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RATIONAL EQUIVALENCE ON SINGULAR VARIETIES (1)

by William FULTON

CONTENTS

(1) Appendix to *Riemann-Roch for Singular Varieties*, by P. Baum, W. Fulton and R. MacPherson.

0. Introduction.

For a non-singular quasi-projective variety X the Chow ring A(X) provides simultaneously a covariant functor for proper morphisms and a contravariant functor for arbitrary morphisms (cf. [AC]). This appendix develops a " homology " theory $A_{\bullet}$ and a " cohomology " theory $A^{\bullet}$ for arbitrary quasi-projective schemes over a field which agree with the Chow ring for non-singular varieties.

The definition of $A_{\bullet}X$ as algebraic cycles modulo rational equivalence, on a possibly singular variety X, has been known for some time ([AC; 4-30], [S; p. V-29]), although a systematic account has never appeared. In § 1 we construct this homology theory for the category of algebraic schemes over a field.

The construction of a corresponding cohomology theory $A^{\bullet}$ (§ 3), for quasi-projective schemes over a field, is based on Serre's intersection theory [S] and Chow's moving lemma (cf. [AC], [R]). The two theories have the usual formal properties: cap products $A^{\bullet}\otimes A_{\bullet}\xrightarrow{\frown} A_{\bullet}$, a projection formula, Chern classes, etc.

We have been able to show (§ 4) that rational equivalence specializes. If $f: X\to C$ is a flat morphism to a regular curve C, and $t\in C$, $i: X_t\to X$ the inclusion of the fibre, we construct the Gysin map

$$i^{*}: A_k X\to A_{k-1}X_t.$$

From this one deduces that the Chow group of the general fibre specializes to the Chow group of the special fibre. Even for non-singular varieties this question has been open for some time (cf. Grothendieck's discussion in [SGA 6; X, 7]).

The final § 5 contains A. Landman's result that, modulo torsion, there are only the obvious natural transformations from $A_{\bullet}$ to itself. This was used to prove the uniqueness of the Riemann-Roch map in [B-F-M]. We also thank him for help in constructing Chern classes (cf. § 3.2).

Invariants of singular varieties tend to lie in homology rather than cohomology (cf. [B-F-M], [M], [F]), with cohomology playing an auxiliary role. Our Chow cohomology is constructed in this spirit by passing to a limit over non-singular varieties containing the given variety. This gives a theory which is as fine as possible, in that it maps to any other theory with Chern classes and Poincaré duality; it is also probably the correct theory up to torsion (cf. § 3.2, 3.3).

R. MacPherson, to whom we are grateful for many stimulating conversations on these topics, has suggested constructing the Chow cohomology of a singular variety by taking as " cocycles " those cycles which intersect the singular locus nicely, with a similar restriction on the rational equivalence. Although this method would depend on a more general moving lemma than is now available, it would be better than our theory in those situations where the invariants do lie in cohomology (cf. [B-F-M; IV, § 5]).

We also thank R. Piene and others who read and commented on a preliminary version.

1. The Chow Homology Groups $A_.$.

In this section we construct the Chow homology groups for algebraic schemes over a field, and study some of their basic properties. However, most of the results extend without difficulty at least to the category of excellent schemes [EGA IV], and our proofs are designed for this generality.

1.1. Cycles and Sheaves.

For a (noetherian) scheme X, we denote by $\mathscr{Z}(X)$ the group of algebraic cycles on X; $\mathscr{Z}(X)$ is the free abelian group on the set of integral (reduced and irreducible) closed subschemes of X. We write $\mathscr{Z}(X)=\mathscr{Z}_{\bullet}X=\bigoplus_k \mathscr{Z}_k X$, where $\mathscr{Z}_k X$ consists of cycles of dimension k. We may also write $\mathscr{Z}(X)=\mathscr{Z}^{\bullet}X=\bigoplus_k \mathscr{Z}^k X$, where $\mathscr{Z}^k X$ consists of cycles of codimension k.

If $\mathscr{F}$ is a coherent sheaf on X, let

$$Z(\mathscr{F})=\sum_i \ell_{\mathcal{O}_{x_i}}(\mathscr{F}_{x_i})\cdot W_i.$$

(The sum is over the components W_i of the support of $\mathscr{F}$, x_i is a generic point of W_i, $\mathcal{O}_{x_i}$ is the local ring of X at x_i, $\mathscr{F}_{x_i}$ is the stalk of $\mathscr{F}$ at x_i, and ℓ denotes length of an artinian module.) If $\mathscr{F}$ has support of dimension $\leq k$, we let $Z_k\mathscr{F}\in\mathscr{Z}_k X$ be the part of $Z(\mathscr{F})$ of dimension k. Similarly if $\operatorname{codim}(\operatorname{Supp}\mathscr{F})\geq k$, $Z^k\mathscr{F}\in\mathscr{Z}^k X$ is the part of $Z(\mathscr{F})$ of codimension k.

Any closed subscheme Y of X determines a cycle

$$[Y]=Z(\mathcal{O}_Y) \quad \text{in } \mathscr{Z}(X)$$

where $\mathcal{O}_Y$ is the structure sheaf of Y (extended by 0 to X). If the components Y_i of Y have multiplicities m_i, then $[Y]=\sum_i m_i[Y_i]$. In particular, X has a *fundamental cycle* [X].

1.2. Pushing Cycles Forward.

Let $f: X\to Y$ be a proper morphism. To define $f_*: \mathscr{Z}_{\bullet}X\to\mathscr{Z}_{\bullet}Y$ it is enough to define $f_*[V]$ for an integral closed subscheme V of X. Let $f(V)=W$. If $\dim W<\dim V$, set $f_*[V]=0$. If $\dim W=\dim V$, set $f_*[V]=d[W]$, where $d=[R(V):R(W)]$ is the degree of the function field extension (cf. [EGA IV, 5.6.6]). If we extend by linearity, $\mathscr{Z}_{\bullet}$ becomes a covariant functor from (noetherian) schemes and proper morphisms to graded abelian groups.

If $z\in\mathscr{Z}_k X$, $z=Z_k(\mathscr{F})$, and $f: X\to Y$ is proper, then

$$f_*z=Z_k(R^0f_*\mathscr{F})=\sum_i(-1)^i Z_k(R^i f_*\mathscr{F})$$

in $\mathscr{Z}_k Y$ (cf. [S; V, § 6]).

1.3. Divisors.

If D is an effective Cartier divisor on X, it determines a Weil divisor $[D] \in \mathscr{Z}^1X$ by § 1.1. Since $[D+E]=[D]+[E]$, this extends to give a homomorphism

$$\mathrm{Div}(X) \to \mathscr{Z}^1X$$

from the group of Cartier divisors to the group of Weil divisors [EGA IV, 21.6.7]; we write [D] for the Weil divisor determined by the Cartier divisor D.

If r is a non-zero element in the function field R(X) of an integral scheme X, or more generally, a " regular meromorphic function " [EGA IV, 20.1.8] on a general scheme X, we write $\mathrm{div}(r)$ for the principal Cartier divisor determined by r, and $[\mathrm{div}(r)]$ for the corresponding Weil divisor.

1.4. An Algebraic Lemma.

Lemma. — *Let* A *be a one-dimensional local noetherian domain with maximal ideal* P *and quotient field* K. *Let* L *be a finite extension of* K, B *a finite* A*-algebra whose quotient field is* L. *Let* $P_1, \ldots, P_r$ *be the prime ideals of* B *lying over* P, $B_i = B_{P_i}$. *Suppose* $t \in B$ *and* $N(t) \in A$, *where* $N : L^* \to K^*$ *is the norm. Then*

$$\ell_A(A/N(t)A) = \sum_i [B_i/P_iB_i : A/P]\ell_{B_i}(B_i/tB_i).$$

Proof. — The right-hand side is equal to $\ell_A(B/tB)$, so we are reduced to showing $\ell_A(B/tB) = \ell_A(A/N(t)A)$. If there is a free A-submodule F of B such that $tF \subset F$ and $F \otimes_A K = L$, then $\ell_A(A/N(t)A) = \ell_A(F/tF)$ [EGA IV, 21.10.17.3], and since $\ell_A(B/F) < \infty$, $\ell_A(F/tF) = \ell_A(B/tB)$ (cf. [EGA IV, 21.10.13]).

In the general case choose any free A-submodule F of B so that $F \otimes_A K = L$. Then $tF \subset \frac{1}{s}F$ for some $s \in A$. We know the result for st and s by the previous case. Since both sides take products to sums, the result follows for t by subtraction.

1.5. Divisors and Mappings.

Proposition 1. — *Let* X, Y *be integral schemes of the same dimension,* $f : X \to Y$ *a proper, surjective morphism. Let* $[R(X) : R(Y)] = n$, *and let* $N : R(X)^* \to R(Y)^*$ *be the norm.*

(1) *If* $r \in R(X)^*$, *then*

$$f_*[\mathrm{div}(r)] = [\mathrm{div}(N(r))].$$

(2) *If* D *is a Cartier divisor on* Y, *then* $f_*[f^*D] = n[D]$.

Proof. — Let W be an integral subscheme of Y of codimension 1, w a generic point of W, $A = \mathcal{O}_{Y,w}$. We may take the base change $\mathrm{Spec}(A) \to Y$, and so assume $Y = \mathrm{Spec}\, A$ Then $X = \mathrm{Spec}\, B$, where B is a finite A-algebra [EGA III, 4.4.2].

To prove (1), we may multiply r by $s \in A$ to achieve the situation where $r \in B$ and $N(r) \in A$. Then the result follows from the lemma of § 1.4. (This is also proved in [AC; 2-12].)

Assertion (2) follows from the same lemma, and the fact that $N(t) = t^n$ if $t \in A$ (cf. also [EGA IV, 21.10.18]).

Proposition 2. — Let X, Y *be integral schemes,* $f : X \to Y$ *proper,* $\dim X > \dim Y$. *Then*

$$f_*[\operatorname{div}(r)] = 0$$

for all $r \in R(X)^*$.

Proof. — We may assume $\dim Y = \dim X - 1$, and make the base change $\operatorname{Spec}(R(Y)) \to Y$ to calculate the multiplicity of $[Y]$ in $f_*[\operatorname{div}(r)]$. By Proposition 1, we may assume X is a normal curve over $Y = \operatorname{Spec} K$. Factor f into a finite map $X \to \mathbf{P}^1_Y$ followed by the projection to Y. Applying Proposition 1 again reduces it to the case $X = \mathbf{P}^1_K$, where it is obvious.

Proposition 3. — Let X *be a scheme,* $[X] = \sum_i m_i [X_i]$, *with* $\varphi_i : X_i \to X$ *the inclusions of the irreducible components into* X. *Let* D *be a Cartier divisor on* X. *Then* $\varphi_i^* D$ *is a Cartier divisor on* X_i, *and*

$$[D] = \sum_i m_i \varphi_{i*}[\varphi_i^* D] \quad \text{in } \mathscr{Z}^1(X).$$

Proof. — As in the proof of Proposition 1, we may assume $X = \operatorname{Spec} A$, where A is a one-dimensional local ring. We may assume D is effective, with local equation $t \in A$. Then the result reduces to an algebraic lemma [EGA IV, 21.10.17.7].

1.6. A Gysin Map for Flat Morphisms.

Let $f : X \to Y$ be a flat morphism. Then we define the " Gysin " homomorphism $f^* : \mathscr{Z}^k Y \to \mathscr{Z}^k X$ as follows: If V is a closed, integral subscheme of Y, let $f^*[V] = [f^{-1}V]$ where $f^{-1}V$ is the scheme-theoretic inverse image of V; set $f^*[V] = 0$ if $f^{-1}V$ is empty. If $z = Z^k \mathscr{F}$ for a coherent sheaf $\mathscr{F}$ on Y, then $f^* z = Z^k(f^* \mathscr{F})$.

Proposition. — Let $f : X \to Y$ *be a flat morphism of relative dimension d.*

(1) *If* D *is a Cartier divisor on* Y, *then* $f^*[D] = [f^* D]$.

(2) *If* $g : Y' \to Y$ *is proper, and we form the fibre square*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Y \end{array}$$

then $g'_* f'^* = f^* g_*$ *from* $\mathscr{Z}(Y')$ *to* $\mathscr{Z}(X)$.

Proof. — (1) follows easily from the definition (cf. [EGA IV, 21.10.6]). For (2) let $y'=\mathrm{Z}_k(\mathscr{F}')\in\mathscr{Z}_k\mathrm{Y}'$; then

$$f^*g_*y'=\sum_i(-1)^i\mathrm{Z}_{k+d}(f^*\mathrm{R}^ig_*\mathscr{F}')$$

and
$$g'_*f'^*y'=\sum_i(-1)^i\mathrm{Z}_{k+d}(\mathrm{R}^ig'_*(f'^*\mathscr{F}')).$$

But since f is flat, $f^*\mathrm{R}^ig_*\mathscr{F}'\cong\mathrm{R}^ig'_*(f'^*\mathscr{F}')$ [EGA III, 1.4.15], so the two cycles are equal.

1.7. A Gysin Map for Divisors.

Let $\mathrm{D}=\mathrm{div}(t)$ be a principal effective Cartier divisor on X, $i:\mathrm{D}\to\mathrm{X}$ the inclusion. We define a Gysin map $i^*:\mathscr{Z}^k\mathrm{X}\to\mathscr{Z}^k\mathrm{D}$ as follows: if V is an integral closed subscheme of X, let

$$i^*[\mathrm{V}]=\begin{cases}0 & \text{if } \mathrm{V}\subset\mathrm{D}\\ [\mathrm{V}_t] & \text{if } \mathrm{V}\not\subset\mathrm{D}\end{cases}$$

where V_t is the subscheme of V defined by the function t.

If $\mathscr{F}$ is a coherent sheaf on X and $\mathrm{Z}^k\mathscr{F}=z$, and the support of $i^*\mathscr{F}=\mathscr{F}\otimes\mathscr{O}_\mathrm{D}$ has codimension $\geq k$, then $i^*z=\sum_{i=0}^{1}(-1)^i\mathrm{Z}^k(\mathrm{Tor}_i^{\mathscr{O}_\mathrm{X}}(\mathscr{F},\mathscr{O}_\mathrm{D}))$. This follows from the fact that if $\mathscr{F}$ has support in D, and $\mathrm{codim}(\mathrm{Supp}\,\mathscr{F})\geq k+1$, then

$$\sum_i(-1)^i\mathrm{Z}^k(\mathrm{Tor}_i^{\mathscr{O}_\mathrm{X}}(\mathscr{F},\mathscr{O}_\mathrm{D}))=0$$

(cf. [EGA IV, 21.10.13]).

Proposition. — Let $f:\mathrm{X}\to\mathrm{Y}$ *be a morphism,* D *a principal effective Cartier divisor on* Y *such that the Cartier divisor* $f^*\mathrm{D}$ *is defined. Let* $i:\mathrm{D}\to\mathrm{Y}$, $j:f^*\mathrm{D}\to\mathrm{X}$ *be the inclusions,* $g:f^*\mathrm{D}\to\mathrm{D}$ *the morphism induced by* f.

(1) *If* f *is proper, then the diagram*

$$\begin{array}{ccc}\mathscr{Z}(\mathrm{X}) & \xrightarrow{f_*} & \mathscr{Z}(\mathrm{Y})\\ \Big\downarrow{\scriptstyle j^*} & & \Big\downarrow{\scriptstyle i^*}\\ \mathscr{Z}(f^*\mathrm{D}) & \xrightarrow{g_*} & \mathscr{Z}(\mathrm{D})\end{array}$$

commutes.

(2) *If* f *is flat, then the diagram*

$$\begin{array}{ccc}\mathscr{Z}(\mathrm{Y}) & \xrightarrow{f^*} & \mathscr{Z}(\mathrm{X})\\ {\scriptstyle i^*}\Big\downarrow & & \Big\downarrow{\scriptstyle j^*}\\ \mathscr{Z}(\mathrm{D}) & \xrightarrow{g_*} & \mathscr{Z}(f^*\mathrm{D})\end{array}$$

commutes.

Proof. — (1) Let V be an integral closed subscheme of X, $f_*[V]=n[W]$. We want to show $g_*[V_t]=n[W_t]$, in case $W\not\subset D$. Then $g^*W_t=V_t$, and the result follows from Proposition 1 (2) of § 1.5. (2) is clear from the definition.

1.8. Definition of the Chow Groups.

Proposition. — *Let* $z\in\mathscr{Z}(X)$. *The following are equivalent:*

(1) *There is a scheme* Y, *a principal Cartier divisor* D *on* Y, *and a proper morphism* $\pi: Y\to X$ *such that* $\pi_*[D]=z$.

(2) *There are integral schemes* Y_i, *rational functions* $r_i\in R(Y_i)^*$ *and proper morphisms* $\pi_i: Y_i\to X$ *so that* $z=\sum_i \pi_{i*}[\mathrm{div}(r_i)]$.

(3) *There are closed integral subschemes* Y_i *of* X, *and* $r_i\in R(Y_i)^*$, *so that* $z=\sum_i[\mathrm{div}(r_i)]$ *in* $\mathscr{Z}(X)$.

Proof. — (1) ⇒ (2) follows from § 1.5, Proposition 3. (2) ⇒ (3) follows from § 1.5, Propositions 1 and 2: if $\pi_i: Y_i\to X$ is proper we may replace Y_i by $\pi_i(Y_i)\subset X$.

Remarks. — (1) If $z\in\mathscr{Z}_k X$, we may choose the Y_i in (2) or (3) to have dimension $k+1$.

(2) We may replace Y_i in (2) by any Y_i' for which there is a birational proper morphism $Y_i'\to Y_i$ (§ 1.5, Proposition 1). Thus for example we may replace Y_i by the closure of the graph of r_i to assume r_i gives a section of $\mathbf{P}^1_{Y_i}$ over Y_i, or a morphism to the projective line if the Y_i are algebraic varieties. Or we may assume each Y_i is normal.

Definition. — A cycle z in $\mathscr{Z}(X)$ is *rationally equivalent to zero*, $z\sim 0$, if it satisfies the conditions of the proposition. The cycles rationally equivalent to zero form a graded subgroup of $\mathscr{Z}_\bullet X$, and the quotient group

$$A_\bullet X=\mathscr{Z}_\bullet X/\sim$$

is called the *Chow* (homology) *group* of X.

We may use the same notation for a cycle and its equivalence class in $A_\bullet X$. For example, if Y is a closed subscheme of X, we say " [Y] in $A_\bullet X$ " to denote the equivalence class of the cycle [Y] modulo rational equivalence.

Corollary. — *Let* X *be an integral scheme of dimension* n. *Then*

(1) $A_n X\cong\mathbf{Z}$, *with generator* [X].

(2) *A Weil divisor* $z\in\mathscr{Z}^1 X$ *is rationally equivalent to zero if and only if it is the divisor of a rational function on* X.

1.9. Properties of the Chow Homology.

If $f: X \to Y$ is proper, and $z \sim 0$ on X, it follows from the definition that $f_* z \sim 0$ on Y. So f induces

$$f_* : A_\bullet X \to A_\bullet Y$$

and $A_\bullet$ becomes a covariant functor for proper morphisms.

If $f: X \to Y$ is a flat morphism, and $z \sim 0$ on Y, then $f^* z \sim 0$ on X. For if $g: Y' \to Y$ is proper, and $z = g_*[D']$ for a principal Cartier divisor on Y', then $f^* z = g'_*[f'^* D]$ is the image of a principal Cartier divisor on $X \times_Y Y'$ (Proposition (2) of § 1.6). So f induces a Gysin map

$$f^* : A_\bullet Y \to A_\bullet X$$

and $A_\bullet$ is contravariant for flat morphisms; if f is of relative dimension d, f^* raises degrees by d.

In particular, if U is an open subscheme of X, $j: U \to X$ the inclusion, we have a restriction homomorphism

$$A_\bullet X \xrightarrow{j^*} A_\bullet U.$$

Proposition (cf. [AC; 4, § 4]). — *Let* $i: X - U \to X$ *be the inclusion. Then the sequence*

$$A_\bullet(X-U) \xrightarrow{i_*} A_\bullet X \xrightarrow{j^*} A_\bullet U \to 0$$

is exact.

Proof. — If Y_0 is a closed integral subscheme of U, and $r_0 \in R(Y_0)^*$, then $Y = \overline{Y}_0$ is a closed integral subscheme of X, and r_0 determines a rational function r in $R(Y) = R(Y_0)$, so $j^*[\operatorname{div} r] = [\operatorname{div} r_0]$. Exactness follows easily from this, for if a cycle z on X becomes rationally equivalent to zero on U, we can find $Y_i \subset X$ and $r_i \in R(Y_i)$ so $z - \sum_i [\operatorname{div}(r_i)]$ has support on $X - U$.

Other identities we proved for cycles, as in § 1.5 and § 1.6, carry over to the Chow group. We will return to the Gysin map of § 1.7 in § 4.

Remark. — It follows from the definition of rational equivalence that the mapping $\mathscr{Z}_\bullet X \to \operatorname{Gr}_\bullet X$ which takes a subvariety V of dimension k of X to its structure sheaf $\mathcal{O}_V \in \operatorname{Filt}_k K_0 X$ modulo $\operatorname{Filt}_{k+1} K_0 X$ (cf. [SGA 6; X], [B-F-M; III, § 1]) induces a homomorphism

$$A_\bullet X \to \operatorname{Gr}_\bullet X$$

which is covariant for proper morphisms.

2. Intersections.

In this section we work in the category of algebraic schemes over a field k.

2.1. Serre's Intersection Theory.

If $f: X \to Y$ is a morphism, and Y is *non-singular*, Serre [S; V, § 7] has defined an intersection of cycles as follows. If $x \in \mathscr{Z}_p X$, $y \in \mathscr{Z}^q Y$, let $|x|$, $|y|$ be the supports of x and y. We say that x and y *intersect properly* (along f) if all components of $|x| \cap f^{-1}(|y|)$ have dimension $p-q$. In this case the intersection cycle

$$x \bullet_f y \in \mathscr{Z}_{p-q} X$$

is defined. If $x = Z_p(\mathscr{F})$ and $y = Z^q \mathscr{G}$, then

$$x \bullet_f y = \sum_i (-1)^i Z_{p-q}(\mathrm{Tor}_i^{\mathcal{O}_Y}(\mathscr{F}, \mathscr{G})).$$

If $X = Y$ we write just $x \bullet y$.

We write $f^* y$ instead of $[X] \bullet_f y$. If $f: X \to Y$ is flat, this definition agrees with that given in § 1.6. For example, if $f: X \to \mathbf{P}^1$, and $y = [0] - [\infty]$, then $f^* y = [\mathrm{div}(f)]$.

Proposition. — *Let* X *be an algebraic scheme,* $p: X \times \mathbf{P}^1 \to X$, $q: X \times \mathbf{P}^1 \to \mathbf{P}^1$ *the projections. A cycle* $x \in \mathscr{Z}_k X$ *is rationally equivalent to zero if and only if there is a cycle* $z \in \mathscr{Z}_{k+1}(X \times \mathbf{P}^1)$ *such that* $z \bullet_q([0] - [\infty])$ *is defined, and* $x = p_*(z \bullet_q([0] - [\infty]))$.

Proof. — If $x = \sum_i [\mathrm{div}(r_i)]$, with r_i rational functions on $(k+1)$-dimensional subvarieties Y_i of X, let Γ_i be the closure of the graph of r_i in $X \times \mathbf{P}^1$. Then $z = \sum_i [\Gamma_i]$ will work, since $[\mathrm{div}(r_i)] = p_*(\Gamma_i \bullet_q([0] - [\infty]))$ by § 1.5, Proposition 1.

2.2. Basic Identities.

Lemma [S; V-30]. — (1) *Let* $X \xrightarrow{f} Y \xrightarrow{g} Z$, x, y, z *cycles on* X, Y, Z *respectively. Assume* Y *and* Z *are non-singular, and all the intersections are proper. Then*

$$x \bullet_f (y \bullet_g z) = (x \bullet_f y) \bullet_{gf} z = (x \bullet_{gf} z) \bullet_f y.$$

(2) *Let* $X \xrightarrow{f} Y \xrightarrow{g} Z$, x, z *cycles on* X, Z *respectively. Assume* Z *is non-singular,* f *is a proper morphism, and all the intersections are proper. Then*

$$f_*(x \bullet_{gf} z) = f_* x \bullet_g z.$$

(3) *Let* $f_i : X \to Y_i$, Y_i *non-singular,* y_i *cycles on* Y_i, $i = 1, 2$, x *a cycle on* X. *Let* $= (f_1, f_2) : X \to Y_1 \times_k Y_2$. *If all intersections are proper, then*

$$(x \bullet_{f_1} y_1) \bullet_{f_2} y_2 = (x \bullet_{f_2} y_2) \bullet_{f_1} y_1 = x \bullet_f (y_1 \times y_2).$$

(4) *Let*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

be a fibre square, with g proper, Y *and* Y' *non-singular. If* $y' \in \mathscr{Z}(Y')$, *and both sides are defined, then*

$$g'_* f'^* y' = f^* g_* y' \quad \textit{in } \mathscr{Z}(X).$$

Proofs. — (1) Let $x = Z_p \mathscr{F}$, $y = Z^q \mathscr{G}$, $z = Z^r \mathscr{H}$. Then (1) follows from the spectral sequence of triple Tor, as in [S; V-30]. Similarly (2) follows from a spectral sequence relating $R^{\bullet} f_*(\mathrm{Tor}_{\bullet}^{\mathcal{O}_Y}(\mathscr{F}, \mathscr{H}))$ and $\mathrm{Tor}_{\bullet}^{\mathcal{O}_Y}(R^{\bullet} f_* \mathscr{F}, \mathscr{G})$ (cf. [S; V-29, 30] and [EGA III, 6.9.8]). The proof of (4) uses the same spectral sequence. For (3), if $x = Z_k \mathscr{F}$, $y_i = Z^{j_i} \mathscr{G}_i$, use the spectral sequence with E_2-term $\mathrm{Tor}_{\bullet}^{\mathcal{O}_{Y_2}}(\mathrm{Tor}_{\bullet}^{\mathcal{O}_{Y_1}}(\mathscr{F}, \mathscr{G}_1), \mathscr{G}_2)$ converging to $\mathrm{Tor}_{\bullet}^{\mathcal{O}_{Y_1} \otimes \mathcal{O}_{Y_2}}(\mathscr{F}, \mathscr{G}_1 \otimes \mathscr{G}_2)$.

This lemma generalizes the usual associativity and commutativity properties of intersections on non-singular varieties, as well as the fact that f^* is multiplicative and functorial on non-singular varieties, and the projection formula. We will use these identities quite freely in what follows. As an application we verify that our definition of rational equivalence agrees with the more usual definition [AC] for non-singular varieties.

Proposition. — *If* X *is non-singular, a cycle* x *in* $\mathscr{Z}_k X$ *is rationally equivalent to zero if and only if there is a cycle* $z \in \mathscr{Z}_{k+1}(X \times \mathbf{P}^1)$ *such that* z *intersects* $X \times \{0\}$ *and* $X \times \{\infty\}$ *properly, and* $x = p_*(z \bullet (X \times \{0\} - X \times \{\infty\}))$, *where* $p = X \times \mathbf{P}^1 \to X$ *is the projection.*

Proof. — This follows from the proposition in § 2.1, together with the fact that, from Lemma (1), $z \bullet_q([0] - [\infty]) = z \bullet (X \times \{0\} - X \times \{\infty\})$.

2.3. Moving Lemma.

Let Y *be non-singular and quasi-projective,* $f_i : X_i \to Y$ *morphisms,* x_i *cycles on* X_i, $i = 1, \ldots, m$, y *a cycle on* Y. *Then there is a cycle* y' *on* Y, *rationally equivalent to* y, *such that* x_i *and* y' *intersect properly along* f_i *for all* $i = 1, \ldots, m$.

Proof. — By looking at the components of the cycles x_i, we are reduced to the case where the X_i are varieties and $x_i = [X_i]$. Stratify Y into a disjoint union of locally closed subsets W_j so that the restriction of each f_i to each W_j is equidimensional. If y' is a cycle on Y which intersects all the W_j properly, then y' intersects all the $[X_i]$ properly along f_i. So it suffices to apply the usual moving lemma ([AC], [R]) to y and the W_j.

Proposition. — *Let* $f: X \to Y$, Y *non-singular and quasi-projective,* $x \in \mathscr{Z}_\bullet X$, $y \in \mathscr{Z}^\bullet Y$ *cycles which intersect properly along* f.

(1) *If* $y \sim 0$, *then* $x \bullet_f y \sim 0$.
(2) *If* $x \sim 0$, *then* $x \bullet_f y \sim 0$.

Proof. — (1) If $x = \sum_i n_i[V_i]$, it suffices to show $[V_i] \bullet_{f_i} y \sim 0$ on V_i, where $f_i : V_i \to Y$ is the induced map (this follows from Lemma (2) of § 2.1 applied to $V_i \to X \to Y$). Thus we may assume $x = [X]$, where X is irreducible, and we want to show $f^* y \sim 0$.

By the proposition of § 2.2, there is a cycle z on $Y \times \mathbf{P}^1$ so that if $D = [0] - [\infty]$ on $\mathbf{P}^1$ and $q: Y \times \mathbf{P}^1 \to \mathbf{P}^1$ is the projection, then $y = p_*(z \bullet_q D)$.

Consider the fibre square

$$\begin{array}{ccc} X \times \mathbf{P}^1 & \xrightarrow{f'} & Y \times \mathbf{P}^1 \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & Y \end{array}$$

Then by § 2.2, Lemma (4), $f^* y = f^* p_*(z \bullet_q D) = p'_* f'^*(z \bullet_q D)$, so it suffices to show $f'^*(z \bullet_q D)$ is rationally equivalent to zero. But $f'^*(z \bullet_q D) = f'^*(z) \bullet_{q'} D$ by § 2.2, Lemma (1), where $q' : X \times \mathbf{P}^1 \to \mathbf{P}^1$ is the projection, and $f'^*(z) \bullet_{q'} D = [\mathrm{div}(qf')]$.

(2) We may assume $x = \pi_*[\mathrm{div}(r)]$, where $\pi : X' \to X$ is proper, $r \in R(X')$. Then $x \bullet_f y = \pi_*([\mathrm{div}(r)] \bullet_{f\pi} y)$ by § 2.2, Lemma (2), so we may assume X is irreducible and $x = [\mathrm{div}(r)]$. As usual, we may assume r is a morphism from X to $\mathbf{P}^1$, so $x = [X] \bullet_r D$, $D = [0] - [\infty]$. By § 2.2, Lemma (3), $x \bullet_f y = ([X] \bullet_r D) \bullet_f y = ([X] \bullet_f y) \bullet_r D \sim 0$.

3. The Chow Cohomology Groups $A^\bullet$.

In this section we work in the category of quasi-projective schemes over a field k.

3.1. Definition and Basic Properties.

If Y is a non-singular variety, define $A^q Y$ to be $\mathscr{Z}^q Y$ modulo the cycles rationally equivalent to zero. It follows from the results of § 2 that $A^\bullet Y = \bigoplus_q A^q Y$ is a graded ring, and that $Y \to A^\bullet Y$ is a contravariant functor from non-singular quasi-projective varieties to graded rings (cf. [AC]).

If X is an arbitrary quasi-projective scheme, let $\mathscr{C}(X)$ denote the category of pairs (Y, f), where Y is non-singular and f is a morphism from X to Y. A morphism from (Y, f) to (Y', f') in $\mathscr{C}(X)$ is a morphism $g : Y \to Y'$ such that $g \circ f = f'$.

Assigning $A^{\bullet}Y$ to (Y, f) gives a contravariant functor from $\mathscr{C}(X)$ to rings, and we define

$$A^{\bullet}X = \varinjlim_{\mathscr{C}(X)} A^{\bullet}Y.$$

More concretely, $A^q X$ is the disjoint union of the $A^q Y$ for all $f: X \to Y$, Y non-singular, modulo the equivalence relation generated by setting $g^* y' = y$ whenever g is a morphism from (Y, f) to (Y', f'), and $y' \in A^q Y'$. If $f_i : X \to Y_i$ and $y_i \in A^{\bullet} Y_i$, $i = 1, 2$, we add (resp. multiply) the classes represented by y_1 and y_2 by setting $Y = Y_1 \times Y_2$, $p_i : Y \to Y_i$ the projections; then $p_1^* y_1 + p_2^* y_2$ (resp. $p_1^* y_1 \bullet p_2^* y_2$) represents the sum (resp. product) of y_1 and y_2.

A morphism $h : X_1 \to X_2$ induces a map $(Y, f) \to (Y, f \circ h)$ from $\mathscr{C}(X_2)$ to $\mathscr{C}(X_1)$, and hence a morphism $h^* : A^{\bullet}X_2 \to A^{\bullet}X_1$. We see that $A^{\bullet}$ is a contravariant functor from quasi-projective schemes to graded rings. Note that if X is non-singular $\mathscr{C}(X)$ has an initial object, so the two definitions of $A^{\bullet}X$ agree.

The cap product

$$A^q X \otimes A_p X \xrightarrow{\frown} A_{p-q} X$$

is defined as follows. If $f: X \to Y$, with Y non-singular, and $x \in A_p X$, $y \in A^q Y$, then $x \bullet_f y \in A_{p-q} X$ is well-defined by § 2. This definition is compatible with maps in $\mathscr{C}(X)$ by Lemma (1) of § 2.2, and so it passes to the limit to give the desired cap product. This makes $A_{\bullet}X$ into a module over $A^{\bullet}X$.

From Lemma (2) of § 2.2 we deduce the

Projection formula. — If $f: X_1 \to X_2$ is proper, and $a \in A_{\bullet} X_1$, $b \in A^{\bullet} X_2$, then

$$f_*(f^* b \frown a) = b \frown f_* a.$$

Two other properties relate the Chow cohomology groups to the Gysin map in the Chow homology (§ 1.9).

Proposition. — *Let*

$$\begin{array}{ccc} Q & \xhookrightarrow{j} & P \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xhookrightarrow{i} & Y \end{array}$$

be a fibre square, with i, j closed immersions, and p, q flat.

(1) *If* $x \in A_{\bullet}X$, $y \in A^{\bullet}Y$, *then*

$$q^*(y \frown x) = p^* y \frown q^* x \quad \textit{in } A_{\bullet}Q.$$

(2) *If* Y *is non-singular and p is smooth,* $x \in A_{\bullet}X$, $z \in A^{\bullet}P$, *then*

$$q_*(j^* z \frown q^* x) = i^*(p_* z) \frown x \quad \textit{in } A_{\bullet}X.$$

Here $q^* : A_\bullet X \to A_\bullet Q$ is the Gysin map of § 1.9, and $p_* : A^\bullet P \to A^\bullet Y$ is the Gysin map that always exists (by Poincaré duality) in the non-singular case.

Both parts are proved by reducing to the case when $x = [X]$, and using the lemma in § 2.

3.2. Chern Classes.

To extend the theory of Chern classes from non-singular varieties to singular varieties we need the following lemma. A. Landman showed us the proof of (3).

Lemma. — (1) *Let* E *be a vector-bundle on a quasi-projective scheme* X. *Then there is a non-singular variety* M, *an imbedding* $i : X \to M$, *and a vector-bundle* F *on* M *so that* $i^* F \cong E$.

(2) *If* $0 \to E' \to E \to E'' \to 0$ *is an exact sequence of bundles on* X, *there is a non-singular* M, *an imbedding* $i : X \to M$, *and an exact sequence*

$$0 \to F' \to F \to F'' \to 0$$

on M *so that* $0 \to i^* F' \to i^* F \to i^* F'' \to 0$ *is isomorphic to the given sequence on* X.

(3) *If* $f : X \to Y$, Y *non-singular, and* E_1, E_2 *are vector-bundles on* Y *such that* $f^* E_1 \cong f^* E_2$, *then there is a factorization* $f = g \circ f'$ *of* f, $f' : X \to Y'$, $g : Y' \to Y$, *with* Y' *non-singular, such that* $g^* E_1 \cong g^* E_2$.

Proof. — Since (1) is a special case of (2), we prove (2). Imbed X in a projective space $P = \mathbf{P}^n$. For m sufficiently large there is a surjection $\varepsilon^N \to E(m) \to 0$ from a trivial bundle onto $E(m) = E \otimes \mathcal{O}(m)$. Let G be the flag manifold classifying successive quotients of ε^N of ranks $e = \text{rank } E$, $e'' = \text{rank } E''$, and let

$$\varepsilon^N \to \xi \to \xi''$$

be the universal example of successive quotients on G. G is a Grassmann-bundle over a Grassmannian, so G is non-singular. There is a morphism $f : X \to G$ so that $\varepsilon^N \to \xi \to \xi''$ pulls back to $\varepsilon^N \to E(m) \to E''(m)$.

Let $M = P \times G$, $i(x) = (x, f(x))$, and let $F = p_1^* \mathcal{O}(-m) \otimes p_2^* \xi$, $F'' = p_1^* \mathcal{O}(-m) \otimes p_2^* \xi''$ (where p_1, p_2 are the projections), and $F' = \text{Ker}(F \to F'')$. It is clear that this restricts to the given sequence on X.

To prove (3), let $Y' = \text{Isom}(E_1, E_2)$ be the open subscheme of the vector-bundle $\text{Hom}(E_1, E_2)$ over Y consisting of isomorphisms, and let $g : Y' \to Y$ be the projection. Since g is locally a bundle with the general linear group for fibre, g is smooth, so Y' is non-singular. There is a one-to-one correspondence between bundle maps from $f^* E_1$ to $f^* E_2$ and factorizations of f through $\text{Hom}(E_1, E_2)$; under this correspondence the isomorphisms correspond to factorizations through Y', which proves (3).

Définition. — For non-singular quasi-projective varieties there is a theory of Chern classes of vector-bundles with the usual formal properties [G]. If E is a bundle on a non-singular Y, we let $c(E)=1+c_1(E)+\dots$ be the total Chern class, $c_i(E)\in A^iY$ the i-th Chern class.

If X is singular and E is a bundle on X, choose a non-singular variety Y, a morphism $f: X\to Y$, and a bundle F on Y so that $f^*F\cong E$. Then $c(F)\in A^{\bullet}Y$ defines an element $c(E)$ in $A^{\bullet}X$, which is independent of choices by the construction of $A^{\bullet}X$ and Lemma (3), and is called the *total Chern class* of E.

Proposition. — (1) *If* $f: X'\to X$, $c(f^*E)=f^*c(E)$.

(2) $c_1: \mathrm{Pic}(X)\to A^1X$ *is an isomorphism.*

(3) *If* D *is a Cartier divisor on* X, *then*

$$c_1(\mathcal{O}(D))\frown[X]=[D] \quad \textit{in } A_{\bullet}X.$$

(4) *If* $0\to E'\to E\to E''\to 0$ *is exact, then*

$$c(E)=c(E').c(E'').$$

(5) *The usual formulas* [G] *for Chern classes of dual bundles, exterior powers, and tensor products hold.*

Proof. — (1) is clear. (2) follows from the non-singular case by passing to the limit.

Given L on X, choose $f: X\to Y$, Y non-singular, and a Cartier divisor $\widetilde{D}$ on Y so that $f^*\widetilde{D}=D$ is defined, and $\mathcal{O}(D)\cong L$ (for example, $Y=\mathbf{P}^n$, $\widetilde{D}=$ the difference of two hypersurfaces). Then $c_1(L)\frown[X]=[X]\bullet_f\widetilde{D}=[D]$ by definition of the intersection cycle. This proves (3).

The additivity follows from the non-singular case and the Lemma (2). The formulas referred to in (5) likewise pull back from the non-singular case.

3.3. The Chern Character.

The construction of Chern classes gives rise to a Chern character

$$\mathrm{ch}: K^0X\to A^{\bullet}X_{\mathbf{Q}}$$

which is a homomorphism of rings.

Proposition. — $\mathrm{ch}_{\mathbf{Q}}: K^{\bullet}X_{\mathbf{Q}}\to A^{\bullet}X_{\mathbf{Q}}$ *is an isomorphism for all* X.

Proof. — It follows from the Riemann-Roch theorem ([SGA 6] or [B-F-M; III, § 1]) that the assertion is true when X is non-singular.

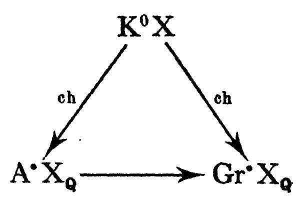
It follows from the lemma in § 3.2 that $K^0X=\varinjlim K^0Y$ where the limit is over all $f: X\to Y$, Y non-singular. Thus the general case follows from the non-singular case.

Corollary. — *There is a natural (contravariant) isomorphism* $A^{\bullet}X_{\mathbf{Q}}\cong \mathrm{Gr}^{\bullet}X_{\mathbf{Q}}$ *of graded rings obtained by filtering* K^0X *by the* λ*-filtration* [SGA 6].

Proof. — If X is non-singular, the mapping is the composite

$$A^{\bullet}X_{\mathbf{Q}}\longrightarrow \mathrm{Gr}^{\bullet}_{\mathrm{top}}X_{\mathbf{Q}}\xleftarrow{\cong}\mathrm{Gr}^{\bullet}X_{\mathbf{Q}}$$

where $\mathrm{Gr}^{\bullet}_{\mathrm{top}}X$ is the graded ring obtained from the topological filtration of K^0X (cf. [SGA 6; VII, 4.11]). Since Chern classes correspond in this isomorphism, we may pass to the limit (for general X, and $f: X \to Y$) to get a homomorphism $A^{\bullet}X_{\mathbf{Q}} \to \mathrm{Gr}^{\bullet}X_{\mathbf{Q}}$, so that the diagram



commutes. Since both Chern characters $\mathrm{ch}_{\mathbf{Q}}$ are isomorphisms (cf. [Yu. I. Manin, Lectures on the K-functor in algebraic geometry, *Russ. Math. Surveys*, 24 (1969), p. 49] for the second), the bottom is also an isomorphism.

Remark. — Grothendieck *et al.* have defined Gysin homomorphisms

$$f_* : \mathrm{Gr}^{\bullet}X_{\mathbf{Q}} \to \mathrm{Gr}^{\bullet}Y_{\mathbf{Q}}$$

for proper complete intersection morphisms $f: X \to Y$ [SGA 6]. So there are corresponding Gysin homomorphisms $A^{\bullet}X_{\mathbf{Q}} \to A^{\bullet}Y_{\mathbf{Q}}$. It is not clear how to define these maps without rational coefficients; even if f is a smooth morphism the definition of $A^{\bullet}$ given here is not amenable to pushing forward.

4. A Gysin Map; Specialization.

In § 4.1-4.3 we remain in the category of quasi-projective schemes over a field.

4.1. Rational Equivalence Specializes.

Let $f: X \to C$ be a flat morphism from a scheme X to a non-singular curve C. Let t be a closed point in C, and let $X_t = f^{-1}(t)$ be the scheme-theoretic fibre, $i: X_t \to X$ the inclusion. We will define a " Gysin homomorphism " [1]

$$i^* : A_k X \to A_{k-1} X_t.$$

The map $i^* : \mathscr{Z}_k X \to \mathscr{Z}_{k-1} X_t$ has already been defined (§ 1.7): $i^*[V] = 0$ if $V \subset X_t$, $i^*[V] = [V_t]$ otherwise. Note that we may replace C by an open neighborhood of $\{t\}$, so we may assume $\{t\}$ is a principal Cartier divisor on C, so $X_t = f^{-1}\{t\}$ is principal on X. The problem is to show that rational equivalence is preserved by i^*. Since this Gysin map is compatible with pushing forward (§ 1.7, Proposition (1)), we are reduced to proving the following case.

(1) *Note added in proof.* J.-L. Verdier has used this to define Gysin homomorphisms for arbitrary complete intersection morphisms [Séminaire Bourbaki, n° 464, Feb. 1975].

Lemma. — *Suppose* X *is integral, and* $r \in R(X)^*$. *Then* $i^*[\operatorname{div}(r)]$ *is rationally equivalent to zero on* X_t.

Proof. — As in the remark in § 1.8, we may assume r is a morphism from X to $\mathbf{P}^1$. Then (f, r) is a morphism from X to $C \times \mathbf{P}^1$. If (f, r) were not dominant, $[\operatorname{div}(r)]$ would lie in a finite number of fibres of f, and then $i^*[\operatorname{div}(r)] = 0$. So we may assume (f, r) is dominant. As in [B-F-M; II, § 2.5], we may find proper, birational maps

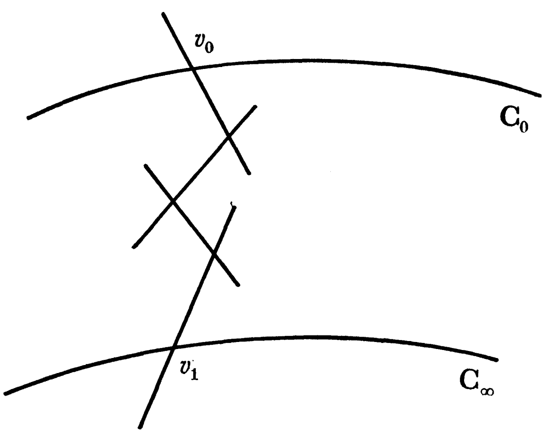
$$\rho : V \to C \times \mathbf{P}^1, \quad \psi : X' \to X,$$

where V is non-singular, and a flat morphism $F : X' \to V$ so that the diagram

$$\begin{array}{ccc} X' & \xrightarrow{F} & V \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\rho} \\ X & \xrightarrow{(f,r)} & C \times \mathbf{P}^1 \end{array}$$

is commutative. Since we may replace X by X′, we may assume (f, r) factors into $X \xrightarrow{F} V \xrightarrow{\rho} C \times \mathbf{P}^1$, where F is flat.

Let $E = \rho^{-1}(\{t\} \times \mathbf{P}^1)$, and let C_0 and C_∞ be the non-singular curves on V that map isomorphically by ρ to $C \times \{0\}$ and $C \times \{\infty\}$. E consists of a connected collection of non-singular rational curves intersecting transversally. Blowing up more points if necessary, we may assume C_0 and C_∞ meet E transversally at points v_0, v_∞ of V.



For any curve D on V, let X_D be the fibre over D, $[X_D]$ the corresponding Weil divisor. Then $[\operatorname{div}(r)] = [X_{C_0}] - [X_{C_\infty}]$ plus components that lie in fibres of f. So the lemma reduces to showing that $i^*[X_{C_0}] \sim i^*[X_{C_\infty}]$ in $\mathscr{Z}_{\bullet} X_t$.

Let D be a smooth curve on V which intersects E transversally in a simple point v; let L be the irreducible component of E which contains v, and let $F_L : X_L \to L$ be the morphism induced by F. We claim that $i^*[X_D] = F_L^*[v]$ in $\mathscr{Z}_{\bullet} X_t$. Since $X_D = F^*[D]$, this follows from Proposition (2) of § 1.7; note that [D] pulls back to $[v]$ on $L \subset E$.

To finish the proof we must show that all the cycles $F_L^*[v]$, $v \in L \subset E$ are rationally equivalent. This is clear for fixed L as v varies in L, since $L \cong \mathbf{P}^1$. Since E is connected, we need only show that in case v is the point of transversal intersection of two components L_1 and L_2 of E, then $F_{L_1}^*[v] = F_{L_2}^*[v]$. The argument for this is the same as in the preceding paragraph.

4.2. Properties of the Gysin Map.

This shows that the Gysin map

$$i^* : A_\bullet X \to A_\bullet X_t$$

is well-defined on the Chow groups. From the Proposition in § 1.7 it follows that if X and Y are flat over C, and $g : X \to Y$ is proper, then the Gysin maps commute with pushing forward. To call i^* a Gysin map, one should check that it is compatible with the cohomology map $i^* : A^\bullet X \to A^\bullet X_t$.

Proposition. — The diagram

$$\begin{array}{ccc} A^\bullet X \otimes A_\bullet X & \xrightarrow{\frown} & A_\bullet X \\ \downarrow{\scriptstyle i^* \otimes i^*} & & \downarrow{\scriptstyle i^*} \\ A^\bullet X_t \otimes A_\bullet X_t & \xrightarrow{\frown} & A_\bullet X_t \end{array}$$

commutes.

Proof. — We must show if $g : X \to Y$, Y non-singular, $x \in A_\bullet X$, $y \in A^\bullet Y$, then $i^*(x \bullet_g y) = i^* x \bullet_{gi} y$. By looking at the components of x, we may assume $x = [X]$, X integral; and we may move y so all the intersections are proper. Then

$$i^*(x \bullet_g y) = [X_t] \bullet_i (x \bullet_g y) = [X_t \bullet_i x] \bullet_{gi} y = i^* x \bullet_{gi} y$$

as in § 2.2, Lemma (1).

4.3. Products.

If X and Y are schemes, there is a Künneth map $\mathscr{Z}_k X \otimes \mathscr{Z}_\ell Y \to \mathscr{Z}_{k+\ell}(X \times Y)$ which takes $[V] \otimes [W]$ to $[V \times W]$ for V, W irreducible subvarieties of X, Y respectively. This is covariant for proper maps, and passes to the Chow groups, giving a Künneth map

$$A_\bullet X \otimes A_\bullet Y \to A_\bullet(X \times Y).$$

Proposition. — If $Y = \mathbf{A}^n$ *is affine space, then*

$$A_\bullet X \otimes A_\bullet Y \to A_\bullet(X \times Y)$$

is an isomorphism for all X.

Proof. — We may assume $n=1$. The surjectivity of the mapping follows by induction on the dimension of X, using the exact sequence of § 1.9 (cf. [AC; 4, § 4]). The injectivity follows from the fact that if $i(x)=(x, 0)$, then

$$i^*(x \times [Y]) = x \quad \text{for all} \quad x \in A_{\bullet}X.$$

This also proves the following fact:

Corollary. — *If* C *is a non-singular rational curve, and* $i_t : X \to X \times C$ *is the imbedding* $x \to (x, t)$, *then the Gysin maps* $i_t^* : A_{\bullet}(X \times C) \to A_{\bullet}X$ *are the same for all k-rational points* $t \in C$.

4.4. Specialization.

The existence of the Gysin map leads easily to a specialization map (cf. [SGA 6; X, 7]). In this paragraph all rings and schemes are noetherian and excellent.

Let R be a discrete valuation ring with residue field $R/\mathfrak{m}=k$, and quotient field K. Let X be a scheme which is flat and quasi-projective over R, and write $X_K = X \otimes_R K$ and $X_k = X \otimes_R k$ for the generic and special fibres, $i : X_k \to X$, $j : X_K \to X$ the inclusions. From § 1.9 we have the exact sequence

$$A_{p+1}X_k \xrightarrow{i_*} A_{p+1}X \xrightarrow{j^*} A_pX_K \longrightarrow 0.$$

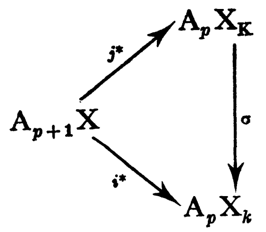
We remark first that the argument of § 4.2 extends to the case where $C = \operatorname{Spec} R$, and $C \times \mathbf{P}^1 = \mathbf{P}^1_R$ and V are regarded as arithmetic surfaces. (Note that a suitable V for the specialization lemma may be constructed by successively blowing up k-rational points, and that only k-rational points need be considered in the proof of the lemma). Thus we obtain a Gysin map

$$A_{p+1}X \xrightarrow{i^*} A_pX_k.$$

Since $i^*i_* = 0$ (even on the cycle level), we conclude that there is a unique map

$$\sigma = \sigma_X : A_pX_K \to A_pX_k,$$

the *specialization homomorphism*, such that the diagram



commutes.

Proposition. — (1) *Let* $f: X \to Y$ *be a proper morphism of flat quasi-projective* R-*schemes. Then the diagram*

$$\begin{array}{ccc} A_\bullet X_K & \xrightarrow{f_{K*}} & A_\bullet Y_K \\ \downarrow{\scriptstyle \sigma_X} & & \downarrow{\scriptstyle \sigma_Y} \\ A_\bullet X_k & \xrightarrow{f_{k*}} & A_\bullet Y_k \end{array}$$

commutes.

(2) *If* $f: X \to Y$ *is a flat morphism of flat quasi-projective* R-*schemes, then the diagram*

$$\begin{array}{ccc} A_\bullet Y_K & \xrightarrow{f_K^*} & A_\bullet X_K \\ \downarrow{\scriptstyle \sigma_Y} & & \downarrow{\scriptstyle \sigma_X} \\ A_\bullet Y_k & \xrightarrow{f_k^*} & A_\bullet X_k \end{array}$$

commutes, where the horizontal maps are the Gysin maps of § 1.9.

Proof. — These follow from the proposition in § 1.7.

If R is henselian (say complete), let $\overline{K}$ (resp. $\bar{k}$) be the algebraic closure of K (resp. k). One may pass to the limit over all finite extensions R′ of R in $\overline{K}$ (using the Gysin maps $A_\bullet X \to A_\bullet X_{R'}$ for the flat morphism $X_{R'} \to X$) and arrive at a specialization homomorphism

$$A_\bullet X_{\overline{K}} \to A_\bullet X_{\bar{k}}$$

of geometric fibres.

As explained by Grothendieck [SGA 6; X, 7.13-7.16], the existence of these specialization maps implies that if X is proper over $C = \operatorname{Spec} R$, with regular fibres, there is a commutative diagram

$$\begin{array}{ccc} A^i X_K & \xrightarrow{cl} & H^{2i}(X_K, \mathbf{Z}_\ell(i)) \\ \downarrow{\scriptstyle \sigma} & & \uparrow \\ A^i X_k & \xrightarrow{cl} & H^{2i}(X_k, \mathbf{Z}_\ell(i)) \end{array}$$

which passes to the limit to give

$$\begin{array}{ccc} A^i X_{\bar{K}} & \xrightarrow{cl} & H^{2i}(X_{\bar{K}}, \mathbf{Z}_\ell(i)) \\ \downarrow & & \| \\ A^i X_{\bar{k}} & \xrightarrow{cl} & H^{2i}(X_{\bar{k}}, \mathbf{Z}_\ell(i)) \end{array}$$

Here $\ell \neq \operatorname{char} k$, and $H^{2i}(\ , \mathbf{Z}_\ell(i))$ is the ℓ-adic cohomology.

5. Natural Transformations.

In this section we work in the category of projective varieties over a field. Let $H_\bullet X = A_\bullet X_{\mathbf{Q}} = A_\bullet X \otimes \mathbf{Q}$. Regard $H_\bullet$ as a covariant functor from projective varieties to abelian groups. We thank A. Landman for the proof of the following proposition.

Proposition. — Let $\alpha : H_\bullet \to H_\bullet$ *be a natural transformation of functors. If for each projective space* $\mathbf{P}^n$, $n = 0, 1, 2, \ldots$

$$\alpha[\mathbf{P}^n] = [\mathbf{P}^n] + \text{terms of degree} \neq n$$

then α *is the identity.*

Proof. — Let $\beta = \alpha - I$, where I is the identity transformation. It suffices to show $\beta[X] = 0$ for all varieties X, since $H_\bullet X$ is generated by [V] for V a subvariety of X, and we can apply naturality to the inclusion of V in X.

We claim first that $\beta[\mathbf{P}^n] = 0$. For suppose the coefficient of [H] in $\beta[\mathbf{P}^n]$ were non-zero, where H is a k-plane in $\mathbf{P}^n$; $k \neq n$ by hypothesis. Choose a morphism $f: \mathbf{P}^n \to \mathbf{P}^n$ such that $f_*[\mathbf{P}^n] = d[\mathbf{P}^n]$, $f_*[H] = e[H]$, and $d \neq e$. Such a morphism can be obtained by composing the Veronese imbedding by a projection. Then apply naturality to f to get a contradiction.

Now given an n-dimensional variety X, choose a separable finite morphism $f: X \to \mathbf{P}^n$. Since it is enough to show $\beta[X'] = 0$ for any X′ for which there is a finite morphism from X′ to X (apply naturality to this morphism), we may assume $f: X \to \mathbf{P}^n$ is a Galois (branched) covering, with Galois group G, and X is normal.

By naturality with respect to the automorphisms in G, $\beta[X]$ must belong to the fixed part $H_\bullet X^G$ of $H_\bullet X$. Finally, applying naturality to the morphism $f: X \to \mathbf{P}^n$ it is enough to check that f_* maps $H_\bullet X^G$ isomorphically to $H_\bullet \mathbf{P}^n$, since we know $\beta[\mathbf{P}^n] = 0$. And this follows easily from the identity

$$f^* f_* c = \sum_{g \in G} g_* c$$

for a cycle c on X. This identity can be seen by applying f_* to both sides and using the projection formula to count the number of times cycles must occur on both sides.

Remark. — If X is a complex projective variety, then there is a homomorphism

$$\mathscr{Z}_{\bullet}X \xrightarrow{c} H_{\bullet}(X;\mathbf{Z})$$

which assigns to each subvariety V of X its homology class $c[V]$ (say by triangulation or resolution of singularities). This is a natural transformation of covariant functors. If $r: X \to \mathbf{P}^1$ is a morphism, then $c[\mathrm{div}(r)]=0$ in $H_{\bullet}(X;\mathbf{Z})$, so c induces a natural transformation

$$A_{\bullet} \xrightarrow{c} H_{\bullet}(\ ;\mathbf{Z})$$

from complex projective varieties to abelian groups.

The proof of the proposition extends to this case to show that c gives the only natural transformation from $A_{\bullet\mathbf{Q}}=A_{\bullet}\otimes\mathbf{Q}$ to $H_{\bullet}(\ ;\mathbf{Q})$ which takes $[\mathbf{P}^n]$ to $[\mathbf{P}^n]+$lower terms for each projective space $\mathbf{P}^n$. In the last step of the proof it is necessary to know that if $X/G=\mathbf{P}^n$, then $H_{\bullet}(X;\mathbf{Q})^G \cong H_{\bullet}(\mathbf{P}^n;\mathbf{Q})$. This follows by suitably triangulating the map from X to $\mathbf{P}^n$ [B. Giesecke, Simpliziale Zerlegung abzählbarer analytischer Räume, *Math. Zeit.*, 83 (1964), 177-213, Satz 7].

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