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ERIC M. FRIEDLANDER

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FIBRATIONS IN ETALE HOMOTOPY THEORY

by ERIC M. FRIEDLANDER

Let $f: X \to Y$ be a map of geometrically pointed, locally noetherian schemes and let A be a locally constant, abelian sheaf on X. We derive the Leray spectral sequence

$$E_2^{p,q} = H^p(Y, R^q f_* A) \Rightarrow H^{p+q}(X, A)$$

as a direct limit of spectral sequences obtained from the simplicial pairs constituting f_{et}, the etale homotopy type of f. This " simplicial Leray spectral sequence " can be naturally compared to the cohomological Serre spectral sequence for f_{et} and A. We employ this comparison to investigate the map in cohomology induced by the canonical map $(X_y)_{et} \to \mathfrak{h}(f_{et}^r)$, where $\mathfrak{h}(f_{et}^r)$ is the homotopy theoretic fibre of f_{et} and X_y is the geometric fibre of f.

Of particular interest are the special cases: 1) $f: X \to Y$ is proper and smooth, and 2) $f: X \to Y$ is the structure map of an algebraic vector bundle minus its zero section. In these cases,

$$H^*(\mathfrak{h}(f_{et}^r), A) \tilde{\to} H^*((X_y)_{et}, A)$$

for any locally constant, abelian sheaf A on X with finite fibre whose order is not divisible by the residue characteristics of Y. With certain hypotheses on the fundamental groups involved, we obtain a long exact sequence of homotopy pro-groups:

$$\dots \to \pi_n(\widehat{(X_y)_{et}}) \to \pi_n(\hat{X}_{et}) \to \pi_n(\hat{Y}_{et}) \to \pi_{n-1}(\widehat{(X_y)_{et}}) \to \dots$$

where $\widehat{(\ )}$ denotes profinite completion " away from the residue characteristics of Y ".

We conclude with a proof of Adams' Conjecture concerning the kernel of the J-homomorphism on complex vector bundles over a finite C-W complex [1]. The proof is a completion of a proof sketched by D. Quillen [13], employing the result that an algebraic vector bundle minus its zero section has completed etale homotopy type equal to a completed sphere fibration.

We caution the reader to remember that throughout this paper, a sheaf F on a scheme X will be a sheaf on the etale site of X; thus, the cohomology on X with coefficients in F, $H^*(X, F)$, will be etale cohomology [3].

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1. The Category J_f Associated to a Map of Schemes

After establishing some notation, we recall the definition of a *special map* of simplicial schemes and of an *etale hypercovering* of a scheme. Relativizing the category of etale hypercoverings of a scheme, we introduce the category J_f associated to a map $f: X \to Y$ of schemes. This category J_f is seen to be codirected with cofiltering homotopy category, Ho J_f. In subsequent sections, we shall study various functors defined on J_f and Ho J_f; in particular, the etale homotopy type of f is a pro-object indexed by Ho J_f.

Let a scheme X be given. Denote by (Et/X) the category of schemes etale over X, with maps in (Et/X) covering the identity map of X. Let $\Delta^0(\mathrm{Et}/X)$ denote the category of simplicial objects $\overline{U}_{\bullet}$ of (Et/X). If X is a "pointed scheme" with given geometric point $x: \operatorname{Spec} \Omega_x \to X$, let $\Delta^0(\mathrm{Et}/X)$ denote the category whose objects $\overline{U}_{\bullet}$ each have a chosen geometric point $u: \operatorname{Spec} \Omega_x \to (\overline{U}_{\bullet})_0 = \overline{U}_0$ above x, and whose maps $\overline{U}'_{\bullet} \to \overline{U}_{\bullet}$ are pointed, simplicial maps.

Given a (pointed) scheme X and a (pointed) simplicial set $S_{\bullet}$, let $\overline{S}_{\bullet}$ in $\Delta^0(\mathrm{Et}/X)$ denote the naturally associated (pointed) simplicial scheme: $(\overline{S}_{\bullet})_n$ is given as the disjoint sum of copies of X indexed by the set S_n, and the face and degeneracy maps of $\overline{S}_{\bullet}$ are induced by the face and degeneracy maps of $S_{\bullet}$. In particular, $\overline{\Delta[0]}$ denotes the final object of $\Delta^0(\mathrm{Et}/X)$. Given two maps $f, g: \overline{U}'_{\bullet} \rightrightarrows \overline{U}_{\bullet}$ in $\Delta^0(\mathrm{Et}/X)$, a categorical homotopy connecting f and g is a map $\overline{U}'_{\bullet} \times_X \overline{\Delta[1]} \to \overline{U}_{\bullet}$ in $\Delta^0(\mathrm{Et}/X)$ restricting to f and g (and furthermore, sending $(s_0 u', 01)$ to $s_0 u$ if X is pointed).

A map $\overline{U}'_{\bullet} \to \overline{U}_{\bullet}$ in $\Delta^0(\mathrm{Et}/X)$ is *special* provided that $\overline{U}'_0 \to \overline{U}_0$ is surjective and provided that for each $n > 0$

$$\overline{U}'_n \to (\operatorname{cosk}_{n-1} \overline{U}'_{\bullet})_n \underset{(\operatorname{cosk}_{n-1} \overline{U}_{\bullet})_n}{\times} \overline{U}_n$$

is surjective (recall that the functor $\operatorname{cosk}_n(\ )$ is right adjoint to the truncation functor). One can readily check that the composition of special maps is special; moreover, the pull-back of a special map by any simplicial map is special. An object $\overline{U}_{\bullet}$ of $\Delta^0(\mathrm{Et}/X)$ is called an *etale hypercovering* of X provided that the unique map $\overline{U}_{\bullet} \to \overline{\Delta[0]}$ is special.

We denote by J_X the category whose objects are etale hypercoverings of X and whose maps are special maps in $\Delta^0(\mathrm{Et}/X)$. We denote by Ho J_X the homotopy category of J_X: a map in Ho J_X is an equivalence class of maps in J_X, where the equivalence relation is generated by pairs of maps connected by a categorical homotopy.

Definition (1.1). — Let $f: X \to Y$ be a (pointed) map of schemes. Define J_f to be the following category: the objects of J_f are pairs $\overline{U}_\bullet \to \overline{V}_\bullet$, with $\overline{V}_\bullet$ in J_Y, $\overline{U}_\bullet$ in J_X, and the induced simplicial map $\overline{U}_\bullet \to \overline{V}_\bullet \times_Y X$ in J_X (*i.e.*, special); a map

$$\begin{array}{ccc} \overline{U}'_\bullet & & \overline{U}_\bullet \\ \downarrow & \rightarrow & \downarrow \\ \overline{V}'_\bullet & & \overline{V}_\bullet \end{array}$$

in J_f is a pair of maps, $\overline{V}'_\bullet \to \overline{V}_\bullet$ in J_Y and $\overline{U}'_\bullet \to \overline{U}_\bullet \times_{\overline{V}_\bullet} \overline{V}'_\bullet$ in J_X.

A connected category J is said to be *codirected* provided that any diagram $\begin{array}{ccc} i & & \\ & \searrow & \\ & & k \\ & \nearrow & \\ j & & \end{array}$ in J can be completed to a commutative square $\begin{array}{ccccc} & & i & & \\ & \nearrow & & \searrow & \\ \ell & & & & k \\ & \searrow & & \nearrow & \\ & & j & & \end{array}$.

A codirected category J is said to be *cofiltering* provided that for any pair of maps $j \rightrightarrows k$ in J, there exists a map $i \to j$ in J such that the compositions $i \to j \rightrightarrows k$ are equal.

We proceed to relativize Verdier's proof that J_X is codirected and $\mathrm{Ho}\, J_X$ is cofiltering ([3], V, App.).

Proposition (1.2). — *Let $f: X \to Y$ be a (pointed) map of schemes. Then J_f is codirected.*

Proof. — J_f is connected: for given $\overline{U}'_\bullet \to \overline{V}'_\bullet$ and $\overline{U}_\bullet \to \overline{V}_\bullet$ in J_f, then $\overline{U}'_\bullet \times_X \overline{U}_\bullet \to \overline{V}'_\bullet \times_Y \overline{V}_\bullet$ is in J_f and the projection maps are maps in J_f.

Let the diagram

$$\begin{array}{ccccc} \overline{U}''_\bullet & \rightarrow & \overline{U}'_\bullet & \leftarrow & \overline{U}_\bullet \\ | & & | & & | \\ \overline{V}''_\bullet & \rightarrow & \overline{V}'_\bullet & \leftarrow & \overline{V}_\bullet \end{array}$$

in J_f be given. Then $\overline{U}''_\bullet \times_{\overline{U}'_\bullet} \overline{U}_\bullet \to \overline{V}''_\bullet \times_{\overline{V}'_\bullet} \overline{V}_\bullet$ is in J_f and the projection maps are maps in J_f, as can be immediately checked using the following sublemma.

Sublemma. — *If*

$$\begin{array}{ccccc} \overline{U}''_\bullet & \rightarrow & \overline{U}'_\bullet & \leftarrow & \overline{U}_\bullet \\ | & & | & & | \\ \overline{W}''_\bullet & \rightarrow & \overline{W}'_\bullet & \leftarrow & \overline{W}_\bullet \end{array}$$

is a commutative diagram in $\Delta^0(\mathrm{Et}/X)$ such that $\overline{U}''_\bullet \to \overline{W}''_\bullet$ and $\overline{U}_\bullet \to \overline{U}'_\bullet \times_{\overline{W}'_\bullet} \overline{W}_\bullet$ are special, then $\overline{U}''_\bullet \times_{\overline{U}'_\bullet} \overline{U}_\bullet \to \overline{W}''_\bullet \times_{\overline{W}'_\bullet} \overline{W}_\bullet$ is special.

Proof of sublemma is immediate upon observing that the maps

$$\overline{U}''_{\bullet}\underset{\overline{U}'_{\bullet}}{\times}\overline{U}_{\bullet}\rightarrow\overline{U}''_{\bullet}\underset{\overline{U}'_{\bullet}}{\times}(\overline{U}'_{\bullet}\underset{\overline{W}'_{\bullet}}{\times}\overline{W}_{\bullet})\quad\text{and}\quad\overline{U}''_{\bullet}\underset{\overline{U}'_{\bullet}}{\times}(\overline{U}'_{\bullet}\underset{\overline{W}'_{\bullet}}{\times}\overline{W}_{\bullet})=\overline{U}''_{\bullet}\underset{\overline{W}'_{\bullet}}{\times}\overline{W}_{\bullet}\rightarrow\overline{W}''_{\bullet}\underset{\overline{W}'_{\bullet}}{\times}\overline{W}_{\bullet}$$

are special.

We denote by Ho J_f the " homotopy category " of J_f: the objects of Ho J_f are those of J_f; a map in Ho J_f is an equivalence class of maps in J_f, where the equivalence relation is generated by doubly commutative squares

$$\begin{array}{ccc}\overline{U}'_{\bullet} & \rightrightarrows & \overline{U}_{\bullet}\\ \downarrow & & \downarrow\\ \overline{V}'_{\bullet} & \rightrightarrows & \overline{V}_{\bullet}\end{array}$$

connected by a " categorical homotopy of pairs ": namely, a commutative diagram

$$\begin{array}{ccc}\overline{U}'_{\bullet}\underset{X}{\times}\overline{\Delta[1]} & \longrightarrow & \overline{U}_{\bullet}\\ \downarrow & & \downarrow\\ \overline{V}'_{\bullet}\underset{Y}{\times}\overline{\Delta[1]} & \longrightarrow & \overline{V}_{\bullet}\end{array}$$

such that $\overline{U}'_{\bullet}\underset{X}{\times}\overline{\Delta[1]}\rightarrow\overline{U}_{\bullet}$ is a categorical homotopy in $\Delta^0(\mathrm{Et}/X)$, $\overline{V}'_{\bullet}\underset{Y}{\times}\overline{\Delta[1]}\rightarrow\overline{V}_{\bullet}$ is a categorical homotopy in $\Delta^0(\mathrm{Et}/Y)$, and $\overline{U}'_{\bullet}\underset{X}{\times}\overline{\Delta[1]}\rightarrow\overline{V}'_{\bullet}\underset{Y}{\times}\overline{\Delta[1]}$ is induced by $\overline{U}'_{\bullet}\rightarrow\overline{V}'_{\bullet}$.

Employing the techniques Verdier used to prove that Ho J_X is cofiltering, we shall verify that the " relative homotopy category " Ho J_f is likewise cofiltering.

Given $\overline{U}'_{\bullet}$ and $\overline{U}_{\bullet}$ in $\Delta^0(\mathrm{Et}/X)$, define the contravariant set-valued functor $\overline{\mathrm{Hom}}_{\bullet}(\overline{U}'_{\bullet},\overline{U}_{\bullet})$ on $\Delta^0(\mathrm{Et}/X)$ by

$$\overline{\mathrm{Hom}}_{\bullet}(\overline{U}'_{\bullet},\overline{U}_{\bullet})(\overline{U}''_{\bullet})=\mathrm{Hom}(\overline{U}''_{\bullet}\underset{X}{\times}\overline{U}'_{\bullet},\overline{U}_{\bullet})$$

for any $\overline{U}''_{\bullet}$ in $\Delta^0(\mathrm{Et}/X)$. If $\overline{\mathrm{Hom}}_{\bullet}(\overline{U}'_{\bullet},\overline{U}_{\bullet})$ is representable as a simplicial scheme in $\Delta^0(\mathrm{Et}/X)$ (also denoted by $\overline{\mathrm{Hom}}_{\bullet}(\overline{U}'_{\bullet},\overline{U}_{\bullet})$), then

$$\mathrm{Hom}(\overline{U}''_{\bullet},\overline{\mathrm{Hom}}_{\bullet}(\overline{U}'_{\bullet},\overline{U}_{\bullet}))=\overline{\mathrm{Hom}}(\overline{U}''_{\bullet}\underset{X}{\times}\overline{U}'_{\bullet},\overline{U}_{\bullet}),$$

the usual adjoint relation between Hom(,) and categorical products. Observe that $\overline{\mathrm{Hom}}_0(\overline{\Delta[n]},\overline{U}_{\bullet})=\overline{U}_n$ and $\overline{\mathrm{Hom}}_0(\mathrm{sk}_{n-1}\overline{\Delta[n]},\overline{U}_{\bullet})=(\mathrm{cosk}_{n-1}\overline{U}_{\bullet})_n$ for any $\overline{U}_{\bullet}$ in $\Delta^0(\mathrm{Et}/X)$ and integer $n>0$.

If $S_{\bullet}$ is a simplicial set with finitely many non-degenerate simplices, $\overline{\mathrm{Hom}}_{\bullet}(\overline{S}_{\bullet},\overline{U}_{\bullet})$ is representable for any $\overline{U}_{\bullet}$ in $\Delta^0(\mathrm{Et}/X)$; for each $n\geq 0$, $\overline{\mathrm{Hom}}_n(\overline{S}_{\bullet},\overline{U}_{\bullet})=\overline{\mathrm{Hom}}_0(\overline{\Delta[n]}\underset{X}{\times}\overline{S}_{\bullet},\overline{U}_{\bullet})$ is a finite projective limit of components of $\overline{U}_{\bullet}$ occurring in dimensions $\leq n+m$, where m is the maximum of the dimensions of the non-degenerate simplices of S. In particular, categorical homotopies into $\overline{U}_\bullet$ are represented by the simplicial scheme $\overline{\mathrm{Hom}}_\bullet(\overline{\Delta[1]}, \overline{U}_\bullet)$. If $\overline{U}_\bullet$ in $\Delta^0(\mathrm{Et}/X)$ is pointed by u, then $\overline{\mathrm{Hom}}_\bullet(\overline{\Delta[1]}, \overline{U}_\bullet)$ is pointed by

$$s_0 u : \mathrm{Spec}\,\Omega_x \to \overline{U}_1 = \overline{\mathrm{Hom}}_0(\overline{\Delta[1]}, \overline{U}_\bullet)$$

and represents pointed categorical homotopies into $\overline{U}$.

The key step in Verdier's proof that $\mathrm{Ho}\,J_X$ is cofiltering is the verification that $\overline{\mathrm{Hom}}_\bullet(\overline{\Delta[1]}, \overline{U}'_\bullet) \to \overline{U}'_\bullet \times_X \overline{U}'_\bullet$ is special whenever $\overline{U}'_\bullet$ is a hypercovering of X. The following lemma relativizes this result (which the lemma implies by setting $\overline{U}_\bullet$ equal to $\overline{\Delta[0]}$).

Lemma (**1.3**). — *Let* X *be a (pointed) scheme and let* $\overline{U}'_\bullet \to \overline{U}_\bullet$ *be a special map in* $\Delta^0(\mathrm{Et}/X)$. *Then the induced map*

$$\overline{\mathrm{Hom}}_\bullet(\overline{\Delta[1]}, \overline{U}'_\bullet) \to (\overline{U}'_\bullet \times_X \overline{U}'_\bullet) \times_{(\overline{U}_\bullet \times_X \overline{U}_\bullet)} \overline{\mathrm{Hom}}_\bullet(\overline{\Delta[1]}, \overline{U}_\bullet)$$

is special in $\Delta^0(\mathrm{Et}/X)$.

Proof. — In dimension 0, $\overline{\mathrm{Hom}}_0(\overline{\Delta[1]}, \overline{U}'_\bullet) = \overline{U}'_1$ maps surjectively onto

$$(\overline{U}'_0 \times_X \overline{U}'_0) \times_{(\overline{U}_0 \times_X \overline{U}_0)} \overline{\mathrm{Hom}}_0(\overline{\Delta[1]}, \overline{U}_\bullet) = (\overline{U}'_0 \times_X \overline{U}'_0) \times_{(\overline{U}_0 \times_X \overline{U}_0)} \overline{U}_1$$

since $\overline{U}'_\bullet \to \overline{U}_\bullet$ is special.

In dimension $k > 0$, we use the notation "$\mathrm{ck}_k \overline{U}_\bullet$" and "$\overline{H}_\bullet(\overline{U}_\bullet)$" to denote $(\mathrm{cosk}_{k-1}\overline{U}_\bullet)_k$ and $\overline{\mathrm{Hom}}_\bullet(\overline{\Delta[1]}, \overline{U}_\bullet)$ respectively, for any $\overline{U}_\bullet$ in $\Delta^0(\mathrm{Et}/X)$. We must prove that $\overline{\mathrm{Hom}}_k(\overline{\Delta[1]}, \overline{U}'_\bullet) = \overline{\mathrm{Hom}}_0(\overline{\Delta[k] \times \Delta[1]}, \overline{U}'_\bullet)$ maps onto the following projective limit:

$$\mathrm{ck}_k \overline{H}_\bullet(\overline{U}'_\bullet) \times_{\mathrm{ck}_k(\overline{U}'_\bullet \times_X \overline{U}'_\bullet) \times_{\mathrm{ck}_k(\overline{U}_\bullet \times_X \overline{U}_\bullet)} \mathrm{ck}_k \overline{H}_\bullet(\overline{U}_\bullet)} (\overline{U}'_k \times_X \overline{U}'_k) \times_{(\overline{U}_k \times_X \overline{U}_k)} \overline{H}_k(\overline{U}_\bullet).$$

We shall denote this projective limit by P_k.

Let M_0 denote the simplicial set defined as the fibre sum

$$(\mathrm{sk}_{k-1}\Delta[k] \times \Delta[1]) \coprod_{(\mathrm{sk}_{k-1}\Delta[k] \times \mathrm{sk}_0\Delta[1])} (\Delta[k] \times \mathrm{sk}_0\Delta[1]).$$

$M_0 \to \mathrm{sk}_k(\Delta[k] \times \Delta[1]) \to \Delta[k] \times \Delta[1]$ can be factored as

$$M_0 \to \ldots \to M_k = \mathrm{sk}_k(\Delta[k] \times \Delta[1]) \to \ldots \to M_{2k+1} = \Delta[k] \times \Delta[1].$$

For $0 \le i < k$, M_{i+1} is obtained from M_i by adjoining a k-simplex (so that

$$\begin{array}{ccc} \mathrm{sk}_{k-1}\Delta[k] & \longrightarrow & \Delta[k] \\ \downarrow & & \downarrow \\ M_i & \longrightarrow & M_{i+1} \end{array}$$

is co-cartesian); and for $k \le i \le 2k$, M_{i+1} is obtained from M_i by adjoining a $(k+1)$-simplex.

For any $\overline{U}_\bullet$ in $\Delta^0(Et/X)$, $\overline{Hom}_0(\overline{M}_0, \overline{U}_\bullet) = ck_k\overline{H}_\bullet(\overline{U}_\bullet) \underset{ck_k(\overline{U}_\bullet \times_X \overline{U}_\bullet)}{\times} (\overline{U}_k \times_X \overline{U}_k)$. Therefore,

$$P_k = \overline{Hom}_0(\overline{M}_0, \overline{U}'_\bullet) \underset{\overline{Hom}_0(\overline{M}_0, \overline{U}_\bullet)}{\times} \overline{Hom}_0(\overline{\Delta[k]} \times \overline{\Delta[1]}, \overline{U}_\bullet).$$

To verify that $\overline{Hom}_0(\overline{\Delta[k]} \times \overline{\Delta[1]}, \overline{U}'_\bullet) \to P_k$ is surjective, it suffices to check for each i, $0 \leq i \leq 2k$, that

$$\overline{Hom}_0(\overline{M}_{i+1}, \overline{U}'_\bullet) \to \overline{Hom}_0(\overline{M}_i, \overline{U}'_\bullet) \underset{\overline{Hom}_0(\overline{M}_i, \overline{U}_\bullet)}{\times} \overline{Hom}_0(\overline{M}_{i+1}, \overline{U}_\bullet)$$

is surjective. This follows directly from the hypothesis that $\overline{U}'_\bullet \to \overline{U}_\bullet$ is special and from the fact that M_{i+1} is obtained from M_i by adjoining a simplex to its skeleton.

The following proposition verifies that $Ho\, J_f$ is a good relativization of $Ho\, J_X$.

Proposition (**1.4**). — *Let* $f : X \to Y$ *be a (pointed) map of schemes. Then* $Ho\, J_f$ *is cofiltering and the source and range functors,* $s : Ho\, J_f \to Ho\, J_X$ *and* $r : Ho\, J_f \to Ho\, J_Y$, *are cofinal.*

Proof. — Provided that $Ho\, J_f$ if shown to be cofiltering, cofinality of the source and range functors is easily checked. One simply observes that if $\overline{U}'_\bullet \to s(\overline{U}_\bullet \to \overline{V}_\bullet)$ is a map in $Ho\, J_X$ then there is a natural lifting

$$\begin{array}{ccc} \overline{U}'_\bullet & & \overline{U}_\bullet \\ \downarrow & \to & \downarrow \\ \overline{V}_\bullet & & \overline{V}_\bullet \end{array}$$

to a map in $Ho\, J_f$; similarly, if $\overline{V}'_\bullet \to r(\overline{U}_\bullet \to \overline{V}_\bullet)$ is a map in $Ho\, J_Y$, then there is a natural lifting

$$\begin{array}{ccc} \overline{U}_\bullet \times_{\overline{V}_\bullet} \overline{V}'_\bullet & & \overline{U}_\bullet \\ \downarrow & \longrightarrow & \downarrow \\ \overline{V}'_\bullet & & \overline{V}_\bullet \end{array}$$

To prove that $Ho\, J_f$ if cofiltering, let

$$\begin{array}{ccc} \overline{U}'_\bullet & & \overline{U}_\bullet \\ \downarrow & \rightrightarrows & \downarrow \\ \overline{V}'_\bullet & & \overline{V}_\bullet \end{array}$$

be two given maps in $Ho\, J_f$. Define

$$\begin{array}{ccc} \overline{U}''_\bullet & & \overline{Hom}_\bullet(\overline{\Delta[1]}, \overline{U}_\bullet) \underset{\overline{U}_\bullet \times_X \overline{U}_\bullet}{\times} \overline{U}'_\bullet \\ \downarrow & = & \downarrow \\ \overline{V}''_\bullet & & \overline{Hom}_\bullet(\overline{\Delta[1]}, \overline{V}_\bullet) \underset{\overline{V}_\bullet \times_Y \overline{V}_\bullet}{\times} \overline{V}'_\bullet \end{array}$$

Since the compositions

$$\begin{matrix} \overline{U}''_\bullet & & \overline{U}'_\bullet & & \overline{U}_\bullet \\ \downarrow & \underset{\mathrm{pr}}{\rightarrow} & \downarrow & \rightrightarrows & \downarrow \\ \overline{V}''_\bullet & & \overline{V}'_\bullet & & \overline{V}_\bullet \end{matrix}$$

are clearly equal in $\mathrm{Ho}\,J_f$, it suffices to prove that $\overline{U}''_\bullet \to \overline{V}''_\bullet$ is an object of $\mathrm{Ho}\,J_f$ and that the projection map

$$\begin{matrix} \overline{U}''_\bullet & & \overline{U}'_\bullet \\ \downarrow & \rightarrow & \downarrow \\ \overline{V}''_\bullet & & \overline{V}'_\bullet \end{matrix}$$

is a map of $\mathrm{Ho}\,J_f$.

By Lemma (1.3), $\overline{\mathrm{Hom}}_\bullet(\overline{\Delta[1]}, \overline{V}_\bullet) \to \overline{V}_\bullet \times_Y \overline{V}_\bullet$ is special; therefore, $\overline{V}''_\bullet \to \overline{V}'_\bullet$ is special and consequently $\overline{V}''_\bullet$ is an etale hypercovering of Y. One readily checks that $\overline{\mathrm{Hom}}_\bullet(\overline{\Delta[1]}, \overline{V}_\bullet \times_Y X) = \overline{\mathrm{Hom}}_\bullet(\overline{\Delta[1]}, \overline{V}_\bullet) \times_Y X$. Hence, $\overline{U}''_\bullet \to \overline{V}''_\bullet \times_Y X$ factors as the composition of the two maps

$$\overline{\mathrm{Hom}}_\bullet(\overline{\Delta[1]}, \overline{U}_\bullet) \underset{\overline{U}_\bullet \times_X \overline{U}_\bullet}{\times} \overline{U}'_\bullet \longrightarrow \overline{\mathrm{Hom}}_\bullet(\overline{\Delta[1]}, \overline{V}_\bullet \times_Y X) \underset{(\overline{V}_\bullet \times_Y \overline{V}_\bullet) \times_Y X}{\times} \overline{U}_\bullet \times_X \overline{U}_\bullet \underset{\overline{U}_\bullet \times_X \overline{U}_\bullet}{\times} \overline{U}'_\bullet$$

$$\overline{\mathrm{Hom}}_\bullet(\overline{\Delta[1]}, \overline{V}_\bullet \times_Y X) \underset{(\overline{V}_\bullet \times_Y \overline{V}_\bullet) \times_Y X}{\times} \overline{U}'_\bullet \longrightarrow \overline{\mathrm{Hom}}_\bullet(\overline{\Delta[1]}, \overline{V}_\bullet \times_Y X) \underset{(\overline{V}_\bullet \times_Y \overline{V}_\bullet) \times_Y X}{\times} \overline{V}'_\bullet \times_Y X.$$

Using Lemma (1.3), we conclude that $\overline{U}''_\bullet \to \overline{V}''_\bullet \times_Y X$ is special.

In order to prove that

$$\begin{matrix} \overline{U}''_\bullet & & \overline{U}'_\bullet \\ \downarrow & \underset{\mathrm{pr}}{\rightarrow} & \downarrow \\ \overline{V}''_\bullet & & \overline{V}'_\bullet \end{matrix}$$

is in $\mathrm{Ho}\,J_f$, it suffices to prove that $\overline{U}''_\bullet \to \overline{U}'_\bullet \times_{\overline{V}'_\bullet} \overline{V}''_\bullet$ is special. This is immediate from Lemma (1.3).

We remark that the range functor $r : J_f \to J_Y$ is fibrant, with fibre above $\overline{V}_\bullet$ denoted by $J_{f,\overline{V}_\bullet}$ ([8], VI.6.1). Let $\mathrm{Ho}\,J_{f,\overline{V}_\bullet}$ denote the homotopy category of $J_{f,\overline{V}_\bullet}$ whose homotopy relation is generated by pairs of maps in $J_{f,\overline{V}_\bullet}$ related by a categorical homotopy

$$\begin{matrix} \overline{U}'_\bullet \times \Delta[1] & \rightarrow & \overline{U}_\bullet \\ \downarrow & & \downarrow \\ \overline{V}_\bullet \times \Delta[1] & \xrightarrow{\mathrm{pr}_1} & \overline{V}_\bullet \end{matrix}$$

With notation as above, the two compositions

$$\begin{array}{ccccc} \overline{\mathrm{U}}''_{\bullet} \underset{\overline{\mathrm{V}}''_{\bullet}}{\times} \overline{\mathrm{V}}_{\bullet} & & \overline{\mathrm{U}}'_{\bullet} & \underset{g'}{\overset{g}{\rightrightarrows}} & \overline{\mathrm{U}}_{\bullet} \\ \downarrow & \longrightarrow & \downarrow & & \downarrow \\ \overline{\mathrm{V}}_{\bullet} & & \overline{\mathrm{V}}_{\bullet} & & \overline{\mathrm{V}}_{\bullet} \end{array}$$

are equal in $\mathrm{Ho}\,\mathrm{J}_{f,\overline{\mathrm{V}}_{\bullet}}$, where g, g' are objects in $\mathrm{J}_{f,\overline{\mathrm{V}}_{\bullet}}$ and where $\overline{\mathrm{V}}_{\bullet} \to \overline{\mathrm{V}}''_{\bullet}$ is the canonical map. Thus, $\mathrm{Ho}\,\mathrm{J}_{f,\overline{\mathrm{V}}_{\bullet}}$ is cofiltering.

Given a scheme Y, let ET/Y denote the category whose objects are pointed, connected galois covers $h' : \mathrm{Y}' \to \mathrm{Y}$ (h' is finite, etale and $\mathrm{Y}' \underset{\mathrm{Y}}{\times} \mathrm{Y}' = \coprod \mathrm{Y}'$) and whose maps $h'' \to h'$ are commutative triangles. Since there exists at most one map $h'' \to h'$ between objects of ET/Y, ET/Y is cofiltering. Recall that the Grothendieck fundamental group $\pi_1(\mathrm{Y})$ of Y is the pro-group $\{\mathrm{Gal}(\mathrm{Y}'/\mathrm{Y})\}_{\mathrm{ET/Y}}$; thus, the cokernel of $\pi_1(\mathrm{Y}') \to \pi_1(\mathrm{Y})$ is $\mathrm{Gal}(\mathrm{Y}'/\mathrm{Y})$, for $\mathrm{Y}' \to \mathrm{Y}$ in ET/Y. (For details, see [8].)

Given a pointed map $f : \mathrm{X} \to \mathrm{Y}$ of schemes, we define the category $\mathrm{J}_{\tilde{f}}$ as follows. An object $\bar{g}' : \overline{\mathrm{U}}'_{\bullet} \to \overline{\mathrm{V}}'_{\bullet}$ of $\mathrm{J}_{\tilde{f}}$ is an object of $\mathrm{J}_{f \times h'}$, for some $h' : \mathrm{Y}' \to \mathrm{Y}$ in ET/Y. A map $\bar{g}'' \to \bar{g}'$ in $\mathrm{J}_{\tilde{f}}$ is a map $\bar{g}'' \to \bar{g}' \underset{\mathrm{Y}'}{\times} \mathrm{Y}''$ in $\mathrm{J}_{f \times h''}$. The homotopy category of $\mathrm{J}_{\tilde{f}}$ is denoted $\mathrm{Ho}\,\mathrm{J}_{\tilde{f}}$: a map $\bar{g}'' \to \bar{g}'$ in $\mathrm{Ho}\,\mathrm{J}_{\tilde{f}}$ is a map $\bar{g}'' \to \bar{g}' \underset{\mathrm{Y}'}{\times} \mathrm{Y}''$ in $\mathrm{Ho}\,\mathrm{J}_{f \times h''}$.

Corollary (**1.5**). — *Let* $f : \mathrm{X} \to \mathrm{Y}$ *be a pointed map of schemes. As defined above,* $\mathrm{J}_{\tilde{f}}$ *is codirected and* $\mathrm{Ho}\,\mathrm{J}_{\tilde{f}}$ *is cofiltering. Furthermore,* $\mathrm{J}_{\tilde{f}}$ *and* $\mathrm{Ho}\,\mathrm{J}_{\tilde{f}}$ *are fibre categories over* ET/Y, *with respective fibres* $\mathrm{J}_{f \times h'}$ *and* $\mathrm{Ho}\,\mathrm{J}_{f \times h'}$ *over* h' *in* ET/Y.

Proof. — Products and fibre products exist in $\mathrm{J}_{\tilde{f}}$ because they exist in ET/Y and in each $\mathrm{J}_{f \times h'}$. The existence of coequalizers in each $\mathrm{Ho}\,\mathrm{J}_{f \times h'}$ implies the existence of coequalizers in $\mathrm{Ho}\,\mathrm{J}_{\tilde{f}}$. By definition of $\mathrm{J}_{\tilde{f}}$ and $\mathrm{Ho}\,\mathrm{J}_{\tilde{f}}$, the inverse image functor exists: sending $\mathrm{Y}'' \to \mathrm{Y}'$ in ET/Y to the functor $\bar{g}' \mapsto \bar{g}' \underset{\mathrm{Y}'}{\times} \mathrm{Y}''$.

We conclude this section by remarking that if a functor $\varphi : \mathrm{F} \to \mathrm{G}$ is fibrant, then $\varinjlim_{\mathrm{F}} = \varinjlim_{\mathrm{G}} . \varinjlim_{\mathrm{F}_g}$ on functors $\mathrm{F}^0 \to (\mathrm{Sets})$.

2. Fibres of f_{et}.

Having recalled the " extended homotopy categories " of simplicial sets, $\mathscr{K}_0$ and $\mathscr{K}_{0,\mathrm{pairs}}$ ([4], § 1), we define the *etale homotopy type* of a map of locally noetherian schemes, $f_{\mathrm{et}} : \mathrm{Ho}\,\mathrm{J}_f \to \mathscr{K}_{0,\mathrm{pairs}}$. We introduce canonical maps $(\mathrm{X}_y)_{\mathrm{et}} \to \mathfrak{h}(f_{\mathrm{et}}) \to \mathfrak{h}(f^r_{\mathrm{et}})$ between the " geometric, " " naive, " and " homotopy-theoretic " fibres of f_{et}. We then identify the map on cohomology induced by $(\mathrm{X}_y)_{\mathrm{et}} \to \mathfrak{h}(f_{\mathrm{et}})$. In subsequent sections, we shall study the map $\mathfrak{h}(f_{\mathrm{et}}) \to \mathfrak{h}(f^r_{\mathrm{et}})$.

Given a locally noetherian scheme U, we denote by $\pi(\mathrm{U})$ the set of connected components of U. If $\overline{\mathrm{U}}_{\bullet}$ in $\Delta^0(\mathrm{Et/X})$ is a simplicial scheme over a locally noetherian scheme X, then we denote by $\pi(\overline{\mathrm{U}}_{\bullet})$ the simplicial set given by $\pi(\overline{\mathrm{U}}_{\bullet})_n = \pi(\overline{\mathrm{U}}_n)$. Clearly, the association of $\pi(\overline{U}_\bullet)$ to $\overline{U}_\bullet$ is functorial on $\Delta^0(Et/X)$ and sends categorical homotopies to simplicial homotopies.

We recall the following " extended homotopy categories " $\mathscr{H}_0$ and $\mathscr{H}_{0,\,pairs}$, which are geometrically realized as the homotopy category of pointed C-W complexes and the homotopy category of pointed pairs of C-W complexes. Let $Ex^\infty(\ )$ denote the left adjoint to the inclusion functor of the full subcategory of Kan complexes in the homotopy category of simplicial sets [10]. Then $\mathscr{H}_0$ denotes the category whose objects are pointed simplicial sets $S_\bullet$, and whose maps from $S_\bullet$ to $T_\bullet$ are homotopy equivalence classes of pointed simplicial maps $S_\bullet \to Ex^\infty(T_\bullet)$. The objects of $\mathscr{H}_{0,\,pairs}$ are pointed simplicial maps $S_\bullet \to T_\bullet$; the maps of $\mathscr{H}_{0,\,pairs}$ are equivalence classes of pointed commutative squares

$$\begin{array}{ccc} S'_\bullet & \longrightarrow & S_\bullet \\ \downarrow & & \downarrow \\ Ex^\infty T'_\bullet & \longrightarrow & Ex^\infty T_\bullet \end{array}$$

where the equivalence relation is generated by doubly commutative squares connected by a pointed homotopy of pairs.

Definition **(2.1)**. — The *etale homotopy type* of a locally noetherian, pointed scheme X is the pro-object in pro-$\mathscr{H}_0$

$$X_{et} : \operatorname{Ho} J_X \to \mathscr{H}_0$$

induced by the connected component functor.

The *etale homotopy type* of a pointed map $f: X \to Y$ of locally noetherian schemes is the pro-object in pro-$\mathscr{H}_{0,\,pairs}$

$$f_{et} : \operatorname{Ho} J_f \to \mathscr{H}_{0,\,pairs}$$

induced by the connected component functor.

We remark that Proposition (1.4) implies that f_{et} determines a map $X_{et} \to Y_{et}$ in pro-$\mathscr{H}_0$.

Given a simplicial map $f: S_\bullet \to T_\bullet$ of simplicial sets, we functorially associate a commutative triangle

$$\begin{array}{ccc} S_\bullet & \longrightarrow & S^r_\bullet \\ & {}_f\searrow \quad \swarrow_{f^r} & \\ & T_\bullet & \end{array}$$

such that the inclusion $S_\bullet \to S^r_\bullet$ is a weak homotopy equivalence and such that f^r is a Kan fibration ([7], VI). Furthermore, given a categorical homotopy of pairs

$$\begin{array}{ccc} S'_\bullet \times \Delta[1] & \longrightarrow & S_\bullet \\ \downarrow & & \downarrow \\ T'_\bullet \times \Delta[1] & \longrightarrow & T_\bullet \end{array}$$

there is induced a categorical homotopy of triangles

$$\begin{array}{ccccc} S'_{\bullet}\times\Delta[1] & & \longrightarrow & & S_{\bullet} \\ \downarrow & \searrow & & & \searrow \\ & & (S'_{\bullet})^r\times\Delta[1] & \longrightarrow & S^r_{\bullet} \\ & \swarrow & & & \swarrow \\ T'_{\bullet}\times\Delta[1] & & \longrightarrow & & T_{\bullet} \end{array}$$

This " fibre resolution functor " therefore induces functors $\mathscr{K}_{0,\text{pairs}} \to \mathscr{K}_{0,\text{pairs}}$ and $\text{pro-}\mathscr{K}_{0,\text{pairs}} \to \text{pro-}\mathscr{K}_{0,\text{pairs}}$.

Given a pointed simplicial map $f: S_{\bullet}\to T_{\bullet}$, let $\mathfrak{h}(f) = S_{\bullet}\times_{T_{\bullet}} t$ be the pointed fibre over the distinguished point t of T. This definition determines a functor

$$\mathfrak{h} : \text{pro-}\mathscr{K}_{0,\text{pairs}} \to \text{pro-}\mathscr{K}_0.$$

The following proposition introduces the various fibres associated to f_{et}.

Proposition (**2.2**). — *Given a pointed map $f: X\to Y$ of locally noetherian schemes, with geometric fibre $i: X_y\to X$. Then there exist canonical maps in pro-$\mathscr{K}_0$*

$$(X_y)_{\text{et}} \to \mathfrak{h}(f_{\text{et}}) \to \mathfrak{h}(f^r_{\text{et}})$$

from the " geometric fibre " to the " naive fibre " to the " homotopy fibre " of f_{et}. Furthermore, these maps fit in a commutative diagram in pro-$\mathscr{K}_0$:

$$\begin{array}{ccccc} (X_y)_{\text{et}} & \longrightarrow & \mathfrak{h}(f_{\text{et}}) & \longrightarrow & \mathfrak{h}(f^r_{\text{et}}) \\ & {\scriptstyle i_{\text{et}}}\searrow & \swarrow & & \swarrow \\ & & X_{\text{et}} & \longrightarrow & X^r_{\text{et}} \end{array}$$

Proof. — The map $\mathfrak{h}(f_{\text{et}})\to\mathfrak{h}(f^r_{\text{et}})$ is induced by the natural map in $\text{pro-}\mathscr{K}_{0,\text{pairs}}$, $f_{\text{et}}\to f^r_{\text{et}}$.

To define $(X_y)_{\text{et}}\to\mathfrak{h}(f_{\text{et}})$ in pro-$\mathscr{K}_0$, we associate to each object $\overline{U}_{\bullet}\to\overline{V}_{\bullet}$ in $\text{Ho}\,J_f$ an etale hypercovering of X_y, which we denote by $\varphi(\overline{U}_{\bullet}\to\overline{V}_{\bullet})$, and a map $\pi(\varphi(\overline{U}_{\bullet}\to\overline{V}_{\bullet}))\to\mathfrak{h}(\pi(\overline{U}_{\bullet}\to\overline{V}_{\bullet}))$.

Given $\overline{U}_{\bullet}\to\overline{V}_{\bullet}$ in $\text{Ho}\,J_f$, we let $\bar{v}$ denote the trivial simplicial sub-object of $\overline{V}_{\bullet}$ generated by the distinguished component v of $\overline{V}_0$. We observe that $\overline{U}_{\bullet}\times_{\overline{V}_{\bullet}}\bar{v} \to v\times_Y X = u$ is special and that $\mathfrak{h}(\pi(\overline{U}_{\bullet}\to\overline{V}_{\bullet})) = \pi(\overline{U}_{\bullet}\times_{\overline{V}_{\bullet}}\bar{v})$. We define $\varphi(\overline{U}_{\bullet}\to\overline{V}_{\bullet})$ to be $\overline{U}_{\bullet}\times_{\overline{V}_{\bullet}}\bar{v}\times_u X_y$ and define $\pi(\varphi(\overline{U}_{\bullet}\to\overline{V}_{\bullet}))\to\mathfrak{h}(\pi(\overline{U}_{\bullet}\to\overline{V}_{\bullet}))$ to be the map induced by the simplicial map $\overline{U}_{\bullet}\times_{\overline{V}_{\bullet}}\bar{v}\times_u X_y\to\overline{U}_{\bullet}\times_{\overline{V}_{\bullet}}\bar{v}$.

If

$$\begin{array}{ccc} \overline{U}'_{\bullet} & & \overline{U}_{\bullet} \\ \downarrow & \to & \downarrow \\ \overline{V}'_{\bullet} & & \overline{V}_{\bullet} \end{array}$$

is a map in J_f, then

$$\varphi(\overline{U}'_\bullet \to \overline{V}'_\bullet) = \overline{U}'_\bullet \underset{\overline{V}'_\bullet}{\times} \overline{v}' \underset{u'}{\times} X_y \to \overline{U}_\bullet \underset{\overline{V}_\bullet}{\times} \overline{v} \underset{\overline{v}}{\times} \overline{v}' \underset{u'}{\times} X_y = \overline{U}_\bullet \underset{\overline{V}_\bullet}{\times} \overline{v} \underset{u}{\times} X_y = \varphi(\overline{U}_\bullet \to \overline{V}_\bullet)$$

is a map in J_{X_y} (*i.e.*, special) and

$$\begin{array}{ccc} \pi(\varphi(\overline{U}'_\bullet \to \overline{V}'_\bullet)) & \to & \mathfrak{h}(\pi(\overline{U}'_\bullet \to \overline{V}'_\bullet)) \\ \downarrow & & \downarrow \\ \pi(\varphi(\overline{U}_\bullet \to \overline{V}_\bullet)) & \to & \mathfrak{h}(\pi(\overline{U}_\bullet \to \overline{V}_\bullet)) \end{array}$$

commutes. Thus the pairs $\pi(\varphi(\overline{U}_\bullet \to \overline{V}_\bullet)) \to \mathfrak{h}(\pi(\overline{U}_\bullet \to \overline{V}_\bullet))$ define a map $(X_y)_{et} \to \mathfrak{h}(f_{et})$.

The maps $\mathfrak{h}(f_{et}) \to X_{et}$ and $\mathfrak{h}(f^r_{et}) \to X^r_{et}$ are given as the compositions $\mathfrak{h}(f_{et}) \to s \circ f_{et} \xleftarrow{\sim} X_{et}$ and $\mathfrak{h}(f^r_{et}) \to s \circ f^r_{et} \xleftarrow{\sim} X^r_{et}$ (where $s : \mathscr{K}_{0,\text{pairs}} \to \mathscr{K}_0$ is the "source" functor). We must check the commutativity of the square

$$\begin{array}{ccc} (X_y)_{et} & \longrightarrow & \mathfrak{h}(f_{et}) \\ \downarrow & & \downarrow \\ X_{et} & \longrightarrow & s \circ f_{et} \end{array}$$

The composition $(X_y)_{et} \to X_{et} \to s \circ f_{et}$ in $\underset{\text{Ho}\, J_f}{\varprojlim}\ \underset{\text{Ho}\, J_{X_y}}{\varinjlim} [\pi(\overline{W}_\bullet), s(\pi(\overline{U}_\bullet \to \overline{V}_\bullet))]$ is given by pairs $\pi(\overline{U}_\bullet \underset{X}{\times} X_y) \to \pi(\overline{U}_\bullet)$; whereas the composition $(X_y)_{et} \to \mathfrak{h}(f_{et}) \to s \circ f_{et}$ is given by pairs $\pi(\overline{U}_\bullet \underset{\overline{V}_\bullet}{\times} \overline{v} \underset{u}{\times} X_y) \to \pi(\overline{U}_\bullet)$.

To check that these pairs determine the same map $(X_y)_{et} \to s \circ f_{et}$, we observe that $\overline{U}_\bullet \underset{\overline{V}_\bullet}{\times} \overline{v} \underset{u}{\times} X_y \to \overline{U}_\bullet$ factors through the simplicial (though not special) map $\overline{U}_\bullet \underset{\overline{V}_\bullet}{\times} \overline{v} \underset{u}{\times} X_y \to \overline{U}_\bullet \underset{X}{\times} X_y$ of etale hypercoverings of X_y. Let $\overline{W}_\bullet$ be the product in J_{X_y} of $\overline{U}_\bullet \underset{\overline{V}_\bullet}{\times} \overline{v} \underset{u}{\times} X_y$ and $\overline{U}_\bullet \underset{X}{\times} X_y$. As in Proposition (1.4), we find $\overline{W}'_\bullet \to \overline{W}_\bullet$ in J_{X_y} with $\overline{W}'_\bullet \to \overline{W}_\bullet \rightrightarrows \overline{U}_\bullet \underset{X}{\times} X_y$ categorically homotopic. Hence, the pairs $\pi(\overline{U}_\bullet \underset{X}{\times} X_y) \to \pi(\overline{U}_\bullet)$ and $\pi(\overline{U}_\bullet \underset{\overline{V}_\bullet}{\times} \overline{v} \underset{u}{\times} X_y) \to \pi(\overline{U}_\bullet)$ induce homotopic maps $\pi(\overline{W}'_\bullet) \to \pi(\overline{U}_\bullet)$.

In order to verify that $\overline{U}_p$ becomes "arbitrarily fine" for $\overline{U}_\bullet$ in $\text{Ho}\, J_X$, we verify the following lemma (asserted in [3], V, App.).

Lemma **(2.3)**. — *For any scheme* X *and any integer* $p \geq 0$, *there exists a functor* $R^p_\bullet(\ ) : (\text{Et}/X) \to \Delta^0(\text{Et}/X)$ *satisfying:*

a) $R^p_\bullet(\ )$ *sends surjective maps to special maps.*

b) $R^p_\bullet(\ )$ *is right adjoint to the functor sending* $\overline{U}_\bullet$ *to* $\overline{U}_p$.

c) *If* $\overline{U}_\bullet$ *is* $\Delta^0(\text{Et}/X)$, $Z \to \overline{U}_p$ *is a map in* (Et/X), *and* $\overline{U}'_\bullet = U_\bullet \underset{R^p_\bullet(\overline{U}_p)}{\times} R^p_\bullet(Z)$, *then* $\text{pr}_\bullet : \overline{U}'_p \to \overline{U}_p$ *factors through* $Z \to \overline{U}_p$.

Proof. — Define $t(p, m)$ to be the set of non-decreasing maps from $\{0, \ldots, p\}$ to $\{0, \ldots, m\}$ for any $m \geq 0$. The $t(p, m)$ naturally determine a co-simplicial set, denoted $\nabla(p)$, with $\nabla(p)^m = t(p, m)$. We define $R^p_\bullet(Z)$ to be the composition of $\nabla(p) : \Delta^0 \to (\text{Finite sets})^0$ with the functor $\pi_Z : (\text{Finite sets})^0 \to (\text{Et}/X)$ given by sending a set S to the fibre product over X of Z with itself $\sharp(S)$-times.

For any $m \geq 1$, $(\operatorname{cosk}_{m-1} R^p_\bullet(Z))_m$ is the fibre product over X of copies of Z indexed by "$\deg^m(\nabla(p))$," those co-simplices in the image of $\nabla(p)^{m-1}$ under some co-face map. Hence, $R^p_\bullet(Z)_m \to (\operatorname{cosk}_{m-1} R^p_\bullet(Z))_m$ is surjective, arising from an injective map of indexing sets $\deg^m(\nabla(p)) \to \nabla(p)^m$. We may thus readily check that

$$R^p_m(Z') \to (\operatorname{cosk}_{m-1} R^p_\bullet(Z'))_m \underset{(\operatorname{cosk}_{m-1} R^p_\bullet(Z))_m}{\times} R^p_m(Z)$$

is surjective whenever $Z' \to Z$ is surjective and $m > 0$; thus, *a)* is valid. Namely, let any geometric point

$$x \underset{y}{\times} z : \operatorname{Spec} \Omega \to (\operatorname{cosk}_{m-1} R^p_\bullet(Z'))_m \underset{(\operatorname{cosk}_{m-1} R^p_\bullet(Z))_m}{\times} R^p_m(Z)$$

be given; we lift $x \underset{y}{\times} z$ to $x \times w : \operatorname{Spec} \Omega \to R^p_m(Z')$ by defining w to be some lifting of those factors of $z : \operatorname{Spec} \Omega \to R^p_m(Z)$ not determined by $y : \operatorname{Spec} \Omega \to (\operatorname{cosk}_{m-1} R^p_\bullet(Z))_m$.

To check *b)*, let $f : \overline{U}_p \to Z$ in (Et/X) be given. Define $\overline{U}_\bullet \to R^p_\bullet(Z)$ by sending $\overline{U}_k$ into the factor of $(R^p_\bullet(Z))_k$ corresponding to $\partial^{i_s} \circ \ldots \partial^{i_1} \circ \sigma^{j_t} \circ \ldots \sigma^{j_1} : \Delta[p] \to \Delta[p-t] \to \Delta[k]$ *via* the composition $f \circ s_{j_1} \circ \ldots \circ s_{j_t} \circ d_{i_1} \circ \ldots \circ d_{i_s} : \overline{U}_k \to \overline{U}_{k-s} \to \overline{U}_p \to Z$. One may readily check that this establishes a $1-1$ correspondence between $\operatorname{Hom}(\overline{U}_p, Z)$ and $\operatorname{Hom}(\overline{U}_\bullet, R^p_\bullet(Z))$.

To check *c)*, observe that the projection map $\operatorname{pr}_1 : \overline{U}'_p \to \overline{U}_p$ factors through any factor $\overline{U}_p \underset{\overline{U}_p}{\times} Z$ of $\overline{U}'_p$. Yet the factor corresponding to the identity map in $\nabla^{(p)}$ equals Z, since $\overline{U}_p$ into the factor of $R^p_p(\overline{U}_p)$ corresponding to the identity map is the identity.

The following proposition explicates the map on abelian cohomology induced by $(X_y)_{et} \to \mathfrak{h}(f_{et})$. A similar statement can be made for non-abelian finite groups G and integers $p = 0, 1$.

Let X be a scheme and let F be an abelian sheaf on the etale site of X. We recall the Verdier isomorphism

$$\varinjlim_{\operatorname{Ho} J_X} H^p(\overline{U}_\bullet, F) \xrightarrow{\sim} H^q(X, F)$$

where $H^*(X, F)$ is the etale cohomology of X with values in F, where $\overline{U}_\bullet$ runs over etale hypercoverings in $\operatorname{Ho} J_X$, and where $H^*(\overline{U}_\bullet, F)$ is the cohomology of the cochain complex $F(\overline{U}_\bullet)$ ([3], V, App.; or see Proposition (3.7) below). In particular, let A be a locally constant, abelian sheaf on the etale site of a locally noetherian scheme X, with corresponding local coefficient system on X_{et} also denoted by A ([4], § 10). Then $H^*(X_{et}, A) = \varinjlim_{\operatorname{Ho} J_X} H^*(\pi(\overline{U}_\bullet), A) = \varinjlim_{\operatorname{Ho} J_X} H^*(\overline{U}_\bullet, A) \xrightarrow{\sim} H^*(X, A)$.

Proposition (**2.4**). — *Let $f: X \to Y$ be a pointed map of locally noetherian schemes, with geometric fibre $i: X_y \to X$. Let A be a locally constant, abelian sheaf on X. Then for each $p \geq 0$, there exists an isomorphism*

$$H^p(\mathfrak{h}(f_{et}), A) \xrightarrow{\sim} (R^p f_* A)_y$$

which fits in the commutative square

$$\begin{array}{ccc} H^p(\mathfrak{h}(f_{et}), A) & \xrightarrow{\sim} & (R^p f_* A)_y \\ \downarrow & & \downarrow \\ H^p((X_y)_{et}, i^* A) & \xrightarrow{\sim} & H^p(X_y, i^* A) \end{array}$$

where the left vertical arrow is induced by $(X_y)_{et} \to \mathfrak{h}(f_{et})$, the right vertical arrow is the canonical map, and the bottom arrow is the Verdier isomorphism.

Proof. — We recall that if $\overline{U}_\bullet$ is an etale hypercovering of a scheme Z, if F is a sheaf on the etale site of Z, and if $F \to I^*$ is an injective resolution of sheaves, then the spectral sequence of the bicomplex $I^*(\overline{U}_\bullet)$ can be written

$$E_2^{p,q} = H^p(\overline{U}_\bullet, \mathscr{H}^q(F)) \Rightarrow H^N(Z, F),$$

since $\overline{U}_\bullet$ is acyclic at each geometric point of Z. The association of

$$\varphi(\overline{U}_\bullet \to \overline{V}_\bullet) = \overline{U}_\bullet \underset{\overline{V}_\bullet}{\times} \bar{v} \underset{u}{\times} X_y$$

in $\text{Ho}\, J_{X_y}$ to $\overline{U}_\bullet \to \overline{V}_\bullet$ in $\text{Ho}\, J_f$ induces a map of spectral sequences:

$$\begin{array}{ccc} E_2^{p,q} = \varinjlim_{\text{Ho}\, J_f} H^p(\overline{U}_\bullet \underset{\overline{V}_\bullet}{\times} \bar{v}, \mathscr{H}^q(F)) & \Rightarrow & \varinjlim_{\text{Ho}\, J_f} H^N(v \underset{Y}{\times} X, F) \\ & & \downarrow \\ 'E_2^{p,q} = \varinjlim_{\text{Ho}\, J_{X_y}} H^p(\overline{W}_\bullet, \mathscr{H}^q(i^* F)) & \Rightarrow & H^N(X_y, i^* F) \end{array}$$

The map on abutments is precisely the natural map $(R^N f_* F)_y \to H^N(X_y, i^* F)$. Taking F equal to A, we obtain $E_2^{p,0} = H^p(\mathfrak{h}(f_{et}), A)$ and $'E_2^{p,0} = H^p((X_y)_{et}, A)$; moreover $E_2^{p,0} \to {}'E_2^{p,0}$ is induced by $(X_y)_{et} \to \mathfrak{h}(f_{et})$. The Verdier isomorphism is the edge homomorphism of the $'\mathbf{E}^{p,q}$ spectral sequence. If we define $H^p(\mathfrak{h}(f_{et}), A) \to (R^p f_* A)_y$ to be the edge homomorphism of the $\mathbf{E}^{p,q}$ spectral sequence, the square

$$\begin{array}{ccc} H^p(\mathfrak{h}(f_{et}), A) & \longrightarrow & (R^p f_* A)_y \\ \downarrow & & \downarrow \\ H^p((X_y)_{et}, i^* A) & \longrightarrow & H^p(X_y, i^* A) \end{array}$$

commutes, for any $p \geq 0$.

To show that $H^p(\mathfrak{h}(f_{et}), A) \to (R^p f_* A)_y$ is an isomorphism, it suffices to verify that $E_2^{p,q}=0$ for $q>0$. Let $q>0$ be given and let $\bar{c}$ be a cohomology class in $H^p(\overline{U}_\bullet \underset{\overline{V}_\bullet}{\times} \bar{v}, \mathscr{H}^q(A))$, represented by a cocycle c in $\mathscr{H}^q(A)(\overline{U}_\bullet \underset{\overline{V}_\bullet}{\times} \bar{v})_p$. Let c' in $\mathscr{H}^q(A)(\overline{U}_p)$ be the cochain obtained from c by extension by zero. Since $\mathscr{H}^q(A)$ vanishes at every geometric point, there exists an etale surjective map $Z \to \overline{U}_p$ such that c' goes to 0 in $\mathscr{H}^q(A)(Z)$. By Lemma (2.3), the special map $\overline{U}'_\bullet = R^p_\bullet(Z) \underset{R^p_\bullet(\overline{U}_p)}{\times} \overline{U}_\bullet \to \overline{U}_\bullet$ satisfies the condition that $\overline{U}'_p \to \overline{U}_p$ factors through $Z \to \overline{U}_p$. Hence, $\bar{c}$ in $H^q(\overline{U}_\bullet \underset{\overline{V}_\bullet}{\times} \bar{v}, \mathscr{H}^q(A))$ goes to 0 in $H^q(\overline{U}'_\bullet \underset{\overline{V}_\bullet}{\times} \bar{v}, \mathscr{H}^q(A))$. Thus $E_2^{p,q}=0$ for $q>0$.

3. The Simplicial Leray Spectral Sequence.

In this section, we study a naturally constructed spectral sequence $\mathbf{E}^{p,q}(g, A)$ associated to a map $g: U_\bullet \to V_\bullet$ of simplicial sets and to a local system A on $U_\bullet$. After checking functoriality of this construction and after verifying that $\mathbf{E}^{p,q}(g, A)$ is the usual Serre spectral sequence whenever g is a Kan fibration, we proceed to identify $\mathbf{E}^{p,q}(f_{et}, A)$ for $f: X \to Y$ a (possibly pointed) map of locally noetherian schemes and A a locally constant, abelian sheaf on X. Theorem (3.9) asserts that $\mathbf{E}^{p,q}(f_{et}, A)$ takes the form of the Leray spectral sequence. This " simplicial Leray spectral sequence " is by construction readily compared to the Serre spectral sequence for f, $\mathbf{E}^{p,q}(f^r_{et}, A)$.

For notational convenience, we shall denote $sk_{p-1}\Delta[p]$ by $sk(p)$.

Construction (3.1). — Let $g: U_\bullet \to V_\bullet$ be a map of simplicial sets and let A be a contravariant local system on $U_\bullet$. We construct a spectral sequence

$$E_1^{p,q}(g, A) = H^{p+q}(g^{-1}V_{(p)}, g^{-1}V_{(p-1)}; A) \Rightarrow H^{p+q}(U_\bullet, A)$$

where $V_{(p)}$ denotes $sk_p V_\bullet$ and where $g^{-1}V_{(p)}$ denotes $V_{(p)} \underset{V_\bullet}{\times} U_\bullet$.

Namely, let $A(U_\bullet)$ be the complex of cochains on $U_\bullet$ with values in A. Define a decreasing filtration on $A(U_\bullet)$ by $F^pA(U_\bullet) = \ker(A(U_\bullet) \to A(g^{-1}V_{(p)}))$. We obtain a spectral sequence $(E_0^{p,q}, d_0)$ with $E_0^{p,q} = F^{p-1}A^{p+q}(U_\bullet)/F^pA^{p+q}(U_\bullet)$ and $d_0: E_0^{p,q} \to E_0^{p,q+1}$ the differential of $F^{p-1}A(U_\bullet)/F^pA(U_\bullet)$. Equivalently, $d_0: E_0^{p,q} \to E_0^{p,q+1}$ is the differential of $\ker(A(g^{-1}V_{(p)}) \to A(g^{-1}V_{(p-1)}))$.

Taking cohomology, $E_1^{p,q} = H^{p+q}(g^{-1}V_{(p)}, g^{-1}V_{(p-1)}; A)$ and $d_1: E_1^{p,q} \to E_1^{p+1,q}$ is the connecting homomorphism in cohomology for the short exact sequence

$$0 \to A(g^{-1}V_{(p+1)}, g^{-1}V_{(p)}) \to A(g^{-1}V_{(p+1)}, g^{-1}V_{(p-1)}) \to A(g^{-1}V_{(p)}, g^{-1}V_{(p-1)}) \to 0.$$

The following lemma explicates the functoriality of $\mathbf{E}^{p,q}(g, A)$ as constructed above.

Lemma (3.2). — *Let*

$$\begin{array}{ccc} U'_\bullet & & U_\bullet \\ {\scriptstyle g'}\downarrow & \xrightarrow{G} & \downarrow{\scriptstyle g} \\ V'_\bullet & & V_\bullet \end{array}$$

be a map of simplicial pairs, let A *be a (contravariant) local system on* $U_\bullet$, *and let* $G^*A \to A'$ *be a map of local systems on* $U'_\bullet$. *Then* G *induces a map* $\mathbf{E}^{p,q}(G) : \mathbf{E}^{p,q}(g, A) \to \mathbf{E}^{p,q}(g', A')$.

Let

$$\begin{array}{ccc} U'_\bullet \times \Delta[1] & \xrightarrow{h} & U_\bullet \\ \downarrow{\scriptstyle g'\times 1} & & \downarrow{\scriptstyle g} \\ V'_\bullet \times \Delta[1] & \xrightarrow{pr_1} & V_\bullet \end{array}$$

be given together with a map $h^*A \to pr_1^*A'$ *of local systems on* $U'_\bullet \times \Delta[1]$. *Then the induced maps* $\mathbf{E}^{p,q}(g, A) \rightrightarrows \mathbf{E}^{p,q}(g', A')$ *are equal, beginning with the* E_1 *term. Furthermore, let*

$$\begin{array}{ccc} U'_\bullet \times \Delta[1] & \xrightarrow{h} & U_\bullet \\ \downarrow{\scriptstyle g'\times 1} & & \downarrow{\scriptstyle g} \\ V'_\bullet \times \Delta[1] & \xrightarrow{k} & V_\bullet \end{array}$$

be given together with a map $h^*A \to pr_1^*A'$ *of local systems on* $U'_\bullet \times \Delta[1]$. *Then the induced maps* $\mathbf{E}^{p,q}(g, A) \rightrightarrows \mathbf{E}^{p,q}(g', A')$ *are equal, beginning with the* E_2 *term.*

Proof. — To verify the first assertion, we observe that

$$\begin{array}{ccc} U'_\bullet & & U_\bullet \\ \downarrow{\scriptstyle g'} & \xrightarrow{G} & \downarrow{\scriptstyle g} \\ V'_\bullet & & V_\bullet \end{array}$$

and $G^*A \to A'$ induce maps of filtered, graded complexes:

$$F^*(A(U_\bullet)) \to F^*(G^*A(U'_\bullet)) \to F^*(A'(U'_\bullet)).$$

The homotopy $h : U'_\bullet \times \Delta[1] \to U_\bullet$ covering the trivial homotopy induces homotopies $g'^{-1}V_{(p)} \times \Delta[1] \to g^{-1}V_{(p)}$. Therefore, the maps

$$\begin{aligned} A(g^{-1}V_{(p)}, g^{-1}V_{(p-1)}) &\to h^*A((g'\times 1)^{-1}V_{(p)}, (g'\times 1)^{-1}V_{(p-1)}) \\ &\to pr_1^*A'((g'\times 1)^{-1}V_{(p)}, (g'\times 1)^{-1}V_{(p-1)}) \rightrightarrows A'(g'^{-1}V_{(p)}, g'^{-1}V_{(p-1)}) \end{aligned}$$

are homotopic.

Finally,

$$\begin{array}{ccc} U'_\bullet \times \Delta[1] & \xrightarrow{h} & U_\bullet \\ \downarrow{\scriptstyle g'\times 1} & & \downarrow{\scriptstyle g} \\ V'_\bullet \times \Delta[1] & \xrightarrow{k} & V_\bullet \end{array}$$

restricts to give homotopies

$$g'^{-1}V'_{(p)} \times \Delta[1] \to g^{-1}V_{(p+1)}.$$

Represent an element x of $E_2^{p,q}(g, A)$ by a cocycle in

$$Z_2^{p,q}(g, A) = \mathrm{Im}(H^{p+q}(g^{-1}V_{(p+1)}, g^{-1}V_{(p-1)}; A) \to H^{p+q}(g^{-1}V_{(p)}, g^{-1}V_{(p-1)}; A)).$$

We must show that $h_0^*(x) - h_1^*(x)$ lies in

$$B_2^{p,q}(g', A') = \mathrm{Ker}(H^{p+q}(g'^{-1}V'_{(p)}, g'^{-1}V'_{(p-1)}; A') \to H^{p+q}(g'^{-1}V'_{(p)}, g'^{-1}V_{(p-2)}); A')),$$

where $h_i = h \circ \varepsilon_i : U'_{\bullet} \to U'_{\bullet} \times \Delta[1] \to U_{\bullet}$. This follows from the observation that the maps

$$A(g^{-1}V_{(p+1)}, g^{-1}V_{(p-1)}) \to h^* A(g'^{-1}V'_{(p)} \times \Delta[1], g'^{-1}V'_{(p-2)} \times \Delta[1])$$
$$\to \mathrm{pr}_1^* A'(g'^{-1}V'_{(p)} \times \Delta[1], g'^{-1}V'_{(p-2)} \times \Delta[1]) \rightrightarrows A'(g'^{-1}V'_{(p)}, g'^{-1}V'_{(p-2)})$$

are homotopic.

The application of Lemma (3.2) which we envisage is the following: A is a locally constant, abelian sheaf on X and $f : X \to Y$ is a map of locally noetherian schemes. We observe that A determines a local system on $U_{\bullet} = \pi(\overline{U}_{\bullet})$, $\overline{U}_{\bullet}$ in J_X, whenever A is constant on each component of $\overline{U}_0$. Furthermore, if $\bar{g}' \to \bar{g}$ is a map in J_f, then the local system on $\overline{U}'_{\bullet}$ determined by A is simply the pull-back of the local system on $\overline{U}_{\bullet}$ determined by A. By an abuse of terminology, we shall denote by A each of these local systems determined by the sheaf A.

In the following proposition we verify that $\mathbf{E}^{p,q}(g, A)$ for a Kan fibration g is the cohomological Serre spectral sequence for g and A. We first observe the following useful fact. Let $g : U_{\bullet} \to V_{\bullet}$ be a map of simplicial sets and let A be a local system on $U_{\bullet}$. For each simplex v of $V_{\bullet}$, let

$$g^{-1}(v) = U_{\bullet} \underset{V_{\bullet}}{\times} \Delta[p] \qquad \text{and} \qquad g^{-1}(\dot{v}) = U_{\bullet} \underset{V_{\bullet}}{\times} \mathrm{sk}(p),$$

where $\Delta[p] \to V_{\bullet}$ is given by v. Since

$$(g^{-1}V_{(p)}, g^{-1}V_{(p-1)}) \to \coprod_{V_p^+} (g^{-1}(v), g^{-1}(\dot{v}))$$

is an excision map, we have

$$\prod_{V_p^+} H^*(g^{-1}(v), g^{-1}(\dot{v}); A) \xrightarrow{\sim} H^*(g^{-1}V_{(p)}, g^{-1}V_{(p-1)}; A)$$

where V_p^+ consists in the non-degenerate p-simplices of V.

Proposition **(3.3)**. — *If* $g : U_{\bullet} \to V_{\bullet}$ *is a Kan fibration and if* A *is a local system on* $U_{\bullet}$, *then* $E_1^{p,q}(g, A) = \prod_{V_p^+} H^q(g^{-1}(v), A)$ *and* (E_1, d_1) *is the complex of normalized cochains on* $V_{\bullet}$ *with coefficients in the local system* $v \mapsto H^q(g^{-1}(v), A)$.

Hence $E_2^{p,q}(g, A) = H^p(V_{\bullet}, v \mapsto H^q(g^{-1}(v), A)) \Rightarrow H^{p+q}(U_{\bullet}, A)$.

Proof. — Since g is a fibration, g contains a Kan fibre bundle h as a strong deformation retract over each connected component of $V_{\bullet}$ ([16], (11.12)). Hence, $g^{-1}(v) \to \Delta[p]$ is fibre homotopy equivalent to a product $h^{-1}(v) = F \times \Delta[p] \to \Delta[p]$. Therefore,

$$H^{p+q}(g^{-1}(v), g^{-1}(\dot{v}); A) \xrightarrow{\sim} H^{p+q}(F \times (\Delta[p], \mathrm{sk}(p)); A) \xleftarrow{\sim} H^q(F, A)$$
$$\xrightarrow{\sim} H^q(F \times \Delta[p], A) \xleftarrow{\sim} H^q(g^{-1}(v), A).$$

Furthermore, $g^{-1}(d_i v) \to g^{-1}(v)$ and $g^{-1}(s_j v) \to g^{-1}(v)$ may be viewed as

$$F \times \Delta[p-1] \xrightarrow{\partial_i} F \times \Delta[p] \quad \text{and} \quad F \times \Delta[p+1] \xrightarrow{\sigma_j} F \times \Delta[p];$$

thus, $H^*(g^{-1}(v), A) \xrightarrow{\sim} H^*(g^{-1}(d_i v), A)$ and $H^*(g^{-1}(v), A) \xrightarrow{\sim} H^*(g^{-1}(s_j v), A)$. We conclude that $v \mapsto H^q(g^{-1}(v), A)$ is a covariant local system on $V_\bullet$ (cf. [7], App. II, (4.6) for more details).

As remarked in Construction (3.1), $d_1 : E_1^{p,q}(g, A) \to E_1^{p+1,q}(g, A)$ is the connecting homomorphism for

$$0 \to A(g^{-1}V_{(p+1)}, g^{-1}V_{(p)}) \to A(g^{-1}V_{(p+1)}, g^{-1}V_{(p-1)}) \to A(g^{-1}V_{(p)}, g^{-1}V_{(p-1)}) \to 0.$$

To identify d_1 into some factor $H^q(g^{-1}(v), A)$ of $E_1^{p+1,q}(g, A) = \prod_{V^+_{p+1}} H^q(g^{-1}(v), A)$, it suffices to examine

$$0 \to A(g^{-1}(v), g^{-1}(\dot{v})) \to A(g^{-1}(v), g^{-1}(\ddot{v})) \to A(g^{-1}(\dot{v}), g^{-1}(\ddot{v})) \to 0$$

where $g^{-1}(\ddot{v}) = U_\bullet \underset{V_\bullet}{\times} \Delta[p+1]_{(p-1)}$.

In this case, a cohomology class σ in

$$H^{p+q}(g^{-1}(\dot{v}), g^{-1}(\ddot{v}); A) = \prod_{0 \leq i \leq p+1} H^{p+q}(g^{-1}(d_i v), g^{-1}(\dot{d}_i v); A)$$

goes to $\delta\sigma = \sum_i (-1)^i \partial_i \sigma$ in $H^{p+q+1}(g^{-1}(v), g^{-1}(\dot{v}); A)$, where ∂_i is defined by extending a cocycle s in $A^{p+q}(g^{-1}(d_i v), g^{-1}(\dot{d}_i v))$ to some cochain $\bar{s}$ in $A^{p+q}(g^{-1}(v), g^{-1}(\dot{d}_i v))$, then restricting $\partial_i \bar{s}$ to a cocycle in $A^{p+q+1}(g^{-1}(v), g^{-1}(\dot{v}))$. Replacing $g^{-1}(v)$ by a strong deformation retract $F \times \Delta[p]$, we readily check the commutativity of

$$\begin{array}{ccc} H^{p+q}(g^{-1}(d_i v), g^{-1}(\dot{d}_i v); A) & \xrightarrow{\partial_i} & H^{p+q+1}(g^{-1}(v), g^{-1}(\dot{v}); A) \\ \downarrow & & \downarrow \\ H^q(g^{-1}(d_i v), A) & \longleftarrow & H^q(g^{-1}(v), A) \end{array}$$

where the bottom row is induced by $g^{-1}(d_i v) \to g^{-1}(v)$.

We conclude that $d_1 : \prod_{V^+_p} H^q(g^{-1}(v'), A) \to \prod_{V^+_{p+1}} H^q(g^{-1}(v), A)$ into some factor $H^q(g^{-1}(v), A)$ is $\sum_i{}' (-1)^i \partial_i \circ \mathrm{pr}_{d_i v}$, where the sum $\sum_i{}'$ is taken over these i, $0 \leq i \leq p+1$, with $d_i v$ in V^+_p. Hence $\{\prod_{V^+_p} H^q(g^{-1}(v), A), d_1\}$ is the complex of normalized cochains on $V_\bullet$ with coefficients in $v \mapsto H^q(g^{-1}(v), A)$.

The map of spectral sequences $\mathbf{E}^{p,q}(f^r_{et}, A) \to \mathbf{E}^{p,q}(f_{et}, A)$ will be our main tool in comparing the cohomology of $\mathfrak{h}(f^r_{et})$ and $\mathfrak{h}(f_{et})$.

Proposition **(3.4)**. — *Let* $f: X \to Y$ *be a (pointed) map of locally noetherian schemes and let* A *be a locally constant, abelian sheaf on* X. *For each* $\bar{g}: \overline{U}_\bullet \to \overline{V}_\bullet$ *in* J_f *with* A *constant on* $\overline{U}_0$, *Construction* (3.1) *provides a map of spectral sequences*

$$\mathbf{E}^{p,q}(g^r, A) \to \mathbf{E}^{p,q}(g, A)$$

with common abutment, where $g^r: U^r_\bullet \to V_\bullet$ *is the fibre resolution of* $g: U_\bullet \to V_\bullet$.

Therefore, we obtain a map of spectral sequences with common abutment:

$$\begin{array}{ccc} E_2^{p,q}(f^r_{et}, A) = \varinjlim_{\mathrm{Ho}\,J_f} H^p(V_\bullet, v \longmapsto H^q(g^{r-1}(v), A)) & \Rightarrow & H^{p+q}(X, A) \\ \downarrow & & \\ E_2^{p,q}(f_{et}, A) = \varinjlim_{\mathrm{Ho}\,J_f} H^*(H^{p+q}(g^{-1}V_{(p)}, g^{-1}V_{(p-1)}; A); d_1) & \Rightarrow & H^{p+q}(X, A) \end{array}$$

Proof. — By Lemma (3.2), the commutative triangle

$$\begin{array}{ccc} U_\bullet & \longrightarrow & U^r_\bullet \\ & \searrow \quad \swarrow & \\ & V_\bullet & \end{array}$$

induces the map $\mathbf{E}^{p,q}(g^r, A) \to \mathbf{E}^{p,q}(g, A)$. Since homotopies in $\mathrm{Ho}\,J_f$ give homotopies of pairs of simplicial sets, the limit spectral sequences beginning with the E_2 term are well-defined by Lemma (3.2). By Proposition (3.3), $E_2^{p,q}(f^r_{et}, A)$ has its stated form. Finally, the abutment $\varinjlim_{\mathrm{Ho}\,J_f} H^*(U^r_\bullet, A) = \varinjlim_{\mathrm{Ho}\,J_f} H^*(U_\bullet, A)$ is isomorphic to $H^*(X, A)$ by Verdier's theorem ([3], § V, App.).

Since any etale map of schemes $U \to X$ with section $s: X \to U$ splits as $(U - s(X)) \amalg X \to X$, $\overline{U}_\bullet$ in $\Delta^0(ET/X)$ admits a unique splitting ([4], § 8). Hence, $\overline{U}_n$ is a disjoint union of copies of $\overline{U}_k^+$, the non-degenerate part of $\overline{U}_k$, for $0 \leq k \leq n$.

Lemma **(3.5)**. — *Let* Y *be a locally noetherian scheme, let* $\overline{V}_\bullet$ *be a simplicial scheme in* $\Delta^0(Et/Y)$, *and let* $V_\bullet = \pi(\overline{V}_\bullet)$. *For each simplex* v *of* $V_\bullet$ *of dimension* d, *corresponding to a map* $v: \Delta[d] \to V_\bullet$, *there exists a unique map* $\overline{S}(v) \to \overline{V}_\bullet$ *in* $\Delta^0(Et/Y)$ *satisfying:*

a) $\overline{S}(v) \to \overline{V}_\bullet$ *is componentwise the identity; and*

b) $\pi(\overline{S}(v) \to \overline{V}_\bullet) = v: \Delta[d] \to V$.

Furthermore, let $f: X \to Y$ *be a (pointed) map of locally noetherian schemes and let* $\bar{g}: \overline{U}_\bullet \to \overline{V}_\bullet$ *be an object of* J_f. *For each d-simplex* v *of* $V_\bullet$, *define* $\bar{g}^{-1}(v) = \overline{U}_\bullet \times_{\overline{V}} \overline{S}(v)$ *and* $\bar{g}^{-1}(\dot{v}) = \overline{U}_\bullet \times_{\overline{V}_\bullet} \mathrm{sk}_{d-1}\overline{S}(v)$. *Then* $\pi(\bar{g}^{-1}(v))$ *equals* $g^{-1}(v)$ *and*

$$\pi(\bar{g}^{-1}(v), \bar{g}^{-1}(\dot{v})) = (g^{-1}(v), g^{-1}(\dot{v})).$$

Therefore, for any locally constant, abelian sheaf A *on* X *which is constant on* $\overline{U}_0$, $H^*(\bar{g}^{-1}(v), A)$ *equals* $H^*(g^{-1}(v), A)$ *and* $H^*(\bar{g}^{-1}(v), \bar{g}^{-1}(\dot{v}); A) = H^*(\ker(A(\bar{g}^{-1}(v)) \to A(\bar{g}^{-1}(\dot{v}))))$ *equals* $H^*(g^{-1}(v), g^{-1}(\dot{v}); A)$.

Proof. — Properties *a)* and *b)* serve to define $\overline{S}(v) \to \overline{V}_{\bullet}$. The equalities $\pi(\bar{g}^{-1}(v)) = g^{-1}(v)$ and $\pi(\bar{g}^{-1}(v), \bar{g}^{-1}(\dot{v})) = (g^{-1}(v), g^{-1}(\dot{v}))$ follow immediately from *a)* and *b)*, and directly imply the equalities

$$H^*(\bar{g}^{-1}(v), A) = H^*(g^{-1}(v), A) \quad \text{and} \quad H^*(\bar{g}^{-1}(v), \bar{g}^{-1}(\dot{v}); A) = H^*(g^{-1}(v), g^{-1}(\dot{v}); A).$$

We proceed to analyze $E_2^{p,q}(f_{et}, A)$.

Lemma (3.6). — *Let* X *be a scheme and let* $\overline{U}_{\bullet} \to \overline{\Delta[d]}$ *be a special map in* $\Delta^0(\mathrm{Et}/X)$. *Let* F *be an abelian sheaf on* X *and let* $F \to I^*$ *be an injective resolution of sheaves. Then the bicomplex*

$$\ker\{I^*(\overline{U}_{\bullet}) \to I^*(\overline{U_{\bullet} \underset{\Delta[d]}{\times} \mathrm{sk}(d)})\}$$

yields the spectral sequence

$$E_1^{p,q} = (\ker(\mathscr{H}^q F(\overline{U}_{\bullet}) \to \mathscr{H}^q F(\overline{U_{\bullet} \underset{\Delta[d]}{\times} \mathrm{sk}(d)})))_p \Rightarrow H^{p+q-d}(X, F).$$

Proof. — Given a scheme W etale over X, we denote by $\mathbf{Z}_W$ the abelian sheaf on X given by $\mathrm{Hom}(\mathbf{Z}_W, F) = F(W)$, for any abelian sheaf F on X. We first verify that $H_*(\mathbf{Z}_{\overline{U}_{\bullet}}/\mathbf{Z}_{\overline{U_{\bullet} \times_{\Delta[d]} \mathrm{sk}(d)}}) = (\mathbf{Z}_X, d)$, the complex of sheaves which is $\mathbf{Z}_X$ in dimension d and 0 elsewhere. It suffices to prove that the canonical maps

$$\mathbf{Z}_{\overline{U}_{\bullet}}/\mathbf{Z}_{\overline{U_{\bullet} \times_{\Delta[d]} \mathrm{sk}(d)}} \to \mathbf{Z}_{\overline{\Delta[d]}}/\mathbf{Z}_{\overline{\mathrm{sk}(d)}} \to (\mathbf{Z}_X, d)$$

induces isomorphisms $H_*(\mathbf{Z}_{\overline{U}_{\bullet}}/\mathbf{Z}_{\overline{U_{\bullet} \times_{\Delta[d]} \mathrm{sk}(d)}})_x \to (\mathbf{Z}_X, d)_x$ at every geometric point x of X. Since $\overline{U}_{\bullet} \underset{X}{\times} x \to \Delta[d]$ and $\overline{U_{\bullet} \underset{\Delta[d]}{\times} \mathrm{sk}(d)} \underset{X}{\times} x \to \mathrm{sk}(d)$ are special maps in Δ^0 (Sets), these are fibrations with contractible fibre. Therefore, the maps

$$H_*(\mathbf{Z}_{\overline{U}_{\bullet}}/\mathbf{Z}_{\overline{U_{\bullet} \times_{\Delta[d]} \mathrm{sk}(d)}})_x = H_*(\mathbf{Z}_{\overline{U}_{\bullet} \times_X x}/\mathbf{Z}_{\overline{U_{\bullet} \times_{\Delta[d]} \mathrm{sk}(d)} \times_X x}) \to H_*(\Delta[d], \mathrm{sk}(d)) = (\mathbf{Z}_X, d)_x$$

are isomorphisms.

We observe that $\ker(I^*(\overline{U}_{\bullet}) \to I^*(\overline{U_{\bullet} \underset{\Delta[d]}{\times} \mathrm{sk}(d)}))$ equals $\mathrm{Hom}(\mathbf{Z}_{\overline{U}_{\bullet}}/\mathbf{Z}_{\overline{U_{\bullet} \times_{\Delta[d]} \mathrm{sk}(d)}}, I^*)$. Therefore,

$$H^q(H^p(\ker(I^*(\overline{U}_{\bullet}) \to I^*(\overline{U_{\bullet} \underset{\Delta[d]}{\times} \mathrm{sk}(d)}))))$$
$$= H^q(\mathrm{Hom}(H_p(\mathbf{Z}_{\overline{U_{\bullet} \times_{\Delta[d]} \mathrm{sk}(d)}}), I^*)) = \begin{cases} H^q(X, F) & \text{if } p = d \\ 0 & p \neq d \end{cases}$$

so that the total cohomology of the bicomplex $\ker(I^*(\overline{U}_{\bullet}) \to I^*(\overline{U}_{\bullet} \underset{\Delta[d]}{\times} \mathrm{sk}(d)))$ equals $H^{*-d}(X, F)$.

Since $\overline{U}_{\bullet} \underset{\Delta[d]}{\times} \overline{\mathrm{sk}(d)} \to \overline{U}_{\bullet}$ is componentwise the identity map,

$$H^q(\ker(I^*(\overline{U}_{\bullet}) \to I^*(\overline{U_{\bullet} \underset{\Delta[d]}{\times} \mathrm{sk}(d)}))) = \mathrm{Hom}(\mathbf{Z}_{\overline{U}_{\bullet}}/\mathbf{Z}_{\overline{U_{\bullet} \times_{\Delta[d]} \mathrm{sk}(d)}}, \mathscr{H}^q(F))$$

equals $\ker(\mathscr{H}^q(F)(\overline{U}_{\bullet}) \to \mathscr{H}^q(F)(\overline{U}_{\bullet} \underset{\Delta[d]}{\times} \mathrm{sk}(d)))$.

The proposition below relativizes Verdier's theorem (which can be obtained by setting $d=0$, $\overline{V}_{\bullet}=\overline{\Delta[0]}$, and $f=\mathrm{id}:X\to X$).

Proposition (3.7). — *Let* $f:X\to Y$ *be a (pointed) map of locally noetherian schemes and let* F *be an abelian sheaf on* X. *Then for any* $\overline{V}_{\bullet}$ *in* J_Y, *all* $d, q\geq 0$

$$\varinjlim_{\mathrm{Ho}\,J_{f,\overline{V}_{\bullet}}} \prod_{V_d^+} H^{d+q}(\bar{g}^{-1}(v), \bar{g}^{-1}(\dot{v}); F) \xrightarrow{\sim} \prod_{V_d^+} H^q(v\underset{Y}{\times}X, F).$$

Proof. — Let P be an abelian presheaf on X which commutes with direct sums: $P(U\amalg W)=P(U)\times P(W)$. For any v in V_d^+, $\ker(P(\bar{g}^{-1}(v))\to P(\bar{g}^{-1}(\dot{v})))$ equals $\ker(P(\bar{g}^{-1}(v)\underset{\overline{S}(v)}{\times}\overline{\Delta[d]}\underset{Y}{\times}v)\to P(\bar{g}^{-1}(v)\underset{\overline{S}(v)}{\times}\overline{\mathrm{sk}(d)}\underset{Y}{\times}v))$, since the components of $\bar{g}^{-1}(v)$ not in $\bar{g}^{-1}(\dot{v})$ are exactly the components of $\bar{g}^{-1}(v)\underset{\overline{S}(v)}{\times}\overline{\Delta[d]}\underset{Y}{\times}v$ not in $\bar{g}^{-1}(v)\underset{\overline{S}(v)}{\times}\overline{\mathrm{sk}(d)}\underset{Y}{\times}v$. Let $F\to I^{\bullet}$ be an injective resolution of abelian sheaves on X. Applying Lemma (3.6) with X replaced by $v\underset{Y}{\times}X$ and $\overline{U}_{\bullet}$ replaced by $\bar{g}^{-1}(v)\underset{\overline{S}(v)}{\times}\overline{\Delta[d]}\underset{Y}{\times}v$, we conclude that the spectral sequence for the bicomplex $\ker(I^{\bullet}(\bar{g}^{-1}(v))\to I^{\bullet}(\bar{g}^{-1}(\dot{v})))$ can be written

$$E_1^{p,q}(\bar{g}, F, v)=(\ker(\mathscr{H}^qF(\bar{g}^{-1}(v))\to\mathscr{H}^qF(\bar{g}^{-1}(\dot{v}))))_p \Rightarrow H^{p+q-d}(v\underset{Y}{\times}X, F).$$

We define the map $\prod_{V_d^+} H^{d+q}(\bar{g}^{-1}(v), \bar{g}^{-1}(\dot{v}); F)\to\prod_{V_d^+} H^q(v\underset{Y}{\times}X, F)$ to be the edge homomorphism of the product spectral sequence $\prod_{V_d^+}\mathbf{E}^{p,q}(g, F, v)$. Since homotopies in $J_{f,\overline{V}_{\bullet}}$ between $\bar{g}'\rightrightarrows\bar{g}$ induce homotopies between $\mathscr{H}^qF(\bar{g}^{-1}(v))\rightrightarrows\mathscr{H}^qF(\bar{g}'^{-1}(v))$ and $\mathscr{H}^qF(\bar{g}^{-1}(\dot{v}))\rightrightarrows\mathscr{H}^qF(\bar{g}'^{-1}(\dot{v}))$, $\prod_{V_d^+}\mathbf{E}^{p,q}(\bar{g}, F, v)$ is functorial on $\mathrm{Ho}\,J_{f,\overline{V}_{\bullet}}$. Moreover, if $\bar{g}'\to\bar{g}$ is a map in $J_{f,\overline{V}_{\bullet}}$ and $v\in V_d^+$, then

$$\begin{array}{ccc} \bar{g}'^{-1}(v)\underset{\overline{S}(v)}{\times}\overline{\Delta[d]}\underset{Y}{\times}v & \longrightarrow & \bar{g}^{-1}(v)\underset{\overline{S}(v)}{\times}\overline{\Delta[d]}\underset{Y}{\times}v \\ & \searrow\quad\swarrow & \\ & \overline{\Delta[d]}\underset{Y}{\times}v & \end{array} \quad\text{and}\quad \begin{array}{ccc} \bar{g}'^{-1}(v)\underset{\overline{S}(v)}{\times}\overline{\mathrm{sk}(d)}\underset{Y}{\times}v & \longrightarrow & \bar{g}^{-1}(v)\times\mathrm{sk}(d)\underset{Y}{\times}v \\ & \searrow\quad\swarrow & \\ & \overline{\mathrm{sk}(d)}\underset{Y}{\times}v & \end{array}$$

commute; hence, $\bar{g}'\to\bar{g}$ induces an isomorphism on abutments for

$$\prod_{V_d^+}\mathbf{E}^{p,q}(\bar{g}, F, v)\to\prod_{V_d^+}\mathbf{E}^{p,q}(\bar{g}', F, v).$$

It suffices to prove that $\varinjlim_{\mathrm{Ho}\,J_{f,\overline{V}_{\bullet}}}\prod_{V_d^+}E_1^{p,q}(\bar{g}, F, v)=0$ for $q>0$ in order to complete the proof. Let $\times\alpha_v$ be an n-cochain in $\prod_{V_d^+}\ker(\mathscr{H}^qF(\bar{g}^{-1}(v))\to\mathscr{H}^qF(\bar{g}^{-1}(\dot{v})))$. For each α_v in

$$\ker((\mathscr{H}^qF(\bar{g}^{-1}(v))\to\mathscr{H}^qF(\bar{g}^{-1}(\dot{v}))))_n=\mathscr{H}^qF(\bar{g}^{-1}(v)_n-\bar{g}^{-1}(\dot{v})_n),$$

let $\varphi_v:W_v\to\bar{g}^{-1}(v)_n-\bar{g}^{-1}(\dot{v})_n$ be some etale surjective map such that $\varphi_v^*(\alpha_v)=0$ in $\mathscr{H}^qF(W_v)$. Since $\bar{g}^{-1}(v)_n-\bar{g}^{-1}(\dot{v})_n$ embeds in $\overline{U}_n=s(\bar{g})_n$ such that $\bar{g}^{-1}(v)_n-\bar{g}^{-1}(\dot{v})_n$ and

$\bar{g}^{-1}(v')_n - \bar{g}^{-1}(\dot{v}')_n$ are disjoint for $v \neq v'$ in V_d^+, $\amalg \varphi_v$ extends to an etale surjective map $\varphi : W = (\coprod_{V_d^+} W_v) \amalg (\bar{U}_n - \bar{g}^{-1}(V_d^+)_n) \to \bar{U}_n$, where $\bar{g}^{-1}(V_d^+)_n$ consists of the components of $\bar{U}_n$ mapping to some non-degenerate component of $\bar{V}_d^+$.

Using Lemma (2.3), find a special map $\psi : \bar{U}'_\bullet \to \bar{U}_\bullet$ such that $\bar{U}'_n \to \bar{U}_n$ factors through $\varphi : W \to \bar{U}_n$ and let $\bar{g}' : \bar{U}'_\bullet \to \bar{V}_\bullet$ be the composition $g \circ \psi$. By construction, $\psi^*(\times \alpha_v) = 0$ in $\prod_{V_d^+} E_1^{p,q}(\bar{g}', F, v) = \prod_{V_d^+} \ker(\mathscr{H}^q F(\bar{g}'^{-1}(v) \to \mathscr{H}^q F(\bar{g}'^{-1}(\dot{v})))$.

As an immediate corollary of Proposition (3.7), we obtain the following comparison of simplicial and algebraic cohomology.

Corollary **(3.8)**. — *Let* $f : X \to Y$ *be a (pointed) map of locally noetherian schemes and let* A *be a locally constant, abelian sheaf on* X. *For all* $\bar{V}_\bullet$ *in* J_Y, *all* $d, q \geq 0$

$$\varinjlim_{\mathrm{Ho}\, J_{f,\bar{V}_\bullet}} H^{d+q}(g^{-1}V_{(d)}, g^{-1}V_{(d-1)}; A) \xrightarrow{\sim} \prod_{V_d^+} H^q(v \times_Y X, A).$$

Proof. — By excision, $H^{d+q}(g^{-1}V_{(d)}, g^{-1}V_{(d-1)}; A) = \prod_{V_d^+} H^{d+q}(g^{-1}(v), g^{-1}(\dot{v}); A)$. Apply Proposition (3.7).

Corollary (3.8) enables us to interpret $\mathbf{E}^{p,q}(f_{et}, A)$ as a simplicially derived Leray spectral sequence, as explicated in Theorem (3.9) below. In conjunction with Proposition (3.4), this enables us to readily compare the Leray spectral sequence with the Serre spectral sequence for f_{et}^r.

Theorem **(3.9)**. — *Let* $f : X \to Y$ *be a (pointed) map of locally noetherian schemes and let* A *be a locally constant abelian sheaf on* X. *For the spectral sequence* $\mathbf{E}^{p,q}(f_{et}, A)$ *of Proposition* (3.4), $E_2^{p,q}(f_{et}, A)$ *equals* $H^p(Y, R^q f_* A)$. *Therefore,* $\mathbf{E}^{p,q}(f_{et}, A)$ *may be written*

$$E_2^{p,q}(f_{et}, A) = H^p(Y, R^q f_* A) \Rightarrow H^{p+q}(X, A).$$

Proof. — As given in Proposition (3.4),

$$E_2^{p,q}(f_{et}, A) = \varinjlim_{\mathrm{Ho}\, J_f} H^*(H^{p+q}(g^{-1}V_{(p)}, g^{-1}V_{(p-1)}; A); d_1).$$

Since $J_f \to J_Y$ is fibrant with fibre $J_{f,\bar{V}_\bullet}$,

$$\varinjlim_{\mathrm{Ho}\, J_f} H^*(H^{p+q}(g^{-1}V_{(p)}, g^{-1}V_{(p-1)}; A); d_1) = \varinjlim_{\mathrm{Ho}\, J_Y} \varinjlim_{\mathrm{Ho}\, J_{f,\bar{V}_\bullet}} H^*(H^{p+q}(g^{-1}V_{(p)}, g^{-1}V_{(p-1)}; A); d_1).$$

As in Lemma (3.2), $H^*(g^{-1}V_{(p)}, g^{-1}V_{(p-1)}; A)$ determines a functor on $\mathrm{Ho}\, J_{f,\bar{V}_\bullet}$; thus, $E_2^{p,q}(f_{et}, A)$ equals $\varinjlim_{\mathrm{Ho}\, J_Y} H^*(\varinjlim_{\mathrm{Ho}\, J_{f,\bar{V}_\bullet}} H^{p+q}(g^{-1}V_{(p)}, g^{-1}V_{(p-1)}; A); d_1)$

By Corollary (3.8) :

$$\varinjlim_{\mathrm{Ho}\, J_{f,\bar{V}_\bullet}} E_1^{p,q}(\bar{g}, A) \xrightarrow{\sim} \prod_{V_p^+} H^q(v \times_Y X, A).$$

Moreover, d_1 equals $\varinjlim_{\mathrm{Ho}\,J_{f,\overline{V}.}} d_1(\bar{g})$, where

$$d_1(\bar{g}) : \prod_{V_p^+} H^{p+q}(g^{-1}(v'), g^{-1}(\dot{v}'); A) \rightrightarrows \prod_{V_{p+1}^+} H^{p+q+1}(g^{-1}(v), g^{-1}(\dot{v}); A)$$

is the coboundary map for

$$0 \to A(g^{-1}V_{(p+1)}, g^{-1}V_{(p)}) \to A(g^{-1}V_{(p+1)}, g^{-1}V_{(p-1)}) \to A(g^{-1}V_{(p)}, g^{-1}V_{(p-1)}) \to 0.$$

Let $\delta(\bar{g})$ be an "un-normalized" restriction of $d_1(\bar{g})$:

$$\delta(\bar{g}) : \prod_{0 \leqslant i \leqslant p+1} H^{p+q}(g^{-1}(d_i v), g^{-1}(d_i^{\cdot} v); A) \to H^{p+q+1}(g^{-1}(v), g^{-1}(\dot{v}); A).$$

We view $\delta(\bar{g})$ as the coboundary in $H^{*,0}$ cohomology for

$$0 \to I^*(\bar{g}^{-1}(v), \bar{g}^{-1}(\dot{v})) \to I^*(\bar{g}^{-1}(v), \bar{g}^{-1}(\ddot{v})) \to I^*(\bar{g}^{-1}(\dot{v}), \bar{g}^{-1}(\ddot{v})) \to 0$$

where $A \to I^*$ is an injective resolution of abelian sheaves on X. As checked in Proposition (3.7), the associated $\mathbf{E}^{p,q}$ spectral sequences of these bicomplexes degenerate in the limit at $\varinjlim_{\mathrm{Ho}\,J_{f,\overline{V}.}} E_2^{p,q}$. Therefore, $\varinjlim \delta(\bar{g}) = \delta$ may be viewed as a map on abutments.

Since the $\mathbf{E}^{q,p}$ spectral sequences for $I^*(\bar{g}^{-1}(v), \bar{g}^{-1}(\dot{v}))$ and $I^*(\bar{g}^{-1}(\dot{v}), \bar{g}^{-1}(\ddot{v}))$ degenerate at the $E_2^{p,q}$ level as checked in Lemma 3.6, we may view δ as a map

$$\prod_{0 \leqslant i \leqslant p+1} H^{p+q}(I^*(\mathbf{Z}_{d_i v \times_Y X}, p)) \to H^{p+q+1}(I^*(\mathbf{Z}_{v \times_Y X}, p+1)).$$

We readily conclude that this map

$$\prod_{0 \leqslant i \leqslant p+1} H^q(d_i v \times_Y X, A) \to H^q(v \times_Y X, A)$$

is simply the alternating face map induced by the maps of schemes $v \to d_i v$. For σ in $\prod_{V_p^+} H^q(v' \times_Y X, A)$, $d_1\sigma$ in $H^q(v \times_Y X, A)$ is therefore the sum $\sum_i{}' (-1)^i d_i^* \sigma$, where the sum is taken over those i, $0 \leq i \leq p+1$, such that $d_i v$ is in V_p^+.

We conclude that $\{ \varinjlim_{\mathrm{Ho}\,J_{f,\overline{V}.}} H^{p+q}(g^{-1}V_{(p)}, g^{-1}V_{(p-1)}; A); d_1 \}$ is the normalized complex of the co-simplicial group: $n \mapsto H^q(\overline{V}_n \times_Y X, A)$. By Verdier's theorem, $E_2^{p,q}(f_{\mathrm{et}}, A) = \varinjlim_{\mathrm{Ho}J_Y} H^p(n \mapsto H^q(V_n \times_Y X, A))$ equals $H^p(Y, R^q f_* A)$: the cohomology of Y with coefficients in the sheaf associated to the presheaf $S \mapsto H^q(S \times_Y X, A)$.

We remark that $\mathbf{E}^{p,q}(f_{\mathrm{et}}, A)$ can also be written

$$E_2^{p,q}(f_{\mathrm{et}}, A) = \varinjlim_{\mathrm{Ho}J_Y} H^p(\overline{V}_., v \mapsto R^q f_* A(v)) \Rightarrow H^{p+q}(X, A),$$

since Verdier's theorem asserts that

$$\varinjlim_{\mathrm{Ho}J_Y} H^p(n \mapsto H^q(\overline{V}_n \times_Y X, A) \quad \text{equals} \quad \varinjlim_{\mathrm{Ho}J_Y} H^p(n \mapsto R^q f_* A(\overline{V}_n)).$$

4. Applications to a Proper, Smooth Map.

In this section, we apply Proposition (3.4) and Theorem (3.9) to the special case of a proper, smooth, pointed map of noetherian schemes, $f: X \to Y$. The Zeeman comparison theorem [15] enables us to compare the cohomology of the geometric and homotopy theoretic fibres of f_{et}. Under additional hypotheses on the fundamental groups $\pi_1((X_y)_{et})$ and $\pi_1(Y_{et})$, we obtain the long exact sequence of homotopy pro-groups asserted in Corollary (4.8).

If Y is a pointed, connected, noetherian scheme, the Grothendieck fundamental group $\pi_1(Y)$ of Y is the profinite completion of $\pi_1(Y_{et})$ ([4], (10.7)). If Y is in addition geometrically unibranched and if $\overline{V}_\bullet$ is an etale hypercovering of Y with $\overline{V}_n$ noetherian for each n, then $\pi_n(V_\bullet)$ is finite for all $n \geq 1$ ([4], § 11). For such Y, $\pi_n(Y_{et})$ is therefore profinite; in particular, $\pi_1(Y_{et})$ equals $\pi_1(Y)$.

We begin by observing (in Corollary (4.2)) that if $f: X \to Y$ is a pointed map of locally noetherian schemes with Y noetherian, we need only consider those $\bar{g}: \overline{U}_\bullet \to \overline{V}_\bullet$ in Ho J_f with $\overline{V}_n$ noetherian for each $n \geq 0$. We shall frequently use this property of $\overline{V}_\bullet$ to conclude that $\varinjlim H^q(V_\bullet, A_i) = H^q(V_\bullet, \varinjlim A_i)$.

Proposition (4.1). — *Let* Y *be a noetherian scheme and let* $\overline{V}'_\bullet \to \overline{V}_\bullet$ *be a special map in* $\Delta^0(\mathrm{Et}/Y)$ *with* $\overline{V}_n$ *noetherian,* $n \geq 0$. *Then there exists a map* $\overline{V}''_\bullet \to \overline{V}'_\bullet$ *in* $\Delta^0(\mathrm{Et}/Y)$ *such that the composition* $\overline{V}''_\bullet \to \overline{V}'_\bullet \to \overline{V}_\bullet$ *is special and* $\overline{V}''_n \to \overline{V}_n$ *is of finite type,* $n \geq 0$.

Proof. — Let $f: W \to Z$ be an etale, surjective map of schemes with Z of finite type over Y. Since $f: W \to Z$ is locally of finite type, there exists an affine open W_w of W which is of finite type over some affine open Z_w of Z for every point w of W. Since Z is noetherian, each $Z_w \to Z$ is of finite type, so that $W_w \to Z$ is likewise of finite type. Since $f: W \to Z$ is an open mapping, some finite union W′ of W_w's satisfies the condition that $f(W') = Z$. We observe that $W' \to W$ is etale and that the composition $W' \to W \to Z$ is etale, surjective, and of finite type.

To obtain $\overline{V}''_\bullet$, we first apply the above observation to the map $\overline{V}'_0 \to \overline{V}_0$ to obtain $\overline{V}''_0$. Having defined the k-th truncation, $(\overline{V}''_\bullet)^{(k)} \to (\overline{V}'_\bullet)^{(k)}$, we obtain $\overline{V}''_{k+1}$ by applying the above observation to

$$\overline{V}'_{k+1} \underset{(\mathrm{cosk}_k \overline{V}'_\bullet)_{k+1}}{\times} (\mathrm{cosk}_k \overline{V}''_\bullet)_{k+1} \to \overline{V}_{k+1} \underset{(\mathrm{cosk}_k \overline{V}_\bullet)_{k+1}}{\times} (\mathrm{cosk}_k \overline{V}''_\bullet)_{k+1},$$

which is surjective since $\overline{V}'_{k+1} \to \overline{V}_{k+1} \underset{(\mathrm{cosk}_k \overline{V}_\bullet)_{k+1}}{\times} (\mathrm{cosk}_k \overline{V}'_\bullet)_{k+1}$ is surjective. By construction, $(\overline{V}''_\bullet)^{(k)} \to (\overline{V}'_\bullet)^{(k)}$ extends to $(\overline{V}''_\bullet)^{(k+1)} \to (\overline{V}'_\bullet)^{(k+1)}$. By induction, we conclude that $\overline{V}''_{k+1} \to \overline{V}_{k+1} \underset{(\mathrm{cosk}_k \overline{V}_\bullet)_{k+1}}{\times} (\mathrm{cosk}_k \overline{V}''_\bullet)_{k+1}$ is finite type and surjective and $\overline{V}''_{k+1} \to \overline{V}_{k+1}$ is of finite type.

Given a pointed map $f: X \to Y$ of locally noetherian schemes with Y noetherian, we define various codirected and cofiltering categories associated to f. Let $'J_f$ be the full subcategory of J_f consisting of pairs $\overline{U}_\bullet \to \overline{V}_\bullet$ with $\overline{V}_n$ noetherian for all $n \geq 0$. Let $''J_f$ be the category whose objects are those of J_f and whose maps are simplicial squares. Let $\mathrm{Ho}'J_f$ and $\mathrm{Ho}''J_f$ denote the homotopy categories (with respect to pointed categorical homotopies of simplicial pairs) of $'J_f$ and $''J_f$ respectively. Arguing as in Propositions (1.2) and (1.4), we conclude that $'J_f$ and J_f are codirected and that $\mathrm{Ho}'J_f$ and $\mathrm{Ho}''J_f$ are cofiltering.

Corollary **(4.2)**. — *Let* $f: X \to Y$ *be a map of locally noetherian schemes with* Y *noetherian. Then* $\mathrm{Ho}'J_f$ *and* $\mathrm{Ho}\, J_f$ *are cofinal subcategories of* $\mathrm{Ho}''J_f$. *Therefore,* $f_{\mathrm{et}} = \{g\}_{\mathrm{Ho}\, J_f}$ *is canonically isomorphic to* $\{g\}_{\mathrm{Ho}'J_f}$ *in pro-*$\mathcal{K}_{0,\mathrm{pairs}}$.

Proof. — Proposition (4.1) implies that $\mathrm{Ho}'J_f \to \mathrm{Ho}''J_f$ is cofinal provided that $\mathrm{Ho}\, J_f \to \mathrm{Ho}''J_f$ is cofinal. Given maps

$$\begin{array}{ccc} \overline{U}'_\bullet & & \overline{U}_\bullet \\ \downarrow & \rightrightarrows & \downarrow \\ \overline{V}'_\bullet & & \overline{V}_\bullet \end{array}$$

in $''J_f$, we obtain as in Proposition (1.4) a map

$$\begin{array}{ccc} \overline{U}''_\bullet & & \overline{U}'_\bullet \\ \downarrow & \to & \downarrow \\ \overline{V}''_\bullet & & \overline{V}'_\bullet \end{array}$$

in J_f such that the compositions are categorically homotopic. Therefore, $\mathrm{Ho}\, J_f \to \mathrm{Ho}''J_f$ is cofinal. Moreover, the cofinal maps $\mathrm{Ho}'J_f \to \mathrm{Ho}''J_f$ and $\mathrm{Ho}\, J_f \to \mathrm{Ho}''J_f$ induce isomorphisms $\{g\}_{\mathrm{Ho}''J_f} \to \{g\}_{\mathrm{Ho}'J_f}$ and $\{g\}_{\mathrm{Ho}''J_f} \to \{g\}_{\mathrm{Ho}\, J_f}$ in pro-$\mathcal{K}_{0,\mathrm{pairs}}$.

In the following proposition, we assume that $\pi_1(Y_{\mathrm{et}}) = 0$. This simplifies the proof, while providing a good example of how Theorem (3.9) can be applied.

Proposition **(4.3)**. — *Let* $f: X \to Y$ *be a pointed map of locally noetherian schemes with* Y *noetherian. Assume that* Y *is connected and* $\pi_1(Y_{\mathrm{et}}) = 0$. *Let* A *be a locally constant abelian sheaf on* X *such that* $R^q f_* A$ *is constant on* Y *and* $(R^q f_* A)_y \xrightarrow{\sim} H^q(X_y, A)$ *for all* $q \geq 0$, *where* X_y *is the geometric fibre of* f.

Then $(X_y)_{\mathrm{et}} \to \mathfrak{h}(f^r_{\mathrm{et}})$ *induces isomorphisms for all* $q \geq 0$:

$$H^q(\mathfrak{h}(f^r_{\mathrm{et}}), A) \xrightarrow{\sim} H^q((X_y)_{\mathrm{et}}, A).$$

Proof. — Since $\pi_1(V_\bullet) = 0$ for all $\overline{V}_\bullet$ in J_Y ([4], § 10), the local system $v \mapsto H^q((g^r)^{-1}(v), A)$ may be identified with its value, $H^q(\mathfrak{h}(g^r), A)$, on the distinguished vertex of $V_\bullet$. Since $\overline{V}_\bullet$ in $'J_f$ is of finite type in each dimension,

$$\varinjlim_{\mathrm{Ho}\, J_f} H^p(V_\bullet, H^q(\mathfrak{h}(g^r), A)) = \varinjlim_{\mathrm{Ho}'J_f} H^p(V_\bullet, H^q(\mathfrak{h}(g^r), A))$$

equals

$$\varinjlim_{\mathrm{Ho}'J_f} H^p(V_\bullet, H^q(\mathfrak{h}(f^r_{\mathrm{et}}), A)) = \varinjlim_{\mathrm{Ho}\, J_f} H^p(V_\bullet, H^p(\mathfrak{h}(f^r_{\mathrm{et}}), A)).$$

Therefore, $\mathbf{E}^{p,q}(f^r_{et}, A) \to \mathbf{E}^{p,q}(f_{et}, A)$ of Proposition (3.4) may be written

$$\begin{array}{ccc} E_2^{p,q}(f^r_{et}, A) = H^p(Y_{et}, H^q(\mathfrak{h}(f^r_{et}), A)) & \Rightarrow & H^{p+q}(X, A) \\ \downarrow \Phi & & \\ E_2^{p,q}(f_{et}, A) = H^p(Y_{et}, (R^q f_* A)_y) & \Rightarrow & H^{p+q}(X, A) \end{array}$$

where

$$E_2^{p,q}(f_{et}, A) = \varinjlim_{\mathrm{Ho}\,J_Y} H^p(n \mapsto H^q(\overline{V}_n \underset{Y}{\times} X, A)) = \varinjlim_{\mathrm{Ho}\,J_Y} H^p(n \mapsto R^p f_* A(\overline{V}_n)) = H^p(Y_{et}, (R^q f_* A)_y).$$

For each $\overline{V}_\bullet$ in J_Y, the composition

$$\varinjlim_{\mathrm{Ho}\,J_{f,\overline{V}_\bullet}} \prod_{V_p^+} H^q(\mathfrak{h}(g^r,)\ A) \xrightarrow{\sim} \varinjlim_{\mathrm{Ho}\,J_{f,\overline{V}_\bullet}} H^q((g^r)^{-1}V_{(p)}, (g^r)^{-1}V_{(p-1)}; A)$$
$$\to \varinjlim_{\mathrm{Ho}\,J_{f,\overline{V}_\bullet}} H^q(g^{-1}V_{(p)}, g^{-1}V_{(p-1)}; A) \xrightarrow{\sim} \prod_{V_p^+} H^q(v \underset{Y}{\times} X, A) \to \prod_{V_p^+} R^q f_* A(v)$$

is induced by the composition map of groups

$$\varinjlim_{\mathrm{Ho}\,J_{f,\overline{V}_\bullet}} H^q(\mathfrak{h}(g^r), A) \to \varinjlim_{\mathrm{Ho}\,J_{f,\overline{V}_\bullet}} H^q(\mathfrak{h}(g), A) \xrightarrow{\sim} H^q(v_0 \underset{Y}{\times} X, A) \to (R^q f_* A)_y.$$

Therefore, the limit map $\Phi : \mathbf{E}^{p,q}(f^r_{et}, A) \to \mathbf{E}^{p,q}(f_{et}, A)$ is induced by the composition map of groups $H^q(\mathfrak{h}(f^r_{et}), A) \to H^q(\mathfrak{h}(f_{et}), A) \to (R^q f_* A)_y$. Applying the Zeeman comparison theorem [15], we conclude that $H^q(\mathfrak{h}(f^r_{et}), A) \xrightarrow{\sim} (R^q f_* A)_y$, all $q \geq 0$. Applying Proposition (2.4), we conclude that $X_{et} \to \mathfrak{h}(f^r_{et})$ induces isomorphisms $H^*(\mathfrak{h}(f^r_{et}), A) \xrightarrow{\sim} H^*((X_y)_{et}, A)$.

We recall that a map $\{X_i\} \to \{Y_j\}$ in pro-$\mathscr{K}_0$ is said to be a $\natural$-isomorphism provided that the induced maps $\mathrm{cosk}_n\{X_i\} \to \mathrm{cosk}_n\{Y_j\}$ are isomorphisms for all $n \geq 0$. A map $f : \{X_i\} \to \{Y_j\}$ in pro-$\mathscr{K}_0$ is a $\natural$-isomorphism of connected pro-objects if and only if $f_* : \pi_1(\{X_i\}) \xrightarrow{\sim} \pi_1(\{Y_j\})$ and for every $\pi_1(\{Y_j\})$-module M, $H^*(\{Y_j\}, M) \xrightarrow{\sim} H^*(\{X_i\}, M)$ ([4], Theorem (4.3)).

The following lemma will enable us to relax the hypothesis on $f : X \to Y$ that $\pi_1(Y_{et}) = 0$.

Lemma **(4.4)**. — *Let* $f : X \to Y$ *be a pointed, galois cover of connected, locally noetherian schemes. For all* $n \geq 2$,

$$f_* : \pi_n(X_{et}) \xrightarrow{\sim} \pi_n(Y_{et}).$$

Proof. — Let K denote the full subcategory of $\mathrm{Ho}\,J_Y$ consisting of pointed, etale hypercoverings $\overline{V}_\bullet$ of Y such that $\overline{V}_0 \to Y$ factors through $f : X \to Y$. Given $\overline{V}_\bullet$ in $\mathrm{Ho}\,J_Y$, $\overline{V}'_\bullet = \mathrm{cosk}_0 X \underset{\mathrm{cosk}_0 Y}{\times} \overline{V}_\bullet \to \overline{V}_\bullet$ is special (where $(\mathrm{cosk}_0 X)_k$ is the k-fold fibre product over Y of X with itself). Hence, $K \to \mathrm{Ho}\,J_Y$ is cofinal.

Represent Y_{et} as $\{\pi(\overline{V}_{\bullet})\}_K$ in pro-$\mathscr{K}_0$ and let $Z_{et} = \{\pi(\overline{V}_{\bullet} \underset{Y}{\times} X)\}_K$. Then $f_{et} : X_{et} \to Y_{et}$ in pro-$\mathscr{K}_0$ factors as $h_{et} \circ g_{et} : X_{et} \to Z_{et} \to Y_{et}$ in the obvious way.

Since X/Y is galois, $X \underset{Y}{\times} X \to X$ is the trivial d-fold cover $\amalg X \to X$, where d equals the degree of f. For each $\overline{V}_{\bullet}$ in K and each $k \geq 0$, there exists a commutative diagram with cartesian squares:

$$\begin{array}{ccccc} \overline{V}_k \underset{Y}{\times} X & \longrightarrow & \amalg X & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ \overline{V}_k & \longrightarrow & X & \longrightarrow & Y \end{array}$$

Hence, $\pi(\overline{V}_{\bullet} \underset{Y}{\times} X) \to \pi(\overline{V}_{\bullet})$ is a simplicial covering map. We conclude that $h_{et} : Z_{et} \to Y_{et}$, given by $\{\pi(\overline{V}_{\bullet} \underset{Y}{\times} X) \to \pi(\overline{V}_{\bullet})\}_K$ in pro-$\mathscr{K}_{0,\text{pairs}}$, induces isomorphisms $h_* : \pi_n(Z_{et}) \xrightarrow{\sim} \pi_n(Y_{et})$ for $n \geq 2$.

For any abelian sheaf F on X, $g_{et} : X_{et} \to Z_{et}$ induces a map of spectral sequences

$$\begin{array}{ccc} {}'E_2^{p,q} = \varinjlim_{K} H^p(\overline{V}_{\bullet} \underset{Y}{\times} X, \mathscr{H}^q(F)) & \Rightarrow & H^N(X, F) \\ \downarrow & & \\ E_2^{p,q} = \varinjlim_{\text{HoJ}_X} H^p(\overline{U}_{\bullet}, \mathscr{H}^q(F)) & \Rightarrow & H^N(X, F) \end{array}$$

inducing an isomorphism on abutments. We observe that $\varinjlim_{K} C^p(\overline{V}_{\bullet} \underset{Y}{\times} X, \mathscr{H}^q(F))$ equals $\varinjlim_{K} \Pi \mathscr{H}^q(F)(\overline{V}_p)$ for all $p, q \geq 0$, since $(\overline{V}_{\bullet} \underset{Y}{\times} X)_p = \amalg \overline{V}_p$. Using Lemma (2.3), we conclude that ${}'E_2^{p,q} = 0 = E_2^{p,q}$ for all $p \geq 0$, $q > 0$. Therefore, g_{et} induces isomorphisms for any locally constant abelian sheaf F:

$$\varinjlim_{K} H^*(\overline{V}_{\bullet} \underset{Y}{\times} X, F) = H^*(Z_{et}, F) \xrightarrow{\sim} H^*(X_{et}, F) = \varinjlim_{\text{HoJ}_X} H^*(\overline{U}_{\bullet}, F).$$

Furthermore, $\{\operatorname{cosk}_0 \overline{U}_{\bullet}\}_{\text{HoJ}_X} \to \{\operatorname{cosk}_0(\overline{V}_{\bullet} \underset{Y}{\times} X)\}_K$ is clearly an isomorphism; thus $\pi_1(X_{et}) \xrightarrow{\sim} \pi_1(Z_{et})$. Therefore, $g_{et} : X_{et} \to Z_{et}$ is a $\natural$-isomorphism. We conclude that the compositions $\pi_n(X_{et}) \to \pi_n(Z_{et}) \to \pi_n(Y_{et})$ are isomorphisms, for $n \geq 2$.

In the following theorem, we eliminate the hypothesis that $\pi_1(Y_{et})$ be 0 by examining the maps $\mathbf{E}^{p,q}(f_{et}'^{r}, A) \to \mathbf{E}^{p,q}(f_{et}', A)$ for each $f' : X' \to Y'$, the pull-back of f by some pointed galois cover $h' : Y' \to Y$. Thus, we are led to consider direct limits of spectral sequences on the cofiltering category Ho $J_{\tilde{f}}$ (Corollary (1.5)).

Theorem **(4.5)**. — *Let* $f : X \to Y$ *be a pointed map of noetherian schemes such that the geometric fibre* X_y *is connected. Assume* $\pi_1(Y_{et})$ *is pro-finite and* Y *is connected. Let* A *be a locally constant abelian sheaf on* X *whose fibre* A_x *is finite with order relatively prime to the residue* *characteristics of* Y. *Assume* $R^q f_* A$ *is locally constant on* Y *and* $(R^q f_* A)_y \xrightarrow{\sim} H^q(X_y, A)$, *for all* $q \geq 0$.

Then $(X_y)_{et} \to \mathfrak{h}(f^r_{et})$ *induces isomorphisms for all* $q \geq 0$:

$$H^q(\mathfrak{h}(f^r_{et}), A) \xrightarrow{\sim} H^q((X_y)_{et}, A).$$

Proof. — Let $h'' \to h'$ be a map in ET/Y and consider the commutative diagram

$$\begin{array}{ccccccc} (X''_y)_{et} & \longrightarrow & \mathfrak{h}(f''^r_{et}) & \longrightarrow & (X''_{et})^r & \longrightarrow & Y''_{et} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ (X'_y)_{et} & \longrightarrow & \mathfrak{h}(f'^r_{et}) & \longrightarrow & (X'_{et})^r & \longrightarrow & Y'_{et} \end{array}$$

where $f' = f \times h' : X' = X \times_Y Y' \to Y'$ and $f'' : X'' \to Y''$. Since $X_y = X'_y = X''_y$ is connected, X″ and X′ are connected; moreover $\pi_1(X'') \to \pi_1(Y'')$ and $\pi_1(X') \to \pi_1(Y')$ are surjective. Moreover, $\pi_1(X'') \to \pi_1(X')$ and $\pi_1(Y'') \to \pi_1(Y')$ are injective with cokernel $\mathrm{Gal}(X''/X') = \mathrm{Gal}(Y''/Y')$. By Lemma (4.4) and the 5-lemma, $\mathfrak{h}(f''^r_{et}) \to \mathfrak{h}(f'^r_{et})$ is a $\sharp$-isomorphism. We conclude that $H^*(\mathfrak{h}(f^r_{et}), A) \xrightarrow{\sim} \varinjlim_{(ET/Y)} H^*(\mathfrak{h}(f'^r_{et}), A)$, as well as $H^*((X_y)_{et}, A) \xrightarrow{\sim} \varinjlim_{(ET/Y)} H^*((X'_y)_{et}, A)$.

We consider the map $\widetilde{\Phi} : \varinjlim_{\mathrm{Ho}J_{\tilde{f}}} (\mathbf{E}^{p,q}(g^r, A) \to \mathbf{E}^{p,q}(g, A))$. Since $\varinjlim_{\mathrm{Ho}J_{\tilde{f}}} = \varinjlim_{ET/Y} \varinjlim_{\mathrm{Ho}J_{Y'}} \varinjlim_{\mathrm{Ho}J_{f'}, \overline{V}_\bullet}$ and since each $\overline{V}_\bullet$ in some $\mathrm{Ho}\,J_{Y'}$ is of finite type in each dimension, $\widetilde{\Phi}$ can be written

$$\begin{array}{c} \varinjlim_{\mathrm{Ho}J_{\tilde{f}}} E_2^{p,q}(g^r, A) = \varinjlim_{ET/Y} \varinjlim_{\mathrm{Ho}J_{Y'}} H^p(V_\bullet, v \mapsto \varinjlim_{\mathrm{Ho}J_{f'}, \overline{V}_\bullet} H^q(g^{r^{-1}}(v), A)) \Rightarrow \varinjlim_{ET/Y} H^{p+q}(X', A) \\ \downarrow \widetilde{\Phi} \\ \varinjlim_{\mathrm{Ho}J_{\tilde{f}}} E_2^{p,q}(g, A) = \varinjlim_{ET/Y} \varinjlim_{\mathrm{Ho}J_{Y'}} H^p(V_\bullet, v \mapsto R^q f'_* A(v)) \Rightarrow \varinjlim_{ET/Y} H^{p+q}(X', A). \end{array}$$

For A satisfying the hypotheses of the theorem, $R^q f'_* A$ equals $g'^* R^q f_* A$ restricted to Y′; thus, $R^q f'_* A$ is locally constant and $(R^q f_* A)_y = (R^q f'_* A)_y$ ([3], XVI).

Given any g in $\mathrm{Ho}\,J_{\tilde{f}}$, we can find $g' \to g$ in $\mathrm{Ho}\,J_{\tilde{f}}$ such that the induced systems $v' \mapsto H^q((g^r)^{-1}(v'), A)$ and $v' \mapsto R^q f_* A(v')$ are constant on $V'_\bullet$. Hence,

$$\begin{array}{ccc} H^q(V_\bullet, v \mapsto H^q((g^r)^{-1}(v), A)) & & H^q(V'_\bullet, v' \mapsto H^q((g^r)^{-1}(v'), A)) \\ \downarrow & \longrightarrow & \downarrow \\ H^q(V_\bullet, v \mapsto R^q f_* A(v)) & & H^q(V'_\bullet, v' \mapsto R^q f_* A(v')) \end{array}$$

factors through

$$\begin{array}{c} H^q(V'_\bullet, H^q(\mathfrak{h}(g^r), A)) \\ \Big\downarrow \\ H^q(V'_\bullet, (R^q f_* A)_y), \end{array}$$

where the last pair is induced by the map of coefficients evaluated at the distinguished vertex v_0 of $V_\bullet$:

$$H^q(\mathfrak{h}(g^r), A) \to H^q(\mathfrak{h}(g), A) \to H^q(v_0 \underset{Y}{\times} X, A) \to R^q f_* A(v_0).$$

Therefore, $\widetilde{\Phi}$ may be written

$$\begin{array}{ccc} \varinjlim_{\mathrm{HoJ}_{\tilde f}} E_2^{p,q}(g^r, A) = \varinjlim_{\mathrm{ET/Y}} \varinjlim_{\mathrm{HoJ}_{Y'}} H^p(V_\bullet, \varinjlim_{\mathrm{ET/Y}} H^q(\mathfrak{h}(f'^r_{\mathrm{et}}), A)) & \Rightarrow & \varinjlim_{\mathrm{ET/Y}} H^{p+q}(X', A) \\ \Big\downarrow & & \\ \varinjlim_{\mathrm{HoJ}_{\tilde f}} E_2^{p,q}(g, A) = \varinjlim_{\mathrm{ET/Y}} \varinjlim_{\mathrm{HoJ}_{Y'}} H^p(V_\bullet, (R^q f_* A)_y) & \Rightarrow & \varinjlim_{\mathrm{ET/Y}} H^{p+q}(X', A) \end{array}$$

with the map induced by

$$H^q(\mathfrak{h}(f^r_{\mathrm{et}}), A) = \varinjlim_{\mathrm{ET/Y}} H^q(\mathfrak{h}(f'^r_{\mathrm{et}}), A) \to \varinjlim_{\mathrm{ET/Y}} H^q(\mathfrak{h}(f'_{\mathrm{et}}), A) \to \varinjlim_{\mathrm{ET/Y}} (R^q f'_* A)_y = (R^q f_* A)_y.$$

Applying the Zeeman comparison theorem [15] and Proposition (2.4), we conclude that $(X_y)_{\mathrm{et}} \to \mathfrak{h}(f^r_{\mathrm{et}})$ induces isomorphisms $H^*(\mathfrak{h}(f^r_{\mathrm{et}}), A) \stackrel{\sim}{\to} H^*((X_y)_{\mathrm{et}}, A)$.

The " proper, smooth base change theorem " ([3], XVI, Corollary (2.2)) in conjunction with Theorem (4.5) immediately imply the following:

Theorem **(4.6)**. — *Let $f : X \to Y$ be a proper, smooth, pointed map of connected noetherian schemes such that the geometric fibre X_y is connected and $\pi_1(Y_{\mathrm{et}})$ is profinite. Let A be a locally constant, abelian sheaf on X whose fibre A_x is finite with order relatively prime to all residue characteristics of Y.*

Then $(X_y)_{\mathrm{et}} \to \mathfrak{h}(f^r_{\mathrm{et}})$ induces isomorphisms $H^*(\mathfrak{h}(f^r_{\mathrm{et}}), A) \stackrel{\sim}{\to} H^*((X_y)_{\mathrm{et}}, A)$.

Although the assertion of the following lemma is made for a pointed map $f : X \to Y$ of schemes, the conclusion is a statement concerning only the associated pro-object, f_{et}, in pro-$\mathscr{K}_{0\text{-pairs}}$. Such pro-pairs arising from a pointed map of schemes are simpler to consider, since the pointed etale cover $Y_H \to Y$ provides a canonical representative of the covering space $(Y_{\mathrm{et}})_H$ of Y_{et} associated to a subgroup H of $\pi_1(Y)$.

For any set L of primes, we denote by C_L the class of finite groups G satisfying the condition that the prime factors of the order of G lie in L.

Lemma (4.7). — *Let* $f : X \to Y$ *be a pointed map of connected, locally noetherian schemes such that* Y *is noetherian and* $\pi_1(X) \to \pi_1(Y)$ *is surjective. Let* L *be a given set of primes such that* $\pi_1(Y_{et})$ *is in pro-*C_L. *Let* $f_{et} \to \hat{f}_{et}$ *be the canonical map of pro-*$\mathscr{K}_{0,\text{pairs}}$, *where* $(\hat{\ })$ *denotes completion with respect to* C_L ([4], § 5).

Then the induced map $\mathfrak{h}(f^r_{et}) \to \mathfrak{h}(\hat{f}^r_{et})$ *in pro-*$\mathscr{K}_0$ *induces isomorphisms for all abelian* A *in* C_L, *all* $q \geq 0$:

$$H^q(\mathfrak{h}(\hat{f}^r_{et}), A) \xrightarrow{\sim} H^q(\mathfrak{h}(f^r_{et}), A).$$

Proof. — For each $\bar{g}$ of $\text{Ho}\, J_{\tilde{f}}$, represent $\hat{g}$ in pro-$\mathscr{K}_{0,\text{pairs}}$ by $\{g' : U'_\bullet \to V'_\bullet\}_{I_g}$. Then the maps $\{f'^r_{et}\}_{ET/Y} = \{g^r\}_{\text{Ho}J_{\tilde{f}}} \to \{\{g'^r\}_{I_g}\}_{\text{Ho}J_{\tilde{f}}} = \{\hat{f}'^r_{et}\}_{ET/Y}$ induce

$$\tilde{\Phi} : \mathbf{E}^{p,q}(\{\hat{f}'^r_{et}\}_{ET/Y}, A) \to \mathbf{E}^{p,q}(\{f'^r_{et}\}_{ET/Y}, A).$$

As in the proof of Theorem (4.5)

$$E_2^{p,q}(\{\hat{f}'^r_{et}\}_{ET/Y}, A) = \varinjlim_{\text{Ho}J_{\tilde{f}}} \varinjlim_{I_g} H^p(V'_\bullet, v' \mapsto H^q((g'^r)^{-1}(v'), A))$$

is isomorphic to

$$\varinjlim_{\text{Ho}J_{\tilde{f}}} \varinjlim_{I_g} H^p(V'_\bullet, H^q(\mathfrak{h}(g'^r), A)) \xrightarrow{\sim} \varinjlim_{\text{Ho}J_{\tilde{f}}} \varinjlim_{I_g} H^p(\hat{V}_\bullet, H^q(\mathfrak{h}(g'^r), A)).$$

Since $\mathfrak{h}(g'^r)$ has L-primary finite homotopy groups, $H^q(\mathfrak{h}(g'^r), A)$ is in C_L. Therefore,

$$E_2^{p,q}(\{\hat{f}'^r_{et}\}_{ET/Y}, A) \xrightarrow{\sim} \varinjlim_{\text{Ho}J_{\tilde{f}}} \varinjlim_{I_g} H^p(V_\bullet, H^q(\mathfrak{h}(g'^r), A)) \xrightarrow{\sim} \varinjlim_{\text{Ho}J_{\tilde{f}}} H^p(V_\bullet, H^q(\mathfrak{h}(\hat{g}^r), A))$$

since V_n is finite for $n \geq 0$, if $\overline{V}_\bullet$ is in $\text{Ho}'J_Y$.

We conclude that $\tilde{\Phi} : \mathbf{E}^{p,q}(\{\hat{f}'^r_{et}\}_{ET/Y}, A) \to \mathbf{E}^{p,q}(\{f'^r_{et}\}_{ET/Y}, A)$ may be written

$$\begin{array}{ccc} \varinjlim_{ET/Y} H^p(Y'_{et}, H^q(\mathfrak{h}(\hat{f}'^r_{et}), A)) & \Rightarrow & \varinjlim_{ET/Y} H^{p+q}(\widehat{X'_{et}}, A) \\ & \downarrow & \\ \varinjlim_{ET/Y} H^p(Y'_{et}, H^q(\mathfrak{h}(f'^r_{et}), A)) & \Rightarrow & \varinjlim_{ET/Y} H^{p+q}(X', A). \end{array}$$

Since A is in C_L, $\tilde{\Phi}$ is an isomorphism on abutments.

As in the proof of Theorem (4.5), a map $g'' \to g'$ in ET/Y induces isomorphisms $H^*(\mathfrak{h}(f'^r_{et}), A) \xrightarrow{\sim} H^*(\mathfrak{h}(f''^r_{et}), A)$. In order to conclude the lemma by applying the Zeeman comparison theorem, it suffices to verify that $H^*(\mathfrak{h}(\hat{f}'^r_{et}), A) \xrightarrow{\sim} H^*(\mathfrak{h}(\hat{f}''^r_{et}), A)$ whenever $g'' \to g'$ is a map in ET/Y.

Consider the following commutative diagram in pro-$\mathscr{K}_0$:

$$\begin{array}{ccccc} \mathfrak{h}(\hat{f}''^r_{et}) & \longrightarrow & \widehat{X''^r_{et}} & \longrightarrow & \widehat{Y''_{et}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathfrak{h}(\hat{f}'^r_{et}) & \longrightarrow & \widehat{X'^r_{et}} & \longrightarrow & \widehat{Y'_{et}} \end{array}$$

Since $\pi_1(Y')/\pi_1(Y'')=\pi_1(X')/\pi_1(X'')$ is in C_L, $\widehat{Y''_{et}}\to\widehat{Y'_{et}}$ and $\widehat{X''_{et}}\to\widehat{X'_{et}}$ are covering spaces ([4], Theorem (4.11)). Moreover, $\pi_1(\widehat{X'_{et}})/\pi_1(\widehat{X''_{et}})=\pi_1(\widehat{Y'_{et}})/\pi_1(\widehat{Y''_{et}})=\pi_1(Y')/\pi_1(Y'')$. Using the 5-Lemma, we conclude that $\mathfrak{h}(\hat{f}''^{r}_{et})\to\mathfrak{h}(\hat{f}'^{r}_{et})$ is a $\sharp$-isomorphism; hence, that $H^*(\mathfrak{h}(\hat{f}'^{r}_{et}),A)\xrightarrow{\sim}H^*(\mathfrak{h}(\hat{f}''^{r}_{et}),A)$.

Combining Theorem (4.6) and Lemma (4.7), we obtain the following homotopy sequence for a proper, smooth map.

Corollary **(4.8)**. — *Let $f:X\to Y$ be a proper, smooth, pointed map of connected, noetherian schemes. Let L be a set of primes not occurring as residue characteristics of Y and satisfying the condition that $\pi_1(Y_{et})$ is in pro-C_L. Assume that the geometric fibre X_y of f is connected and that the L-completion $\widehat{\pi_1(X_y)}$ of $\pi_1(X_y)$ is o.*

Then there exists a long exact sequence of pro-groups

$$\ldots\to\pi_n((\widehat{X_y})_{et})\to\pi_n(\widehat{X_{et}})\to\pi_n(\widehat{Y_{et}})\to\pi_{n-1}((\widehat{X_y})_{et})\to\ldots$$

where the maps $\pi_n((\widehat{X_y})_{et})\to\pi_n(\widehat{X_{et}})$ and $\pi_n(\widehat{X_{et}})\to\pi_n(\widehat{Y_{et}})$ are induced by the L-completions of the etale homotopy type of the pointed maps $i:X_y\to X$ and $f:X\to Y$.

Proof. — By Proposition (2.2), $i_*:\pi_*((\widehat{X_y})_{et})\to\pi_*(\widehat{X_{et}})$ factors as

$$\pi_*((\widehat{X_y})_{et})\to\pi_*(\widehat{\mathfrak{h}(\hat{f}^r_{et})})\to\pi_*(\mathfrak{h}(\hat{f}^r_{et}))\to\pi_*(\widehat{X_{et}}).$$

Since $\mathrm{Ex}^\infty(\mathfrak{h}(\hat{f}^r_{et}))\to\mathrm{Ex}^\infty(\widehat{X^r_{et}})\to\mathrm{Ex}^\infty(\widehat{Y_{et}})$ is a pro-fibre triple ([10], Theorem (4.3)), the triple $\mathfrak{h}(\hat{f}^r_{et})\to\widehat{X^r_{et}}\to\widehat{Y_{et}}$ induces a long exact sequence of pro-groups

$$\ldots\to\pi_n(\mathfrak{h}(\hat{f}^r_{et}))\to\pi_n((\widehat{X_y})_{et})\to\pi_n(\widehat{Y_{et}})\to\pi_{n-1}(\mathfrak{h}(\hat{f}^r_{et}))\to\ldots$$

To verify the corollary, it suffices to prove that the composition $(\widehat{X_y})_{et}\to\widehat{\mathfrak{h}(\hat{f}^r_{et})}\to\mathfrak{h}(\hat{f}^r_{et})$ is a $\sharp$-isomorphism.

Recall that $\pi_1(X_y)\to\pi_1(X)\to\pi_1(Y)\to e$ is exact ([8], § X). Since completion with respect to C_L is right exact, $\pi_1(\widehat{X_{et}})=\widehat{\pi_1(X)}\to\widehat{\pi_1(Y)}=\pi_1(\widehat{Y_{et}})$ is an isomorphism. Therefore, $\pi_2(\widehat{Y_{et}})\to\pi_1(\mathfrak{h}(\hat{f}^r_{et}))$ is surjective, implying that $\pi_1(\mathfrak{h}(\hat{f}^r_{et}))$ is pro-abelian. Since $\pi_1((X_y)_{et})=0$ and $\pi_1(\mathfrak{h}(\hat{f}^r_{et}))$ is pro-abelian, it suffices to check that the composition $(X_y)_{et}\to\widehat{\mathfrak{h}(\hat{f}^r_{et})}\to\mathfrak{h}(\hat{f}^r_{et})$ induces isomorphisms $H^*(\mathfrak{h}(\hat{f}^r_{et}),A)\xrightarrow{\sim}H^*((X_y)_{et},A)$, any abelian group A in C_L ([4], Theorem (4.3)). This follows directly from Theorem (4.6) and Lemma (4.7).

5. Applications to a Vector Bundle minus its o-section.

Given a locally free sheaf M on a scheme Y, the (algebraic) vector bundle $\mathbf{V}(M)$ associated to M is the spectrum of the symmetric $\mathcal{O}_Y$-algebra on M. In this section, we investigate the structure map $f:\mathbf{V}(M)-o(Y)\to Y$, where $\mathbf{V}(M)-o(Y)$ is the vector bundle minus its o-section. We conclude that $\mathfrak{h}(f^{r}_{\mathrm{et}})$ completed away from the residue characteristics of Y is a completed sphere.

A commutative triangle

$$\begin{array}{ccc} Z & \xrightarrow{i} & V \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle g} & \\ & S & \end{array}$$

of schemes is said to be a smooth S-pair of codimension c provided that i is a closed immersion, h and g are smooth, and for every geometric point s of S, $Z_s \rightarrow V_s$ is of codimension c. In particular, if M is a locally free sheaf of rank r on a scheme Y, then

$$\begin{array}{ccc} Y & \xrightarrow{0} & \mathbf{V}(M) \\ & {\scriptstyle =}\searrow \quad \swarrow{\scriptstyle g} & \\ & Y & \end{array}$$

is a smooth Y-pair of codimension r.

Lemma (5.1). — *Given a smooth* S-*pair*

$$\begin{array}{ccc} Z & \xrightarrow{i} & V \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle g} & \\ & S & \end{array}$$

of codimension $c>0$, *with* Z *simply connected. Let* L *be any set of primes not occurring as residue characteristics of* S. *Denote* $V-Z$ *by* X.

Then for any abelian group A *in* C_L, *there exists the following " Gysin " long exact sequence in cohomology*:

$$\ldots \rightarrow H^{n-2c}(Z, A) \rightarrow H^{n}(V, A) \rightarrow H^{n}(X, A) \rightarrow H^{n-2c+1}(Z, A) \rightarrow \ldots.$$

Proof. — Consider the Leray spectral sequence for the map $j: X \rightarrow V$ and the constant sheaf A on X:

$$E_2^{p,q} = H^p(V, R^q j_* A) \Rightarrow H^{p+q}(X, A).$$

Recall that for $j: X \rightarrow V$ associated to a smooth S-pair as in the present situation, j_*A equals the constant sheaf A on V; $R^q j_* A = 0$ for $q \neq 0$, $2c-1$; and $i^*(R^{2c-1}j_*A)$ is locally isomorphic to A on Z ([3], XVI, Theorem (3.7)). Since Z is simply connected, $i^*(R^{2c-1}j_*A)$ equals the constant sheaf A. Hence $R^{2c-1}j_*A = i_*A$ ([2], Theorem (2.2)). Thus $E_2^{p,q}$ reduces to two non-vanishing rows: $H^*(V, R^0 j_* A) = H^*(V, A)$ and $H^*(V, R^{2c-1}j_*A) \xrightarrow{\sim} H^{*-2c+1}(V, i_*A) \xrightarrow{\sim} H^{*-2c+1}(Z, A)$. This spectral sequence therefore reduces to the asserted long exact sequence.

The following proposition verifies the cohomological conditions on $\mathbf{V}(\mathrm{M})-\mathrm{o}(\mathrm{Y})\to\mathrm{Y}$ required in order to employ the simplicial Leray spectral sequence.

Proposition **(5.2)**. — *Let* Y *be a connected, noetherian scheme; let* M *be a locally free, rank r sheaf on* Y; *and let* $f:\mathrm{X}=\mathbf{V}(\mathrm{M})-\mathrm{o}(\mathrm{Y})\to\mathrm{Y}$ *be the restriction of the structure map* $g:\mathbf{V}(\mathrm{M})\to\mathrm{Y}$. *Let* L *be the set of primes not occurring as residue characteristics of* Y. *Then for any abelian group* A *in* $\mathrm{C_L}$, $\mathrm{R}^q f_*\mathrm{A}$ *is locally constant on* Y; *and* $(\mathrm{R}^q f_*\mathrm{A})_y \xrightarrow{\sim} \mathrm{H}^q(\mathrm{X}_y, \mathrm{A})$ *for all* $q\geq 0$, *all geometric points y of* Y.

Furthermore,

$$\mathrm{H}^q(\mathrm{X}_y, \mathrm{A}) = \begin{cases} \mathrm{A} & \text{if } q=0,\ 2r-1 \\ 0 & \text{otherwise} \end{cases}$$

for any abelian A *in* $\mathrm{C_L}$, *any geometric point y of* Y.

Proof. — We view $f:\mathrm{X}\to\mathrm{Y}$ as the composition of g with $j:\mathrm{X}\to\mathbf{V}(\mathrm{M})$. Consider the spectral sequence for the composite left exact functor f_* and the constant sheaf A on the etale site of X:

$$\mathrm{E}_2^{s,t}=\mathrm{R}^s g_*(\mathrm{R}^t j_*\mathrm{A}) \Rightarrow \mathrm{R}^{\mathrm{N}} f_*\mathrm{A}.$$

As in the proof of Lemma (5.1), $\mathrm{R}^0 j_*\mathrm{A}=\mathrm{A}$ on $\mathbf{V}(\mathrm{M})$; $\mathrm{R}^t j_*\mathrm{A}=0$ for $t\neq 0,\ 2r-1$; and $\mathrm{o}^*(\mathrm{R}^{2r-1}j_*\mathrm{A})$ is locally isomorphic to the constant sheaf A on Y. Since $g:\mathbf{V}(\mathrm{M})\to\mathrm{Y}$ is acyclic for L (g is smooth with L-acyclic fibres), we conclude that $\mathrm{R}^s g_*\mathrm{R}^0 j_*\mathrm{A}=0$ for $s\neq 0$. Furthermore, for $s\neq 0$,

$$\mathrm{R}^s g_*(\mathrm{R}^{2r-1}j_*\mathrm{A})=\mathrm{R}^s g_*(\mathrm{o}_*\mathrm{o}^*\mathrm{R}^{2r-1}j_*\mathrm{A})=\mathrm{R}^s \mathrm{id}_*(\mathrm{o}^*\mathrm{R}^{2r-1}j_*\mathrm{A})=0.$$

Hence, $\mathrm{R}^{2r-1}f_*\mathrm{A}=g_*(\mathrm{R}^{2r-1}j_*\mathrm{A})=(g\circ\mathrm{o})_*\mathrm{o}^*(\mathrm{R}^{2r-1}j_*\mathrm{A})$ is locally isomorphic to $\mathrm{o}^*\mathrm{A}=\mathrm{A}$ on Y, whereas $\mathrm{R}^0 f_*\mathrm{A}=g_*(j_*\mathrm{A})=\mathrm{A}$. Thus, $\mathrm{R}^q f_*\mathrm{A}$ is locally constant on Y for all $q\geq 0$.

Since $(\mathrm{R}^q f_*\mathrm{A})_y \xrightarrow{\sim} \mathrm{H}^q(\mathrm{X}_\mathfrak{y}, \mathrm{A})$, where $\mathfrak{y}$ denotes the strict local ring at a geometric point y, it suffices to prove that the natural homomorphisms $\mathrm{H}^q(\mathrm{X}_\mathfrak{y}, \mathrm{A})\to\mathrm{H}^q(\mathrm{X}_y, \mathrm{A})$ are isomorphisms to complete the proof. The residue map $y\to\mathfrak{y}$ induces a map of Gysin sequences for the smooth pairs of codimension r given in Lemma (5.1):

$$\begin{array}{ccc} y & \longrightarrow & /\mathbf{A}^r_y=(\mathbf{V}(\mathrm{M}))_y \\ & {\scriptstyle =}\searrow \quad \swarrow & \\ & y & \end{array} \qquad \begin{array}{ccc} \mathfrak{y} & \longrightarrow & /\mathbf{A}^r_\mathfrak{y}=(\mathbf{V}(\mathrm{M}))_\mathfrak{y} \\ & {\scriptstyle =}\searrow \quad \swarrow & \\ & \mathfrak{y} & \end{array}$$

Applying the 5-Lemma, we conclude that $y\to\mathfrak{y}$ induces isomorphisms

$$\mathrm{H}^q(\mathrm{X}_\mathfrak{y}, \mathrm{A}) \xrightarrow{\sim} \mathrm{H}^q(\mathrm{X}_y, \mathrm{A})$$

for all $q\geq 0$, all geometric points y.

Proposition (5.2) provides the hypotheses needed to prove the analogue of Theorem (4.6) and Corollary (4.8) for $f:\mathrm{X}=\mathbf{V}(\mathrm{M})-\mathrm{o}(\mathrm{Y})\to\mathrm{Y}$.

Theorem **(5.3).** — *Let* Y *be a pointed, connected, noetherian scheme such that* $\pi_1(Y_{\text{et}})$ *is pro-finite. Let* M *be a locally free, rank* $r \geq 2$ *sheaf on* Y. *Let* $f : X = \mathbf{V}(M) - o(Y) \to Y$ *be the restriction of the structure map* $g : \mathbf{V}(M) \to Y$, *pointed above the pointing of* Y. *Let* L *be some set of primes excluding all residue characteristics of* Y *and let* $(\hat{\ })$ *denote completion with respect to* C_L.

Then there exist $\sharp$*-isomorphisms in pro-*$\mathcal{K}_0$:

$$\widehat{S^{2r-1}} \to \widehat{(X_y)}_{\text{et}} \quad \textit{and} \quad \widehat{(X_y)}_{\text{et}} \to \widehat{\mathfrak{h}(f^r_{\text{et}})}.$$

Furthermore, if $\pi_1(Y)$ *is in pro-*C_L, *there then exists a* $\sharp$*-isomorphism* $\widehat{(X_y)}_{\text{et}} \to \mathfrak{h}(\hat{f}^r_{\text{et}})$; *consequently, there exists a long exact sequence of pro-groups*:

$$\dots \to \pi_n(\widehat{(X_y)}_{\text{et}}) \to \pi_n(\widehat{X_{\text{et}}}) \to \pi_n(\widehat{Y_{\text{et}}}) \to \pi_{n-1}(\widehat{(X_y)}_{\text{et}}) \to \dots$$

where the maps $\pi_n(\widehat{(X_y)}_{\text{et}}) \to \pi_n(\widehat{X_{\text{et}}})$ *and* $\pi_n(\widehat{X_{\text{et}}}) \to \pi_n(\widehat{Y_{\text{et}}})$ *are induced by the* L*-completions of the etale homotopy types of the pointed maps* $i : X_y \to X$ *and* $f : X \to Y$.

Proof. — To verify the existence of a $\sharp$-isomorphism $\widehat{S^{2r-1}} \to \widehat{(X_y)}_{\text{et}}$, it suffices to verify that $\pi_1(\widehat{(X_y)}_{\text{et}}) = 0$ and that

$$H_q(\widehat{(X_y)}_{\text{et}}, \mathbf{Z}) = \begin{cases} \hat{\mathbf{Z}} & q = 2r-1 \\ 0 & q \neq 0, 2r-1 \end{cases} \qquad \text{([4], Corollary (4.15)).}$$

Yet $\pi_1(X_y) = \pi_1(\mathbf{V}(M)_y) = \pi_1(\mathbf{A}^r_y)$ by the " Purity Theorem " ([3], XVI, Theorem (3.3)), so that $\pi_1(\widehat{(X_y)}_{\text{et}}) = 0$. Serre class theory implies that $H_q((X_y)_{\text{et}}, \mathbf{Z})$ is in pro-C_L for all $q > 0$. Since $H^*(X_y, A) = H^*(\widehat{(X_y)}_{\text{et}}, A)$ for abelian groups A in C_L, Proposition (5.2) and the universal coefficient theorem imply that

$$H_q(\widehat{(X_y)}_{\text{et}}, \mathbf{Z}) = \begin{cases} \hat{\mathbf{Z}} & q = 2r-1 \\ 0 & q \neq 0, 2r-1 \end{cases}$$

as required.

To verify that $(X_y)_{\text{et}} \to \mathfrak{h}(f^r_{\text{et}})$ induces a $\sharp$-isomorphism $\widehat{(X_y)}_{\text{et}} \to \widehat{\mathfrak{h}(f^r_{\text{et}})}$, it suffices to prove that $\pi_1(\widehat{(X_y)}_{\text{et}}) = 0 = \pi_1(\widehat{\mathfrak{h}(f^r_{\text{et}})})$, and that for any abelian group A in C_L, $H^*(\mathfrak{h}(f^r_{\text{et}}), A) \xrightarrow{\sim} H^*((X_y)_{\text{et}}, A)$.

Observe that if $Y' \to Y$ is a pointed, galois cover in ET/Y, then

$$\begin{array}{ccc} X' = \mathbf{V}(M \underset{\mathcal{O}_Y}{\otimes} \mathcal{O}_{Y'}) - o(Y') & \longrightarrow & \mathbf{V}(M) - o(Y) \\ \Big\downarrow{\scriptstyle f'} & & \Big\downarrow{\scriptstyle f} \\ Y' & \longrightarrow & Y \end{array}$$

is cartesian. Furthermore, since $\pi_1(X) \xrightarrow{\sim} \pi_1(\mathbf{V}(M))$ by the Purity Theorem and since $\mathbf{V}(M) \to Y$ has a section, $\pi_1(X) \to \pi_1(Y)$ is surjective; therefore, X' is connected. Applying the 5-Lemma plus Lemma (4.4), we conclude that the square

$$\begin{array}{ccc} (X'_y)_{et} & \xrightarrow{\sim} & (X_y)_{et} \\ \downarrow & & \downarrow \\ \mathfrak{h}(f'^{r}_{et}) & \longrightarrow & \mathfrak{h}(f^{r}_{et}) \end{array}$$

commutes in pro-$\mathscr{K}_0$, with $\mathfrak{h}(f'^{r}_{et}) \to \mathfrak{h}(f^{r}_{et})$ a $\#$-isomorphism for any map $Y' \to Y$ in ET/Y.

We consider the long exact sequence of pro-groups

$$\ldots \to \pi_n(\mathfrak{h}(f^{r}_{et})) \to \{\pi_n(X'_{et})\}_{ET/Y} \to \{\pi_n(Y'_{et})\}_{ET/Y} \to \ldots$$

In particular, $\{\pi_2(Y'_{et})\}_{ET/Y} \to \pi_1(\mathfrak{h}(f^{r}_{et})) \to \{\pi_1(X'_{et})\}_{ET/Y} \to e$ is exact, since

$$\{\pi_1(Y'_{et})\}_{ET/Y} = \{\pi_1(Y')\}_{ET/Y} = e.$$

For any $Y' \to Y$ in ET/Y, $f' : X' \to Y'$ is 1-aspherical for L; therefore, $\pi_1(\widehat{X'}) = \pi_1(\widehat{Y'})$. Since $(\hat{\ })$ is right exact, we conclude that $\{\widehat{\pi_2(Y_{et})}\}_{ET/Y} \to \pi_1(\widehat{\mathfrak{h}(f^{r}_{et})})$ is surjective; thus $\pi_1(\widehat{\mathfrak{h}(f^{r}_{et})})$ is pro-abelian. We conclude that it suffices to prove that $(X_y)_{et} \to \mathfrak{h}(f^{r}_{et})$ induces isomorphisms for any abelian A in $C_L : H^*(\mathfrak{h}(f^{r}_{et}), A) \xrightarrow{\sim} H^*((X_y)_{et}, A)$. This is immediate by Theorem (4.5) and Proposition (5.2).

Finally, to conclude the existence of the asserted long exact sequence provided $\pi_1(Y)$ is in pro-C_L, we apply Lemma (4.7) as in the proof of Corollary (4.8) to prove that the composition $\widehat{(X_y)}_{et} \to \widehat{\mathfrak{h}(f^{r}_{et})} \to \mathfrak{h}(\hat{f}^{r}_{et})$ is a $\#$-isomorphism.

6. Adams' Conjecture.

In this section, we provide a completion of D. Quillen's sketch of Adams' conjecture for complex vector bundles [13]. The arguments and results of Quillen's discussion are freely employed, together with Theorem (5.3) of the previous section. An additional ingredient is D. Sullivan's observation that pro-finite pro-objects of the homotopy category admit inverse limits.

Let $\mathscr{H}$ denote the homotopy category of topological spaces having the homotopy type of C-W complexes and let $[.,.]$ denote maps in $\mathscr{H}$. We recall that any pair in $\mathscr{H}$ is represented by a Hurewicz fibration (namely, the mapping path fibration of any representative), unique up to fibre homotopy equivalence ([6], Theorem (6.1)). Thus, the fibrewise join $E \underset{Y}{*} E' \to Y$ of pairs $E \to Y$ and $E' \to Y$ in $\mathscr{H}$ is well-defined up to fibre homotopy equivalence [9]. Moreover, the fibre $\mathfrak{h}(E \to Y)$ of a pair $E \to Y$ in $\mathscr{H}$

is thereby defined up to homotopy equivalence; furthermore, $\mathfrak{h}(E \to Y)$ is an object of $\mathscr{H}$ [11]. We use $||.||$ to denote the geometric realization functor from (pointed) simplicial sets to $\mathscr{H}$. If $S_\bullet \to T_\bullet$ is a Kan fibration of simplicial sets, then $||\mathfrak{h}(S_\bullet \to T_\bullet)||$ is isomorphic to $\mathfrak{h}(||S_\bullet|| \to ||T_\bullet||)$ in $\mathscr{H}$ [12].

We begin by recalling the representability of pro-finite inverse limits of connected objects of $\mathscr{H}$ ([14], Propositions (3.1) and (3.3)). For the sake of completeness, we include Sullivan's proof.

Proposition **(6.1)** (Sullivan). — *Let* $\{X_i\}$ *in pro-*$\mathscr{H}$ *satisfy the conditions that* X_i *is (arcwise) connected and* $\pi_n(X_i)$ *is finite for each* $n \geq 1$, i *in* I. *Then the functor*

$$\varprojlim [\ , X_i] : \mathscr{H}^0 \to (\text{Sets})$$

is representable in $\mathscr{H}$ *by a* C-W *complex, denoted* $\varprojlim X_i$.

Proof. — A functor $F : \mathscr{H}^0 \to$ (Compact Hausdorff spaces) is a " compact representable functor " provided that the composition with the stripping functor

(Compact Hausdorff spaces) $\to$ (Sets)

is representable in $\mathscr{H}$ by a connected C-W complex. Using Brown's representability axioms [5], we verify the following: if $\{F_i\}_I$ is a system of compact representable functors indexed by a cofiltering indexing category I, then $\varprojlim F_i$ is a compact representable functor.

Since $\varprojlim$ commutes with inverse limits, $\varprojlim F_i(\vee Y_j)$ equals $\prod_j \varprojlim F_i(Y_j)$, where $\vee Y_j$ is an arbitrary wedge product of connected objects of $\mathscr{H}$ (each pointed by a non-degenerate base point); furthermore, by the compactness of F_i, $\varprojlim F_i(Y)$ maps surjectively onto $\varprojlim_{i,n} F_i(Y_n)$, where $Y_n \to Y$ satisfies the condition that $\mathrm{sk}_n(Y_n) \xrightarrow{\sim} \mathrm{sk}_n(Y)$ in $\mathscr{H}$ for any non-negative integer n. Let $Y = S \cup T$ and let $Z = S \cap T$ be given in $\mathscr{H}$, with Z a subcomplex of both S and T. Let $\varprojlim s_i \times \varprojlim t_i$ be an element of $\varprojlim F_i(S) \times \varprojlim F_i(T)$ such that the image of $\varprojlim s_i$ equals the image of $\varprojlim t_i$ in $\varprojlim F_i(Z)$. Let C_i denote the closed subset of $F_i(Y)$ given as the inverse image of $s_i \times t_i$ in $F_i(S) \times F_i(T)$. Since the inverse limit of non-empty compact sets is non-empty, there exists an element $\varprojlim y_i$ in $\varprojlim C_i \subset \varprojlim F_i(Y)$ which maps to $\varprojlim s_i \times \varprojlim t_i$.

Thus, it suffices to prove that $\{[\ , X_i]\}_I$ is a system of compact representable functors. Observe that if Y_c is a finite subcomplex of Y, then $[Y_c, X_i]$ is a finite set. For any two complexes X and Y, the natural map $[Y, X] \to \varprojlim_c [Y_c, X]$ is surjective, where Y_c runs through the finite subcomplexes of Y. We use the finiteness of homotopy classes of homotopies $Y_c \times I \to X_i$ restricting to given maps on $Y_c \times 0$ and $Y_c \times 1$ in order to verify that $[Y, X_i] \to \varprojlim_c [Y_c, X_i]$ is injective. If $f, g : Y \rightrightarrows X_i$ map to the same element of $\varprojlim_c [Y_c, X_i]$, we obtain an inverse system of non-empty finite sets of homotopy classes of homotopies between the restrictions of f and g. An element of the inverse limit yields compatible homotopies which patch together to yield a homotopy between f and g.

We conclude that $[Y, X_i] = \varprojlim_c [Y_c, X_i]$ admits the structure of a (totally disconnected) compact Hausdorff space. We readily check that any $Y' \to Y$ induces a continuous map $[Y, X_i] \to [Y', X_i]$. Moreover, whenever $X_i \to X_j$ in $\{X_i\}_I$, then $[Y, X_i] \to [Y, X_j]$ is continuous for any Y. Hence, $\{[\ , X_i]\}_I$ is a system of compact representable functors.

This inverse limit of pro-finite pro-objects preserves fibrations in the following sense.

Lemma (**6.2**). — *Let* $\{E_i \to Y_i\}_I$ *be an object of pro-*$\mathscr{H}_{\text{pairs}}$ *such that* E_i, Y_i, *and* $\mathfrak{h}(E_i \to Y_i)$ *are connected and* $\pi_n(E_i)$ *and* $\pi_n(Y_i)$ *are finite for every* $n \geq 1$, i *in* I. *Then*

$$\mathfrak{h}(\varprojlim E_i \to \varprojlim Y_i) \xrightarrow{\sim} \varprojlim \mathfrak{h}(E_i \to Y_i) \quad \text{in } \mathscr{H}.$$

Proof. — By autoduality of finite abelian groups, $\varprojlim$ is exact on functors from I to (Finite abelian groups). Hence the long exact sequences associated to the triples

$$\begin{array}{ccccc} \mathfrak{h}(\varprojlim E_i \longrightarrow \varprojlim Y_i) & \longrightarrow & \varprojlim E_i & \longrightarrow & \varprojlim Y_i \\ \downarrow & & \downarrow & & \downarrow \\ \mathfrak{h}(E_i \longrightarrow Y_i) & \longrightarrow & E_i & \longrightarrow & Y_i \end{array}$$

imply isomorphisms $\pi_n(\mathfrak{h}(\varprojlim E_i \to \varprojlim Y_i)) \xrightarrow{\sim} \pi_n(\varprojlim \mathfrak{h}(E_i \to Y_i))$ for $n \geq 2$.

For $n = 1$, we let $E_i' \to Y_i'$ denote the pull-back of $E_i \to Y_i$ to the universal covering space Y_i' of Y_i. Exactness of $\varprojlim$ on finite abelian groups and left exactness on arbitrary groups imply the exactness of $\varprojlim \pi_2(E_i') \to \varprojlim \pi_2(Y_i') \to \varprojlim \pi_1(\mathfrak{h}(E_i' \to Y_i')) \to \varprojlim \pi_1(E_i')$. Upon applying the 5-Lemma to the commutative diagram

$$\begin{array}{ccccccccc} \pi_2(\varprojlim E_i') & \longrightarrow & \pi_2(\varprojlim Y_i') & \longrightarrow & \pi_1(\mathfrak{h}(\varprojlim E_i' \longrightarrow \varprojlim Y_i')) & \longrightarrow & \pi_1(\varprojlim E_i') & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \varprojlim \pi_2(E_i') & \longrightarrow & \varprojlim \pi_2(Y_i') & \longrightarrow & \varprojlim \pi_1(\mathfrak{h}(E_i' \longrightarrow Y_i')) & \longrightarrow & \varprojlim \pi_1(E_i') & & \end{array}$$

we conclude that $\pi_1(\mathfrak{h}(\varprojlim E_i' \to \varprojlim Y_i')) \xrightarrow{\sim} \varprojlim \pi_1(\mathfrak{h}(E_i' \to Y_i'))$. Since

$$\begin{array}{ccc} \mathfrak{h}(\varprojlim E_i' \longrightarrow \varprojlim Y_i') & \longrightarrow & \varprojlim \mathfrak{h}(E_i' \longrightarrow Y_i') \\ \downarrow & & \downarrow \\ \mathfrak{h}(\varprojlim E_i \longrightarrow \varprojlim Y_i) & \longrightarrow & \varprojlim \mathfrak{h}(E_i \longrightarrow Y_i) \end{array}$$

commutes in $\mathscr{H}$, $\mathfrak{h}(\varprojlim E_i \to \varprojlim Y_i) \to \varprojlim \mathfrak{h}(E_i \to Y_i)$ is an isomorphism in $\mathscr{H}$.

The following lemma will enable us to define the fibrewise join of " L-completed sphere fibrations."

Lemma **(6.3)**. — *Let* (^) *denote completion with respect to the class* C_L *of finite groups with prime factors in a given set* L *of primes. Let* S *and* S' *be* m *and* n *spheres, with* $m, n \geq 1$, *respectively. Then the natural map* $S * S' \to \varprojlim \hat{S} * \varprojlim \hat{S}'$ *induces a* $\natural$*-isomorphism*

$$(S * S')^{\wedge} \to (\varprojlim \hat{S} * \varprojlim \hat{S}')^{\wedge}.$$

Proof. — Since the join of pathwise connected spaces is simply connected, it suffices to prove that for all $q \in L$, $S * S' \to \varprojlim \hat{S} * \varprojlim \hat{S}'$ induces isomorphisms $H^*(\varprojlim \hat{S} * \varprojlim \hat{S}', \mathbf{Z}/q) \xrightarrow{\sim} H^*(S * S', \mathbf{Z}/q)$. We recall that for any pair of connected spaces Z and Z'

$$(CZ, Z) \times (CZ', Z') = (CZ \times CZ', Z * Z')$$

where CZ denotes the cone on Z. Furthermore, we recall that $H^*(\varprojlim \hat{S}, \mathbf{Z}/q) = H^*(S, \mathbf{Z}/q)$ ([14], " complement " Theorem (3.9) for $m > 1$; for $m = 1$, use the fact that $\mathbf{Z}$ and $\varprojlim \hat{\mathbf{Z}}$ have same $\mathbf{Z}/q$ cohomology), thus finite. Applying the Künneth theorem, we conclude that $H^*(C \varprojlim \hat{S}, \varprojlim \hat{S}; \mathbf{Z}/q) \otimes H^*(C \varprojlim \hat{S}', \varprojlim \hat{S}'; \mathbf{Z}/q)$ is isomorphic to $H^*(C \varprojlim \hat{S} \times C \varprojlim \hat{S}', \varprojlim \hat{S} * \varprojlim \hat{S}'; \mathbf{Z}/q)$. Therefore,

$$\begin{array}{ccc} H^*(\varprojlim \hat{S}, \mathbf{Z}/q) \otimes H^*(\varprojlim \hat{S}', \mathbf{Z}/q) & \xrightarrow{\sim} & H^*(S, \mathbf{Z}/q) \otimes H^*(S', \mathbf{Z}/q) \\ \downarrow \wr & & \downarrow \wr \\ H^{*+1}(\varprojlim \hat{S} * \varprojlim \hat{S}', \mathbf{Z}/q) & \longrightarrow & H^{*+1}(S * S', \mathbf{Z}/q). \end{array}$$

If $\{X_i\} \to \{Y_j\}$ is a map in pro-$\mathscr{H}$ with $\{Y_j\}$ 1-connected, then

$$\mathfrak{h}(\{X_i\} \to \{Y_j\})^{\wedge} \to \mathfrak{h}(\widehat{\{X_i\}} \to \widehat{\{Y_j\}})$$

is a $\natural$-isomorphism in pro-$\mathscr{H}$ ([4], Theorem (5.9)).

Let Y in $\mathscr{H}$ be 1-connected. If $E \to Y$ and $E' \to Y$ are pairs in $\mathscr{H}$ having fibres $\varprojlim S^{m-1}$ and $\varprojlim S^{n-1}$ respectively, then

$$\mathfrak{h}(E \underset{Y}{*} E' \to Y)^{\wedge} = (\varprojlim \widehat{S^{m-1}} * \varprojlim \widehat{S^{n-1}})^{\wedge} \to \mathfrak{h}(\widehat{E \underset{Y}{*} E'} \to \hat{Y})$$

is a $\natural$-isomorphism. By Lemma (6.3), $(S^{m+n-1})^{\wedge} \to \mathfrak{h}(E \underset{Y}{*} E' \to Y)^{\wedge}$ is a $\natural$-isomorphism. Applying Lemma (6.2), we conclude that $\varprojlim \widehat{E \underset{Y}{*} E'} \to \varprojlim \hat{Y}$ has fibre $\varprojlim (S^{m+n-1})^{\wedge}$.

Definition **(6.4)**. — Let L be a set of primes, (^) denote completion with respect to C_L. A 1-connected C-W-complex Y is L-*good* provided that the canonical map $Y \to \varprojlim \hat{Y}$ in $\mathscr{H}$ is an isomorphism.

If Y is L-good and $m > 1$, $SF^L_m(Y)$ is defined to be the set of fibre homotopy equivalence classes of Hurewicz fibrations with fibre $\varprojlim S^{m-1}$. The symmetric operation

$$+ : SF^L_m(Y) \times SF^L_n(Y) \to SF^L_{m+n}(Y)$$

is defined by sending $p : E \to Y$, $p' : E' \to Y$ to the fibration $p + p' : \varprojlim \widehat{E \underset{Y}{*} E'} \to Y$ induced by $\widehat{E \underset{Y}{*} E'} \to \hat{Y}$.

If Y' is a 1-connected, finite type C-W complex (*i.e.*, $H_n(Y')$ is finitely generated for each n), then for any L, $\varprojlim \widehat{Y'} = Y$ is L-good, where $(\hat{\ })$ is the completion with respect to this L ([14], " complement " Theorem (3.9)).

The following proposition introduces the monoid of stable, L-completed sphere fibrations over an L-good complex.

Proposition **(6.5)**. — *Let* Y *be an* L-*good complex, let* $e_n : \varprojlim S^{m-1} \times Y \to Y$ *denote the trivial fibration of* $SF^L_m(Y)$, *and let* ε_n *denote*

$$. + e_n : SF^L_m(Y) \to SF^L_{m+n}(Y).$$

Then $\{SF^L_m(Y), \varepsilon_n\}$ *is filtering, with set-theoretic direct limit denoted* $SF^L(Y)$. *Moreover,*

$$+ : SF^L_m(Y) \times SF^L_n(Y) \to SF^L_{m+n}(Y)$$

provides the set $SF^L(Y)$ *with the structure of a commutative monoid, natural with respect to homotopy classes of maps* $X \to Y$ *of* L-*good* C-W *complexes.*

Proof. — The projection maps for $((\varprojlim \widehat{S^{m-1}} * \varprojlim \widehat{S^{n-1}}) \times Y)\hat{\ }$ induce a fibre homotopy equivalence $\varprojlim((\varprojlim \widehat{S^{m-1}} * \varprojlim \widehat{S^{n-1}}) \times Y)\hat{\ } \to \varprojlim \widehat{S^{m+n-1}} \times \varprojlim \hat{Y}$ over $\varprojlim \hat{Y}$ by Lemma (6.2) and the fact that $\mathfrak{h}(E \to Y)\hat{\ } \to \mathfrak{h}(\hat{E} \to \hat{Y})$ is a $\natural$-isomorphism. Hence $\varepsilon_n(e_m)$ equals e_{m+n}. To prove that $\{SF^L_m(Y), \varepsilon_n\}$ is filtering and that $\{SF^L(Y), +\}$ is a well-defined commutative monoid, it suffices to check the associativity of $+$. There are natural maps $E \underset{Y}{*} E' \underset{Y}{*} E'' \to \varprojlim(\varprojlim \widehat{E \underset{Y}{*} E'} \underset{Y}{*} E'')\hat{\ }$ and $E \underset{Y}{*} E' \underset{Y}{*} E'' \to \varprojlim(E \underset{Y}{*} \varprojlim \widehat{E' \underset{Y}{*} E''})\hat{\ }$ covering Y. By examining the induced maps of fibres, we conclude that $(p + p') + p''$ and $p + (p' + p'')$ are fibre homotopy equivalent to $\varprojlim(p \underset{Y}{*} p' \underset{Y}{*} p'')\hat{\ }$.

Let $f : X \to Y$ represent a homotopy class of maps of L-good complexes, let $p : E \to Y$ be in $SF^L_m(Y)$, and let $p' : E' \to Y$ be in $SF^L_{m'}(Y)$. Using the natural maps $(E \underset{Y}{\times} X) \underset{X}{*} (E' \underset{Y}{\times} X) \to \varprojlim((E \underset{Y}{\times} X) \underset{Y}{*} (E' \underset{Y}{\times} X))\hat{\ }$ and $(E \underset{Y}{*} E') \underset{Y}{\times} X \to \varprojlim \widehat{E \underset{Y}{*} E'} \underset{Y}{\times} X$, we readily conclude that $(f^*p) + (f^*p')$ is fibre homotopy equivalent to $f^*(p + p')$. Hence, f induces a homomorphism $SF^L(Y) \to SF^L(X)$.

We recall the C-W complex B_G, the classifying space for sphere fibrations. For a finite dimensional C-W complex Z, the group $[Z, B_G]$ is the group of stable fibre homotopy equivalence classes of sphere fibrations over Z. The following proposition relates $[Z, B_G]$ to $SF^L(\varprojlim \hat{Z})$.

Proposition **(6.6).** — *Let* Z *be a* 1-*connected, finite* C-W *complex, let* L *be a set of primes, let* (^) *denote completion with respect to* L, *and let* $Y = \varprojlim \hat{Z}$. *The association of* $\varprojlim \hat{p} : \varprojlim \hat{E} \to Y$ *to a sphere fibration* $p : E \to Z$ *defines a homomorphism of monoids*

$$\theta_L : [Z, B_G] \to SF^L(Y), \quad \textit{any set } L \textit{ of primes.}$$

Furthermore, if $p : E \to Z$ *is a sphere fibration in the kernel of* θ_L, *then the order of* p *in* $[Z, B_G]$ *has no prime factor in* L.

Proof. — Let $p : E \to Z$, $p' : E' \to Z$ be sphere fibrations over Z, inducing a (not necessarily unique) map $p \underset{Z}{*} p' \to \varprojlim \hat{p} \underset{Y}{*} \varprojlim \hat{p}'$. Applying the Künneth formula to the map on the cohomology of the fibres, we conclude that the induced map

$$\varprojlim \widehat{p \underset{Z}{*} p'} \to \varprojlim \hat{p} + \varprojlim \hat{p}'$$

is a fibre homotopy equivalence by Lemma (6.3). Moreover, if $p : S^{m-1} \times Z \to Z$ is the trivial S^{m-1} fibration, then the natural map $\varprojlim \hat{p} \to e_m$ is a fibre homotopy equivalence of fibrations over Y. Hence, θ_L is defined on stable classes and is a homomorphism of monoids.

Let $p : E \to Z$ represent a class in the kernel of θ_L. We may assume that there exists a map $\varprojlim \hat{E} \to \varprojlim \hat{S}^m$ whose composition with the inclusion of the fibre $\varprojlim \hat{S}^m \to \varprojlim \hat{E}$ is an isomorphism in $\mathscr{H}$.

Observe that $[\hat{E}, \hat{S}^m]$ equals $[E, \hat{S}^m]$ equals $[\varprojlim \hat{E}, \hat{S}^m]$ since E is L-good, which by definition equals $[\varprojlim \hat{E}, \varprojlim \hat{S}^m]$. Similarly, $[\hat{S}^m, \hat{S}^m]$ equals $[\varprojlim \hat{S}^m, \varprojlim \hat{S}^m]$. We conclude that $[\hat{E}, \hat{S}^m] \to [\hat{S}^m, \hat{S}^m]$ is surjective, where this map is given by composition with the fibre inclusion $\hat{S}^m \to \hat{E}$ in pro-$\mathscr{H}$. Therefore, $(\hat{E}, \hat{S}^m) \to (\hat{S}^m, \hat{S}^m)$ is likewise surjective, where $(.,.)$ denotes the abelian pro-group of stable maps in pro-$\mathscr{H}$ ([4], § 7). Furthermore, $(\hat{E}, \hat{S}^m) = (E, S^m)\hat{}$ and $(\hat{S}^m, \hat{S}^m) = (S^m, S^m)\hat{} = \hat{\mathbf{Z}}$ ([4], Corollary (7.5)). In other words, the inclusion of the fibre $S^m \to E$ induces a homomorphism $(E, S^m) \to (S^m, S^m)$ whose cokernel has L-completion 0.

For some $n > 0$, there thus exists a map $S^n E \to S^{n+m}$ whose composition with $S^{n+m} \to S^n E$ is a map $S^{n+m} \to S^{n+m}$ of degree k, with k having no factor in L. By Adam's " mod. k Dold Theorem " ([1], Theorem (1.1)), applied to $(S^{n-1} \times Y) * E \to S^{n+m} \times Y$ we conclude that $p : E \to Z$ represents a stable class in $[Z, B_G]$ whose order has no factor in L.

The following proposition, proposed by Quillen ([13], § 10), will suffice to prove Adams' conjecture. We recall that an algebraic variety V over the complex numbers can be triangulated; thus if V is projective, it is a finite C-W complex.

Proposition **(6.7).** — *Let* p *be a prime number, let* L *be the set of all primes except* p, *and let* (^) *denote completion with respect to* C_L. *Let* R *denote the local ring given as the strict loca-*

lization of $\mathbf{Z}$ *at* p, *and let* k *denote the residue field of* R. *Let* Y_R *be a scheme projective and smooth over* Spec R, *with generic geometric fibre* $Y_{\mathbf{C}}$ *over* Spec $\mathbf{C}$ ($\mathbf{C}$ = *complex numbers*) *and special geometric fibre* Y_k *over* Spec k. *Let* Y_{cl} *denote the "classical" topological space of complex points of* $Y_{\mathbf{C}}$. *Let* $K^{top}(Y_{cl})$ *denote the Grothendieck group of complex vector bundles over* Y_{cl} *and let* $K^{alg}(Y_*)$ *denote the Grothendieck group of locally free, finite rank sheaves over* Y_*.

If Y_{cl} *is* 1-*connected, then there exists a commutative diagram of monoids*

$$\begin{array}{ccccccc} K^{top}(Y_{cl}) & \xleftarrow{cl} & K^{alg}(Y_{\mathbf{C}}) & \xleftarrow{j^*} & K^{alg}(Y_R) & \xrightarrow{i^*} & K^{alg}(Y_k) \\ \downarrow{\scriptstyle \theta_L \circ J} & & \downarrow{\scriptstyle \mathscr{J}_{\mathbf{C}}} & & \downarrow{\scriptstyle \mathscr{J}_R} & & \downarrow{\scriptstyle \mathscr{J}_k} \\ SF^L(\varprojlim \hat{Y}_{cl}) & \stackrel{\sim}{\leftarrow} & SF^L(\varprojlim \| \hat{Y}_{\mathbf{C},et} \|) & \stackrel{\sim}{\leftarrow} & SF^L(\varprojlim \| \hat{Y}_{R,et} \|) & \stackrel{\sim}{\rightarrow} & SF^L(\varprojlim \| \hat{Y}_{k,et} \|) \end{array}$$

where $J : K^{top}(Y_{cl}) \to [Y_{cl}, B_G]$ *is the usual* J *homomorphism, where*

$$\mathscr{J}_*(E_*) = \varprojlim \| (\mathbf{V}(E_*) - o(Y_*))^{\wedge}_{et} \| \to \varprojlim \| \hat{Y}_{*,et} \|$$

for E_* *locally free, of rank* ≥ 2 *over* Y_* (*see Theorem* (5.3)), *and where the lower horizontal arrows are induced by the isomorphisms in pro-*$\mathscr{H}$,

$$\hat{Y}_{cl} \xrightarrow{\hat{\gamma}} \| \hat{Y}_{\mathbf{C},et} \| \xrightarrow{\hat{j}} \| \hat{Y}_{R,et} \| \xleftarrow{\hat{i}} \| \hat{Y}_{k,et} \|.$$

Proof. — To prove that $\mathscr{J}_*$ extends to a homomorphism on $K^{alg}(Y_*)$, it suffices to prove that $\mathscr{J}_*$ is additive with invertible image. For $\mathscr{J}_{\mathbf{C}}$, these properties and the commutativity of the left-most square follows from the observation that $\hat{\gamma}^* \mathscr{J}_{\mathbf{C}}(E_{\mathbf{C}}) = \theta_L \circ J \circ cl(E_{\mathbf{C}})$ in $SF^L(\varprojlim \hat{Y}_{cl})$, by the "generalized Riemann existence theorem" ([4], Theorem (12.9)). For $\mathscr{J}_R$, we observe that $\hat{j}^* \mathscr{J}_R(E_R) = \mathscr{J}_{\mathbf{C}}(j^* E_R)$ in $SF^L(\varprojlim \| \hat{Y}_{\mathbf{C},et} \|)$ by employing the proper, smooth base change theorem and lemma (5.1). Similarly, the proper base change theorem implies that the right-hand square commutes, provided $\mathscr{J}_k$ extends to a homomorphism on $K^{alg}(Y_k)$. There remains to prove the following:

1) for any locally free, rank ≥ 2 sheaf E_k over Y_k, $\mathscr{J}(E_k)$ is invertible, and

2) for any short exact sequence

$$(*) \qquad 0 \to E'_k \to E_k \to E''_k \to 0$$

of locally free, rank ≥ 2 sheaves over Y_k, $\mathscr{J}(E_k)$ equals $\mathscr{J}_k(E'_k) + \mathscr{J}_k(E''_k)$.

Let $(*)$ be a short exact sequence as above such that E_k is generated by its global sections. Let n be the rank of E_k, m be the rank of E''_k, and let N be an integer such that E is generated by N global sections. Let $D_{\mathbf{Z}}$ denote the 2-stage flag manifold of type N, n, m. Then D_R is projective and smooth over Spec R and D_{cl} is 1-connected. Hence, $\mathscr{J}_D : K^{alg}(D_R) \to SF^L(\varprojlim \| \hat{D}_{R,et} \|)$ is well-defined. Moreover, $(*)$ arises by pull-back from the universal short exact sequence over D_R. Hence, $\mathscr{J}_k(E_k) = \mathscr{J}_k(E'_k) + \mathscr{J}_k(E''_k)$

and each of $\mathscr{J}_k(E_k)$, $\mathscr{J}_k(E'_k)$, and $\mathscr{J}_k(E''_k)$ are invertible, since $\mathscr{J}_k \circ i^* = \hat{i}^* \circ \mathscr{J}_R$ on actual bundles.

More generally, any locally free, finite rank sheaf E_k over Y_k is embedded (locally as a direct summand) in $bL^{\otimes a}$, where L is a very ample bundle on Y_k and a, b are positive integers. By the above argument, $\mathscr{J}_k(E_k)$ is invertible.

Finally, given any short exact sequence $(*)$ of locally free, rank ≥ 2 sheaves over Y_k, there exists a commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & F_3 & = & F_3 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & E'_k & \longrightarrow & F_1 & \longrightarrow & F_2 & \longrightarrow & 0\\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & E'_k & \longrightarrow & E_k & \longrightarrow & E''_k & \longrightarrow & 0\\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

with F_1 and F_2 generated by global sections. Hence

$$\begin{aligned}\mathscr{J}_k(E_k) &= \mathscr{J}_k(F_1) - \mathscr{J}_k(F_3) = (\mathscr{J}_k(E'_k) + \mathscr{J}_k(F_2)) - (\mathscr{J}_k(F_2) - \mathscr{J}(E''_k))\\ &= \mathscr{J}_k(E'_k) + \mathscr{J}_k(E''_k).\end{aligned}$$

In conclusion, we complete Quillen's sketch of Adams' conjecture.

Theorem **(6.8)** (Adams' Conjecture). — *Let* E *be a complex vector bundle over a finite complex* X *and let* k *be a positive integer. For some positive integer* n, *the stable sphere fibration associated to* $k^n(\psi^k E - E)$ *in* $K^{top}(X)$ *is fibre homotopically trivial.*

Proof. — It suffices to prove the theorem for the canonical m-dimensional quotient bundle Q over the Grassmannian G of complex N planes in $m+N$ space (for arbitrary $m, N > 1$) and for $k=p$, a prime. Let L be the set of all primes except p and let R denote the strict localization of $\mathbf{Z}$ at p.

Let $G_{\mathbf{Z}}$ denote the Grassmannian scheme over Spec $\mathbf{Z}$ representing the functor " isomorphism classes of m-dimensional vector bundles with $m+N$ generating sections. " The pull-back of $G_{\mathbf{Z}}$ to Spec R, denoted G_R, satisfies the hypotheses of Proposition (6.7). Let Q_R denote the canonical locally free, rank m sheaf over G_R; Q is the topological vector bundle over $G = G_{cl}$ associated to $Q_R \underset{\mathcal{O}_{G_R}}{\otimes} \mathcal{O}_{G_{\mathbf{C}}}$ over $G_{\mathbf{C}}$.

To verify the theorem, it suffices by Proposition (6.6) to prove that $\theta_L \circ J(\psi^p(Q) - Q) = 0$ in $SF^L(\varprojlim \hat{G})$. By Proposition (6.7) it suffices to prove that $\mathscr{J}_k(\psi^p(Q_k) - Q_k) = 0$, or equivalently that $\mathscr{J}_k(\psi^p(Q_k)) = \mathscr{J}_k(Q_k)$, in $SF^L(\varprojlim || \hat{G}_{k,\,et} ||)$.

We recall that $\psi^p(Q_k)=Q_k^{(p)}$ in $K^{alg}(G_k)$, where $Q^{(p)}$ is the pull-back of Q_k by the Frobenius map $\varphi : G_k \to G_k$. Furthermore, there exists a purely inseparable map

$$\mathrm{Spec}(\mathrm{Sym}_{\mathcal{O}_{G_k}}(Q_k)) = \mathbf{V}(Q_k) \to \mathbf{V}(Q_k^{(p)}) = \mathrm{Spec}(\mathrm{Sym}_{\mathcal{O}_{G_k}}(Q_k^{(p)}))$$

of schemes over G_k, given in local coordinates by (Z_i) goes into (Z_i^p). Therefore, the induced map in pro-$\mathscr{H}_0$ $(\mathbf{V}(Q_k)-o(G_k))_{et} \to (\mathbf{V}(Q_k^{(p)})-o(G_k))_{et}$ is an isomorphism of pro-objects over $(G_k)_{et}$ ([8], Theorem (4.10)). By definition of $\mathscr{J}_k$, we conclude that $\mathscr{J}_k(\psi^p(Q_k))=\mathscr{J}_k(Q_k)$.

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