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EXPANDING ATTRACTORS

by R. F. WILLIAMS

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INTRODUCTION

The purpose of this paper is to present complete proofs of the results toward characterizing attractors with hyperbolic structure (see Smale [9]), as announced in [13] and [14]. The crucial additional assumption (beyond hyperbolic structure) is that the attractor is *expanding* (that is, its set-theoretic dimension is equal to the dimension of the fiber of the unstable bundle). One other (technical) assumption is made that

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the stable manifold foliation $\{W^s(x);\ x$ is in the attractor$\}$ is of class C^r, $r \geq 1$. We expect that this assumption can be dropped with additional work.

Thus this paper carries out a portion of the program we felt was possible in [15]. In some sections (e.g. the one on periodic points) more detailed proofs are given than in [15]. In turn, this is a part of a large program evolving from the work of many authors, with S. Smale being the prime mover. See the important survey paper of Smale [9], and the volume [10] on Global Analysis.

It is a pleasure to acknowledge the help and encouragement given by many colleagues, including A. Haefliger, M. Hirsch, S. Narasimhan, S. Smale, R. Thom, C. Robinson, S. Newhouse, G. Reeb, H. Rosenberg, and M. Shub.

BASIC CONCEPTS

Suppose $f \in \mathrm{Diff}^r(M)$, $r \geq 1$ where M is a compact manifold.

Definition. — For an endomorphism $f : M \to M$ one says that $x \in M$ is a *non-wandering point* of f (notation: $x \in \Omega(f)$) provided that for each neighborhood N of x there is a positive integer n such that $f^n(N) \cap N \neq \emptyset$. For $f \in \mathrm{Diff}^r(M)$, $x \in M$ define the *(generalized) stable* [resp. *unstable*] *manifold at f relative to x*, $W^s(x,f)$, [resp. $W^u(x,f)$], by

$$W^s(x,f) = \{y \in M : \lim_{n \to \infty} \mathrm{dist}(f^n x, f^n y) = 0\}$$
$$W^u(x,f) = W^s(x, f^{-1}).$$

A closed invariant subset Λ of M has a *hyperbolic structure* $E^u + E^s$ provided the tangent bundle TM restricted to Λ splits as a direct sum, $TM|\Lambda = E^u + E^s$, which is invariant under the derivative Tf of f and such that $Tf|E^u$ is an expansion and $Tf|E^s$ is a contraction. This last means that there are numbers $A, B > 0$ and $\mu > 1$ such that $|Tf^n(v)| \leq A\mu^n|v|$ for $v \in E^u$ and $n > 0$ and $|Tf^n(w)| \leq B\mu^{-n}|w|$ for $w \in E^s$ and $n > 0$. The symbols u, s are also used to denote the dimensions of the fibers of the bundles E^u, E^s.

Definition. — A subset $\Lambda \subset M$ is an *attractor for f* provided there is a closed neighborhood N of Λ such that:

1) $f(N) \subset \mathrm{Int}\, N$;
2) $\Lambda = \bigcap_{i \geq 0} f^i(N)$; and
3) $\Lambda = \Omega(f|N)$.

A *hyperbolic attractor* satisfies 1)-3) plus

4) Λ has a hyperbolic structure, $E^u + E^s$.

An *expanding attractor* satisfies 1)-4) plus

5) $\dim \Lambda = u$, the dimension of a fibre of E^u.

Now assume that Λ is an attractor for f with a hyperbolic structure and use the notations above. We formulate a *part* of the Generalized stable manifold theorem as formulated by Smale [9] and proved by Hirsch-Pugh [5], with addendum by Hirsch-Palis-Pugh-Shub [4].

Generalized Stable Manifold Theorem (Smale, Hirsch-Pugh). — *For* $x \in \Lambda$, $W^s(x,f)$, $W^u(x,f)$ *are injectively immersed euclidean spaces of dimensions s and u.* $\{W^s(x,f); x \in \Lambda\}$ *is a foliation of a neighborhood of* Λ *(at times* C^1 *as a foliation)*

$$f(W^s(x,f)) = W^s(fx,f) \quad \textit{and} \quad \Lambda = \Omega(f|N).$$

We proceed to define branched-n-manifolds in § 1 and n-solenoids Σ as the inverse limits of a sequence

$$K \underset{g}{\leftarrow} K \underset{g}{\leftarrow} K \underset{g}{\leftarrow} \dots$$

in which the same map $g : K \to K$ is used repeatedly. g is an *immersion* of the branched n-manifold K and satisfies certain axioms. There is the coordinate shift map $h : \Sigma \to \Sigma$. These concepts generalize those of [15]. It is interesting to note that Alekseev's definition of " topological Markov chain " [0; p. 104] is general enough to include such inverse limits.

STATEMENT OF RESULTS

Theorem A. — *Assume* Λ *is an expanding attractor for* $f \in \mathrm{Diff}(M)$ *and that the foliation* $\{W^s(x,f) : x \in \Lambda\}$ *is* C^1 *on some neighborhood of* Λ. *Then* $f|\Lambda$ *is conjugate to the shift map h of an n-solenoid* Σ, i.e. *there is a homeomorphism* $\varphi : \Lambda \to \Sigma$ *such that* $f|\Lambda = \varphi^{-1} h \varphi$.

Theorem B. — *Assume* $h : \Sigma \to \Sigma$ *is a shift map of an n-solenoid. Then there is a manifold* M *and* $f \in \mathrm{Diff}^r(M)$ *having an expanding attractor* Λ *such that* $f|\Lambda$ *is conjugate to h.*

Theorem C. — *Each point of an n-solenoid has a neighborhood of the form (Cantor set)* $\times$ *(n-disk).*

Theorem D. — *The periodic points of a shift map of an n-solenoid are dense.*

Theorem E. — *The reduced cohomology over* **Z** *(or even* **R***) of an n-solenoid is not* 0.

This last is all the author has proved at this time toward a very strong

Conjecture. — $H^n(\Sigma; \mathbf{Z}) \neq 0$, $H^1(\Sigma; \mathbf{Z}) \neq 0$, and $H^1(\Sigma; \mathbf{Z})$ generates $H^*(\Sigma; \mathbf{Z})$ as a ring. If orientable Σ is a fiber bundle over an n-manifold with Cantor set as fiber.

§ 1. BRANCHED N-MANIFOLDS : DEFINITIONS, EXAMPLES AND ELEMENTARY PROPERTIES

Smooth branched n-manifolds are perhaps simplest imagined as " complexes " embedded in some higher dimensional Euclidean space, in such manner that there is a unique tangent n-plane at each point. See figures 1 and 2. We also need an abstract definition, as our branched n-manifolds arise quite naturally as quotients of foliations (see figures 5 and 6), but with no given embedding.

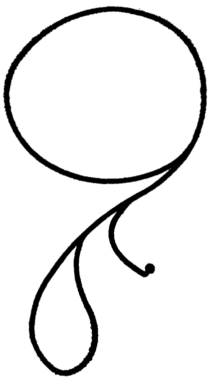


FIG. 1

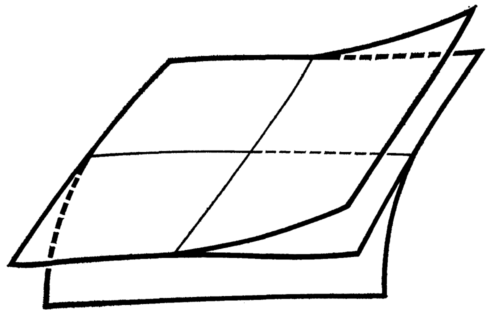


FIG. 2

Note that the " local chart " U_1 in figure 2 can be described as the union of three overlapping 2-disks: a flat one in the middle D_{12}, one branching up to the right D_{11}, and the third down in a forward direction D_{13}. The vertical projection of this chart onto a horizontal $\mathbf{R}^2$ would map each of these 2-disks diffeomorphically onto a square 2-disk D_1 in $\mathbf{R}^2$. (The corners are just for easy visualization). With this in mind, we give a preliminary version of our definition.

Definition (**1.0** ns). — By a *non-singular branched n-manifold* of class C^k is meant a metrizable space K together with:

(i) a collection $\{U_i\}$ of closed subsets of K;

(ii) for each U_i a finite collection $\{D_{ij}\}$ of closed subsets of U_i; and

(iii) for each i, a map $\pi_i : U_i \to D_i^n$, D_i^n a closed, n-disk of class C^k in $\mathbf{R}^n$;

subject to the following axioms:

a) $\bigcup_j D_{ij} = U_i$ and $\bigcup_i \operatorname{Int} U_i = K$;

*b*ns) $\pi_i | D_{ij}$ is a homeomorphism onto D_i^n;

c) there is a " cocycle " of diffeomorphisms $\{\alpha_{i'i}\}$ of class C^k such that $\pi_{i'} = \alpha_{i'i} \circ \pi_i$ when defined. The domain of $\alpha_{i'i}$ is $\pi_i(U_i \cap U_{i'})$.

Note. — The $\alpha_{i'i}$ satisfy the standard identities $\alpha_{i''i'}\alpha_{i'i} = \alpha_{i''i}$ and $\alpha_{ii} = 1$, as one can verify.

Note that this definition allows locally infinite branching, as in figure 3. This is all right for our purposes, at first. Later (§ 8), when we have to carefully imbed our branched manifolds in Euclidean spaces, we almost literally " iron out " this pathology by the collapsing technique introduced in (2.2) and used in (5.5).

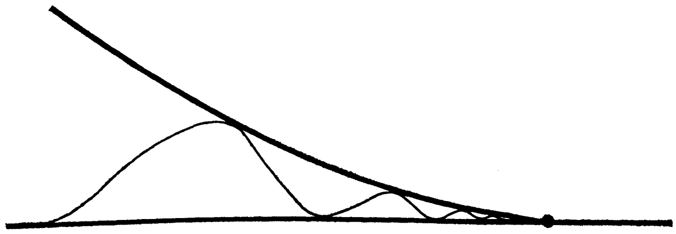


Fig. 3

But there is a more troublesome anomaly, which occurs in particular in the two-dimensional example of figure 4. The fully assembled branched 2-manifold is indicated at the right, embedded in three dimensional space. At the left, we have indicated the three disks T, M and B of which it is composed. One is to identify T and M along

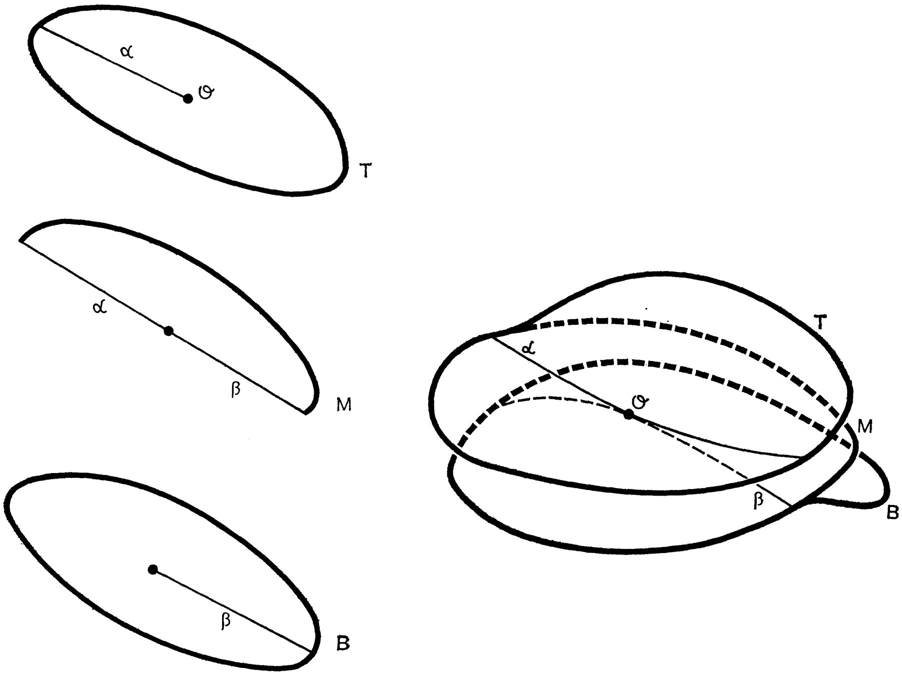


Fig. 4

the radius α and M and B along the radius β. Note that $\mathcal{O}$ (which lies on all three disks) is singular in that it has no neighborhood which is the union of smooth 2-disks D_{ij} each of which has $\mathcal{O} \in \operatorname{Int} D_{ij}$ (interior as a 2-disk).

Thus, this example cannot be given the structure of a non-singular branched 2-manifold, because of axiom *b*ns) of 1.0 ns). " Boundary points " are also singular by this definition, which suits our purposes, as we will eventually want to rule them both out. However, as our first task is to prove certain spaces *are* branched n-manifolds, we want to begin with a weaker version of this axiom.

Definition **(1.0)**. — A space K is a *branched n-manifold* if axiom (*b*ns) of (1.0 ns) above is replaced with:

b) $\pi_i | D_{ij}$ is a homeomorphism onto its image $\pi_i(D_{ij})$ which is a closed C^k n-disk relative to ∂D_i^n.

Note 1. — The phrase " relative to ∂D_i^n " means that $\pi_i(D_{ij})$ can have corners only on the boundary ∂D_i^n of D_i^n. Of course, such a " corner ", say $\pi_i(x) \in \partial D_i^n$, is not really a corner point of K itself, because for some different chart, say $U_{i'}$, $x \in U_{i'}$ and $\pi_{i'}(x) \in \operatorname{Int} D_{i'}^n$.

Note 2. — Each D_{ij} inherits the structure of a C^k n-disk because:

a) π_i is a homeomorphism into a smooth disk D_i^n; and

b) any other relevant $\pi_{i'}$ is smoothly related to π_i via the $\alpha_{i'i}$.

To help fix the idea of branched manifolds, and to motivate their introduction, we will give some examples of:

a) a compact neighborhood N in a foliated manifold; and

b) the quotient space this structure gives rise to.

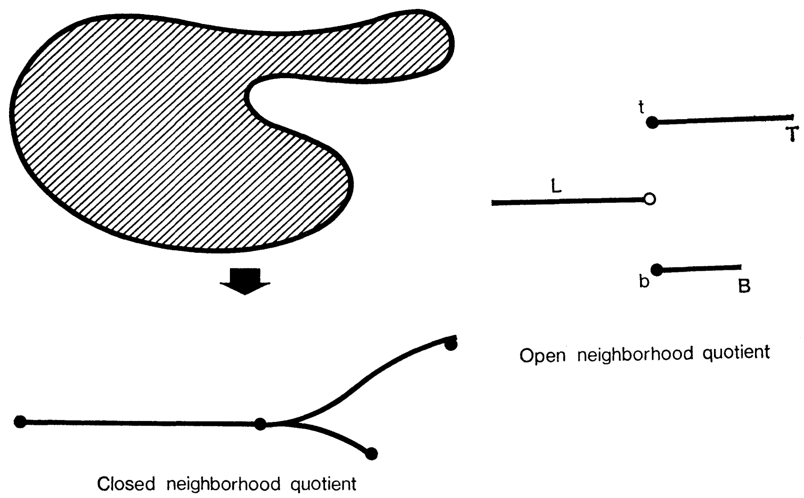


FIG. 5

In figure 5, N is a smooth disk in $\mathbf{R}^2$ which in turn is foliated by the family $\mathcal{F}$ of all vertical lines. Then one considers the collection $\mathcal{C}$ of all (connected) components C of $F \cap N$, for all $F \in \mathcal{F}$. The quotient is formed by collapsing each component $C \in \mathcal{C}$ to a point (i.e. the quotient corresponding to the partition $\mathcal{C}$ of N). The branch point is formed by the component C_0 which is tangent internally to the boundary of N. Whereas, to the left of C_0, each $F \cap N$ is in a single piece, to the right, each $F \cap N$ has two components. These determine different " branches " of the quotient space. It is always such special fibers which give rise to branching. Note that both of these branches are smooth continuations of the left hand line interval, reflecting a similar fact about the foliation.

Note. — If one takes N to be open, one gets an unbranched manifold (the right hand side of figure 5) which is not Hausdorff at the two points corresponding to the branch point. Thus in figure 5, neighborhoods of t and b would intersect along L.

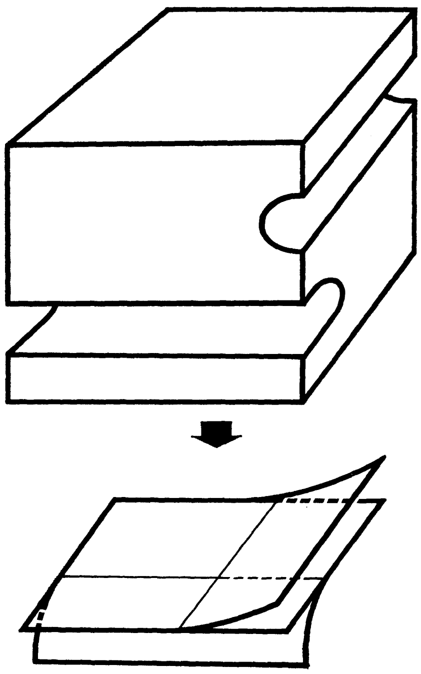


Fig. 6

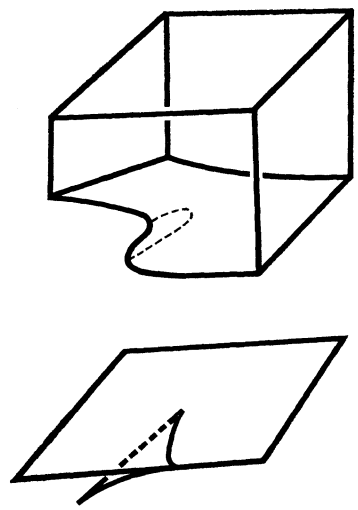


Fig. 7

Figures 6 and 7 are similar, except N is in $\mathbf{R}^3$, which is again foliated by vertical lines. Figure 6 is quite " generic " and has a quotient just like figure 3, the branching lines corresponding to leaves which are tangent (internally) to the boundary of N. Note that it is natural to have the branch lines in figure 6 intersecting transversely because this corresponds to the holes in the cube being cut transversely.

In figure 7, the Whitney cusp-singularity is the lower boundary of the neighborhood N. The quotient space has a " boundary ", including the awkward point at the cusp.

(1.1) Branched manifolds of class C^k, $k \geq 1$, have tangent bundles (and jet bundles) defined as follows: one has the induced bundles $(\pi_i \mid D_{ij})^* \mathbf{TR}^n$ over D_{ij} for all i, j. To wit $(\pi_i | D_{ij})^* = \{(x, v) \in D_{ij} \times T\mathbf{R}^n : v$ is tangent to $\mathbf{R}^n$ at $\pi_x\}$. In the disjoint union $\bigcup_{ij} (\pi_i | D_{ij})^* T\mathbf{R}^n$ introduce the equivalence relation $(x, v, i, j) \sim (x', v', i', j')$ if:

α) $x = x'$; and

β) $T\alpha_{i'i} v = v'$.

In particular, if $i = i'$, $(x, v, i, j) \sim (x, v', i, j')$ provided $v = v'$. Similarly for higher jets.

Definition (**1.2**). — If K, L are branched manifolds of class C^k and $f : K \to L$ is a map, one says that f is of class C^k provided:

(i) $f^j_{\ell i} : \pi_i(D_{ij}) \xrightarrow{(\pi_i | D_{ij})^{-1}} D_{ij} \xrightarrow{f | (D_{ij} \cap f^{-1}(U_\ell))} U_\ell \xrightarrow{\pi_\ell} \mathbf{R}^n$ is of class C^k for each j, ℓ and i.

(ii) For each $x \in K$, setting $y = \pi_i x$, the $f^j_{\ell i}$ for various j have the same germ at y.

Remark (**1.2.1**). — Condition (ii) suffices for our purposes as it is very natural in terms of foliations. But one could contrive a theory in which one replaced (ii) by the weaker

(ii′) The various $f^j_{\ell i}$ have the same k-jet at y.

Lemma (**1.3**). — *If $f : K \to K'$ is of class C^r, $r \geq 1$, then there is an induced map $Tf : TK \to TK'$ covering f (and similarly, an induced map of k-jets).*

Proof. — Given (x, v, i, j) in $(\pi_i | D_{ij})^* \mathbf{R}^n$, we define

$$(Tf)^j_{\ell i}(x, v, i, j) = (fx, Tf^j_{\ell i}(v), \ell) \quad \text{if} \quad fx \in U_\ell.$$

We must first show that $(Tf)^{j'}_{\ell i} = (Tf)^j_{\ell i}$. But (1.2) (ii) or even the weaker (1.2) (ii′) says this. Now if $(x, v, i, j) \sim (x, v', i', j')$ and if $fx \in U_{\ell'}$ as well, then

$$\begin{aligned} Tf^{j'}_{\ell' i'}(v') &= Tf^{j'}_{\ell' i'} . T\alpha_{i' i}(v) \\ &= Tf^{j'}_{\ell' i}(v) \\ &= T\alpha_{\ell' \ell} . Tf^{j'}_{\ell i}(v) \\ &= T\alpha_{\ell' \ell} . Tf^{j}_{\ell i}(v) \end{aligned}$$

so that $(Tf)^j_{\ell i} = (Tf)^{j'}_{\ell' i'}$.

Definition (**1.4**). — A C^k-map $f : K \to L$, $k \geq 1$, of one branched manifold to another is an *immersion* provided Tf is a monomorphism on each fiber.

For example, $\pi_j : U_j \to \mathbf{R}^n$ is an immersion. Thus immersions are not in general locally one-to-one. However, note that an immersion *is* a local diffeomorphism on each *smooth* submanifold—e.g. each smooth sub-disk. (Branched manifolds generally contain sub-disks which are not smooth.) We need the following corollary of this fact about smooth submanifolds:

Corollary (**1.5**). — *If $f : K \to L$ is an immersion of branched manifolds and K is compact, then there is an $\varepsilon > 0$ such that $f | D$ is 1-to-1 on each smooth subdisk of diameter $< \varepsilon$.*

Proof. — Let ε_0 be the Lebesgue number of the covering $\{U_i\}$. Now for each i, j there is an $\varepsilon_{ij}>0$ such that the corollary is true for $K=D_{ij}$, by standard results on manifolds. Then $\varepsilon<\varepsilon_0$ and $\varepsilon<\varepsilon_{ij}$ for all i, j works.

Lemma **(1.6)**. — *If* K *is a branched manifold and* $\mathscr{V}$ *is an open cover of* K, *then we may take the structure* (1.0) *so that* $\{U_i\}$ *refines* $\mathscr{V}$.

Proof. — First, we may choose closed sets $E_i\subset\pi_i(\operatorname{Int} U_i)$ such that $\{\pi_i^{-1}(E_i)\}$ covers K. Let x be a point of K and choose ε so that $N_\varepsilon(x)$ lies in an element of $\mathscr{V}$. Choose i so that $\pi_i(x)\in E_i$ and let r denote the number of disks D_{ij} making up U_i. Then there is a closed disk $D\subset\mathbf{R}^n$ with $\pi_i x$ in its interior such that $(\pi_i|D_{ij})^{-1}(D)$ has diameter $<\varepsilon/r$ for each j, by uniform continuity. Let U_x be the component of $\pi^{-1}(D)$ which contains x. Then U_x is closed and is the union of at most r disks, each mapping into D, and each having diameter $<\varepsilon/r$. As U_x is connected, it has diameter $<\varepsilon$ and hence lies in an element of $\mathscr{V}$.

Lemma **(1.7)**. — *Every branched n-manifold is n-dimensional.*

Proof. — Each U_i is n-dimensional as it is the union of finitely many n-disks. Thus each point has an n-dimensional neighborhood which is enough.

The following result is by no means definitive; it is proved in this weak form because it suffices for our immediate purposes and is elementary. See below (§ 8).

Lemma **(1.8)**. — *If* K *is a compact* C^k *branched manifold, then there is a* C^k *embedding of* K *in a euclidean space of finite dimension.*

Proof. — That is, we seek a finite set of C^k maps $K\to\mathbf{R}$ which " distinguish points " of K. To this end, let $\{U_i, \pi_i, D_{ij}\}$ be a system satisfying (1.0), where $\{U_i\}$ is finite. Also choose closed disks $E_i\subset\mathbf{R}^n$ so that $\pi_i^{-1}(E_i)\subset\operatorname{Int} U_i$ and $\{\pi_i^{-1}(E_i)\}$ covers K.

Lemma **(1.8.1)**. — *Suppose* $U=D_1\cup\ldots\cup D_m$ *and* $\pi: U\to\mathbf{R}^n$ *is a chart for a* C^k *branched n-manifold. Let* $X=D_1\cup\ldots\cup D_i$ *and* $Y=D_{i+1}\cup\ldots\cup D_m$. *Then there is a* C^k *map* $g: U\to\mathbf{R}$ *such that*

1) $gx=0$ *for* $x\in X$, *and*

2) $gx>0$ *for* $x\in Y$ *and* $\pi x\notin\pi(X\cap Y)$.

Proof. — There is a C^k map $g_0:\pi(U)\to\mathbf{R}$ such that $g_0=0$ on $\pi(X\cap Y)$ and $g_0>0$ otherwise. Now define

$$g(x)=\begin{cases}0, & x\in X\\ g_0(\pi x), & x\in Y\end{cases}$$

This is well defined; to see that it is C^k we must show that $g\circ(\pi|D_j)^{-1}:\pi(D_j)\to\mathbf{R}$ is C^k for each j. First suppose $j\le i$. Then $g\circ(\pi|D_j)^{-1}\equiv 0$, so this is C^k. Next, suppose $j\ge i+1$. Then

$$g\circ(\pi|D_j)^{-1}=g_0\circ\pi_0\circ(\pi|D_j)^{-1}=g_0$$

so that this is also C^k, and the proof of (1.8.1) is complete.

Lemma **(1.8.2).** — *Under the hypothesis of* (1.8.1), *there is a* C^k *embedding*

$$\varphi : U \to \mathbf{R}^n \times \mathbf{R}^{2^m-1},$$

of the form $\varphi = \pi \times h$.

Proof. — Note that $2^m - 1$ is the number of non-empty subsets σ of $\{D_1, \ldots, D_m\}$. For such a σ, define $X_\sigma = \bigcup_{D_j \in \sigma} D_j$, $Y_\sigma = \bigcup_{D_j \notin \sigma} D_j$. Then let $g_\sigma : U \to \mathbf{R}$ be a C^k map for X_σ and Y_σ as guaranteed by (1.8.1), let $h : U \to \mathbf{R}^{2^m-1}$ be the cartesian product of all the φ_σ's, and let $\varphi = \pi \times h$.

Then φ is C^k, so we need only show φ is 1-to-1. So assume $x, x' \in U$, $x \neq x'$ but $\varphi x = \varphi x'$. Then $\pi x = \pi x'$. Then define σ by $\sigma = \{D_i : x \in D_i\}$. Then $x \in X_\sigma$ and $x' \in Y_\sigma$: as no disk containing x can also contain x', another point which maps onto $\pi x = \pi x'$, we must have $\pi x' \notin \pi(X_\sigma \cap Y_\sigma)$. Thus $0 = g_\sigma x = g_\sigma x' > 0$, a contradiction, which completes the proof of (1.8.2).

Proof of (1.8) (continued). — (1.8) now follows by the usual partition of unity argument: let $\lambda_i : K \to \mathbf{R}$ be positive on $\pi_i^{-1}(E_i)$, and 0 outside of U_i. Define

$$f = \prod_i \lambda_i (\pi_i \times h_i)$$

where $\pi_i \times h_i$ is an embedding of U_i as given by (1.8.2). This is an embedding, but the image space is of very large dimension.

Corollary **(1.9).** — *Let* K *be a compact* C^k *branched manifold,* $k \geq 1$, *and let* d *be the metric induced upon* K *by an embedding as given above. Let* E_1, E_2 *be two smooth disks in some* U_i, *let* $\theta \in E_1 \cap E_2$, *and* $x_{j\alpha} \in E_j$, $j = 1, 2$ *be a sequence of points converging to* θ, *as* $\alpha \to \infty$. *Then*

$$\lim_{\alpha \to \infty} \frac{d(x_{1\alpha}, x_{2\alpha})}{d(x_{1\alpha}, \theta)} = 0.$$

Proof. — Assume, as case 1, that K consists entirely of one U_i. Then the metric d is that induced by the embedding $\varphi : U_i \to \mathbf{R}^n \times \mathbf{R}^N$ given in (1.8.2). Let E_3 be the n-space tangent to E_1 and E_2 at θ. But then the projection $\mathbf{R}^n \times \mathbf{R}^N \to \mathbf{R}^n$ maps E_1, E_2 and E_3 diffeomorphically, and sends $x_{1\alpha}$, $x_{2\alpha}$ and, say, t_α, to the same point, say y_α, $\alpha = 1, 2, \ldots$ Then

$$\lim_{\alpha \to \infty} \frac{d(x_{1\alpha}, t_\alpha)}{d(x_{1\alpha}, \theta)} = \dim_{\alpha \to \infty} \frac{d(x_{2\alpha}, t_\alpha)}{d(x_{2\alpha}, \theta)} = 0$$

by well known properties of tangent planes. Combining these produces the required result in this case.

General case. — The embedding is given by $\prod_i \lambda_i \varphi_i : K \to \prod_i \mathbf{R}_i^n \times \mathbf{R}^{N_i}$. Now suppose $\theta \in U_i$. Then case one shows that the appropriate limit is zero for the projection of U_i onto $\mathbf{R}_i^n \times \mathbf{R}^{N_i}$. Then the general case follows from standard considerations.

§ 2. HOW BRANCHED MANIFOLDS ARISE

The purpose of this section is to prove two lemmas (needed below) which motivate the introduction of branched manifolds. For the first, see also Haefliger [2].

Haefliger, Reeb and others have considered quotients of foliations but in a slightly different way. By taking the neighborhood N to be open, the quotient $N/\sim$ is unbranched, but not Hausdorff. These two points of view are of course equivalent. But as we are primarily concerned with the " branch set "—i.e. arranging for its simplification (§§ 2, 5) it seems better to have it (than to speak of " the points at which $N/\sim$ is not Hausdorff "). In addition, in § 8 we are concerned with embedding $N/\sim$ in a manifold. Though this of course has its counterpart in the other point of view, actually embedding $N/\sim$ (which requires Hausdorff) along with " tubular neighborhoods " seems simpler to us.

Suppose M is a smooth $(m+n)$-manifold, $\mathscr{F}$ is a C^k foliation of M of codimension n, and X is a compact subset of M. Suppose X has a compact neighborhood N_0 such that

(2.0) Each component of $N_0 \cap F$, F a leaf of $\mathscr{F}$, lies in an m-disk lying in F.

For perhaps a smaller neighborhood N of X, introduce the equivalence relation $x \sim y$ iff x and y lie in the same component of $N \cap F$ for some leaf F of $\mathscr{F}$.

Let $N/\sim$ be the quotient space and $q : N \to N/\sim$ the quotient map.

Lemma **(2.1)**. — *A compact neighborhood* $N \subset \operatorname{Int} N_0$ *can be so chosen that* $N/\sim$ *is a* C^k *branched manifold and* $q : N \to N/\sim$ *is of class* C^k.

Note. — We conjecture that N can be taken to be a smooth manifold with boundary. The only problem would be to choose N so that (2.1.1) (below) is true.

Proof. — Let $x \in X$. Then there is a C^k foliation box $D^n \times D^m \subset \operatorname{Int} N_0$ having x in its interior. In particular, $x = y \times z$ where $y \in \operatorname{Int} D^n$ and $z \in \operatorname{Int} D^m$ and each $w \times D^m$ is a smooth disk with boundary lying in some leaf of $\mathscr{F}$.

Then as X is compact, there is a finite collection $\{D_j^n \times D_j^m\}$ of such C^k foliation boxes whose interiors cover X. We let $N = \bigcup_j D_j^n \times D_j^m$. Then first note

(2.1.1) If C is a component of $N_0 \cap F$, $F \in \mathscr{F}$ and G_0 is the disk guaranteed by (2.0), then G_0 intersects only finitely many components C of $N \cap F$.

Proof of (2.1.1). — If C is a component of $N \cap F$, lying in G_0, it contains a set of the form $Y_C \times D_{jC}^m$. And for distinct C's, the corresponding disks $Y_C \times D_{jC}^m$ are disjoint. But G_0, being bounded, cannot contain infinitely many $Y_C \times D_{jC}^m$'s as they come in only finitely many different " sizes ", i.e. one for each j.

Now let C be a component of $F \cap N$, for some $F \in \mathscr{F}$, and let G_0 be the disk guaranteed by (2.0), so that $C \subset G_0 \subset F$. Then there is a separation $G_0 \cap N = C \cup D$, where C and D are closed and disjoint. Hence there is a compact, C^k, bounded m-manifold,

$B^m \subset G_0$, such that $C \subset \operatorname{Int} B^m$, and $B^m \cap D = \emptyset$. Then of course, $\partial B^m \cap N = \emptyset$. As G_0 is a disk (thus there is no holonomy near G_0) we can extend B^m to a neighborhood $D^n \times B^m$ such that $(D^n \times \partial B^m) \cap N = \emptyset$. Hence by compactness again, there is a finite collection $\{D_i^n \times B_i^m\}$, whose interiors cover N, each satisfying

(2.1.2) $$(D_i^n \cap \partial B_i^m) \cap N = \emptyset.$$

As we have room to shrink any of this structure, we may assume certain generic properties. In particular, that

(2.1.3) $$(D_j^n \times D_j^m) \cap (D_i^n \times B_i^m) = X_{ij}$$

has only finitely many components, each the closure of its interior.

Next, consider the diagram

$$\begin{tikzcd}
(D_i^n \times B_i^m) \cap N \arrow[d, "\pi_i'"'] \arrow[dr, "q_i"] & \\
\mathbf{R}^n \supset D_i^n & U_i \subset N/{\sim} \arrow[l, dashed, "\pi_i"]
\end{tikzcd}$$

in which q_i is the restriction of q, U_i the image of q_i and π_i' is the restriction of the projection onto its first factor, D_i^n.

(2.1.4) *a)* $\pi_i = \pi_i' \circ q_i^{-1}$ is well defined.

b) For $w \in N/{\sim}$ and $C = q^{-1}(w)$, and any i such that $C \cap (D_i^n \times B_i^m) \neq \emptyset$ we have $C \subset y \times \operatorname{Int} B_i^m$, for some $y \in D_i^n$.

Proof of (2.1.4). — First note *b)* implies *a)*, because then $q_i^{-1}(w) = C \subset y \times \operatorname{Int} B_i^m$ so that $\pi_i(w) = y$.

To prove part *b)*, let F be the leaf such that C is a component of $F \cap N$, and let $y \in D_i^n$ be a part with $(y \times B_i^m) \cap C \neq \emptyset$. Then C, being connected and containing a point of $y \times B_i^m$ but no point of $y \times \partial B_i^m$ (2.1.2), the boundary of $y \times B_i^m$, relative to the leaf F, must lie entirely in $y \times \operatorname{Int} B_i^m$.

Though the fact that $N/{\sim}$ is compact and metrizable is geometrically obvious, we include a proof. It suffices to show that the partition

$$\mathscr{C} = \{q^{-1}(w) : w \in N/{\sim}\} = \{C \mid C \text{ is a component of } F \cap N, \text{ for some } F \in \mathscr{F}\}$$

is upper-semi-continuous (see, e.g. Kuratowski [18; p. 42]). That is, for A closed in N, the ("saturation") $B = \bigcup \{C \in \mathscr{C} \mid C \cap A = \emptyset\}$ is also closed. We need only show this for A small, so we suppose $A \subset \operatorname{Int}(D_i^n \times B_i^m)$ for some i. The assumption that B is not closed leads to the existence of a sequence of $C_\alpha \in \mathscr{C}$, $\alpha = 1, 2, \ldots$ such that

a) $C_\alpha \cap A \neq \emptyset$;

b) $C_\alpha \subset y_\alpha \times B_i^m$, $y_\alpha \in D_i^n$ (by (2.1.4 *b)*));

c) $y_\alpha \to y_0 \in \operatorname{Int} D_i^n$;

d) $C_\alpha \to D$, a closed subset of $y_0 \times B_i^m$; but

e) $D \not\subset B$.

Then D is connected by *d)* and $D \subset N$, so that D lies in some $C_0 \in \mathscr{C}$. But by *a)* $D \cap A \neq \emptyset$ so that $B \subset C_0 \subset B$, contradicting *e)*, and finishing the proof that $N/\sim$ is compact, metrizable.

Next let $X_{ij} = (D_j^n \times D_j^m) \cap (D_i^n \times B_i^m)$, $M_{ij} = q(X_{ij})$, $\pi_{ij} = \pi_i \mid M_{ij}$, $\pi'_{ij} = \pi'_i \mid X_{ij}$, $q_{ij} = q \mid X_{ij}$ and $\pi_j : X_{ij} \to M_j$, be the restriction of the projection map $D_j^n \times D_j^m \to D_j^n$. Then

Lemma **(2.1.5)**. — *There is the commutative diagram*

$$\begin{tikzcd}
 & X_{ij} \arrow[dl, "\pi'_{ij}"'] \arrow[d, "\pi_j"] \arrow[dr, "q_{ij}"] & \\
 & M_j \arrow[dl, dashed, "\alpha"] \arrow[dr, dashed, "\beta"'] & \\
D_i^n & & M_{ij} \subset U_i \arrow[ll, "\pi_{ij}"]
\end{tikzcd}$$

in which α is a C^k immersion relative to the induced structures, β is onto and (consequently) π_{ij} is locally 1-to-1.

Proof of **(2.1.5)**. — By (2.1.4 *a)*), $\pi_j^{-1}(\mathrm{pt}) = y \times D_j^m \subset y' \times \operatorname{Int} B_i^m$ for some $y \in D_j^n$ and $y' \in D_i^n$. Hence $\beta(\mathrm{pt}) = q \circ \pi_j^{-1}(\mathrm{pt})$ is well defined and, by definition, β is onto.

Now since the kernels of the two differentials, $T\pi_j$ and $T\pi'_{ij}$ are the same, to wit the tangent space of the leaf F at any point where they are both defined, it follows that π'_{ij} and π_j are submersions and α an immersion. This last together with the fact that β is onto implies π_{ij} is locally 1-to-1, and finishes the proof of (2.1.5).

We now have the ingredients required, $K = N/\sim$, U_i, $\pi_i : U_i \to D_i$, $M_{ij} \subset U_i$ needed for a branched manifold, although we still need to subdivide M_{ij} to get the disks, " D_{ij} ".

We proceed to verify the axioms:

(iii *a)*) As $\bigcup_i \operatorname{Int}(D_i^n \times B_i^m) \supset N$, it follows that $\bigcup_i \operatorname{Int} U_i \supset K$. Also, as

$$\bigcup_j X_{ij} \supset N \cap (D_i^n \times B_i^m)$$

it follows that $\bigcup_j M_{ij} = U_i$.

(iii *b)*) We have shown $\pi_i \mid M_{ij}$ is locally 1-to-1, and its image is a C^k immersed manifold with boundary, possibly with " corners ". But these corners occur only because we may have an intersection of $\partial(D_j^n \times D_j^m) \cap (D_i^n \times B_i^m)$ which lies in $(\partial D_i^n) \times B_i^m$ by (2.1.2). Thus the only possible corners are in ∂D_i^n.

We can now subdivide M_j so that via β we get a subdivision of M_{ij} into disks $\{D_{ij*}\}$ satisfying our condition. This uses the fact (2.1.5) that $\pi_i | M_{ij} = \pi_{ij}$ is locally 1-to-1.

(iii *c*)) Now since the maps π_i' are the projections of C^k foliation boxes, it follows that there is a C^k " cocycle "

$$\{\alpha_{ii'} : D_i^n \to D_i^n\}$$

relative to $\{\pi_i'\}$. But these serve just as well for $\{\pi_i\}$, of course.

Thus K is a C^k branched n-manifold. It remains only to remark that $q : N \to K$ is also C^k. But this reduces to the statement that $\{\alpha_{ii'}\}$ is a C^k cocycle for $\{\pi_i'\}$, so that the proof is complete.

Collapsing lemma **(2.2)**. — *Suppose* K *is a branched manifold,* $A \subset U \subset K$ *where* A *is closed and* U *open, and that* $\varphi : U \to \mathbf{R}^n$ *is an immersion. Let* $K' = K/\sim$ *where* $x \sim y$ *iff* $x = y$ *or* $x, y \in A$ *and* $\varphi x = \varphi y$. *Then* K′ *is a branched manifold and the natural map* $\pi : K \to K'$ *is an immersion.*

Proof. — There is an $\varepsilon > 0$ such that each smooth subdisk $D \subset K$ which hits A lies in U and is embedded by φ in L. Let $\{U_i, \pi_i, D_{ij}\}$ be the charts for K, and suppose (1.5) that the diameter of U_i is $< \varepsilon$ for each i.

Let $\{U_i'\} = \{\varphi(U_i) : \text{for all } i\}$. Then define $\pi_i' : U_i' \to \mathbf{R}^n$ by $\pi_i \circ (\varphi^{-1} | U_i')$ and $D_{ij}' = \varphi(D_{ij})$. The axioms are trivially verified, and since φ is smooth and imbeds small disks it is an immersion.

Definition **(2.3)**. — We say K′ is obtained from K by *collapsing the points of* A *under* φ.

§ 3. N-SOLENOIDS

Suppose K is a compact branched C^r n-manifold and $g : K \to K$ is a C^r immersion. We note the following.

Axioms **(3.0)**

1) the nonwandering set of g, $\Omega(g)$, is all of K;

2) for each $x \in K$ there is a neighborhood N of x and an integer j such that $g^j(N)$ is a subset of a smooth n-cell;

3^+) g is an expansion.

Then define Σ to be the inverse limit of the sequence

$$K \xleftarrow{g} K \xleftarrow{g} K \leftarrow \dots$$

and $h : \Sigma \to E$ to be the shift, i.e.

$$\begin{aligned} h(x_0, x_1, x_2, \dots) &= (gx_0, gx_1, gx_2, \dots) \\ &= (gx_0, x_0, x_1, \dots). \end{aligned}$$

Definition **(3.1)**. — If $g : K \to K$ satisfies Axioms 1), 2), 3^+), one says that Σ is an *n-solenoid* and h a *shift map* of Σ. One calls $g : K \to K$ or the pair (K, g) a *presentation* of $h : \Sigma \to \Sigma$.

The name n-solenoid is justified we feel by the regularity of such Σ's.

The first part of the proof that n-solenoids are " nice " is that branched manifolds must be " nice " in order to admit a self immersion satisfying the axioms. Note that we have allowed branched manifolds to have boundaries and corners. In dimensions ≥ 2 they can have " helical " points—see (3.3) below.

Definition **(3.2)**. — A point p of a branched n-manifold is said to be a *regular* point provided the union of the smooth disks which have p as interior point (as disks) contains a neighborhood of p. Otherwise p is a *singular* point. A point $c \in K$ is a *non-branch* point of K provided it has an open smooth n-disk as a neighborhood; otherwise it is a *branch* point.

Notation. — $\beta K =$ set of all branch points of K. A closed disk $D \subset K$ is said to be *enlargeable* provided there is a disk $E \subset K$ such that $D \subset$ Interior of E.

Remark **(3.2.1)**. — βK is closed and of dimension $\leq n-1$ (equivalently, nowhere dense).

Proof. — By definition the non-branch points form an open set so that βK is closed. Now if D_1 and D_2 are two smooth n-disks in K, then $\beta(D_1 \cup D_2) \subset \text{boundary}(D_1 \cap D_2)$, which has dimension $\leq n-1$. But $\beta X \subset \bigcup_j \beta(D_j \cap D_j')$, where the union is over all pairs which intersect.

Example **(3.3)**. — There is a branched 2-manifold with an isolated singular point.

Proof. — Let $K = U_1 = D_1$ be the unit disk in the complex plane and let $\pi_1 : U_1 \to D_1$ be defined by $z \mapsto z^2$. Then K can be written as the union of 3 closed disks D_{1i}, $i = 1, 2, 3$ which map 1-to-1 under π_1. Thus π_1 induces a differentiable structure in D_{1j}, $i = 1, 2, 3$, which describes K as a branched 2-manifold. Note that the singular set of K consists of the boundary and the center of the unit disk.

Remark **(3.4)**. — The singular set of a branched manifold is closed.

Proof. — If $p \in K$ is regular, it has a neighborhood which is the union of a finite set of n-disks, each having p as interior point. Then an open set about p is covered by these disks; each point of such an open set is regular so that the regular points of K form an open set.

However, these pathologies do not occur in presentations of n-solenoids.

Lemma **(3.5)**. — *If* K *is a compact branched manifold and* $g : K \to K$ *is an immersion which satisfies axioms* 1), 2), 3^+), *then the singular set of* K *is empty.*

The proof is surprisingly different from the 1-dimensional case [14; (3.1)], and proceeds in two steps. The first is itself often useful below; we assume $g : K \to K$ satisfies axioms 1), 2), 3^+).

Lemma (**3.5.1**). — *K has a covering by finitely many enlargeable disks.*

Proof. — For $x \in K$, there is a neighborhood U_x and an integer $m = m_x$ such that $g^m(U_x) = D_x$, a disk. If $i \geq m$, then $D_x = D_{x1} \cup \ldots \cup D_{xr}$ where each D_{xj} is an open disk, and $g^i | D_{xj}$ is 1-to-1. Thus $U_x = (g | U_x)^{-m}(D_{x1}) \cup \ldots \cup (g | U_x)^{-m}(D_{xr})$, where each of these sets is open and maps onto an open disk under g^{m+i}.

Thus take a finite cover by U_x's and let $p \geq$ all the corresponding $(m_x + i)$'s. Then each such U_x is the finite union $\{U_{xi}\}$ of open sets each of which maps onto a disk under g^p. Thus the U_{xi}'s for the finite number of U_x's cover K. We rename them $V_1, V_2, \ldots, V_s$. Then $g^p(V_1), \ldots, g^p(V_s)$ is a cover of K by open disks, as g is onto. Thus if we take D_i to be a slightly smaller closed disk than $g^p(V_i)$, then $\{D_1, \ldots, D_s\}$ is a covering of K by enlargeable disks.

Proof of (**3.5**). — For $x \in K$, $\bigcup_i \{g^p(V_i) : x \in D_i\}$ is a neighborhood of x and each of the disks $g^p(V_i)$ has x in its interior. Hence x is not a singular point.

Definition (**3.6**). — Given branched manifolds K and L and an immersion $g : K \to L$, a neighborhood U in K is said to be a *disk neighborhood* (relative to g) provided there is an integer p and a smooth closed disk $D \subset L$ such that U is the union of finitely many disks $E_1, \ldots, E_q$, each of which maps diffeomorphically onto D under g^p, and such that $\partial U = \bigcup_i \partial E_i$.

Lemma (**3.7**). — *If* $g : K \to K$ *satisfies axioms* 1)-2)-3^+) *and* $x \in K$, *then there is a disk neighborhood of* x.

Proof. — By axiom 2), there is a disk D_0, a neighborhood U_0 of x, and an integer p such that g^p immerses U_0 in D. Then U_0 is a subset of the union of finitely many disks which by (3.5) can be taken so that none has x on its boundary. Let $D_1, \ldots, D_q$ denote the disks which contain x. Then $D_1 \cup \ldots \cup D_q$ is a neighborhood of x and each $g^p(D_j)$ contains a neighborhood of $g^p(x)$ in D_0. Thus there is a disk $D \subset D_0$, having x in its interior, such that $g^p(D_j) \supset D$. If we set $E_j = (g^p | D_j)^{-1}(D)$ we have $E_1 \cup \ldots \cup E_q$ which is a disk neighborhood.

Corollary (**3.8**). — *If* $g : K \to K$ *satisfies Axioms* 1)-2)-3^+), *then there is a finite cover of* K *by disk neighborhoods all having the same* " p " (*see* (3.6)).

Proof. — First, if $x \in K$, then there is an open set U containing x which is immersed in a disk by g^p, for some p. But then if $i \geq p$, g^i immerses a neighborhood of each point of U in some, perhaps smaller, disk. Thus by compactness, there is a p which works for all points in K. But in the proof of (3.7) p was picked at the beginning, with this requirement above. Thus (3.8) is proved by (3.7).

§ 4. EXPANDING ATTRACTORS

If Λ is an attractor with hyperbolic structure $E^u + E^s$, then there is the elementary inequality (u and s are the dimensions of the fibers of E^u and E^s):

(4.0) $\dim \Lambda \geq u$.

This follows from the

Lemma **(4.1)**. — $W^u(x) \subset \Lambda$ *for any* $x \in \Lambda$.

Proof. — Λ has a neighborhood N such that $f(N) \subset N$ and $\Lambda = \bigcap_{i>0} f^i(N)$. Let $x \in \Lambda$ and $y \in W^u(x)$. Then $\lim_{i \to \infty} \rho(f^{-i}x, f^{-i}y) = 0$ so that $f^{-i}y \in N$ for i big enough. That is $y \in f^i(N)$ for i big enough so that $y \in f^i(N)$ for all $i > 0$. Hence $y \in \Lambda$.

If $\dim \Lambda = 0$, then $u = 0$ and it follows that Λ is a finite set. If $\dim \Lambda = 1$, then $u \neq 0$ as then Λ would be finite so that $\dim \Lambda = u = 1$. This fact was exploited in the papers [15], [16], in which one-dimensional attractors were characterized and classified.

Definition **(4.2)**. — By an *expanding attractor* is meant an attractor Λ with hyperbolic structure where $\dim \Lambda = u$.

(All of the usual definitions of dimension agree on attractors as they are compact and metric.) It should be pointed out that this is a special case. That is, not all attractors are expanding. For example, an Anosov map $f : M \to M$, has M as an attractor where $\dim M > u$.

Lemma **(4.3)**. — *If* Λ *is an expanding attractor, then* $\dim(\Lambda \cap W^s(x)) = 0$ *for any* $x \in \Lambda$.

Proof. — For a neighborhood N such that $f(N) \subset N$ and $\bigcap_{i>0} f^i(N) = \Lambda$, we may take *local stable manifolds* $W_0^s(x)$, $x \in \Lambda$ so that $N = \bigcup_{x \in \Lambda} W_0^s(x)$, by Hirsch-Pugh, as

$$W^s(x) = \bigcup_{i<0} f^i(W_0^s(x)),$$

it suffices to show $\dim \Lambda \cap W_0^s(x) = 0$ and $\Lambda \cap W_0^s(x)$ is compact. (In fact $\Lambda \cap W_0^s(x)$ is a Cantor set as is seen from Theorem C.) Meanwhile, $\bigcup_{y \in \Lambda \cap W_0^s(x)} W_0^u(y)$ is homeomorphic to the abstract product $(\Lambda \cap W_0^s(x)) \times W_0^u(y_0)$ for some fixed y_0, by Hirsch-Pugh. Thus $\dim(\Lambda \cap W_0^s(x)) = 0$ as dimension *is* additive for cartesian products in which one of the factors is euclidean.

Note that this proves that for attractors in general one has the formula

(4.3.1) $\dim(\Lambda \cap W^s(x)) = \dim \Lambda - u$, *for each* $x \in \Lambda$.

The remainder of this section is devoted to the proof of

Theorem **(4.4)**. — *Suppose* Λ *is an expanding attractor of* $f \in \mathrm{Diff}(\mathrm{M})$ *and that the foliation* $\{\mathrm{W}^s(x) : x \in \Lambda\}$ *is of class* C^1 *on some neighborhood of* Λ. *Then* $f \mid \Lambda$ *is* C^1 *conjugate to the shift map of an n-solenoid, where* $n = \dim \Lambda$.

Proof. — There is a closed neighborhood N_0 of Λ such that

$$(*) \qquad f(\mathrm{N}_0) \subset \mathrm{Int}\, \mathrm{N}_0 \quad \text{and} \quad \Lambda = \bigcap_{i>0} f^i(\mathrm{N}_0).$$

In particular we can take N_0 to be a manifold with boundary; let N_1 be a slight enlargement of N_0. Then condition $(*)$ still holds for any neighborhood N between N_0 and N_1. Notice that condition (2.0) holds so that lemma (2.1) applies, yielding a closed neighborhood N where $\mathrm{N}_0 \subset \mathrm{N} \subset \mathrm{N}_1$, so that N also satisfies $(*)$.

Let K be the quotient given by (2.1) and $q : \mathrm{N} \to \mathrm{K}$ the natural map. We then have the commutative diagram (as in [15]):

$$(4.4.1) \qquad \begin{tikzcd}
\vdots \arrow[d] & \vdots & \vdots & \\
f^2(\mathrm{N}) \arrow[d] & f(\mathrm{N}) \arrow[l, "f"'] \arrow[d] & \mathrm{N} \arrow[l, "f"'] \arrow[ddd, "q"] & \cdots \\
f(\mathrm{N}) \arrow[d] & \mathrm{N} \arrow[l, "f"'] \arrow[dd, "q"] & & \\
\mathrm{N} \arrow[d, "q"'] & & & \\
\mathrm{K} & \mathrm{K} \arrow[l, dotted] & \mathrm{K} \arrow[l, dotted] & \cdots \arrow[l, dotted, "g"]
\end{tikzcd}$$

Here, each unlabeled vertical map is an inclusion and one defines $g = qfq^{-1}$. To see g is well defined, let $q^{-1}(\text{point}) = \mathrm{C}$, a component of $\mathrm{F} \cap \mathrm{N}$ where F is a leaf of the foliation $\mathscr{F}$. Then $f(\mathrm{C}) \subset f(\mathrm{F})$, which is also a leaf F' (or in one leaf) of $\mathscr{F}$ so that $f(\mathrm{C}) \subset \mathrm{C}'$, a component of $\mathrm{F}' \cap \mathrm{N}$. Thus $g(\mathrm{pt}) = qf(\mathrm{C}) = q(\mathrm{C}') =$ a point. The diagram is commutative as all the bottom rectangles are the same.

Each vertical inverse limit is just the intersection $\bigcap_{i>0} f^i(\mathrm{N}) = \Lambda$. The horizontal inverse limit yields Σ with shift map h. The diagram defines a map $\mathrm{R} : \Lambda \to \Sigma$ such that

$$\begin{tikzcd}
\Lambda \arrow[d, "\mathrm{R}"'] & \Lambda \arrow[l, "f \mid \Lambda"'] \arrow[d, "\mathrm{R}"] \\
\Sigma & \Sigma \arrow[l, "h"]
\end{tikzcd}$$

is commutative. R is defined by

$$\mathrm{R}(x) = (qx, qf^{-1}x, qf^{-2}x, \ldots)$$

as one sees from the diagram. Notice the diagonal inverse limit

$$N \xleftarrow{f_1} N \xleftarrow{f_1} N \leftarrow \dots$$

which is also convenient for some purposes. This gives the limit Σ_1 and shift map h_1, and one has the diagram

$$\begin{tikzcd}
\Lambda \arrow[dr, "R_1"] \arrow[dd, "R"'] & & \Lambda \arrow[ll, "f|\Lambda"'] \arrow[dr, "R_1"] \arrow[dd, "R"] & \\
& \Sigma_1 \arrow[dl, "Q"] & & \Sigma_1 \arrow[ll, "h_1"'] \arrow[dl, "Q"] \\
\Sigma & & \Sigma \arrow[ll, "h"] &
\end{tikzcd}$$

where $R_1(x) = (x, f^{-1}x, f^{-2}x, \dots)$ and

$$Q(y_0, y_1, y_2, \dots) = (qy_0, qy_1, qy_2, \dots).$$

To finish the proof of (4.4) we need to show that g is an immersion satisfying Axioms 1), 2), 3^+) and that R is a homeomorphism.

First suppose x, x' are distinct points of Λ such that $qx = qx'$. Then $x' \in W^s(x)$ and moreover $x' \in C$, the component of $W^s(x) \cap N$ which contains x. Note that C is in a local stable manifold of f at x. But there is an integer i, such that $f^{-i}(x')$ is so far away from x, with respect to $W^s(x)$, that it is not in C, so that $qf^{-i}(x) \neq qf^{-i}(x')$. That is $R(x) \neq R(x')$, so that R is a monomorphism.

But R is not necessarily epic as we have not been able to choose N for this specific purpose. Rather, we chose N so that the quotient space be a branched manifold. Thus let $K_0 = q(\Lambda)$.

(4.4.2) We claim that there is an integer r such that $K_0 = g^r(K)$.

Proof. — In [5], Hirsch-Pugh show that local stable manifolds $W^s_{\mathrm{loc}}(x)$, $x \in \Lambda$ can be chosen so that they are disks, they lie in N, and their union is a neighborhood N′ of Λ. Hence for each $x \in N$, there is a neighborhood V_x of x and an integer $n = n_x$ such that $f^n(V_x) \subset N'$. By compactness there is an integer r such that $f^r(N) \subset N'$. Now as $W^s_{\mathrm{loc}}(x)$ is connected and lies in $W^s(x)$, $q(W^s_{\mathrm{loc}}(x)) = q(x)$, for each $x \in \Lambda$. Therefore $K_0 = q(\Lambda) = q(N') = qf^r(N) = g^r q(N) = g^r(K)$.

Thus, as Λ is invariant under f, it follows that R is onto the inverse limit Σ, as the two maps $g_0 = g \mid K_0$ and g have the same inverse limit. Thus R is a homeomorphism and similarly, so is R_1, and thus also Q. It is g_0 which we deal with, as it will satisfy Axioms 1)-2)-3^+).

Axiom 1. — Let $x \in K_0$ and V be a neighborhood of x in K_0. Then $q^{-1}(V) \cap N'$ is a neighborhood of some point $y \in q^{-1}(x) \cap \Lambda$. Therefore, for some m, there is a point $z \in f^m(q^{-1}(V) \cap N') \cap (q^{-1}(V) \cap N')$. Then

$$q(z) \in V \cap qf^m(q^{-1}(v)) \cap q(N') = V \cap g^m qq^{-1}(v) \cap K_0 = V \cap g^m(v) \cap K_0$$

which proves axiom 1 for g_0.

Axiom 2. — Let $y_0 \in \mathrm{K}$ and $\mathrm{C} = q^{-1}(y_0)$. Then there is a box neighborhood $\mathrm{D}^s \times \mathrm{D}^u$ such that for each $x \in \mathrm{D}^u$, $\mathrm{D}^s \times x$ lies in a leaf of the foliation $\{\mathrm{W}^s(x') : x' \in \Lambda\}$, and $\mathrm{C} \subset \mathrm{D}^s \times x_0$. If the unstable factor D^u is taken small enough, then $q^{-1}(y) \subset$ some $\mathrm{D}^s \times x'$ for each y in some neighborhood V of y. (Here V is a neighborhood in K as g itself will be shown to satisfy axioms 2) and 3^+).) Now it follows, just as in the proof of (4.4.2), that $f^v(\mathrm{D}^s \times \mathrm{D}^u) \subset \mathrm{N}'$ for some integer v. That is, each $f^v(\mathrm{D}^s \times x)$ lies in a $\mathrm{W}^s_{\mathrm{loc}}(x')$ for some $x' \in \Lambda$. Hence $q^{-1}q(x') \supset f^v(\mathrm{D}^s \times x)$ as this last is connected. This says that each of the stable disks $\mathrm{D}^s \times x$ is mapped by qf^v into a single point of K. That is, if the unstable factor D^u is taken small enough, $qf^v(\mathrm{D}^s \times \mathrm{D}^u)$ is conjugate to the projection $\mathrm{D}^s \times \mathrm{D}^u \to \mathrm{D}^u$. But $qf^v = g^v q$ so that $q((\mathrm{D}^s \times \mathrm{D}^u) \cap \mathrm{N}) = \mathrm{V}$ is a neighborhood of y_0 such that $g^v(\mathrm{V}) = qf^v(\mathrm{D}^s \times \mathrm{D}^u)$, a smooth n-disk.

Axiom 3⁺. — Recall that a Riemannian metric is put on a paracompact space by putting one on locally and then using a partition of unity to fit these together. The metric on K is, locally, induced on U_i by the map $\pi_i : \mathrm{U}_i \to \mathbf{R}^n$ which defines the branched manifold structure on K.

Now if $x \in \Lambda \cap q^{-1}(\mathrm{V}_i)$ then $\mathrm{W}^u_{\mathrm{loc}}(x)$ is transverse to the foliation $\{\mathrm{W}^s(x) : x \in \Lambda\}$ so that the metric induced on $\mathrm{W}^u_{\mathrm{loc}}(x)$ by $\pi_i q : \mathrm{U}_i \to \mathbf{R}^n$ is equivalent to the given one. That is, the metric induced by $\pi_i q$ on $\mathrm{U}_i \cap \Lambda$ is equivalent to the given one on $\mathrm{E}^u \mid \mathrm{U}_i$. But f is an expansion relative to E^u so that g is an expansion on U_i.

This completes the verification of Axioms 1-2-3^+ for $g_0 : \mathrm{K}_0 \to \mathrm{K}_0$ and finishes the proof of (4.4).

§ 5. SHIFT EQUIVALENCE

Definition. — Two immersions $g_i : \mathrm{K}_i \to \mathrm{K}_i$, $i = 1, 2$ are *shift* equivalent (see [16]) provided there are immersions $r : \mathrm{K}_i \to \mathrm{K}'_i$, $s : \mathrm{K}'_i \to \mathrm{K}_i$ and an integer m such that $g_2 r = r g_1$, $g_1 s = s g_2$ and $sr = g_1^m$, $rs = g_2^m$.

Remark (**5.1**). — This definition agrees with that given in [16], where shift equivalence was introduced, except that r and s are required to be immersions; this is no additional assumption in view of the equation $rs = g_2^m$.

The following theorem shows that shift equivalence classes are really what concern us here. Part *a)* is valid in any category and was proved in [16].

Theorem (**5.2**). — *If* $g : \mathrm{K} \to \mathrm{K}$ *and* $g' : \mathrm{K}' \to \mathrm{K}'$ *are shift equivalent immersions, then :*

a) *they present topologically conjugate shift maps*;

b) *if* g *satisfies one of the axioms so does* g';

c) g *and* g' *have the same zeta function.*

Proof. — Let $h : \Sigma \to \Sigma$ and $h' : \Sigma' \to \Sigma'$ be the shift maps that g and g' represent, and let $r : \mathrm{K} \to \mathrm{K}'$, $s : \mathrm{K}' \to \mathrm{K}$, and $m \in \mathbf{N}$ be as in the definitions of shift equivalence. Now for $x = (x_0, x_1, \ldots) \in \Sigma$, define $\mathrm{R}(x) = (rx_0, rx_1, \ldots)$. Then $\mathrm{R} : \Sigma \to \Sigma'$, because

$g' r x_{i+1} = r g x_{i+1} = r x_i$. If $S : \Sigma' \to \Sigma$ is the map induced by s, then $SR = h^m$ and $RS = h'^m$ so that R is a homeomorphism. Clearly $Rh = h'R$ which proves *a)*.

To prove *b)* suppose first that $x \in K$ is a non-wandering point of g and let $y = rx$. We claim y is a non-wandering point of g', for if N is a neighborhood of y, then $r^{-1}(N)$ is a neighborhood of x. Thus $g^n(r^{-1}(N)) \cap r^{-1}(N) \neq \emptyset$ for some n so that

$$rg^n(r^{-1}(N)) \cap N = g'^n(N) \cap N \neq \emptyset.$$

Thus y is a non-wandering point of g'.

For Axiom 2, suppose $y \in K'$; let $x = sy$ and let U be a neighborhood of x and n an integer such that g^n immerses U in a disk. Then rg^{nm} also immerses some perhaps smaller neighborhood V in a disk. Now y has a neighborhood W such that $s(W) \subset V$ and we see that $g'^{(n+1)m} = rg^{nm} s$ immerses W in a disk, provided V is sufficiently small.

For Axiom 3^+, note that $g'^{n+m} = g'^n rs = rg^n s$ for all n. Thus if g is an expansion so also is g'.

To show g and g' have the same zeta function, suppose $g^n x = x$. Then

$$rx = rg^n x = g'^n rx$$

so that $g'^n(rx) = rx$. Furthermore, if $x \neq y$ and $g^n x = x$ and $y = g^n y$, then $rx \neq ry$, for otherwise $x = g^{nm} x = (sr)^n x = s(rs)^{n-1} rx = s(rs)^{n-1} ry = g^{nm} y = y$. Thus $\# \operatorname{Fix} g^n \leq \# \operatorname{Fix} g'^n$. By symmetry one has the other inequality so that the zeta functions of g and g' are equal.

Lemma **(5.3)**. — *If $g : K \to K$ is a presentation of $h : \Sigma \to \Sigma$, then g and h have the same zeta function.* (*See* [15, 16].)

Proof. — For $x \in \operatorname{Fix} g^n$, let $a(x) = (x, g^{n-1}x, g^{n-2}x, \ldots, x, g^{n-1}, g^{n-2}x, \ldots)$. Then $a(x) \in \operatorname{Fix} h^n$; clearly $a : \operatorname{Fix} g^n \to \operatorname{Fix} h^n$ is also 1-1 and onto.

Recall that in (2.3) we introduced the notion of "collapsing" for branched manifolds.

Lemma **(5.4)**. — *If $g : K \to K$ satisfies Axioms 1-2-3^+, $A \subset K$ is a closed set and m a positive integer, then collapsing the points of A under g^m yields a branched manifold K' and g induces $g' : K' \to K'$ which is shift equivalent to g. The given m is the "m" in the definition of shift equivalence.*

Proof. — In (2.2) we showed that K' is a branched manifold. The balance of (5.4) was proved in [16; (5.3)], so formally as to be valid here (and in most categories).

The balance of this section is devoted to finding, for each $g : K \to K$ satisfying Axioms 1-2-3^+, a shift equivalent $g' : K' \to K'$ where K' is a smoothly triangulated n-complex with $(n-1)$-dimensional branch set. This requires big ammunition: the relative triangulation theorem of J. H. C. Whitehead [12]. However, the result which we actually need is that which one obtains just *before* applying Whitehead's Theorem, so that we state it separately as (5.5).

Definition (5.5). — By an *n-cross* is meant a space differentiably equivalent to the union of the $(n-1)$-dimensional coordinate planes in $\mathbf{R}^n$. We say that a branched manifold K is *normally branched* at $x \in K$ if x has a neighborhood U such that the branch set of U is contained in an n-cross B, which in turn maps diffeomorphically onto $\pi_i(B)$ by one of the chart maps π_i of K. Finally, K is *normally branched* if it is normally branched at each of its points.

Lemma (5.6). — *Given an immersion* $g : K \to K$ *which satisfies Axioms* 1-2-3$^+$, *there is a shift equivalent* $g' : K' \to K'$ *where* K' *is normally branched.*

Proof. — By (3.8) there is a finite set of disk neighborhoods $\{N_i, D_i, m\}$, $i = 1, \ldots, \alpha$, covering K. That is, each N_i is the union of disks each mapping diffeomorphically onto D_i under g^m. We also take these D_i so small that $g^m | D_i$ is a diffeomorphism as well. Then $\{\partial D_i\}$ is a collection of $(n-1)$-spheres, and if we slightly vary our disk neighborhoods, these can be taken to be in general position. That is, they intersect locally in n-crosses, or subsets of n-crosses.

Now define K_1 by collapsing the points of N_1 under g^m. Then by (5.4) there are also immersions $p_1 : K \to K_1$, $q_1 : K_1 \to K$, which together with m express the fact that g' is shift equivalent to g.

Continuing, form K_2 by collapsing the points of $p_1(N_2)$ under g_1^m, then collapsing the points of N_3, etc., getting, finally, $g' : K' \to K'$, shift equivalent to g. Note that we can form K' in one step, by collapsing the N_i's all at once under g^m. Thus the proof of (5.4) given in [16] shows that there are maps $p : K \to K'$, $q : K' \to K$ such that $qp = q'^m$ and $pq = g^m$.

For each i, let $V_i = p(N_i)$. Then V_i is a smooth n-disk and ∂V_i its boundary.

Let $\mathscr{E}_0$ be the collection of all components of all intersections of the form

$$V_{i_1} \cap \ldots \cap V_{i_v} \cap \partial V_{i_{v+1}} \cap \ldots \cap \partial V_{i_s}$$

where $s, v \geq 1$ and all of the indices $i_1, \ldots, i_s$ are distinct. Now as the V_i's are in general position, each $E \in \mathscr{E}_0$ has an interior point, say e. That is $e \in \operatorname{Int}(V_{i_1} \cap \ldots \cap V_{i_v})$ which is n-dimensional and $e \in \operatorname{Int}(\partial V_{i_{v+1}} \cap \ldots \cap \partial V_{i_s})$ which is of dimension $n-(s-v)$.

Define $\mathscr{E} \subset \mathscr{E}_0$ by

$$\mathscr{E} = \{E \in \mathscr{E}_0 : E \text{ has an interior point in } \beta K'\}.$$

(Here and below we use $\beta K'$ to denote the branch set of K'.) First, note that each element of $\mathscr{E}$ is a smooth manifold with (usually) boundary and that its boundary is likewise made up of elements of $\mathscr{E}$.

Next, we claim $\beta K' \subset \bigcup \mathscr{E}$. To see this note that in K_1, βK_1 lies in the exterior of $p_1(N_1)$, possibly intersecting its boundary. Similarly

$$\beta K_2 \subset (K_2 - p_2(N_1) \cup p_2(N_2)) \cup p_2(\partial N_1) \cup p_2(\partial N_2),$$

so that by induction, one proves $\beta K' \subset \bigcup \mathscr{E}$ as claimed.

Thus to complete the proof of (5.5), it suffices to prove the

Lemma (**5.6.1**). — *If* $E \in \mathscr{E}$, *then* $E \subset \beta K'$.

Proof. — Let $E \in \mathscr{E}$ and let $y \in E$ be one of its interior points. Say E is a component of

$$V_{i_1} \cap \ldots \cap V_{i_v} \cap \partial V_{i_{v+1}} \cap \ldots \cap \partial V_{i_s}.$$

Now suppose $x \in p^{-1}(y)$. Then x has a disk neighborhood W so that $p(W)$ is a disk $\subset \operatorname{Int}(V_{i_1} \cap \ldots \cap V_{i_v})$. Now $x \in \operatorname{Int} N_i$ for some $i = i_1, \ldots, i_v$ so that we may suppose that $W \subset \operatorname{Int} N_i$. Hence there are disk neighborhoods $W_1, \ldots, W_w$ such that:

0) $p^{-1}(y) \subset \bigcup_j W_j$;

1) $p(W_j)$ is a disk G_j and $G_j \subset \operatorname{Int}(V_{i_1} \cap \ldots \cap V_{i_v})$;

2) if for some i, $W_j \cap N_i \neq \emptyset$, then either

a) i is one of $i_1, \ldots, i_v$ and $W_j \subset \operatorname{Int} N_i$, or

b) i is one of $i_{v+1}, \ldots, i_s$;

3) each N_j contains exactly one of the W_j, $i = i_1, \ldots, i_v$.

Parts 1) and 2) are clear and 3) can be brought about by amalgamating two or more W_j's.

Now, the sequence of collapsings, first the points of N_1 under g^m, then the points of N_2, etc. flatten the W_j in turn to disks D_j, according as to case *a)* happens in 2) above. The only further identifications among these disks occur at states where case *b)* of 2) applies.

Therefore, $p(\bigcup_j W_j)$ near y can be found by beginning with a set of w n-disks and identifying them along one side of $(n-1)$-cells through x, two or more at a time. Each of these $(n-1)$-cells corresponds to one of the ∂V_i, $i = i_{v+1}, \ldots, i_s$. We do not know that every one of these i's occurs, but this is all right, because this shows that a neighborhood of y in a larger intersection than

$$\partial V_{i_{s+1}} \cap \ldots \cap \partial V_{i_s}$$

lies in $\beta K'$. In particular, we have shown that $\beta K' \cap \operatorname{Int} E$ is open in Int E. But obviously $\beta K' \cap \operatorname{Int} E$ is closed in Int E so that $\operatorname{Int} E \subset \beta K'$. Again $\beta K'$ is closed so that $E \subset \beta K'$ completing the proof of (5.6.1) and (5.6).

Finally, by using the relative version of the powerful triangulation theorem of J. H. C. Whitehead [12], we can proceed through the skeleta of $\beta K'$ by induction and obtain

Lemma (**5.7**). — *If* K' *is as in the proof of* (5.5), *then* K' *has a smooth triangulation as an n-complex, with* βK *lying in the* $(n-1)$*-skeleton.*

Lemma (**5.8**). — *If* K' *is as in the proof of* (5.5), *then each point of* K' *has a neighborhood which can be smoothly embedded in* $\mathbf{R}^{n+2}$.

Proof. — For $x \in K'$ we may take a neighborhood $N = D_1 \cup \ldots \cup D_m$, the union of m smooth n-disks intersecting only along the branch set $\beta K'$ of K', and the (" π_i ") map $\pi : N \to \mathbf{R}^n$, where $\pi(\beta K')$ lies in an n-cross in $\mathbf{R}^n$. Choose a smooth function $\alpha_i : D_i \to \mathbf{R}$ so that $\alpha_i(x) = 0$ if $x \in \beta K'$ and $\alpha_i(x) > 0$ otherwise. Note that we may extend α_i to all of N so that $\alpha_i(x) = 0$ if $x \notin D_i$.

Next choose m distinct unit vectors $v_1, \ldots, v_m$ in $\mathbf{R}^2$ and define $f : N \to \mathbf{R}^n \times \mathbf{R}^2$ by

$$x \mapsto (\pi x, \sum_i \alpha_i(x) v_i).$$

Then f is the required smooth embedding.

§ 6. PERIODIC POINTS; THEOREM D

Lemma **(6.1)**. — *If* $g : K \to K$ *satisfies Axioms* 1), 2), 3^+), *then the periodic points of* g *are dense in* K.

Proof. — We need to show that each non-empty open set U in K contains a periodic point. Suppose, as *case* 1, that U is a (smooth) open n-disk. As pointed out above, there is a positive number Δ_1 such that if E is a smooth n-disk in K of diameter $< \Delta_1$, then $g | E$ is 1-to-1. There is an integer $m > 0$ such that if $i \geq m$, g^i increases all small distances in smooth n-disks by a factor $\geqslant 3$. That is, for $\Delta_2 > 0$ small enough, and $x, y \in D$, a smooth disk of diameter $< \Delta_2$, $d(g^i x, g^i y) \geq 3d(x, y)$, for $i \geq m$, provided $g^i | D$ is 1-1 for $j \leq i$.

Let a be the center of U and let D_ε denote the disk centered at a and of radius ε for $\varepsilon \leq \Delta_1, \Delta_2$, and the radius of U. Let $d > 0$ be such that D_d, D_{2d}, and D_{3d} are defined; choose $r > m$ so that $g^r(D_d) \cap D_d \neq \emptyset$, say $\mathcal{O} \in g^r(D_d) \cap D_d$.

We claim that $g^r(D_{2d}) \supset D_{2d}$. Even more, we claim that $D_{2d} \supset E$, a disk with

$$D_d \subset g^r(E) \subset D_{3d}.$$

To prove this, introduce polar coordinates with $\mathcal{O}$ as pole and $\theta \in S$ a parameter $(n-1)$-sphere.

Define $e : S \to D_{2d}$ by specifying that $[\mathcal{O}, e(\theta)]$ is the maximal arc in direction θ such that $g^r([\mathcal{O}, e(\theta)]) \subset D_{3d}$. Then $e : S \to D_{3d}$ is continuous and obviously 1-1, so that $E = \bigcup_\theta [\mathcal{O}, e(\theta)]$ is an n-cell. By induction and the properties of Δ_1 and Δ_2 we find that $g^i | E$ is 1-1 for all $i \leq r$ so that in particular $g^r | E$ is 1-1.

Now $\partial g^r(E) = g^r(e(s)) \subset D_{3d} - D_{2d}$ by choice of $e(\theta)$, so that, as $g^r(E)$ contains $\mathcal{O}$, $g^r(E) \supset D_{2d}$. Thus in particular $g^{-r} | D_{2d}$ is well defined and maps D_{2d} into D_{2d}. Hence g^{-r}, and therefore g^r, has a fixed point in $D_{2d} \subset U$. Thus in case 1, we have shown that the periodic points are dense in U.

General Case. — Choose $a \in U$ and let $V \subset U$ be a neighborhood of a and m an integer such that g^m immerses V in some disk, say D. Then " collapse " the points of K under g^m (see (2.2)) getting the shift equivalent $g' : K' \to K'$ and maps $r : K \to K'$, $s : K' \to K$ and integer m expressing this shift equivalence.

Let $b=r(a)$. Then b has a neighborhood $W=r(V)$ which is a disk so that case 1 applies. By case 1, the periodic points of g' are dense in W; by (5.2) the periodic points of g are dense in $s(W)$, a subdisk of D. Note that $s(W)$ contains $sr(a)=g^m(a)$ in its interior (as a disk), as s is an immersion, and that $s(W)$ is an open subset of D. As g^m immerses some neighborhood of a in D, it immerses some (smaller) neighborhood V′ of a in $s(W)$. But then as $g^m(V')$ is a (relative) neighborhood of a in D, it follows that the periodic points of g are dense in $g^m(V')$. Finally, as $g^{m+i}(V')\cap V'\neq\emptyset$ for some $i>0$, V′ contains a periodic point of g, which completes the proof of (6.1).

Corollary **(6.2)** (Theorem D). — *If h is a shift map of an n-solenoid Σ, then the periodic points of h are dense in Σ.*

Proof. — Let $g:K\to K$ be a presentation of $h:\Sigma\to\Sigma$ and let U be non empty and open in Σ. Then there is a V non empty and open in K and an integer n such that $\{x\in\Sigma : x_n\in V\}\subset U$. For $a_n\in V$, periodic under g, say $g^m a_n=a_n$ we have

$$(\ldots, ga_n, a_n, g^{m-1}a_n, g^{m-2}a_n, \ldots)$$

is periodic under h and is in U.

§ 7. LOCAL HOMOGENEITY OF n-SOLENOIDS

Theorem. — *Each point of an n-solenoid has a neighborhood of the form* (n-disk) × (Cantor set).

Proof. — Let $g:K\to K$ be a presentation of an n-solenoid Σ. As $g^i:K\to K$ presents Σ also (the shift map presented by g^i is different) for $i=1, 2, \ldots,$ there are $\varepsilon>0$ and an integer $r_0>1$ for which we may suppose:

1) K is covered by disk neighborhoods $U_i=\bigcup_{j=1}^{r_i} D_{ij}$ where g maps D_{ij} diffeomorphically onto D_i a disk, where $r_i\leq r_0$;

2) the Lebesgue number of the covering in 1) is $>\varepsilon$;

3) g is at least 2-to-1;

4) g increases distances $\leq r_0\varepsilon$ on n-disks by a factor $\geq r_0$.

To see this note that the ε can be chosen last. Next, having picked a finite cover as in 1), the bound r_0 obtained can be retained for a finer cover, which may be required for a bigger power of g needed to obtain 3) and especially 4).

Now choose $a=(a_0, a_1, \ldots)\in\Sigma$ and let N_0 be a disk neighborhood of a_0 of diameter $<\varepsilon$ so that g maps each disk of N_0 onto a disk E_0.

Lemma. — *Each component of $g^{-1}(N_0)$ is a disk neighborhood of diameter $<\varepsilon/2$.*

Proof. — Let $x_1\in C$, a component of $g^{-1}(N_0)$. Choose U_1 as in 1) with $d(x_1, \partial U_1)>\varepsilon$. Say $U_1=D_{11}\cup\ldots\cup D_{1r}$, $r\leq r_0$, where g maps each D_{1j} 1-to-1 onto a disk D_1. Then $d(gx_1, \partial D_1)\geq 2r_0\varepsilon$ so that $C\cap\partial U_i=\emptyset$. Thus $C\subset U_1$ as it is connected.

Now $g^2|D_{1j}$ maps D_{1j} onto a disk containing E_0 so that $g(D_{1j})\cap N_0=D_1\cap N_0=D_1'$, a disk mapping onto E_0 under g. Thus let $E_j=(g|D_{1j})^{-1}(D_1')$. Then E_j is a disk of diameter $\leq \varepsilon/2r_0 \leq \varepsilon/2s$ and as the E_j are connected and cover C, they form a coherent collection. That is, if $y, y' \in \bigcup_j E_j = C$, then there is a sequence of E_j's " chaining " y to y', so that adjacent elements of the sequence intersect. It follows that

$$\text{diameter of } C \leq s.(\varepsilon/2s)=\varepsilon/2.$$

This proves the lemma.

Let $N_i=g^{-i}(N_0)$ for $i=1, 2, \ldots$ and consider the diagram

$$\begin{tikzcd}
 & N_0 \arrow[dl, dashed] & & N_1 \arrow[ll, "g"'] \arrow[dl, dashed] & & N_2 \arrow[ll] \arrow[dl, dashed] & & \cdots \arrow[ll] \\
E_0 & & E_1\times F_1 \arrow[ll, dashed] \arrow[ul, dashed] & & E_2\times F_2 \arrow[ll, dashed] \arrow[ul, dashed] & & E_3\times F_3 \arrow[ll, dashed] \arrow[ul, dashed] &
\end{tikzcd}$$

The factorization of $g|N_i$ is through $E_i\times F_i$, where E_i is an n-disk and F_i is a finite set having as many points as N_i has components. This factorization was shown to exist in the proof of the lemma. Note that each component of N_i maps onto E_0 under g^i, by induction. Thus each of the disks $E_i\times(\mathrm{pt})$ maps onto a disk of the form $E_{i-1}\times(\text{a point})$. The top sequence describes a neighborhood N of $a\in\Sigma$. The bottom sequence which describes N as well, clearly has a limit of the form

$$(n\text{-disk})\times C$$

where C is the inverse limit of the induced sequence

$$(**) \qquad F_1 \leftarrow F_2 \leftarrow F_3 \leftarrow \ldots$$

Thus to show C is a Cantor set it will suffice to show that the maps in $(**)$ are at least 2-to-1 after some finite stage. But

a) there is a $\delta>0$ so that $g^{-1}(x)$ contains at least two points with mutual distance $\geq\delta$ for all $x\in K$; and

b) by the lemma there is an I such that for $i>I$, each component of N_i has diameter $<\delta$.

That is, for $i\geq I_0$, there are at least two components of N_{i+1} which map onto each component of N_i, or the map $F_{i+1}\to F_i$ is at least 2-to-1.

§ 8. REALIZATION OF n-SOLENOIDS AS EXPANDING ATTRACTORS

In order to prove Theorem B we suppose $h:\Sigma\to\Sigma$ is an n-solenoid and choose a representative $g:K\to K$ such that K is normally branched (see (5.5) and following).

Lemma **(8.1)**. — *There is a smooth embedding of* K *in some euclidean space.*

Proof. — Cover K with neighborhoods $\{N_i\}_{i=1}^m$ as in (5.7). We suppose the covering so chosen that it still covers when shrunk slightly to, say, $V_i\subset N_i$. Choose an embedding $\varphi_i : V_i \to \mathbf{R}^{n+2}$ which in turn is extended to $\varphi_i : N_i \to \mathbf{R}^{n+2}$, so that φ_i sends a neighborhood of ∂N_i to $0 \in \mathbf{R}^{n+2}$. Thus we may assume φ_i defined on all of K by setting $\varphi_i(x) = 0$, for $x \in K - N_i$. This yields a smooth map

$$\varphi_1 \times \varphi_2 \times \ldots \times \varphi_m : K \to \mathbf{R}^{m(n+2)}$$

which is 1-to-1 on each V_i and hence an embedding.

The notion of " tubular neighborhood " of an embedded branched n-manifold is unfortunately a bit awkward:

Definition (**8.2**). — Given a branched n-manifold K embedded in a manifold M^{n+k}, a *tubular neighborhood* of K is a neighborhood N together with a finite *ordered* covering $N = N_1 \cup N_2 \cup \ldots \cup N_r$ and diffeomorphisms $\varphi_i : D^n \times D^k \to N_i$, $i = 1, \ldots, r$, such that:

1) There is a system of charts $\{N_i \cap K; \pi_i, \alpha_{ii'}\}$ where
   a) $N_i \cap K$ is a disk neighborhood $= \bigcup_j D_{ij}$; and
   b) $\pi_i = \psi \circ \varphi_i^{-1} | (N_i \cap K)$ where $\psi : D^n \times D^k \to D^n$ is the projection.

2) If two fibers $\varphi_i(a \times D^k)$, $\varphi_j(b \times D^k)$ intersect, then one contains the other. Specifically $\varphi_i(a \times D^k) \supset \varphi_j(b \times D^k)$ if $i < j$.

Note that for unbranched manifolds, condition 2) would read $\varphi_i(a \times D^k) = \varphi_j(b \times D^k)$.

Remark (**8.2.1**). — In the notation of (8.2) each of the disks D_{ij} making up U_i has the form $D_{ij} = \{\varphi_i(x, \psi_{ij}x) : x \in D^n\}$, where $\psi_{ij} : D^n \to D^k$ is a smooth map.

Proof. — For each $x \in D^n$, $\varphi_i^{-1}(D_{ij}) \cap (x \times D^k)$ is a single point of the form $(x, \psi_{ij}(x)) \in D^n \times D^k$. We can also write

$$\psi_{ij} = \pi \circ \varphi_i^{-1} \circ (\pi_i | D_{ij})^{-1} : D^n \to D^k$$

where $\pi : D^n \times D^k \to D^n$ is the projection onto the second factor. This is well defined and smooth, as $(\pi_i | D_{ij})$ and φ_i are diffeomorphisms.

The complication of our tubular neighborhood occurs only at the branch set. We prove next that we may assume that the branch set of K lies in an actual manifold, V.

Lemma (**8.3**). — *Given a normally branched, n-manifold* K, *there is an n-manifold* V *with boundary* ∂V *such that* $K \cup V$ *is a branched manifold and* $V - \partial V$ *contains the branch set of* $K \cup V$.

Proof. — Given any point $b \in \beta K$, the branch set of K, there is a U_i with $b \in U_i$, such that $\beta K \cap U_i$ is mapped diffeomorphically onto $B_i \subset \mathbf{R}^n$ by $\pi_i : U_i \to \mathbf{R}_n$. Let V_i be a neighborhood of B_i in $\mathbf{R}^n$. Then we can think of $\pi_i \cup 1$ as immersing the disjoint union $U_i \cup V_i$ into $\mathbf{R}^n$. Thus pinching along $\beta K \cup B_i$ (see (2.2)), we obtain the conclusion locally, for each coordinate chart.

Next, if $U_i \cap U_j \neq \emptyset$, then $\pi_i | ((U_i \cup V_i) \cap (U_j \cup V_j))$ immerses this set in $\mathbf{R}^n$. Thus pinching along $V_i \cup V_j$ puts these pieces together. The resulting quotient V' of $\bigcup_i V_i$ is a neighborhood of the branch set, but is apt to have a poor boundary. However,

$\operatorname{Int} V'$ is certainly a smooth manifold and contains $\beta K = \beta(U \cup V')$. Thus there is a compact smooth manifold V with boundary ∂V lying in V' and containing βK in its interior. This proves (8.3).

Lemma **(8.4)**. — *Given an embedding h of the branched manifold* $K \cup V$ *of* (8.3), *there is a nearby embedding* (C^r *topology*) *of* $K \cup V_0$ *which has a tubular neighborhood* (8.2). V_0 *is a (perhaps) smaller version of* V.

Proof. — As V is an actual manifold, it has a tubular neighborhood in the standard sense—that is, a neighborhood N_V along with a projection $\varphi_V : N_V \to V$. By taking V smaller if necessary but without decreasing the radius of the fibers, we may assume that $(K \cap V) \cap \varphi_V^{-1}(\mathrm{pt})$ lies in the interior of the disk $\varphi_V^{-1}(\mathrm{pt})$. Pick a local chart (U_i, π_i) where $V_i = U_i \cap N_V \neq \emptyset$ and consider the two maps

$$(*)\qquad \begin{aligned} &N_V \cap U_i \xrightarrow{\varphi_V} V_i \xrightarrow{\pi_i} \mathbf{R}^n \\ &N_V \cap U_i \xrightarrow{\qquad\pi_i\qquad} \mathbf{R}^n. \end{aligned}$$

These agree on V_i and are smooth. Thus for each of the disks D_{ij} making up U_i, the map $x \mapsto \varphi_V x \mapsto (\pi_i | D_{ij})^{-1} \pi_i \varphi_V x$, $x \in D_{ij}$, is a C^r diffeomorphism near the branch set βK.

It follows that for the inverse map $x \mapsto \theta_{ij} x$, $\theta_{ij} : D_{ij} \to D_{ij}$, defined perhaps only near βK, is a C^r diffeomorphism. As they agree on overlaps, the θ_{ij} define a diffeomorphism $\theta_0 : W_0 \to K$, where W_0 is a neighborhood of βK in K. θ_0 is a diffeomorphism, is the identity on $W_0 \cap V$ and C^r-near the identity on W_0.

Hence there is an isotopy between θ_0 and the identity on W_0, and using this isotopy we can define a diffeomorphism $\theta : K \to K$, where $\theta = \theta_0$ near βK, say on the neighborhood W of βK, and $\theta =$ the identity of a small neighborhood of βK. Now compose the original embedding of K with θ and we have a new embedding which is C^r-near the original. Next, let V' be a smaller version of V and $N' = N_V$ cut down (as a bundle) to V', where V' is chosen small enough so that $N' \cap (K \cup V) \subset W$. Let $\varphi' = \varphi_V | N'$. Now the maps π_i and $\pi_i \varphi'$ agree on $N' \cap U_i$ as $N' \cap U_i \subset W$.

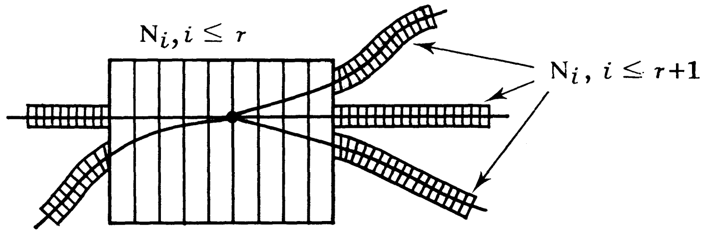
Let $\{D_i\}_{i=1}^{v}$ be a covering of V' by smooth disks and let $N_i = \varphi'^{-1}(N_i)$. Then there are diffeomorphisms $\varphi_i : D^n \times D^k \to N_i$ sending the center disk $D^n \times \mathcal{O}$ to D_i, $\mathcal{O} \in D_i$, in such manner that $\varphi' \circ \varphi_i$ sends each fiber $a \times D^k$ to the point $\varphi_i(a \times \mathcal{O}) \in D_i$. One easily checks that $N_1 \cup \ldots \cup N_r$ is a tubular neighborhood of a part of $K \cup V$; in particular the inclusion of part 2) of (8.2) is an equality so far.

Furthermore, for each point $y \in V'$, $(K \cup V) \cap \varphi'^{-1}(y)$ is finite and lies in the interior of $\varphi'^{-1}(y)$ as pointed out in the first paragraph of this proof. Then

$$K' = K \cup V - \varphi'^{-1}(\operatorname{Int} V)$$

is an actual manifold, and can be written as $D_{r+1} \cup \ldots \cup D_s$, where these D_i's overlap only as in (unbranched) manifolds. Note that the fibers of φ' which intersect D_i for

$r+1\leq i\leq s$ are C^r-near the normal bundle of D_i. Thus for $i=r+1, \ldots, s$ we may choose $\varphi_i : D^n\times D^k\to N_i$ for these disks, to be consistent with one another, and with the fibers of φ'. This is done one disk at a time, and at each stage, the bundle already chosen over part of D_i is so near to the normal bundle of D_i as to be trivial and hence extendable. By cutting down the fibers $\varphi_i(\mathrm{pt}\times D^k)$ if necessary, we may assume $N_i\cap N_j\neq\emptyset$, implies $D_i\cap D_j\neq\emptyset$, for $r+1\leq i<j\leq s$. (See figure 8.)



One easily checks that the N_i so described satisfy the definition (8.2).

Lemma **(8.5)**. — *There is a diffeomorphism* $f : N\to f(N)\subset N$ *such that for each* $x\in K$, f *sends the fiber at* x *by a contraction into the fiber at* gx.

Proof. — We will use two versions of tubular neighborhoods of K, one corresponding to the domain of f, the other to the range. The fibers of the first neighborhood will be larger, but the " horizontal " disks will be bigger in the second neighborhood. This corresponds, of course, to the two parts of the hyperbolic structure. These neighborhoods are easily obtained from the result (8.4) by standard topological devices.

Thus by (8.4) there exists

1) a tubular neighborhood $N=N_1\cup\ldots\cup N_r$ given by

$$\varphi_i : D^n\times D^k\to N_i, \quad i=1, \ldots, r,$$

where $\bigcup_i \operatorname{Int} N_i\supset K$;

2) a tubular neighborhood $M=M_1\cup\ldots\cup M_s$ given by

$$\theta_j : D^n\times D^k\to M_j, \quad j=1, \ldots, s,$$

such that

a) Each of the fibers, $\theta_i(a\times D^k)$ of M, lies in one of the fibers, $\varphi_j(b\times D^k)$, of N.

b) Each of the disks D_{ij} making up $U_i=N_i\cap K$ is mapped diffeomorphically by g into $g(D_{ij})$.

c) Each $g(D_{ij})$ as in *b)* lies in a single M_ℓ.

For each $i=1, \ldots, r$, choose D_{i1}, one of the D_{ij}'s making up $N_i\cap K$. We define inductively $f_i : N\to M$ as follows. For some ℓ, the box $M_\ell\supset g(D_{11})$. Using the structure $\theta_\ell : D^u\times D^k\to M_\ell$, there is a diffeomorphism $f_1 : N_1\to M_\ell$, such that f_1 extends $g|D_{11}$ and maps the normal fiber at x by a contraction into the normal fiber at gx. Now consider N_2. As N_2 may intersect N_1, f_1 may already be defined on a portion of D_{21}. But f_1x is in the same fiber as gx and as this fiber is small, f_1x is near gx. Now $g(D_{21})\subset M_m$

for some m. Thus it is easy to find a diffeomorphism $f_2 : D_{21} \to M_m$, so that $f_2 = f_1$ on $D_{21} \cap N_1$ and f_{2x} lies in the same fiber as gx. Now f_2 is defined on D_{21}, which may be considered for the moment as the " center " of N_2, and agrees with f_1 on the overlap $N_1 \cap D_{21}$. Thus it is an easy matter to extend f_1 to all of N_2, so as to send fibers into fibers by contractions and agree with f_1 on $N_1 \cap N_2$.

Continuing in this manner, one defines $f_3, \ldots$ until one has f_r defined on all of N. The crucial fact at each step is that though f_{i-1} has been defined on a portion of N_i, it must map this into a trivial bundle, as it is a diffeomorphism of a trivial bundle.

Proof of Theorem B **(8.6)**. — We now have $K \subset N$, a tubular neighborhood and a map $f : N \to \operatorname{Int} N$ such that:

1) for each $x \in K$, $f(x)$ is in the same fiber as $g(x)$; and

2) f sends each fiber into a fiber by a contraction.

Now consider the projection $q : N \to K'$ as in the proof of (4.4) of Theorem A. As in (4.4) we obtain a commutative diagram

$$\begin{tikzcd} f(N) \arrow[d, "i"] & N \arrow[l] \arrow[dd, "q"] \\ N \arrow[d, "q"'] & \\ K' & K' \arrow[l, "g'"] \end{tikzcd}$$

where g' exists by the fact that f preserves fibers (property 2). Thus, as above, $\Lambda = \Omega(f) = \bigcap_{i=0}^{\infty} f^i(N)$ and $f | \Lambda$ is topologically (actually C^r) conjugate to the shift map $h' : \Sigma' \to \Sigma'$ determined by $g' : K' \to K'$.

But we also have the diagram

$$\begin{tikzcd} K \arrow[d, "i"] & K \arrow[l, "g^m"'] \arrow[d, "i"] \\ N \arrow[d, "q"'] & N \arrow[ul, "f^m"] \arrow[d, "q"] \\ K' & K' \arrow[l, "g'm"] \end{tikzcd}$$

where m is chosen via Axiom 2) (see (3.8)). The top triangle does not commute, but the large rectangle does. Whereas the map q^{-1} sends a point of K' to a fiber of N which intersects finitely many (perhaps more than one) cells in K, g^m collapses all of these to a single cell, so that $g^m q^{-1}$ is well defined. Let $r = g^m q^{-1}$. Then we have

$$q_0 = q | K : K \to K', \quad r : K' \to K,$$

and the integer m giving a shift equivalence between g and g'. Hence $f|\Lambda$ is topologically conjugate to the shift map $h:\Sigma\to\Sigma$ presented by $g:K\to K$.

The fibers of N provide $f|\Lambda$ with the stable part of a hyperbolic structure, say E^s, which is defined on the neighborhood N and behaves well under f. The existence of the unstable part of the hyperbolic structure follows by techniques which are now fairly well known (see e.g. Smale [11], Alekseev [0; p. 119 (4)] or Hirsch-Pugh [5]).

§ 9. PROOF OF THEOREM E

Theorem **(9.0)**. — *For some* $j>0$, $\hat{H}_j(\Sigma;\mathbf{R})\neq 0$, *for each n-solenoid* ($\hat{H}$ *is Čech-theory*).

Proof. — Let $g:K\to K$ be a presentation of $h:\Sigma\to\Sigma$ satisfying axioms 1), 2), 3^+) and 4). (Axiom 4) allows us to apply the Lefchetz trace formula. One could, without Axiom 4), proceed as in [17]). If K (that is, the tangent bundle TK of K) is orientable, then there is a formula due to Smale ([9] or [17]):

$$N_i(g)=\pm\sum_j(-1)^j \operatorname{trace} g^i_{*j} \tag{9.1}$$

where $N_i(g)$ is the number of points left fixed by g^i, and $g_{*j}:H_j(K;\mathbf{R})\to H_j(K;\mathbf{R})$ is the map induced by g. (9.1) applies here, as TK plays the role of E^u, by Axiom 3^+). But (6.1) we know the periodic points of g are dense in K. Hence the set of numbers $\{\operatorname{tr} g^i_{*j}\}$ is unbounded, for some $j>0$, because

$$\operatorname{tr} g^i_{*0}\leq (\text{the number of components of K})<\infty.$$

Note that if $h:V\to V$ is an endomorphism of a real finite dimensional vector space where $\{\operatorname{tr} h^i\}$ is unbounded, then the inverse limit of

$$V\xleftarrow[h]{}V\xleftarrow[h]{}V\xleftarrow[h]{}\dots$$

is non-zero. Hence $\hat{H}_j(\Sigma;\mathbf{R})\neq 0$.

In case K is not orientable, we consider the orienting double cover $\tilde{K}\to K$ and note that the map g can be lifted so as to preserve a (chosen) orientation. Then the involution $T:\tilde{K}\to\tilde{K}$ exchanging the two points of the fiber, reverses this orientation. We proved the formula

$$2N_i(g)=\pm\sum_j(-1)^j \operatorname{trace}(g^i_{*j}(1-T_{*j})) \tag{9.2}$$

in the paper [17]. It follows as before that $\hat{H}_j(\tilde{\Sigma}:\mathbf{R})\neq 0$, for some $j>0$, where $\tilde{\Sigma}$ is the inverse limit of

$$\tilde{K}\xleftarrow[\tilde{g}]{}\tilde{K}\xleftarrow[\tilde{g}]{}\tilde{K}\xleftarrow[\tilde{g}]{}\dots$$

and hence is the oriented double cover of Σ. It is well known that the covering map induces an isomorphism

$$\hat{H}_*(\tilde{\Sigma}:\ F)\to H(\Sigma;F)$$

over any field F of characteristic $\neq 2$ or more directly, either $\widetilde{\Sigma}\to\Sigma$ is non-trivial, in which case $H_1(\widetilde{\Sigma};\mathbf{R})\neq 0$, or $\widetilde{\Sigma}=\Sigma\times\mathbf{Z}_2$, in which case $\hat{H}_j(\Sigma;\mathbf{R})\neq 0$, being "half" of $\hat{H}_j(\widetilde{\Sigma};\mathbf{R})$.

§ 10. A CRITERION FOR AXIOM 1)

In this section we give an alternative way of checking Axiom 1), for an immersion satisfying Axioms 2), 3^+), as well as 0), 4) which we introduce now. This essentially repeats a part of [16, § 1, pp. 343-344].

Axiom 0). — $g:K\to K$ is indecomposable. That is, K is not the union of two disjoint closed invariant subsets.

Axiom 4). — K is triangulated as an n-complex whose $(n-1)$-skeleton contains its branch set.

Remark. — Given $g:K\to K$ satisfying Axiom 2), 3^+), one can subdivide K into finitely many disjoint subsets $K_1,\ldots,K_r$ so that each of $g|K_i:K_i\to K_i$ satisfies Axioms 0), 2), 3^+). We proved in (5.6) that any $g:K\to K$ satisfying 1), 2), 3^+), is shift equivalent to one satisfying 1), 2), 3^+) and 4).

Criterion (**10.1**). — *If* $g:K\to K$ *satisfies Axioms* 0), 2), 3^+), *and* 4) *then it satisfies Axiom* 1) *if and only if there is an integer m such that* $g^m(\sigma)\supset K$ *for each n-simplex* σ *of* K.

Proof. — First, if g satisfies this criterion, and N is a neighborhood of a point p, then for some i, $g^i(N)$ contains a simplex σ, by Axiom 3^+). Then, applying the criterion, $g^{i+m}(N)\supset K\supset N$ so that p is a non-wandering point, and Axiom 1) is verified.

The other way requires several steps: the first is essentially trivial (see also [16; p. 344]).

Lemma (**10.2**). — *If g satisfies Axioms* 1), 2), 3^+), *then so does* g^i *for* $i=2, 3, \ldots$

Lemma (**10.3**). — *If* $g:K\to K$ *satisfies Axioms* 1), 2), 3^+) *and* D *is any n-cell in* K, *then there is an integer n such that* $g^m(D)\supset D$.

Proof. — D is a subset of the union of finitely many n-simplexes, say $\sigma_1,\sigma_2,\ldots,\sigma_r$. Choose a periodic point, say x_i, in the interior of $\sigma_i\cap D$, $i=1,2,\ldots,r$, and let

$$\lambda=\max\rho(x_i,\partial\sigma_i)/\min(\rho(x_i,\partial\sigma_i\cup\partial D)).$$

Then $\lambda>0$ and for some integer m, g^m increases distances by more than λ and $g^m(x_i)=x_i$. By an argument as in the proof of case 1 of (6.1), $g^m(D\cap\sigma_i)\supset\sigma_i$ so that $g^m(D)\supset D$.

Lemma (**10.4**). — *Let* $g:K\to K$ *satisfy Axioms* 0), 1), 2), 3^+), 4) *and let* D *be an n-disk in* K. *Then* $\bigcup_{i=1}^{\infty}g^i(D)$ *is dense in* K.

Proof. — Assume false, let D^0 be the interior of D and let A be the closure of $\bigcup_{i=1}^{\infty} g^i(D)$ and B the closure of $K-A$. Note that A is invariant under g. Likewise B, for if $b \in K-A$, $gb \notin \bigcup_{i=1}^{\infty} g^i(D^0)$, for otherwise b would be a wandering point. That is

$$g(K-A) \subset \overline{K-A}.$$

Hence $g(B) \subset B$.

By Axiom 0), there is a point $x \in A \cap B$. Let E be a small open n-disk containing x. Then E intersects both $\bigcup_{i=1}^{\infty} g^i(D^0)$ and $K-A$. Therefore there is a periodic point $a \in g^i(D^0) \cap E$ for some i. We may suppose a is not a branch point. If m is a period of a, then $a \in g^{m+i}(D^0)$ so that if m is also sufficiently large, $g^{m+i}(D^0)$ contains the closed n-simplex containing a, which in turn contains x. In particular $g^{m+i}(D^0)$ contains a disk E' having x as interior point.

By Axiom 2), there is an integer r and a neighborhood N of x such that $g^r(N)$ lies in a disk. Thus $g^r(N) \subset g^r(E')$. There is a point $b \in N \cap (K-A)$; but then

$$g^r(b) \in g^r(E') \subset g^{m+i+n}(D^0) \subset \text{Interior of A}.$$

As this is absurd, this completes the proof of (10.4).

Lemma **(10.5)**. — *If* $g : K \to K$ *satisfies Axioms* 0), 1), 2), 3^+), 4) *and* D *is an n-disk in* K, *then* $g^m(D) \supset K$ *for some m.*

Proof. — By (10.3) there is an r such that $g^r(D) \supset D$. Then

$$D \subset g^r(D) \subset g^{2r}(D) \subset \ldots$$

Now suppose E is an n-simplex in K. Then there is a periodic point $a \in (\text{Int } E) \cap g^{rk}(D)$ for some k, by (9.4). Thus if m is a large multiple of rk and of the period of a, then $E \subset g^m(D)$. As there are only finitely many n-simplexes (as E), the lemma follows.

Lemma **(10.6)**. — *For lemma* (10.4) *we need only assume Axioms* 0), 1), 2), 3^+).

Proof. — For assume $g : K \to K$ satisfies Axioms 0), 1), 2), 3^+) and let D be an n-disk in K. By (5.6) there is a shift equivalent $g'K' \to K'$ which satisfies Axioms 0), 1), 2), 3^+), 4). Say $r: K \to K'$, $s : K' \to K$ and the integer p give the shift equivalence. Then $r(D)$ contains a disk, say D', so that, by (10.5), $g'(D') \supset K'$, for some m. But then

$$g^{(p+1)m}(D) = s(rs)^{pm} r(D) \supset sg'^m(D) \supset s(K') = K.$$

This uses the fact that g is onto, which follows from Axiom 1).

§ 11. NEIGHBORHOODS OF COMPACTA, NICE RELATIVE TO A FOLIATION

Such neighborhoods always exist, with various meanings put to " nice ". In this section, " nice " will be given in terms of a triangulation (see also Munkres [6]) though we could proceed using " foliation boxes " as in Gromov [1; Th. 4]. Such neighborhoods give insight into the proof of (2.1) above.

Definition (**11.0**). — If $\mathscr{F}$ is a C^r foliation of a manifold M and σ is a smooth simplex lying in M, σ is said to be *transverse to* $\mathscr{F}$ provided there is a chart $\varphi : U \to \mathbf{R}^n$ defining $\mathscr{F}$ locally where $\sigma \subset U$, its vertices lie on distinct fibers, and $\varphi | \sigma$ is smoothly conjugate to a linear map. A C^r triangulation of a subset of M is *transverse to* $\mathscr{F}$ if each of its simplexes is.

The principal result of this section is the

Lemma (**11.1**). — *If* K *is a compact set in a manifold* M *with a* C^r *foliation* $\mathscr{F}$, *then there is a neighborhood* N *of* K *and a* C^r *triangulation of* N *which is transverse to* $\mathscr{F}$ $(r \geq 1)$.

Proof. — First triangulate a neighborhood of K so that each simplex lies in an open set U of a chart giving the foliation. We may assume that N is compact and that each simplex of N lies in a simplex of (top) dimension m. Linearly order the m-simplexes, $\sigma_1, \sigma_2, \ldots, \sigma_s$.

There is an $\varepsilon > 0$ such that any piecewise smooth map within ε of the identity in the C^r metric is a homeomorphism. Let $\sigma_1 \subset U$, $\varphi : U_1 \to \mathbf{R}^n$ be a chart for the foliation $\mathscr{F}$. We think of $U \subset \mathbf{R}^n \times \mathbf{R}^{m-n}$ with φ the vertical projection. Choose a bump function $\psi : N \to \mathbf{R}$ where $\psi = 1$ on σ, $\psi = 0$ outside a small neighborhood N_1 of σ_1, and $||\varphi||_r < \infty$.

There is a subdivision σ_1' of σ_1 such that:

a) The vertices of each simplex lie on distinct fibers of $\varphi_1 : U_1 \to \mathbf{R}^n$.

b) The natural map $\varphi_\sigma : \sigma \to \tau_\sigma$ from a simplex $\sigma \in \sigma_1'$ to the rectilinear simplex τ_σ spanning its vertices is within $\varepsilon / 3 ||\varphi||$ of the identity in the C^r metric.

Part *b)* requires some explanation. First there is the natural map $\sigma \to \tau_\sigma$ given by their respective barycentric coordinates. As these are C^∞ functions, this map is C^r. Secondly, this map will be C^r-close to the identity provided the tangent spaces of the various faces of σ vary only a little within any one face.

Now define $f_1 : U_1 \to U_1$ by

$$x \mapsto \psi(x) \sum_{\sigma \in \sigma_1'} \lambda_\sigma(x) \varphi_\sigma(x) + (1 - \psi(x)) x$$

where λ_σ is a partition of unity associated with a slight enlargement of the m-simplexes $\sigma \in \sigma_1'$. Then in the C^r-metric, $||f - \mathrm{id}|| \leq ||\psi|| \cdot \sum_\sigma \lambda_\sigma ||\varphi_\sigma - \mathrm{id}|| \leq \varepsilon$. One needs to note as well that for a face τ of several σ, the various φ_σ agree on τ, as their barycentric coordinates do. Then f_1 is well defined. This gives us a new triangulation in which the simplexes σ of σ_1' are " straight " relative to U_1.

Note that the local projection $\varphi_1 | \sigma$, for any simplex $\sigma \in \sigma_1'$, is linear and has distinct values on distinct vertices of σ. Thus there is an $\varepsilon_1 > 0$ such that if f is a piecewise smooth map C^r-within ε of the identity, then the local projections $\varphi_1 | f(\sigma)$ are still conjugate to $\varphi_1 | \sigma$.

Thus we proceed to U_2, subdivide σ_2 to σ_2' compatibly with the given subdivision of σ_1, and straighten the simplexes of σ_2', moving nothing further than ε_1, so that now the local projections are conjugate to linear ones on all simplexes lying in $\sigma_1 \cup \sigma_2$. Continuing in this way, we find the desired triangulated neighborhood, N.

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