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GENERIC ONE-PARAMETER FAMILIES OF VECTOR FIELDS ON TWO-DIMENSIONAL MANIFOLDS

by J. SOTOMAYOR

INTRODUCTION

In this paper we present a study on the theory of the topological variation of the phase space of one-parameter families of vector fields (differential equations, flows). This theory, also called bifurcation theory, has been developed since H. Poincaré from several points of view; see, for example, [1, 2, 3, 4]. Here, we will be mainly interested in a collection of one-parameter families of vector fields which has the following properties: *a)* it is large with respect to all the families, and *b)* its elements exhibit a topological variation which is amenable to simple description.

Collections with properties *a)* and *b)* are currently called " generic "; they were introduced in the global qualitative analysis of differential equations by M. Peixoto [7], S. Smale [9] and I. Kupka [12]. See S. Smale [10] for a thorough survey on this topic.

In this work we restrict ourselves to the case of two-dimensional compact manifolds, where a very complete characterization of the set Σ of structurally stable vector fields has been given by M. Peixoto [8]. The way Σ is imbedded in the space $\mathfrak{X}$ of all vector fields and the study of " generic " one-parameter families of vector fields are closely related. A vector field is structurally stable if its phase space does not change topologically under small perturbations; a one-parameter family of vector fields exhibits the simpler phase space topological variation the larger the intersection it has with Σ, or equivalently, the smaller the intersection it has with its complement $\mathfrak{X}-\Sigma$.

In this paper, in Part I, we define a set Σ_1, densely contained in $\mathfrak{X}-\Sigma$. We prove that Σ_1 is an immersed Banach submanifold of codimension one in the Banach manifold $\mathfrak{X}$. Also, we describe the variation of the phase space of vector fields in a neighborhood of Σ_1. In Part II, we prove that the " generic " one-parameter families of vector fields meet Σ_1 transversally at points where they are not vector fields of Kupka-Smale [12, 11]. See Theorems 1 (5, Part I) and 2 (1, Part II) for a precise and complete statement of these results.

Whether or not Σ_1 coincides with the " regular (differentiable, or even Hölder) part " of codimension one of $\mathfrak{X}-\Sigma$ immersed in $\mathfrak{X}$, remains an important non-trivial problem, related to delicate questions of " closing lemma " type [8, 20], involving the innermost structure of recurent trajectories.

The conception of the submanifold Σ_1 was motivated by [14], where we treated the concept of first order structural stability, introduced by Andronov and Leontovich [13]. Part I extends the results of [14].

In section 6 of Part I we comment on first order structural stability and relate it to Σ_1. In section 3 of Part II we define the concept of structural stability for parametrized families of vector fields and formulate some related conjectures.

In a forthcoming paper some of the methods, results and concepts of the present paper are pursued to manifolds of higher dimension.

The main results of this work answer questions raised by M. Peixoto. We are most grateful to him, to S. Smale and to I. Kupka for fruitful conversations and manifold aid.

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Pictures and references appear at the end of the paper.

I. — THE SUBMANIFOLD Σ_1^r

1. Preliminaries

Let $\mathfrak{X}$ be a Banach manifold of class C^∞ defined as in [15, p. 16], i.e., $\mathfrak{X}$ is locally homeomorphic to an open set of some Banach space, the changes of coordinates being C^∞ functions.

Definition (1.1). — A subset $S \subset \mathfrak{X}$ is said to be an *imbedded* Banach submanifold of class C^s and codimension k of $\mathfrak{X}$ if every $p \in S$ has a neighborhood N where a C^s-function $f : N \to \mathbf{R}^k$ is defined so that:

a) $Df_p : \mathfrak{X}_p \to \mathbf{R}^k$, the derivative of f at p, is onto, and

b) $f^{-1}(0) = N \cap S$.

Definition (1.2). — $S \subset \mathfrak{X}$ is said to be an *immersed* Banach submanifold of class C^s and codimension k of $\mathfrak{X}$ if there is a sequence $\{S_i\}$, $i = 1, 2, \ldots,$ of imbedded Banach submanifolds of class C^s and codimension k of $\mathfrak{X}$ such that $S_i \subset S_{i+1}$ and $S = \bigcup_{i=1}^{\infty} S_i$.

It follows from the Implicit Function Theorem [15, p. 15] that a submanifold S, as defined in (1.1), has an atlas of class C^s which makes the inclusion $S \to \mathfrak{X}$ an imbedding in the usual sense [15, p. 20]. Also, if S is an immersed submanifold in the sense of (1.2), the union of the atlases of S_i defines on S an atlas, which makes it a manifold and makes the inclusion $S \to \mathfrak{X}$ a one-to-one immersion in the usual sense [15, p. 19]. In this work the Banach submanifolds will be defined through (1.1) and (1.2).

Let M^2 be a compact two-dimensional C^∞ differentiable manifold. Denote by $\mathfrak{X}^r$ the space of tangent vector fields of class C^r defined on M^2, endowed with the C^r-topology. $\mathfrak{X}^r$ is a Banach manifold in the sense of [15]; its atlas is given by the collection of identity mappings of $\mathfrak{X}^r$ Banached by the C^r-norms associated to finite coverings of M^2 by compact coordinate neighborhoods.

If $X \in \mathfrak{X}^r$, $\varphi_X : M^2 \times \mathbf{R} \to M^2$ will denote the flow generated by X; φ_X is characterized by $\frac{\partial}{\partial t}\varphi_X(p, t) = X(\varphi_X(p, t))$, $(p, t) \in M^2 \times \mathbf{R}$ and $\varphi_X(p, 0) = p$. $\varphi_X(p, \ ) : \mathbf{R} \to M^2$ is the *orbit* of X passing through p; the image of an orbit, oriented but with no distinguished parametrization, is a *trajectory* of X.

Definition (1.3). — X and $Y \in \mathfrak{X}^r$ are said to be *topologically equivalent* if there is a homeomorphism of M^2 onto itself mapping trajectories of X onto trajectories of Y. If

X has a neighborhood N in $\mathfrak{X}^r$ such that X is topologically equivalent to every $Y \in N$, then it is called *structurally stable.*

The set of structurally stable vector fields will be denoted by Σ^r; its complement in $\mathfrak{X}^r$ will be denoted by $\mathfrak{X}_1^r$. It has been shown by M. Peixoto [8] that Σ^r coincides with the collection of vector fields X such that

a) X has all its singular points and periodic trajectories generic;

b) X does not have saddle connections; and

c) the α and ω-limit sets of every trajectory of X are singular points or periodic trajectories.

The collection of vector fields X satisfying *a)* and *b)* have been studied by I. Kupka [12] and S. Smale [11] in a more general context; it will be denoted by $[\text{K-S}]^r$.

For future reference we recall some definitions of [5].

Definition **(1.4)**. — *a)* A trajectory of X is called *ordinary* if it has a neighborhood N in M^2 such that $X|N$ is topologically equivalent to the horizontal field $\dfrac{\partial}{\partial x_1}$ in $\mathbf{R}^2$. A connected component of the (open) set of ordinary trajectories of X is called a *canonical region* of X.

b) A *critical region* of X is a neighborhood N of a generic critical element (i.e. singular point or periodic trajectory of X) δ_X, such that for Y close to X, Y has only one critical element δ_Y in N and δ_Y is generic and of the same type of δ. See [5, p. 144].

2. Periodic trajectories

Since the evaluation map $(X, p) \mapsto X(p)$ is of class C^r on $\mathfrak{X}^r \times M^2$ [16, p. 25], it follows from [15, p. 94], taking X as parameter, that $\varphi : \mathfrak{X}^r \times M^2 \times \mathbf{R} \to M^2$ defined by $(x, p, t) \to \varphi_X(p, t)$ is of class C^r.

Preliminary definitions **(2.1)**. — Let U and S be C^∞ arcs transversal to $X \in \mathfrak{X}^r$; i.e., $U = u(I)$, $S = s(I)$, where u, s are C^∞ imbeddings of $I = [-1, 1]$ into M^2 such that $u'(x)$ and $X(u(x))$, (resp. $s'(x)$ and $X(s(x))$) are linearly independent. Assume that $u(0) = p$, $s(0) = q$ and $\varphi_X(p, \tau) = q$. Let (x_1, x_2) be a system of coordinates around q; assume that $x_1(q) = x_2(q) = 0$, $\dfrac{\partial}{\partial x_1} = X$, $x_2 \circ s = \text{Id}$, and $x_1 \circ s \equiv 0$. By continuity, $x_1(\varphi_X(u, t))$ is defined in a neighborhood of $(X, p, \tau) \in \mathfrak{X}^r \times U \times \mathbf{R}$; also, $x_1(\varphi_X(p, \tau)) = 0$ and:

$$\frac{\partial}{\partial t} x_1(\varphi_X(p, \tau)) = 1.$$

By the Implicit Function Theorem, there is a unique function $T : B_0 \times U_0 \to \mathbf{R}$ such that $T(X, p) = \tau$ and $x_1(\varphi_Y(u, t)) = 0$ for $(Y, u) \in B_0 \times U_0$ only if $t = T(Y, u)$. Define

$\pi : B_0 \times U_0 \to S$ by $\pi(Y, u) = \varphi_Y(u, T(U, u))$; thus, π as well as $\pi_Y = \pi(Y, \ ) : U_0 \to S$ are of class C^r. If γ is a periodic trajectory of period τ of $X \in \mathfrak{X}^r$, $p = q \in \gamma$, and $U = S$, $\pi_X : U_0 \to U$ is called the *Poincaré transformation* associated to U_0, U, p, γ. γ is called *generic* if $|\pi'_X(0)| \neq 1$; if $\pi'_X(0) = 1$ and $\pi_X^{(2)}(0) \neq 0$, or if $\pi'_X(0) = -1$ and $(\pi_X^2)^{(3)}(0) \neq 0$, γ is called *quasi-generic*. The derivatives of π_X are computed in u-coordinates of U. It is easy to verify that the above definitions do not depend either on u or $p \in \gamma$. Also, γ is two sided (i.e. has a trivial normal bundle) if and only if $\pi'_X(0) > 0$.

Proposition **(2.2)**. — *Denote by* Q_2 *the set of vector fields* $X \in \mathfrak{X}^r$, $r \geq 3$ *such that:*

1) X *has one quasi-generic periodic trajectory as unique non-generic periodic trajectory.*
2) X *has only generic singular points and does not have saddle connections.*
3) *The* α *and* ω*-limit sets of any trajectory of* X *are singular points or periodic trajectories.*

Then, Q_2 *is an immersed Banach submanifold of class* C^{r-1} *and codimension one of* $\mathfrak{X}^r$; *furthermore, every* $X \in Q_2$ *has a neighborhood* B_1 *in* Q_2 *so that every* $Y \in B_1$ *is topologically equivalent to* X.

For the sake of reference, the concepts of generic singular points and saddle connection involved in the statement of (2.2), are reviewed in (3.1) and (3.4). The proof of (2.2) depends on several lemmas.

Lemma **(2.3)**. — *Let* γ *be a quasi-generic periodic trajectory of* X. *Then* γ *has a fundamental system of closed neighborhoods* $\{N_\Theta\}$, *where* Θ *is a small real number. If* γ *is one-sided* (resp. *two-sided*) ∂N_Θ *is a closed curve* (resp. *the union of two closed curves*) *transversal to* X.

Proof. — If γ is two sided, it has a tubular neighborhood diffeomorphic to a plane annulus N. Therefore X may be assumed to be a (plane) vector field on N. The conditions $\pi'_X(0) = 1$, $\pi_X^{(2)}(0) \neq 0$ imply that γ is orbitally semi-stable, i.e. γ is the α-limit set of the trajectories on one of its sides and the ω-limit set of trajectories on the other side. By properly rotating X in N by a angle Θ, two periodic trajectories of the rotated vector field are obtained. These trajectories are obviously transversal to X and bound a neighborhood N_Θ of γ. This follows from [1, p. 18].

If γ is one sided, it has a tubular neighborhood diffeomorphic to a Moebius band N, with orientable double covering $P : \widetilde{N} \to N$, where N is a plane ring. Call $\widetilde{\gamma}$ and $\widetilde{X}$ the liftings of γ and X; γ as well as $\widetilde{\gamma}$ are orbitally stable or unstable depending on $(\pi_X^2)^{(3)}(0)$ being negative or positive. In either case, by rotating $\widetilde{X}$ of an angle Θ, a periodic trajectory of the rotated vector field is obtained [1]. This trajectory and $\widetilde{\gamma}$ bound an open set $\widetilde{N}_\Theta$. The $N_\Theta = \mathrm{Int}(\overline{P(\widetilde{N}_\Theta)})$ give the desired system of neighborhoods. ∂N_Θ is transversal to X by construction.

Lemma **(2.4)**. — *Let* $X \in \mathfrak{X}^r$, $r \geq 2$, *have a quasi-generic periodic trajectory* γ_X *of period* $\tau(X)$ *such that* $\pi'_X(0) = 1$ *and* $\pi_X^{(2)}(0) \neq 0$.

Let ε and T_0 be given positive numbers. Then there are neighborhoods B *of* X *and* N *of* γ_X, *and a* C^{r-1} *function* $f: B \to \mathbf{R}$ *such that*

1) ∂N *is union of two closed curves* C_1 *and* C_2, *transversal to every* $Y \in B$.
2) $Y \in B$ *has one periodic trajectory which is quasi-generic, contained in* N *if and only if* $f(Y)=0$; *if* $f(Y)<0$, Y *has two periodic trajectories, both generic, contained in* N; *if* $f(Y)>0$, Y *has no periodic trajectory in* N. *Furthermore,* $f(X)=0$ *and* $df_X \neq 0$.
3) *The period of any periodic trajectory of* $Y \in B$ *contained in* N *is within* ε *of* $\tau(X)$. *Also, every trajectory of* $Y \in B$ *meeting* N *spends there a time greater than* T_0.

Proof. — Define $G_1 : B_0 \times U_0 \to \mathbf{R}$ by $G_1(Y, u)=\pi(Y, u)-u$, where π, B_0 and U_0 are defined in (2.1). $\dfrac{\partial G_1}{\partial u}(X, 0)=\pi'_X(0)-1=0$ and $\dfrac{\partial^2 G_1}{\partial u^2}(X, 0)=\pi_X^{(2)}(0)\neq 0$; therefore, by the Implicit Function Theorem, there is a neighborhood B of X, $B\subset B_0$, and a unique C^{r-1} function $G_2 : B \to U \subset U_0$ such that $G_2(X)=0$ and $\dfrac{\partial G_1}{\partial u}(Y, u)=\pi'_Y(u)-1=0$ for $Y \in B$, only if $u=G_2(Y)$.

For definiteness assume $\pi_X^{(2)}(0)>0$; the case $\pi_X^{(2)}(0)<0$ is similar. By continuity, it is possible to assume that B and U_1 satisfy $\dfrac{\partial^2 G_1}{\partial u^2}(Y, u)>0$, for $(Y, u) \in B \times U_1$, and $G_1(Y, x)>0$, for $x \in \partial U_1$.

Furthermore, U_1 may be taken so that $U_1=U_0 \cap N$ where $N=N_\Theta$ (see (2.3)) for some small Θ; B may be taken so that every $Y \in B$ is transversal to ∂N.

Define $f(Y)=G_1(Y, G_2(Y))$; from the construction above, it follows that $f(Y)$ is the minimum of $\pi_Y(u)-u$, $u \in U_1$; also, $\pi'_Y(x)<1$ for $x<G_2(Y)$ and $\pi'_Y(x)>1$ for $x>G_2(Y)$. Thus, π_Y has one fixed point, $G_2(Y)$, only if $f(Y)=0$; if $f(Y)>0$, it has no fixed point; if $f(Y)<0$, by the Intermediate Value Theorem, it has two fixed points, both generic, one on each side of $G_2(Y)$.

Obviously $f(X)=0$; we prove that $df_X \neq 0$.

$$df_X(V)=\frac{\partial \pi}{\partial V}(X, 0)+\frac{\partial \pi}{\partial u}(X, 0)\frac{\partial G_2}{\partial V}(X)-\frac{\partial G_2}{\partial V}(X)$$

$$=\frac{\partial \pi}{\partial V}(X, 0), \quad \text{since} \quad \frac{\partial \pi}{\partial u}(X, 0)=1.$$

For $V=g\dfrac{\partial}{\partial x_2}$, where (x_1, x_2) is the coordinate system in (2.1) and g is a bump function with support in $|x_i|<\delta$, $df_X(V)=\int_{-\delta}^{\delta} g(x_1, 0)\,dx_1 \neq 0$. In fact,

$$\frac{\partial \pi}{\partial V}(X, 0)=\frac{d}{d\lambda}\pi(X+\lambda V, 0)\,|_{\lambda=0}=\frac{d}{d\lambda}\pi_X(\beta(\lambda))\,|_{\lambda=0}=\beta'(0),$$

where $\beta(\lambda)$ is the solution of $\dfrac{dx_2}{dx_1}=\lambda g(x_1, x_2)$ passing through $x_1=-\delta$, $x_2=0$; the expression for $df_X(V)=\beta'(0)$ follows from a known formula for the derivative of solutions of differential equations depending on parameters [15, p. 94].

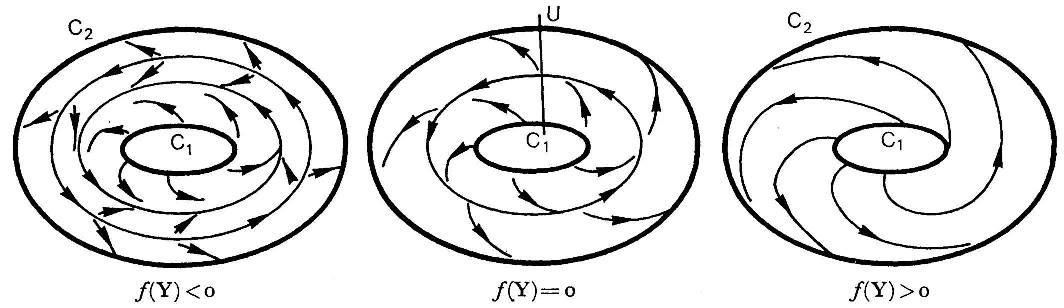


FIG. 2.1. — Quasi generic periodic orbit

This proves (1) and (2), since the fixed points of π_Y and the periodic trajectories of Y contained in N are in one-to-one correspondence; (3) is immediate by continuity, since it is satisfied for $Y=X$. See Fig. (2.1) for a graphical illustration.

Remarks **(2.4.1)**. — *a)* Assume $Y \in B$ points inward (resp. outward) N on C_1 (resp. C_2). Thus for $f(Y)>0$ when Y has no periodic trajectory, φ_Y defines a C^r-mapping $\delta_Y : C_1 \to C_2$; (3) implies that the arc of trajectory of Y joining m to $\delta_Y(m)$, $m \in C_1$, spends in N a time greater than T_0. See Fig. (2.1).

b) If $f(Y) \leq 0$, the ω (resp. α)-limit set of every trajectory of Y passing through C_1 (resp. C_2), is a periodic trajectory of Y contained in N. This is obvious by the Poincaré-Bendixon Theorem.

c) If M^2 is endowed with a Riemannian metric and $L_0>0$ is given, B of (2.4) may be taken so that the length of every arc of trajectory of Y, $f(Y)>0$ joining m to $\delta_Y(m)$, $m \in C_1$, is greater than L_0. This is obvious since the length of $Y(p)$ is bounded away from zero in N, say greater than $K>0$, and the length of the trajectory is greater than T_0K.

Lemma **(2.5)**. — *Call* $Q_2(n)$ *the set of* $X \in Q_2$ *(notation of* (2.2), (2.3)*) such that its quasi-generic periodic trajectory,* γ_X, *is two-sided and has period* $\tau(X)<n$.

$Q_2(n)$ *is an imbedded Banach submanifold of class* C^{r-1} *and codimension one of* $\mathfrak{X}^r$. *Also, every* $X \in Q_2(n)$ *has a neighborhood* B_1 *in* $Q_2(n)$ *such that every* $Y \in B_1$ *is topologically equivalent to* X.

Proof. — Assume the notation in (2.4). Take $\varepsilon<n-\tau(X)$ and $T_0>n$. Call M_1^2 the manifold with boundary $M^2-\text{Int } N$. For $Y \in \mathfrak{X}^r$, call $Y_1=Y|M_1^2$. X_1 is transversal to $\partial M_1^2=\partial N$, has only generic periodic trajectories, and satisfies conditions (2) and (3) of (2.2). Since these conditions are open and characterize Σ^r in M_1^2, B may be taken so that Y_1, $Y \in B$, is topologically equivalent to X_1; denote by $h_1(Y)$ the homomorphism of M_1^2 onto itself mapping trajectories of X_1 onto those of Y_1; $h_1(Y)$ can be arbitrarily close to the identity of M_1^2 by properly reducing B. The above assertions follow from [5, 8].

Thus if $f(Y)<0$, from (2.4) and the characterization of Σ^r it follows that $Y \in \Sigma^r$.

If $f(Y)>0$ every periodic trajectory of Y (if any) which meets N has, by (2.4), period greater than $T_0>n$. Therefore, $f^{-1}(0)=B\cap Q_2(n)$. If $Y\in B\cap Q_2(n)=B_1$, $h_1(Y)$ can be extended to M^2, mapping trajectories of X onto trajectories of Y.

This is done below, following [5, p. 153]. Call $p_i\in C_i$, $i=1, 2$, the points of intersection of ∂N and U. Call $\tilde{p}_i=h_1(Y)(p_i)$; let $\tilde{U}$ be a C^1 arc close to U joining $\tilde{p}_1$ to $\tilde{p}_2$. $\tilde{U}$ is transversal to ∂N and Y, for Y close to X, since then $\tilde{p}_i$ is close to p_i.

To extend $h_1(Y)$ to $h(Y)$ defined in Int N, map the trajectory of X through $n_0\in C_1$ (resp. C_2) onto the trajectory of Y through $\tilde{n}_0=h_1(Y)(n_0)$ in the following way. $\varphi_X(n_0, t)$ and $\varphi_Y(\tilde{n}_0, t)$ meet, for $t>0$ (resp. $t<0$), Int U and Int $\tilde{U}$ respectively in monotonic sequences n_i and $\tilde{n}_i$, $i=1, 2, \ldots$, tending respectively to $p=\gamma_X\cap U$ and $\tilde{p}=\gamma_Y\cap\tilde{U}$. Map the arc $\overset{\frown}{n_i n_{i+1}}$ (resp. $\overset{\frown}{n_{i-1} n_i}$) onto $\overset{\frown}{\tilde{n}_i\tilde{n}_{i+1}}$ (resp. $\overset{\frown}{\tilde{n}_{i-1}\tilde{n}_i}$) by ratio of arc length, i.e., n is mapped to $\tilde{n}$ if $|\overset{\frown}{n_i n}|/|\overset{\frown}{n_i n_{i+1}}|=|\overset{\frown}{\tilde{n}_i\tilde{n}}|/|\overset{\frown}{\tilde{n}_i\tilde{n}_{i+1}}|$ where the bars indicate arc length of the corresponding arc, measured in the positive sense from the left extreme of the arc. Finally, map $\gamma_X=\overset{\frown}{pq}$ onto $\gamma_Y=\overset{\frown}{\tilde{p}\tilde{q}}$ by ratio of arc length. Since every point of N belongs to one trajectory, $h(Y)$ is a one-to-one mapping of N onto itself, sending trajectories of X onto those of Y. $h(Y)$ is a homeomorphism; it is continuous outside of γ_X by standard continuity of trajectories on initial data, it is continuous on γ_X as in [5, p. 153] by a lemma in [5, p. 153] (this lemma will be used several times in this work, for the sake of reference it is stated in (3.9.1 *b*)). This ends the proof of (2.5).

Lemma **(2.6)**. — *Let* $X\in\mathfrak{X}^r$, $r\geq 3$, *have a quasi-generic periodic trajectory* γ_X *of period* $\tau(X)$ *such that* $\pi_X^1(0)=-1$ *and* $(\pi_X^2)^{(3)}(0)\neq 0$. *Then, given* $\varepsilon>0$, *there are neighborhoods* B *of* X *and* N *of* γ_X *and a* C^{r-1} *function* $f: B\to\mathbf{R}$ *such that*:

1) ∂N *is a curve transversal to every* $Y\in B$.

2) $Y\in B$ *has one periodic trajectory, which is quasi-generic and one-sided, contained in* N *if and only if* $f(Y)=0$; *if* $f(Y)>0$, Y *has two periodic trajectories both generic, only one being one-sided, contained in* N; *if* $f(Y)<0$, $Y\in B$ *has one one-sided periodic trajectory, which is generic, contained in* N. *Furthermore*, $f(X)=0$ *and* $df_X\neq 0$.

3) *A periodic trajectory of* $Y\in B$ *contained in* N *has period within* ε *of* $\tau(X)$ *if it is one-sided, and within* ε *of* $2\tau(X)$ *if it is two-sided.*

Proof. — Assume that $(\pi_X^2)^{(3)}(0)<0$; the case $(\pi_X^2)^{(3)}(0)>0$ is similar. Let $G_1: B_0\times U_0\to\mathbf{R}$ be defined, as in (2.4), by $G_1(Y, u)=\pi(Y, u)-u$. $G_1(X, 0)=0$ and $\frac{\partial G_1}{\partial u}(X, 0)=\pi_X'(0)-1=-2$. Therefore, by the Implicit Function Theorem, there is a neighborhood B of X, $B\subset B_0$, and a C^r function $k: B\to U_1\subset U_0$ such that $k(X)=0$ and $G_1(Y, k(Y))=\pi_Y(k(Y))-k(Y)=0$, for $Y\in B$. Thus $k(Y)$ is the unique fixed point of π_Y contained in U_1.

By continuity, B and U_1 can be taken so that $\pi'(u)<0$ and $(\pi_Y^2)^{(3)}(Y)<0$ for $Y\in B$ and $u\in U_1$, and $\pi_Y^2(U_1)\subset U_1$. The last choice of U_1 is possible since $\pi_X'=-1$

implies $(\pi_X^2)^{(2)}(0)=0$, and π_X^2 and π_X are (topologically) contractions since $(\pi_X^2)^{(3)}(0)<0$. U_1 can be taken so that $U_1=N\cap U_0$, where $N=N_\Theta$ for some small Θ (see (2.3)).

Define $f(Y)=\pi'_Y(k(Y))+1$. If $f(Y)\geq 0$, π_Y and π_Y^2 have $k(Y)$ as unique fixed point; $k(Y)$ is a generic fixed point of π_Y only if $f(X)>0$. If $f(Y)<0$, π_Y^2 has three fixed points: $k(Y)$, $\ell_1(X)$ and $\ell_2(Y)=\pi_Y(\ell_1(Y))$, all generic. The negation of any of these assertions is not compatible with $(\pi_Y^2)^{(3)}<0$.

For $V\in\mathfrak{X}^r$, $df_X(V)=\dfrac{\partial^2\pi}{\partial u\,\partial V}(X,0)+\dfrac{\partial^2\pi}{\partial u^2}(V).dk_X(V)$. Let $V=g(x_1,x_2)x_2\dfrac{\partial}{\partial x_2}$ where (x_1,x_2) is the coordinate system of (2.1) and g is a bump function with support $|x_i|\leq\delta$. A straightforward computation similar to that in (2.4) shows that $dk_X(V)=0$ and:

$$\frac{\partial^2\pi}{\partial u\,\partial V}(X,0)=-\int_{-\delta}^{\delta}g(x_1,0)\,dx_1\neq 0.$$

Thus $df_X(V)\neq 0$.

The last assertion of (2.6) is immediate, by continuity of T defined in (2.1).

Lemma **(2.7)**. — *Call* $Q'_2(n)$ *the set of* $X\in Q_2$ (Prop. (2.2)) *such that its quasi-generic periodic trajectory,* γ_X, *is one-sided, with period* $\tau(X)<n$.

$Q'_2(n)$ *is an imbedded Banach submanifold of class* C^{r-1} *and codimension one of* $\mathfrak{X}^r$.

Furthermore, $Q'_2(n)$ *is open in* $\mathfrak{X}_1^r$ *and every* $X\in Q'_2(n)$ *has a neighborhood* B_1 *in* $Q'_2(n)$ *such that every* $Y\in B_1$ *is topologically equivalent to* X.

Proof. — Similar to the proof of (2.5), using (2.6) in this case. The construction of the topological equivalence is formally that of (2.5), but in the present case $\partial N=C$ and the trajectory through $n_0\in C$ meets Int U in a sequence $\{n_i\}$ such that $\{n_{2i}\}$ is decreasing and $\{n_{2i+1}\}$ is increasing, both converging monotonically to $p=\gamma_X\cap U$. The same holds for Y, $f(Y)=0$, and its corresponding sequence $\{\tilde{n}_i\}$ in $\tilde{U}$; the map of $\overset{\frown}{n_{2i}n_{2i+1}}$ onto $\overset{\frown}{\tilde{n}_{2i}\tilde{n}_{2i+1}}$ and γ_X onto γ_Y by ratio of arc length produces the desired topological equivalence in N. The openness of $Q_2(n)$ follows from the fact that every Y close to X, $f(Y)\neq 0$, is in Σ^r, since Y is so in M_1^2 and N, N being an attractive region (sink).

Proof of Proposition **(2.2)**. — Take $S_i=Q'_2(i)\cup Q_2(i)$ for $i=1,2,\ldots$; by (2.4) Remark *a)*, (2.5), and (2.7), S_i is an imbedded submanifold of class C^{r-1} and codimension one of $\mathfrak{X}^r$. Since $Q_2=\bigcup_{i=1}^{\infty}S_i$, (2.2) follows (see (1.2)).

Remarks **(2.8)**. — *a)* Since each $Q'_2(i)$ is open in $\mathfrak{X}_1^r$, $Q_2(0)=\bigcup_i Q'_2(i)$ is an imbedded submanifold of class C^{r-1} and codimension one of $\mathfrak{X}^r$, open in $\mathfrak{X}$.

b) Call $\tilde{Q}_2$ the subset of Q_2, of fields X which satisfy the additional following axiom:

4) The quasi-generic periodic trajectory of X is not both α and ω-limit set of either saddle separatrices or of any trajectory different from itself.

Obviously,

$$Q_2(0) \subset \widetilde{Q}_2.$$

Proposition (2.2) *holds for* $\widetilde{Q}_2$, *changing immersed by imbedded. Furthermore,* $\widetilde{Q}_2$ *is open in* $\mathfrak{X}_1^r$.

This follows from the openness of each $\widetilde{Q}_2(n) = \widetilde{Q}_2 \cap Q_2(n)$, and the openness of $Q_2(0)$. In fact, if $X \in \widetilde{Q}_2(n)$ and γ_X is, say, the ω-limit set of saddle separatrices, which *a fortiori* meet C_1, then all the trajectories through C_2 have the same ω-limit set, a generic singular point or periodic trajectory L_X contained in a critical region N^1, with ∂N^1 transversal to X (see [5], or (1.4) for the definition of critical region). We can assume in this case that C_2 is part of ∂N^1. Therefore, when $f(Y)>0$, $\delta_Y : C_1 \to C_2$ is defined and L_Y, the generic singular point or periodic trajectory of Y in N^1, is the ω-limit set of all trajectories through $N \cup N^1 = N^2$, which works as a critical region for L_Y. Thus, since Y is in Σ^r, in $M^2 - \text{Int } N^2$ (X is so), it is in Σ^r in M^2; the decomposition of M^2 in critical and canonical regions of Y is the same as that for Y in $M^2 - \text{Int } N^2$ plus the critical region N^2.

When $f(Y)<0$, $Y \in \Sigma^r$, also when $X \in Q_2(n)$. This follows from a similar analysis using $N^2 = N$ and taking into account (2.4) and Remark *b)* in (2.4.1). This shows that $B \cap \mathfrak{X}_1^r = B \cap Q_2(n) = f^{-1}(0)$; hence the assertion above is proved.

c) If γ_X is both the α and ω-limit of saddle separatrices it can be shown that there is Y, $f(Y)>0$, arbitrarily close to X, which has saddle connections meeting N which, by Remark *c)* after (2.4) have length arbitrarily large.

d) If there is a trajectory η of X which has γ_X as α and ω-limit set, either all trajectories of X have this property and $M^2 = \mathbf{T}^2$ or K^2, or X has saddle separatrices which have γ_X as α and ω-limit set. This is shown by looking at the canonical region R of X which contains η; R is either a cylinder with boundary $C_1 \cup C_2$ where the flow is parallel, or is a region bounded by arcs of C_1 and C_2 and saddle separatrices meeting C_1 and C_2.

In the first case, it can be shown that there is Y, $f(Y)>0$, arbitrarily close to X, which has non-generic periodic trajectories meeting N. When $M^2 = \mathbf{T}^2$, Y can be found with irrational rotation number, thus exhibiting recurrent orbits dense in $\mathbf{T}^2$. This is shown by considering the rotation number ρ_Y of Y relative to C_2, which is defined when $f(Y)>0$, and showing that $\rho_Y \to \infty$ when $Y \to X$, thus passing through irrational values and also through rational values for Y at the boundary of Σ^r, and the assertion follows for $M^2 = \mathbf{T}^2$. For $M^2 = K^2$, the assertion, left as an open question in [14], has a more delicate proof communicated to us by I. Kupka (unpublished work).

e) We summarize *d)*. $\widetilde{Q}_2^1 = Q_2 - \widetilde{Q}_2$ is open in Q_2 and its intrinsic topology is finer (has more open sets) than its ambient topology.

The fact that for $X \in \widetilde{Q}_2^1$ and $\varepsilon>0$ small $f^{-1}((-\varepsilon, 0)) \subset \Sigma^r$, while $f^{-1}((0, \varepsilon))$ is not completely contained in Σ^r, can be expressed by asserting that $\Sigma^r \cup \widetilde{Q}_2^1$ is a submanifold of $\mathfrak{X}^r$ with boundary $\widetilde{Q}_2^1$.

3. Singular Points.

Preliminary Definitions (**3.1**). — $[V, X]$ stands for the Lie bracket of V and X. Let $p \in M^2$ be a singular point of $X \in \mathfrak{X}^r$, $r \geq 1$. For any $V \in \mathfrak{X}^r$, $[V, X](p)$ depends only on $V(p)$, as follows from a straightforward computation taking into account that $X(p) = 0$. Thus, it is possible to define an endomorphism DX_p of the tangent space T_p of M^2 at p; $DX_p(v) = [V, X](p)$, where $V(p) = v$. The determinant and trace of DX_p will be denoted respectively by $\Delta(X, p)$ and $\sigma(X, p)$.

A singular point p of X is called *simple* if DX_p is an isomorphism, i.e. if $\Delta(X, p) \neq 0$. It is called *generic* if DX_p has eigenvalues with nonvanishing real parts. If the eigenvalues are real and have opposite sign, p is called a *saddle*; if they have equal sign, p is called a *node*. If the eigenvalues of DX_p are complex conjugate, p is called a *focus*.

Assume $r \geq 2$. Call λ_1 and λ_2 the eigenvalues of DX_p. Let $\lambda_1 = 0$ and $\lambda_2 \neq 0$. Denote by T_1 and T_2 the eigenspaces of DX_p, associated respectively to λ_1 and λ_2. Call $\pi_1 : T_p \to T_1$ the projection of T_p onto T_1 parallel to T_2. For $v \in T_1$, $v \neq 0$, define $\Delta_1(X, p, v)$ by $\pi_1[V, [V, X]](p) = \Delta_1(X, p, v)v$, where $V \in \mathfrak{X}^r$ is an extension of v.

$\Delta_1(X, p, v)$ does not depend on V, as it is easy to show. Also,

$$\Delta_1(X, p, kv) = k\Delta_1(X, p, v),$$

for any $k \neq 0$. If $\Delta_1(X, p, v) \neq 0$ for some (and for all) $v \neq 0$, p is called a *saddle-node* of X.

Assume the notation above. Denote by u the covector on T_p such that $\pi_1 = vu$; denote by X^i, v^i and u_i, respectively, the components of X, v and u, with respect to a system of coordinates, (x_1, x_2), around p. Then:

$$\Delta_1(X, p, v) = u[V, [V, X]](p) = \sum_{i,j,k} \frac{\partial^2 X^i}{\partial x_j \partial x_k}(p) v^j v^k u_i.$$

In particular, Δ_1 does not depend on V. This follows from a straightforward computation.

Lemma (**3.2**). — *Let p be a saddle-node of* $X \in \mathfrak{X}^r$, $r \geq 2$. *Then there is a neighborhood* B *of* X, *a neighborhood* N *of p, and a* C^{r-1} *function* $f : B \to \mathbf{R}$ *such that*:

1) $Y \in B$ *has a saddle-node as unique singular point in* N *if and only if* $f(Y) = 0$; *if* $f(Y) < 0$, Y *has two singular points, both generic, one saddle and one node, in* N; *if* $f(Y) > 0$, Y *has no singular point in* N. *See* Fig. (3.1).

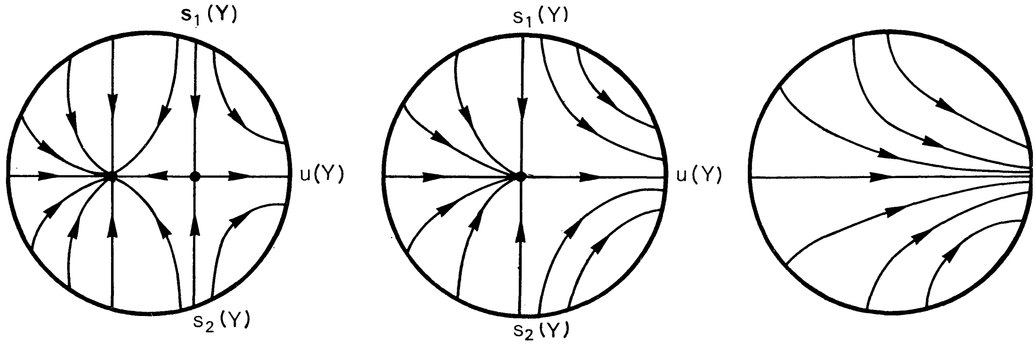
2) $f(X) = 0$ *and* $df_X \neq 0$.

Proof. — Let (x_1, x_2) be a system of coordinates around p; assume that:

$$x_1(p) = x_2(p) = 0$$

and $\dfrac{\partial}{\partial x_i}(p) \in T_i$ (notation of (3.1)). In these coordinates the components of X, X^1 and X^2, satisfy $\dfrac{\partial X^1}{\partial x_1}(0, 0) = \dfrac{\partial X^1}{\partial x_2}(0, 0) = 0$, $\dfrac{\partial X^2}{\partial x_2}(0, 0) = \sigma(X, p)$, and:

$$\frac{\partial^2 X^1}{\partial x_1^2}(0, 0) = \Delta_1\left(X, p, \frac{\partial}{\partial x_1}(p)\right).$$



$f(Y)<0$ $\qquad$ $f(Y)=0$ $\qquad$ $f(Y)>0$

FIG. 3.1. — Saddle-node

In other terms:

(3.2.1)
$$X^1(x_1, x_2)=\Delta_1 x_1^2+bx_1x_2+cx_2^2+M^1(x_1, x_2)$$
$$X^2(x_1, x_2)=\sigma x_2+\alpha x_1^2+\beta x_1x_2+\gamma x_2^2+M^2(x_1, x_2),$$
where:
$$M^i(x_1, x_2)=o(x_1^2+x_2^2).$$

Assume for definiteness that $\sigma(X, p)<0$ and $\Delta_1\left(X, p, \frac{\partial}{\partial x_1}(p)\right)>0$. Let N_0 and B_0 be neighborhoods of p and X such that for $Y=\sum_i Y^i\frac{\partial}{\partial x_i}\in B$; the following relations are verified in N_0.

a) $\dfrac{\partial Y^2}{\partial x_2}<0$;

b) $\Delta_1(Y, v_Y)=\sum_{i,j,k}\dfrac{\partial^2 Y^i}{\partial x_j\,\partial x_k}\,v_Y^j\,v_Y^k\,u_i^Y>0$; here, $v_Y^1=1$, $v_Y^2=-\left(\dfrac{\partial Y^2}{\partial x_2}\right)^{-1}\dfrac{\partial Y^2}{\partial x_1}$,

$$u_1^Y=\left(1+\left(\frac{\partial Y^2}{\partial x_2}\right)^{-2}\frac{\partial Y^2}{\partial x_1}\,\frac{\partial Y^1}{\partial x_2}\right)^{-1}>0,\quad\text{and}\quad u_2^Y=-\frac{\partial Y^1}{\partial x_2}\left(\frac{\partial Y^2}{\partial x_2}\right)^{-1}\left(1+\left(\frac{\partial Y^2}{\partial x_2}\right)^{-2}\left(\frac{\partial Y^1}{\partial x_2}\right)^2\right)^{-1};$$

finally,

c) $\sigma(Y)=\sum_i\dfrac{\partial Y^i}{\partial x_i}<0.$

The existence of the neighborhoods N_0 and B_0 for which the above relations are satisfied follows from continuity, since they are satisfied for X at p.

Take $v_Y=\sum_i v_Y^i\frac{\partial}{\partial x_i}$, $w_Y=\sum_i w_Y^i\frac{\partial}{\partial x_i}$, and $u^Y=\sum_i u_i^Y\,dx_i$. Here $w_Y^1=\left(\dfrac{\partial Y^2}{\partial x_2}\right)^{-1}\dfrac{\partial Y^1}{\partial x_2}$, $w_Y^2=1$. If $q\in N_0$ is a singular point of Y and $\Delta(Y, q)=0$, then $v_Y(q)$ is an eigenvector associated to the zero eigenvalue of DY_q; w_Y is an eigenvector associated to $\sigma(Y, q)\neq 0$; also, $u^Y(q)$ is the covector in (3.1) $(u^Y(v_Y)=1,\ u^Y(w_Y)=0)$. These assertions follow from a straightforward computation. Thus, by *b)* and *c)*, any non-generic singular point $q \in N_0$ of $Y \in B_0$ is such that $\sigma(Y, q) < 0$ and $\Delta_1(Y, q, v_Y) > 0$, i.e. q is a saddle-node of Y.

Define $F : B_0 \times N_0 \to \mathbf{R}$ by $F(Y; x_1, x_2) = Y^2(x_1, x_2)$. F is of class C^r since it is an evaluation map [16]; also, $F(X; 0, 0) = 0$ and $\dfrac{\partial F}{\partial x_2}(X; 0, 0) = \dfrac{\partial Y^2}{\partial x_2}(0, 0) = \sigma(X, p) < 0$. By the Implicit Function Theorem, there are neighborhoods $B_1 \times I_1$ of $(X, 0)$ and I_2 of 0 and a unique C^r function $F_1 : B_1 \times I_1 \to I_2$, such that:

$$F_1(X, 0) = 0 \quad \text{and} \quad F(Y; x_1, x_2) = Y^2(x_1, x_2) = 0$$

for $(Y, x_1) \in B_1$ and $x_2 \in I_2$ only if $x_2 = F_1(Y, x_1)$. Define:

$$F_2 : B_1 \times I_1 \to \mathbf{R} \quad \text{by} \quad F_2(Y, x_1) = Y^1(x_1, F_1(x_1, Y)).$$

A straight forward computation shows that:

d) $\dfrac{\partial F_2}{\partial x_1} = \left(\dfrac{\partial Y^2}{\partial x_2}\right)^{-1} \Delta(Y)$;

e) $\dfrac{\partial^2 F_2}{\partial x_1^2} = (u_1^Y)^{-1} \Delta_1(Y, v_Y) > 0$.

Since $\dfrac{\partial F_2}{\partial x_1}$ is of class C^{r-1}, $\dfrac{\partial F_2}{\partial x_1}(X, 0) = 0$, and $\dfrac{\partial^2 F_2}{\partial x_1^2}(X, 0) \neq 0$, there is a neighborhood B of X, $B \subset B_0$, and a unique C^{r-1} function $F_3 : B \to I_1$ such that $F(X) = 0$ and $\dfrac{\partial F_2}{\partial x_1}(Y, x_1) = 0$ for $Y \in B$, $x_1 \in I$ only if $x_1 = F_3(Y)$. This follows from the Implicit Function Theorem.

Define $f : B \to \mathbf{R}$ by $f(Y) = F_2(Y, F_3(Y)) = Y^1(F_3(Y), F_1(Y, F_3(Y)))$. From the definition of F_i, $i = 1, 2, 3$, $Y \in B$ has a singular point $(x_1, x_2) \in N = I_1 \times I_2$ if and only if $x_2 = F_1(Y, x_1)$ and $F_2 = (Y, x_1) = 0$. Since $\Delta_1 > 0$ and $\sigma < 0$, *d)* and *c)* imply that $f(Y)$ is the minimum of $F_2(Y, x_1)$, $x_1 \in I_1$. Thus, if $f(Y) > 0$, Y has no singular point in N; if $f(Y) = 0$, Y has a saddle-node as unique singular point in N. If $f(Y) < 0$, the Intermediate Value Theorem implies that $F_2(Y, x_2)$ has two zeros $r(Y)$ and $q(Y)$, $r(Y) < F_2(Y) < q(Y)$; by *d)*, the first corresponds to a node $\Delta(Y) > 0$ $\left(\dfrac{\partial F_2}{\partial x_1} < 0\right)$, and the second corresponds to a saddle, $\Delta(Y) < 0$ $\left(\dfrac{\partial F_2}{\partial x_1} > 0\right)$. This holds because $\Delta(Y)(x_1)$ is decreasing since $\dfrac{\partial F_2}{\partial x_1}$ is increasing, by *c)*, and $\dfrac{\partial Y^2}{\partial x_2} < 0$, by *a)*. This proves 1). A straightforward computation shows that $df_X(Z) = Z^2(0, 0)$, for $Z = \sum_i Z^i \dfrac{\partial}{\partial x_i}$, and 2) follows.

Lemma **(3.3)**. — *Let p be a saddle-node of $X \in \mathfrak{X}^r$, $r \geq 2$. Assume that $\sigma(X, p) < 0$ (the case $\sigma(X, p) > 0$ is similar). The neighborhoods* N *and* B *of* (3.2) *can be chosen so that for* $Y \in B$ *with* $f(Y) \leq 0$ *the following assertions hold.*

1) *There is a unique point* $u(\mathrm{Y})\in\partial\mathrm{N}$ *such that* $\varphi_{\mathrm{Y}}(u(\mathrm{Y}),t)\in\mathrm{N}$ *for* $t<0$; *the set* $s(\mathrm{Y})$ *of points* $q\in\partial\mathrm{N}$ *such that* $\varphi_{\mathrm{Y}}(q,t)\in\mathrm{N}$ *for* $t>0$ *is an arc whose extremes we call* $s_1(\mathrm{Y})$, $s_2(\mathrm{Y})$.

2) $\partial\mathrm{N}$ *is a differentiable curve, transversal to every* $\mathrm{Y}\in\mathrm{B}$ *at points of neighborhoods* U *of* $u(\mathrm{X})$ *and* S *of* $s(\mathrm{X})$.

3) $u(\mathrm{Y})$, $s_1(\mathrm{Y})$ *and* $s_2(\mathrm{Y})$ *depend continuously on* Y.

Proof. — From (3.2.1), the coordinate expression for X in (3.2), and [17, p. 319], it follows that X has one separatrix, γ, whose α-limit set is p, and is tangent to T_1 at p; also X has two separatrices δ_1, δ_2 whose ω-limit set is p and are tangent to T_2 at p. See Fig. (3.1). Take $\mathrm{N}_r=\{(x_1,x_2);\, x_1^2+x_2^2\leq r\}$; $\partial\mathrm{N}_r$ is given by $x_1=r\cos\theta$, $x_2=r\sin\theta$, $\theta\in[-\pi,\pi]$. Since T_1, T_2 are transversal to $\partial\mathrm{N}$, so are the separatrices, provided r is small; γ meets $\partial\mathrm{N}$ at a point we call $u(\mathrm{X})$; δ_1, δ_2 meet $\partial\mathrm{N}$ at points we call $s_1(\mathrm{X})$, $s_2(\mathrm{X})$. The existence and continuity of u follows from the continuity on Y of neighboring trajectories, as for the case of saddle points [5, p. 147]; the continuity of s_1 follows [16, p. 137], where the trajectory tangent to the eigenspace of smallest (negative) eigenvalue is given by an integral equation which depends continuously on the field.

On $\partial\mathrm{N}_r$:

$$\frac{1}{2}\frac{\partial(x_1^2+x_2^2)}{\partial\mathrm{X}}=r^3\left(\Delta_1\cos^3\theta+\frac{\sigma\sin^2\theta}{r}+(b+\alpha)\sin\theta\cos^2\theta+(c+\beta)\cos\theta\sin^2\theta\right.$$
$$\left.+\gamma\sin^3\theta+\frac{\mathrm{M}^1\cos\theta+\mathrm{M}^2\sin\theta}{r^2}\right).$$

Since for $\theta=\pi$ the expression in brackets is equal to $-\Delta_1+\dfrac{\mathrm{M}^1}{r^2}$, there are ν and ρ so that if $r\leq\rho$ and $|\theta-\pi|<\nu$, it is less than $-\dfrac{\Delta_1}{r^2}$. For $\pi-\nu>|\theta|>\pi/4$, the expression in brackets is negative since $\sigma<0$ and $\dfrac{\sin^2\theta}{r}$ is unbounded for these values of θ, while all the other terms are bounded. Thus, for r small, X is transversal to $\partial\mathrm{N}$ and points inward N on $|\theta|\geq\pi/4$. The arc joining $s_1(\mathrm{X})$ to $s_2(\mathrm{X})$, contained in $|\theta|>\pi/4$ is defined to be $s(\mathrm{X})$. This shows the existence of $s(\mathrm{X})$; the existence of U, S, $s(\mathrm{Y})$ follows by continuity.

Remark. — If p is a saddle-node of X with $\sigma(\mathrm{X},p)<0$, the stable manifold of p is a two-dimensional manifold with boundary tangent to T_2 at p. The unstable manifold is one-dimensional with boundary p, tangent to T_2 at p. If $\sigma(\mathrm{X},p)>0$, the remark holds with the obvious change of stable for unstable.

Definition **(3.4).** — A *saddle connection* is a trajectory whose α and ω-limit sets are saddle or saddle-note singular points and is not interior to the two-dimensional invariant manifold of the saddle-node.

In terms of transversality, a saddle connection is a trajectory along which the invariant manifolds of saddle and saddle-node singular points fail to meet transversally.

Now we state one of the main results of this section.

Proposition **(3.5)**. — *Denote by* Q_1^1 *the collection of* $X \in \mathfrak{X}^r$, $r \geq 2$, *such that*:

1) X *has a saddle-node as unique non-generic singular point.*

2) X *has only generic periodic trajectories.*

3) *The* α *and* ω*-limit sets of any trajectory of* X *are singular points or periodic trajectories.*

4) X *has no saddle connections.*

Then:

a) Q_1^1 *is open in* $\mathfrak{X}_1^r$.

b) *It is an imbedded Banach submanifold of class* C^{r-1} *and codimension one of* $\mathfrak{X}^r$; *and*

c) *Every* $X \in Q_1^1$ *has a neighborhood* B_1 *in* Q_1^1 *such that every* $Y \in B_1$ *is topologically equivalent to* X.

The proof of (3.5) depends on some lemmas.

Lemma **(3.6)**. — *Assume the hypothesis and notation in* (3.2), (3.3). *Let* $U_1 \subset U$ *be a neighborhood of* $u(X)$. *Then,* S *and* B *can be chosen so that for* $Y \in B$ *with* $f(Y) > 0$, φ_Y *defines a* C^r *mapping* $h_Y : S \to U_1$; $h_Y(q)$ *is the point where* $\varphi_Y(t, q)$, $t > 0$, *meets* U_1 *for the first time. Moreover, if* S_1 *is a closed arc contained in* $\operatorname{Int} s(X)$, *given* $\varepsilon > 0$, B *can be chosen so that* $|h'_Y(q)| < \varepsilon$ *for* $q \in S_1$ *and* $Y \in B$.

Proof. — Let S be an arc so that $s(X) \subset \operatorname{Int} S$ and the trajectories of X through $S - s(X)$ meet U_1. This choice of S is possible by a continuity property at $s_i(X)$ on hyperbolic sectors [19, p. 167]. The first assertion of (3.6) follows from continuity on Y of the trajectories passing through the extremes of S; for Y with $f(Y) > 0$, no trajectory through S remains in N, since this would imply the existence of a singular point in N, by the Poincaré-Bendixon theorem. Thus for $q \in S$ there is a $t_q(Y)$ such that $\varphi_Y(q, t) \in N$ for $0 < t < t_q(Y)$ and $\varphi_Y(q, t_q(Y)) \in U_1$. $h_Y(q)$ is defined to be $\varphi_Y(q, t_q(Y))$; by (2.1) h_Y is of class C^r.

A known formula in differential equations [17, p. 204] implies that:

$$h'_Y(q) = L_q(Y) \exp\left(\int_0^{t_q(Y)} \sigma(Y, \varphi_Y(q, t))\, dt\right).$$

$L_q(Y)$ only depends on the angles between Y and ∂N at q and $h_Y(q)$:

$$|\sin \alpha_1| / |\sin \alpha_2| = L_q(Y),$$

where $\alpha_1 = \text{angle}\,(Y(q), \partial N)$, $\alpha_2 = \text{angle}\,(Y(h_Y(q)), \partial N)$. Since $\sigma = \sigma(X, p) < 0$, we may assume that the integrand in the expression for h'_Y is less than $\sigma/2 < 0$ in N for every $Y \in B$; also, we may assume that for $q \in S_1$ and Y, $f(Y) > 0$, $t_q(Y) > k = \frac{2}{\sigma} \log\left(\frac{\varepsilon}{L}\right)$ where $|L_q(Y)| \leq L$.

The last inequality for $t_q(Y)$ is justified as follows. For $q \in S_1$ there are neighborhoods I_q of q and B_q of X such that $\varphi_Y(r, t) \in \operatorname{Int} N$ for $0 < t \leq k$, $r \in I_q$, $Y \in B_q$. This follows by continuity since it is obvious for $Y = X$ on S_1. Compacity of S_1 ends the argument. A straight forward computation, replacing $t_q(Y) > k$ into the integrand above, shows that $|h'_Y(q)| < \varepsilon$.

Lemma **(3.7)**. — *Assume the hypothesis and notation in* (3.3) *and call* L_X *the* ω-*limit set of* γ_X, *the unstable separatrix of* p $(\gamma_X = \varphi_X(u(X), t),\ |t| < \infty)$. *If* $L_X \neq p$, *let* L_X *be contained in a neighborhood* N′ *whose boundary is transversal to* X, X *pointing inward* N′; *if* $L_X = p$, *let* γ_X *be interior to the stable manifold of* p. *Then* γ_X *has a neighborhood* N^2 *which contains* L_X, *whose boundary is transversal to* X.

Proof. — Take U_1, of (3.6), small so that every trajectory of X passing through it meets $\partial N'$ transversally at points of an arc A; if $L_X = p$, A is assumed to be contained in $\operatorname{Int} s(X)$. Call $p_i \in \partial N$, the extremes of S (3.6), and call A_i the arc of trajectories of X joining p_i to $q_i \in A$. S together with A_i and $\partial N'$ (when $L_X \neq p$), bound a neighborhood of γ_X whose boundary is transversal to X except on A_i. Replacing A_i by arcs A'_i, C^1-close to them, joining p_i to A, and smoothing corners at the extremes of A'_i, the desired neighborhood N^2 is obtained. The change of A_i by the arcs A'_i is possible since X is parallel, in suitable local coordinates, in a neighborhood of A_i.

Lemma **(3.8)**. — *Assume that in* (3.7) L_X *is*:

a) *a generic singular point or periodic trajectory*; *or*

b) *a saddle-node* $L_X = p$.

In case a), assume that N *is a critical region associated to* L_X (see (1.4), *b)*).

Then B *of* (3.2) *can be taken so that for* $f(Y) \neq 0$, Y *is structurally stable in* N^2, Y *being transversal to* ∂N^2. *If* $f(Y) > 0$, *for case a)*, L_Y, *the only generic singular point or periodic trajectory of* Y *in* N′ *corresponding to* L_X, *is the* ω-*limit set of every trajectory of* Y *passing through* N^2; *for case b), there is in* N^2 *one periodic trajectory of* Y, *generic and orbitally stable which is the* ω-*limit set of every trajectory of* Y *meeting* N^2.

If $f(Y) < 0$ (resp. $f(Y) = 0$), *call* $r(Y)$ *and* $q(Y)$ (resp. $p(Y)$) *the nodal and saddle points* (resp. *the saddle-node*) *of* Y *in* N (3.2). *In case a)* $r(Y)$ (resp. $p(Y)$) *is the* ω-*limit set of every trajectory of* Y *meeting* $\operatorname{Int} s(Y)$ *and of one unstable separatrix of* $q(Y)$ (resp. *of all trajectories of* Y *meeting* $s(Y)$); L_Y *is the* ω-*limit set of every trajectory of* Y *meeting* $\partial N^2 - s(Y)$ *and the unstable separatrix of* $q(Y)$ (resp. *of* $p(Y)$) *passing through* $u(Y)$ (3.2). $q(Y)$ *is the* ω-*limit set of its stable separatrices passing through* $s_1(Y)$, $s_2(Y)$. *In case b)*, $r(Y)$ (resp. $p(Y)$) *is the* ω-*limit set of every trajectory meeting* N^2 *except* $q(Y)$ *and its stable separatrices through* $s_1(Y)$, $s_2(Y)$ (resp. *of every trajectory meeting* N^2).

Proof. — For Y close to X, the mapping $\pi_Y : U_1 = U \cap N^2 \to S_1 \subset A$ (notation, Proof of (3.1)), is defined. For case *a)*, $f(Y) > 0$, every trajectory through $\partial N^2 - S$ meets ∂N^1, for Y near X (since they do so for X), and the trajectories through S define the map $\pi_Y \circ h_Y : S \to S_1 \subset A \subset \partial N^1$, by (3.6); therefore, every trajectory through ∂N^1 must have L_Y as ω-limit set. For case *b)*, $f(Y)>0$, $S_1 \subset A \subset \operatorname{Int} s(X)$, and $g_Y = \pi_Y \circ h_Y : U_1 \to U_1$ is defined.

Taking B so that $|\pi'_Y|<k$ in U_1 and $|h'_Y|<\frac{1}{2}k^{-1}$, by (3.6), g_Y is a contraction having in U one fixed point, generic and orbitally stable. Since every trajectory through N^2 meets U_1, its ω-limit must be the generic periodic trajectory through the fixed point of g_Y.

For $f(Y)\leq 0$, *a)* and *b)* follow directly from continuity of $u(Y)$, $s_1(Y)$, $s_2(Y)$, and standard continuity of trajectories with respect to Y and initial data.

Lemma **(3.9)**. — *Assume the notation in* (3.2). *Then, given* $\varepsilon>0$, B *and* N *may be chosen so that every arc of trajectory of* Y *contained in* N *has length less than* ε, *provided* $f(Y)=0$.

Proof. — In the coordinate expression (3.2.1) for X, in Proof of (3.2), α and Δ_1 are taken so that $\dfrac{|\alpha|}{\Delta_1}<\dfrac{1}{2}$; this is obtained by changing coordinates x_1 to μx_1, x_2 to x_2, where μ satisfies $\mu\dfrac{|\alpha|}{\Delta_1}<\dfrac{1}{2}$. Call $P(Y)=(x_1(Y), x_2(Y))$ the saddle-node of Y in N, call $(1, v)$ the components of v_Y, the eigenvector associated to the zero eigenvalue of DY at P(Y).

Denoting $x_i - x_i(Y)$ by ξ_i, $i=1, 2$, Y can be written:

$$Y^1(x_1, x_2)=\bar{a}(\xi_2 - v\xi_1)+\bar{\Delta}_1\xi_1^2+\bar{b}\xi_1\xi_2+\bar{c}\xi_2^2$$
$$Y^2(x_1, x_2)=\bar{\sigma}(\xi_2 - v\xi_1)+\bar{\alpha}\xi_1^2\ \ +\bar{\beta}\xi_1\xi_2+\bar{\gamma}\xi_2^2,$$

where $\bar{\Delta}_1-\Delta_1$, $\bar{\sigma}-\sigma$, v, $x_i(Y)$ tend to zero as Y tends to X; $\bar{b}$, $\bar{c}$, $\bar{\beta}$, $\bar{\gamma}$ are functions uniformly bounded on N.

Divide N into two regions:

$$N_1=\{|\bar{\Delta}_1\xi_2^2|\geq|\bar{\sigma}(\xi_2 - v\xi_1)\} \quad \text{and} \quad N_2=\{|\bar{\Delta}_1\xi_1^2|\leq|\bar{\sigma}(\xi_2 - v\xi_1)|\}.$$

On N_1 the trajectories of Y satisfy the following equation:

$$\frac{dx_2}{dx_1}=\frac{\bar{\sigma}(\xi_2 - v\xi_1)/\bar{\Delta}_1\xi_1^2+\bar{\alpha}/\Delta_1+\bar{\beta}\xi_2/\bar{\Delta}_1\xi_1+\bar{\gamma}\xi_2^2/\bar{\Delta}_1\xi_1}{\bar{a}(\xi_2 - v\xi_1)/\bar{\Delta}_1\xi_1^2+1+\bar{b}\xi_2/\bar{\Delta}_1\xi_1+\bar{c}\xi_2^2/\bar{\Delta}_1\xi_1^2}.$$

Since in N_1, $|\xi_2|/|\xi_1|=|\xi_2 - v\xi_1+v\xi_1|/|\xi_1|\leq\left|\dfrac{\bar{\Delta}_1}{\bar{\sigma}}\right||\xi_1|+|v|$ by making N and B small, the numerator of $|dx_2/dx_1|$ can be made less than 2 and the denominator greater than $1/2$. Thus, $|dx_2/dx_1|<4$.

On N_2 the trajectories of Y satisfy:

$$\frac{dx_1}{dx_2}=\frac{\bar{a}/\bar{\sigma}+\bar{\Delta}_1\xi_1^2/\bar{\sigma}(\xi_2 - v\xi_1)+\bar{b}\xi_1\xi_2/\bar{\sigma}(\xi_2 - v\xi_1)+\bar{c}\xi_2^2/\bar{\sigma}(\xi_2 - v\xi_1)}{1+(\bar{\alpha}/\bar{\Delta}_1)(\bar{\Delta}_1\xi_1^2/\bar{\sigma}(\xi_2 - v\xi_1))+\bar{\beta}\xi_1\xi_2/\bar{\sigma}(\xi_2 - v\xi_1)+\bar{\gamma}\xi_2^2/\bar{\sigma}(\xi_2 - v\xi_1)}.$$

Since on N_2, $|\xi_2|/|\xi_2-v\xi_1|\leq 1+|v||\xi_1|/|\xi_2-v\xi_1|$, $|\xi_1\xi_2|/|\xi_2-v\xi_1|\leq|\xi_1|+|v||\bar{\sigma}|/|\bar{\Delta}_1|$, and $|\xi_2^2|/|\xi_2-v\xi_1|\leq|\xi_2|+|v||\xi_1|+|v|^2|\bar{\sigma}|/|\bar{\Delta}_1|$, by making N and B small, $\left|\dfrac{dx_1}{dx_2}\right|$ can be made less than 4, making its numerator less than 2 and its denominator greater than 1/2, in absolute value. The lemma follows immediately from the expression for the arc length of a curve, taking account that the interval of integration does not exceed the diameter of N.

Remark **(3.9.1)**. — *a)* Lemma (3.9) is similar to [5, Lemma 7, p. 143], proved for the generic saddle singular points. (3.9), and the next result *b)* also due to [5], are important tools for the construction of topological equivalences in canonical regions which contain saddles, saddle-nodes, or periodic trajectories in their closure.

b) [5, p. 150]. Let $\overset{\frown}{A_0B_0}$ be an arc and $\overset{\frown}{A_iB_i}$, $i=1, 2, \ldots$ be a sequence of arcs converging uniformly to $\overset{\frown}{A_0B_0}$ in such a way that $|\overset{\frown}{A_iB_i}|\to|\overset{\frown}{A_0B_0}|$, when $i\to\infty$. Then:

1) A point $M_i\in\overset{\frown}{A_iB_i}$ with ratio of arc length $z_i=|\overset{\frown}{A_iM_i}|/|\overset{\frown}{A_iB_i}|$ converges to a point $M_0\in\overset{\frown}{A_0B_0}$ if and only if $z_i\to z_0=|\overset{\frown}{A_0M_0}|/|\overset{\frown}{A_0B_0}|$.

2) Considering $\overset{\frown}{A_iB_i}$ and $\overset{\frown}{A_0B_0}$ parametrized by ratio of arc length, $\overset{\frown}{A_iB_i}$ converges uniformly to $\overset{\frown}{A_0B_0}$ when $i\to\infty$.

Proof of **(3.5)**. — Take $X\in Q_1^1$ and assume the notation in (3.8). Call:

$$M_1^2=M^2-\text{Int}\,N^2,$$

and take B such that $Y_1=Y|M_1^2$, $Y\in B$, belongs to Σ^r in M_1^2, and no saddle separatrices of Y_1 meet $s_1(Y)$, $s_2(Y)\in\partial N^2$. This choice of B is possible since the conditions imposed are open and hold for $Y=X$. Hence, by (3.8) and (3.2), $Y\in\Sigma^r$ in M^2 if and only if $f(Y)\neq 0$, and therefore $\mathfrak{X}_1^r\cap B=Q_1^1\cap B=f^{-1}(0)$. This proves *a)* and *b)* of (3.5).

To prove *c)*, a topological equivalence between X and $Y\in B_1=B\cap Q_1^1$ must be constructed. Obviously a topological equivalence $h_1=h_1(Y)$ between X_1 and Y_1 can be constructed; here, care is taken so that $h_1|\partial N^2$ maps $s_i(X)$ to $s_i(Y)$, $i=1, 2$. We proceed to show how to extend h_1 to $h=h(Y)$ defined on M^2.

For case *a)*, (3.8), N^2 is divided into two canonical regions $R_1(Y)$ and $R_2(Y)$ of Y and one critical region N_1^1 which contains L_Y. See Fig. (3.2).

The construction of h from $R_2(X)$ onto $R_2(Y)$ is performed in [5, p. 152] for the case where p is a saddle point; such construction is carried *mutatis mutandis* for the present case, by (3.9). See Remark (3.9.1). The construction of h from N_1^1 onto itself is done in [5, p. 154]. We proceed to define h from $R_1(X)$ onto $R_1(Y)$: Map the arc of trajectory of X, $\overset{\frown}{mp}$, passing through $m\in s(X)$, onto the arc of trajectory of Y, $\overset{\frown}{\tilde{m}p_Y}$ passing through $\tilde{m}=h_1(m)\in s(Y)$, by ratio of arc length (see proof of (2.5)). Since every point

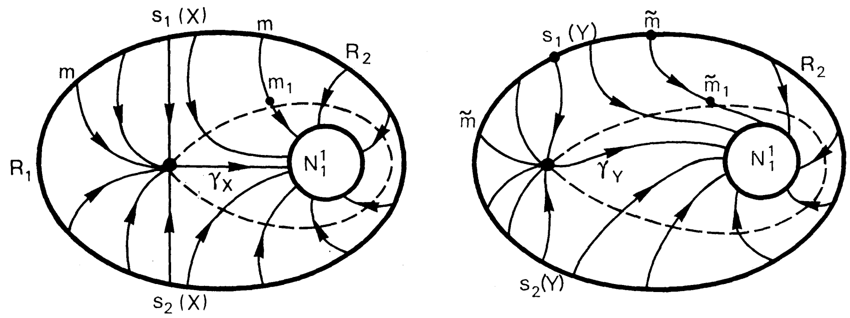


FIG. 3.2

of $R_1(X)$ belongs to a unique trajectory, this defines a one-to-one map of $R_1(X)$ onto $R_1(Y)$, which by (3.9.1) is a homeomorphism. In fact (3.9) implies that $|\overrightarrow{m_1p}|$ is close to $|\overrightarrow{m_2p}|$ and $|\overrightarrow{\tilde{m}_1p_Y}|$ is close to $|\overrightarrow{\tilde{m}_2p_Y}|$ provided $\overrightarrow{m_1p}$ is uniformly close to $\overrightarrow{m_2p}$ and $\overrightarrow{\tilde{m}_1p_Y}$ is uniformly close to $\overrightarrow{\tilde{m}_2p_Y}$; in our case this holds when m_1 is close to m_2, by continuity of h_1 and standard continuity of trajectories on initial data. That is, the hypothesis of (3.9.1, *b)*) is satisfied for these arcs. This implies continuity of h and h^{-1}, since they preserve ratio of arc length and uniform convergence on arcs of trajectories, which by (3.9.1, *b)*) amounts to preservation of convergence. Finally, we remark that the definition of h on $R_2(X)$ mentioned above coincides with our construction on the common boundary, $\overrightarrow{s_1(X)p}\cup\overrightarrow{s_2(X)p}$, with $R_1(X)$, since there it is performed by ratio of arc length.

For case *b)* and $Y\in B_1$, N^2 is divided into two canonical regions $R_1(Y)$, $R_2(Y)$ of the same type. See Fig. (3.3), where (i, j) is (1, 2) or (2, 1), according to $\gamma_X\cup\{p\}$ being a two-sided or one-sided curve.

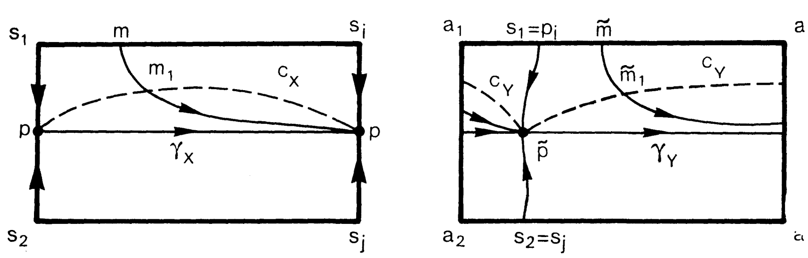


FIG. 3.3

We proceed to define h from $R_1(X)$ onto $R_1(Y)$. Map $\overrightarrow{s_1(X)p}$, $\overrightarrow{s_i(X)p}$ and $\gamma_X=\overrightarrow{pp}$ respectively onto $\overrightarrow{s_1(Y)p_Y}$, $\overrightarrow{s_i(Y)p_Y}$ and $\gamma_Y=\overrightarrow{p_Yp_Y}$, by ratio of arc length. Let η be a continuous monotonic increasing function from $\overrightarrow{s_1(X)s_i(X)}$ onto [0, 1]; let:

$$a(Y)=|\overrightarrow{s_1(Y)p_Y}|/(|\overrightarrow{s_1(Y)p_Y}|+|\gamma_Y|).$$

We define a closed curve C_Y in $R_1(Y)$ from p_Y to p_Y as follows: take $m \in \operatorname{Int} \overrightarrow{s_1(Y)s_i(Y)}$ and take the point m_1 on the arc of trajectory through m such that

$$|\overrightarrow{mm_1}|/|\overrightarrow{mp_Y}| = (1-\eta(h^{-1}(Y)(m)))a(Y) + \eta(h_1^{-1}(Y)(m)).$$

C_Y is the curve which assigns m_1 to m, and p_Y to $s(Y)$ and $s_i(Y)$; it is continuous, on $\operatorname{Int} \overrightarrow{s_1(Y)s_i(Y)}$ by continuity of trajectories on initial data and at $s_1(Y)$, $s_i(Y)$ by (3.9). C_Y divides $R_1(Y)$ into two regions $R_1^1(Y)$ and $R_1^2(Y)$. Map the arc of trajectory of X through $m \in \operatorname{Int} \overrightarrow{s_1(X)s_i(X)}$ onto the arc of trajectory of Y through $\widetilde{m} = h_1(Y)(m) \in \operatorname{Int} \overrightarrow{s_1(Y)s_i(Y)}$, as follows: map $\overrightarrow{mm_1}$ onto $\overrightarrow{\widetilde{m}\widetilde{m}_1}$ and $\overrightarrow{m_1 p}$ onto $\overrightarrow{\widetilde{m}_1 p_Y}$, respectively, by ratio of arc length. This defines a one-to-one map of $R_1^i(X)$ onto $R_1^i(Y)$, $i=1, 2$, which by (3.9) is a topological equivalence, as follows from an analysis similar to that performed in case *a)*. An identical construction works for $R_2(X)$. This ends the proof of (3.5).

The composed focus **(3.10)**. — Let p be a singular point of $X \in \mathfrak{X}^r$; assume that the eigenvalues of DX_p have non vanishing imaginary parts (i.e., $(\sigma(X, p))^2 - 4\Delta(X, p) < 0$). Let (x_1, x_2) be a coordinate system on a neighborhood U of p; assume that $x_1(p) = x_2(p) = 0$. Define $G : \mathfrak{X}^r \times U \to \mathbf{R}^2$ by $G(Y, q) = (Y^1(q), Y^2(q))$, $Y = \sum_i Y^i \frac{\partial}{\partial x_i}$. G is of class C^r since it is an evaluation mapping [16, p. 25]; also $G(x, p) = (0, 0)$ and $\frac{\partial G}{\partial v}(X, p) = DX(v)$. Since $\det DX = \Delta(X, p) \neq 0$, there is a unique C^r U-valued function P defined on a neighborhood B of X such that $P(X) = p$ and $G(Y, q) = (0, 0)$ for $Y \in B$ only if $q = P(Y)$. This follows from the Implicit Function Theorem.

Define $f : B \to \mathbf{R}$ by $f(Y) = \sigma(Y, P(y)) = \frac{\partial Y^1}{\partial x_1}(P(y)) + \frac{\partial Y^2}{\partial x_2}(P(y))$: f is of class C^{r-1} and:

$$df_X(z) = d\left(\frac{\partial X^1}{\partial x_1} + \frac{\partial X^2}{\partial x_2}\right)_p . dP_X(z) + \frac{\partial z^1}{\partial x_1}(p) + \frac{\partial z^2}{\partial x_2}(p),$$

as follows from a straightforward computation; in particular, if $\sigma(Z, p) \neq 0$ and $Z(P) = 0$, $dP_X(Z) = 0$ and $df_X(Z) = \sigma(Z, p) \neq 0$.

Let $P(Y) = (P_1(Y), P_2(Y))$, and take polar coordinates ρ, θ: $x_1 - P_1(Y) = \rho \cos\theta$, $x_2 - P_2(Y) = \rho \sin\theta$. The orbits of Y satisfy the following equations:

$$\frac{d\rho}{dt} = Y^1 \cos\theta + Y^2 \sin\theta = R_Y(\rho, \theta) \quad \text{and} \quad \rho\frac{d\theta}{dt} = Y^2 \cos\theta - Y^1 \sin\theta = \Theta_Y(\rho, \theta),$$

where $Y^i = Y^i(P_1(Y) + \rho\cos\theta, P_2(Y) + \rho\sin\theta)$ and $\Theta_Y(\rho, \theta)$ are of class C^r in $B \times I \times \mathbf{R}$, where $I = [-a, a]$, a small; also, they are periodic of period 2π in θ. The hypothesis $\sigma - 4\Delta < 0$ implies that $\frac{\partial \Theta_X}{\partial \rho}(0, \theta) \neq 0$, for all θ. By continuity we may assume that

$\dfrac{\partial\Theta_X}{\partial\rho}(\rho, \theta) \neq 0$ in $B \times I \times \mathbf{R}$. Define $\overline{\Theta}_Y$ by $\overline{\Theta}_Y(\rho, \theta) = \int_0^1 \dfrac{\partial\Theta_Y}{\partial\rho}(\rho s, \theta)ds$. $\overline{\Theta}$ is of class C^{r-1} and $\overline{\Theta}(\rho, \theta) = \dfrac{1}{\rho}\Theta(\rho, \theta)$, for $\rho \neq 0$; also:

$$\overline{\Theta}_Y(-\rho, \theta+\pi) = \overline{\Theta}_Y(\rho, \theta) \quad \text{and} \quad R_Y(-\rho, \theta+\pi) = -R_Y(\rho, \theta).$$

This implies that $(R_Y, \overline{\Theta}_Y)$, for $Y \in B$, is a vector field in $I \times \mathbf{R}$, invariant under the mapping $\mu : (\rho, \theta) \to (-\rho, \theta+\pi)$. Since $R_Y(0, \theta) = 0$, $\rho = 0$ is a trajectory of $(R_Y, \overline{\Theta}_Y)$.

Let $u(x) = (x, 0)$, $U = I \times \{0\}$, and $s(x) = (x, 2\pi)$, $S = I \times \{2\pi\}$. Call $\rho_Y : U_0 \to S$ the mapping associated to u, s and $\tau = 2\pi$ defined by the flow $(R_Y, \overline{\Theta}_Y)$ as in (2.1). $(Y, x) \mapsto \rho_Y(x)$ is of class C^{r-1} in $B \times U_0$. Also, as a straightforward computation shows, $\rho'_Y(0) = 1$ if and only if $\sigma(Y, P(Y)) = 0$.

Definition **(3.11)**. — Assume that $X \in \mathfrak{X}^r$, $r \geq 4$, has a singular point p with $\sigma(X, p) = 0$ and $\Delta(X, p) > 0$. If, with the notation above, $(\rho_X)^{(3)}(0) \neq 0$, p is called a *composed focus*.

Proposition **(3.12)**. — *Denote by* Q_1^2 *the set of vector fields* $X \in \mathfrak{X}^r$, $r \geq 4$ *such that:*

1) X *has a composed focus as unique non-generic singular point.*
2) X *has only generic periodic trajectories.*
3) *The* α *and* ω*-limit sets of any trajectory of* X *are singular points or periodic trajectories.*
4) X *has no saddle connections.*

Then:

a) Q_1^2 *is open in* $\mathfrak{X}_1^r$.

b) *It is an imbedded Banach submanifold of class* C^{r-1} *and codimension one of* $\mathfrak{X}^r$; *and*

c) *Every* $X \in Q_1^2$ *has a neighborhood* B_1 *in* Q_1^2 *so that it is topologically equivalent to every* $Y \in B_1$.

The proof of (3.11) depends on the following

Lemma **(3.12)**. — *Let* $X \in \mathfrak{X}^r$, $r \geq 4$, *have a composed focus* p. *Assume that* $(\rho_X)^{(3)}(0) < 0$. *Then there is a neighborhood* B *of* X, *a neighborhood* N *of* p *and a* C^{r-1} *function* $f : B \to \mathbf{R}$ *such that:*

1) ∂N *is a closed curve transversal to every* $Y \in B$.

2) $Y \in B$ *has one singular point* $P(Y) \in N$. $P(Y)$ *is generic if and only if* $f(Y) \neq 0$, *it is asymptotically stable* (resp. *unstable*) *if* $f(Y) < 0$ (resp. $f(Y) > 0$).

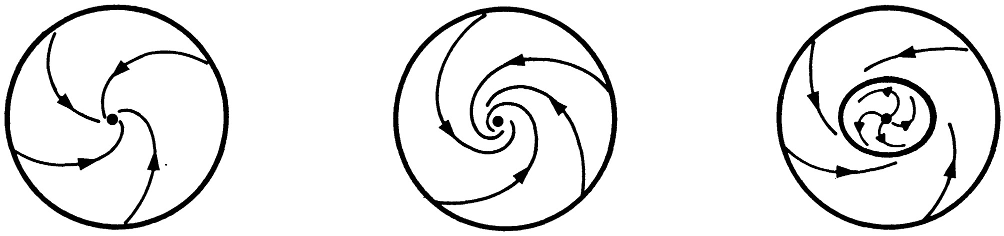
3) Y *has one periodic trajectory, generic and orbitally stable, in* N *only when* $f(Y) > 0$. See Fig. (3.4).

Proof. — Assume the notation of (3.10). Call N_0 the quotient manifold:

$$(I \times \mathbf{R})/\mu \to \mathbf{R}/\mu;$$

N_0 is a Moebius band. Call $\overline{\mu}$ the quotient mapping $I \times \mathbf{R} \to N_0$.

Let $\overline{Y} = D\mu(R_Y, \overline{\Theta}_Y)$ and let $\overline{u} = \mu \circ s = \mu \circ u$. $\rho_Y : U_0 \to U$ is equal to the square



$f(\mathrm{Y})>0$ $f(\mathrm{Y})=0$ $f(\mathrm{Y})<0$

FIG. 3.4. — Composed focus

of the Poincaré transformation associated to the periodic trajectory $\gamma_{\mathrm{Y}}=\mu\{\{0\}\times\mathbf{R}\}$ of period π of $\overline{\mathrm{Y}}$. Now, the proof is reduced to (2.6), with $f(\mathrm{Y})=\sigma(\mathrm{Y}, \mathrm{P}(\mathrm{Y}))$ and N the neighborhood of p bounded by $\mathrm{C}=\{x_1=\rho(\theta)\cos\theta,\ x_2=\rho(\theta)\sin\theta;\ \theta\in[0, 2\pi]\}$ where $\rho=\rho(\theta)$ is so that $\{\rho(\theta), \theta\}$ is the lifting to $\mathrm{I}\times\mathbf{R}$ of the boundary of the neighborhood N of γ_{X} given in (2.6).

Proof of **(3.12)**. — Similar to (2.5). Assume the notation of (3.12). Call M_1^2 the manifold with boundary $\mathrm{M}^2-\mathrm{Int}\,\mathrm{N}$. $\mathrm{X}_1=\mathrm{X}\,|\,\mathrm{M}_1^2$ is structurally stable, and B can be taken so that every $\mathrm{Y}\in\mathrm{B}$ is such that $\mathrm{Y}_1=\mathrm{Y}\,|\,\mathrm{M}_1^2$ is topologically equivalent to X_1; $h_1(y)$, the homeomorphism of M_1^2 mapping trajectories of X_1 onto those of Y_1, can be made arbitrarily close to the identity of M_1^2 by properly reducing B.

By openness of Σ^r in M_1^2, when $f(\mathrm{Y})\neq 0$, $\mathrm{Y}\in\Sigma^r$ in M^2, by (3.12). Thus $f^{-1}(0)=\mathrm{Q}_1^2\cap\mathrm{B}$. For $\mathrm{Y}\in\mathrm{B}_1=\mathrm{Q}_1^2\cap\mathrm{B}$, $h_1(\mathrm{Y})$ can be extended to a topological equivalence between X and Y. This is done as for the case of generic focus [5, p. 153]. This proves (3.12).

Remark **(3.13)**. — By (3.5) and (3.12), $\mathrm{Q}_1=\mathrm{Q}_1^1\cup\mathrm{Q}_1^2$ is an imbedded submanifold, open in $\mathfrak{X}_1^r$.

Calling the saddle-node and composed focus *quasi-generic singular points*, (3.5) and (3.12) can be stated in one Proposition changing in condition 1), in either one, saddle-node or composed focus by quasi-generic singular point.

4. Saddle Connections.

Definition **(4.1)**. — A saddle connection γ of X (see (3.4)) whose α and ω-limit sets coincide with a saddle point p is called a *loop*; it is called a *simple loop* if $\sigma(\mathrm{X}, p)\neq 0$.

Proposition **(4.2)**. — *Let* Q_3 *denote the set of vector fields* $\mathrm{X}\in\mathfrak{X}^r$, $r\geq 2$ *such that:*

1) X *has one saddle connection, which in case of being a loop is a simple loop.*

2) X *has only generic singular points and generic periodic trajectories.*

3) *The* α *and* ω*-limit sets of every trajectory of* X *are singular points, periodic trajectories, or loops.*

Then Q_3 *is a Banach submanifold of class* C^{r-1} *and codimension one immersed in* $\mathfrak{X}^r$; *furthermore, every* $X \in Q_3$ *has a neighborhood* B_1 *in* Q_3 *such that every* $Y \in B_1$ *is topologically equivalent to* X.

The proof of this proposition depends on several preliminary lemmas.

Lemma **(4.3)**. — *Let* p *be a saddle point of* $X \in \mathfrak{X}^r$, $r \geq 1$. *There is a neighborhood* B *of* X *and a neighborhood* N *of* p *such that*:

1) $Y \in B$ *has one singular point* $p(Y)$, *which is a saddle point, in* N; ∂N *is a differentiable curve.*

2) *The stable* (resp. *unstable*) *separatrices of* $p(Y)$ *for* $Y \mid N$ *meet* ∂N *in two points* $s_1(Y)$, $s_2(Y)$ (resp. $u_1(Y)$, $u_2(Y)$) *so that the functions* $s_i : B \to N$ (resp. $u_i : B \to N$) *are of class* C^{r-1}. *Also, there are closed arcs* S_i (resp. U_i), *which contain* $s_i(B)$ (resp. $u_i(B)$), *on which* $Y \in B$ *is transversal to* ∂N.

Proof. — 1) Follows as in (3.10) from the fact that $\Delta(X, p) \neq 0$, by the Implicit Function Theorem. If N is small, 2) is valid for X, since the stable and unstable manifolds are tangent, at p, to the eigenspaces of DX_p (see (3.1)), which are transversal to ∂N; the continuity of $s_i(Y)$ (resp. $u_i(Y)$) is proved in [5, p. 147]; differentiability relative to a parameter is shown in [16, p. 151]; 2) follows taking Y as parameter; the existence of S_i, U_i follow from continuity.

A construction **(4.4)**. — Assume the notation and hypothesis in (4.3).

a) The point $s_i(Y)$ divides S_i into two closed arcs $S_i^1(Y)$ and $S_i^2(Y)$, which have $s_i(Y)$ as unique common point. See Fig. (4.1). S_i is taken small so that every trajectory of $Y \in B$ which enters N through $x \in S_i^j(Y) - \{s_i(Y)\}$, leaves N through a point $k_Y^{ij}(x) \in U_j$. This follows as in the first part of Proof (3.6), by continuity. Furthermore, the mapping $k_Y^{ij} : S_i^j(Y) \to U_j$, defined above for $x \neq s_i(Y)$ and equal $u_j(Y)$ for $s_i(Y)$ is continuous. This follows from the continuity property on hyperbolic sectors [19, p. 167].

b) The mapping k_Y^{ij} is differentiable of class C^r in $S_i^j(Y) - \{s_i(Y)\}$, and if $\sigma(X, p) < 0$, for any given $\varepsilon > 0$, N, S_i and B may be taken so small that $\left| \frac{dk_Y^{ij}}{dx}(x) \right| < \varepsilon$.

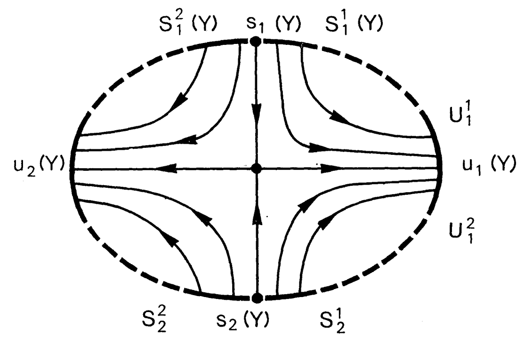


FIG. 4.1

This follows from a well-known formula for the derivative in terms of σ, in the same way as in (3.6).

c) Finally, the length of the arc $\overrightarrow{xk_Y^{ij}(x)}$ contained in N tends to the sum of the arc lengths of separatrices in N, $\overrightarrow{pu_j(X)}$ and $\overrightarrow{s_i(X)p}$, as $Y \to X$ and $x \to s_i(X)$. See [5, p. 149] for a proof of this fact.

Let p_1 and p_2 be two saddle points of X (the case $p_1 = p_2$ is not excluded). Let $m > 0$ be less than the lengths of the saddle separatrices leaving or approaching p_1 and p_2. Denote by B^i, N^i, U_j^i, S_j^i, $u_j^i(Y)$, $s_j^i(Y)$, $j = 1, 2$, the objects associated to $p = p_i$, $i = 1, 2$, by (4.3). Let X have a saddle connection γ_X joining $u_i^1(X)$ to $s_j^2(X)$, with length $\ell > 0$. For $Y \in B_1 \cap B_2$, call $\pi_Y = \pi(Y, \ ) : U_i^1 \to S_j^2$, the map defined by the flow of Y (see (2.1)). Define $f(Y) = \pi(Y, u_j^1(Y)) - s_j^2(Y)$.

Lemma **(4.5)**. — *Assume the notation above. Given* $0 < \varepsilon < m$, *Y has a saddle connection* γ_Y *joining* $u_i^1(Y)$ *and* $s_j^2(Y)$, *with length within* ε *of* ℓ, *if and only if* $f(Y) = 0$; *otherwise any saddle separatrix passing through any of these points has length greater than* $\ell + m$, *for* $Y \in B = B_1 \cap B_2$ *small. Furthermore,* $df_X \neq 0$.

Proof. — The first part follows from continuity (on Y) of the length of arcs of trajectories far from singularities, and from the continuity property (4.4) *c)* in N_1, N_2. If V is defined as in the proof of (2.4) in a small neighborhood of $s_1(X)$, $df_X(V) \neq 0$, as follows similarly to (2.4).

Remark **(4.5.1)**. — Trajectories of Y passing near $u_1^1(Y)$ or $s_j^2(Y)$ which do not connect them also have length greater than $\ell + m$ by the same arguments as in the first part of proof of (4.5).

On simple loops **(4.6)**. — Assume the notation in (4.4) and (4.5) and suppose that γ_X is a loop of $X \in \mathfrak{X}^r$, $r > 1$, $p_1 = p_2 = p$. Let $\sigma(X, p) < 0$ and take $N_1 = N_2 = N$, $u(X) = u_i^1(X)$, $s(X) = s_j^2(X)$, and $B \subset B_1 \cap B_2$ small so that for $Y \in B$, $|\pi'_Y| < K$ in $U = U_1$. Also, take $\varepsilon = (1/2)K^{-1}$ in (4.4, *b)*), so that $k_Y = k_Y^{11}$ satisfies $|k'_Y| < (1/2)K^{-1}$ in $S(Y) = S_1^1(Y) \subset S_1 = S$.

Take some orientation in ∂N, say, counterclockwise in Fig. (4.1); thus, k_Y reverses orientation. Define $\rho_Y = \pi_Y \circ k_Y : S(Y) \to S$.

There are two cases: *a)* π_Y reverses orientation, and *b)* π_Y preserves orientation.

Assume first case *a)*, where ρ_Y preserves orientation. If $f(Y) = 0$, Y has one loop γ_Y joining $u(Y)$ to $s(Y)$, which is the ω-limit set of all trajectories of Y meeting $S(Y) - \{u(Y)\}$. This follows from the fact that ρ_Y is a contraction, i.e.:

$$|\rho_Y(x) - \rho_Y(y)| < (1/2)|x - y|, \qquad x, y \in S(Y).$$

If $f(Y) < 0$, obviously $\rho_Y(S(Y)) \subset \operatorname{Int} S(Y)$, and ρ_Y has one fixed point, $P(Y)$, generic and orbitally stable; thus, through $P(Y)$ passes a periodic trajectory, Γ_Y, which is the ω-limit set of all trajectories of Y meeting $S(Y) - \{u(Y)\}$ and of the saddle separatrices through $u(Y)$. Moreover, $|P(Y)-u(Y)|<2f(Y)$, as follows from the evaluation of $P(Y)$ as the limit of iterates of ρ_Y.

The separatrix through $s(Y)$ meets Int $U_1^2(Y)$ in $s^1(Y)=\pi_Y^{-1}(s(Y))$, and the closed arc $\overrightarrow{s_1(Y)u(Y)}$ is mapped into $S(Y)$ by π_Y; thus Γ_Y is also the ω-limit set of trajectories of Y passing through the arc $\overrightarrow{s_1(Y)u(Y)}$, open at $s_1(Y)$.

If $f(Y)>0$, ρ_Y has no periodic point; in this case, the separatrix through $u(Y)$ meets Int $S_2^1(Y)$ at $u'(Y)=\pi_Y(u(Y))$, the separatrix through $s(Y)$ meets successively $U_1^1(Y)$ and $S(Y)$ at points $s^1(Y)=\pi_Y^{-1}(s(Y))$ and $s^2(Y)=\rho_Y^{-1}(s(Y))$. The closed arc $\overrightarrow{s^2(Y)s(Y)}$ is mapped by ρ_Y onto $\overrightarrow{s(Y)u^1(Y)}$. See Fig. (4.5) for a graphical illustration of case *a)*.

Consider now case *b)*, where ρ_Y reverses orientation. If $f(Y)=0$:

$$\rho_Y : S(Y) \to S_1^2(Y),$$

has $s(Y)$ as unique fixed point, and $k_Y^{12}\circ\rho_Y : S(Y) \to U_2(Y)$ is defined. Call γ_Y the one-sided loop through $s(Y)$.

If $f(Y)<0$, $\rho_Y(\overrightarrow{\pi_Y(u(Y))s(Y)}) \subset \pi_Y\overrightarrow{(u(Y))s(Y)}$ since:

$$\rho_Y(s(Y))=\pi_Y\circ k_Y(s(Y))=\pi_Y(u(Y)),$$

and hence $|\rho_Y(\pi_Y(u(Y)))-\pi_Y(u(Y))|=|\rho_Y(\pi_Y(u(Y)))-\rho_Y(s(Y))|<(1/2)|\pi_Y(u(Y))-s(Y)|$. Therefore, since ρ_Y is a contraction, it has a unique fixed point $P(Y)\in\text{Int}\,\overrightarrow{\pi_Y(u(X))s(Y)}$, since $|P(Y)-s(Y)|<f(Y)$. The separatrix through $s(Y)$ meets successively $U_1^1(Y)$, $S(Y)$, and $U_1^2(Y)$ at points $s^1(Y)=\pi_Y^{-1}(s(Y))$, $s^2(Y)=\rho_Y^{-1}(u(Y))$, and $s^3(Y)=\pi^{-1}(s^2(Y))$. The arc $\overrightarrow{s^2(Y)s^1(Y)}$ is mapped by π_Y onto $\overrightarrow{s^2(Y)s(Y)}$ which is mapped by ρ_Y onto $\overrightarrow{\pi_Y(u(Y))s(Y)}$. Thus the periodic trajectory Γ_Y of Y, passing through $P(Y)$, which obviously is generic and one-sided, is the ω-limit set of all trajectories through $\overrightarrow{s^3(Y)s^1(Y)}$.

If $f(Y)>0$, $\rho_Y(S(Y))\subset\text{Int}\,S_1^2(Y)$, and ρ_Y has no periodic points. The separatrices through $s(Y)$ and $u(Y)$ meet $U_1^2(Y)$ and $S_1^2(Y)$ at points $s^1(Y)=\pi_Y^{-1}(s(Y))$ and $u_1(Y)=\pi_Y(u(Y))$ respectively. ρ_Y maps $\overrightarrow{s^1(Y)u(Y)}$ onto $\overrightarrow{s(Y)u^1(Y)}$; $S(Y)$ is mapped into Int $U_1^2(Y)$ by π_Y^{-1}. See Fig. (4.6).

Canonical Regions for fields in Q_3 **(4.7).** — Take $X\in Q_3$. In case *a)* of (4.6), $\gamma_Y\cup\{p\}$, which is a two-sided loop, has on its (orbitally) stable region a differentiable closed curve C, arbitrarily close to the loop, transversal to X, which together with $\gamma_Y\cup\{p_Y\}$, when $f(Y)=0$, bound a region $N(Y)$ homeomorphic to a cylinder. C meets $S=S(X)$ transversally in a point m_0, which we regard as the lower extreme of S. Furthermore, $\gamma_Y\cup\{p_Y\}$ is the ω-limit set of trajectories of Y meeting Int $N(Y)$. See Fig. (4.2) I'.

For X, these assertions follow from [1], taking $C=\Gamma_Z$, where Z is a vector field

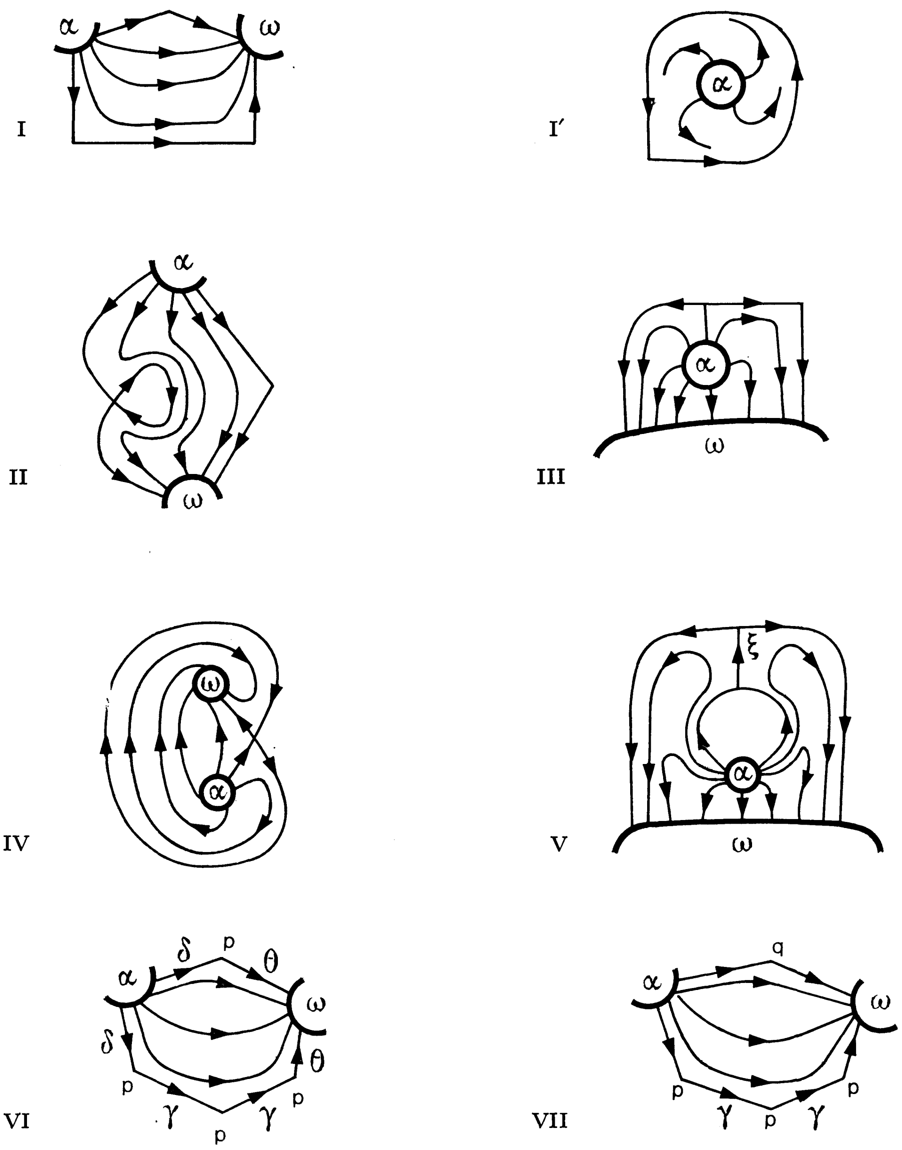


FIG. 4.2. — Canonical Regions in Q_3

obtained from X by a small rotation (in a neighborhood of $\gamma_X \cup \{p\}$ diffeomorphic to a plane region). For Y close to X, they follow from continuity and results in (4.6), case *a)*. Obviously m_0 is taken to be P(Z).

For future reference we will distinguish two cases.

A) All the trajectories of X meeting C have the same α-limit, which *a fortiori* must be a generic singular point of nodal or focal type, or a generic periodic trajectory.

B) There is some saddle separatrix of X which meets C.

A) and B) are the unique, and mutually exclusive possibilities; in either case, $N(X)$ will be regarded as a critical region associated to the loop $\gamma_X \cup \{p\}$.

The other canonical regions that contain $\gamma_X \cup \{p\}$ on their closure and are possible for $X \in Q_3$ are shown in Fig. (4.2).

This follows from making all the compatible identifications of edges and/or vertices in the fundamental polygons in Fig. (4.3).

For instance, II is obtained from *a)*, identifying p_1 and p_2; III is obtained from *a)*, identifying θ_1 and δ_1, and p_1 and q; IV is obtained from *a)* identifying δ_i with θ_i, $i=1, 2$.

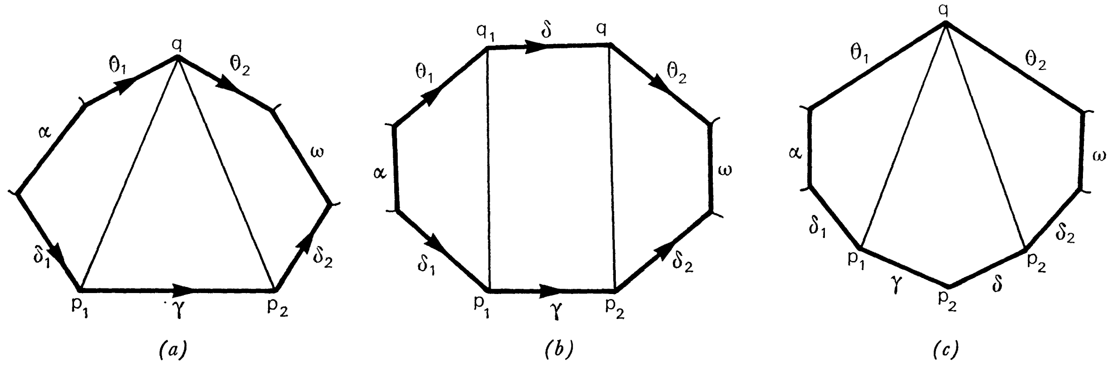


FIG. 4.3

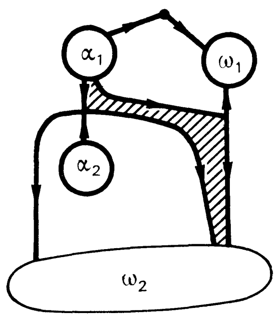
V is obtained from *b)* identifying δ and γ; VI and VII are obtained from *c)* making the identifications indicated in Fig. (4.2).

Consider the decomposition of M^2 into canonical and critical regions of X. γ_Y belongs to the common boundary of two such regions, except in cases V, VI, VII, Fig. (4.2), where it belongs to only one; call $M(X)$ the union of the (closed) regions which contain γ_X. Call $\widetilde{M}(X)$ the union of $M(X)$ and the critical regions of X which intersect saddle separatrices on the boundary of $M(X)$.

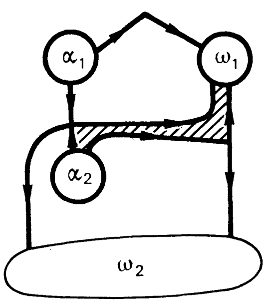
The complement of Int $M(X)$, denoted $\widetilde{N}(X)$, is the union of a finite number of critical and canonical regions of X; these regions are of structurally stable type and such that, for Y close to X, to each canonical region of X corresponds one of Y of the same type; the critical regions of Y are the same as those of X. Call $\widetilde{N}(Y)$ the union of such canonical regions of Y. Following [5], each canonical region of $\widetilde{N}(X)$ is mapped by a topological equivalence onto its corresponding canonical region of $\widetilde{N}(Y)$; gluing these partial mappings, a topological equivalence results, defined from the complement of all critical regions of $\widetilde{N}(X)$ onto the complement of all critical regions of $\widetilde{N}(Y)$; this topological equivalence is defined on the boundary of all critical regions, except on that of those contained in $\widetilde{M}(X)$, where it is defined only on the boundary of $\widetilde{M}(X)$. Below we show that when $f(Y)=0$, a topological equivalence can be defined from $M(X)$ onto $M(Y)=M^2-\text{Int}\,\widetilde{N}(Y)$, extending the above mentioned equivalence, which thus becomes defined on the boundary of all critical regions of X. This topological equivalence is extended to the interior of the critical regions by the method of [5].

We proceed to show how define a topological equivalence between M(X) and M(Y). In Fig. (4.4), M(X) is made up of one region of type I and one of type III, Fig. (4.2); $\widetilde{M}(X)$ is the union of M(X) and the critical regions of sources α_1, α_2 and sinks ω_1, ω_2 of generic type.

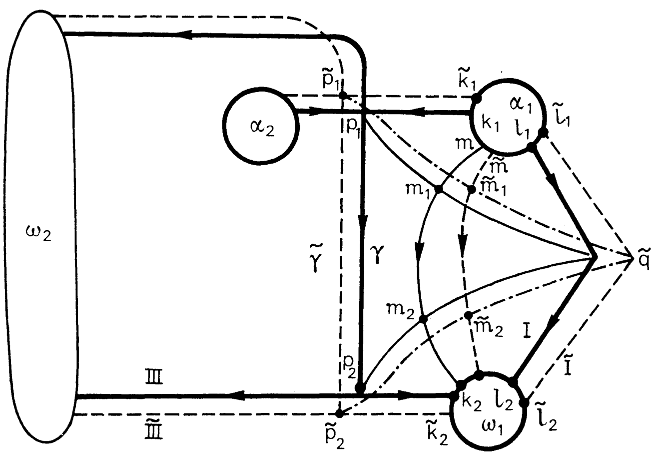
For region I, map by means of a homeomorphism h_1, $\overset{\frown}{k_1\ell_1}$ onto $\overset{\frown}{\tilde{k}_1\tilde{\ell}_1}$; also map by ratio of arc length $\delta_1=\overset{\frown}{k_1p_1}$, $\gamma=\overset{\frown}{p_1p_2}$, $\delta_2=\overset{\frown}{p_2k_2}$, $\Theta_1=\overset{\frown}{\ell_1q}$, $\Theta_2=\overset{\frown}{q\ell_2}$ onto their correspondents in $\tilde{\mathrm{I}}$, $\tilde{\delta}_1$, $\tilde{\delta}_2$, $\tilde{\Theta}_1$, $\tilde{\Theta}_2$; it should be remarked that this definition coincides with the above mentioned topological equivalence, which, following [5], takes saddle separatrices onto saddle separatrices by ratio of arc length. Divide every arc of trajectory of X (resp. Y) joining $m\in\mathrm{Int}\,\overset{\frown}{k_1\ell_1}$ (resp. $\tilde{m}=h_1(m)\in\mathrm{Int}\,\overset{\frown}{\tilde{k}_1\tilde{\ell}_1}$) to $n\in\overset{\frown}{k_2\ell_2}$ (resp. $\tilde{n}\in\overset{\frown}{\tilde{k}_2\tilde{\ell}_2}$) into three arcs $\delta_1(m)=\overset{\frown}{mm_1}$, $\gamma(m)=\overset{\frown}{m_1m_2}$, $\delta_2(m)=\overset{\frown}{m_2n}$ (resp. $\tilde{\delta}_1(\tilde{m})=\overset{\frown}{\tilde{m}\tilde{m}_1}$, $\tilde{\gamma}(\tilde{m})=\overset{\frown}{\tilde{m}_1\tilde{m}_2}$.



$f(\mathrm{Y})>0$



$f(\mathrm{Y})<0$



$f(\mathrm{X})=0$
$f(\mathrm{Y})=0$

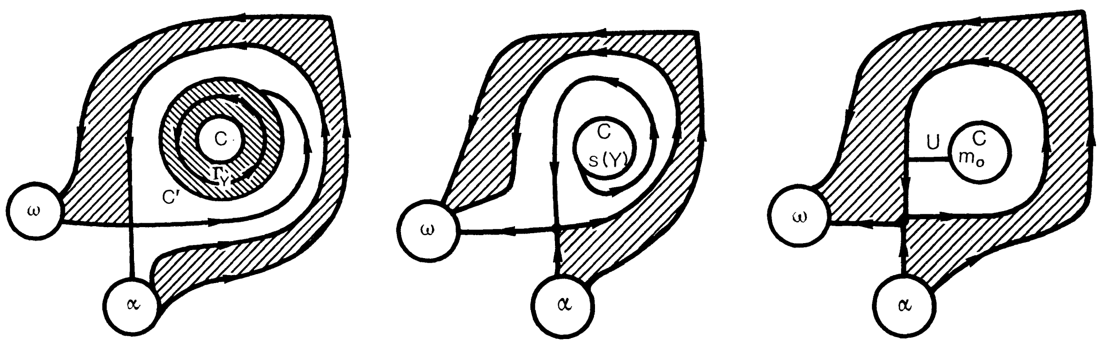
FIG. 4.4. — Saddle connection

$\tilde{\delta}_2(\tilde{m})=\overset{\curvearrowright}{\tilde{m}_2\tilde{n}}$), in the following way. Take a continuous monotonic increasing function η from $\overset{\curvearrowright}{k_1\ell_1}$ onto $[0, 1]$ (resp. $\tilde{\eta}=\eta\circ h_1^{-1}:\overset{\curvearrowright}{\tilde{k}_1\tilde{\ell}_1}\to[0, 1]$); call $a_1=|\delta_1|(|\delta_1|+|\gamma|+|\delta_2|)^{-1}$, $b_1=|\Theta_1|(|\Theta_1|+|\Theta_2|)^{-1}$, $a_2=|\gamma|(|\gamma|+|\delta_2|)^{-1}$, $b_2=1$ (resp. call $\tilde{a}_1, \tilde{b}_1, \tilde{a}_2, \tilde{b}_2=1$, the obvious analogous for $\tilde{\mathrm{I}}$). Take m_1 such that $|\overset{\curvearrowright}{mm_1}|(|\overset{\curvearrowright}{mn}|)^{-1}=a_1(1-\eta(m))+b_1\eta(m)$ and take m_2 such that $|\overset{\curvearrowright}{m_1m_2}|(|\overset{\curvearrowright}{m_1n}|)^{-1}=a_1(1-\eta(m))+b_2\eta(m)$ (resp. take $\tilde{m}_1$ and $\tilde{m}_2$ in the analogous way). Map $\delta_1(m)$, $\gamma(m)$ and $\delta_2(m)$, respectively onto $\tilde{\delta}_1(\tilde{m})$, $\tilde{\gamma}(\tilde{m})$ and $\tilde{\delta}_2(\tilde{m})$, by ratio of arc length. Thus we have defined a one-to-one map from I to $\tilde{\mathrm{I}}$ which, by (3.9.1) and the same arguments in the proof of (3.5), is a topological equivalence between $\mathrm{X}|\mathrm{I}$ and $\mathrm{Y}|\tilde{\mathrm{I}}$ that can be made arbitrarlly close to the identity for Y close to X [5], and extends to I the above mentioned topological equivalence. Of course this construction works for regions II, III, and IV, obtained from I by proper identifications. Also, when region I is modified to having three saddle points p_1, p_2, p_3 joined by saddle separatrices γ_1, γ_2, or two saddle points q_1, q_2 joined by a saddle separatrix ξ, which, respectively, are the cases of VI and VII, and V, it is clear how to construct the topological equivalence.

The extension of this map, now defined in $\partial\mathrm{I}'$, to Int I' is done in a similar way as in the case of the stable part of a periodic trajectory (2.5). (Here, $\mathrm{C}_1=\mathrm{C}$, $\gamma_{\mathrm{X}}=\gamma_{\mathrm{X}}\cup\{p\}$, and $\mathrm{U}=\mathrm{S}(\mathrm{X})=\overset{\curvearrowright}{m_0 s(\mathrm{X})}$.) See Fig. (4.5).

For $f(\mathrm{Y})\neq 0$, $\mathrm{Y}\in\Sigma^r$ except in case B), when $\gamma_{\mathrm{X}}\cup\{p\}$ is the ω-limit set of saddle separatrices: for the case where γ_{X} is not a loop, this follows by continuity of saddle separatrices and maps k_{Y}^{ij} in (4.4); in this case M(Y) has three canonical regions respectively joining α_1 to ω_1, α_2 to ω_2, and α_i to ω_j, $(i,j)=(1, 2)$ or $(2, 1)$ according to the sign of $f(\mathrm{Y})$. See Fig. (4.4) (of course in the case of region V, $\alpha=\alpha_1, \alpha_2$ and $\omega=\omega_1, \omega_2$).

For the case of two-sided loops, following (4.6) *a)*, we have that when $f(\mathrm{Y})>0$, M(Y) has a region of type $\mathrm{R}_2(\mathrm{Y})$, Fig. (3.2), with the sense on the trajectories reversed; the separatrix through $s(\mathrm{Y})$ meets C at a point $\tilde{s}(\mathrm{Y})$. The other region is bounded



$f(\mathrm{Y})<0$ $f(\mathrm{Y})>0$ $f(\mathrm{Y})=0$

Fig. 4.5. — Two sided loop

by the separatrices through $s_2(Y)$ and $u(Y)$ of p_Y and the separatrices entering and leaving q_Y (the correspondent of q). See Fig. (4.5). When $f(Y)<0$, Γ_Y (see (4.6)) is contained in a critical region bounded by C and $C'=\Gamma_{Z'}$, where Z′ is a rotated field like the one used to construct C; the canonical regions of M(Y) are similar to those for the case $f(Y)>0$, with the sense of the trajectories reversed, replacing C by C′. See Fig. (4.5).

Thus in case A), $f(Y)\neq 0$, and in case B), $f(Y)<0$, Y is in Σ^r as follows from the above assertions and arguments similar to those in (2.8). For the case B), $f(Y)>0$,

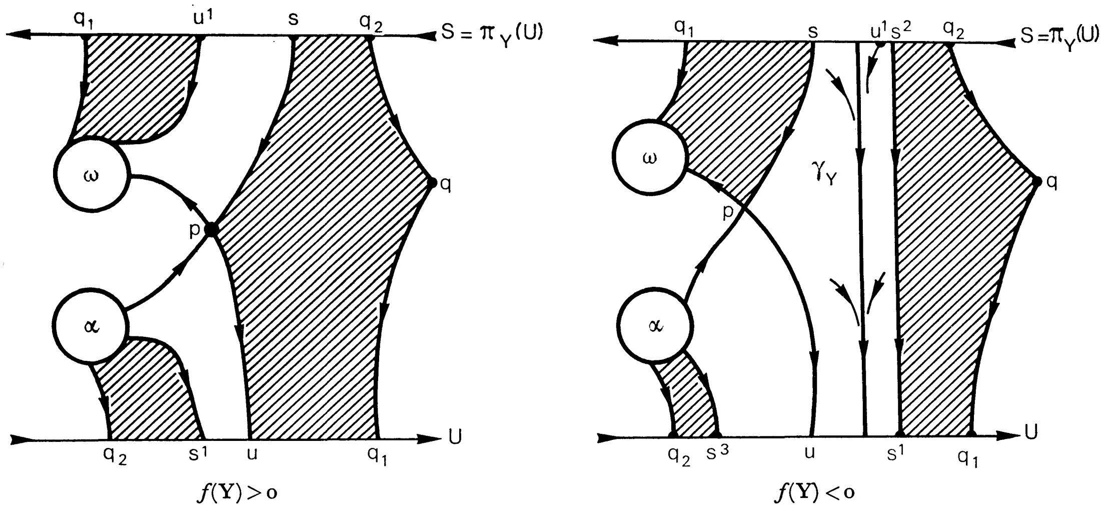


FIG. 4.6. — One sided loop after perturbation

$\mathfrak{T}(Y)$ winds around C when $Y\to X$ and meets infinitely many times, for fields Y arbitrarily close to X, all the unstable separatrices which (by hypothesis) intersect C.

For the case of one-sided loops following (4.6) *b)*, we have the canonical and critical regions on M(Y) as shown in Fig. (4.6).

Remark **(4.7.1)**. — Given any number $L>0$ (resp. $T>0$), B can be taken so that any trajectory of Y meeting C has length (resp. spends a time) greater than L (resp. T before closing, if it closes at all). This assertion is obvious by continuity arguments since it holds for X.

We summarize (4.7) in the following lemma.

Lemma **(4.8)**. — *Call* $Q_3(n)$ *the set of* $X\in Q_3$ *of Proposition* (4.2) *whose saddle connection* γ_X *has length less than n. Then:*

a) $Q_3(n)$ *is a submanifold of class* C^{r-1} *and codimension one imbedded in* $\mathfrak{X}^r$, *and*

b) *every* $X\in Q_3(n)$ *has a neighborhood* B_1 *in* $Q_3(n)$ *such that every* $Y\in B$ *is topologically equivalent to* X.

Proof. — Take $L>n$ in (4.8.1) and $\varepsilon<n-|\gamma_X|$ in (4.5). Take B as in (4.8), (4.8.1) and (4.5); *a)* follows from (4.8.1) since $f^{-1}(0)=Q_3(n)\cap B=B_1$, for all saddle connections, if any, of Y, $f(Y)>0$, must have length greater than $L>n$; *b)* is proved in (4.7).

Proof of Proposition (4.2). — Immediate by (4.8) since $Q_3=\bigcup_n Q_3(n)$ and $Q_3(n)\subset Q_3(n+1)$.

Remarks **(4.8.1).** — Call $\tilde{Q}_3$ the subset of Q_3 consisting of fields X which present case A) defined in (4.7). The following is proved in (4.7).

a) Proposition (2.2) *holds for* $\tilde{Q}_3$, *changing immersed by imbedded. Furthermore* $\tilde{Q}_3$ *is open in* $\mathfrak{X}_1^r$.

b) $\tilde{Q}_3^1=Q_3-\tilde{Q}_3$ is open in Q_3 and its intrinsic topology it finer than its ambient topology.

c) The fact that for $X\in\tilde{Q}_3^1$ and $\varepsilon>0$ small, $f^{-1}((-\varepsilon, 0))\subset\Sigma^r$, while $f^{-1}((0, \varepsilon))$ is not completely contained in Σ^r, can be expressed by asserting that $\Sigma^r\cup\tilde{Q}_3^1$ is a submanifold of $\mathfrak{X}^r$ with boundary $\tilde{Q}_3^1$.

d) From (2.4.1) and (4.7.1) it follows that $Q_2(n)\cup Q_2'(n)\cup Q_3(n)$ is an *imbedded* submanifold of $\mathfrak{X}^r$.

5. The Manifold Σ_1^r.

We define $S_i=Q_1\cup Q_2(i)\cup Q_2'(i)\cup Q_3(i)$. By (3.13) and (4.8.1) *d)*, S_i is an imbedded submanifold of $\mathfrak{X}^r$. Hence, $\Sigma_1^r=\bigcup_i S_i$ is an immersed submanifold of $\mathfrak{X}^r$.

Theorem 1. — a) Σ_1^r *defined above is an immersed Banach submanifold of class* C^{r-1} *and codimension one of* $\mathfrak{X}^r$, $r\geq 4$.

b) Σ_1^r *is dense in* $\mathfrak{X}_1^r$.

c) *Every* $X\in\Sigma_1^r$ *has a* Σ_1^r*-neighborhood* B_1, i.e., *a neighborhood in the intrinsic topology of* Σ_1^r, *such that* X *is topologically equivalent to every* $Y\in B_1$.

Proof. — Part *a)* follows from definition of Σ_1^r; part *c)* follows from Propositions (2.2), (3.13), (4.8.1) *d)*. Part *b)* follows from a sequence of approximations similar to those used in [8] to get density of Σ; the steps leading to *b)* are more suitably stated in Part II, Remarks (2.1.1), (2.2.3), (2.3.1).

6. On First Order Structural Stability.

A field $X\in\mathfrak{X}_1^r$ is said to be *first order structurally stable* if there is a neighborhood N of X in the subspace $\mathfrak{X}_1^r$ with the induced C^r-topology, such that every $Y\in N$ is topologically equivalent to X.

This concept is due to A. Andronov and E. Leontovich, see [12]. We will denote by $\tilde{\Sigma}_1^r$ the set of first order structurally stable vector fields.

After (2.8), (3.13), (4.8.1) it follows that Q_1, $\widetilde{Q}_2$, $\widetilde{Q}_3$ are contained in $\widetilde{\Sigma}_1^r$, and since each one is open in $\mathfrak{X}_1^r$, $Q_1 \cup \widetilde{Q}_2 \cup \widetilde{Q}_3 \subset \widetilde{\Sigma}_1^r$. By suitable C^r-approximations, it is not hard to show [14] that no field of $\mathfrak{X}_1^r$, outside $Q_1 \cup \widetilde{Q}_2 \cup \widetilde{Q}_3$, can be in $\widetilde{\Sigma}_1^r$. That is, $\widetilde{\Sigma}_1^r = Q_1 \cup \widetilde{Q}_2 \cup \widetilde{Q}_3$. Thus, since each Q_i is also an imbedded submanifold, $\widetilde{\Sigma}_1^r$ *is an imbedded Banach submanifold of class* C^{r-1} *and codimension one of* $\mathfrak{X}^r$, $r \geq 4$, open in $\mathfrak{X}_1^r$.

It is obvious how to define the set $\widetilde{\Sigma}_n^r$ of n-th order structurally stable vector fields as well as Σ_n^r (an n-dimensional version of Σ_1^r); the characterization of these sets seems most important for a generic theory of families of vector fields depending on n parameters.

II. — GENERIC ONE-PARAMETER FAMILIES OF VECTOR FIELDS

1. Preliminaries.

Let $J=[a, b]$ be a closed interval. Denote by Φ^r the space of C^1 mappings $\xi : J \to \mathfrak{X}^r$. Under the C^1 topology, Φ^r is a Banach manifold; its elements will be called *one-parameter families of vector fields.* $\lambda_0 \in J$ is called an *ordinary value* of $\xi \in \Phi^r$ if there is a neighborhood N of λ_0 such that $\xi(\lambda)$ is topologically equivalent to $\xi(\lambda_0)$ for every $\lambda \in N$; if λ_0 is not an ordinary value of ξ, it is called a *bifurcation value* of ξ. Obviously, if $\xi(\lambda_0) \in \Sigma^r$, λ_0 is an ordinary value of ξ; equivalently, if λ_0 is a bifurcation value of ξ, then $\xi(\lambda_0) \in \mathfrak{X}_1^r$.

Examples **(1.1)**. — *a)* Let $\xi(\lambda)=(1, \lambda)$ in $M^2=T^2=\mathbf{R}^2/\mathbf{Z}^2$. Every $\lambda \in [a, b]$ is a bifurcation value of ξ. This follows from the fact that the rotation number of $\xi(\lambda)$, which in this case is λ itself, is a topological invariant of $\xi(\lambda)$.

b) Let ξ be transversal to Σ_1^r. Every $\lambda_0 \in \xi^{-1}(\Sigma_1^r)$ is a bifurcation value of ξ. This follows from the results in Part II, where the topological change of the phase space of $Y=\xi(\lambda)$ is described in a neighborhood of $X=\xi(\lambda_0)$, according to the sign of $f(Y)$ defined there; the transversality condition implies that $f \circ \xi$ is monotonic on any neighborhood of λ_0, on which, therefore, we find λ's for which $\xi(\lambda)$ is not topologically equivalent to $\xi(\lambda_0)$.

Two preliminary lemmas **(1.2)**. — The following lemmas have a straightforward verification. We recall that, since J is manifold with boundary, $\{a, b\}$, ξ is transversal to Q if it is so when restricted to (a, b) and also when restricted to $\{a, b\}$ (i.e., $\xi(a), \xi(b) \notin Q$, if Q has codimension>0).

Lemma a). — *Let Q be an imbedded Banach submanifold of* $\mathfrak{X}^r$. *Call* $\Phi(Q)$ *the collection of* $\xi \in \Phi^r$ *such that*:

1) $\xi(J)$ *and* $\partial Q = (\text{Clos } Q) - Q$ *are disjoint, and*

2) ξ *is transversal to* Q.

Then $\Phi(Q)$ *is open in* Φ^r.

Lemma b). — *Call Φ_1^r the space of C^{r+1} mappings $\xi_1 : J\times M^2 \to T(M^2)$ such that $\pi(\xi_1(\lambda, p))=p$ for all λ, p; π stands for the projection of $T(M^2)$ onto M^2. Φ_1^r is endowed with the C^{r+1} topology. For $\xi_1\in\Phi_1^r$ define $\xi(\lambda)=\xi_1(\lambda, \ ) : M^2\to T(M^2)$. Then, $\xi\in\Phi^r$, and $\xi_1\mapsto\xi$ is a continuous linear mapping whose image is dense in Φ^r.*

Theorem 2. — *Assume $r\geq 4$. Call Γ^r the set of one-parameter families of vector fields ξ such that:*

1) $\xi(J)\subset[K\text{-}S]^r\cup\Sigma_1^r$.
2) ξ *is transversal to* Σ_1^r.
3) *The set of ordinary values of ξ is open and dense in J and coincides with $\xi^{-1}(\Sigma^r)$.*

Then Γ^r contains a Baire subset of Φ^r, in particular, Γ^r is dense in Φ^r.

2. Proof of Theorem 2.

The proof of Theorem 2 depends on several propositions.

Proposition **(2.1)**. — *Denote by $\Phi(Q_1)$ the set of $\xi\in\Phi^r$ such that:*

1) $\xi(J)$ *and* $\partial Q_1=(\text{Clos } Q_1)-Q_1$ *are disjoint and*
2) ξ *is transversal to* Q_1.

Then $\Phi(Q_1)$ is open and dense in Φ^r.

Proof. — The openness of $\Phi(Q_1)$ follows from (1.2) *a)*. Let $\xi\in\Phi^r$; we will show that it can be approximated by $\eta\in\Phi(Q_1)$; this will prove the density of $\Phi(Q_1)$. By (1.2) *b)*, we may assume that $\xi(\lambda)(x)=\xi_1(\lambda, x)$ for $\xi_1\in\Phi_1^r$. By density of transversality and density of Σ^r, we may assume that ξ_1 is transversal to M_0^2, the zero section of $T(M^2)$, and that $\xi(a), \xi(b)\in\Sigma^r$. $\xi_1^{-1}(M_0^2)=S(\xi_1)$ is a one-dimensional C^{r+1} submanifold of $J\times M^2$, which depends continuously on ξ_1 (in the C^{r+1} sense); $S(\xi_1)$ is transversal to $\{\lambda_0\}\times M^2$ at $(\lambda_0, p_0)\in S(\xi_1)$ if and only if p_0 is a simple singular point of $\xi(\lambda_0)$; since $\xi(a), \xi(b)\in\Sigma^r$, $S(\xi_1)$ is transversal to $\{a\}\times M^2$ and $\{b\}\times M^2$.

Let p_0 be a singular point of $\xi(\lambda_0)$, call $\xi_1^i(\lambda, x_1, x_2)$, $i=1, 2$, the components of ξ_1 in a coordinate system (x_1, x_2) around p_0.

ξ_1 is transversal to M_0^2 at (λ_0, p_0) if and only if the Jacobian matrix of $\xi_1^i(\lambda, x_1, x_2)$ has rank 2 at (λ_0, p_0). When p_0 is not a simple singular point of $\xi(\lambda_0)$, the coordinates (x_1, x_2) may be taken such that $x_1(p_0)=x_2(p_0)=0$ and the Jacobian matrix of $\xi_1^i(\lambda, x_1, x_2)$ has one of the following forms:

$$\textbf{(2.1.1)} \qquad \begin{pmatrix} c_1 & c & 0 \\ c_2 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} c_2 & 0 & 0 \\ c_1 & c & 0 \end{pmatrix}.$$

In either case there is a neighborhood $N_\delta=\{|\lambda-\lambda_0|<\delta, |x_i|<\delta\}$ such that $S(\xi_1)\cap N_\delta$ is given by $\lambda=\lambda(x_2)$, $x_1=x_1(x_2)$ for $|x_2|<\delta$, with $\dfrac{d\lambda}{dx_2}(0)=\dfrac{dx_1}{dx_2}(0)=0$. Thus, it may be assumed that $|\lambda(x_2)-\lambda_0|, |x_1(x_2)|<\delta_0$ for $|x_2|<\delta$, $\delta_0<\delta$.

Let ε be a regular value of $\lambda^1(x_2)$; let φ be a bump function: $\varphi=1$, for $|x_2|\leq\delta_0$; $\varphi=0$, for $|x_2|\geq\delta_1$; $\delta_0<\delta_1<\delta$. Define $\lambda_1(x_2)$ by $\lambda_1(x_2)=\lambda(x_2)-\varepsilon\varphi(x_2)x_2$; by Sard's Theorem, ε can be taken so small that $|\lambda_1(x_2)-\lambda_0|<\delta$ for $|x_2|<\delta$. Define $\xi_1^{(1)}\in\Phi_1^r$ by $\xi_1^{(1)}(\lambda, x_1, x_2)=\xi_1(\lambda+\varepsilon\varphi(x_2)x_2, x_1, x_2)$. For ε small, $\xi_1^{(1)}$ is C^{r+1} close to ξ_1; also, in $\operatorname{Clos} N_{\delta_0}$, $S(\xi_1^{(1)})$ is given by $\lambda=\lambda_1(x_2)$, $x_1=x_1(x_2)$, $x_2=x_2$; outside N_{δ_1}, $\xi_1^{(1)}=\xi_1$; thus, if $|x_2|\leq\delta_0$ and $\dfrac{d\lambda_1}{dx_2}(x_2)=0$, then $\dfrac{d^2\lambda_1}{dx_2^2}(x_2)\neq0$, by construction of λ_1. Therefore, in $\operatorname{Clos} N_{\delta_0}$, $S(\xi_1^{(1)})$ has only a finite number of non-simple singular points, one corresponding to each critical point of λ_1. Since these critical points are non-degenerate, this situation is not changed by small perturbations of $\xi_1^{(1)}$.

The set of non-simple singular points of ξ_1 is compact and can be covered by a finite number of neighborhoods $N_{\delta_0^1}, N_{\delta_0^2}, \ldots, N_{\delta_0^k}$ with $N_{\delta_0^i}\subset N_{\delta_1^i}\subset N_{\delta^i}$, $i=1, 2, \ldots, k$. As indicated above, we approximate ξ_1 by $\xi_1^{(1)}$ on $N_{\delta_1^1}$; then with the same criterion, we approximate $\xi_1^{(1)}$ by $\xi_1^{(2)}$ on $N_{\delta_1^2}$, without destroying the non-degeneracy conditions (which are open) already obtained in $\operatorname{Clos} N_{\delta_0^1}$; next we approximate $\xi_1^{(2)}$ by $\xi_1^{(3)}$ on $N_{\delta_1^3}$ without destroying what was already obtained in $\operatorname{Clos}\{N_{\delta_0^1}\cup N_{\delta_0^2}\}$, and so on. In this way we obtain $\xi_1^{(k)}$ with only a finite number of non-simple singular points (λ_1, p_1), (λ_2, p_2), $\ldots$, (λ_n, p_n), corresponding to the critical (non-degenerate) points of the projection of $S(\xi_1^{(k)})$ onto J. By further modification, if necessary, we get:

$$a<\lambda_1<\lambda_2<\ldots<\lambda_n<b.$$

Next we modify $\xi_1^{(k)}$ to make p_i a saddle-node of $\xi^{(k)}(\lambda_i)$; this is done by a small perturbation on the linear and quadratic terms of $\xi^{(k)}$ around p_i. Call η_1^0 the family thus obtained; $\eta^0(\lambda_i)$, $i=1, 2, \ldots, n$, has the saddle-node p_i as unique non-simple singular point. We approximate η_1^0 by $\eta_1^{(1)}$ which, at λ_i, satisfies condition (1) of Prop. (3.6), Part I; the other conditions, 2), 3), 4), of this proposition are obtained for $\eta^{(1)}(\lambda_i)$ by the approximation techniques introduced by M. Peixoto [8] to obtain the same conditions.

Thus, $\eta^{(1)}(\lambda_i)\in Q_1^1$; furthermore, at the saddle-nodes p_i of $\eta^{(1)}(\lambda_i)$, the transversality of $\eta_1^{(1)}$ to M_0^2 at (λ_i, p_i) is equivalent to the transversality of $\eta^{(1)}$ to the local submanifold associated to p_i and $\eta^{(1)}(\lambda_i)$ defined in (3.2), Part I; actually, c_2 of the first expression in (2.1.1) (which corresponds to saddle-nodes), is such that $df_X(V)=c_2\neq0$, for $X=\eta^{(1)}(\lambda_i)$, $V=\dfrac{\partial\eta^{(1)}}{\partial\lambda}(\lambda_i)$ where f is the function defined in (3.2), Part I.

Obviously, $\eta^{(1)}\in\Phi(Q_1^1)$. In fact, $\eta^{(1)}(\bar\lambda)\in\partial Q_1^1$, $\bar\lambda\neq\lambda_i$, implies that in every neighborhood of $\eta^1(\bar\lambda)$ there are fields of Q_1^1, which is not possible since for $\bar\lambda\neq\lambda_i$, $\eta^{(1)}(\bar\lambda)$ has only simple singular points (and this holds for fields in a neighborhood of $\eta^{(1)}(\bar\lambda)$).

Now we show how to approximate $\eta^{(1)}$ by $\eta\in\Phi(Q_1^2)$.

Let p_0 be a simple singular point of $\eta^{(1)}(\lambda_0)$ as in (3.10), Part I. Let K be a neighborhood of λ_0 such that $\eta^{(1)}(K)\subset B$. Let $P(\lambda)=P(\eta^{(1)}(\lambda))$, (see (3.10), Part I) and let $\sigma(\eta^{(1)})(\lambda)=\sigma(\eta^{(1)}(\lambda))$; obviously, $S(\eta^{(1)})\cap(K\times U)=\{(\lambda, P(\lambda)); \lambda\in K\}$.

Take neighborhoods $K_0^1 \subset K_0 \subset K$ of λ_0, and $U_0^1 \subset U_0 \subset U$ of p, such that:

$$\operatorname{Clos} K_0^1 \subset \operatorname{Int} K_0, \quad \operatorname{Clos} K_0 \subset \operatorname{Int} K, \quad \operatorname{Clos} U_0^1 \subset \operatorname{Int} U_0,$$
$$\operatorname{Clos} U_0 \subset \operatorname{Int} U, \quad P(K_0) \subset U_0^1.$$

Take bump functions $\nu, \varphi : \nu = 1$ on K_0^1, $\nu = 0$ outside K_0; $\varphi = 1$ on U_0^1 and $\varphi = 0$ outside U_0. Take coordinates (x_1, x_2) in U and define:

$$\delta(\lambda)(x_1, x_2) = -c(x_1 - P_1(\lambda))\varphi(x_1, x_2)\nu(\lambda),$$

where $P_1(\lambda)$ is the first coordinate of $P(\lambda)$, and c is a regular value of $\sigma(\eta^{(1)})$. For ε small, $\eta^{(2)} = \eta^{(1)} + \delta$ is close to $\eta^{(1)}$; also $\eta^{(2)} = \eta^{(1)}$ outside $K_0 \times U_0$, $S(\eta^{(2)}) = S(\eta^{(1)})$, and $\sigma(\eta^{(2)})(\lambda) = \sigma(\eta^{(1)})(\lambda) - \varepsilon$ for $\lambda \in \operatorname{Clos} K_0^1$.

Thus, when $\sigma(\eta^{(2)})(\lambda) = 0$, then $\sigma(\eta^{(2)})'(\lambda) \neq 0$, and $\eta^{(2)}$ has only a finite number of non-generic singular points on $S(\eta^{(2)}) \cap ((\operatorname{Clos} K_0^1) \times U_0^1)$; $\eta^{(2)} | \operatorname{Clos} K_0^1$ is transversal to the local submanifold $f = 0$ defined in (3.10), Part I.

The set of simple non-generic singular points of $\eta^{(1)}$ is compact and can be covered by a finite number of neighborhoods $K_0^1(i) \times U_0^1(i)$, $i = 1, 2, \ldots, m$, with the properties of $K_0^1 \times U_0^1$ above; obviously $K(i)$, $i = 1, 2, \ldots, m$, is disjoint from $(\eta^{(1)})^{-1}(Q_1^1)$. We approximate $\eta^{(1)}$ by $\eta^{(2)}$, as above, on $\operatorname{Clos} K_0^1(1) \times U_0^1(1)$; next, in the same fashion, we approximate $\eta^{(2)}$ by $\eta^{(3)}$ in $\operatorname{Clos} K_0^1(2) \times U_0^1(2)$ without breaking the transversality conditions (which are open) obtained in $K_0^1(1) \times U_0^1(1)$; we repeat this process on $K_0^1(3) \times U_0^1(3), \ldots, K_0^1(m) \times U_0^1(m)$ and obtain $\eta^{(m+1)}$ with finitely many non-generic simple singular points $(\lambda_1, p_1), (\lambda_2, p_2), \ldots, (\lambda_\ell, p_\ell)$, $\eta^{(m+1)}$ being transversal to the local submanifolds $f = 0$ associated to $p = p_i$ and $X = \eta^{(m+1)}(\lambda_i)$ of (3.10), Part I. After a further small modification, we may assume that $a < \lambda_1 < \lambda_2 < \ldots < \lambda_\ell < b$. Now we approximate $\eta^{(m+1)}$ by η which at λ_i has p_i as a composed-focus; this is done by a small change in the coefficients of the terms of second, third, and fourth order at p_i (see [21] for a coordinate expression of $\rho_X^{(3)}(0)$ defined in (3.11), Part I). Further modification leads to $\eta(\lambda_i) \in Q_1^2$; this is done as indicated above for the case of saddle-nodes, using the approximation techniques in [8] to obtain conditions 2), 3), and 4) of (3.12), Part I (condition 1) is already satisfied) for $\eta(\lambda_i)$. As in the case of Q_1^1, it follows that $\eta \in \Phi(Q_1^2) \cap \Phi(Q_1^1) = \Phi(Q_1^1 \cup Q_1^2) = \Phi(Q_1)$. This ends the proof of (2.1).

Remark **(2.1.1)**. — Call Q_1^0 the set of vector fields in $\mathfrak{X}^r$, which have non-generic singular points. Then Q_1 is dense in Q_1^0.

For instance, if p_0 is a non-generic singular point of $X \in Q_1^0$, we can find X_1 C^r-close to X_1 which has a quasi-generic singular point at p_0 as unique non-generic singular point; if p_0 is a saddle-node (resp. composed-focus) of X_1, there is an X_2, C^r-close to X_1, which belongs to Q_1^1 (resp. Q_1^2). This follows from arguments similar to those in the proof of (2.1), using [8].

Remark **(2.1.2)**. — If $\xi \in \Phi(Q_1)$, $\xi^{-1}(Q_1)$ has a finite number of points $\lambda_1, \ldots, \lambda_k$; we may assume that λ_i has a neighborhood K_i such that $\xi(K_i) \subset B_i$, a neighborhood of $X_i=\xi(\lambda_i)$ which, by (3.13), Part I, can be taken disjoint with $\mathfrak{X}_1^r - Q_1$. Thus, ξ has a neighborhood $\vartheta \subset \Phi(Q_1)$ such that every $\eta \in \vartheta$ is such that $\eta(K_i)\subset B_i$, and hence $\eta \mid K_i \in \Phi(Q_2(n)) \cap \Phi(Q_2'(n)) \cap \Phi(Q_3(n))$, for every $n=1, 2\ldots$ Therefore, to approximate ξ by $\eta \in \Phi(Q_2(n)) \cap \Phi(Q_2'(n)) \cap \Phi(Q_3(n))$, it is sufficient to do so on $J - \bigcup_i K_i \subset \bigcup_j J_j$, where J_j are closed intervals whose extremities are a, b, or the extremities of K_i. By continuity, every $\eta \in \vartheta$ can be assumed to have only generic singular points on J_j and to be structurally stable on the extremities of J_j.

Remark (**2.1.3**). — *a)* Let $\mathcal{O}$ be an open set of $\mathfrak{X}^r$ such that every $Y \in \mathcal{O}$ has only generic singular points. Call $\tau(Y)$ the minimum of the periods of all periodic trajectories of Y; $\tau(Y)=\infty$, if Y has no periodic trajectory.

It follows easily that τ is a positive lower semicontinuous function; see, for example, [6, p. 219].

b) Under the same hypothesis in *a)*, the minimum of the length of saddle separatrices (resp. connections) of $Y \in \mathcal{O}$, $\ell(Y)$ (resp. $\ell_1(Y)$), is a positive lower semicontinuous function, as follows from (4.5), Part I. Here we are assuming that a saddle separatrix whose α or ω-limit set is a generic node or focus has infinite length, and that $\ell(Y)$ (resp. $\ell_1(Y)$) is infinite when X has no saddle separatrix (resp. no saddle connection). Obviously, $\ell_1 \leq \ell$.

Proposition (**2.2**). — *$\Phi(Q_2(n))$ and $\Phi(Q_2'(n))$, defined as in* (2.1) *according to* (1.2) *a), are open and dense in Φ^r.*

The proof of this proposition depends on two preliminary results. Some notation is introduced first.

Assume that $\xi \in \Phi_1^r$; let γ be a periodic trajectory of period τ of $X=\xi(\lambda_0)$ and let $\pi: B_0 \times U_0 \to U$ be the mapping defined (in (2.1), Part I) in a neighborhood of $\{X\}\times\{p\}$, $p \in \gamma$. Suppose that $\varepsilon > 0$, a neighborhood N of γ, and a positive integer $\bar{n}$ are given; then B_0 and U_0 can be taken so that every arc of trajectory of $Y \in B_0$ joining $u \in U_0$ to $\pi_Y^{\bar{n}}(u) \in U$ spends a time within ε of $\tau\bar{n}$ and is contained in Int N. Take neighborhoods N_0' and N_0 of γ and $U_0' \subset U_0^2$ of p, such that $\operatorname{Clos} N_0' \subset \operatorname{Int} N_0$, $\operatorname{Clos} N_0 \subset \operatorname{Int} N$, $N_0' \cap U_0 = U_0'$, and $\operatorname{Clos} U_0' \subset \operatorname{Int} U_0^2 \subset U_0$; also take neighborhoods $K_0' \subset K_0 \subset K$ of λ_0 such that $\operatorname{Clos} K_0^1 \subset \operatorname{Int} K_0$, $\operatorname{Clos} K_0 \subset \operatorname{Int} K$, and $\xi(K) \subset B_0$. Define $\pi_\xi : K \times U_0 \to U$ by $\pi_\xi(\lambda, u) = \pi(\xi(\lambda), u)$.

Lemma (**2.2.1**). — *Assume the notation above.*

a) *If γ is two-sided, π_ξ can be C^{r+1} approximated by π_1 such that $\pi_1=\pi_\xi$ outside $K_0 \times U_0^2$, $\pi_1(\lambda, u) - u$ restricted to $\operatorname{Clos} K_0' \times U_0'$ has zero as regular value, and when $\pi_1(\lambda, u)=u$ and $\dfrac{\partial \pi_1}{\partial u}(\lambda, u)=1$, then $\dfrac{\partial^2 \pi_1}{\partial u^2}(\lambda, u) \neq 0$.*

b) *If γ is one-sided, we may assume that $\xi(\lambda)$, $\lambda \in K$ has only one one-sided periodic trajectory meeting U_0' at $a(\lambda)$ $(=k(\xi(\lambda))$ of* (2.6), *Part I*). *π_ξ can be C^{r+1} approximated by π_1 such* *that* $\pi_1(\lambda, u) = u$ *only for* $u = a(\lambda)$, $\pi_1 = \pi_\xi$ *outside* $K_0 \times U_0^2$, *and at every* $\lambda \in \operatorname{Clos} K_0'$ *with* $\dfrac{\partial \pi_1}{\partial u}(\lambda, u) = -1$ *at* $u = a(\lambda)$, *then* $\dfrac{\partial^2 \pi_1}{\partial \lambda \, \partial u}(\lambda, u) \neq 0$ *and* $\dfrac{\partial^3 \pi_1}{\partial u^3}(\lambda, \pi_1(\lambda, u)) \neq 0$ *at* $u = a(\lambda)$.

Proof. — Follows from Sard's Theorem.

Lemma **(2.2.2)** (Kupka). — *In the space of* C^{r+1} *functions from* $K_0 \times U_0$ *to* U, *with the* C^{r+1} *topology, there is a neighborhood* V *of* π_ξ *where a continuous* Φ_1^r*-valued mapping* $\pi \to \xi_\pi$ *is defined so that* $\pi_{\xi_\pi} = \pi$ *in* $\operatorname{Clos} K_0' \times U_0'$, $\pi_{\xi_\pi} = \pi_\xi$ *outside* $K_0 \times U_0^2$ *and* $\xi_\pi = \xi$ *outside* $K_0 \times N_0$.

Proof. — Similar to [12, p. 464].

Proof of (2.2). — Given $\xi \in \Phi^r$, we will approximate it by $\eta \in \Phi(Q_2(n)) \cap \Phi(Q_2'(n))$. We may assume that ξ has only generic singular points and that $\xi(a), \xi(b) \in \Sigma^r$, by Remark (2.1.2); also, we may assume that every periodic trajectory of ξ has period greater than $\tau_0 > 0$, by Remark (2.1.3), *a)*. Call $P(n)$ the following set:

$\{(\lambda, p) \in J \times M^2$, such that $\xi(\lambda)$ has a non-generic periodic trajectory γ of period $\leq n$ through $p\}$.

$P(n)$ is a compact subset contained in $\operatorname{Int} J \times M^2$; the subset $P_1(n) \subset P(n)$ of points for which γ is one-sided, is also compact.

First we will approximate ξ by $\bar{\eta} \in \Phi(Q_2'(n))$. For $\lambda_0 \in J$, $\xi(\lambda_0)$ has at most a finite number of one-sided periodic trajectories $\gamma_1, \gamma_2, \ldots, \gamma_k$. Take $\bar{n} = 2$, $\varepsilon < \tau_0$, and $N(i) = N(\gamma_i)$ disjoint neighborhoods of γ_i, $i = 1, 2, \ldots, k$. B_0 is taken as above with the additional conditions that every periodic trajectory of $Y \in B_0$ has period $> \tau_0$ (2.1.3), *a)*, and that, on $M^2 - \bigcup_i N_0'(i)$, $Y \in B_0$ has only either periodic trajectories of period $> n$ or two-sided periodic trajectories of period $\leq n$. $K = K(\lambda_0)$, $K_0' = K_0'(\lambda_0)$, etc., are taken as above. Take a finite covering $K_0'(\lambda_1), K_0'(\lambda_2), \ldots, K_0'(\lambda_m)$ of the projection of $P_1(n)$ on J, and take $N_0'(\lambda_i)(1), N_0'(\lambda_i)(2), \ldots, N_0'(\lambda_i)(k_i)$, the corresponding neighborhoods of the one-sided periodic trajectories of $\xi(\lambda_i)$. On each $K_0(\lambda_1) \times N_0(\lambda_1)(i)$, $i = 1, 2, \ldots, k_1$, approximate ξ (using (2.2.2)) by $\bar{\eta}_1$ such that $\pi_{\bar{\eta}_1} = \pi_1$ of (2.2.1), *b)* on $K_0(\lambda_1) \times U_0^2(i)$ and $\bar{\eta}_1 = \xi$ outside $K_0(\lambda_1) \times (\bigcup_i N_0(\lambda_1)(i))$. Then, with the same criterion, approximate $\bar{\eta}_1$ by $\bar{\eta}_2$ on each $K_0(\lambda_2) \times N_0(\lambda_2)(i)$, $i = 1, 2, \ldots, k_2$, without breaking the regularity conditions (which are open) obtained for $\pi_{\bar{\eta}_1}$ on $\operatorname{Clos} K_0'(\lambda_1)$. Iterating this procedure for $K_0(\lambda_3), K_0(\lambda_4), \ldots, K_0(\lambda_m)$, we obtain $\bar{\eta}_m = \bar{\eta}$ which has finitely many non-generic one-sided periodic trajectories of period $\leq n$, which are quasi-generic; furthermore, if $\bar{\eta}(\lambda^i)$, $i = 1, 2, \ldots, k$, has one such trajectory γ^i, $\bar{\eta}$ is transversal at γ^i to the local manifold $f = 0$ defined in (2.6), Part I, associated to $X = \bar{\eta}(\lambda^i)$ and $\gamma_X = \gamma^i$; thus we may assume (after a small change, if necessary) that $a < \lambda^1 < \lambda^2 < \ldots < \lambda^k < b$ and that $\bar{\eta}(i)$ has γ^i as unique quasi-generic periodic trajectory, of period $\leq n$. By a further small change on $\bar{\eta}$ to $(1 + \Theta)\bar{\eta}$, we get period $\gamma^i < n$; if $\Theta > 0$ is small, no new non-generic one-sided periodic trajectory of period $\leq n$ is created. Finally, the approximation techniques of [7] lead to $\bar{\eta}$ satisfying 1), 2), 3) of (2.2), Part I, at λ^i, $i=1, 2, \ldots, k$. Obviously $\bar{\eta}\in\Phi(Q'_2(n))$.

It follows from (2.7), Part I, that there are neighborhoods J^i of λ^i such that $\bar{\eta}|J^i\in\Phi(Q_2(n))$; hence it is sufficient to approximate $\bar{\eta}$ by $\eta\in\Phi(Q_2(n))$ on intervals $\bar{J}$ contained in the complement of $\bigcup_i J^i$, where $\bar{\eta}$ has only generic one-sided periodic trajectories of period $\leq n$. Take now $\bar{n}>n/\tau_0$, and $\varepsilon<\tau_0\bar{n}-n$. For $\lambda_0\in\bar{J}$, consider a finite covering $N'_0(\gamma_i)$ of the compact set of two-sided periodic trajectories of $\bar{\eta}(\lambda_0)$ of period $\leq n$; the neighborhoods $N'_0(\gamma_i)$ are taken as at the beginning of this section. B_0 is taken so that $Y\in B_0$ has only periodic trajectories of period $>\tau_0$ and, through $M^2-\bigcup_i N'_0(\gamma_i)$, Y has only periodic trajectories of period $>n$ or generic one-sided periodic trajectories of period $\leq n$; $K_0=K(\lambda_0)$, $K'_0=K'_0(\lambda_0)$, etc., are taken as above. Take a finite covering $K'_0(\lambda_1)$, $K'(\lambda_2)$, $\ldots$, $K'_0(\lambda_m)$ of the projection of $P(n)$ into J. For $K_0(\lambda_i)$, $i=1, \ldots, m$, take the corresponding neighborhoods $N'_0(\gamma_1)(i)$, $N'_0(\gamma_2)(i)$, $\ldots$, $N'_0(\gamma_{k_i})(i)$ which cover the two-sided periodic trajectories of $\bar{\eta}(\lambda_i)$ of period $\leq n$. Start with $K_0(\lambda_1)$. On $K_0(\lambda)\times N(\gamma_1)(1)$ approximate $\bar{\eta}$ (using (2.2.2)) by $\bar{\eta}_1$ such that $\pi_{\bar{\eta}_1}=\pi_1$ of (2.2.1), *a)* on $K_0(\lambda_1)\times U^2_0(1)$, and $\bar{\eta}_1=\bar{\eta}$ outside $K_0(\lambda_1)\times N_0(\gamma_1)(1)$; then with the same criterion, approximate $\bar{\eta}_1$ by $\bar{\eta}_2$ on $K_0(\lambda_1)\times N_0(\lambda_2)(1)$, without breaking the regularity conditions obtained for $\pi_{\bar{\eta}_1}$ on $\operatorname{Clos} K'_0(\lambda_1)\times N_0(\gamma_2)(1)$, and so on for $K_0(\lambda_1)\times N_0(\gamma_i)(1)$, $i=3, 4, \ldots, k_1$, and afterwards for $K_0(\lambda_2)$, $K(\lambda_3)$, $\ldots$, $K_0(\lambda_m)$, thus obtaining η which has finitely many non-generic periodic trajectories of period $\leq n$, which are two-sided and quasi-generic. This last assertion can be shown as follows. Every two-sided non-generic periodic trajectory of period $\leq n$ of η must be contained, for some i, in $\bigcup_j K'_0(\lambda_i)\times N'_0(\gamma_j)(i)$ (otherwise it would have period $>n$) and therefore corresponds to a fixed point of $\pi_\eta=\pi_1$ on $K'_0(\lambda_i)\times U'_0(j)$ otherwise, since π_1 has no periodic points of period >1, for it is orientation preserving, it would contain a simple arc which spends a time greater than:

$$n.(\text{period } \gamma_j)-\varepsilon>n\tau_0-\varepsilon>n$$

and hence it would have period greater than n.

Now, further small modifications of η (which is also transversal to the local manifolds $f=0$ of (2.4), Part I) similar to those indicated above for $\bar{\eta}$, lead to $\eta\in\Phi(Q_2(n))$ on $\bar{J}$ and therefore on J. Thus $\eta\in\Phi(Q_2(n))\cap\Phi(Q'_2(n))$ and approximates ξ.

Remark **(2.2.3).** — Call Q^0_2 the set of vector fields $X\in\mathfrak{X}^r_1$ which have non-generic periodic trajectories.

Approximation arguments similar to those used in the proof of (2.2) show that Q_2 (defined in (2.2), Part I) is dense in Q^0_2.

If X has one non-generic periodic trajectory γ, we first make it quasi-generic for X_1 C^r-close to X, using adequate versions of (2.2.1) and (2.2.2). Then we use the approximation techniques of [8] to get $X_2\in Q_2$, C^r-close to X_1, with γ as quasi-generic periodic trajectory.

Proposition **(2.3)**. — $\Phi(Q_3(n))$, *defined as in* (2.1) *according to* (1.2), *a), is open and dense in* Φ^r.

Proof. — Openness is obvious, by (1.1), *a)*; we prove density. Let $\xi\in\Phi^r$; by (1.1), *b)* and Remark (2.1.2) we may assume that $\xi\in\Phi_1^r$ and that all its singular points are generic. Also we assume that $\xi(a), \xi(b)\in\Sigma^r$.

Let $m>0$ be less than the length of any saddle separatrix of $\xi(\lambda)$, $\lambda\in J$; the existence of m follows from (2.1.3), *b)*.

Let $A(\ell)=\{\lambda\in J;\ \xi(\lambda)$ has some saddle connection with length $\leq\ell\}$. $A(\ell)\subset \operatorname{Int} J$ is compact. For $\lambda_0\in A(\ell)$, let $\{\gamma_i\}$ be the saddle connections with length $\leq\ell$ of $X=\xi(\lambda_0)$; for γ_i consider the neighborhoods N_1^i, N_2^i and B_i, of the saddle points connected by γ_i and $X=\xi(\lambda_0)$, so that $f^i(Y)=\pi_Y^i(u^i(Y))-s^i(Y)$ is defined for $Y\in B_i$ by (4.4), Part I; $\pi_Y^i : U^i\subset \partial N_1^i\to S^i\subset\partial N_2^i$.

Also consider neighborhoods N'_{0i}, N_{0i}, N_i of the arcs of γ^i joining $u^i(X)$ to $s^i(X)$; assume that $\operatorname{Clos} N'_{0i}\subset \operatorname{Int} N_{0i}$, $\operatorname{Clos} N_{0i}\subset N^i$, and that the arcs of separatrices $\overrightarrow{u'(Y)\pi_Y^i(u(Y))}$ and $\overrightarrow{(\pi_Y^i)^{-1}s^i(Y)s^i(Y)}$ are contained in $\operatorname{Int} N'_{0i}$, for $Y\in B_i$. The N_i's are taken disjoint. Take neighborhoods $K'_0(\lambda_0)$, $K_0(\lambda_0)$, $K(\lambda_0)$ of λ, with $\operatorname{Clos} K'_0(\lambda_0)\subset \operatorname{Int} K_0(\lambda_0)$ and $\operatorname{Clos} K_0(\lambda_0)\subset \operatorname{Int} K(\lambda_0)$ such that $\xi(K(\lambda_0))\subset B\subset\bigcap_i B_i$. Assume that B is such that all saddle separatrices of $Y\in B$, different from those through $u^i(Y)$, $s^i(Y)$, have length greater than ℓ, and also that the saddle separatrices through $s^i(Y)$, $u^i(Y)$ for $f^i(Y)\neq 0$ have length greater than ℓ_i+m, where $\ell_i=\text{length}\,\gamma^i$. See (4.5), Part I, and (2.1.3), *b)*.

The $K'_0(\lambda_0)$'s form an open covering of $A(\ell)$; select a finite subcovering $K'_0(\lambda_1)$, $K'_0(\lambda_2)$, ..., $K'_0(\lambda_n)$.

Call $\gamma_1^j, \gamma_2^j, \ldots, \gamma_{n_j}^j$ the saddle connections of $\xi(\lambda_j)$, with length $\leq\ell$. Take $K(\lambda_1)$ and γ_i^1 and approximate ξ by $\xi^{(1)}$ such that $\xi^{(1)}(K(\lambda_1))\subset B$, $\xi^{(1)}=\xi$ outside $K_0(\lambda_1)\times(\bigcup_i N_{0i})$ and such that zero is a regular value of $f^i(\xi^{(1)}(\lambda))$, for $\lambda\in\operatorname{Clos} K_0^1(\lambda_1)$. This is achieved by a procedure similar to that described in the proof of (2.1), using here a version of (2.2.2) suited for saddle connections [6, p. 221]. For $K(\lambda_2)$ approximate $\xi^{(1)}$ by $\xi^{(2)}$ as above, taking care not to destroy the regularity conditions (which are open) obtained in $\operatorname{Clos} K'_0(\lambda_1)$. Do the same for $\lambda_3, \ldots, \lambda_n$ and obtain $\xi^{(n)}=\eta$.

Start with $\ell=3m/2$, then η obtained above has a finite number of λ's:

$$a<\bar\lambda_1<\ldots<\bar\lambda_{\bar n}<b,$$

such that, after a small change on η, $\eta(\bar\lambda_i)$ has only one saddle connection γ^i with length $\leq\ell$; each one corresponding to a zero of $f^i(\eta(\lambda))$; hence η is transversal at $\bar\lambda_i$ to the local manifolds $f^i=0$ defined in 4, Part I. Notice that there may be other saddle connections $\bar\gamma$ for $\eta(\bar\lambda_i)$ but, by construction of η, they will have length greater than:

$$m+\text{length}\,\bar\gamma>2m>\ell.$$

By a further small change, we may assume that $\eta(\bar\lambda_i)\in Q_3(\ell)$. This, as an (2.1) and (2.3), is achieved by the approximation techniques in [8]. Now, all the saddle separatrices of $\eta(\lambda)$, $\lambda \neq \bar{\lambda}_i$, have length greater than $3m/2$. Each $\bar{\lambda}_i$ has a neighborhood J_i such that $\eta(\lambda)$, for $\lambda \in J_i$, $\lambda \neq \bar{\lambda}_i$, has only saddle connections (if any) with length greater than n, (4.7.1), Part I. Hence, it is sufficient to approximate η restricted to the complement of $\bigcup_i J_i$, where all saddle separatrices have length greater than $m_1 = 3m/2$.

Repeat the above procedure for each of the intervals J on the complement of $\bigcup_i J_i$, now for $\ell = 3m_1/2 = (3/2)^2 m$, and so on. Thus after $k-1$ steps we obtain $\ell = (3/2)^k m > n$, for k big enough. It follows as in (2.1) and (2.3) that the one parameter family thus obtained belongs to $\Phi(Q_3(n))$.

Remark **(2.3.1)**. — Call Q_3^0 the set of vector fields $X \in \mathfrak{X}_1^r$ which have saddle connections or non trivial recurrent orbits and all its singular points and periodic orbits are generic. The set $Q_2 \cup Q_3$ (defined in (2.2) and (4.2), Part I) is dense in Q_3^0, as follows from arguments similar to those employed to prove (2.3). In fact, if $X \in Q_3^0$ has a saddle connection γ, it is C^r-approximated by $X_1 \in Q_3$ that have the same saddle connection, which is a simple loop in case γ is a loop. This is done by a local perturbation of X around the saddle points. If X has a recurrent orbit, it is approximated by X_1, C^r-close to it, which has either a saddle connection, if X has some recurrent saddle separatrix, or a quasi-generic periodic orbit, if X has none. The first alternative follows from the " closing lemma " in [8, p. 114]; the second happens only if $M^2 = \mathbf{T}^2$ (torus) and X has no singular point.

The first case was treated just above, the second is handled as follows.

There is a cycle S^1 transversal to every Y in a small ball V centered at X. Let $Y_1 \in \Sigma^r \cap V$ and call $\rho(s)$ the rotation number of $X(s) = sY_1 + (1-s)X$, relative to S^1. Notice that $\rho(0)$ is irrational and $\rho(1)$ is rational. Call s_1 the g.l.b. of:

$$\{s \in [0, 1];\ \rho([s, 1]) = \rho(1)\}.$$

Clearly $0 < s_1 < 1$.

Since ρ is continuous $\rho(s_1) = \rho(1)$ is rational, and $X(s_1)$ has periodic orbits. These orbits are necessarily non generic, otherwise for all small non negative ε, $X(s_1 - \varepsilon)$ will have generic periodic orbits and $\rho(s_1 - \varepsilon)$ will also be equal to $\rho(1)$. Contradiction. Now we approximate $X(s_1)$ by $X_1 \in V \cap Q_2$, according to Remark (2.2.3).

Notice that $\mathfrak{X}_1^r = Q_1^0 \cup Q_2^0 \cup Q_3^0$. Remarks (2.1.1), (2.2.3), (2.3.1) indicate how to approximate fields in $\mathfrak{X}_1^r$ by fields in $\Sigma_1^r = Q_1 \cup Q_2 \cup Q_3$.

Proof of Theorem 2. — Take a countable dense set of J, $\{a_i\}$, $i \in \mathbf{N}$, which contains the extremes a, b; call $\Phi(a_i)$ the set $\{\xi \in \Phi^r;\ \xi(a_i) \in \Sigma^r\}$. $\Phi(a_i)$ is open and dense in Φ; $\Phi(S_j) = \Phi(Q_1) \cap \Phi(Q_2(j)) \cap \Phi(Q_3(j))$ is also open and dense in Φ^r, by (2.1), (2.2), (2.3). Thus $\mathscr{B} = \bigcap_{i,j} (\Phi(a_i) \cap \Phi(S_j))$ is a Baire set; we show that $\mathscr{B} \subset \Gamma^r$. In fact, if $\xi \in \mathscr{B}$, $\xi^{-1}(\Sigma^r)$ is open and dense, since it contains $\{a_i\}$ and ξ is transversal to Σ_1^r; this proves that ξ satisfies 2) and part of 3) of Theorem 2. We show that it satisfies 1); in fact, if $\lambda \notin \xi^{-1}(\Sigma^r)$ and $\xi(\lambda) \notin [K\text{-}S]^r$, $\xi(\lambda)$ has a non-generic singular point, a non-generic periodic trajectory, or a saddle connection and then $\xi(\lambda) \in \Sigma_1^r$. To complete the proof that ξ satisfies 3), it is sufficient to observe that every $\lambda_0 \notin \xi^{-1}(\Sigma^r)$ is a bifurcation value; if $\xi(\lambda_0) \in \Sigma_1^r$ this is obvious by (1.1); if $\xi(\lambda_0) \in [\text{K-S}]^r$, it has a non trivial recurrent trajectory and can not be topologically equivalent to $\xi(a_i)$ for a_i close to λ_0.

3. Structural Stability.

In this section we formulate the concept of structural stability for vector fields depending on a parameter, and state some related conjectures.

Definition (3.1). — *a)* $\xi, \eta \in \Phi^r$ are said to be *topologically equivalent* if there is a homeomorphism $h : J \to J$ and a continuous family of homeomorphisms, $H : J \to \text{Hom}\, M^2$, of M^2 such that for very $\lambda \in J$, $H(\lambda)$ is a topological equivalence between $\xi(\lambda)$ and $\eta(h(\lambda))$.

b) $\xi \in \Phi^r$ is *structurally stable* if it has a neighborhood N such that ξ is topologically equivalent to every $\eta \in N$.

Obviously this definition makes sense when J is any manifold. When $J = \{a\}$, a point, this definition reduces to plain structural stability, (1.3), Part I.

Also we may require that N be such that h and H be ε-close to the identity (of J and M^2 respectively), for ε given beforehand.

Call $\Sigma(J)$ the set of structurally stable elements of Φ^r.

It seems quite possible to show that $\Sigma(J) \subset \Gamma^r$. Also that $\Gamma_1 \subset \Sigma(J)$, where $\Gamma_1 = \{\xi \in \Gamma^r;\ \xi(J) \subset \Sigma^r \cup \widetilde{\Sigma}_1^r\}$.

More delicate questions are the following:

a) Prove that $\Gamma_2 \cap \Sigma(J)$ is open in Φ^r and dense in $\Gamma_2 = \{\xi \in \Gamma^r;\ \xi(J) \subset \Sigma^r \cup \Sigma_1^r\}$.

b) Prove (or disprove) that there are elements $\xi \in \Sigma(J)$ such that:

$$\xi(J) \cap \{[\text{K-S}]^r - \Sigma^r\} \neq \emptyset.$$

c) Characterize $\Sigma(J)$. Is it dense in Φ^r?

An answer for *b)* and *c)* should require a deep understanding of the " generic " type of non trivial recurrent orbits and of the " part of codimension one " of $[\text{K-S}]^r - \Sigma^r$. A basic question in this direction is if Q_ρ, the set of vector fields in $\mathbf{T}^2$ without singularities and irrational rotation number ρ, contains an open dense manifold of codimension one.

REFERENCES

[1] G. F. Duff, Limit cycles and vector fields, *Ann. of Math.*, vol. 57 (1953), 15-31.

[2] M. Urabe, Infinitesimal deformation of cycles, *Jour. Sci. Hiroshima University*, 1954, 37-53.

[3] L. Marcus and A. Aeppli, Integral equivalence of vector fields on manifolds and bifurcation of differential systems, *American Journal of Math.*, vol. 85 (1963), 633-654.

[4] R. Sacker, *Invariant surfaces and bifurcation of periodic solutions of ordinary differential equations*, New York University, Courant Inst. Math. Sci. Tech. Rep. IMM-NYU 333, 1964.

[5] M. C. Peixoto and M. Peixoto, Structural Stability in the plane with enlarged boundary conditions, *Ann. Acad. Bras. Sci.*, vol. 81 (1959), 135-160.

[6] M. Peixoto, An approximation theorem of Kupka and Smale, *Jour. of Diff. Eq.*, vol. 3 (1967), 214-227.

[7] M. Peixoto, On structural stability, *Ann. of Math.*, vol. 69 (1959), 199-222.

[8] M. Peixoto, Structural stability on two dimensional manifolds, *Topology*, vol. 1 (1962), 101-120.

[9] S. Smale, On dynamical Systems, *Boletín de la Soc. Mat. Mexicana*, vol. 66 (1960), 195-198.

[10] S. Smale, Differentiable dynamical Systems, *BAMS*, vol. 73 (1967), 747-817.

[11] S. Smale, Stable manifolds for differential equations and diffeomorphisms, *Annali Scuola Normale de Pisa*, serie III, XVII (1963), 97-116.

[12] I. Kupka, Contribution à la théorie des champs génériques, *Cont. to Diff. Eq.*, vol. 2 (1963), 451-487.

[13] A. Andronov and E. Leontovich, Sur la théorie de la variation de la structure qualitative de la division du plan en trajectoires, *Dokl. Akad. Nauk.*, vol. 21 (1938), 427-430.

[14] J. Sotomayor, *Estabilidade estrutural de primeira ordem em variedades de Banach*, Thesis, Rio de Janeiro, IMPA, 1964.

[15] S. Lang, *Introduction to differentiable manifolds*, New York, Interscience, 1962.

[16] R. Abraham, *Transversal mappings and flows*, New York, Benjamin, 1967.

[17] G. Sansone and R. Conti, *Equazione differentiali non lineari*, Rome, Ed. Cremonese, 1956.

[18] E. Coddington and N. Levinson, *Theory of ordinary differential equations*, New York, McGraw Hill, 1955.

[19] P. Hartman, *Ordinary differential equations*, New York, John Wiley, 1964.

[20] C. Pugh, The closing lemma, *Amer. J. Math.*, vol. 89 (1967), 956-1009.

[21] W. S. Loud, Behavior of the period of solutions of certain plane autonomous systems near centers, *Contr. to Diff. Eq.*, vol. 3 (1963), 21-36.

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