A}, \mathcal{O}_{X_{1,A}}) \\ \| & & \\ H^0(X_1, \mathcal{O}_{X_1}) \otimes_k \mathbf{W}(A)/(p) & & \end{array}$$

Since $X_{1,A} = X_1 \otimes_k \mathbf{W}(A)/(p)$, ω is bijective [**EGA** III, (1.4.15)]. So it suffices to show that

$$u_k : H^0(X, \mathcal{O}_X) \to H^0(X_1, \mathcal{O}_{X_1})$$

is surjective, i.e. that $H^0(X, \mathcal{O}_X)_{\text{red}} = H^0(X_1, \mathcal{O}_{X_1})$.

Let $\bar{k}$ be an algebraic closure of k, and let $\overline{X}=X\otimes_{W(k)}W(\bar{k})$. k being perfect, we have that

$$\overline{X}_1=X_1\otimes_{W(k)}W(\bar{k})=X_1\otimes_k\bar{k}$$

is reduced, and hence

$$\overline{X}_1=\overline{X}_{red}.$$

Since, clearly, $\mathbf{W}(\bar{k})$ is flat over $\mathbf{W}(k)$, therefore [**EGA** III, (1.4.15)]

$$H^0(\overline{X}, \mathcal{O}_{\overline{X}})=H^0(X, \mathcal{O}_X)\otimes_{W(k)}\mathbf{W}(\bar{k})$$

and hence $\quad H^0(\overline{X}, \mathcal{O}_{\overline{X}})_{red}=H^0(X, \mathcal{O}_X)_{red}\otimes_k\bar{k}.$

Consequently

$[H^0(X, \mathcal{O}_X)_{red} : k]=[H^0(\overline{X}, \mathcal{O}_{\overline{X}})_{red} : \bar{k}]=$ number of connected components of $\overline{X}$,

and similarly

$[H^0(X_1, \mathcal{O}_{X_1}) : k]=[H^0(X_1, \mathcal{O}_{X_1})_{red} : k]=$ number of connected components of $\overline{X}_1$.

Thus $\quad [H^0(X, \mathcal{O}_X)_{red} : k]=[H^0(X_1, \mathcal{O}_{X_1}) : k]$

and the conclusion follows. Q.E.D.

Finally, we prove (*).

Since we are dealing only with H^0 and H^1 we can use Čech cohomology. If $g\in H^0(\overline{\mathcal{O}})=H^0(\mathcal{O}/\mathcal{J})$, then, by *c')* above, $g=\gamma(h)+\bar{n}$, with $h\in H^0(\mathcal{O})$ and $\bar{n}\in H^0(\mathcal{N}/\mathcal{J})$, and so $\delta g=\delta\bar{n}$. Let $\{U_i\}$ be an open cover of Y such that $\bar{n}|U_i$ lifts to $n_i\in\Gamma(U_i,\mathcal{N})$. Then $\delta\bar{n}$ is represented by the cocycle $\{n_{ij}\}$, with

$$n_{ij}=(n_i|U_i\cap U_j)-(n_j|U_i\cap U_j)\in\Gamma(U_i\cap U_j,\mathcal{J}).$$

Also, $1+\bar{n}\in H^0((\mathcal{O}/\mathcal{J})^*)$ (since $\mathcal{N}$ is a Nil-ideal, cf. *a)* above), and $\delta^*(1+\bar{n})$ is represented by the cocycle $\{n^*_{ij}\}$, with

$$n^*_{ij}=\frac{1+n_i|U_i\cap U_j}{1+n_j|U_i\cap U_j}=1+n_{ij}=\exp(n_{ij}).$$

(The second equality holds because $\mathcal{N}\mathcal{J}=0$ (cf. *b)* above), so that $(n_j|U_i\cap U_j)n_{ij}=0$). Thus $e(\delta\bar{n})=\delta^*(1+\bar{n})$, and so we have

$$e(\mathrm{im}(\delta))\subset\mathrm{im}(\delta^*).$$

Conversely, suppose that $g\in H^0(\overline{\mathcal{O}}^*)$. As above, $g=\gamma(h)+\bar{n}$, and now $\gamma(h)=g-\bar{n}$ is a unit in $H^0(\overline{\mathcal{O}})$ since g is a unit and $\bar{n}$ is locally nilpotent. Since γ has nilpotent kernel $H^0(\mathcal{J})$ ($\mathcal{J}^2=(0)$), it follows that h is a unit in $H^0(\mathcal{O})$. Hence, if $n'=\bar{n}/\gamma(h)$, we have

$$\delta^*(g)=\delta^*(\gamma(h)(1+n'))=\delta^*(1+n')=e(\delta n')$$

(cf. preceding paragraph), and so

$$\mathrm{im}(\delta^*) \subset e(\mathrm{im}(\delta)).$$

This completes the proof of (6.4), and of Proposition (6.2).

Corollary **(6.7)**. — *Let* B *be a* k*-algebra such that* $X_1 \otimes_k B$ *has a section over* $k_1 \otimes_k B$ $(k_1 = H^0(X_1, \mathcal{O}_{X_1}))$. *Then for any* B*-algebra* A *such that* $A^p = A$, *the canonical map* $\mathrm{Pic}(X_A) \to \mathbf{P}(A)$ *is* surjective, *its kernel being (functorially) isomorphic to* $\mathrm{Pic}(\mathrm{Spec}(k_1 \otimes_k A))$.

Proof. — By (6.2) (and in view of (6.1)), *we may assume that* $X = X_1$. Then $\mathbf{P}$ can be identified with the usual Picard functor $\mathbf{Pic}_{X/k}$ (cf. middle of § 2), so it will suffice to show that the natural sequence

$$0 \to \mathrm{Pic}(\mathrm{Spec}(k_1 \otimes_k A)) \to \mathrm{Pic}(X \otimes_k A) \to \mathbf{Pic}_{X/k}(A) \to 0$$

is *exact* whenever A is a B-algebra. But then $X \otimes_k A$ has a section over $k_1 \otimes_k A$, and

$$H^0(X \otimes_k A, \mathcal{O}_{X \otimes_k A}) = H^0(X, \mathcal{O}_X) \otimes_k A = k_1 \otimes_k A$$

[**EGA** III, (1.4.15)]; so from [**FGA**, p. 232-05, Cor. (2.4)] it follows that the sequence

$$\begin{array}{c} 0 \to \mathrm{Pic}(\mathrm{Spec}(k_1 \otimes_k A)) \to \mathrm{Pic}(X \otimes_{k_1} (k_1 \otimes_k A)) \to \mathbf{Pic}_{X/k_1}(k_1 \otimes_k A) \to 0 \\ \Vert \\ \mathrm{Pic}(X \otimes_k A) \end{array}$$

is exact; and the desired conclusion is given by [**FGA**, p. 232-15, Prop. (6.1)], which tells us that $\mathbf{Pic}_{X/k_1}(k_1 \otimes_k A)$ is functorially isomorphic to $\mathbf{Pic}_{X/k}(A)$.

(No proof of the cited Prop. (6.1) is given, but one can proceed (for example) as in the proof of Corollary (6.13) below.)

Corollary **(6.8)**. — *Let* k' *be a* k_1*-algebra such that* k' *is a free* k_1*-module of finite rank* r, *and such that* $X_1 \otimes_{k_1} k'$ *has a section over* k'. *Then for any* k*-algebra* A *such that* $A^p = A$, *the cokernel of* $\mathrm{Pic}(X_A) \to \mathbf{P}(A)$ *is annihilated by* r.

In particular, if X *is connected, so that* k_1 *is a field, then this cokernel is annihilated by the greatest common divisor of the degrees (over* k_1*) of all the zero-cycles on* X_1.

Proof. — As in the proof of (6.7), we may assume that $X = X_1$ and $\mathbf{P} = \mathbf{Pic}_{X/k}$; and furthermore, since then $\mathrm{Pic}(X_A) \to \mathbf{Pic}_{X/k}(A)$ can be identified with

$$\mathrm{Pic}(X \otimes_{k_1} (k_1 \otimes_k A)) \to \mathbf{Pic}_{X/k_1}(k_1 \otimes_k A),$$

we are reduced to a well-known statement about $\mathbf{Pic}_{X/k_1}$: with $B = k_1 \otimes_k A$, if $\xi \in \mathbf{Pic}_{X/k_1}(B)$, then the image of ξ in $\mathbf{Pic}_{X/k_1}(B')$ $(B' = B \otimes_{k_1} k')$ is given by an invertible sheaf $\mathscr{L}$ on $X' = X \otimes_{k_1} B'$ (since X' has a section over $B' = H^0(X', \mathcal{O}_{X'})$), and one checks that $r\xi$ is given by the *norm* of $\mathscr{L}$, which is an invertible sheaf on $X \otimes_{k_1} B$ [**EGA** II, § 6.5]. Q.E.D.

We see next (Corollary (6.10)) how $\mathbf{P}(A)$ can be described by " Galois descent ".

Lemma **(6.9)**. — *Let* Y *be a reduced scheme proper over a field* F, *and set* $F_1 = H^0(Y, \mathcal{O}_Y)$. *Let* K *be a Galois field extension of* F *such that each connected component* Z *of* Y *has a* K-*rational point* (i.e. *there exists an* F-*morphism* $\mathrm{Spec}(K) \to Z$). *Then* $Y \otimes_F K$ *has a section over* $F_1 \otimes_F K$.

Proof. — We reduce at once to the case where Y is connected, so that F_1 is a finite field extension of F. Then we have, by assumption, an F-morphism $\mathrm{Spec}(K) \to Y$, which, composed with the canonical map $Y \to \mathrm{Spec}(F_1)$, gives rise to an F-homomorphism $F_1 \to K$; so we may identify F_1 with a field between F and K.

It will suffice to show the existence of an F_1-*homomorphism* $K \to F_1 \otimes_F K$, *i.e.* an F_1-morphism $\mathrm{Spec}(F_1 \otimes_F K) \to \mathrm{Spec}(K)$; for then, composing with $\mathrm{Spec}(K) \to Y$, we will have an F_1-morphism $\mathrm{Spec}(F_1 \otimes_F K) \to Y$, whence a section of $Y \otimes_F K$ $(= Y \otimes_{F_1} (F_1 \otimes_F K))$ over $F_1 \otimes_F K$, as desired.

Since F_1 is a subfield of finite degree of the Galois extension K/F, we have that

$$F_1 = F[X]/(P(X)), \quad \text{where} \quad P(X) \in F[X]$$

is a polynomial which splits into linear factors over K; hence $F_1 \otimes_F K$ is a direct product of $[F_1 : F]$ F_1-algebras K_i $(i = 1, 2, \ldots, [F_1 : F])$, each of which is F-isomorphic to K. But since K is Galois over F, each K_i is actually F_1-isomorphic to K, and consequently there exists an F_1-homomorphism $K \to F_1 \otimes_F K$. Q.E.D.

Corollary **(6.10)**. — *Let* K *be a Galois field extension of k such that each connected component of* X_1 *has a* K-*rational point. Let* A *be a k-algebra such that* $A^p = A$, *and set* $A_K = A \otimes_k K$. *Then*:

(i) *The canonical map* $\mathrm{Pic}(X_{A_K}) \to \mathbf{P}(A_K)$ *is surjective, with kernel isomorphic to* $\mathrm{Pic}(\mathrm{Spec}(k_1 \otimes_k A_K))$.

(ii) *The obvious map* $\mathbf{P}(A) \to \mathbf{P}(A_K)$ *takes* $\mathbf{P}(A)$ *isomorphically onto the subset of* $\mathbf{P}(A_K)$ *consisting of those elements which are invariant under the natural action of the Galois group of* K/*k*.

Proof. — (i) follows from (6.9) and (6.7).

(ii) is given, when $[K : k] < \infty$, by a standard—and straightforward—interpretation of the *exactness* of the diagram

$$\mathbf{P}(A) \to \mathbf{P}(A_K) \rightrightarrows \mathbf{P}(A_K \otimes_A A_K).$$

(Exactness holds because **P** is a sheaf and A_K is faithfully flat over A.) If K is not finite over K, the conclusion follows easily from the facts that $K = \varinjlim K_\alpha$ as K_α runs through all Galois subfields of K/*k* finite over *k*, and that **P** commutes with filtered direct limits (for example because **P** is locally algebraic over *k*, by Theorem (1.2)).

Corollary **(6.11)**. — *Let* $\mathbf{P}^{et}$ *be the* étale sheaf *associated to the functor* $\mathrm{Pic}(X_A)$. *If* $A^p = A$, *then the canonical map*

$$\mathbf{P}^{et}(A) \to \mathbf{P}(A)$$

is bijective.

Proof. — Let K be as in (6.10), with $[K:k]<\infty$. For any k-algebra A, $A\otimes_k K$ is an étale A-algebra, $\mathrm{Spec}(A\otimes_k K)\to\mathrm{Spec}(A)$ is surjective, and $(A\otimes_k K)^p=A\otimes_k K$. It follows easily, since $\mathbf{P}^{et}$ and $\mathbf{P}$ are both étale sheaves, that in proving (6.11) we may assume that A is a K-algebra. Then, by (6.9) and (6.7), $\mathrm{Pic}(X_A)\to\mathbf{P}(A)$ is surjective, so the same is true of $\mathbf{P}^{et}(A)\to\mathbf{P}(A)$. For injectivity we need:

Lemma **(6.12).** — *If* $A^p=A$ *and* B *is an étale* A-*algebra, then* $B^p=B$.

Granting this for the moment, we see, by (6.9) and (6.7), that there is a B-functorial exact sequence

$$0\to\mathrm{Pic}(\mathrm{Spec}(k_1\otimes_k B))\to\mathrm{Pic}(X_B)\to\mathbf{P}(B)$$

(B an étale A-algebra), and passing to associated étale sheaves, we see that the kernel of $\mathbf{P}^{et}(A)\to\mathbf{P}(A)$ is $\mathbf{P}^*(A)$, where $\mathbf{P}^*$ is the étale sheaf associated to the functor

$$B\mapsto\mathrm{Pic}(\mathrm{Spec}(k_1\otimes_k B)).$$

But any element of $\mathrm{Pic}(\mathrm{Spec}(k_1\otimes_k B))$ is *locally trivial* on Spec(B), even for the Zariski topology [**EGA** IV, (21.8.1)]. Thus $\mathbf{P}^*=0$, and so $\mathbf{P}^{et}(A)\to\mathbf{P}(A)$ is injective.

Proof of (6.12). — Since $A=A^p$, the structure map $A\to B$ factors through B^p

$$\begin{array}{ccc} & A & \\ \swarrow & & \searrow \\ B^p & \hookrightarrow & B \end{array}$$

and since B is étale (i.e. flat, unramified, and finitely presented) over A, it follows that B is unramified over B^p, and that B is a finitely generated B^p-algebra, hence a finite B^p-module (since B is integral over B^p). Localizing at the maximal ideals of B^p and using Nakayama's Lemma, we find then that $B^p=B$. Q.E.D.

For the last result in this section, we consider, as in remark (1.5), a commutative diagram

$$(\#)\qquad \begin{array}{ccc} X & = & X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ \mathrm{Spec}(\mathbf{W}(k)) & \xleftarrow[\mathrm{Spec}(\mathbf{W}(\theta))]{} & \mathrm{Spec}(\mathbf{W}(k')) \end{array}$$

where $\theta: k\to k'$ is a homomorphism of fields. The corresponding commutative diagram

$$\begin{array}{ccc} k & \xrightarrow{\ \theta\ } & k'=(\mathbf{W}(k')/(p^N))_{red} \qquad (p^N\mathcal{O}_X=(0))\ ^{(1)} \\ {\scriptstyle\iota}\searrow & & \swarrow{\scriptstyle\iota'} \\ & H^0(X,\mathcal{O}_X)_{red} & \end{array}$$

(1) For any k-algebra A let L_A be the kernel of the truncation $\mathbf{W}(A)\to A$. Writing (as we may)

$$(0,a_1,a_2,\ldots)=p(a_1^{1/p},a_2^{1/p},\ldots),$$

we find easily that $L_A^2=pL_A$, whence $L_A^{N+1}=p^N L_A$ for any $N>0$. Thus $A_{red}=(\mathbf{W}(A)/(p^N))_{red}$.

shows that k' is a finite algebraic extension of k (so that k' is perfect of characteristic p), and that $H^0(X, \mathcal{O}_X)_{red}$ is a finite k'-module via ι'. So we can set

$$\mathbf{P}' = \mathbf{P}(X, k', \iota') = fpqc \text{ sheaf associated to the functor}$$
$$B \mapsto \mathrm{Pic}(X \otimes_{\mathbf{W}(k')} \mathbf{W}(B)) \text{ of } k'\text{-algebras } B.$$

Then for any k-algebra A, setting $A' = k' \otimes_k A$, we have an A-functorial map (as described in remark (1.6))

$$\delta_A : \mathbf{P}(A) \to \theta_* \mathbf{P}'(A) = \mathbf{P}'(A').$$

As is easily seen, since $\theta_* \mathbf{P}'$ is an *fpqc* sheaf, $\delta : \mathbf{P} \to \theta_* \mathbf{P}'$ is the unique map such that the following diagram commutes for all A:

$$\begin{array}{ccc} \mathrm{Pic}(X \otimes_{\mathbf{W}(k)} \mathbf{W}(A)) & \xrightarrow{\gamma_A} & \mathrm{Pic}(X \otimes_{\mathbf{W}(k')} \mathbf{W}(A')) \\ \downarrow{\scriptstyle \lambda_A} & & \downarrow{\scriptstyle \lambda'_A} \\ \mathbf{P}(A) & \xrightarrow{\delta_A} & \mathbf{P}'(A') \end{array}$$

(Here γ_A is defined in the obvious way; and the vertical arrows are the canonical maps.)

Corollary (**6.13**). — *The above map δ_A is bijective for all k-algebras* A. (In other words

$$\mathbf{P} = \theta_* \mathbf{P}' = \prod_{k'/k} \mathbf{P}'$$

where $\prod_{k'/k}$ is Weil's " restriction of scalars ".)

Proof. — Since $\mathbf{P}$ and $\theta_* \mathbf{P}'$ are *fpqc* sheaves, we may assume that $A^p = A$ (Corollary (0.2)). As in the proof of (6.11), we may further assume that there exist exact sequences

$$0 \to \mathrm{Pic}(\mathrm{Spec}(k_1 \otimes_k A)) \to \mathrm{Pic}(X \otimes_{\mathbf{W}(k)} \mathbf{W}(A)) \xrightarrow{\lambda_A} \mathbf{P}(A) \to 0$$
$$0 \to \mathrm{Pic}(\mathrm{Spec}(k_1 \otimes_{k'} A') \to \mathrm{Pic}(X \otimes_{\mathbf{W}(k')} \mathbf{W}(A')) \xrightarrow{\lambda'_A} \mathbf{P}'(A') \to 0.$$

Since $A^p = A$, we have $X \otimes_{\mathbf{W}(k)} \mathbf{W}(A) = X \otimes_{\mathbf{W}_N(k)} \mathbf{W}_N(A)$ where N is such that $p^N \mathcal{O}_X = (0)$; similarly $X \otimes_{\mathbf{W}(k')} \mathbf{W}(A') = X \otimes_{\mathbf{W}_N(k')} \mathbf{W}_N(A')$. Now Theorem (C.5) (Appendix C), shows that $\mathbf{W}_N(A')$ is canonically isomorphic to $\mathbf{W}_N(k') \otimes_{\mathbf{W}_N(k)} \mathbf{W}_N(A)$, and so γ_A *is an isomorphism.* One checks that γ_A maps the kernel of λ_A *isomorphically onto* the kernel of λ'_A (since $k_1 \otimes_k A = k_1 \otimes_{k'} k' \otimes_k A = k_1 \otimes_{k'} A'$). The conclusion follows.

7. An exact sequence.

The point of this section is to establish the exact sequence (7.5), which carries much information about the difference between $\mathrm{Pic}(X_A)$ and $\mathbf{P}(A)$ when $A^p = A$

(cf. remark (7.7)). This exact sequence is deduced from the exact sequence (7.4), which is essentially well-known (cf. Corollaire (5.3) in Grothendieck's exposé " Groupe de Brauer III " [$\mathbf{G}_2$]). For the convenience of the reader (and to satisfy the author) we begin by reviewing the derivation of (7.4), elaborating on some details which are taken for granted in *loc. cit.*

Let $h : X \to \mathrm{Spec}(R)$ be a proper map, where R is a local Artin ring. The category Aff/R of affine R-schemes and the category Sch/X of schemes over X can both be given the étale topology, and then h defines a morphism of sites

$$\mathrm{Spec}(T) \mapsto X_T = X \otimes_R T$$

(T an R-algebra) (1). We have the left-exact functor h_* from étale sheaves G on Sch/X to étale sheaves on Aff/R, namely

$$h_* G(\mathrm{Spec}(T)) = G(X_T).$$

The category of sheaves on Aff/R is contained in the category of presheaves; let υ be the corresponding (left-exact) inclusion functor. Let $\mathscr{H}^n$ (resp. $\mathscr{R}^n$) be the n-th right derived functor of υ (resp. $\upsilon \circ h_*$). For any abelian sheaf F on Aff/R, $\mathscr{H}^n F$ can be thought of as a covariant functor of R-algebras T, namely

$$\mathscr{H}^n F(T) = H^n(\mathrm{Spec}(T), F) = H^n(T, F)$$

(where H^n denotes étale cohomology). Similarly for an abelian sheaf G on X we can write

$$\mathscr{R}^n G(T) = H^n(X_T, G).$$

The étale *sheaf* associated to $\mathscr{R}^n G$ is just the higher direct image $R^n h_* G$. In particular (take $X = \mathrm{Spec}(R)$ and $h = \text{identity}$) *the sheaf associated to* $\mathscr{H}^n F$ *vanishes for* $n \geq 1$. (For more details cf. (for example) [$\mathbf{A}_2$, chap. II, § 4].)

Let $\mathbf{G}_X$ be the *multiplicative group* on Sch/X, i.e. the étale sheaf given by

$$\mathbf{G}_X(Y \to X) = H^0(Y, \mathcal{O}_Y^*).$$

The spectral sequence for the composite functor $\upsilon \circ h_*$ gives rise to the exact sequence of *presheaves* on Aff/R (i.e. of covariant functors of R-algebras)

$$(7.1) \qquad \begin{aligned} &0 \to \mathscr{H}^1(h_*\mathbf{G}_X) \to \mathscr{R}^1(\mathbf{G}_X) \xrightarrow{\tau} R^1 h_*(\mathbf{G}_X) \\ &\to \mathscr{H}^2(h_*\mathbf{G}_X) \to \mathscr{R}^2(\mathbf{G}_X). \end{aligned}$$

Let us make more explicit the terms in this sequence.

According to the above remarks, $\mathscr{R}^n \mathbf{G}_X(T) = H^n(X_T, \mathbf{G}_X)$ and in particular

$$\mathscr{R}^1 \mathbf{G}_X(T) = H^1(X_T, \mathbf{G}_X) = \mathrm{Pic}(X_T)$$

(1) On Aff/R, the étale topology can be described, as in § 0, in terms of covering algebras; and " locally " the same is true for Sch/X.

(the equality being functorial in T, cf. [$\mathbf{A}_2$, chap. IV, p. 102, Prop. (1.2)]), and

$$\mathscr{R}^2\mathbf{G}_X(T) = H^2(X_T, \mathbf{G}_X) = \mathrm{Br}(X_T)$$

(where " Br " denotes " *cohomological* Brauer group ").

Furthermore *the map* $\tau : \mathscr{R}^1(\mathbf{G}_X) \to R^1h_*(\mathbf{G}_X)$ *in* (7.1) *can be identified with the canonical map of the presheaf* $\mathscr{R}^1(\mathbf{G}_X)$ *into its associated étale sheaf (which we will denote, as before, by* $\mathbf{Pic}_{X/R}$). (*Proof*: Denoting associated sheaves with " $\sim$ ", we obtain from (7.1) a commutative diagram

$$\begin{array}{ccccccc}
\mathscr{H}^1(h_*\mathbf{G}_X) & \longrightarrow & \mathscr{R}^1(\mathbf{G}_X) & \xrightarrow{\tau} & R^1h_*(\mathbf{G}_X) & \longrightarrow & \mathscr{H}^2(h_*\mathbf{G}_X) \\
\downarrow & & \downarrow & & \| & & \downarrow \\
(\mathscr{H}^1(h_*\mathbf{G}_X))^\sim & \longrightarrow & (\mathscr{R}^1(\mathbf{G}_X))^\sim & \xrightarrow{\tau^\sim} & (R^1h_*(\mathbf{G}_X))^\sim & \longrightarrow & (\mathscr{H}^2(h_*\mathbf{G}_X))^\sim \\
\| & & & & & & \| \\
0 & & & & & & 0
\end{array}$$

where the vertical maps are canonical maps of presheaves into their associated sheaves. The top row is exact in the category of presheaves, so the bottom row is exact in the category of sheaves, i.e. $\tau^\sim$ *is an isomorphism of sheaves*; our assertion follows.)

Next look at the functor

$$\mathscr{H}^n(h_*\mathbf{G}_X)(T) = H^n(T, h_*\mathbf{G}_X).$$

For any R-algebra S, let $S_0 = H^0(X_S, \mathcal{O}_{X_S})$. Then $h_*\mathbf{G}_X(S)$ is the group of units S_0^* in S_0. If T is an R-algebra and S is an *étale* T-algebra then $S_0 = S \otimes_T T_0$ [**EGA** III, (1.4.15)], and so we see that the restriction of $h_*\mathbf{G}_X$ to the site $T_{\text{ét}}$ consisting of spectra of étale T-algebras (with the étale topology) is equal to $h_{0*}\mathbf{G}_{T_0}$, where $\mathbf{G}_{T_0}$ is the multiplicative group on $(T_0)_{\text{ét}}$ and $h_0 : \mathrm{Spec}(T_0) \to \mathrm{Spec}(T)$ is the canonical map. Hence we have natural homomorphisms

$$H^n(T, h_*\mathbf{G}_X) = H^n(T, h_{0*}\mathbf{G}_{T_0}) \to H^n(T_0, \mathbf{G}_{T_0}).$$

These homomorphisms are bijective. (Indeed, by [**SGA** 4, VIII, Cor. (5.6)] it is enough to check that T_0 is integral over T; since

$$H^0(X_T, \mathcal{O}_{X_T}) = \varinjlim_{S} H^0(X_S, \mathcal{O}_{X_S})$$

as S runs through all finitely generated R-subalgebras of T [**EGA** IV, (8.5.4)], we may assume that T is finitely generated over R; in this case T is noetherian, and since X_T is proper over T, T_0 is actually a *finite* T-module.) Moreover, *the edge homomorphisms*

$$\mathscr{H}^i(h_*\mathbf{G}_X)(T) \to \mathscr{R}^i(\mathbf{G}_X)(T) \qquad (i = 1, 2)$$

in (7.1) *can be identified with the* usual *cohomology maps*

$$H^i(T_0, \mathbf{G}_{T_0}) \to H^i(X_T, \mathbf{G}_X)$$

(cf. [**EGA** 0_{III}, (12.1.7)]).

We have, finally:

Lemma **(7.2).** — *For any ring* S, *let* $\mathbf{G}_S$ *be the multiplicative group on the site* $S_{\text{ét}}$. *If* $\mathfrak{J}$ *is a nilideal in* S, *then the canonical maps*

$$H^n(S, \mathbf{G}_S) \to H^n(S/\mathfrak{J}, \mathbf{G}_{S/\mathfrak{J}}) \quad (n \geq 1)$$

are bijective.

Proof. — By [**SGA** 4, VIII, Cor. (1.2), or Cor. (5.6)] we have *isomorphisms*

$$H^n(S, i_{\mathfrak{J}*}(\mathbf{G}_{S/\mathfrak{J}})) \xrightarrow{\approx} H^n(S/\mathfrak{J}, \mathbf{G}_{S/\mathfrak{J}})$$

where $i_{\mathfrak{J}} : \mathrm{Spec}(S/\mathfrak{J}) \to \mathrm{Spec}(S)$ is the inclusion map. Since, clearly

$$i_{\mathfrak{J}*}(\mathbf{G}_{S/\mathfrak{J}}) = \varinjlim_{J} i_{J*}(\mathbf{G}_{S/J})$$

as J runs through all *finitely generated* ideals of S contained in $\mathfrak{J}$, and since cohomology on $S_{\text{ét}}$ commutes with direct limits [**SGA** 4, VII, Prop. (3.3)] it will suffice to show that the natural map

$$H^n(S, \mathbf{G}_S) \to H^n(S, i_{\mathfrak{J}*}(\mathbf{G}_{S/\mathfrak{J}})) \quad (n \geq 1) \tag{7.3}$$

is bijective whenever $\mathfrak{J}$ is *finitely generated*; so we may assume that $\mathfrak{J}^N = (0)$ for some integer N. By an obvious induction, we need only consider the case $\mathfrak{J}^2 = (0)$.

For any étale S-algebra S′ we have an exact sequence of multiplicative abelian groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & 1 + \mathfrak{J}S' & \longrightarrow & (S')^* & \longrightarrow & (S'/\mathfrak{J}S')^* & \longrightarrow & 1 \\ & & & & \| & & \| & & \\ & & & & \mathbf{G}_S(S') & & (i_{\mathfrak{J}*}(\mathbf{G}_{S/\mathfrak{J}}))(S') & & \end{array}$$

Since $\mathfrak{J}^2 = (0)$, the truncated exponential map $x \to 1 + x$ gives an isomorphism of the additive group $\mathfrak{J}S'$ onto the multiplicative group $1 + \mathfrak{J}S'$. Thus we have a surjective homomorphism of sheaves $\mathbf{G}_S \to i_{\mathfrak{J}*}(\mathbf{G}_{S/\mathfrak{J}})$ whose kernel is isomorphic to the sheaf $\tilde{\mathfrak{J}}$ given as a functor by $\tilde{\mathfrak{J}}(S') = \mathfrak{J}S'$, S′ as above. The bijectivity of (7.3) follows now from the fact that $H^n(S, \tilde{\mathfrak{J}}) = (0)$ $(n \geq 1)$ [**SGA** 4, VII, Cor. (4.4)]. Q.E.D.

In summary: for a proper map $h : X \to \mathrm{Spec}(R)$, *with* R *a local Artin ring, we have an exact sequence of functors of* R-*algebras* T

$$\begin{aligned} 0 &\longrightarrow \mathrm{Pic}(T_{0,\mathrm{red}}) \xrightarrow{\alpha} \mathrm{Pic}(X_T) \xrightarrow{\beta} \mathbf{Pic}_{X/R}(T) \\ &\longrightarrow \mathrm{Br}(T_{0,\mathrm{red}}) \xrightarrow{\gamma} \mathrm{Br}(X_T) \end{aligned} \tag{7.4}$$

where

— $X_T = X \otimes_R T$

— $T_0 = H^0(X_T, \mathcal{O}_{X_T})$

— $T_{0,\mathrm{red}} = T_0/(\text{nilradical of } T_0)$

— $\mathrm{Pic}(T_{0,\mathrm{red}}) = \mathrm{Pic}(\mathrm{Spec}(T_{0,\mathrm{red}}))$, and similarly for $\mathrm{Br}(T_{0,\mathrm{red}})$

— Br = cohomological Brauer group $= H^2_{\acute{e}t}(\cdot, \mathbf{G}_\bullet)$
— α and γ are the natural maps arising from the map $T_0 \to T_{0,\mathrm{red}}$ (which induces isomorphisms of cohomology, cf. Lemma (7.3)), and from $X_T \to \mathrm{Spec}(T_0)$.
— β is the natural map of the functor $\mathrm{Pic}(X_T)$ into its associated étale sheaf.

* * *

We shall apply (7.4) in the case $R = \mathbf{W}_N(k)$ $(N \geq 1)$, where k is, as usual, a perfect field of characteristic $p > 0$. We need two non-trivial observations. First of all, Corollary (C.6) (ii) (Appendix C) shows that *the functor of k-algebras*

$$A \to \mathbf{Pic}_{X/R}(\mathbf{W}_N(A))$$

is the étale sheaf associated to the functor

$$A \to \mathrm{Pic}(X \otimes_R \mathbf{W}_N(A)).$$

Secondly, it follows, by Corollary (6.11), that *if* $A^p = A$, *then the canonical map*

$$\mathbf{Pic}_{X/R}(\mathbf{W}_N(A)) \to \mathbf{P}(A)$$

is bijective. (Let $f : X \to \mathrm{Spec}(\mathbf{W}(k))$ be obtained from $h : X \to \mathrm{Spec}(R) = \mathrm{Spec}(\mathbf{W}_N(k))$ in the obvious way; and note that when $A^p = A$, then $X \otimes_{\mathbf{W}_N(k)} \mathbf{W}_N(A) = X \otimes_{\mathbf{W}(k)} \mathbf{W}(A)$, and furthermore if B is an étale A-algebra then $B^p = B$ (Lemma (6.12)); consequently, with notation as in (6.11)

$$\mathbf{Pic}_{X/R}(\mathbf{W}_N(A)) = \mathbf{P}^{\acute{e}t}(A).)$$

Now, as in § 6, we set

$$k_1 = H^0(X, \mathcal{O}_X)_{\mathrm{red}} = H^0(X_{\mathrm{red}}, \mathcal{O}_{X_{\mathrm{red}}})$$

(cf. proof of Lemma (6.6)), and for any k-algebra A, we set

$$X_A = X \otimes_{\mathbf{W}(k)} \mathbf{W}(A) \,(= X \otimes_R \mathbf{W}_N(A) \quad \text{when} \quad A^p = A)$$
$$\overline{A} = A_{\mathrm{red}} = A/(\text{nilradical of } A).$$

Theorem **(7.5).** — *If* $A^p = A$, *then we have an exact* A*-functorial sequence*

$$0 \to \mathrm{Pic}(k_1 \otimes_k \overline{A}) \to \mathrm{Pic}(X_A) \to \mathbf{P}(A)$$
$$\to \mathrm{Br}(k_1 \otimes_k \overline{A}) \to \mathrm{Br}(X_A).$$

Proof. — In view of the preceding remarks, (7.5) follows from (7.4) (with $R = \mathbf{W}_N(k)$, $T = \mathbf{W}_N(A)$) as soon as we can show that

$$H^0(X_A, \mathcal{O}_{X_A})_{\mathrm{red}} = k_1 \otimes_k \overline{A}.$$

But by Lemma (6.6) and its proof, we have a surjective homomorphism

$$H^0(X_A, \mathcal{O}_{X_A}) \to k_1 \otimes_k (\mathbf{W}(A)/(p)) = k_1 \otimes_k A,$$

with nilpotent kernel; hence we have a surjective map, with kernel consisting of nilpotent elements:

$$H^0(X_A, \mathcal{O}_{X_A}) \to k_1 \otimes_k \overline{A}.$$

Since k_1 and $\overline{A}$ are reduced, and k is perfect, therefore $k_1 \otimes_k \overline{A}$ is reduced, and the proof is complete.

Corollary **(7.6)**. — *Let* K *be as in Corollary* (6.10). *Then, for* $A^p = A$, *the cokernel of* $\mathrm{Pic}(X_A) \to \mathbf{P}(A)$ *is naturally contained in the kernel of* $\mathrm{Br}(k_1 \otimes_k \overline{A}) \to \mathrm{Br}(k_1 \otimes_k K \otimes_k \overline{A})$.

Remark **(7.7)**. — The main results of § 6 are all easy consequences of Theorem (7.5). (But note that (6.11)—and hence, *via* (6.7), (6.2)—was used in the proof of (7.5)!)

III. — SOME LIE ALGEBRAS

8. The Lie algebra of H; conditions for H and P to be smooth.

The main result of § 8 is Theorem (8.1), which describes completely the Lie algebra of the Greenberg module defined in Theorem (2.4). We deduce sufficient conditions for the smoothness of **P** (Proposition (8.5)); in case X is a scheme over k (i.e. $p\mathcal{O}_X = (0)$), these conditions reduce to the classical condition $H^2(X, \mathcal{O}_X) = (0)$.

Let i be a fixed integer, let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-Module, and let $\mathbf{H} = \mathbf{H}(X, \mathcal{F}, i)$ be the *fpqc sheaf* associated to the functor $H^i(X_A, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_A})$ of k-algebras A. As in § 4, **H** is a Greenberg module, and (Corollary (4.4)) $\mathbf{H}(A) = H^i(X_A, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_A})$ whenever $A^p = A$. By Proposition (A.3) (Appendix A) the dimension of **H** as a k-scheme is equal to the length of the $\mathbf{W}(k)$-module $\mathbf{H}(k) = H^i(X, \mathcal{F})$. (But in general **H** is *not* a reduced scheme, cf. Corollary (8.4).) As in Appendix B, we have a natural grading

$$\mathrm{Lie}(\mathbf{H}) = \bigoplus_{t \geq 0} \mathrm{Lie}^{p^t}(\mathbf{H}).$$

For any k-vector space V, and $t \geq 0$, $V^{(-t)}$ will denote the k-vector space with the same underlying abelian group as V, the $V^{(-t)}$-product of $a \in k$ and $v \in V$ being $a^{p^{-t}}v$ (the product, in the vector space V, of $a^{p^{-t}}$ and v). Any basis of V is also a basis of $V^{(-t)}$, so V and $V^{(-t)}$ are isomorphic (but not canonically!). $V \mapsto V^{(-t)}$ is clearly an exact functor.

In the next theorem, we will refer to the following canonical commutative diagram of $\mathbf{W}(k)$-module homomorphisms (where $H^i(\mathcal{F}) = H^i(X, \mathcal{F})$, etc.):

$$
\begin{array}{ccccccc}
 & & \mathrm{H}^i(p^{t+1}\mathcal{F}) & = & \mathrm{H}^i(p^{t+1}\mathcal{F}) & & \\
 & & \downarrow & & \downarrow{\scriptstyle \lambda_{t+1}} & & \\
\mathrm{H}^{i-1}(\mathcal{F}/p^t\mathcal{F}) & \longrightarrow & \mathrm{H}^i(p^t\mathcal{F}) & \xrightarrow{\lambda_t} & \mathrm{H}^i(\mathcal{F}) & \xrightarrow{\sigma_t} & \mathrm{H}^i(\mathcal{F}/p^t\mathcal{F}) \\
\| & & \downarrow{\scriptstyle \nu_t} & & \downarrow{\scriptstyle \sigma_{t+1}} & & \| \\
\mathrm{H}^{i-1}(\mathcal{F}/p^t\mathcal{F}) & \xrightarrow[\delta_t]{} & \mathrm{H}^i(p^t\mathcal{F}/p^{t+1}\mathcal{F}) & \xrightarrow[\mu_t]{} & \mathrm{H}^i(\mathcal{F}/p^{t+1}\mathcal{F}) & \xrightarrow[\rho_t]{} & \mathrm{H}^i(\mathcal{F}/p^t\mathcal{F})
\end{array}
$$

Theorem **(8.1)** (1). — *With the preceding notation, there exists a natural (functorial in* X, $\mathcal{F}$*) isomorphism of k-vector spaces*

(i) $$\varphi : \mathrm{Lie}^{p^t}(\mathbf{H}) \xrightarrow{\approx} \mathrm{im}(\mu_t)^{(-t)} = \ker(\rho_t)^{(-t)}$$

which induces

(ii) $$\mathrm{Lie}^{p^t}(\mathbf{H}_{\mathrm{red}}) \xrightarrow{\approx} \mathrm{im}(\mu_t \circ \nu_t)^{(-t)} = (\mathrm{im}(\lambda_t)/\mathrm{im}(\lambda_{t+1}))^{(-t)}$$

and

(iii) $$\mathrm{Lie}^{p^t}(\mathbf{H}/\mathbf{H}_{\mathrm{red}}) = \mathrm{Lie}^{p^t}(\mathbf{H})/\mathrm{Lie}^{p^t}(\mathbf{H}_{\mathrm{red}}) \xrightarrow{\approx} \mathrm{coker}(\nu_t)^{(-t)}$$

(Note that $\mathrm{im}(\mu_t)$ (= image of μ_t) is a $\mathbf{W}(k)$-module annihilated by p, hence a k-vector space, so the notation makes sense. Note also that $\mathbf{H}_{\mathrm{red}}$ is a Greenberg submodule of $\mathbf{H}$, cf. proof of Proposition (A.3).)

Proof. — To begin with, the equality in (ii) holds because

$$\mathrm{im}(\lambda_t) \supseteq \mathrm{im}(\lambda_{t+1}) = \ker(\sigma_{t+1})$$

and so $$\mathrm{im}(\mu_t \circ \nu_t) = \mathrm{im}(\sigma_{t+1} \circ \lambda_t) \cong \mathrm{im}(\lambda_t)/\mathrm{im}(\lambda_{t+1}).$$

The equality in (iii) will come out explicitly from the proof of (ii) (2). For the isomorphism in (iii), once (i) and (ii) have been established we need only note that

$$\ker(\mu_t) = \mathrm{im}(\delta_t) \subseteq \mathrm{im}(\nu_t)$$

so that $$\mathrm{im}(\mu_t)/\mathrm{im}(\mu_t \circ \nu_t) \cong \mathrm{coker}(\nu_t).$$

Let us now define the isomorphism φ in (i). Let $\mathfrak{U}$ be a finite affine open covering of X, and let $\mathrm{C}_\bullet = \mathrm{C}_\bullet(\mathfrak{U}, \mathcal{F})$ be the corresponding Čech complex. As in § 4, there

---

(1) A more comprehensive statement describing **H** itself is given in remark (8.8) below.

(2) More generally, with the notation and assumptions of Lemma (6.5), if **E** is also *smooth*, then

$$\mathrm{H}^1(k, \mathbf{E}) = \mathrm{H}^1(k[\varepsilon], \mathbf{E}) = (0) \qquad (\varepsilon^2 = 0),$$

and it follows easily that $\mathrm{Lie}(\mathbf{I}) = \mathrm{Lie}(\mathbf{H})/\mathrm{Lie}(\mathbf{E})$.

exists a complex $Q_{\bullet}$ of finite-length $\mathbf{W}(k)$-modules, and a $\mathbf{W}(k)$-homomorphism of complexes $u : Q_{\bullet} \to C_{\bullet}$ which induces, for k-algebras A, an A-functorial isomorphism of $\mathbf{W}(A)$-modules

$$H_{-i}(Q_{\bullet} \otimes_{\mathbf{W}(k)} \mathbf{W}(A)) \xrightarrow{\approx} H_{-i}(C_{\bullet} \otimes_{\mathbf{W}(k)} \mathbf{W}(A)) = H^i(X_A, \mathscr{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_A}).$$

Furthermore, the construction of $Q_{\bullet}$ and u is such that the homology maps induced by u

$$H_{-i}(p^r Q_{\bullet}) \to H_{-i}(p^r C_{\bullet}) \qquad (r \geq 0)$$

are *isomorphisms* (cf. first two paragraphs in the proof of Proposition (3.1)). Using the "five-lemma", we deduce isomorphisms, for $s \geq r \geq 0$

(8.2) $$H_{-i}(p^r Q_{\bullet}/p^s Q_{\bullet}) \xrightarrow{\approx} H_{-i}(p^r C_{\bullet}/p^s C_{\bullet}) = H^i(X, p^r \mathscr{F}/p^s \mathscr{F})$$

where the equality holds because

$$p^r C_{\bullet}/p^s C_{\bullet} \cong C_{\bullet}(\mathfrak{U}, p^r \mathscr{F}/p^s \mathscr{F}).$$

For convenience, we denote the piece

$$Q_{-i+1} \to Q_{-i} \to Q_{-i-1}$$

of $Q_{\bullet}$ by $E \xrightarrow{\alpha} F \xrightarrow{\beta} G$ so that, with $I = \alpha(E)$, (8.2) becomes:

(8.3) $$(p^r F \cap \beta^{-1}(p^s G))/(p^s F + p^r I) \xrightarrow{\approx} H^i(X, p^r \mathscr{F}/p^s \mathscr{F}) \qquad (s \geq r \geq 0).$$

As in the proof of (4.1), we see that if $F' = F/I$, so that there is a natural map $\beta' : F' \to G$ induced by β, then our functor $\mathbf{H}$ is just the kernel of the corresponding map of Greenberg modules $\boldsymbol{\beta}' : \mathbf{F}' \to \mathbf{G}$. In view of Proposition (B.2) (Appendix B), we have therefore the isomorphisms

$$\begin{aligned} \mathrm{Lie}^{p^t}(\mathbf{H}) &\cong \ker((p^t F'/p^{t+1} F')^{(-t)} \to (p^t G/p^{t+1} G)^{(-t)}) \\ &\cong (\ker((p^t F + I)/(p^{t+1} F + I) \to p^t G/p^{t+1} G))^{(-t)} \\ &= ((p^t F \cap \beta^{-1}(p^{t+1} G) + I)/(p^{t+1} F + I))^{(-t)} \\ &\cong \mathrm{im}(\mu_t)^{(-t)} \end{aligned}$$

where the last isomorphism results easily from (8.3). This gives us the desired φ (depending, for the moment, on the choice of $\mathfrak{U}$, $Q_{\bullet}$, and u).

To check (ii), we set $J = \beta(F) = \beta'(F')$ and show that

$$\mathbf{H}_{\mathrm{red}} = \ker(\mathbf{F}' \to \mathbf{J}).$$

(Replacing G by J in the preceding paragraph, we see then that

$$\varphi(\mathrm{Lie}^{p^t}(\mathbf{H}_{\mathrm{red}})) = ((p^t F \cap (p^{t+1} F + \beta^{-1}(0)) + I)/(p^{t+1} F + I))^{(-t)}$$

whence, using (8.3), we find that

$$\varphi(\mathrm{Lie}^{p^t}(\mathbf{H}_{\mathrm{red}})) = \mathrm{im}(\mu_t \circ \nu_t)^{(-t)}$$

as desired.) Setting $K = \ker(\beta')$, we obtain, for any k-algebra A, an exact A-functorial sequence

$$K \otimes_{\mathbf{W}(k)} \mathbf{W}(A) \to F' \otimes_{\mathbf{W}(k)} \mathbf{W}(A) \to J \otimes_{\mathbf{W}(k)} \mathbf{W}(A) \to 0$$

and then, upon passage to associated *fpqc* sheaves, an exact sequence

$$\mathbf{K} \to \mathbf{F}' \to \mathbf{J} \to 0.$$

So the kernel $\mathbf{H}_*$ of $\mathbf{F}' \to \mathbf{J}$, being a homomorphic image of the smooth Greenberg module $\mathbf{K}$, is itself *smooth*. Moreover, since $\mathbf{H} = \ker(\mathbf{F}' \to \mathbf{G})$, we have that

$$\mathbf{H}/\mathbf{H}_* \cong \ker(\mathbf{J} \to \mathbf{G}).$$

But since $\mathbf{J}(k) \to \mathbf{G}(k)$ is just the inclusion map of J into G, we have that

$$(\mathbf{H}/\mathbf{H}_*)(k) = (0)$$

so that $\mathbf{H}/\mathbf{H}_*$ is *infinitesimal* [**DG**, p. 601, (1.2)]. Thus $\mathbf{H}_* = \mathbf{H}_{\mathrm{red}}$, and (ii) is proved.

Finally, we have now, as above

$$\begin{aligned}\mathrm{Lie}^{p^t}(\mathbf{H}/\mathbf{H}_{\mathrm{red}}) = \mathrm{Lie}^{p^t}(\mathbf{H}/\mathbf{H}_*) &= \ker((p^t\mathrm{J}/p^{t+1}\mathrm{J})^{(-t)} \to (p^t\mathrm{G}/p^{t+1}\mathrm{G})^{(-t)}) \\ &= \ker((p^t\mathrm{F}'/p^{t+1}\mathrm{F}')^{(-t)} \to (p^t\mathrm{G}/p^{t+1}\mathrm{G})^{(-t)})/\ker((p^t\mathrm{F}'/p^{t+1}\mathrm{F}')^{(-t)} \\ &\to (p^t\mathrm{J}/p^{t+1}\mathrm{J})^{(-t)}) \\ &= \mathrm{Lie}^{p^t}(\mathbf{H})/\mathrm{Lie}^{p^t}(\mathbf{H}_{\mathrm{red}}),\end{aligned}$$

and this completes the proof of (iii).

It remains to be shown that φ is natural. Let $(\mathrm{Y}, \mathscr{G})$ be a pair satisfying the same conditions as $(\mathrm{X}, \mathscr{F})$, let $\psi : \mathrm{Y} \to \mathrm{X}$ be a $\mathbf{W}(k)$-morphism, and let $\theta : \mathscr{F} \to \psi_*\mathscr{G}$ be a homomorphism of $\mathscr{O}_{\mathrm{X}}$-modules. Choose a finite affine open covering $\mathfrak{U}'$ of Y, and, as before, a map of complexes

$$u' : \mathrm{Q}'_\bullet \to \mathrm{C}'_\bullet = \mathrm{C}_\bullet(\mathfrak{U}', \mathscr{G}),$$

and let φ' be the isomorphism defined as above, but relative to $(\mathfrak{U}', \mathrm{Q}'_\bullet, u')$. Next, choose a finite affine open covering $\mathfrak{U}''$ of Y which refines *both* $\mathfrak{U}'$ *and* $\psi^{-1}(\mathfrak{U})$. Then, if $\mathrm{C}''_\bullet = \mathrm{C}_\bullet(\mathfrak{U}'', \mathscr{G})$ we get a $\mathbf{W}(k)$-homomorphism of complexes $\mathrm{C}'_\bullet \to \mathrm{C}''_\bullet$, unique up to homotopy. Similarly, via (ψ, θ), we get a map $\mathrm{C}_\bullet(= \mathrm{C}_\bullet(\mathfrak{U}, \mathscr{F})) \to \mathrm{C}''_\bullet$, unique up to homotopy. Hence we have a composed map

$$v : \mathrm{Q}_\bullet \oplus \mathrm{Q}'_\bullet \to \mathrm{C}_\bullet \oplus \mathrm{C}'_\bullet \to \mathrm{C}''_\bullet.$$

Now according to remark (3.9), we can choose $u'' : \mathrm{Q}''_\bullet \to \mathrm{C}''_\bullet$ satisfying the usual conditions, and such that furthermore $v = u'' \circ w$ for a suitable $w : \mathrm{Q}_\bullet \oplus \mathrm{Q}'_\bullet \to \mathrm{Q}''_\bullet$. So we have a diagram

$$\begin{array}{ccc} \mathrm{Q}_\bullet & \longrightarrow & \mathrm{C}_\bullet \\ \downarrow & & \downarrow \\ \mathrm{Q}''_\bullet & \longrightarrow & \mathrm{C}''_\bullet \\ \uparrow & & \uparrow \\ \mathrm{Q}'_\bullet & \longrightarrow & \mathrm{C}'_\bullet \end{array}$$

by means of which we can relate the isomorphisms φ, φ' to the isomorphism φ'' (defined as before, relative to $(\mathfrak{U}'', Q''_\bullet, u'')$). In this way we can reach the desired conclusion; details are left to the reader. Q.E.D.

Remark. — The following consequence of (8.1) seems worth noting. Is there a simple direct proof? *For each* $t \geq 0$, *let* $\mathbf{H}_t = \mathbf{H}(X, \mathscr{F}/p^t\mathscr{F}, i)$, *and let* $\mathbf{K}_t$ *be the kernel of the obvious map* $\mathbf{H}_{t+1} \to \mathbf{H}_t$. *Then the canonical sequence*

$$0 \to \mathrm{Lie}(\mathbf{K}_t) \to \mathrm{Lie}(\mathbf{H}_{t+1}) \to \mathrm{Lie}(\mathbf{H}_t) \to 0$$

is exact, and has a natural *splitting*; *furthermore we have natural isomorphisms*

$$\mathrm{Lie}(\mathbf{K}_t) = \mathrm{Lie}^{p^t}(\mathbf{H}) = (\mathbf{K}_t(k))^{(-t)}.$$

Corollary **(8.4)**. — **H** *is smooth (as a scheme) if and only if for all* $t > 0$ *the canonical map*

$$\sigma_t : \mathrm{H}^i(X, \mathscr{F}) \to \mathrm{H}^i(X, \mathscr{F}/p^t\mathscr{F})$$

is surjective.

Proof. — Since $\mathbf{H}/\mathbf{H}_{\mathrm{red}}$ is connected, therefore:

$\mathbf{H}/\mathbf{H}_{\mathrm{red}} = (0)$ (i.e. **H** is smooth)

$\Leftrightarrow \mathrm{Lie}(\mathbf{H}/\mathbf{H}_{\mathrm{red}}) = (0)$ [**DG**, p. 236, (1.4) (v)]

$\Leftrightarrow \nu_t$ is surjective for all $t \geq 0$ (cf. (8.1) (iii))

$\Leftrightarrow \mathrm{H}^{i+1}(p^{t+1}\mathscr{F}) \to \mathrm{H}^{i+1}(p^t\mathscr{F})$ is injective for all $t \geq 0$

$\Leftrightarrow \mathrm{H}^{i+1}(p^{t+1}\mathscr{F}) \to \mathrm{H}^{i+1}(\mathscr{F})$ is injective for all $t \geq 0$

$\Leftrightarrow \sigma_{t+1}$ is surjective for all $t \geq 0$. Q.E.D.

Proposition **(8.5)**. — **P** *is* smooth *if any one of the following (equivalent) conditions hold*:

(i) *For all* $t \geq 0$, $\mathrm{H}^2(p^t\mathcal{O}_X) = \mathrm{H}^2(\mathcal{O}_X/p^t\mathcal{O}_X) = (0)$.

(ii) *For all* $t \geq 0$, $\mathrm{H}^2(p^t\mathcal{O}_X/p^{t+1}\mathcal{O}_X) = (0)$.

(iii) *The scheme* $\mathbf{H}^1 = \mathbf{H}(X, \mathcal{O}_X, 1)$ *is smooth and the scheme* $\mathbf{H}^2 = \mathbf{H}(X, \mathcal{O}_X, 2)$ *is trivial* (i.e. *isomorphic to* $\mathrm{Spec}(k)$).

Remark. — If the conditions of Proposition (8.5) hold, then:

dimension of $\mathbf{P}$ = dimension of the k-vector space $\mathrm{Lie}(\mathbf{P})$

$= \lambda(\mathrm{H}^1(\mathcal{O}_X))$ (cf. Theorem (9.1)).

where "λ" denotes the length of a $\mathbf{W}(k)$-module.

Corollary **(8.6)**. — a) *If the dimension of* X *is* 1, *then* **P** *is smooth, of dimension* $\lambda(\mathrm{H}^1(\mathcal{O}_X))$.

b) *If the dimension of* X *is* 2, *and if* $\mathrm{H}^2(\mathcal{O}_X) = (0)$, *then* **P** *is smooth, of dimension* $\lambda(\mathrm{H}^1(\mathcal{O}_X))$.

(*Proof*: If dim. X=1, then condition (ii) of (8.5) clearly holds. The same is true if dim. X=2 and $\mathrm{H}^2(\mathcal{O}_X) = 0$, since $p^t\mathcal{O}_X/p^{t+1}\mathcal{O}_X$ is a homomorphic image of $\mathcal{O}_X$.)

Proof of (8.5). — We first show that (i), (ii), and (iii) are equivalent.

(i) ⇒ (ii): It suffices to show that for all s with $0 < s \leq t$

$$(*) \qquad H^2(p^s\mathcal{O}_X/p^{t+1}\mathcal{O}_X) \to H^2(p^{s-1}\mathcal{O}_X/p^{t+1}\mathcal{O}_X)$$

is *injective*; or, equivalently,

$$(**) \qquad H^1(p^{s-1}\mathcal{O}_X/p^{t+1}\mathcal{O}_X) \to H^1(p^{s-1}\mathcal{O}_X/p^s\mathcal{O}_X)$$

is *surjective*; since $(*)$ holds by assumption when t is large enough (so that $p^{t+1}\mathcal{O}_X=(0)$) so then does $(**)$, and it follows that $(**)$ holds for all t.

(ii) ⇒ (i): For $0 < s \leq t$ we have an exact sequence

$$H^2(p^s\mathcal{O}_X/p^t\mathcal{O}_X) \to H^2(p^{s-1}\mathcal{O}_X/p^t\mathcal{O}_X) \to H^2(p^{s-1}\mathcal{O}_X/p^s\mathcal{O}_X).$$

Hence, by induction on $t-s$, we find that

$$H^2(p^s\mathcal{O}_X/p^t\mathcal{O}_X)=(0).$$

(iii) ⇔ (i). Since $\dim.\mathbf{H}^2=\lambda(H^2(\mathcal{O}_X))$, Corollary (8.4) shows that $\mathbf{H}^2$ is trivial if and only if $H^2(\mathcal{O}_X/p^t\mathcal{O}_X)=(0)$ for all $t \geq 0$. We also deduce from (8.4) that $\mathbf{H}^1$ is smooth if and only if $H^2(p^t\mathcal{O}_X) \to H^2(\mathcal{O}_X)$ is injective for all $t \geq 0$; the equivalence of (i) and (iii) follows.

Now, assuming that (iii) holds, we show that $\mathbf{P}$ is smooth, i.e. for every k-algebra A and every A-ideal $\mathfrak{I}$ with $\mathfrak{I}^2=(0)$, the canonical map $\mathbf{P}(A) \to \mathbf{P}(A/\mathfrak{I})$ is *surjective* (cf. (for example) [**DG**, p. 238, (2.1) (vii)]).

Let us show, to begin with, that *if* $A^p=A$ *then the canonical map* $\mathrm{Pic}(X_A) \to \mathrm{Pic}(X_{A/\mathfrak{I}})$ *is surjective*. Indeed, if $\mathcal{J}$ is the kernel of $\mathcal{O}_{X_A} \to \mathcal{O}_{X_{A/\mathfrak{I}}}$, then we have an exact sequence

$$H^1(\mathcal{O}_{X_A}) \to H^1(\mathcal{O}_{X_{A/\mathfrak{I}}}) \to H^2(\mathcal{J}) \to H^2(\mathcal{O}_{X_A});$$

since $A^p=A$, this sequence can be written

$$\mathbf{H}^1(A) \xrightarrow{\pi} \mathbf{H}^1(A/\mathfrak{I}) \to H^2(\mathcal{J}) \to \mathbf{H}^2(A)$$

(cf. remark (4.4)); since $\mathbf{H}^1$ is smooth, π is surjective, and since $\mathbf{H}^2$ is trivial, $\mathbf{H}^2(A)=(0)$; thus $H^2(\mathcal{J})=0$. But $\mathcal{J}^2=(0)$ (Lemma (8.7) below) and so via the truncated exponential map we have an exact sequence

$$\mathrm{Pic}(X_A) \to \mathrm{Pic}(X_{A/\mathfrak{I}}) \to H^2(\mathcal{J}),$$

whence the assertion.

Next, consider the *fpqc* sheaf $\mathbf{P}_I$ defined on the category of A-algebras B by

$$\mathbf{P}_I(B)=\mathbf{P}(B/\mathfrak{I}B).$$

There is a canonical homomorphism of sheaves $\varphi: \mathbf{P} \to \mathbf{P}_I$, and, I claim, this is *surjective*. For, by (0.1) and (6.10) (i), we can find, for any B, a faithfully flat B-algebra $\overline{B}$ such that $\overline{B}^p=\overline{B}$, and such that there exists a commutative diagram, with exact rows

$$\begin{array}{ccccc} \mathrm{Pic}(X_{\overline{B}}) & \longrightarrow & \mathbf{P}(\overline{B}) & \longrightarrow & 0 \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \\ \mathrm{Pic}(X_{\overline{B}/\mathfrak{I}\overline{B}}) & \longrightarrow & \mathbf{P}_{\mathrm{I}}(\overline{B}) & \longrightarrow & 0 \end{array}$$

But, as we have just seen, α is surjective, whence so is β, so that indeed φ is surjective.

We wish, finally, to show that

$$\varphi(A) : \mathbf{P}(A) \to \mathbf{P}_{\mathrm{I}}(A)$$

is surjective. For this it will suffice to show that $H^1_{fpqc}(A, \mathbf{L}) = 0$, where $\mathbf{L} = \ker(\varphi)$. But for any A-algebra B, we have a canonical isomorphism

$$\mathbf{L}(B) \cong \mathfrak{I}B \otimes_k \mathrm{Lie}(\mathbf{P})$$

(cf. [**DG**, p. 208]). If, furthermore, B is *flat* over A, then clearly

$$\mathbf{L}(B) \cong B \otimes_A \mathbf{L}(A).$$

Since only flat A-algebras enter into the determination of $H^1_{fpqc}(A, \mathbf{L})$, descent theory shows that $H^1_{fpqc}(A, \mathbf{L}) = 0$, as desired [**SGA** 4, Exposé VII, remarque 4.5]. Q.E.D.

In the preceding proof, and also in § 9, we need:

Lemma (**8.7**). — *Let* A *be a k-algebra, let* $\mathfrak{I}$ *be an* A*-ideal with* $\mathfrak{I}^2 = (0)$, *and let* $\mathscr{J}$ *be the kernel of the natural map* $\mathcal{O}_{X_A} \to \mathcal{O}_{X_{A/\mathfrak{I}}}$. *Then*:

(i) $p\mathscr{J} = (0)$.

(ii) $\mathscr{J}^2 = (0)$.

Proof. — $\mathscr{J}$ is generated, as an $\mathcal{O}_{X_A}$-Ideal, by the kernel of $\mathbf{W}(A) \to \mathbf{W}(A/\mathfrak{I})$, *i.e.* by the $\mathbf{W}(A)$-ideal

$$\mathbf{W}(\mathfrak{I}) = \{(x_0, x_1, x_2, \dots) \in \mathbf{W}(A) \mid x_i \in \mathfrak{I} \text{ for all } i\}.$$

So it suffices to show that $p\mathbf{W}(\mathfrak{I}) = \mathbf{W}(\mathfrak{I})^2 = (0)$.

That $p\mathbf{W}(\mathfrak{I}) = (0)$ follows at once (since $\mathfrak{I}^2 = (0)$) from the identity

$$p(x_0, x_1, x_2, \dots) = (0, x_0^p, x_1^p, x_2^p, \dots).$$

Now if $(U_0, U_1, U_2, \dots)$, $(V_0, V_1, V_2, \dots)$ are Witt vectors with indeterminate entries, then their product can be expressed in the form

$$(U_0, U_1, U_2, \dots)(V_0, V_1, V_2, \dots) = (P_0(U, V), P_1(U, V), P_2(U, V), \dots)$$

where the $P_\nu(U, V)$ are polynomials in

$$U = U_0, U_1, U_2, \dots \quad \text{and} \quad V = V_0, V_1, V_2, \dots$$

such that

$$P_\nu(U, 0) = P_\nu(0, V) = 0 \qquad (\nu \geq 0);$$

thus each term in P_ν is divisible by at least one U_i and at least one V_j; specializing

$$U_i \to x_i, \quad V_j \to y_j \quad (x_i, y_j \in I)$$

we conclude, since $\mathfrak{J}^2 = (0)$, that

$$(x_0, x_1, x_2, \ldots)(y_0, y_1, y_2, \ldots) = (0, 0, 0, \ldots).$$

Hence $\mathbf{W}(\mathfrak{J})^2 = (0)$. Q.E.D.

* * *

Remark **(8.8)**. — What follows will not be needed elsewhere; but it serves to put the results of § 8 in better perspective.

There is a *structure theorem* for the category of Greenberg modules, due to C. Schoeller [**Sch**, § 5]. Schoeller's theorem can be reformulated in terms of *Dieudonné modules*. First, some notation. Let W be the ring $\mathbf{W}(k)$. For any $w = (w_0, w_1, w_2, \ldots) \in W$, and any integer t, we set

$$w^{(p^t)} = (w_0^{p^t}, w_1^{p^t}, w_2^{p^t}, \ldots).$$

The *Dieudonné ring* $\mathscr{D}$ (over k) is the (non-commutative) ring generated by W and by two indeterminates F and V subject to the relations

$$\begin{aligned} &Fw = w^{(p)}F, \quad wV = Vw^{(p)} \quad (w \in W) \\ &FV = VF = p. \end{aligned}$$

For a left $\mathscr{D}$-module S, a direct product decomposition of W-modules

$$S = \prod_{n \geq 0} S_n$$

will be called a *cograding* on S if there exist maps

$$V_{n+1} : S_{n+1} \to S_n, \quad F_n : S_n \to S_{n+1} \quad (n \geq 0)$$

such that for $(s_0, s_1, s_2, s_3, \ldots) \in S$, we have

$$\text{(8.9)} \quad \begin{aligned} &V(s_0, s_1, s_2, s_3, \ldots) = (V_1 s_1, V_2 s_2, V_3 s_3, \ldots) \\ &F(s_0, s_1, s_2, s_3, \ldots) = (0, F_0 s_0, F_1 s_1, F_2 s_2, \ldots). \end{aligned}$$

A *morphism* $\varphi : \prod_{n \geq 0} S_n \to \prod_{n \geq 0} T_n$ of such cograded $\mathscr{D}$-modules is a family of maps $\varphi_n : S_n \to T_n$ $(n \geq 0)$ satisfying certain obvious conditions. We say that the cograding on S is of *cofinite type* if S_n is a W-module of *finite length* for all n, and furthermore there is an n_0 such that V_{n+1} is *bijective* for $n \geq n_0$.

To each Greenberg module $\mathbf{M}$, we can associate a cograded $\mathscr{D}$-module $S(\mathbf{M})$ of cofinite type, as follows. Let K be the fraction field of W, and for any W-module M let M′ be the W-module

$$M' = \mathrm{Hom}_W(M, K/W).$$

Also, for any integer t, let $M^{(t)}$ be the W-module with the same underlying abelian group as M, and with scalar multiplication $*$ given by

$$w * m = w^{(p^t)} m \quad (w \in W, \ m \in M).$$

$S(\mathbf{M})$ is defined to be the W-module

$$S(\mathbf{M}) = \prod_{n \geq 0} (\mathrm{Hom}_{\mathbf{W}}(\mathbf{M}, \mathbf{W_{n+1}})')^{(-n)} = \prod_{n \geq 0} S_n(\mathbf{M})$$

together with the maps F_n, V_{n+1} $(n \geq 0)$ given by

$$(F_n f)(\varphi) = f(\rho_{n+1, n+2} \circ \varphi)$$

($f \in S_n(\mathbf{M})$; $\varphi : \mathbf{M} \to \mathbf{W_{n+2}}$; and $\rho_{n+1,\, n+2} : \mathbf{W_{n+2}} \to \mathbf{W_{n+1}}$ is the truncation map)

$$(V_{n+1}g)(\psi) = g(\tau_{n+2,\, n+1} \circ \psi)$$

($g \in S_{n+1}(\mathbf{M})$; $\psi : \mathbf{M} \to \mathbf{W_{n+1}}$; and $\tau_{n+2,\, n+1} : \mathbf{W_{n+1}} \to \mathbf{W_{n+2}}$ is the unique map such that $\tau_{n+2,\, n+1} \circ \rho_{n+1,\, n+2}$ = multiplication by p in $\mathbf{W_{n+2}}$). Multiplication by F and V in S is defined by (8.9). S(**M**) varies functorially with **M**.

Structure theorem. — The functor S *is an* equivalence *from the category of Greenberg modules to the category of cograded* $\mathscr{D}$*-modules of cofinite type* [1].

Complements. — It can be shown that:

a) The kernel of $V : S(\mathbf{M}) \to S(\mathbf{M})$ is a $k[F]$ ($= \mathscr{D}/\mathscr{D}V$)-module, and (with $(V_0 : S_0 \to (0)) =$ null-map)

$$\ker(V) = \prod_{n \geq 0} \ker(V_n) = \bigoplus_{n \geq 0} \ker(V_n).$$

(Since S(**M**) is of cofinite type, $\ker(V_n) = (0)$ for large n.) There exist *natural isomorphisms* of k-vector spaces

$$\ker(V_t) \simeq \mathrm{Lie}^{p^t}(\mathbf{M}) \qquad (t \geq 0)$$

via which multiplication by F in ker(V) corresponds to the standard p-th-power operation in Lie(**M**) [**DG**, p. 273].

b) There is a natural map

$$\sigma_n : \mathbf{M}(k) \to S_n(\mathbf{M})$$

where, for $x \in \mathbf{M}(k)$ and $\varphi \in \mathrm{Hom}_{\mathbf{W}}(\mathbf{M}, \mathbf{W_{n+1}})$

$$(\sigma_n(x))(\varphi) = \text{image of } x \text{ under the composite map}$$

$$\mathbf{M}(k) \xrightarrow{\varphi(k)} \mathbf{W_{n+1}}(k) \xrightarrow{\approx} p^{-n-1}W/W \subseteq K/W.$$

For all $n \geq 0$, we have

$$S_n(\mathbf{M}_{\mathrm{red}}) = \sigma_n \mathbf{M}(k) \subseteq S_n(\mathbf{M}).$$

Now let us determine S(**H**) *for* **H** *as in Theorem* (8.1). If M is a W-module of finite length, and **M** is the associated Greenberg module, then (Proposition (A.1) (iii)), there are canonical isomorphisms

$$\mathrm{Hom}_{\mathbf{W}}(\mathbf{M}, \mathbf{W_{n+1}}) \xrightarrow{\sim} \mathrm{Hom}_W(M, \mathbf{W_{n+1}}(k)) \qquad (n \geq 0).$$

The isomorphisms

$$\mathbf{W_{n+1}}(k) \xrightarrow{\approx} p^{-n-1}W/W \subseteq K/W$$

give isomorphisms

$$\mathrm{Hom}_W(M, \mathbf{W_{n+1}}(k)) \xrightarrow{\approx} (M/p^{n+1}M)'.$$

From this we find that

$$S_n(\mathbf{M}) = (M/p^{n+1}M)^{(-n)}$$

$$V_{n+1} : M/p^{n+2}M \to M/p^{n+1}M \text{ is the canonical map}$$

$$F_n : M/p^{n+1}M \to M/p^{n+2}M \text{ is the map induced by multiplication by } p \text{ in } M.$$

Arguing as in the proof of Theorem (8.1) we deduce that

$$S_n(\mathbf{H}) = (H^i(\mathscr{F}/p^{n+1}\mathscr{F}))^{(-n)}$$

V being induced by the natural maps

$$\mathscr{F}/p^{n+1}\mathscr{F} \to \mathscr{F}/p^n\mathscr{F}$$

and F by the composed maps

$$\mathscr{F}/p^n\mathscr{F} \xrightarrow{p} p\mathscr{F}/p^{n+1}\mathscr{F} \hookrightarrow \mathscr{F}/p^{n+1}\mathscr{F}.$$

This, then, describes the "structure" of the Greenberg module **H**. Theorem (8.1) and Corollary (8.4) follow easily now from *a)* and *b)* above.

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[1] Cf. also remark (8.10) at the end of this section.

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(8.10) We end with an observation which may make the Structure Theorem seem more appealing. The W-module $S(\mathbf{M})'$ can be made into a $\mathcal{D}$-module in a natural way [**DG**, p. 622, (5.2 b)], and one can construct a *natural isomorphism of $\mathcal{D}$-modules* from $S(\mathbf{M})'$ to the $\mathcal{D}$-module $\mathcal{M}$ associated (as in [**DG**, chap. V, § 1, no. 4]) with the unipotent algebraic k-group underlying $\mathbf{M}$. Thus $\mathcal{M}=S(\mathbf{M})'$ is a $\mathcal{D}$-module of *finite type*; and there is furthermore a *grading of* W-*modules*

$$\mathcal{M}=S(M)'=\bigoplus_{n\geq 0} S_n(\mathbf{M})'=\bigoplus_{n\geq 0}\mathcal{M}'_n$$

such that multiplication by F (resp. V) in $\mathcal{M}$ takes $\mathcal{M}'_n$ into $\mathcal{M}'_{n+1}$ (resp. $\mathcal{M}'_{n+1}$ into $\mathcal{M}'_n$). So we have also an *anti-equivalence* from the category of Greenberg modules to the category of graded (as above) $\mathcal{D}$-modules of finite type.

Informally speaking, putting a *Greenberg module structure* on a unipotent algebraic k-group $\mathbf{U}$ is equivalent to putting a *grading* (as above) on the $\mathcal{D}$-module associated with $\mathbf{U}$.

9. The dimension of Lie(P).

This section is devoted entirely to the proof of:

Theorem (**9.1**). — *For each* $t\geq 0$, *let* ν_t *be the canonical map*

$$H^1(X, p^t\mathcal{O}_X)\to H^1(X, p^t\mathcal{O}_X/p^{t+1}\mathcal{O}_X).$$

Let " λ " *denote the length of a* $\mathbf{W}(k)$-*module. With this notation, the dimension of the k-vector space* Lie$(\mathbf{P})$ *is*

$$\lambda(H^1(X, \mathcal{O}_X))+\sum_{t\geq 0}\lambda(\operatorname{coker}(\nu_t)).$$

Proof. — For any k-vector space V, let $\dim_k(V)$ be the dimension of V over k. Note that $\dim_k(V)=\dim_k(V^{(-t)})$ (cf. paragraph preceding Theorem (8.1)). Taking $i=1$, $\mathcal{F}=\mathcal{O}_X$ in Theorem (8.1), so that $\mathbf{H}=\mathbf{H}(X, \mathcal{O}_X, 1)$, we see that

$$\begin{aligned}\dim_k(\operatorname{Lie}(\mathbf{H}))&=\dim_k(\operatorname{Lie}(\mathbf{H}_{red}))+\dim_k(\operatorname{Lie}(\mathbf{H}/\mathbf{H}_{red}))\\&=\lambda(H^1(X, \mathcal{O}_X))+\sum_{t\geq 0}\lambda(\operatorname{coker}(\nu_t)).\end{aligned}$$

(The fact that $\dim_k(\operatorname{Lie}(\mathbf{H}_{red}))=\lambda(H^1(X, \mathcal{O}_X))$ can also be established by noting that $\mathbf{H}_{red}$ is smooth, of dimension $\lambda(H^1(X, \mathcal{O}_X))$, cf. beginning of § 8.) So Theorem (9.1) asserts that Lie$(\mathbf{P})$ *and* Lie$(\mathbf{H})$ *have the same dimension* (where, again, $\mathbf{H}$ is the *fpqc* sheaf associated to the functor $H^1(X_A, \mathcal{O}_{X_A})$ of k-algebras A) ([1]).

Let X′ be the closed subscheme of X defined by the $\mathcal{O}_X$-Ideal $p\mathcal{O}_X$. Let $\mathbf{P}'$ (resp. $\mathbf{H}'$) be the *fpqc* sheaf associated to the functor $\operatorname{Pic}(X'_A)$ (resp. $H^1(X'_A, \mathcal{O}_{X'_A})$) of k-algebras A ($X'_A=X'\otimes_{\mathbf{W}(k)}\mathbf{W}(A)$). X′ is a scheme over k, and, as in the middle of § 2, there is an isomorphism of functors

$$\mathbf{P}'\to\mathbf{Pic}_{X'/k}.$$

([1]) A more satisfying result would be that there is a *natural isomorphism* between Lie$(\mathbf{P})$ and Lie$(\mathbf{H})$. I have not been able to prove—or disprove—this. (Cf. however remark *b)* at the end of § 9.)

It is well-known (and not hard to prove) that there is an isomorphism of k-vector spaces $\mathrm{Lie}(\mathbf{Pic}_{X'/k}) \cong H^1(X', \mathcal{O}_{X'})$. Similarly (or by Theorem (8.1)) $\mathrm{Lie}(\mathbf{H}') \cong H^1(X', \mathcal{O}_{X'})$. Thus

$$\text{(9.2)} \qquad \dim_k(\mathrm{Lie}(\mathbf{H}')) = \dim_k(\mathrm{Lie}(\mathbf{P}')).$$

(Actually, a proof of (9.2) will fall out at the end of the proof of (9.5) and (9.6) below.)

From the exact sequences

$$0 \to p\mathcal{O}_{X_A} \to \mathcal{O}_{X_A} \to \mathcal{O}_{X'_A} \to 0$$
$$1 \to 1 + p\mathcal{O}_{X_A} \to \mathcal{O}^*_{X_A} \to \mathcal{O}^*_{X'_A} \to 1$$

we deduce the exact sequences of *fpqc* sheaves

$$\text{(9.3)} \qquad \begin{array}{l} 0 \to \mathbf{C} \to \mathbf{D} \to \mathbf{H} \to \mathbf{H}' \\ 0 \to \mathbf{C}^* \to \mathbf{D}^* \to \mathbf{P} \to \mathbf{P}' \end{array}$$

where $\mathbf{C}$, $\mathbf{D}$, $\mathbf{C}^*$, $\mathbf{D}^*$ are *fpqc* sheaves associated to certain functors of k-algebras A, viz:

$\mathbf{C}$ — to the cokernel of $H^0(\mathcal{O}_{X_A}) \to H^0(\mathcal{O}_{X'_A})$

$\mathbf{D}$ — to $H^1(p\mathcal{O}_{X_A})$

$\mathbf{C}^*$ — to the cokernel of $H^0(\mathcal{O}^*_{X_A}) \to H^0(\mathcal{O}^*_{X'_A})$

$\mathbf{D}^*$ — to $H^1(1 + p\mathcal{O}_{X_A})$.

Arguing as in § 2, we find that $\mathbf{C}$, $\mathbf{D}$, $\mathbf{C}^*$, $\mathbf{D}^*$ are all affine algebraic k-groups. We will show below that:

(9.4) $\mathbf{C}$ *and* $\mathbf{C}^*$ *are smooth algebraic k-groups, and their dimensions are equal* (*whence* $\dim_k(\mathrm{Lie}(\mathbf{C})) = \dim_k(\mathrm{Lie}(\mathbf{C}^*))$).

(9.5) $\mathrm{Lie}(\mathbf{D})$ *and* $\mathrm{Lie}(\mathbf{D}^*)$ *are naturally isomorphic.*

(9.6) *The following two sequences of k-vector spaces* (derived from (9.3)) *are exact*:

$$0 \to \mathrm{Lie}(\mathbf{C}) \to \mathrm{Lie}(\mathbf{D}) \to \mathrm{Lie}(\mathbf{H}) \to \mathrm{Lie}(\mathbf{H}') \to 0$$
$$0 \to \mathrm{Lie}(\mathbf{C}^*) \to \mathrm{Lie}(\mathbf{D}^*) \to \mathrm{Lie}(\mathbf{P}) \to \mathrm{Lie}(\mathbf{P}') \to 0.$$

In view of (9.2), (9.4) and (9.5), (9.6) implies that

$$\dim_k(\mathrm{Lie}(\mathbf{H})) = \dim_k(\mathrm{Lie}(\mathbf{P}))$$

as required.

Proof of (9.4). — Let $\mathcal{N}$ be the Nilradical of $\mathcal{O}_X$, and let $X_1 = X_{red} = X'_{red}$. We have a commutative A-functorial diagram

$$\text{(9.7)} \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & H^0(\mathcal{N}\mathcal{O}_{X_A}) & \longrightarrow & H^0(\mathcal{O}_{X_A}) & \longrightarrow & H^0(\mathcal{O}_{X_{1,A}}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & H^0(\mathcal{N}\mathcal{O}_{X'_A}) & \longrightarrow & H^0(\mathcal{O}_{X'_A}) & \longrightarrow & H^0(\mathcal{O}_{X_{1,A}}) & \longrightarrow & 0 \end{array}$$

with exact rows (cf. Lemma (6.6)), from which we see that $\mathbf{C}$ is also the *fpqc* sheaf associated to the cokernel of $H^0(\mathcal{N}\mathcal{O}_{X_A}) \to H^0(\mathcal{N}\mathcal{O}_{X'_A})$. So "sheafification" of the exact sequence

$$0 \to H^0(p\mathcal{O}_{X_A}) \to H^0(\mathcal{N}\mathcal{O}_{X_A}) \to H^0(\mathcal{N}\mathcal{O}_{X'_A})$$

gives rise to an exact sequence of *fpqc* sheaves

$$(9.8) \qquad 0 \to \mathbf{F} \to \mathbf{G} \to \mathbf{G}' \to \mathbf{C} \to 0.$$

Each of $\mathbf{F}$, $\mathbf{G}$, $\mathbf{G}'$ *is an affine algebraic k-group*. (To see that $\mathbf{G}$ is affine algebraic, sheafify the first row of (9.7) and use Theorem (2.4); for $\mathbf{G}'$ do the same with the second row of (9.7); finally, note that $\mathbf{F}$ is the kernel of $\mathbf{G} \to \mathbf{G}'$.) Furthermore, $\mathbf{G}'$ is *smooth*: indeed, since $X'_A = X' \otimes_k \mathbf{W}(A)/(p)$, we have

$$\begin{aligned} H^0(\mathcal{N}\mathcal{O}_{X'_A}) &= H^0(\mathcal{N}\mathcal{O}_{X'}) \otimes_k \mathbf{W}(A)/(p) \\ &= H^0(\mathcal{N}\mathcal{O}_{X'}) \otimes_{\mathbf{W}(k)} \mathbf{W}(A); \end{aligned}$$

so $\mathbf{G}'$ is isomorphic, as a scheme, to some affine space (Proposition (A.1) (i), Appendix A). Since $\mathbf{C}$ is a quotient of $\mathbf{G}'$, $\mathbf{C}$ *is smooth*. From (9.8), we have

$$\dim(\mathbf{C}) = \dim(\mathbf{F}) - \dim(\mathbf{G}) + \dim(\mathbf{G}').$$

Similarly, we have an exact sequence

$$0 \to \mathbf{F}^* \to \mathbf{G}^* \to \mathbf{G}'^* \to \mathbf{C}^* \to 0$$

where $\mathbf{F}^*$ (resp. $\mathbf{G}^*$, resp. $\mathbf{G}'^*$) is the *fpqc* sheaf associated to the (multiplicative) group-functor $H^0(1+p\mathcal{O}_{X_A})$ (resp. $H^0(1+\mathcal{N}\mathcal{O}_{X_A})$, resp. $H^0(1+\mathcal{N}\mathcal{O}_{X'_A})$). As *set* functors, $H^0(p\mathcal{O}_{X_A})$ and $H^0(1+p\mathcal{O}_{X_A})$ are clearly isomorphic; hence, *as k-schemes*, $\mathbf{F}$ and $\mathbf{F}^*$ are isomorphic. Similarly, $\mathbf{G} \cong \mathbf{G}^*$, and $\mathbf{G}' \cong \mathbf{G}'^*$. So $\mathbf{G}'^*$ is smooth, and as above, we deduce that $\mathbf{C}^*$ *is smooth*; furthermore,

$$\begin{aligned} \dim(\mathbf{C}^*) &= \dim(\mathbf{F}^*) - \dim(\mathbf{G}^*) + \dim(\mathbf{G}'^*) \\ &= \dim(\mathbf{F}) - \dim(\mathbf{G}) + \dim(\mathbf{G}') \\ &= \dim(\mathbf{C}), \end{aligned}$$

and this proves (9.4). (Note: Since Lie($\mathbf{C}$) is isomorphic to the Zariski tangent space at the origin of $\mathbf{C}$ (cf. Appendix B), and since $\mathbf{C}$ is smooth, we have

$$\dim_k(\mathrm{Lie}(\mathbf{C})) = \dim(\mathbf{C}),$$

and similarly for $\mathbf{C}^*$.)

Proofs of (9.5), (9.6), *and* (9.2). — Let A be a k-algebra and let $\mathfrak{J}$ be a *non-zero* ideal in A such that $\mathfrak{J}^2 = (0)$. Assume that:

(i) $A^p = A$ (whence $(A/\mathfrak{J})^p = A/\mathfrak{J}$), and
(ii) the canonical maps $\mathrm{Pic}(X_A) \to \mathbf{P}(A)$, $\mathrm{Pic}(X'_A) \to \mathbf{P}'(A)$,

are surjective, with kernels isomorphic to

$$\mathrm{Pic}(\mathrm{Spec}(k_1 \otimes_k \mathrm{A})), \qquad \text{where} \qquad k_1 = \mathrm{H}^0(\mathrm{X}, \mathcal{O}_{\mathrm{X}_{\mathrm{red}}}) = \mathrm{H}^0(\mathrm{X}', \mathcal{O}_{\mathrm{X}'_{\mathrm{red}}});$$

and similarly for $\mathrm{Pic}(\mathrm{X}_{\mathrm{A}/\mathfrak{I}}) \to \mathbf{P}(\mathrm{A}/\mathfrak{I})$, $\mathrm{Pic}(\mathrm{X}'_{\mathrm{A}/\mathfrak{I}}) \to \mathbf{P}'(\mathrm{A}/\mathfrak{I})$.

(For example, if K is a normal algebraic field extension of k such that every connected component of X has a K-rational point, if B is a k-algebra and $\mathfrak{J} \neq (0)$ [1] is a B-ideal with $\mathfrak{J}^2 = (0)$, then we can take A to be a faithfully flat $\mathrm{B}\otimes_k \mathrm{K}$-algebra with $\mathrm{A}^p = \mathrm{A}$ (Lemma (0.1)), and $\mathfrak{I}$ to be $\mathfrak{J}\mathrm{A}$. Then (ii) follows from (6.9) and (6.7).)

For convenience, set

$$\mathcal{O}_{\mathrm{A}} = \mathcal{O}_{\mathrm{X}_{\mathrm{A}}} \qquad \mathcal{O}_{\mathrm{A}/\mathfrak{I}} = \mathcal{O}_{\mathrm{X}_{\mathrm{A}/\mathfrak{I}}}$$
$$\mathcal{O}'_{\mathrm{A}} = \mathcal{O}_{\mathrm{X}'_{\mathrm{A}}} \qquad \mathcal{O}'_{\mathrm{A}/\mathfrak{I}} = \mathcal{O}_{\mathrm{X}'_{\mathrm{A}/\mathfrak{I}}}$$

and let $\mathcal{O}^*_{\mathrm{A}}$, $\mathcal{O}'^*_{\mathrm{A}}$, etc., be the corresponding sheaves of units. (These are all sheaves on the topological space underlying X_{A}.)

For any locally algebraic k-group $\mathbf{Q}$ there exists a natural isomorphism

$$\mathrm{Lie}(\mathbf{Q}) \otimes_k \mathfrak{I} \xrightarrow{\approx} \ker(\mathbf{Q}(\mathrm{A}) \to \mathbf{Q}(\mathrm{A}/\mathfrak{I}))$$

(cf. [**DG**, p. 208]). Our first chore, which will be rather dreary, is to show that *with* (A, $\mathfrak{I}$) *as above we have natural isomorphisms*:

a) $\mathrm{Lie}(\mathbf{D}) \otimes_k \mathfrak{I} \cong \ker(\mathrm{H}^1(p\mathcal{O}_{\mathrm{A}}) \to \mathrm{H}^1(p\mathcal{O}_{\mathrm{A}/\mathfrak{I}}))$
b) $\mathrm{Lie}(\mathbf{H}) \otimes_k \mathfrak{I} \cong \ker(\mathrm{H}^1(\mathcal{O}_{\mathrm{A}}) \to \mathrm{H}^1(\mathcal{O}_{\mathrm{A}/\mathfrak{I}}))$
c) $\mathrm{Lie}(\mathbf{H}') \otimes_k \mathfrak{I} \cong \ker(\mathrm{H}^1(\mathcal{O}'_{\mathrm{A}}) \to \mathrm{H}^1(\mathcal{O}'_{\mathrm{A}/\mathfrak{I}}))$
d) $\mathrm{Lie}(\mathbf{D}^*) \otimes_k \mathfrak{I} \cong \ker(\mathrm{H}^1(1 + p\mathcal{O}_{\mathrm{A}}) \to \mathrm{H}^1(1 + p\mathcal{O}_{\mathrm{A}/\mathfrak{I}}))$
e) $\mathrm{Lie}(\mathbf{P}) \otimes_k \mathfrak{I} \cong \ker(\mathrm{Pic}(\mathrm{X}_{\mathrm{A}}) \to \mathrm{Pic}(\mathrm{X}_{\mathrm{A}/\mathfrak{I}}))$
f) $\mathrm{Lie}(\mathbf{P}') \otimes_k \mathfrak{I} \cong \ker(\mathrm{Pic}(\mathrm{X}'_{\mathrm{A}}) \to \mathrm{Pic}(\mathrm{X}'_{\mathrm{A}/\mathfrak{I}}))$.

b) holds because $\mathrm{H}^1(\mathcal{O}_{\mathrm{A}}) = \mathbf{H}(\mathrm{A})$ and $\mathrm{H}^1(\mathcal{O}_{\mathrm{A}/\mathfrak{I}}) = \mathbf{H}(\mathrm{A}/\mathfrak{I})$, cf. (4.4). *c)* holds for a similar reason. As for *a)*, we have (cf. proof of (9.4)) a commutative diagram

$$\begin{array}{ccccccccc}
\mathrm{H}^0(\mathcal{N}\mathcal{O}_{\mathrm{A}}) & \longrightarrow & \mathrm{H}^0(\mathcal{N}\mathcal{O}'_{\mathrm{A}}) & \longrightarrow & \mathrm{H}^1(p\mathcal{O}_{\mathrm{A}}) & \longrightarrow & \mathrm{H}^1(\mathcal{O}_{\mathrm{A}}) & \longrightarrow & \mathrm{H}^1(\mathcal{O}'_{\mathrm{A}}) \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle\gamma} & & \downarrow & & \downarrow \\
\mathbf{G}(\mathrm{A}) & \xrightarrow{\ \alpha\ } & \mathbf{G}'(\mathrm{A}) & \xrightarrow{\ \beta\ } & \mathbf{D}(\mathrm{A}) & \longrightarrow & \mathbf{H}(\mathrm{A}) & \longrightarrow & \mathbf{H}'(\mathrm{A})
\end{array}$$

in which the vertical arrows arise from canonical maps of functors (of k-algebras A) into their associated *fpqc* sheaves. All these vertical arrows, except possibly for γ, are bijective (cf. (4.4) and its proof, and note, for example, that $\mathrm{H}^0(\mathcal{N}\mathcal{O}_{\mathrm{X}_{\mathrm{A}}})$ is the kernel of $\mathrm{H}^0(\mathcal{O}_{\mathrm{X}_{\mathrm{A}}}) \to \mathrm{H}^0(\mathcal{O}_{\mathrm{X}_{\mathrm{red},\mathrm{A}}}) \ldots$); *what we need to show is that* γ *is bijective too.* (A similar

[1] Here—and below—distinguish between "$\mathfrak{I}$" (gothic "I") and "$\mathfrak{J}$" (gothic "J").

argument will show that $H^1(p\mathcal{O}_{A/\mathfrak{J}}) \xrightarrow{\approx} \mathbf{D}(A/\mathfrak{J})$.) The top row is exact, and the composition of any two successive maps in the bottom row is zero; if, furthermore, the bottom row is *exact* at $\mathbf{G}'(A)$ and at $\mathbf{D}(A)$, then by simple diagram chasing γ can be seen to be bijective.

For proving exactness at $\mathbf{G}'(A)$, let $\mathbf{F}$ be the *kernel* of $\mathbf{G}\to\mathbf{G}'$ and let $\mathbf{I}\subseteq\mathbf{G}'$ be the *image* of this map (in the category of *fpqc* sheaves). Since $\mathbf{G}\to\mathbf{G}'$ is actually a homomorphism of *Greenberg modules* (cf. proof of (9.4)), therefore $\mathbf{F}$ and $\mathbf{I}$ can themselves be regarded as Greenberg modules, so they are connected and unipotent [**DG**, p. 601, (1.2)]. Since $\mathbf{F}$ is connected and unipotent, Lemma (6.5) shows that $\mathbf{G}(A)\to\mathbf{I}(A)$ is surjective; but $\mathbf{I}(A)$ is the kernel of β ($\mathbf{I}$ being the kernel of $\mathbf{G}'\to\mathbf{D}$), so we have the desired exactness at $\mathbf{G}'(A)$. Similarly since the kernel $\mathbf{I}$ of $\mathbf{G}'\to\mathbf{D}$ is connected and unipotent, exactness holds at $\mathbf{D}(A)$. This completes the proof of *a)*.

e) and *f)* follow from condition (ii) above, because as in the second half of the proof of Lemma (7.2) [1], the canonical map $\mathrm{Pic}(X_A)\to\mathrm{Pic}(X_{A/\mathfrak{J}})$ takes $\mathrm{Pic}(\mathrm{Spec}(k_1\otimes_k A))$ *isomorphically onto* $\mathrm{Pic}(\mathrm{Spec}(k_1\otimes_k A/\mathfrak{J}))$. As for *d)*, we begin as in the preceding proof of *a)*: there is a commutative diagram

$$\begin{array}{ccccccccc}
H^0(1+\mathscr{N}\mathcal{O}_A) & \longrightarrow & H^0(1+\mathscr{N}\mathcal{O}'_A) & \longrightarrow & H^1(1+p\mathcal{O}_A) & \longrightarrow & \mathrm{Pic}(X_A) & \longrightarrow & \mathrm{Pic}(X'_A) \\
\downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\rho'} & & \downarrow{\scriptstyle\gamma^*} & & \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\tau} \\
\mathbf{G}^*(A) & \longrightarrow & \mathbf{G}'^*(A) & \longrightarrow & \mathbf{D}^*(A) & \longrightarrow & \mathbf{P}(A) & \longrightarrow & \mathbf{P}'(A)
\end{array}$$

and we need to show that γ^* is bijective. σ and τ are *surjective*, and $\mathrm{Pic}(X_A)\to\mathrm{Pic}(X'_A)$ maps the kernel of σ *isomorphically onto* the kernel of τ (cf. (ii) above). ρ is *bijective*. (For, if $\mathbf{E}$ (resp. $\bar{\mathbf{E}}$) is the *fpqc* sheaf associated to the functor $H^0(\mathcal{O}_A)$ (resp. $H^0(\bar{\mathcal{O}}_A)$, $\bar{\mathcal{O}}=\mathcal{O}_{X_{\mathrm{red}}}$) then $\mathbf{E}$ is a k-ring scheme, the group of units $\mathbf{E}^*$ is the *fpqc* sheaf associated to $H^0(\mathcal{O}_A^*)$, and since $H^0(\mathcal{O}_A)\to\mathbf{E}(A)$ is bijective (cf. (4.4)), so also is $H^0(\mathcal{O}_A^*)\to\mathbf{E}^*(A)$; and similarly $H^0(\bar{\mathcal{O}}_A^*)\to\bar{\mathbf{E}}^*(A)$ is bijective; but $H^0(1+\mathscr{N}\mathcal{O}_A)$ is the kernel of

$$H^0(\mathcal{O}_A^*)\to H^0(\bar{\mathcal{O}}_A^*),$$

and $\mathbf{G}^*(A)$ is the kernel of $\mathbf{E}^*(A)\to\bar{\mathbf{E}}^*(A)$ ...) Similarly ρ' is bijective. So, as in the proof of *a)*, diagram chasing reduces us to showing exactness of the bottom row at $\mathbf{G}'^*(A)$ and at $\mathbf{D}^*(A)$, and this can be done by showing that the kernel $\mathbf{F}^*$ and the image $\mathbf{I}^*$ of $\mathbf{G}^*\to\mathbf{G}'^*$ are connected unipotent k-groups.

Let us show that $\mathbf{F}^*$ is connected and unipotent. (A similar proof shows that $\mathbf{G}^*$ is connected and unipotent, whence so is its quotient $\mathbf{I}^*$.) As in the proof of (9.4), $\mathbf{F}^*$ is isomorphic, *as a k-scheme*, to $\mathbf{F}$, and we have already noted that $\mathbf{F}$, being a Greenberg module, is connected. As for unipotence, we have a filtration

$$\mathbf{F}^*=\mathbf{F}^1\supseteq\mathbf{F}^2\supseteq\mathbf{F}^3\supseteq\dots$$

---

[1] Here we can even replace the étale topology by the Zariski topology.

where $\mathbf{F}^i$ $(i \geq 1)$ is the *fpqc* sheaf associated to the functor $H^0(1+p^i\mathcal{O}_A)$. The quotient $\mathbf{F}^i/\mathbf{F}^{i+1}$ is then the *fpqc* sheaf associated to the (multiplicative) group functor

$$H^0(1+p^i\mathcal{O}_{X_A})/H^0(1+p^{i+1}\mathcal{O}_{X_A}).$$

But this functor is isomorphic (via the truncated logarithm $(1+x)\mapsto x$) to the (additive) group functor

$$H^0(p^i\mathcal{O}_{X_A})/H^0(p^{i+1}\mathcal{O}_{X_A}),$$

whose associated *fpqc* sheaf is a Greenberg module, hence is a unipotent algebraic k-group. Since all the quotients $\mathbf{F}^i/\mathbf{F}^{i+1}$ are unipotent, so therefore is $\mathbf{F}^*$ [**DG**, p. 485, (2.3)].

This completes the proof of *d)*.

***

Let $\mathscr{J}$ (resp. $\mathscr{J}'$) be the kernel of the natural surjective map $\mathcal{O}_A \to \mathcal{O}_{A/\mathfrak{J}}$ (resp. $\mathcal{O}'_A \to \mathcal{O}'_{A/\mathfrak{J}}$). Note that $\mathscr{J}' \cong \mathscr{J}/(p\mathcal{O}_A \cap \mathscr{J})$. In view of *a)*, *b)*, *c)*, *d)*, *e)*, *f)* above, we have the *natural* commutative diagrams, with exact rows:

(9.9)
$$\begin{array}{ccccccc}
H^0(p\mathcal{O}_{A/\mathfrak{J}}) & \xrightarrow{\delta} & H^1(p\mathcal{O}_A \cap \mathscr{J}) & \longrightarrow & \mathrm{Lie}(\mathbf{D})\otimes_k \mathfrak{J} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
H^0(\mathcal{O}_{A/\mathfrak{J}}) & \longrightarrow & H^1(\mathscr{J}) & \longrightarrow & \mathrm{Lie}(\mathbf{H})\otimes_k \mathfrak{J} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
H^0(\mathcal{O}'_{A/\mathfrak{J}}) & \xrightarrow{\alpha} & H^1(\mathscr{J}') & \longrightarrow & \mathrm{Lie}(\mathbf{H}')\otimes_k \mathfrak{J} & \longrightarrow & 0
\end{array}$$

(9.10)
$$\begin{array}{ccccccc}
H^0(1+p\mathcal{O}_{A/\mathfrak{J}}) & \xrightarrow{\delta^*} & H^1(1+(p\mathcal{O}_A \cap \mathscr{J})) & \longrightarrow & \mathrm{Lie}(\mathbf{D}^*)\otimes_k \mathfrak{J} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
H^0(\mathcal{O}^*_{A/\mathfrak{J}}) & \longrightarrow & H^1(1+\mathscr{J}) & \longrightarrow & \mathrm{Lie}(\mathbf{P})\otimes_k \mathfrak{J} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
H^0(\mathcal{O}'^*_{A/\mathfrak{J}}) & \xrightarrow{\alpha^*} & H^1(1+\mathscr{J}') & \longrightarrow & \mathrm{Lie}(\mathbf{P}')\otimes_k \mathfrak{J} & \longrightarrow & 0
\end{array}$$

A closer look at these diagrams and some relations between them will yield the desired proofs.

We begin with the proof of (9.6). — Since $\mathbf{C}$ is the kernel of $\mathbf{D}\to\mathbf{H}$, it is immediate that $\mathrm{Lie}(\mathbf{C})$ is the kernel of $\mathrm{Lie}(\mathbf{D}) \to \mathrm{Lie}(\mathbf{H})$. Similarly, $\mathrm{Lie}(\mathbf{C}^*)$ is the kernel of $\mathrm{Lie}(\mathbf{D}^*) \to \mathrm{Lie}(\mathbf{P})$.

Next observe that the map α in (9.9) is the *zero map*; in other words, the canonical map $\beta : H^0(\mathcal{O}'_A) \to H^0(\mathcal{O}'_{A/\mathfrak{J}})$ is *surjective*. (This is because (X' being a scheme over k) β can be identified with the canonical map

$$H^0(X', \mathcal{O}_{X'})\otimes_k \mathbf{W}(A)/(p) \to H^0(X', \mathcal{O}_{X'})\otimes_k \mathbf{W}(A/\mathfrak{J})/(p)$$

which is clearly surjective.) It follows easily that the third column of (9.9) is exact. Similarly $\beta^* : H^0(\mathcal{O}'^*_A) \to H^0(\mathcal{O}'^*_{A/\mathfrak{J}})$ is surjective (since β is), i.e. the map α^* in (9.10) is the zero-map, and so the third column in (9.10) is exact. Consequently (since $I \neq (0)$) the sequences

$$\mathrm{Lie}(\mathbf{D}) \to \mathrm{Lie}(\mathbf{H}) \to \mathrm{Lie}(\mathbf{H}')$$
$$\mathrm{Lie}(\mathbf{D}^*) \to \mathrm{Lie}(\mathbf{P}) \to \mathrm{Lie}(\mathbf{P}')$$

are exact.

Finally, Theorem (8.1) shows that $\mathrm{Lie}(\mathbf{H}) \to \mathrm{Lie}(\mathbf{H}')$ is *surjective*; and this implies that in (9.9) the map $H^1(\mathcal{J}) \to H^1(\mathcal{J}')$ is surjective (recall that $\alpha = 0$). Since $\mathcal{J}^2 = (0)$ (cf. Lemma (8.7)), therefore the truncated exponential $x \mapsto 1 + x$ maps $\mathcal{J}$ (resp. $\mathcal{J}'$) isomorphically onto $1 + \mathcal{J}$ (resp. $1 + \mathcal{J}'$); hence the map $H^1(1+\mathcal{J}) \to H^1(1+\mathcal{J}')$ in (9.10) is also surjective, whence $\mathrm{Lie}(\mathbf{P}) \otimes_k \mathfrak{J} \to \mathrm{Lie}(\mathbf{P}') \otimes_k \mathfrak{J}$ is surjective, i.e.

$$\mathrm{Lie}(\mathbf{P}) \to \mathrm{Lie}(\mathbf{P}')$$

is *surjective*. This completes the proof of (9.6).

As for (9.5) (1), the truncated exponential induces an *isomorphism*

$$H^1(p\mathcal{O}_A \cap \mathcal{J}) \xrightarrow{\approx} H^1(1 + (p\mathcal{O}_A \cap \mathcal{J})).$$

Imitating the proof of (*) following Lemma (6.6) (keeping in mind that $p\mathcal{J} = (0)$, cf. Lemma (8.7)), we see that this isomorphism takes the image of δ (cf. (9.9)) *onto* the image of δ^* (cf. (9.10)). We obtain thereby an *isomorphism of groups*

$$\mathrm{Lie}(\mathbf{D}) \otimes_k \mathfrak{J} \cong \mathrm{Lie}(\mathbf{D}^*) \otimes_k \mathfrak{J},$$

and this isomorphism varies functorially—in the obvious sense—with the pair $(A, \mathfrak{J})$.

We wish to deduce an isomorphism of *k-vector spaces* $\mathrm{Lie}(\mathbf{D}) \cong \mathrm{Lie}(\mathbf{D}^*)$. For this purpose, consider the category of pairs $(B, \mathfrak{J})$, with B a k-algebra and $\mathfrak{J}$ a B-ideal such that $\mathfrak{J} \neq (0)$ and $\mathfrak{J}^2 = (0)$. (A morphism $(B_1, \mathfrak{J}_1) \to (B_2, \mathfrak{J}_2)$ of such pairs is a k-algebra homomorphism $\varphi : B_1 \to B_2$ such that $\varphi(\mathfrak{J}_1) \subseteq \mathfrak{J}_2$.) On this category define the group-valued functors $\mathcal{D}$, $\mathcal{D}^*$, by

$$\mathcal{D}(B, \mathfrak{J}) = \mathrm{Lie}(\mathbf{D}) \otimes_k \mathfrak{J}$$
$$\mathcal{D}^*(B, \mathfrak{J}) = \mathrm{Lie}(\mathbf{D}^*) \otimes_k \mathfrak{J}.$$

I claim that *the functors $\mathcal{D}$, $\mathcal{D}^*$ are isomorphic.*

(From this it will follow that $\mathrm{Lie}(\mathbf{D})$ and $\mathrm{Lie}(\mathbf{D}^*)$ are isomorphic *as k-vector spaces*; for, if $a \in k$, then multiplication by a in $\mathrm{Lie}(\mathbf{D}) = \mathcal{D}(k[\varepsilon], \varepsilon k[\varepsilon])$ $(\varepsilon^2 = 0)$ is induced by the *morphism of pairs*

$$\varphi_a : (k[\varepsilon], \varepsilon k[\varepsilon]) \to (k[\varepsilon], \varepsilon k[\varepsilon])$$

given by $\quad \varphi_a(x + y\varepsilon) = x + ya\varepsilon \quad (x, y \in k)$

and similarly for multiplication by a in $\mathrm{Lie}(\mathbf{D}^*)$ (cf. [**DG**, p. 208, (3.6)]).)

(1) Cf. Remark *a)* at the end of this section.

To see that $\mathscr{D}$ and $\mathscr{D}^*$ are isomorphic, note that if $(B, \mathfrak{J})$ is a pair as above and if A is a *faithfully flat* B-algebra, then the following canonical diagrams are *exact*:

$$\mathscr{D}(B, \mathfrak{J}) \to \mathscr{D}(A, \mathfrak{J}A) \rightrightarrows \mathscr{D}(A\otimes_B A, \mathfrak{J}(A\otimes_B A))$$

$$\mathscr{D}^*(B, \mathfrak{J}) \to \mathscr{D}^*(A, \mathfrak{J}A) \rightrightarrows \mathscr{D}^*(A\otimes_B A, \mathfrak{J}(A\otimes_B A)).$$

But such an A can always be chosen so that, with $\mathfrak{J}=\mathfrak{J}A$, the conditions (i) and (ii) at the beginning of this proof are satisfied (cf. remark immediately following (i) and (ii)). Simple considerations show then that for proving $\mathscr{D}$ and $\mathscr{D}^*$ isomorphic *we may restrict our attention to pairs* $(B, \mathfrak{J})=(A, \mathfrak{J})$ *where* $(A, \mathfrak{J})$ *satisfies* (i) *and* (ii). But for such pairs, we have already given a functorial isomorphism $\mathscr{D}(A, \mathfrak{J}) \to \mathscr{D}^*(A, \mathfrak{J})$. (9.5) is now proved.

The *proof of* (9.2) is similar, and simpler, being based on the isomorphism

$$H^1(\mathscr{J}') \cong H^1(1+\mathscr{J}')$$

induced by the truncated exponential. (Recall that the maps α and α^* are zero-maps, so that $H^1(\mathscr{J}') \cong \mathrm{Lie}(\mathbf{H}')\otimes_k \mathfrak{J}$, and $H^1(1+\mathscr{J}')=\mathrm{Lie}(\mathbf{P}')\otimes_k \mathfrak{J}$.)

This completes the proof of Theorem (9.1).

Remarks. — *a)* For $p>2$, a much simpler proof of (9.5) is obtained by observing that *the abelian sheaves* $p\mathcal{O}_{X_A}$, $1+p\mathcal{O}_{X_A}$ *are naturally isomorphic* (so that the functors $\mathbf{D}$, $\mathbf{D}^*$ themselves are isomorphic!).

Indeed, for $n>0$, we can write $p^n/n! = p^{f(n)}a_n/b_n$, where a_n, b_n are integers not divisible by p, and where $f(n)>0$ *tends to infinity with* n. Hence on the category of $\mathbf{W}_N(k)$-algebras S (N a fixed integer) we can define a *natural group-isomorphism*

$$E : pS \to 1+pS$$

by

$$E(ps)=\sum_{n=0}^{\infty}(p^n/n!)s^n \quad (s\in S;\ p^m s=0 \quad \text{if} \quad m\geq N).$$

(The inverse L of E is given by

$$L(1+ps)=\sum_{n=1}^{\infty}(-1)^{n-1}(p^n/n)s^n.)$$

b) We have canonical maps

$$\mu : \mathrm{Lie}(\mathbf{D}) \to \mathrm{Lie}(\mathbf{H}), \qquad \mu^* : \mathrm{Lie}(\mathbf{D}^*) \to \mathrm{Lie}(\mathbf{P})$$

and, by (9.5), a canonical isomorphism

$$\nu : \mathrm{Lie}(\mathbf{D}) \to \mathrm{Lie}(\mathbf{D}^*).$$

If we could show that $\nu(\ker(\mu))=\ker(\mu^*)$, then we could deduce from (9.9) and (9.10) that there is a *natural* isomorphism $\mathrm{Lie}(\mathbf{H}) \to \mathrm{Lie}(\mathbf{P})$ (induced by the truncated exponential $\mathscr{J}\to 1+\mathscr{J}$).

APPENDICES ON GREENBERG MODULES

We present in these appendices the facts on Greenberg modules needed in the body of the paper. (The definition of " Greenberg modules " is given immediately below.) The material in appendices A and B is either well-known or straightforward, but convenient references seem to be lacking.

As always, let k be a perfect field of characteristic $p>0$. We say that a functor $\mathbf{Q}$ of k-algebras *has a module* (resp. *algebra*) *structure* over a k-ring-scheme $\mathbf{S}$, or simply that $\mathbf{Q}$ *is an* $\mathbf{S}$*-module* (resp. *algebra*), if there is given for each k-algebra A an $\mathbf{S}(A)$-module (resp. algebra) structure on $\mathbf{Q}(A)$, the structure varying functorially with A. Homomorphisms of S-modules or algebras are defined in the obvious way.

We define a **Greenberg module** (resp. **algebra**) over k to be an *affine* k*-scheme of finite type together with a module* (resp. *algebra*) *structure over the Witt vectors* $\mathbf{W}$.

The category of Greenberg modules and their homomorphisms is abelian. (Use the corresponding fact for commutative affine k-groups.)

Appendix A. — The Greenberg module associated to a W(k)-module.

Proposition **(A.1)**. — *Let* M *be a* $\mathbf{W}(k)$*-module* (resp. *algebra*) *of finite length, let* $\mathbf{M}$ *be the fpqc sheaf associated to the functor of* k*-algebras*

$$A \mapsto M \otimes_{\mathbf{W}(k)} \mathbf{W}(A)$$

and for any k*-algebra* A *let*

$$\psi_A : M \otimes_{\mathbf{W}(k)} \mathbf{W}(A) \to \mathbf{M}(A)$$

be the canonical map. Then:

(i) $\mathbf{M}$ *is isomorphic, as a set-valued functor of* k*-algebras, to the affine space*

$$\mathrm{Spec}(k[X_1, X_2, \ldots, X_\lambda]),$$

($\lambda =$ *length of* M; $X_1, X_2, \ldots, X_\lambda$ — *independent indeterminates*). *Thus, with its natural* $\mathbf{W}$*-module* (resp. *algebra*) *structure,* $\mathbf{M}$ *is a Greenberg module* (resp. *algebra*).

(ii) ψ_A *is surjective for every* A, *and even bijective if* $A^p = A$.

(iii) *If* $\mathbf{N}$ *is an fpqc sheaf with a* $\mathbf{W}$*-module* (resp. *algebra*) *structure, then for any* $\mathbf{W}(k)$*-homomorphism* $\varphi : M \to \mathbf{N}(k)$ *there exists a unique* $\mathbf{W}$*-homomorphism* $\boldsymbol{\varphi} : \mathbf{M} \to \mathbf{N}$ *such that*

$$\varphi = \boldsymbol{\varphi}(k) \circ \psi_k$$

Proof. — There exists a $\mathbf{W}(k)$-module isomorphism

$$M \cong \prod_{i=1}^{r} \mathbf{W}_{n_i}(k) = (say) \prod_{i=1}^{r} M_i \qquad (n_i > 0).$$

Then clearly $\mathbf{M} = \prod_{i=1}^{r} \mathbf{M}_i$ (as $\mathbf{W}$-modules), so for proving (i) and (ii) we may assume that $M = \mathbf{W}_n(k)$ for some $n > 0$. In this case we have a canonical surjective A-functorial map

$$M \otimes_{\mathbf{W}(k)} \mathbf{W}(A) = \mathbf{W}(A)/(p^n) \to \mathbf{W}_n(A)$$

which is *bijective* if $A^p = A$. Hence by Corollary (0.2), $\mathbf{M} \cong \mathbf{W}_n$, and (i) and (ii) are proved.

(iii) follows at once from the obvious fact that there is a unique A-functorial $\mathbf{W}(A)$-homomorphism

$$\varphi_A : M \otimes_{\mathbf{W}(k)} \mathbf{W}(A) \to \mathbf{N}(A)$$

such that $\varphi_k = \varphi$. Q.E.D.

Corollary (**A.2**). — *If* R *is a* $\mathbf{W}(k)$*-algebra of finite length, and* M *is a finitely generated* R*-module, and* $\mathbf{R}$, $\mathbf{M}$, ψ *are as in* (A.1), *then* $\mathbf{M}$ *is naturally an* $\mathbf{R}$*-module, and for any* k*-algebra* A, *the map*

$$M \otimes_{\mathbf{R}(k)} \mathbf{R}(A) \to \mathbf{M}(A)$$

(*obtained by extension of scalars from* $M \xrightarrow{\psi_k} \mathbf{M}(k) \to \mathbf{M}(A)$) *is surjective, and even bijective if* $A^p = A$. *Hence* $\mathbf{M}$ *is the fppf sheaf associated to the functor* $A \mapsto M \otimes_{\mathbf{R}(k)} \mathbf{R}(A)$.

Proof. — From the natural $R \otimes_{\mathbf{W}(k)} \mathbf{W}(A)$-module structure of $M \otimes_{\mathbf{W}(k)} \mathbf{W}(A)$, we obtain, by passage to associated *fpqc* sheaves, an $\mathbf{R}$-module structure on $\mathbf{M}$. The next assertion results, in view of (A.1) (ii), from the following commutative diagram:

$$\begin{array}{ccc} M \otimes_{\mathbf{R}(k)} (R \otimes_{\mathbf{W}(k)} \mathbf{W}(A)) & \xrightarrow{\sim} & M \otimes_{\mathbf{W}(k)} \mathbf{W}(A) \\ \downarrow & & \downarrow \\ M \otimes_{\mathbf{R}(k)} \mathbf{R}(A) & \longrightarrow & \mathbf{M}(A) \end{array}$$

The final assertion results from Corollary (0.2) if we show that the functors $M \otimes_{\mathbf{R}(k)} \mathbf{R}(A)$ and $\mathbf{M}(A)$ of k-algebras A commute with filtered direct limits. But this follows easily from (A.1) (i). Q.E.D.

Proposition (**A.3**). — *Let* $\mathfrak{M}$ *be any Greenberg module, and let* M *be the* $\mathbf{W}(k)$*-module* $\mathfrak{M}(k)$. *Then* M *is of finite length (say)* λ, *and as a* k*-scheme*

$$\mathfrak{M}_{\text{red}} = \operatorname{Spec}(k[X_1, X_2, \ldots, X_\lambda]).$$

($X_1, X_2, \ldots, X_\lambda$ *independent indeterminates*).

Proof. — For the fact that λ is finite cf. [**DG**, top of p. 602]. Now, k being perfect, we have that $\mathfrak{M}_{\text{red}} \times_k \mathfrak{M}_{\text{red}}$ and $\mathbf{W} \times_k \mathfrak{M}_{\text{red}}$ are reduced schemes, so that the " addition " map $\mathfrak{M}\times_k\mathfrak{M}\to\mathfrak{M}$ induces $\mathfrak{M}_{\mathrm{red}}\times_k\mathfrak{M}_{\mathrm{red}}\to\mathfrak{M}_{\mathrm{red}}$, and similarly " scalar multiplication " $\mathbf{W}\times_k\mathfrak{M}\to\mathfrak{M}$ induces $\mathbf{W}\times_k\mathfrak{M}_{\mathrm{red}}\to\mathfrak{M}_{\mathrm{red}}$; in other words, $\mathfrak{M}_{\mathrm{red}}$ is a Greenberg submodule of $\mathfrak{M}$. Since clearly $\mathrm{M}=\mathfrak{M}_{\mathrm{red}}(k)$, we may assume that $\mathfrak{M}=\mathfrak{M}_{\mathrm{red}}$.

By Lazard's theorem [**DG**, p. 536, (4.1)], it suffices to show that, as a k-group, $\mathfrak{M}$ has a composition series of length λ with quotients isomorphic to the additive group $\mathbf{W}_1$. This can be done in many ways. We proceed by induction on λ. If $\lambda=0$ there is nothing to prove. If $\lambda=1$, then there is a $\mathbf{W}(k)$-isomorphism $k\xrightarrow{\sim}\mathrm{M}$, which must come from a W-homomorphism $\mathbf{k}=\mathbf{W}_1\to\mathfrak{M}$ (cf. (A.1) (iii)); the cokernel $\mathfrak{C}$ of this map is infinitesimal [**DG**, p. 601, (1.2) *c)* and *d)*]; but since $\mathfrak{M}$ is reduced, this means that $\mathfrak{C}=0$, and hence $\mathfrak{M}\cong\mathbf{W}_1$ as a k-group (cf. [**DG**, p. 483, (1.1)]). Finally, if $\lambda>1$, let N be a submodule of M such that M/N has length 1; by (A.1) (iii), the inclusion map $\mathrm{N}\to\mathrm{M}$ comes from a $\mathbf{W}$-homomorphism $\mathbf{N}\to\mathfrak{M}$, whose cokernel $\mathfrak{C}'$ is reduced and such that $\mathfrak{C}'(k)$ ($\cong\mathrm{M}/\mathrm{N}$) has length 1 [**DG**, p. 601, (1.2) *c)*]; the conclusion follows.

Appendix B. — Lie algebras of Greenberg modules.

We discuss next, for a Greenberg module $\mathbf{M}$, the associated functor $\mathbf{Lie}(\mathbf{M})$ of k-algebras A. First we recall the definition: let X be an indeterminate, set

$$\mathrm{A}[\varepsilon]=\mathrm{A}[\mathrm{X}]/(\mathrm{X}^2),$$

and let $\pi_{\mathrm{A}}:\mathrm{A}[\varepsilon]\to\mathrm{A}$ be the A-algebra homomorphism such that $\pi_{\mathrm{A}}(\varepsilon)=0$; then

$$\mathbf{Lie}(\mathbf{M})(\mathrm{A})=\text{kernel of }\mathbf{M}(\pi_{\mathrm{A}}):\mathbf{M}(\mathrm{A}[\varepsilon])\to\mathbf{M}(\mathrm{A}).$$

To begin with, we think of $\mathbf{Lie}(\mathbf{M})$ as a functor into the category of abelian groups.

Next, for each $a\in\mathrm{A}=\mathbf{W}_1(\mathrm{A})$, the homomorphism $u_a:\mathrm{A}[\varepsilon]\to\mathrm{A}[\varepsilon]$ defined by

$$u_a(\alpha+\beta\varepsilon)=\alpha+\beta a\varepsilon \qquad (\alpha,\ \beta\in\mathrm{A})$$

gives rise, via the commutative diagram

$$\begin{array}{ccc}\mathbf{M}(\mathrm{A}[\varepsilon]) & \xrightarrow{\mathbf{M}(u_a)} & \mathbf{M}(\mathrm{A}[\varepsilon]) \\ & {\scriptstyle\mathbf{M}(\pi_{\mathrm{A}})}\searrow \quad \swarrow{\scriptstyle\mathbf{M}(\pi_{\mathrm{A}})} & \\ & \mathbf{M}(\mathrm{A}) & \end{array}$$

to an endomorphism of the abelian group $\mathbf{Lie}(\mathbf{M})(\mathrm{A})$; in this way, $\mathbf{Lie}(\mathbf{M})$ becomes a $\mathbf{W}_1$-module, varying functorially with $\mathbf{M}$. The $\mathbf{W}_1(k)$ ($=k$)-vector space

$$\mathrm{Lie}(\mathbf{M})=\mathbf{Lie}(\mathbf{M})(k)$$

is canonically isomorphic to the Zariski tangent space $\mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2,k)$ at the zero-point $o_{\mathbf{M}}$ of $\mathbf{M}$ ($\mathfrak{m}=$ maximal ideal of the local ring of $o_{\mathbf{M}}$ on $\mathbf{M}$). More generally, there are isomorphisms of A-modules, functorial in both A and $\mathbf{M}$,

$$\mathbf{Lie}(\mathbf{M})(\mathrm{A})\cong\mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2,\mathrm{A})\cong\mathrm{Lie}(\mathbf{M})\otimes_k\mathrm{A}.$$

(cf. [**DG**, p. 208, (3.6)]).

Furthermore, the canonical map $A \to A[\varepsilon]$ gives a map $\mathbf{W}(A) \to \mathbf{W}(A[\varepsilon])$, via which $\mathbf{M}(A[\varepsilon])$ is a $\mathbf{W}(A)$-module, and then $\mathbf{M}(\pi_A)$ and $\mathbf{M}(u_a)$ are both $\mathbf{W}(A)$-module homomorphisms. Consequently $\mathbf{Lie}(\mathbf{M})(A)$ is a $\mathbf{W}(A)$-submodule of $\mathbf{M}(A[\varepsilon])$, and the action of $\mathbf{W}(A)$ on $\mathbf{Lie}(\mathbf{M})(A)$ commutes with the above action of A. Thus we obtain a ring homomorphism, functorial in **A**

(B.1) $$\mathbf{W}(A) \to \mathrm{End}_A(\mathbf{Lie}(\mathbf{M})(A)) = \mathrm{End}_A(\mathrm{Lie}(\mathbf{M}) \otimes_k A)$$
(End_A = A-endomorphisms).

Since $p(\mathbf{Lie}(\mathbf{M})(A)) = (0)$, therefore the kernel of the truncation map $\mathbf{W}(A) \to \mathbf{W}_1(A)$ annihilates $\mathbf{Lie}(\mathbf{M})(A)$ (to verify this we may assume that $A^p = A$ (cf. Lemma (0.1))...); in other words (B.1) factors uniquely as

$$\mathbf{W}(A) \to \mathbf{W}_1(A) = A \overset{\rho_A}{\to} \mathrm{End}_A(\mathrm{Lie}(\mathbf{M}) \otimes_k A).$$

The maps ρ_A constitute a *linear representation* of the ring-scheme $\mathbf{W}_1$ in $\mathrm{Lie}(\mathbf{M})$, varying functorially with **M** (1).

By [**DG**, p. 176, Example 1] (with Γ = monoid of non-negative integers) we have then that

$$\mathrm{Lie}(\mathbf{M}) = \bigoplus_{n \geq 0} \mathrm{Lie}^n(\mathbf{M})$$

where

$$\mathrm{Lie}^n(\mathbf{M}) = \{x \in \mathrm{Lie}(\mathbf{M}) \mid \text{for all } k\text{-algebras A and } a \in A,\ \rho_A(a)(x \otimes 1) = x \otimes a^n\}.$$

Let A be the field $k(a, b)$ where a and b are independent indeterminates over k. Since $\rho_A(a+b) = \rho_A(a) + \rho_A(b)$, we get for $x \in \mathrm{Lie}^n(\mathbf{M})$

$$x \otimes (a+b)^n = x \otimes a^n + x \otimes b^n = x \otimes (a^n + b^n).$$

It follows, *if* $x \neq 0$, that

$$(a+b)^n = a^n + b^n.$$

Writing $n = qp^t$, $(q, p) = 1$ or $q = 0$, we have

$$(a^{p^t} + b^{p^t})^q = (a^{p^t})^q + (b^{p^t})^q$$

which is possible only if $q = 1$. The conclusion is that *if n is not a power of p, then* $\mathrm{Lie}^n(\mathbf{M}) = 0$.

In summary:

The **W**-*module structure on* **M** *determines a grading, as above, on the k-vector space* $\mathrm{Lie}(\mathbf{M})$:

$$\mathrm{Lie}(\mathbf{M}) = \bigoplus_{t \geq 0} \mathrm{Lie}^{p^t}(\mathbf{M}).$$

This grading is natural, i.e. it varies functorially with **M**.

---

(1) Equivalently, we can interpret the $\rho_A : A \to \mathrm{End}_A(\mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, A))$ as being a representation of $\mathbf{W}_1$ in $\mathfrak{m}/\mathfrak{m}^2$. Actually, in the general case when **M** is not necessarily algebraic over k, one *must* use this last interpretation, so that (cf. following paragraph) the natural grading appears on $\mathfrak{m}/\mathfrak{m}^2$ rather than on its dual $\mathrm{Lie}(\mathbf{M})$.

Now let $\mathbf{M}$ be the Greenberg module associated, as in Appendix A, to a $\mathbf{W}(k)$-module M of finite length (so that

$$\mathbf{M}(k) \cong \mathrm{M} = \prod_{i=1}^{r} \mathbf{W}_{\mathrm{n}_i}(k) \quad \text{and} \quad \mathbf{M} \cong \prod_{i=1}^{r} \mathbf{W}_{\mathrm{n}_i}).$$

In this case we can give an explicit description of the graded k-vector space $\mathrm{Lie}(\mathbf{M})$ in terms of M. For this purpose, we need some notation. For a k-vector space V, and any integer i, we denote by $\mathrm{V}^{(i)}$ the k-vector space obtained from V by pull-back through the automorphism $a \mapsto a^{p^i}$ of k. (So $\mathrm{V}^{(i)}$ is the vector-space whose underlying abelian group is the same as that of V, and whose multiplication $\mu^{(i)} : k \times \mathrm{V} \to \mathrm{V}$ is given by

$$\mu^{(i)}(a, v) = a^{p^i} v.)$$

Clearly any basis of V is also one of $\mathrm{V}^{(i)}$, so that V and $\mathrm{V}^{(i)}$ have the same dimension.

Proposition **(B.2)** [1]. — *If* $\mathbf{M} = \prod_{i=1}^{r} \mathbf{W}_{\mathrm{n}_i}$, *then for each* $t \geq 0$, *there is a* natural *isomorphism of k-vector spaces*

$$\varphi^t_{\mathbf{M}} : \mathrm{Lie}^{p^t}(\mathbf{M}) \xrightarrow{\sim} (p^t(\mathbf{M}(k))/p^{t+1}(\mathbf{M}(k)))^{(-t)}$$

Proof. — There are canonical isomorphisms

$$\bigoplus_{i=1}^{r} \mathrm{Lie}^{p^t}(\mathbf{W}_{\mathrm{n}_i}) \xrightarrow{\sim} \mathrm{Lie}^{p^t}(\mathbf{M}) \xrightarrow{\sim} \prod_{i=1}^{r} \mathrm{Lie}^{p^t}(\mathbf{W}_{\mathrm{n}_i})$$

and (with $\mathrm{W}_{n_i} = \mathbf{W}_{\mathrm{n}_i}(k)$)

$$\bigoplus_{i=1}^{r} p^t \mathrm{W}_{n_i}/p^{t+1}\mathrm{W}_{n_i} \xrightarrow{\sim} p^t(\mathbf{M}(k))/p^{t+1}(\mathbf{M}(k)) \xrightarrow{\sim} \prod_{i=1}^{r} p^t \mathrm{W}_{n_i}/p^{t+1}\mathrm{W}_{n_i}.$$

Using these isomorphisms (and the fact that $\prod_{i=1}^{r} \mathbf{W}_{\mathrm{n}_i} = \bigoplus_{i=1}^{r} \mathbf{W}_{\mathrm{n}_i}$) we reduce the problem to defining $\varphi^t_{\mathbf{W}_\mathrm{n}}$ $(n \geq 1)$ and to checking that for any W-homomorphism $\psi : \mathbf{W}_\mathrm{n} \to \mathbf{W}_\mathrm{m}$, the resulting diagram

$$\textbf{(B.3)} \qquad \begin{array}{ccc} \mathrm{Lie}^{p^t}(\mathbf{W}_\mathrm{n}) & \xrightarrow[\varphi^t_{\mathbf{W}_\mathrm{n}}]{} & (p^t\mathrm{W}_n/p^{t+1}\mathrm{W}_n)^{(-t)} \\ \big\downarrow & & \big\downarrow \\ \mathrm{Lie}^{p^t}(\mathbf{W}_\mathrm{m}) & \xrightarrow[\varphi^t_{\mathbf{W}_\mathrm{m}}]{} & (p^t\mathrm{W}_m/p^{t+1}\mathrm{W}_m)^{(-t)} \end{array}$$

commutes.

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[1] For a stronger result (without proof) cf. remark (B.3) below.

For any k-algebra A, we have

$$\mathbf{Lie}(\mathbf{W}_n)(A)=\{(\varepsilon a_0, \varepsilon a_1, \ldots, \varepsilon a_{n-1}) \mid a_i \in A, \ \varepsilon \text{ as before}\}.$$

Now if $X=(X_0, X_1, \ldots, X_{n-1})$, $Y=(Y_0, Y_1, \ldots, Y_{n-1})$ are two families of independent indeterminates, then, in $\mathbf{W}_n(k[X, Y])$ we have

$$(X_0, \ldots, X_{n-1})+(Y_0, \ldots, Y_{n-1})=(S_0(X, Y), \ldots, S_{n-1}(X, Y))$$

where $S_0, \ldots, S_{n-1}$ are polynomials such that

$$S_i(X, 0)=X_i \quad S_i(0, Y)=Y_i \quad (0\leq i<n)$$

i.e.

$$S_i(X, Y)=X_i+Y_i+(\text{terms of degree}\geq 2).$$

By specializing $X_i \to \varepsilon a_i$, $Y_j \to \varepsilon a'_j$ $(a_i, a'_j \in A)$, and since $\varepsilon^2=0$, we deduce that

$$(\varepsilon a_0, \ldots, \varepsilon a_{n-1})+(\varepsilon a'_0, \ldots, \varepsilon a'_{n-1})=(\varepsilon(a_0+a'_0), \ldots, \varepsilon(a_{n-1}+a'_{n-1})).$$

Furthermore, for $a \in A$, the homomorphism $\mathbf{W}_n(u_a)$ (see above) takes $(\varepsilon a_0, \ldots, \varepsilon a_{n-1})$ to $(\varepsilon a a_0, \ldots, \varepsilon a a_{n-1})$. Thus the A-module $\mathbf{Lie}(\mathbf{W}_n)(A)$ is (functorially) isomorphic to the direct product A^n.

If $(b_0, b_1, \ldots) \in \mathbf{W}(A)$, then, either directly, or because we know (as above) that $\mathbf{Lie}(\mathbf{W}_n)(A)$ is annihilated by the kernel of $\mathbf{W}(A) \to \mathbf{W}_1(A)$, we see that

$$\begin{aligned}(b_0, b_1, \ldots)(\varepsilon a_0, \varepsilon a_1, \ldots, \varepsilon a_{n-1}) &= (b_0, 0, 0, \ldots)(\varepsilon a_0, \varepsilon a_1, \ldots, \varepsilon a_{n-1})\\ &= (\varepsilon b_0 a_0, \varepsilon b_0^p a_1, \ldots, \varepsilon b_0^{p^{n-1}} a_{n-1}).\end{aligned}$$

From these formulas it is immediate that

$$\mathrm{Lie}^{p^t}(\mathbf{W}_n)=\{(\varepsilon x_0, \varepsilon x_1, \ldots, \varepsilon x_{n-1}) \in \mathbf{W}_n(k[\varepsilon]) \mid x_i=0 \text{ for } i \neq t\}.$$

Since, for $t<n$, $p^t W_n$ (resp. $p^{t+1} W_n$) is the kernel of the surjective truncation map $W_n \to W_t$ (resp. $W_n \to W_{t+1}$), therefore

$$p^t W_n / p^{t+1} W_n \cong \text{kernel of } (W_{t+1} \to W_t) \qquad (t<n).$$

Hence $$p^t W_n / p^{t+1} W_n \cong \{(0, 0, \ldots, 0, x) \in W_{t+1}\}$$

with scalar multiplication given by

$$c(0, 0, \ldots, 0, x)=(0, 0, \ldots, c^{p^t} x) \qquad (c \in k).$$

We can then define $\varphi^t_{\mathbf{W}_n} : \mathrm{Lie}^{p^t}(\mathbf{W}_n) \to (p^t W_n / p^{t+1} W_n)^{(-t)}$:

$$\begin{aligned}\varphi^t_{\mathbf{W}_n}(\varepsilon x_0, \ldots, \varepsilon x_{n-1}) &= (0, 0, \ldots, 0, x_t) \qquad (t<n)\\ &= 0 \qquad\qquad\qquad\qquad (t \geq n).\end{aligned}$$

It is easily verified now that the diagram (B.3) commutes. Q.E.D.

Remark **(B.3).** — Let M be a $\mathbf{W}(k)$-module of finite length. We say that a filtration of M by $\mathbf{W}(k)$-submodules

$$M=M^0 \supseteq M^1 \supseteq M^2 \supseteq \ldots$$

is a *p-filtration* if

(i) $pM^t \subseteq M^{t+1}$ for all $t \geq 0$, and
(ii) $M^t = (0)$ for some t (so that $M^n = (0)$ for $n \geq t$).

Homomorphisms of p-filtered $\mathbf{W}(k)$-modules are defined in the obvious way (compatible with filtrations).

From the structure theorem of remark (8.8), one can deduce the following *structure theorem for* smooth *Greenberg modules*:

For each smooth Greenberg module $\mathbf{M}$, *let* $M^0 = \mathbf{M}(k)$ *be filtered by its submodules*

$$M^t = \{ x \in \mathbf{M}(k) \mid \text{for all } \mathbf{W}\text{-module homomorphisms } \varphi : \mathbf{M} \to \mathbf{W_t},\ x \in \ker(\varphi(k) : \mathbf{M}(k) \to \mathbf{W_t}(k)) \} \qquad (t > 0).$$

In this way, we obtain an equivalence *from the category of smooth Greenberg modules to the category of p-filtered* $\mathbf{W}(k)$*-modules of finite length.*

Furthermore:

There are natural isomorphisms of k-vector spaces

$$\mathrm{Lie}^{p^t}(\mathbf{M}) \xrightarrow{\approx} (M^t/M^{t+1})^{(-t)} \qquad (t \geq 0).$$

Via these isomorphisms, the p-th*-power map* [**DG**, p. 273] in $\mathrm{Lie}(\mathbf{M}) = \bigoplus_{t \geq 0} \mathrm{Lie}^{p^t}(\mathbf{M})$ *corresponds to the additive endomorphism of* $\bigoplus_{t \geq 0} (M^t/M^{t+1})^{(-t)}$ *induced by multiplication by p in* $\mathbf{M}(k)$.

Appendix C. — Greenberg modules and étale algebras.

The main result in this appendix is Theorem (C.5), which follows quite directly from its special case Lemma (C.2). In the paper, (C.5) is used mainly via Corollary (C.6).

Let p be, as usual, a positive prime number. Let A be a ring such that $pA = (0)$, and let B be an A-algebra, with structural homomorphism $g : A \to B$. For any positive integer m, the truncation map $\rho = \rho_{1m} : \mathbf{W_m} \to \mathbf{W_1}$ gives the commutative diagram

$$\begin{array}{ccc} \mathbf{W_m}(A) & \xrightarrow{\rho(A)} & \mathbf{W_1}(A) = A \\ {\scriptstyle \mathbf{W_m}(g)} \downarrow & & \downarrow {\scriptstyle g} \\ \mathbf{W_m}(B) & \xrightarrow{\rho(B)} & \mathbf{W_1}(B) = B \end{array}$$

whence a homomorphism

(C.1) $$\mathbf{W_m}(B) \otimes_{\mathbf{W_m}(A)} A \to B.$$

Lemma (**C.2**). — *With the preceding notation, if* B *is an étale* A*-algebra then* $\mathbf{W_m}(B)$ *is an étale* $\mathbf{W_m}(A)$*-algebra (via* $\mathbf{W_m}(g)$*), and the map* (C.1) *is bijective.*

Proof. — We proceed by induction on m. There being nothing to prove when $m = 1$, let us prove the Lemma for $m = n+1$ assuming that it holds for $m = n$ $(n \geq 1)$.

Let $\mathfrak{J}_A$ (resp. $\mathfrak{K}_A$) be the kernel of the truncation map $\mathbf{W}_{n+1}(A) \to \mathbf{W}_n(A)$ (resp. $\mathbf{W}_{n+1}(A) \to A$):

$$\mathfrak{J}_A = \{(0, 0, \dots, 0, a) \mid a \in A\} \qquad (n \text{ zeros})$$
$$\mathfrak{K}_A = \{(0, a_1, a_2, \dots, a_n) \mid a_i \in A \text{ for } 1 \leq i \leq n\}.$$

Let $\mathfrak{J}_B$, $\mathfrak{K}_B$ be similarly defined (with respect to B). Then $\mathbf{W}_{n+1}(g)(\mathfrak{J}_A) \subseteq \mathfrak{J}_B$; $\mathbf{W}_{n+1}(g)(\mathfrak{K}_A) \subseteq \mathfrak{K}_B$; and (C.1) is bijective (for $m = n+1$) $\Leftrightarrow \mathfrak{K}_B = \mathfrak{K}_A \mathbf{W}_{n+1}(B)$.

We have

$$\mathfrak{J}_A \mathfrak{K}_A = (0).$$

(To prove this, we may assume that $A^p = A$ (by Lemma (0.1), for example), and then $\mathfrak{J}_A \subseteq p^n \mathbf{W}_{n+1}(A)$, $\mathfrak{K}_A \subseteq p\mathbf{W}_{n+1}(A)$, so $\mathfrak{J}_A \mathfrak{K}_A \subseteq p^{n+1}\mathbf{W}_{n+1}(A) = (0)$.) Consequently the $\mathbf{W}_{n+1}(A)$-module structure on $\mathfrak{J}_A$ is the pull-back of an A $(= \mathbf{W}_{n+1}(A)/\mathfrak{K}_A)$-module structure, the multiplication $A \times \mathfrak{J}_A \to \mathfrak{J}_A$ being

$$(a_0, (0, 0, \dots, 0, a)) \mapsto (0, 0, \dots, 0, a_0^{p^n} a).$$

Thus, if $A^{(n)}$ is the ring A together with its structure of A-algebra for which the structural map $A \to A^{(n)}$ is the n-th iterate $\mathfrak{F}_A^n$ of the Frobenius endomorphism $\mathfrak{F}_A$ of A $(\mathfrak{F}_A(x) = x^p$ for x in A), then the map $\varphi_A : A^{(n)} \to \mathfrak{J}_A$ given by

$$\varphi_A(a) = (0, 0, \dots, 0, a) \qquad (a \in A)$$

is an *isomorphism of* A-*modules*. (To check that φ_A preserves addition, we may assume again that $A^p = A$ and write $(0, 0, \dots, 0, a) = p^n(a^{1/p^n}, 0, \dots, 0)$, etc.) There is a similarly defined isomorphism of B-modules $\varphi_B : B^{(n)} \to \mathfrak{J}_B$, and a commutative diagram of A-*module homomorphisms* (where B-modules are made into A-modules by means of g)

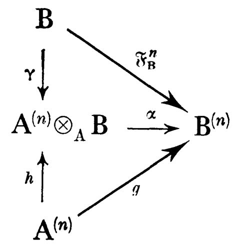
$$\begin{array}{ccc} A^{(n)} & \xrightarrow{\ g\ } & B^{(n)} \\ \varphi_A \downarrow \wr & & \wr \downarrow \varphi_B \\ \mathfrak{J}_A & \xrightarrow[\mathbf{W}_{n+1}(g)]{} & \mathfrak{J}_B \end{array}$$

Hence, by extension of scalars, we have a commutative diagram of B-*module homomorphisms*

$$\text{(C.3)} \qquad \begin{array}{ccc} A^{(n)} \otimes_A B & \xrightarrow{\ \alpha\ } & B^{(n)} \\ \varphi_A \otimes_A B \downarrow \wr & & \wr \downarrow \varphi_B \\ \mathfrak{J}_A \otimes_A B & \xrightarrow[\beta]{} & \mathfrak{J}_B \end{array}$$

in which the vertical arrows are isomorphisms.

The key point is that α *is bijective* (whence, so is β). A proof is given by C. Houzel in [**SGA** 5, Exposé XV, p. 5, Prop. 2 *c)*]. The idea of the proof is as follows: In the first place, α is actually the unique *ring-homomorphism* for which the following diagram commutes:



$$\gamma(b) = 1 \otimes b$$

$$h(a) = a \otimes 1$$

It follows easily that $\operatorname{Spec}(\alpha) : \operatorname{Spec} B^{(n)} \to \operatorname{Spec}(A^{(n)}\otimes_A B)$ is a radicial morphism; furthermore since $A^{(n)}$ is radicial over A, therefore $A^{(n)}\otimes_A B$ is radicial over B, so that $\operatorname{Spec}(\gamma)$ is injective, and since $\operatorname{Spec}(\gamma) \circ \operatorname{Spec}(\alpha) = \operatorname{Spec}(\mathfrak{F}_B^n)$ is surjective, also $\operatorname{Spec}(\alpha)$ is surjective; finally α is an $A^{(n)}$-homomorphism of étale $A^{(n)}$-algebras, and so $\operatorname{Spec}(\alpha)$ is an étale morphism [**EGA** IV, (17.3.5)]; thus $\operatorname{Spec}(\alpha)$ is *étale*, *radicial*, and *surjective*, *i.e.* [**EGA** IV, (17.9.1)] $\operatorname{Spec}(\alpha)$—and hence α—is an isomorphism.

Now since β in (C.3) is surjective, we see that

$$\mathfrak{I}_B = \mathfrak{I}_A \mathbf{W}_{n+1}(B) \subseteq \mathfrak{R}_A \mathbf{W}_{n+1}(B);$$

since the kernel of the truncation map $\mathbf{W}_n(A) \to A$ is clearly $\mathfrak{R}_A \mathbf{W}_n(A)$, the bijectivity of (C.1) for $m = n$ implies that the truncation map $\mathbf{W}_n(B) \to B$ has kernel $\mathfrak{R}_A \mathbf{W}_n(B)$; hence (since $\mathbf{W}_n(B) = \mathbf{W}_{n+1}(B)/\mathfrak{I}_B$)

$$\mathfrak{R}_B = \mathfrak{R}_A \mathbf{W}_{n+1}(B) + \mathfrak{I}_B = \mathfrak{R}_A \mathbf{W}_{n+1}(B);$$

thus (C.1) is bijective for $m = n+1$.

Furthermore, since $\mathfrak{I}_A \mathfrak{R}_A = (0)$, we have that

$$\mathfrak{I}_A \otimes_{\mathbf{W}_{n+1}(A)} \mathbf{W}_{n+1}(B) \cong \mathfrak{I}_A \otimes_A (\mathbf{W}_{n+1}(B)/\mathfrak{R}_A \mathbf{W}_{n+1}(B)) \cong \mathfrak{I}_A \otimes_A B.$$

Since β in (C.3) is bijective, we conclude that the natural map

$$\mathfrak{I}_A \otimes_{\mathbf{W}_{n+1}(A)} \mathbf{W}_{n+1}(B) \to \mathfrak{I}_A \mathbf{W}_{n+1}(B)$$

is bijective. Since $\mathfrak{I}_A^2 = (0)$, and since

$$\mathbf{W}_{n+1}(B)/\mathfrak{I}_A \mathbf{W}_{n+1}(B) = \mathbf{W}_{n+1}(B)/\mathfrak{I}_B = \mathbf{W}_n(B)$$

is, by assumption, étale—and hence *flat*—over $\mathbf{W}_{n+1}(A)/\mathfrak{I}_A$ $(= \mathbf{W}_n(A))$, therefore [**B**, p. 98, Th. 1] shows that $\mathbf{W}_{n+1}(B)$ is *flat* over $\mathbf{W}_{n+1}(A)$.

The proof is completed by the following lemma (with $R = \mathbf{W}_{n+1}(A)$, $S = \mathbf{W}_{n+1}(B)$, $\mathfrak{I} = \mathfrak{I}_A$):

Lemma **(C.4)**. — *Let* R *be a ring, and let* $\mathfrak{I}$ *be a nilpotent ideal in* R (i.e. $\mathfrak{I}^q = (0)$ *for some integer* $q > 0$). *For an* R*-algebra* S, *the following conditions are equivalent*:

(i) S *is an étale* R*-algebra.*

(ii) S *is flat over* R *and* $S/\mathfrak{I}S$ *is an étale* $R/\mathfrak{I}$*-algebra.*

Proof. — (i) $\Rightarrow$ (ii) is left to the reader. Assume that (ii) holds. Then [**EGA** IV, (18.1.2)] there exists an étale R-algebra T together with an $R/\mathfrak{I}$*-isomorphism*

$$\bar{\theta} : T/\mathfrak{I}T \to S/\mathfrak{I}S.$$

$\bar{\theta}$ lifts to a homomorphism of R-algebras $\theta : T \to S$ [**EGA** IV, (17.1.1)]. To show that θ is an isomorphism—whence (i) holds—it suffices to show that the induced map $\mathrm{gr}_{\mathfrak{I}}\theta : \mathrm{gr}_{\mathfrak{I}}T \to \mathrm{gr}_{\mathfrak{I}}S$ is bijective. (Here $\mathrm{gr}_{\mathfrak{I}}T$ is the graded ring

$$(T/\mathfrak{I}T) \oplus (\mathfrak{I}T/\mathfrak{I}^2T) \oplus \ldots \oplus (\mathfrak{I}^{q-1}T/\mathfrak{I}^qT),$$

and similarly for $\mathrm{gr}_{\mathfrak{I}}S$.) We have a natural commutative diagram

$$\begin{array}{ccc} \mathrm{gr}_{\mathfrak{I}}R \otimes_{R/\mathfrak{I}} T/\mathfrak{I}T & \longrightarrow & \mathrm{gr}_{\mathfrak{I}}T \\ {\scriptstyle \mathrm{id}\otimes\bar{\theta}}\downarrow\wr & & \downarrow{\scriptstyle \mathrm{gr}_{\mathfrak{I}}\theta} \\ \mathrm{gr}_{\mathfrak{I}}R \otimes_{R/\mathfrak{I}} S/\mathfrak{I}S & \longrightarrow & \mathrm{gr}_{\mathfrak{I}}S \end{array}$$

(where " id " is the identity map of $\mathrm{gr}_{\mathfrak{I}}R$, so that $\mathrm{id}\otimes\bar{\theta}$ is bijective); since T and S are flat over R, the horizontal arrows are bijective [**B**, p. 98, Th. 1], and so $\mathrm{gr}_{\mathfrak{I}}\theta$ is bijective. Q.E.D.

From Lemma (C.2) we now deduce a more general statement (Theorem (C.5) below). Let k be a perfect field of characteristic p, and let M be a finitely generated $\mathbf{W}_m(k)$-module ($m \geq 1$), with corresponding Greenberg module $\mathbf{M}$ (Appendix A). Let A be a k-algebra and let B, C be two A-algebras. From the canonical map $C \to B \otimes_A C$ we obtain, by functoriality, a homomorphism $\varphi : \mathbf{M}(C) \to \mathbf{M}(B \otimes_A C)$. We have a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & B \otimes_A C \end{array}$$

whence, by functoriality, a commutative diagram

$$\begin{array}{ccc} \mathbf{W}_m(A) & \xrightarrow{\omega_1} & \mathbf{W}_m(B) \\ {\scriptstyle \omega_3}\downarrow & & \downarrow{\scriptstyle \omega_2} \\ \mathbf{W}_m(C) & \xrightarrow[\omega_4]{} & \mathbf{W}_m(B \otimes_A C). \end{array}$$

$\mathbf{M}(C)$ is a $\mathbf{W}_m(A)$-module via ω_3, $\mathbf{M}(B\otimes_A C)$ is a $\mathbf{W}_m(B)$-module via ω_2 and a $\mathbf{W}_m(A)$-module via $\omega_2\circ\omega_1=\omega_4\circ\omega_3$. The map φ is a homomorphism of $\mathbf{W}_m(A)$-modules, and hence by extension of scalars we obtain a $\mathbf{W}_m(B)$-homomorphism

$$\psi : \mathbf{W}_m(B)\otimes_{\mathbf{W}_m(A)}\mathbf{M}(C)\to\mathbf{M}(B\otimes_A C).$$

Theorem **(C.5)**. — *With the preceding notation, if* B *is an* étale A-*algebra, then*:

(i) ψ *is bijective.*

(ii) *If* M *is a* $\mathbf{W}_m(k)$-*algebra* (*so that* $\mathbf{M}$ *is naturally a* $\mathbf{W}_m$-*algebra* (*Corollary* (A.2)), *then* $\mathbf{M}(B\otimes_A C)$ *is an* étale $\mathbf{M}(C)$ *algebra* ([1]).

Proof. — The second assertion follows from the first because $\mathbf{W}_m(B)$ is étale over $\mathbf{W}_m(A)$ (Lemma (C.2)) and because ψ is a homomorphism of $\mathbf{M}(C)$-algebras if M is a $\mathbf{W}_m(k)$-algebra.

For the first assertion we may assume that $\mathbf{M}=\prod_{i=1}^r\mathbf{W}_{m_i}$, $(m_i\leq m)$. This reduces us immediately to the case $\mathbf{M}=\mathbf{W}_n$ $(n\leq m)$. Since B is étale over A, $B\otimes_A C$ is étale over C, and Lemma (C.2) shows then that ψ (with $\mathbf{M}=\mathbf{W}_n$) is a homomorphism of *étale* $\mathbf{W}_n(C)$-algebras. Now the kernel K of the truncation $\mathbf{W}_n(C)\to C$ satisfies $K^n=(0)$ (to see this, we may assume that $C^p=C$ (Lemma (0.1)), in which case $K=p\mathbf{W}_n(C)$), so by [**EGA** IV, (18.1.2)], ψ is bijective if and only if $\psi\otimes_{\mathbf{W}_n(C)}C$ is bijective. But, in view of (C.2)

$$\begin{aligned}\mathbf{W}_m(B)\otimes_{\mathbf{W}_m(A)}\mathbf{W}_n(C)\otimes_{\mathbf{W}_n(C)}C &\cong \mathbf{W}_m(B)\otimes_{\mathbf{W}_m(A)}A\otimes_A C\\ &\cong B\otimes_A C\\ &\cong \mathbf{W}_n(B\otimes_A C)\otimes_{\mathbf{W}_n(C)}C,\end{aligned}$$

and modulo these isomorphisms, we find that $\psi\otimes_{\mathbf{W}_n(C)}C$ is the identity map of $B\otimes_A C$.
Q.E.D.

Corollary **(C.6)**. — *Let* R *be a local Artin* $\mathbf{W}_m(k)$-*algebra such that the natural map of* k *into the residue field of* R *is bijective, and let* $\mathbf{R}$ *be the corresponding* $\mathbf{W}_m$-*algebra. Then for any* k-*algebra* A:

(i) *The functor* $B\mapsto\mathbf{R}(B)$ *is an* equivalence *from the category* Ét_A *of étale* A-*algebras to the category* $\text{Ét}_{\mathbf{R}(A)}$ *of étale* $\mathbf{R}(A)$-*algebras.*

(ii) *If* F *is any functor of* $\mathbf{R}(A)$-*algebras, with associated étale sheaf* $F^\sim$, *then* $F^\sim\circ\mathbf{R}$ (*together with the map* $F\circ\mathbf{R}\to F^\sim\circ\mathbf{R}$ *induced by the canonical map* $F\to F^\sim$) *is the étale sheaf associated to the functor* $F\circ\mathbf{R}$ *of* A-*algebras, i.e.*

$$(F\circ\mathbf{R})^\sim=F^\sim\circ\mathbf{R}.$$

([1]) Theorem (C.5) remains valid for *any* Greenberg module $\mathbf{M}$ annihilated by p^m: this follows from (C.5) as stated and the fact that $(\mathbf{W}_n)_{n>0}$ is a *cogenerating family* for the category of Greenberg modules (cf. [**Sch**, § 5.3, proof of Théorème]).

Proof. — Let $\sigma : R \to k$ be the natural map, and let $\rho_A : \mathbf{W}_m(A) \to A$ be the truncation map. We have a commutative diagram

$$\begin{array}{ccc} R\otimes_{\mathbf{W}_m(k)}\mathbf{W}_m(A) & \xrightarrow{\tau} & \mathbf{R}(A) \\ \Big\downarrow{\scriptstyle \sigma\otimes\rho_A} & & \Big\downarrow{\scriptstyle \sigma_A} \\ k\otimes_k A & \xrightarrow{\approx} & A \end{array}$$

in which the right side is obtained from the left by passing to associated *fppf* sheaves (Corollary (A.2)). Since $\sigma\otimes\rho_A$ is surjective, so is σ_A. Furthermore, the kernel of $\sigma\otimes\rho_A$ is nilpotent, and by Corollary (A.2), τ is surjective; hence *the kernel of the surjective map* σ_A *is nilpotent.*

Now by [**EGA** IV, (18.1.2)], the functor $E \mapsto E\otimes_{\mathbf{R}(A)}A$ from $\text{Ét}_{\mathbf{R}(A)}$ to Ét_A is *fully faithful.* Moreover, for any étale A-algebra B, (C.5) (with $M = R$ and $C = A$) shows that $\mathbf{R}(B)$ is étale over $\mathbf{R}(A)$, and that

$$\mathbf{R}(B)\otimes_{\mathbf{R}(A)}A \cong \mathbf{W}_m(B)\otimes_{\mathbf{W}_m(A)}\mathbf{R}(A)\otimes_{\mathbf{R}(A)}A \cong B \qquad \text{(cf. (C.2))}.$$

This proves (i).

(ii) follows in a straightforward way from (i) in view of the following observations:

a) Since an equivalence of categories takes (categorical) direct sums into direct sums, therefore for any two étale A-algebras B, C we have

$$\mathbf{R}(B\otimes_A C) \cong \mathbf{R}(B)\otimes_{\mathbf{R}(A)}\mathbf{R}(C).$$

(This also follows directly from (C.5)).

b) Since for any étale A-algebra B, $\sigma_B : \mathbf{R}(B) \to B$ has nilpotent kernel, therefore a family $(B_i)_{i\in I}$ of étale A-algebras *covers* A (cf. § 0) if and only if the family $(\mathbf{R}(B_i))_{i\in I}$ *covers* $\mathbf{R}(A)$. Q.E.D.

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