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MINIMAL INJECTIVE RESOLUTIONS WITH APPLICATIONS TO DUALIZING MODULES AND GORENSTEIN MODULES

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Since Bass' original paper [5] on injective resolutions and Gorenstein rings, there has been a number of papers following a similar theme. Perhaps the most notable is the paper of Peskine and Szpiro [33] which blends the algebraic theory begun by M. Auslander, Bass *et al.* together with the more geometric concepts of Serre and Grothendieck to solve many outstanding problems in commutative algebra of a homological flavor as well as obtaining finiteness and vanishing theorems in the cohomology of schemes. It is our intent in this paper to investigate further the properties of minimal injective resolutions and to consider algebraic criteria for the vanishing of local cohomology. We also study the structure of a special class of modules of finite injective dimension introduced by Sharp [40] under the name of Gorenstein modules. And in the same vein, we consider conditions under which a local ring has a dualizing module (that is, a " module of dualizing differentials " in the sense of Grothendieck [19]). In what immediately follows, we describe briefly our paper in a section-by-section account.

In Section 1 we consider, for a finitely generated module M over a commutative noetherian ring A with minimal injective resolution $M \to I^0 \to I^1 \to \ldots$, the question: If $\mathfrak{p}$ is a prime ideal of A and if $j > 0$, when is $\mathfrak{p}$ associated to the injective module I^j? Although many partial answers were obtained by Bass [5] and Foxby [14], by making use of Hochster's construction of maximal Cohen-Macaulay modules [25], we are able to give a complete answer (this answer is independent of the characteristic of A).

We concern ourselves in Section 2 with the cohomology modules of a dualizing complex of a local ring. (These modules are dual via Matlis duality to the local cohomology modules of A.) And we establish a vanishing criteria a form of which

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Hartshorne and Ogus [23] have used in order to show complete local, factorial rings of equi-characteristic zero are Cohen-Macaulay.

In Section 3, after discussing pertinent facts concerning dualizing modules and Gorenstein modules, we turn to a problem related to the central result of Section 1, namely: If A is a local ring with maximal ideal $\mathfrak{m}$ and M is a finitely generated A-module, under what conditions can $\mathrm{Ext}^j(-, M)$ be exact and nonvanishing on the category of A-modules of finite length (denoted by $\mathscr{L}$)? The relation of this question to that of Section 1 is that the maximal ideal $\mathfrak{m}$ is not associated to the j-th injective I^j (in a minimal injective resolution of M) if and only if $\mathrm{Ext}^j(-, M)$ vanishes on $\mathscr{L}$. It turns out that such a module M must be a Gorenstein A-module and that necessarily $j = \dim A$.

In our investigation of the structure of Gorenstein modules in Section 4, we utilize heavily the fact that the endomorphism ring of such a module is an Azumaya algebra. Among other results, we show that the endomorphism ring of a Gorenstein module G over a local ring A necessarily has order one or two in the Brauer group of A. This yields the result: If G has odd rank r (see Section 3 (3.7) for the definition of rank), then A has a dualizing module Ω and $G \cong \Omega^r$.

In Section 5 we obtain complete answers to all questions of Section 3 for one-dimensional local rings. Two facts which emerge are, firstly, that a one-dimensional local ring has a dualizing module if and only if it has Gorenstein formal fibers and, secondly, an analytically irreducible one-dimensional local ring necessarily has a dualizing module.

Finally, in Section 6 we give some classical examples of Cohen-Macaulay rings which are n-Gorenstein but not $(n+1)$-Gorenstein by utilizing the properties of dualizing modules and the Eagon-Northcott resolution [9].

An excellent source for standard but unexplained conventions and terminology is Matsumura's book [30].

1. Minimal injective resolutions.

In this first section of our paper we wish to refine the observations made by Bass [5] concerning minimal injective resolutions of finitely generated modules over a noetherian ring. In this section the word ring always means a commutative noetherian ring containing a multiplicative identity. The notation and definitions which immediately follow are standard and are used without further reference in subsequent sections.

If M is a finitely generated module over a ring A, then M has a minimal injective resolution:

$$0 \to M \to I^0 \to I^1 \to \dots \to I^j \to \dots$$

It is known from Matlis [29] (or Gabriel [17]) that each injective module I^j is a direct sum of indecomposable injectives of the form $E(A/\mathfrak{p})$ for some prime ideal $\mathfrak{p}$ in the support of M, where $E(A/\mathfrak{p})$ denotes the injective envelope of $A/\mathfrak{p}$. The main result of this section (Theorem (1.1)) answers the question: For which integers j is $E(A/\mathfrak{p})$ a direct summand of I^j? The notation $\mu^j(\mathfrak{p}, M)$ denotes the number of copies of $E(A/\mathfrak{p})$ in I^j. Bass [5] shows that $\mu^j(\mathfrak{p}, M)$ depends only on j, $\mathfrak{p}$ and M. If $M_\mathfrak{p}$ is of finite injective dimension over $A_\mathfrak{p}$, say $t = \mathrm{id}_{A_\mathfrak{p}} M_\mathfrak{p} < \infty$, then $\mu^t(\mathfrak{p}, M) \neq 0$ while $\mu^j(\mathfrak{p}, M) = 0$ for $j > t$. The least number for which $\mu^j(\mathfrak{p}, M) \neq 0$ turns out to be the depth of the $A_\mathfrak{p}$-module $M_\mathfrak{p}$ (abbreviated $\mathrm{depth}_{A_\mathfrak{p}} M_\mathfrak{p}$, cf. Bass [5]). So the aforementioned question becomes: Is $\mu^j(\mathfrak{p}, M) \neq 0$ for all j between $\mathrm{depth}_{A_\mathfrak{p}} M_\mathfrak{p}$ and $\mathrm{id}_{A_\mathfrak{p}} M_\mathfrak{p}$ (the latter might be infinite)? Bass gives the following result: If $\mathrm{id}_{A_\mathfrak{p}} M_\mathfrak{p} = \infty$, then $\mu^j(\mathfrak{p}, M) \neq 0$ for $j \geq \dim A_\mathfrak{p}$. In [14] Foxby gives an affirmative answer to the last question in the following special cases:

1) The ring $A_\mathfrak{p}$ or the module $M_\mathfrak{p}$ is Cohen-Macaulay;
2) For the prime $\mathfrak{p}$ we have $\mathrm{depth}_{A_\mathfrak{p}} A_\mathfrak{p} \leq \mathrm{depth}_{A_\mathfrak{p}} M_\mathfrak{p}$; and
3) The module $M_\mathfrak{p}$ has finite injective dimension.

We now provide an affirmative answer in the general case. Our proof relies heavily on Hochster's construction of (possibly non-noetherian) maximal Cohen-Macaulay modules (see [25]), but also uses Bass' result and case 2) mentioned above.

Theorem **(1.1)**. — *Let* A *be a commutative noetherian ring and let* M *be a finitely generated* A-*module with minimal injective resolution*:

$$0 \to M \to I^0 \to I^1 \to \ldots \to I^j \to \ldots$$

Let $\mathfrak{p}$ *be a prime ideal in the support of* M. *Then* $E(A/\mathfrak{p})$ *is a direct summand of* I^j *for all* j *such that* $\mathrm{depth}_{A_\mathfrak{p}} M_\mathfrak{p} \leq j \leq \mathrm{id}_{A_\mathfrak{p}} M_\mathfrak{p}$. (The latter might be infinite.)

The proof of Theorem (1.1) is accomplished via the propositions which follow. The first of these is merely a restatement of Bass' result [5; Lemma 3.5] for modules. Its proof also follows easily from local duality as in Peskine and Szpiro [33].

Proposition **(1.2)** [Bass]. — *If* $\mathrm{id}_{A_\mathfrak{p}} M_\mathfrak{p} = \infty$, *then*

$$\mu^j(\mathfrak{p}, M) > 0, \quad \textit{for all} \quad j \geq \dim A_\mathfrak{p} = \mathrm{ht}\,\mathfrak{p}.$$

For the proof of the next proposition one may consult Foxby's article [14]. An additional remark here is that, if a finitely generated module M over a local ring A satisfies the condition $\mathrm{depth}\, A \leq \mathrm{depth}\, M \leq \mathrm{id}\, A$, then $\mathrm{depth}\, A = \mathrm{depth}\, M = \mathrm{id}\, A$ whenever $\mathrm{id}\, A < \infty$.

Proposition **(1.3)** [Foxby]. — *Let* $(A, \mathfrak{m}, k)$ *be a local ring and let* M *be a finitely generated* A-*module with* $\mathrm{depth}\, M \geq \mathrm{depth}\, A$. *Then* $\mu^j(\mathfrak{m}, M) > 0$ *for* $\mathrm{depth}\, M \leq j \leq \mathrm{id}\, A$.

The following lemma will be useful together with Hochster's construction of maximal Cohen-Macaulay modules. For A a local ring with residue field k and maximal ideal $\mathfrak{m}$, the notation $M^\vee$ denotes the Matlis dual of the A-module M, that is

$$M^\vee = \mathrm{Hom}(M, E(k)).$$

Furthermore, if M is finitely generated, then $\operatorname{Supp} M^{\vee}=\{\mathfrak{m}\}$ and hence $\mathfrak{m}$ is the only possible prime ideal in the support of $\operatorname{Tor}_i(X, M^{\vee})$ for an arbitrary A-module X.

Lemma (**1.4**). — *Let* $(A, \mathfrak{m}, k)$ *be a local ring and let* N *be a* (not necessarily finitely generated) A-*module. Suppose* $x_1, \ldots, x_n$ *is an* N-*regular sequence such that* $\operatorname{ann}(N/\mathfrak{x}_n N)$ *is proper and* $\mathfrak{m}$-*primary* (*where* $x_0=0$ *and* $\mathfrak{x}_j=(x_1, \ldots, x_j)$ *for* $1 \leq j \leq n$). *Furthermore, let* M *be a finitely generated nonzero module of depth zero. Then* $\operatorname{Ext}^i(N/\mathfrak{x}_j N, M) \neq 0$ *for all* i *and* j *with* $0 \leq i \leq j$.

Proof. — The argument goes by induction on j. For $j=0$, we have that $N \otimes k \neq 0$ since $\operatorname{ann}(N/\mathfrak{x}_n N)$ is $\mathfrak{m}$-primary and since $\mathfrak{m}(N/\mathfrak{x}_n N) \neq N/\mathfrak{x}_n N$. Consequently

$$\operatorname{Hom}(N, k)=(N \otimes k)^{\vee} \neq 0.$$

Therefore, since $\operatorname{depth} M=0$ is equivalent to k being isomorphic with a submodule of M, it follows that $\operatorname{Hom}(N, M) \neq 0$ as desired. With regard to the induction step, we first note the standard duality isomorphisms give (since $M=M^{\vee\vee}$):

$$\operatorname{Ext}^i(N/\mathfrak{x}_j N, M) \cong \operatorname{Tor}_i(N/\mathfrak{x}_j N, M^{\vee})^{\vee}$$

for all i and j (see Cartan and Eilenberg [7; Chapter VI]). By hypothesis there is an exact sequence:

$$0 \rightarrow N/\mathfrak{x}_{j-1}N \xrightarrow{x_j} N/\mathfrak{x}_{j-1}N \rightarrow N/\mathfrak{x}_j N \rightarrow 0.$$

Now if $i \leq j$, then $\operatorname{Tor}_{i-1}(N/\mathfrak{x}_i N, M^{\vee}) \neq 0$, by the inductive hypothesis, so x_j is a zero divisor on $\operatorname{Tor}_{i-1}(N/\mathfrak{x}_{j-1}N, M^{\vee})$ by the remark preceding the statement of this lemma. The fact that $\operatorname{Tor}_i(N/\mathfrak{x}_j N, M^{\vee}) \neq 0$ now follows from the exact sequence

$$\operatorname{Tor}_i(N/\mathfrak{x}_j N, M^{\vee}) \rightarrow \operatorname{Tor}_{i-1}(N/\mathfrak{x}_{j-1}N, M^{\vee}) \xrightarrow{x_j} \operatorname{Tor}_{i-1}(N/\mathfrak{x}_{j-1}N, M^{\vee})$$

which is induced by the previous short exact sequence. Q.E.D.

Lemma (**1.5**). — *Let* $(A, \mathfrak{m}, k)$ *be a local ring and let* j *be a nonnegative integer. If* M *is an* A-*module and if* $\mu^j(\mathfrak{m}, M)=0$, *then* $\operatorname{Ext}^j(T, M)=0$ *for all* A-*modules* T *with support in* $\{\mathfrak{m}\}$.

Proof. — By induction on length it follows that $\operatorname{Ext}^j(-, M)$ vanishes on modules of finite length. If T is an A-module with support in $\{\mathfrak{m}\}$, then $T=\varinjlim T_\alpha$ where each T_α is a module of finite length. Since the inverse limit functor is exact on the category of projective systems of modules of finite length, it follows that

$$\operatorname{Ext}^j(T, M) \cong \varprojlim \operatorname{Ext}^j(T_\alpha, M)=0$$

(cf. Roos [36]). Q.E.D.

We are now ready to return to the proof of our main result of this section.

Proof of Theorem (1.1). — Using standard facts concerning the $\mu^j(\mathfrak{p}, M)$ under localization and completion which can be found in Bass [5] or Peskine and Szpiro [33], it is enough to prove that $\mu^j(\mathfrak{m}, M) \neq 0$ for all j with $\operatorname{depth} M \leq j \leq \operatorname{id} M$ under the assumption that A is a complete local ring with maximal ideal $\mathfrak{m}$. Because of this reduction we shall abuse the notation $\mu^j(\mathfrak{m}, M)$ by writing $\mu^j(M)$. To begin we assume the contrary, that is, assume that $\mu^j(M) = 0$ for some j with $\operatorname{depth} M \leq j \leq \operatorname{id} M$. It follows from the result of Bass (Proposition (1.2)) quoted previously that $j < \dim A$ and from Foxby's result (Proposition (1.3)) that depth A>depth M. Now let $a_1, \ldots, a_s$ be a maximal M- and A-regular sequence. Hence $s = \operatorname{depth} M$. We write

$$\bar{C} = C/(a_1, \ldots, a_s)C$$

for an A-module C. We note that $\mu_{\bar{A}}^i(\bar{M}) = \mu_A^{i+s}(M)$ for all i (see Bass [5; Corollary 2.6]). Hence $\mu_{\bar{A}}^{j-s}(\bar{M}) = 0$. We put $d = \dim \bar{A} = -s + \dim A$.

Before concluding with the final step in our proof, we must recall the remarkable result proved by Hochster [25]: If R is an equi-characteristic local ring of dimension t with a parameter system $x_1, \ldots, x_t$, then there exists a (possibly non-noetherian) R-module T such that $x_1, \ldots, x_t$ is a T-regular sequence and such that $T/(x_1, \ldots, x_t)T$ is nonzero. Such a module is called a *maximal Cohen-Macaulay module.*

Once again, returning to the proof of Theorem (1.1), we let k denote the residue field of A and p the characteristic of k (so p is either zero or a prime number). Then $R = \bar{A}/p\bar{A}$ is an equi-characteristic local ring of dimension either $d-1$ or d. If $\dim R = d-1$ (respectively, if $\dim R = d$) let $x_2, \ldots, x_d$ (respectively, $x_1, \ldots, x_d$) be elements in $\bar{A}$ whose images in R form a parameter system for R. In both cases let T be a maximal Cohen-Macaulay R-module as in the preceding paragraph. If

$$\dim R = d-1$$

put $N = T$ and if $\dim R = d$ put $N = T/x_1T$. In each case $x_2, \ldots, x_d$ is an N-regular sequence and $\operatorname{ann}_{\bar{A}}(N/\mathfrak{x}N)$ is $\mathfrak{m}'$-primary, where $\mathfrak{x} = (x_2, \ldots, x_n)$ and $\mathfrak{m}' = \mathfrak{m}\bar{A}$. From Lemma (1.4) we have that $\operatorname{Ext}_{\bar{A}}^{j-s}(N/\mathfrak{x}N, \bar{M}) \neq 0$, since $j-s \leq d-1$. Moreover, by Lemma (1.5), it follows that $\operatorname{Ext}_{\bar{A}}^{j-s}(k, \bar{M}) \neq 0$, that is, $\mu_{\bar{A}}^{j-s}(\bar{M}) \neq 0$ which gives the desired contradiction. Q.E.D. for Theorem

Remark. — It is easy to give examples which show that Theorem (1.1) does not hold for nonfinitely generated modules M, e.g. if N is finitely generated of depth two and $M = N \oplus E(k)$, then $\mu^0(M) = 1$, $\mu^1(M) = 0$ and $\mu^2(M) \neq 0$. However, we can state the following partial result which is easily proved: If A is a local ring, M an A-module and i a number greater than depth A such that $\mu^i(M) \neq 0$, then $\mu^j(M) \neq 0$ for $j \geq i$.

2. On the vanishing of local cohomology.

After recalling some basic facts about dualizing complexes, we establish a representation (Corollary (2.3)) for the functor $\operatorname{Ext}_A^s(E(k), -)$ used so successfully by Peskine and Szpiro [33], where s is the depth of the local ring A. We then concentrate on the vanishing of local cohomology which subsequently yields a result of Hartshorne and Ogus [23] on factorial rings.

Let R be a Gorenstein local ring of dimension d and let A be a homomorphic image of R, say $A = R/\mathfrak{I}$. Let $\mathfrak{P}$ and $\mathfrak{p}$ denote the maximal ideals of R and A, respectively, and let k be the common residue class field of R and A. In the minimal R-injective resolution of R, i.e. the dualizing complex for R (see Grothendieck [19])

$$0 \to R \to I_R^0 \to I_R^1 \to \dots \to I_R^d \to 0,$$

the j-th injective module I_R^j is of the form $\coprod E_R(R/\mathfrak{Q})$, where the sum is taken over all prime ideals $\mathfrak{Q}$ in R of height j, that is (since R is Cohen-Macaulay) all primes $\mathfrak{Q}$ in R with $\dim R/\mathfrak{Q} = d-j$. In particular $I_R^d = E_R(k)$. Applying the functor $\mathrm{Hom}_R(A, -)$ to this injective complex $I_R^{\bullet}$ we obtain a complex $I_A^{\bullet} = \mathrm{Hom}_R(A, I_R^{\bullet})$ of injective A-modules, the dualizing complex for A. We note that, for $j \geq \dim R - \dim A$, the module $I_A^j = \mathrm{Hom}_R(A, I_R^j) = \coprod E_A(A/\mathfrak{q})$, the sum being taken over the prime ideals $\mathfrak{q}$ in A with $\dim A/\mathfrak{q} = \dim A - j$. With regard to the cohomology of this latter complex, we have that $\mathrm{Ext}_R^j(A, R) = 0$ if $j < \dim R - \dim A$ or if $j > \dim R - \mathrm{depth}\, A$ and that

$$\mathrm{Ext}_R^j(A, R) \neq 0$$

in case $j = \dim R - \dim A$ and $j = \dim R - \mathrm{depth}\, A$ (see Grothendieck [19]). Moreover, Grothendieck's local duality theorem which is stated in Section 3 shows that the A-modules $\mathrm{Ext}_R^j(A, R)$ are independent of the Gorenstein ring R (except of course for the superscript "j") of which A is a homomorphic image. Consequently, dividing R by an appropriate R-regular sequence in the ideal $\mathfrak{I}$, we may assume (and do hereafter) that $d = \dim R = \dim A$. Setting $\Omega_A^j = \mathrm{Ext}_R^j(A, R)$, we see from the discussion above that $\Omega_A^0 \neq 0$, $\Omega_A^t \neq 0$, for $t = d - \mathrm{depth}\, A$, and finally that $\Omega_A^j = 0$ for $j > t$. Returning to the dualizing complex of A, $0 \to I_A^0 \to I_A^1 \to \dots \to I_A^d \to 0$, described above and letting B denote the t-th coboundary module in this complex and W_A the corresponding cocycle module, we obtain the following diagram with exact rows and column:

$$\begin{array}{ccccccccccccc}
 & & & & 0 & & & & & & & & \\
 & & & & \downarrow & & & & & & & & \\
I_A^{t-2} & \longrightarrow & I_A^{t-1} & \longrightarrow & B & \longrightarrow & 0 & & & & & & \\
 & & & & \downarrow & & & & & & & & \\
 & & 0 & \longrightarrow & W_A & \longrightarrow & I_A^t & \longrightarrow & \dots & \longrightarrow & I_A^{d-1} & \longrightarrow & E_A(k) \longrightarrow 0 \\
 & & & & \downarrow & & & & & & & & \\
 & & & & \Omega^t & & & & & & & & \\
 & & & & \downarrow & & & & & & & & \\
 & & & & 0. & & & & & & & &
\end{array}$$

Diagram (2.1)

We observe, for $\mathfrak{q}$ a prime ideal in A, that

$$\mu_A^j(\mathfrak{q}, W_A) = \begin{cases} 1, & \text{for} \quad j = d - \dim A/\mathfrak{q} - t \geq 0 \\ 0, & \text{otherwise,} \end{cases}$$

and that $\operatorname{id} W_A = s = \operatorname{depth} A$. In particular, if A is Cohen-Macaulay, then $\Omega_A^0 = W_A$ and $\mu_A^j(\mathfrak{q}, \Omega_A^0) = \delta_{j, \operatorname{ht}\mathfrak{q}}$. In general the Cohen-Macaulay locus of A is the (pairwise) disjoint union of the Zariski open sets

$$U_j = \{\mathfrak{q} \in \operatorname{Spec} A : \Omega_\mathfrak{q}^i = 0 \text{ for } i \neq j\}.$$

Proposition **(2.2)**. — *Let* A *be a complete local ring and let* M *be a finitely generated* A*-module. Then there are functorial isomorphisms*

$$\operatorname{Ext}_A^{j+s}(E(k), M) \cong \operatorname{Ext}_A^j(W_A, M)$$

for all j, *where* $s = \operatorname{depth} A$.

Proof. — Since A is a complete local ring, then via the Cohen structure theorem [8] A is the homomorphic image of a Gorenstein local ring R. (Hence, we shall make use of the notation and general set up preceding the statement of this proposition.) If $\mathfrak{q}$ is a prime ideal in A which is not maximal and if L is an A-module of finite length, then $\operatorname{Ext}_A^i(E(A/\mathfrak{q}), L) = 0$ for all $i \geq 0$, since there must be some $x \in A - \mathfrak{q}$ which acts as an automorphism on $E(A/\mathfrak{q})$ and such that $xL = 0$. Since M is of finite type and complete (so $M = \varprojlim (M/\mathfrak{p}^n M)$, where $\mathfrak{p}$ is the maximal ideal of A), there is an exact sequence

$$0 \to M \to \prod_n (M/\mathfrak{p}^n M) \to \prod_n (M/\mathfrak{p}^n M) \to 0,$$

where here M is identified with $\varprojlim (M/\mathfrak{p}^n M)$. This exact sequence together with the preceding discussion gives the result

$$\operatorname{Ext}^i(E(A/\mathfrak{q}), M) = 0 \quad \text{for all} \quad i \geq 0.$$

We now use the middle row of diagram (2.1) to further obtain the isomorphism $\operatorname{Ext}^{j+s}(E(k), M) \cong \operatorname{Ext}^j(W_A, M)$. Q.E.D.

Corollary **(2.3)**. — *Let* A *be a complete local ring and let* M *be a finitely generated* A*-module. Then:*

$$\operatorname{Ext}_A^j(E(k), M) = \begin{cases} 0, & \textit{if} \quad j < s. \\ \operatorname{Hom}(\Omega_A^t, M), & \textit{for} \quad j = s. \end{cases}$$

Proof. — The fact that $\operatorname{Ext}_A^j(E(k), M) = 0$ for $j < s = \operatorname{depth} M$ is a result of Peskine and Szpiro [33]. By Proposition (2.2) we have that $\operatorname{Ext}_A^s(E(k), M) \cong \operatorname{Hom}(W_A, M)$. Again refering to Diagram (2.1), if $B \neq 0$, then B is the homomorphic image of the injective I_A^{t-1} which is a direct sum of injective A-modules of the form $E(A/\mathfrak{q})$, where

$\mathfrak{q}$ is not maximal. Since $\mathrm{Hom}(\mathrm{E}(\mathrm{A}/\mathfrak{q}), \mathrm{M}) = 0$ as above, it follows that $\mathrm{Hom}(\mathrm{B}, \mathrm{M}) = 0$. Thus, the column of Diagram (2.1) now gives that $\mathrm{Hom}(\Omega^{i}, \mathrm{M}) \cong \mathrm{Hom}(\mathrm{W}_{\mathrm{A}}, \mathrm{M})$.

Q.E.D.

We next turn to the vanishing of the modules Ω^{j}_{A} which in turn implies the vanishing of local cohomology. If A is a local ring with maximal ideal $\mathfrak{m}$ and residue field k which is a homomorphic image of a Gorenstein ring, then the precise connection between the local cohomology of A with respect to $\mathfrak{m}$ and the cohomology modules Ω^{j}_{A} is a consequence of Grothendieck's Theorem on local duality which gives

$$\mathrm{H}^{d-i}_{\mathfrak{m}}(\mathrm{A}) \cong (\Omega^{i}_{\mathrm{A}})^{\vee},$$

where $d = \dim \mathrm{A}$, where $\mathrm{M}^{\vee}$ denotes the Matlis dual of M and where

$$\mathrm{H}^{j}_{\mathfrak{m}}(\mathrm{M}) = \varinjlim_{n} \mathrm{Ext}^{j}(\mathrm{A}/\mathfrak{m}^{n}, \mathrm{M}).$$

If $\Omega^{j}_{\mathrm{A}} = 0$ for $j > 0$, then of course A is Cohen-Macaulay and conversely. In fact, it is our intention here to establish criteria in order that a ring be Cohen-Macaulay (especially in case the ring is factorial).

Lemma **(2.4)**. — *If* A *is a normal local domain which is the homomorphic image of a Gorenstein ring, then* Ω^{0}_{A} *is a divisorial ideal. In particular, if* A *is factorial, then* $\Omega^{0}_{\mathrm{A}} \cong \mathrm{A}$.

Proof. — As usual let $\mathrm{A} = \mathrm{R}/\mathfrak{I}$ where R is a Gorenstein local ring of dimension $d = \dim \mathrm{A}$. Since $\Omega^{0}_{\mathrm{A}} = \mathrm{Hom}_{\mathrm{R}}(\mathrm{A}, \mathrm{R})$ is easily seen to be a reflexive R-module, we have, for $\mathfrak{p}$ a prime ideal, the inequality

$$\mathrm{depth}_{\mathrm{A}_{\mathfrak{p}}}(\Omega^{0}_{\mathrm{A}})_{\mathfrak{p}} = \mathrm{depth}_{\mathrm{R}_{\mathfrak{P}}}(\Omega^{0}_{\mathrm{A}}) \geq \min(2, \dim \mathrm{A}_{\mathfrak{p}}),$$

where $\mathfrak{P}$ is the prime ideal of R such that $\mathfrak{p} = \mathfrak{P}/\mathfrak{I}$. So Ω^{0}_{A} certainly satisfies Samuel's condition (a_2) (that is, every A-regular sequence of length two is an Ω^{0}_{A}-regular sequence) and is therefore a reflexive A-module.

We also remark that Ω^{0}_{A} is necessarily isomorphic to an ideal of A since, if K is the field of quotients of A, then $\Omega^{0}_{\mathrm{A}} \otimes \mathrm{K} \cong \mathrm{I}^{0}_{\mathrm{A}} \otimes_{\mathrm{A}} \mathrm{K} \cong \mathrm{K} \otimes_{\mathrm{A}} \mathrm{K} \cong \mathrm{K}$. Q.E.D.

From Lemma (2.4) one may easily deduce Murthy's result [31] that a factorial ring which is the Cohen-Macaulay homomorphic image of a Gorenstein ring is itself Gorenstein. In fact, one may draw a slightly stronger conclusion, namely: If A is a factorial ring which is the homomorphic image of a Gorenstein ring, and if A has a Cohen-Macaulay ideal, then A is a Gorenstein ring. One need only observe that a Cohen-Macaulay ideal is necessarily divisorial.

Proposition **(2.5)**. — *Let* A *be a local ring which is a factor ring of a Gorenstein ring and which satisfies the following two conditions:*

(1) $\mathrm{Supp}\, \Omega^{0}_{\mathrm{A}} = \mathrm{Spec}\, \mathrm{A}$.

(2) *The localizations* $\mathrm{A}_{\mathfrak{p}}$ *are Cohen-Macaulay for all prime ideals* $\mathfrak{p}$ *with* $\mathrm{depth}\, (\Omega^{0}_{\mathrm{A}})_{\mathfrak{p}} < n$. *Then* $\Omega^{j}_{\mathrm{A}} = 0$ *for* $1 \leq j \leq n-2$.

Proof. — We proceed by induction on n. And since there is nothing to prove for $n \leq 2$, let us assume $n \geq 3$ and (by the inductive hypothesis) $\Omega_A^j = 0$ for $1 \leq j \leq n-3$. Also we may assume that Ω^{n-2} is of finite length; otherwise, we would apply the induction hypothesis on $d = \dim A$ in order to obtain a contradiction. As usual, we assume $A = R/\mathfrak{J}$, where R is a Gorenstein local ring and $\dim A = \dim R$. Let $P_\bullet$ be an R-free resolution of A and let $M^* = \mathrm{Hom}_R(M, R)$ denote the R-dual of M. We obtain a complex of R-modules

$$0 \to \Omega_A^0 \to P_0^* \to \ldots \to P_{n-3}^* \to P_{n-2}^* \to P_{n-1}^*$$

of length n. Here $\mathrm{depth}_R \Omega_A^0 \geq n$ (for otherwise A is Cohen-Macaulay and there is nothing to prove) and $\mathrm{depth}_R P_i^* = d = \dim R$ for all i. Furthermore, the only possible nonzero homology module is Ω_A^{n-2} which is of finite length. Therefore, we can apply the Acyclicity Lemma of Peskine and Szpiro [33; Lemma 1.8] to get $\Omega_A^{n-2} = 0$ as desired. Q.E.D.

Corollary **(2.6)**. — *Suppose* A *is a local ring which is a factor of a Gorenstein ring and let n be a positive integer. Assume that* A *satisfies Serre's condition* (S_n) *and further assume that* Ω_A^0 *satisfies Samuel's condition* (a_n) *and* $\mathrm{Supp}\, \Omega_A^0 = \mathrm{Spec}\, A$ (e.g. if $\Omega_A^0 \cong A$ as is the case when A is a factorial ring). *Then* $\Omega_A^j = 0$ *for* $1 \leq j \leq n-2$. *In particular if it is the case that* $\mathrm{depth}\, A + n - 2 \geq \dim A$, *then* A *is Cohen-Macaulay.*

Although there are local factorial rings in characteristic $p > 0$ (of dimension not exceeding four) which are not Cohen-Macaulay (see Bertin [6] and for complete local rings see Fossum and Griffith [12]), the situation in characteristic zero is quite different. Indeed, Raynaud, Boutot, and Hartshorne and Ogus [23] have shown that a complete local factorial ring with residue field algebraically closed and of characteristic zero necessarily satisfies Serre's condition (S_3). And with the aid of the following lemma, Hartshorne and Ogus [23] show that such a ring is Cohen-Macaulay if its Krull dimension does not exceed four.

Corollary **(2.7)** (Hartshorne and Ogus). — *Assume that* A *is a factor of a Gorenstein ring which satisfies the condition* (S_3). *If* A *is factorial and in addition satisfies the condition*:

$$\mathrm{depth}\, A_\mathfrak{p} \geq \tfrac{1}{2} \mathrm{ht}\, \mathfrak{p} + 1,$$

for any prime ideal $\mathfrak{p}$ *of* A, *then* A *is Cohen-Macaulay* (and thereby Gorenstein).

Proof. — By induction on n it will be proved that A is (S_n) for all n. Since there is nothing to prove for $n \leq 3$, we assume that $n \geq 4$ and that A is (S_{n-1}). Let $\mathfrak{p}$ be a prime ideal in A with $\mathrm{depth}\, A_\mathfrak{p} = n-1$. Then $\mathrm{ht}\, \mathfrak{p} \leq 2(n-1) - 2 = 2n-4$ and therefore $\mathrm{ht}\, \mathfrak{p} - \mathrm{depth}\, A_\mathfrak{p} \leq n-3$; so $\Omega_{A_\mathfrak{p}}^j = 0$ for $j > n-3$. On the other hand $\Omega_{A_\mathfrak{p}}^j = 0$ for $1 \leq j \leq n-3$ by Proposition (2.5) or by Corollary (2.6). Q.E.D.

3. Dualizing modules and Gorenstein modules.

In order to facilitate the discussion that follows, we shall assume until further notice that A is a local ring with maximal ideal $\mathfrak{m}$ and residue class field k. Moreover $\mathscr{L}$ denotes the category of A-modules of finite length. While our main theorem of Section 1 (Theorem (1.1)) can be viewed as conditions as to when $\mathrm{Ext}^j(-, M)$ vanishes on $\mathscr{L}$ (that is, as to when " holes " in the injective resolution of M can appear for lack of $E(k)$ as a direct summand), in this section of our paper we concern ourselves with the more general question of the exactness of $\mathrm{Ext}^j(-, M)$ on $\mathscr{L}$. This question is in turn related to Grothendieck's Theorem [19] on local duality which states that

$$\mathrm{Ext}^d(-, A) = \mathrm{Hom}(-, H^d_{\mathfrak{m}}(A)) = \mathrm{Hom}(-, E(k))$$

is a natural equivalence of functors on $\mathscr{L}$ for A a d-dimensional Gorenstein ring. More generally our discussion here is related to the so-called Gorenstein modules, which are those for which their Cousin complex gives a minimal injective resolution (cf. Sharp [40], but also Hartshorne [22]). The structure and properties of Gorenstein modules have been studied by Hartshorne [22], Sharp [37], [38], [39], Foxby [14], [15] and Reiten [34]. Since we shall be concerned with Gorenstein modules in one fashion or another throughout the remainder of this paper, it is convenient to elaborate here on the properties of Gorenstein modules and dualizing modules needed for subsequent discussion. The definition of a Gorenstein module in terms of its Cousin complex really states that such a module G (always assumed finitely generated and nonzero) has a " nice " injective resolution in that $\mu^j(\mathfrak{p}, G) \neq 0$ if and only if $\mathrm{ht}\,\mathfrak{p} = j$. This property turns out to be equivalent to the following ones:

Definition **(3.1)** [15; Foxby]. — *The module* G *has finite injective dimension and*

$$\mathrm{Ext}^j(G, G) = \begin{cases} \textit{projective}, & \textit{if } j = 0 \\ 0, & \textit{if } j > 0. \end{cases}$$

(3.2) [37; Sharp]. — *The module* G *satisfies* depth G = id G.

(3.3) [37; Sharp] [15; Foxby]. — *The finitely generated* A-*module* G *is Gorenstein if and only if each regular* A-*sequence* $\mathfrak{x}$ *is a regular* G-*sequence and* $G/\mathfrak{x}G$ *is a Gorenstein* $A/\mathfrak{x}A$-*module.*

Some further properties of Gorenstein modules which will prove useful are the following:

(3.4) [37; Sharp]. — *If* A *has a Gorenstein module* G, *then both* A *and* G *are Cohen-Macaulay and* Supp G = Spec A.

(3.5) [15; Foxby]. — *If* G *is a Gorenstein* A-*module and if* $\mathfrak{x}$ *is an* A-*regular sequence, then* $\mathrm{End}_A G/\mathfrak{x}\,\mathrm{End}_A G \cong \mathrm{End}_{A/\mathfrak{x}A}(G/\mathfrak{x}G)$.

(3.6) [37; Sharp] [15; Foxby]. — *If* G *is a Gorenstein* A*-module and if* $\mathfrak{x}$ *is a maximal* A*-regular sequence, then* $G/\mathfrak{x}G \cong E^n$, *where* E *is the* $A/\mathfrak{x}A$*-injective envelope of* k. *Then* $n = (\text{rank End G})^{1/2} = \mu^d(G) = \dim_k \operatorname{Ext}^d(k, G)$, *where* $d = \dim A$.

After Sharp [37] we call this common integer the *rank* of G (as a Gorenstein module). In the situation that A is an integral domain it is easily seen that the usual notion of rank (that is, $\dim_K(K \otimes G)$ where K is the field of quotients of A) is the same as the Gorenstein rank of G. Finally, we mention that it is straightforward to globalize the notion of a Gorenstein module. However, we do require that $G_\mathfrak{m} \neq 0$ for each maximal ideal $\mathfrak{m}$.

Classically, the notion of a Gorenstein module must go back to Grothendieck (see [18; pages 94, 95] and [19]) and the so-called *module of dualizing differentials* (also see Section 5 (Remarks of Serre) of Bass [5] and also Sharp [40; Theorem 3.1]). Indeed, the classical example [19] is simply Ω_A^0 (as defined in Section 2) under the assumption that A is Cohen-Macaulay. In this instance $\Omega_A^j = 0$ for $j > 0$. Moreover, there is a natural equivalence of functors

$$\operatorname{Ext}^d(-, \Omega_A^0) = \operatorname{Hom}(-, E(k))$$

on modules of finite length. For this reason we call such a module a dualizing module for A. One observes from Sharp [37] that this is equivalent to the notion of a Gorenstein module of rank one (if A is Cohen-Macaulay the notion of a dualizing module is also equivalent to that of a canonical module in Herzog and Kunz [24]). We state a final observation concerning dualizing modules before coming to the central theorem of this section.

(3.7) [Reiten [34] and Foxby [15]]. — *The trivial extension* $A \ltimes \Omega$ *is a Gorenstein ring if and only if* A *is Cohen-Macaulay and* Ω *is a dualizing module for* A.

Here $A \ltimes \Omega$ denotes the ring obtained through componentwise addition and multiplication given by: $(a, x) . (b, y) = (ab, ay + bx)$. From this fact and the preceding discussion we see that A has a dualizing module if and only if it is the homomorphic image of a Gorenstein local ring. Thus, via the Cohen structure theory [8] we see that every complete Cohen-Macaulay local ring has a dualizing module. In [13] Fossum, Griffith and Reiten developed and used the theory of trivial extensions to deduce properties of dualizing modules from the properties of Gorenstein rings as in Bass [5]. For a discussion of the global theory of dualizing modules see Sharp [38].

Theorem **(3.8)**. — *Let* A *be a local ring with maximal ideal* $\mathfrak{m}$ *and residue class field* k. *Let* $\mathcal{L}$ *denote the category of* A*-modules of finite length. If* M *is a finitely generated* A*-module and if* t *is a nonnegative integer such that* $\operatorname{Ext}^t(-, M)$ *is a nonzero exact functor on* $\mathcal{L}$, *then* M *is a Gorenstein module and* $t = \dim A$.

Proof. — As usual we may assume that A is complete. Let R be a Gorenstein local ring such that A is a homomorphic image of R, say $A = R/\mathfrak{r}$. We may assume that $\dim R = \dim A = d$. Since $\mathrm{Ext}^t(-, M)$ is left exact on $\mathscr{L}$, we have

$$\mathrm{Ext}^t(L, M) \cong \mathrm{Hom}(L, H^t_{\mathfrak{m}}(M))$$

for all modules L in $\mathscr{L}$, where $H^t_{\mathfrak{m}}(M)$ denotes the t-th right derived functor of the section function with support in $\{\mathfrak{m}\}$, that is, $H^t_{\mathfrak{m}}(M) = \varinjlim_j \mathrm{Ext}^t(A/\mathfrak{m}^j, M)$ (cf. Grothendieck [19]). The identity $\mathrm{Ext}^t(-, M) \cong \mathrm{Hom}(-, H^t_{\mathfrak{m}}(M))$ also holds on all modules with support in $\{\mathfrak{m}\}$ because the inverse limit functor is exact on projective systems over $\mathscr{L}$ and both functors carry direct limits into inverse limits. Now by the local duality theorem (Grothendieck [19; Theorem 6.3]) the local cohomology group $H^t_{\mathfrak{m}}(M) = \mathrm{Ext}^{d-t}_R(M, R)^\vee$. Since $\mathrm{ht}_R(\mathfrak{r}) = 0$ we can choose a prime ideal $\mathfrak{q}$ in R containing $\mathfrak{r}$ and also of height zero. Since $(R/\mathfrak{q})^\vee$ is an artinian A-module, we have the isomorphisms:

$$\begin{aligned}\mathrm{Ext}^t((R/\mathfrak{q})^\vee, M) &\cong \mathrm{Hom}((R/\mathfrak{q})^\vee, H^t_{\mathfrak{m}}(M))\\ &\cong \mathrm{Hom}((R/\mathfrak{q})^\vee, \mathrm{Ext}^{d-t}_R(M, R)^\vee)\\ &\cong ((R/\mathfrak{q})^\vee \otimes \mathrm{Ext}^{d-t}_R(M, R))^\vee\\ &\cong \mathrm{Hom}(\mathrm{Ext}^{d-t}_T(M, R), (R/\mathfrak{q})^{\vee\vee})\\ &\cong \mathrm{Hom}(\mathrm{Ext}^{d-t}_R(M, R), R/\mathfrak{q}).\end{aligned}$$

Since $(R/\mathfrak{q})^\vee$ has support in $\{\mathfrak{m}\}$, there is an exact sequence $0 \to k \to (R/\mathfrak{q})^\vee$, and since $\mathrm{Ext}^t_A(-, M)$ is also right exact on modules with support in $\{\mathfrak{m}\}$ this gives the exact sequence

$$\mathrm{Ext}^t((R/\mathfrak{q})^\vee, M) \to \mathrm{Ext}^t(k, M) \to 0.$$

By our assumption $\mathrm{Ext}^t(k, M) \neq 0$, and hence

$$\mathrm{Hom}(\mathrm{Ext}^{d-t}_R(M, R), R/\mathfrak{q}) = \mathrm{Ext}^t_A((R/\mathfrak{q})^\vee, M) \neq 0,$$

that is, $\mathrm{ann}_R \mathrm{Ext}^{d-t}_R(M, R) \subseteq \mathfrak{q}$ which is of height zero. On the other hand R is Gorenstein and therefore $\mathrm{grade}\, \mathrm{Ext}^{d-t}_R(M, R) \geq d - t$ (Bass [5]). Hence $d \leq t$. Since $\mathrm{Ext}^{t+1}_A(-, M)$ is necessarily left exact on $\mathscr{L}$, we have $\mathrm{Ext}^{t+1}_A(-, M) \cong \mathrm{Hom}(-, H^{t+1}_{\mathfrak{m}}(M))$ on $\mathscr{L}$. But $H^{t+1}_{\mathfrak{m}}(M) = 0$ since $t+1 > d = \dim A$ (cf. again Grothendieck [19; Theorem 6.4]). So $H^{t+1}_{\mathfrak{m}}(M) = 0$, and hence $\mathrm{id}\, M < t+1$ (Bass [5]). Now we know that $t = \mathrm{id}\, M = \mathrm{depth}\, A = \dim M = \dim A = d$. In particular A is Cohen-Macaulay.

To complete our proof, we recall from the proof of Theorem (1.4.10) in Peskine and Szpiro [33] that $\mathrm{Ext}^j(E(k), M) = 0$ for $j < d$. Since $\mathrm{Ext}^{d-1}(-, M)$ is right exact, we deduce from the exact sequence $0 \to k \to E(k)$ that $\mu^{d-1}(M) = 0$. From Theorem (1.1) (or repeating this process) it follows that $\mathrm{depth}\, M = d = \mathrm{id}\, M$. From property (3.2) it follows that M is a Gorenstein module. Q.E.D.

We remark that the converse of the preceding theorem is well-known and easy to prove (see Sharp [37] or Fossum, Griffith and Reiten [13; Chapter 5]).

Corollary (**3.9**). — *Let* M *be a finitely generated module of depth s. Then* M *is a Gorenstein module if and only if* $H^s_{\mathfrak{m}}(M)$ *is an injective module.*

Examples. — It is easy to give examples of modules M that are not Gorenstein but such that $H^t_{\mathfrak{m}}(M)$ is injective (and of course non-zero) for a suitable t. If A is Gorenstein of dimension $d \geq 2$, then $H^d_{\mathfrak{m}}(\mathfrak{m}) \cong \mathrm{Hom}_A(\mathfrak{m}, A)^{\vee} \cong \mathrm{Hom}_A(A, A)^{\vee} \cong E(k)$; or if A is a factorial local domain of dimension d that is not Cohen-Macaulay (and such exist), then $H^d_{\mathfrak{m}}(A) = E(k)$.

4. Structural results on Gorenstein modules.

The investigation into the structure of Gorenstein modules over local rings which was initiated by Sharp [37] has generally been centered around the question: If the local ring A has a Gorenstein module G, must then A have a dualizing module Ω such that $G \cong \Omega^n$ (the direct sum of n copies of Ω) for $n \geq 1$? With regard to this question, Sharp [38] showed that a dualizing module is unique up to isomorphism and that, if the local ring A has a dualizing module Ω, then the Gorenstein A-modules take the simple form Ω^n, for $n > 0$. These facts follow easily from the main result (Theorem (4.5)) of this section. By utilizing the Brauer group of a commutative ring we provide an affirmative answer to the above question for Henselian local rings or under the assumption that the Gorenstein module G has odd rank. In Section 5 we establish an affirmative answer in case A is a local ring with $\dim A \leq 1$. Finally, it is shown here that a local ring A having a Gorenstein module necessarily has a minimal Gorenstein module M (unique up to isomorphism) such that all Gorenstein A-modules take the form M^n, for $n > 0$. We refer the reader to Section 1 and Section 3 for relevant definitions and terminology.

The first glimpse of the role of the Brauer group in determining the structure of Gorenstein modules can be seen in our very first result. We use the notation Br(A) to denote the Brauer group of the commutative ring A and $M_n(A)$ to denote the ring of $n \times n$ matrices over A. We refer the reader to the papers of Auslander and Goldman [3] and Azumaya [4] for basic concepts concerning the Brauer Group.

Theorem (**4.1**). — *Let* A *be a Cohen-Macaulay local ring with maximal ideal* $\mathfrak{m}$ *and residue class field k. If* A *has a Gorenstein module* G, *then the following statements hold for* $\Lambda = \mathrm{End}\, G$:

(1) *The* A*-algebra* Λ *is an Azumaya* A*-algebra.*

(2) *For each* A*-regular sequence* $\mathfrak{x}$, *the residue ring* $\Lambda/\mathfrak{x}\Lambda \cong \mathrm{End}(G/\mathfrak{x}G)$.

(3) *Let* $\mathfrak{x}$ *be a maximal* A*-regular sequence. Then* Λ *represents an element in the kernel of the map* $\mathrm{Br}(A) \to \mathrm{Br}(A/\mathfrak{x}A)$ *and consequently in the kernel of the map* $\mathrm{Br}(A) \to \mathrm{Br}(k)$.

(4) *If any primitive idempotent of* $\Lambda/\mathfrak{m}\Lambda$ *lifts to* Λ, *then* A *has a dualizing module.*

(5) *There is a finite separable free* A*-algebra* S *having connected prime spectrum which splits* Λ *and which consequently has a dualizing module.*

Proof. — Part (2) of this result was observed in Section 3 (3.5). Let n be the rank of the Gorenstein module G. Let $\mathfrak{x}$ be a maximal A-regular sequence and let E be the $A/\mathfrak{x}A$ injective envelope of k. From remarks (3.3) and (3.6), the sequence $\mathfrak{x}$ is a G-regular sequence and $G/\mathfrak{x}G \cong E^n$. Hence by part (2) the residue ring

$$\Lambda/\mathfrak{x}\Lambda \cong \mathrm{End}(E^n) \cong M_n(A/\mathfrak{x}A),$$

since $\mathrm{End}(E) \cong A/\mathfrak{x}A$. Since Λ is a free A-module by (3.1) it follows easily that the natural map $A \to Z(\Lambda)$, where $Z(\Lambda)$ is the center of Λ, is a monomorphism. From the preceding discussion and Nakayama's lemma it follows that $A \to Z(\Lambda)$ is a bijection. Hence Λ is a central A-algebra such that $\Lambda/\mathfrak{m}\Lambda \cong M_n(k)$ is an Azumaya k-algebra. By a result of Auslander and Goldman [3; Theorem 4.7], we have that Λ is an Azumaya A-algebra. It remains to establish parts (4) and (5) But the hypothesis of (4) insures the existence of a Gorenstein module of rank one and thus a dualizing module. Theorem (6.3) of Auslander and Goldman [3] provides the existence of a finite free A-algebra S with connected prime spectrum which splits Λ. We have the isomorphisms $\mathrm{End}_S(S \otimes G) \cong S \otimes_A \mathrm{End}_A G = S \otimes_A \Lambda \cong \mathrm{End}_S(\Lambda^{op})$ and Λ^{op} is a free S-module. Moreover, $\mathrm{id}_S(S \otimes_A G) = \mathrm{id}_A G$ since $\mathrm{rad}\, S = \mathfrak{m}S$ (see Corollary (4.4*(a)*) of this paper.) From these statements we may conclude the existence of an S-direct summand Ω of $S \otimes_A G$ with $\mathrm{End}_S \Omega \cong S$ and with $\mathrm{id}_S \Omega = \mathrm{id}_A G$. Moreover, since S is a finite free A-algebra, it follows that $\mathrm{depth}_{S_\mathfrak{p}}(\Omega_\mathfrak{p}) = \mathrm{depth}_A G = \dim A = \dim S$ for all maximal primes $\mathfrak{p}$ in Spec S. Thus (3.2) and (3.6) show that Ω is an S-dualizing module. Q.E.D.

The corollary that follows provides a characterization of those local rings having Gorenstein modules.

Corollary **(4.2)**. — *The local ring* A *has a Gorenstein module if and only if* A *is Cohen-Macaulay and some finite free* (separable) A-*algebra is a homomorphic image of a Gorenstein ring of finite dimension.*

Proof. — If A has a Gorenstein module, then Theorem (4.1 (5)) guarantees the existence of a finite free and separable A-algebra S which has a dualizing module Ω. As observed in (3.7) $S \ltimes \Omega$ is a Gorenstein ring. Now suppose that A is Cohen-Macaulay and that S is a finite free A-algebra which is the homomorphic image of a Gorenstein ring B of finite dimension. Then $\Omega = \mathrm{Ext}^d_B(S, B)$, for $d = \dim B - \dim S$, is (locally) a dualizing module for S. Furthermore, since S is a finite free A-algebra, it is elementary that $\mathrm{id}_A \Omega < \infty$ and that Ω is a maximal Cohen-Macaulay A-module. From (3.2) we have that Ω is a Gorenstein A-module. Q.E.D.

Before getting to our main theorem of this section we need two elementary results concerning injective dimension. The proof of the first is based on a proof of Auslander and Buchsbaum [2; Proposition 2.2] for commutative rings. We include a proof here for the sake of completeness.

Lemma **(4.3)**. — *Let* A *be a semi-local commutative ring and let* Λ *be a finite* A*-algebra such that* $\operatorname{rad}\Lambda=(\operatorname{rad}A)\Lambda$. *If* M *is a finitely generated left* Λ*-module such that*

$$\operatorname{Ext}^i(\Lambda/\operatorname{rad}\Lambda, M)=0 \quad \textit{for} \quad i>n,$$

then $\operatorname{id}_\Lambda M\leq n$.

Proof. — We suppose that $\operatorname{id}_\Lambda M>n$. Then there is a left ideal $\mathfrak{J}$ of Λ maximal with respect to the property $\operatorname{Ext}^i_\Lambda(\Lambda/\mathfrak{J}, M)\neq 0$ for some $i>n$. Since $\Lambda/\operatorname{rad}\Lambda$ is necessarily semi-simple and $\operatorname{Ext}^i_\Lambda(\Lambda/\operatorname{rad}\Lambda, M)=0$ for $i>n$, it follows that $\operatorname{rad}\Lambda\nsubseteq\mathfrak{J}$. Hence there is some central element $c\in\operatorname{rad}\Lambda=(\operatorname{rad}A)\Lambda$ with $c\notin\mathfrak{J}$. We note that $(\Lambda/\mathfrak{J})/c(\Lambda/\mathfrak{J})\cong\Lambda/(c,\mathfrak{J})$, where $(c,\mathfrak{J})$ denotes the left ideal of Λ generated by c and $\mathfrak{J}$. By definition of $\mathfrak{J}$, we also have $\operatorname{Ext}^i_\Lambda(\Lambda/(c,\mathfrak{J}), M)=0$ for all $i>n$.

If c is not a zero divisor on $\Lambda/\mathfrak{J}$, then the exact sequence

$$0\to\Lambda/\mathfrak{J}\xrightarrow{c}\Lambda/\mathfrak{J}\to\Lambda/(c,\mathfrak{J})\to 0$$

induces the exact sequence

$$\operatorname{Ext}^i_\Lambda(\Lambda/\mathfrak{J}, M)\xrightarrow{c}\operatorname{Ext}^i_\Lambda(\Lambda/\mathfrak{J}, M)\to 0,$$

for $i>n$. By Nakayama's Lemma this last statement implies that $\operatorname{Ext}^i_\Lambda(\Lambda/\mathfrak{J}, M)=0$ for $i>n$. Hence it must be the case that c is a zero divisor on $\Lambda/\mathfrak{J}$. So let

$$\mathfrak{Z}=\{\lambda\in\Lambda : c\lambda\in\mathfrak{J}\}.$$

Since c is central in Λ and since c is a zero divisor on $\Lambda/\mathfrak{J}$, we have that $\mathfrak{Z}$ is a left ideal of Λ which properly contains $\mathfrak{J}$ and that $\Lambda/\mathfrak{Z}\cong(c,\mathfrak{J})/\mathfrak{J}$. Hence we have an exact sequence

$$0\to\Lambda/\mathfrak{Z}\to\Lambda/\mathfrak{J}\to\Lambda/(c,\mathfrak{J})\to 0$$

from which we obtain the exact sequence

$$0=\operatorname{Ext}^i_\Lambda(\Lambda/\mathfrak{Z}, M)\to\operatorname{Ext}^i_\Lambda(\Lambda/\mathfrak{J}, M)\to\operatorname{Ext}^i_\Lambda(\Lambda/(c,\mathfrak{J}), M)=0$$

for $i>n$. Once again we have reached a contradiction. Thus $\operatorname{id}_\Lambda M\leq n$. Q.E.D.

Parts *(a)* and *(b)* of the next corollary are an easy consequence of the preceding lemma and faithfully flat base change while the proof of part *(c)* is identical with Kaplansky's proof [27; Theorem 217].

Corollary **(4.4)**. — *Let* A *be a semi-local ring and let* Λ *be a finite faithfully flat* A*-algebra such that* $\operatorname{rad}\Lambda=(\operatorname{rad}A)\Lambda$.

(a) *If* M *is a finitely generated* A*-module, then* $\operatorname{id}_A M=\operatorname{id}_\Lambda(\Lambda\otimes_A M)$.

(b) *If* M *is a left* Λ*-module with* $\operatorname{id}_\Lambda M<\infty$, *then* $\operatorname{id}_\Lambda M=\operatorname{id}_A M$.

(c) *If* M *is a left* Λ*-module such that* $\operatorname{id}_\Lambda M<\infty$ *and if* N *is a finitely generated left* Λ*-module, then* $\operatorname{Ext}^i_\Lambda(N, M)=0$ *for* $i>\operatorname{depth}A-\operatorname{depth}N$.

The key result of this section now follows.

Theorem **(4.5).** — *Let* $(A, \mathfrak{m}, k)$ *be a Cohen-Macaulay local ring and let* Λ *be an Azumaya* A-*algebra such that* $\Lambda/\mathfrak{x}\Lambda \cong M_r(A/\mathfrak{x}A)$ *for some maximal* A-*regular sequence* $\mathfrak{x}$ *and positive integer* r. *Suppose that* G *and* H *are left* Λ-*modules of finite* Λ-*injective dimension. Further, suppose that* G *and* H *as* A-*modules are Gorenstein of ranks* s *and* t, *respectively, with* $s \leq t$. *Then* $\mathrm{id}_\Lambda G = \mathrm{id}_\Lambda H = \operatorname{depth} A$ *and* G *is isomorphic to a direct summand of* H *as a left* Λ-*module.*

Proof. — The proof will be accomplished via induction on dim A. For $\dim A = 0$ our hypothesis implies that $\Lambda \cong M_r(A)$. Since Λ is Morita equivalent to A, there is an injective left Λ-module E (unique up to isomorphism) such that all finitely generated left Λ-modules of finite injective dimension are of the form E^n, for $n \geq 0$. Thus $G \cong E^s$ and $H \cong E^t$ and $\mathrm{id}_\Lambda G = \mathrm{id}_\Lambda H = \operatorname{depth} A = 0$.

We now suppose that $\dim A > 0$ and let x be the first element in the A-regular sequence $\mathfrak{x}$. First we note that $\mathrm{id}_\Lambda G = \mathrm{id}_\Lambda H = \operatorname{depth} A$ is a consequence of Corollary (4.4 *(b)*) and Bass' result [5; Lemma 3.3]. Since G/xG and H/xH are Gorenstein A/xA-modules (see Section 3 (3.3)) and since the result holds over A/xA by our inductive assumption, it follows that there is a Λ-epimorphism $\varphi : H \to G/xG$. Further

$$\mathrm{Ext}^1_\Lambda(H, G) = 0$$

since $\operatorname{depth} H = \operatorname{depth} A$ (see (3.2) and (3.4)) by Corollary (4.4 *(c)*). Hence, we have an exact sequence

$$0 \to \mathrm{Hom}_\Lambda(H, G) \xrightarrow{x} \mathrm{Hom}_\Lambda(H, G) \to \mathrm{Hom}_\Lambda(H, G/xG) \to \mathrm{Ext}^1_\Lambda(H, G) = 0$$

from which it follows that there is a Λ-homomorphism $\psi : H \to G$ such that $\eta\psi = \varphi$, where $\eta : G \to G/xG$ is the natural map. Clearly $G = xG + \operatorname{Image} \psi$. By Nakayama's Lemma ψ is necessarily an epimorphism and so we obtain the left Λ-exact sequence

$$0 \to K \to H \to G \to 0.$$

The facts $\operatorname{depth}_A H = \operatorname{depth}_A G = \operatorname{depth} A$ and $\mathrm{id}_\Lambda H = \mathrm{id}_\Lambda G = \operatorname{depth} A$ imply that $\operatorname{depth}_A K = \operatorname{depth} A$ and (using Corollary (4.4 *(b)*) that $\mathrm{id}_\Lambda K = \operatorname{depth} A < \infty$. Thus (3.2) of Section 3 gives that K is a Gorenstein A-module. By Corollary (4.4 *(c)*), the group $\mathrm{Ext}^1_\Lambda(G, K) = 0$ and thus G is isomorphic to a left Λ direct summand of H.

Q.E.D.

Corollary **(4.6).** — *Let* A *be a local ring having a Gorenstein module. Then* A *has a minimal Gorenstein module* Ω *which is unique up to isomorphism and such that all Gorenstein* A-*modules are of the form* Ω^n, *for* $n > 0$.

Proof. — Choose Ω to be the Gorenstein A-module of least rank and apply Theorem (4.5) in the situation $\Lambda = A$. Q.E.D.

Corollary **(4.7)** [Sharp, 38]. — *Let* A *be a local Cohen-Macaulay ring which is a homomorphic image of a Gorenstein ring. Then* A *has a dualizing module* Ω *and all Gorenstein* A-*modules are of the form* Ω^n, *for* $n > 0$.

Proof. — This is a special case of Corollary (4.6). Q.E.D.

The next result shows that a Henselian local ring has a Gorenstein module if and only if it has a dualizing module. In the case of complete local rings this result was first observed by Sharp [39]. Let $(A, \mathfrak{m}, k)$ be a local ring. If $f \in A[X]$, then $\bar{f} \in k[X]$ denotes the polynomial obtained by reducing the coefficients of f modulo $\mathfrak{m}$. The Henselization of A will be denoted by A^h. A standard étale neighborhood of A is a local ring of the form $(A[X]/f)_{\mathfrak{p}}$, where f is a monic polynomial in $A[X]$ such that $f'(0)$ is a unit in A (f' is the derivative of f) and where $\mathfrak{p}$ is a prime ideal which corresponds to the kernel of the homomorphism $g \mapsto \bar{g}(y)$, for $y \in k$ a simple root of $\bar{f}(x) \in k[X]$. We note that A^h is a (directed) direct limit of standard étale neighborhoods of A. (For details see Iversen's lecture notes [26]).

Corollary **(4.8)**. — *If the local ring* $(A, \mathfrak{m}, k)$ *has a Gorenstein module, then some standard étale neighborhood and also* A^h *have dualizing modules. In particular,* A^h *is the homomorphic image of a Gorenstein ring.*

Proof. — Suppose that G is a Gorenstein A-module. Then the " usual " base change arguments show that $\operatorname{depth} G = \operatorname{depth} (A^h \otimes G) = \operatorname{id}_A G = \operatorname{id}_{A^h}(A^h \otimes G)$, that is, $A^h \otimes G$ is a Gorenstein module for A^h. Let $\Lambda = \operatorname{End}_{A^h}(A^h \otimes G)$. Since, for Henselian rings, the map $\operatorname{Br}(A^h) \to \operatorname{Br}(k)$ is monic [Azumaya, 4], we have from Theorem (4.1) (3 and 4) that A^h has a dualizing module Ω. Furthermore, since A^h is faithfully flat over the standard étale neighborhoods of A and since A^h is a direct limit of the same, we may descend Ω to Ω' a dualizing module for some standard étale neighborhood of A.

We remark that the example of Ferrand and Raynaud [11] provides an example of a one-dimensional local domain A with no Gorenstein module since the formal fibres of A are not Gorenstein (see Theorem (5.2)). In fact the integral closure of this A is the ring of convergent power series in one variable over the complex number field **C**. Hence the Henzelization A^h is a faithfully flat A-subalgebra of the integral closure of A, and thus $A = A^h$. So A is a Henselian local domain which is not a homomorphic image of a Gorenstein ring.

Our final theorem of this section is a decomposition result for Gorenstein modules of odd rank.

Theorem **(4.9)**. — *Let* A *be a local ring with maximal ideal* $\mathfrak{m}$ *and residue class field* k. *Suppose* A *has a Gorenstein module* G. *Then the order of* $\Lambda = \operatorname{End}_A G$ *in the Brauer group of* A *is one or two. Thus, if the rank of* G *is odd, then* Λ *is trivial in* $\operatorname{Br}(A)$ *and* A *has a dualizing module* Ω *such that* $G \cong \Omega^r$, *where* $r = $ rank of G (as a Gorenstein module).

Proof. — Let $\mathfrak{x}$ be a maximal A-regular sequence. Then $\mathfrak{x}$ is a G-regular sequence (3.3) and $G/\mathfrak{x}G \cong E^r$, where E is the $A/\mathfrak{x}A$-injective envelope of k and $\Lambda/\mathfrak{x}\Lambda \cong \operatorname{End}(E^r) \cong M_r(A/\mathfrak{x}A)$ (see 3.6). It follows that $G/\mathfrak{x}G$ is an injective $\Lambda/\mathfrak{x}\Lambda$-module and via successive applications of Kaplansky's Theorem 172 [27] that $\mathrm{id}_\Lambda G < \infty$. At the same time the Gorenstein A-module $H = \Lambda \otimes_A G$ has the property $\mathrm{id}_\Lambda H = \mathrm{id}_A G < \infty$ by Corollary (4.4 *(a)*). Now the Gorenstein rank of G is r and the Gorenstein rank of $H = \Lambda \otimes G$ is r^3. An application of Theorem (4.5) yields that $\Lambda \otimes G \cong G^{r^2}$ as left Λ-modules. Thus, we have the ring isomorphisms

$$\Lambda \otimes_A \Lambda \cong \Lambda \otimes_A \mathrm{End}_A G \cong \mathrm{End}_\Lambda(\Lambda \otimes_A G) \cong \mathrm{End}_\Lambda(G^{r^2})$$
$$\cong M_{r^2}(\mathrm{End}_\Lambda G) \cong M_{r^2}(A)$$

since it is easily seen that $\mathrm{End}_\Lambda G \cong A$. It follows that $\Lambda \otimes_A \Lambda$ is trivial in the Brauer group of A, that is Λ determines an element of order one or two in Br(A).

If $r = \mathrm{rank}\, G$ is odd, then the preceding discussion shows that $\Lambda = \mathrm{End}\, G$ is trivial in Br(A) since the order of Λ divides r (see Grothendieck [20; Proposition 1.4] and Knus and Ojanguren [28]). In case Λ is trivial in Br(A), Theorem (4.1 (4)) gives the existence of a dualizing module Ω and Corollary (4.7) shows that $G \cong \Omega^r$.

Q.E.D.

5. Descent of dualizing modules for one-dimensional local rings.

Although the results in this section hold for rings of dimension zero, they generally have trivial consequences in this case. Hence, we shall confine ourselves to the one-dimensional case, the higher dimensional cases as yet being unresolved.

A ring A will be called *one-Gorenstein* if its total quotient ring, denoted K(A), is Quasi-Frobenius, that is, K(A) is self injective. For the definition and general properties of n-Gorenstein rings see Fossum and Reiten [35]. If $(A, \mathfrak{m}, k)$ is a local ring and if $\mathfrak{p} \in \mathrm{Spec}\, A$, then $k(\mathfrak{p})$ denotes the residue class field of $A_\mathfrak{p}$ and the ring $\hat{A} \otimes_A k(\mathfrak{p})$ is called the formal fiber of A at $\mathfrak{p}$, where $\hat{A}$ (always) denotes the completion of A in its $\mathfrak{m}$-adic topology.

Lemma **(5.1)**. — *Let* A *be a one-dimensional Cohen-Macaulay local ring. Then* A *is the homomorphic image of a Cohen-Macaulay, one-Gorenstein local ring* B *of dimension one such that* A *and* B *have isomorphic formal fibers.*

Proof. — Let S denote the set of regular elements of A. Then $K(A) = S^{-1}A$ is a commutative artin ring. Hence $S^{-1}A$ has a finitely generated injective module I such that $\mathrm{Hom}_{S^{-1}A}(I, I) \cong S^{-1}A$. Furthermore, there is a finitely generated A-module M such that $S^{-1}M = I$ and such that the natural map $M \to S^{-1}M = I$ is monic. It follows easily that the trivial extension $A \ltimes M$ (see Section 3 (3.7)) has depth one and is therefore a one-dimension Cohen-Macaulay ring. The set $S \times 0$ consists of regular elements on $B = A \ltimes M$ and there is a natural isomorphism

$$(S \times 0)^{-1}B \xrightarrow{\sim} S^{-1}A \ltimes S^{-1}M.$$

A result of Gulliksen [21] shows that B is one-Gorenstein. Finally, since $\hat{B} \cong \hat{A} \ltimes \hat{M}$ and since the primes in B are of the form $\mathfrak{p} \times M$, where $\mathfrak{p} \in \mathrm{Spec}\, A$, one easily observes that the formal fibers of A and B are isomorphic. Q.E.D.

The next theorem was proved by Hartshorne [22] in the case of Gorenstein rings and was generalized by Sharp [41] for the case of Gorenstein modules.

Theorem (5.2). — *If the local ring* A *has a Gorenstein module, then* A *has Gorenstein formal fibers.*

The central result of this section now follows. It gives an affirmative answer to the question concerning the structure of Gorenstein modules mentioned at the beginning of Section 4 for the case of one-dimensional local rings. Moreover, it establishes that the dualizing module for the completion of the local ring A descends to A exactly when the formal fibers are Gorenstein. The latter statement remains open for higher dimensional local rings.

Theorem (5.3). — *Let* $(A, \mathfrak{m}, k)$ *be a one-dimensional Cohen-Macaulay ring. Then the following statements are equivalent:*

(a) *The ring* A *has a Gorenstein module.*

(b) *The formal fibers of* A *are Gorenstein.*

(c) *The formal fibers of* A *are one-Gorenstein.*

(d) *The ring* A *has a dualizing module* Ω *and hence all Gorenstein* A*-modules take the simple form* Ω^n, *for* $n > 0$.

Proof. — The fact that *(a)* implies *(b)* is of course a restatement of Theorem (5.2) for one-dimensional rings and the fact that *(b)* implies *(c)* and *(d)* implies *(a)* is obvious. In view of Corollary (4.7), it remains only to show that A has a dualizing module under the assumption that the formal fibers of A are one-Gorenstein. We begin this proof under the additional assumption that A is a one-Gorenstein ring. By [35; Proposition 1] of Fossum and Reiten, it follows that $\hat{A}$ is also one-Gorenstein. As noted in Section 3 (3.7), the completion $\hat{A}$ has a dualizing module which we denote by Ω. Let S denote the set of regular elements of $\hat{A}$. From Sharp [37] it follows that $S^{-1}\Omega = \coprod_{\operatorname{ht}\mathfrak{p}=0} E(\hat{A}/\mathfrak{p})$ is injective, where as usual $E(\hat{A}/\mathfrak{p})$ denotes the injective envelope of $\hat{A}/\mathfrak{p}$ for $\mathfrak{p} \in \operatorname{Spec} \hat{A}$. Since $\hat{A}$ is one-Gorenstein, we also have that $S^{-1}A \cong \coprod_{\operatorname{ht}\mathfrak{p}=0} E(\hat{A}/\mathfrak{p})$. But the relation $S^{-1}\hat{A} \cong S^{-1}\Omega$ and the fact that Ω is finitely generated imply that Ω can be realized as an ideal in $\hat{A}$. Hence by Herzog and Kunz [24; Lemma 2.10], the ring A also has a dualizing module. (The ideal Ω descends to A since it is necessarily primary to the maximal ideal of $\hat{A}$ when $\dim \hat{A} \leq 1$). In the general case we let B be the one-Gorenstein local ring of dimension one with A as a homomorphic image as described in Lemma (5.1). Then B has one-Gorenstein formal fibers and our preceding discussion shows that B has a dualizing module, that is B is the Cohen-Macaulay homomorphic image of a Gorenstein ring. But then A is also the Cohen-Macaulay homomorphic image of a Gorenstein ring and thus has a dualizing module (see Section 3 (3.7)). Q.E.D.

A somewhat amusing corollary of Theorem (5.3) is the following one.

Corollary **(5.4)**. — *Suppose that* A *is a one-dimensional local Cohen-Macaulay ring such that the dualizing module of its completion* $\hat{A}$ *is an ideal* (e.g. in case $\hat{A}$ is an integral domain). *Then* A *has a dualizing module and consequently Gorenstein formal fibers.*

Proof. — Let Ω be the dualizing module for $\hat{A}$ which is in addition an ideal. As in the proof of Theorem (5.3), the module Ω is primary for the maximal ideal of $\hat{A}$ and thus descends to A. Now apply Theorem (5.3) *(b)* and *(d)*. Q.E.D.

6. Examples and applications.

Finally we give some examples of commutative rings which are m-Gorenstein but not $(m+1)$-Gorenstein. The rings we use are classical. They are the local rings at the vertex of the cone for the Segre embeddings $\mathbf{P}^1 \times \mathbf{P}^m \to \mathbf{P}^{2m+1}$. It is well known that these rings are Cohen-Macaulay of dimension $m+2$, but not Gorenstein if $m>1$ (see Eagon [10] for example). But these rings are regular outside of the maximal ideal so they are $(m+2)$-Gorenstein but not $(m+3)$-Gorenstein. Here we are using the conditions: The (local) Noetherian ring A is k-Gorenstein if and only if for all $\mathfrak{p}$ in Spec A the inequality $\operatorname{depth} A_\mathfrak{p} < k$ implies $A_\mathfrak{p}$ is Gorenstein. (See Fossum and Reiten [35].)

For completeness and in order to demonstrate the usefulness of the module of dualizing differentials or the canonical module, we give a proof of the next proposition which is independent of the proof given in Eagon [10], but which uses Eagon and Northcott's resolution [9]. Before stating the proposition, we establish some notation.

Let $\{X_{ij}\}$ with $1 \leq i \leq s$, $1 \leq j \leq r$, $s<r$, be a set of rs indeterminates over a field k. Let M be the maximal ideal generated by the indeterminates in the ring of polynomials $k[\{X_{ij}\}]$ and let B be the localization $k[\{X_{ij}\}]_M$. Then B is a regular local ring of dimension rs. Let $\mathfrak{I}$ be the ideal generated by the $s \times s$ minors of the matrix (X_{ij}) and let A be the local ring $B/\mathfrak{I}$. According to Eagon and Northcott [9] and Northcott [32], the ideal $\mathfrak{I}$ is perfect, that is

$$\operatorname{depth}_{\mathfrak{I}} B = \operatorname{pd}_B A = r - s + 1$$

and there is a minimal free resolution

$$0 \to P_{r-s+1} \to \ldots \to P_1 \to B \to A \to 0$$

of A by free B-modules (complex 3.4 and Theorem 2 of Eagon and Northcott [9]). Furthermore, the rank of P_{r-s+1} is equal to the order of the set

$$\{(v_1, \ldots, v_s) : v_i \in \mathbf{Z},\ v_i \geq 0 \text{ and } v_1 + \ldots + v_s = r - s\}.$$

Keeping these things in mind, we now state part of the result of Eagon's Theorem [10].

Proposition **(6.1)**. — *If* $1 < s < r$, *then* A *is Cohen-Macaulay but not Gorenstein* ($\dim A = rs - r + s - 1$).

Proof. — The factor ring A is Cohen-Macaulay since $\mathrm{Ext}^i_B(A, B)$ vanishes everywhere except at $i=r-s+1$. If $i<r-s+1$, the extension group vanishes since

$$\mathrm{grade}_{\mathfrak{J}} B = r-s+1,$$

while, for $i>r-s+1$, it vanishes since $\mathrm{pd}_B A = r-s+1$. In fact the dual of the above resolution gives a minimal resolution of the canonical module $\Omega_A = \mathrm{Ext}^{r-s+1}_B(A, B)$ and this complex is:

$$0 \to B^* \to P_i^* \to \ldots \to P^*_{r-s+1} \to \Omega_A \to 0.$$

Now A is Gorenstein if and only if $\Omega_A \cong A$. But Ω_A is cyclic as a B-module if and only if $\mathrm{rk}\, P_{r-s+1} = 1$. But this rank is more than 1 for the given values of r and s.

Q.E.D.

We would like to know the Gorenstein locus of A, which is just the set $\{\mathfrak{p} \in \mathrm{Spec}\, A : (\Omega_A)_\mathfrak{p} \text{ is free}\}$. This also can be determined in general, but for our purposes it is enough to do so for the very special case $s=2$. In this case A is the local ring of the vertex of the cone for the Segre embedding

$$\mathbf{P}^1 \times \mathbf{P}^{r-1} \to \mathbf{P}^{2r-1}.$$

So geometrically it is clear that the singular locus of A is the maximal ideal. However, this fact can also be seen in a purely algebraic fashion.

Suppose $\mathfrak{p} \in \mathrm{Spec}\, A - \{\mathfrak{m}\}$, the ideal $\mathfrak{m}$ being the image of M in A. There is at least one element x_{ij} which does not belong to $\mathfrak{p}$. Let $\mathfrak{P}$ be the inverse image of $\mathfrak{p}$ in B. Then $X_{ij} \notin \mathfrak{P}$. Now let $A_{x_{ij}}$ be the ring of quotients of A with respect to the multiplicatively closed set $\{1, x^1_{ij}, x^2_{ij}, \ldots\}$. Then $A_\mathfrak{p}$ is the localization of $A_{x_{ij}}$ at some prime ideal. Now $A_{x_{ij}} = B_{X_{ij}}/\mathfrak{J}B_{X_{ij}}$. The claim is: The ring $A_{x_{ij}}$ is regular.

The ideal $\mathfrak{J}$ is generated by the elements

$$X_{1j}X_{2k} - X_{1k}X_{2j} \quad \text{for} \quad j \neq k.$$

We can assume $X_{ij} = X_{11}$. The ideal $\mathfrak{J}B_{X_{11}}$ is generated by the elements

$$X_{11}^{-1}X_{2k} - (X_{11}^{-1}X_{1k})(X_{11}^{-1}X_{21}) \quad \text{for} \quad k=1, 2, \ldots, r.$$

Hence the ring $B_{X_{11}}/\mathfrak{J}B_{X_{11}} = k[X_{12}, X_{13}, \ldots, X_{1r}][X_{11}, X_{11}^{-1}]$ which is clearly a regular ring. This establishes our next result:

Proposition **(6.2)**. — *Let* $r>2$. *Then the ring*

$$(k[X_1, \ldots, X_r, Y_1, \ldots, Y_r]/(\{X_iY_j - X_jY_i\}))_{(x_i, y_j)}$$

has dimension $r+1$, *is Cohen-Macaulay, is not Gorenstein* (and hence not $(r+2)$-Gorenstein) *but is* $(r+1)$*-Gorenstein.*

Proof. — We let $X_{1j} = X_j$ and $X_{2j} = Y_j$. Then the ring is the A considered above.

Q.E.D.

Since this ring is regular outside the maximal ideal it is normal but cannot be a unique factorization domain.

So there is a local ring A of dimension n which is Cohen-Macaulay and n-Gorenstein but not $(n+1)$-Gorenstein. (If $\dim A = n$ and A is $(n+1)$-Gorenstein, then it is Gorenstein). Now the polynomial ring A[T] is also n-Gorenstein, but not $(n+1)$-Gorenstein by results in Fossum and Reiten [35]. Thus all possible combinations of Krull dimension and the Gorenstein property appear.

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