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C*-ALGEBRAS OF OPERATORS ON A HALF-SPACE II : INDEX THEORY(1)

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§ 1. Introduction.

If $\varphi \in C(\mathbf{S}^1)$ is a continuous complex valued function on the circle, the restriction of the multiplication operator M_φ on $L^2(\mathbf{S}^1)$ to the Hardy space $H^2(\mathbf{S}^1)$ is often called a Toeplitz operator. Although M_φ is invertible on $L^2(\mathbf{S}^1)$ if and only if φ is everywhere non-zero, the associated Toeplitz operator $W_\varphi = PM_\varphi | H^2(\mathbf{S}^1)$, where P is the orthogonal projection onto $H^2(\mathbf{S}^1)$, is frequently not invertible even when M_φ is. It is elementary that W_φ is a Fredholm operator if and only if φ is nonvanishing, in which case its analytic index:

$$a\text{-ind}(W_\varphi) = \dim \ker(W_\varphi) - \dim \ker(W_\varphi^*)$$

equals minus the winding number of φ with respect to the origin. It is also elementary that $W_{\varphi\psi} - W_\varphi W_\psi$ is a compact operator; it turns out that if $\mathscr{C}$ is the C*-algebra generated by $\{W_\varphi\}_{\varphi \in C(\mathbf{S}^1)}$ and $\mathfrak{K}$ is the ideal of compact operators, then $\mathscr{C}/\mathfrak{K} \simeq C(\mathbf{S}^1)$ (see [8]). The algebra $\mathscr{C}$ is part of the algebra of singular integral operators on $\mathbf{S}^1$ so that for each $A \in \mathscr{C}$ it is natural to call $A + \mathfrak{K} \in C(\mathbf{S}^1)$ the *symbol* $\sigma(A)$ of A. Then A is Fredholm if and only if $\sigma(A)$ never vanishes, and the most elementary of index theorems states that $a\text{-ind}\, A = t\text{-ind}\, A$ where the topological index of A:

$$t\text{-ind}\, A = -\text{winding number } \sigma(A).$$

The principal result in this paper is a continuous analogue of the above index theorem for a class of operators analogous to Toeplitz operators (see [9], [10]). The topological index which appears is real valued and the corresponding analytic index uses M. Breuer's index theory for a II_∞ von Neumann algebra ([5], [6]). For the moment, we restrict our attention to a special case we now describe.

In Fourier transform space, the algebra $\mathscr{C}$ may be alternatively characterized as the C*-algebra generated by translations on $\ell^2(\mathbf{Z}^+)$, with $\mathbf{Z}^+$ the non-negative integers. We now let $\mathscr{A}$ be the C*-algebra generated by translations on $L^2(\mathbf{R}^+)$, with $\mathbf{R}^+$ the

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non-negative reals. (Note that after an inverse Fourier transform, a linear combination of translations $\sum_j a_j T_{\lambda_j}$ on $L^2(\mathbf{R}^+)$ takes the form $PM_\varphi = W_\varphi$ where P is the projection onto the Hardy space $H^2(\mathbf{R})$ and M_φ is multiplication by the almost periodic function $\varphi(t) = \sum_j a_j e^{i\lambda_j t}$.) It was shown ([1]) in [10] that $\mathscr{A}/\mathfrak{J}$, where $\mathfrak{J}$ is the closed ideal generated by all commutators in $\mathscr{A}$, is isomorphic to $AP(\mathbf{R}) = C(\mathbf{R}^B)$, the space of almost periodic functions on $\mathbf{R}$ = the space of continuous functions on the Bohr compactification, $\mathbf{R}^B$. By analogy with the discrete case of $A \in \mathscr{A}$, let the symbol of A denoted by $\sigma(A)$ be $A + \mathfrak{J} \in AP(\mathbf{R})$. If $\sigma(A)$ is bounded away from zero, recall that the mean motion of $\sigma(A)$ is defined by

$$\lim_{T\to\infty} \frac{1}{2T}\{\arg \sigma(A)(T) - \arg \sigma(A)(-T)\} = -(t\text{-ind}(A)).$$

This concept of average winding number provides a real valued topological index on those A with invertible symbols analogous to the winding number for nonvanishing functions in $C(\mathbf{S}^1)$. In this paper we construct a faithful representation ρ of $\mathscr{A}$ into a factor $\mathscr{N}$ of type II_∞ such that for $A \in \mathscr{A}$, $\rho(A)$ is Fredholm relative to $\mathscr{N}$ in the sense of Breuer if and only if $\sigma(A)$ is bounded away from zero; whence one can define the $a\text{-ind}(A) = \dim_{\mathscr{N}}(\ker \rho(A)) - \dim_{\mathscr{N}}(\ker \rho(A^*))$. Then we prove the index theorem: $t\text{-ind}(A) = a\text{-ind}(A)$ for $A \in \text{Fred}(\mathscr{A}, \mathfrak{J})$, the elements in $\mathscr{A}$ invertible mod $\mathfrak{J}$.

The organization of this paper is as follows: in § 2 we prove the index theorem in the special case described above; in § 3 we extend this result to a certain class of locally compact abelian groups; in § 4 we discuss open problems related to this paper.

§ 2. The real case.

We will not distinguish between the two equivalent representations of the C^*-algebra $\mathscr{A}$ introduced in § 1:

(i) The C^*-algebra on $L^2(\mathbf{R}^+)$ generated by the translations $\{T_\lambda; \lambda \geq 0\}$, where $(T_\lambda f)(x) = f(x-\lambda)$ or

(ii) The C^*-algebra on the Hardy space $H^2(\mathbf{R})$ generated by $W_\varphi = PM_\varphi$ for $\varphi \in AP(\mathbf{R})$, where P is projection on $H^2(\mathbf{R})$.

It should be clear from the context which representation is being used. Note that $\sigma(W_\varphi) = \varphi$.

For any Banach algebra τ, let τ^0 denote the group of invertible elements in τ and τ^d the discrete group obtained by dividing τ^0 by its identity component. It is easy to check that the components of $\text{Fred}(\mathscr{A}, \mathfrak{J})$ are in one to one correspondence with $(\mathscr{A}/\mathfrak{J})^d$. It was shown in [10] that the topological index gives an isomorphism of $(\mathscr{A}/\mathfrak{J})^d$ with

(1) See Remark (3.7).

the discrete reals $\mathbf{R}_d$ (1). It is readily checked that t-ind: $\mathrm{Fred}(\mathscr{A}, \mathfrak{J}) \to \mathbf{R}$ possesses the usual properties of an index

$$(2.1)\quad \begin{array}{lll} \text{(i)} & t\text{-ind}(AB) = t\text{-ind}(A) + t\text{-ind}(B) & \\ \text{(ii)} & t\text{-ind}(A^*) = -t\text{-ind}(A) & \\ \text{(iii)} & t\text{-ind}(A+J) = t\text{-ind}(A) & \text{for } J \in \mathfrak{J} \\ \text{(iv)} & t\text{-ind}(A+E) = t\text{-ind}(A) & \text{for } \|E\| \text{ sufficiently small.} \end{array}$$

Let $\widetilde{\mathscr{A}}$ denote the C*-algebra on $L^2(\mathbf{R})$ generated by the multiplications M_φ for $\varphi \in L^\infty(\mathbf{R})$ and by the translations T_λ for $\lambda \in \mathbf{R}$. Clearly $\mathscr{A}$ is isomorphic to the sub-C*-algebra of $\widetilde{\mathscr{A}}$ generated by translations restricted to $\mathbf{R}^+ : \{\chi_{\mathbf{R}^+} T_\lambda \chi_{\mathbf{R}^+}, \lambda \in \mathbf{R}\}$ where $\chi_{\mathbf{R}^+}$ is multiplication by the characteristic function of $\mathbf{R}^+ \subset \mathbf{R}$.

To construct our desired representation ρ of $\mathscr{A}$ into a II_∞ factor $\mathscr{N}$, we first construct a representation $\widetilde{\rho}$ of $\widetilde{\mathscr{A}}$ into a factor $\widetilde{\mathscr{N}}$ and restrict $\widetilde{\rho}$ to $\mathscr{A}$. The II_∞ factor $\widetilde{\mathscr{N}}$ is the group-measure space construction of Murray and von Neumann (see Dixmier [11], p. 136): Let L be the Hilbert space $L^2(\mathbf{R}) \otimes \ell^2(\mathbf{R}_d) \simeq L^2(\mathbf{R} \times \mathbf{R}_d)$ where $\mathbf{R}_d$ denotes the reals in the discrete topology. For $\varphi \in L_\infty(\mathbf{R})$ and $\lambda \in \mathbf{R}$, we define operators on L by:

$$(\widetilde{M}_\varphi f)(x, t) = \varphi(x) f(x, t)$$
$$(\widetilde{T}_\lambda f)(x, t) = f(x-\lambda, t-\lambda).$$

The weakly closed algebra $\widetilde{\mathscr{N}}$ generated by $\{\widetilde{M}_\varphi, \widetilde{T}_\lambda\}$ is a factor of type II_∞. It is customary to normalize the relative dimension function $\dim_{\widetilde{\mathscr{N}}}$ so that the trace on the trace class of $\widetilde{\mathscr{N}}$ satisfies:

if $S = \sum_j \widetilde{M}_{\varphi_j} \widetilde{T}_{\lambda_j}$ with $\varphi_j \in L^\infty(\mathbf{R}) \cap L^1(\mathbf{R})$, then $\mathrm{tr}(S) = \int_{-\infty}^{\infty} \varphi_0(t)\,dt$ where $\lambda_0 = 0$.

On the set of finite sums $\{\sum_j M_{\varphi_j} T_{\lambda_j}, \varphi_j \in L^\infty(\mathbf{R}), \lambda_j \in \mathbf{R}\}$ we define $\widetilde{\rho}$ by: $\widetilde{\rho}(\sum_j M_{\varphi_j} T_{\lambda_j}) = \sum_j \widetilde{M}_{\varphi_j} \widetilde{T}_{\lambda_j}$. According to lemma (2.1) below, $\widetilde{\rho}$ is an isometry on such sums and hence $\widetilde{\rho}$ has a unique continuous extension to the closure, $\widetilde{\mathscr{A}}$. Clearly $\widetilde{\rho}$ is a $*$-homomorphism and hence a C*-isomorphism of $\widetilde{\mathscr{N}}$ into $\widetilde{\mathscr{A}}$.

Lemma (**2.1**). — *If* $A \in \widetilde{\mathscr{A}}$ *is of the form* $\sum_j M_{\varphi_j} T_{\lambda_j}$, *then* $\|\widetilde{\rho}(A)\| = \|A\|$.

Proof. — Note that $\ell^2(\mathbf{R}_d) \simeq L^2(\mathbf{R}^B)$ by a Fourier transform and that T_λ becomes multiplication by $e^{i\lambda(\cdot)}$ on $L^2(\mathbf{R}^B)$. Hence

$L \cong L^2(\mathbf{R}) \otimes L^2(\mathbf{R}^B) = \{L^2$ functions on $\mathbf{R}^B$ with values in $L^2(\mathbf{R})\}$.

So $$\|\widetilde{\rho}(\sum_j M_{\varphi_j} T_{\lambda_j})\| = \|\sum_j M_{\varphi_j} T_{\lambda_j} \otimes e^{i\lambda_j(\cdot)}\| = \sup_{x \in \mathbf{R}^B} \|\sum_j e^{i\lambda_j x} M_{\varphi_j} T_{\lambda_j}\|$$

(1) The isomorphism is induced by $W_\varphi + \mathfrak{J} \to t\text{-ind}(\varphi)$; thus if $\varphi = e^{isx} e^{\psi}$ with $\psi \in AP(\mathbf{R})$, then $t\text{-ind}(W_\varphi) = -s$. It was proved by Arens and Royden ([1], [14]) that if $\mathscr{B}$ is a commutative Banach algebra, then $\mathscr{B}^d = H^1(X, \mathbf{Z})$ where X is the maximal ideal space of $\mathscr{B}$ and H^1 is the first Čech cohomology. Hence $H^1(\mathbf{R}^B, \mathbf{Z}) = \mathbf{R}$.

where the norm on the right is the operator norm for $L^2(\mathbf{R})$. Since $\mathbf{R}$ is dense in $\mathbf{R}^B$, we may replace the supremum over $\mathbf{R}^B$ by one over $\mathbf{R}$ alone.

But for $x \in \mathbf{R}$, let N_x denote multiplication by $e^{ix(\cdot)}$ on $L^2(\mathbf{R})$ so that N_x is unitary and $N_x T_\lambda = e^{i\lambda x} T_\lambda N_x$. Hence,

$$||\tilde{\rho}(\sum_j M_{\varphi_j} T_{\lambda_j})|| = \sup_{x \in \mathbf{R}} ||\{\sum_j e^{i\lambda_j x} M_{\varphi_j} T_{\lambda_j}\} N_x|| = \sup_{x \in \mathbf{R}} ||N_x \{\sum_j M_{\varphi_j} T_{\lambda_j}\}|| = ||\sum_j M_{\varphi_j} T_{\lambda_j}||,$$

which was to be proved.

If we restrict $\tilde{\rho}$ to $\mathscr{A} \subset \tilde{\mathscr{A}}$, we get a C*-algebra imbedding of $\mathscr{A}$ into $\mathscr{N}$ the II_∞ factor $\tilde{\chi}_{\mathbf{R}^+} \tilde{\mathscr{N}} \tilde{\chi}_{\mathbf{R}^+}$ on $\tilde{\chi}_{\mathbf{R}^+} \mathrm{L}$. Let $\mathfrak{K}_{\mathscr{N}}$ denote the closed ideal generated by operators whose range has finite relative dimension. It is the unique nontrivial two-sided ideal in $\mathscr{N}$. Following Breuer [5],

$\mathrm{Fred}(\mathscr{N}, \mathfrak{K}_{\mathscr{N}}) = \{N \in \mathscr{N} \text{ invertible mod } \mathfrak{K}_{\mathscr{N}}\}$,

the set of Fredholm operators relative to $\mathscr{N}$, and if $N \in \mathrm{Fred}(\mathscr{N}, \mathfrak{K}_{\mathscr{N}})$, ind $N = \dim_{\mathscr{N}}(\ker N) - \dim_{\mathscr{N}}(\ker N^*)$. With the definitions of a-ind and t-ind in § 1, we can now prove

Theorem (**2.2**). — $\rho(A) \in \mathrm{Fred}(\mathscr{N}, \mathfrak{K}_{\mathscr{N}})$ *if and only if* $A \in \mathrm{Fred}(\mathscr{A}, \mathfrak{J})$, i.e. $\sigma(A)$ *is invertible in* $\mathrm{AP}(\mathbf{R})$, *in which case*

$$a\text{-ind}(A) = t\text{-ind}(A).$$

Proof. — For $\lambda \in \mathbf{R}$, let $W_\lambda \in \mathscr{A}$ be the restriction of T_λ to $L^2(\mathbf{R}^+)$. A simple calculation shows

$$[W_\lambda, W_\mu] = \begin{cases} 0 & \text{if } \lambda \text{ and } \mu \text{ have the same sign} \\ W_{\lambda+\mu} \chi_{(0,-\lambda)} & \text{if } \lambda < 0,\ \mu > 0 \\ -W_{\lambda+\mu} \chi_{(0,-\mu)} & \text{if } \lambda > 0,\ \mu < 0. \end{cases}$$

So $\rho([W_\lambda, W_\mu])$ belongs to the trace class of $\mathscr{N}$ and hence lies in $\mathfrak{K}_{\mathscr{N}}$. Since $\mathfrak{J}$ is generated by commutators $[W_\lambda, W_\mu]$, it follows that $\rho(\mathfrak{J}) \subset \mathfrak{K}_{\mathscr{N}}$.

To prove the first part of the theorem we must show $\rho^{-1}(\mathfrak{K}_{\mathscr{N}}) = \mathfrak{J}$ (1). For this we need

Lemma (**2.3**). — *If* $\mathscr{A}$ *is a uniformly closed subalgebra of a factor* $\mathscr{N}$ *and* $\mathfrak{K}_1 \subset \mathfrak{K}_{\mathscr{N}}$ *is the trace class, then* $\mathrm{closure}(\mathscr{A} \cap \mathfrak{K}_1) = \mathscr{A} \cap \mathfrak{K}_{\mathscr{N}}$.

Proof. — Using linear combinations, we need only show that if $A \geq 0$ and $A \in \mathscr{A} \cap \mathfrak{K}_{\mathscr{N}}$, then $A \in \mathrm{closure}(\mathscr{A} \cap \mathfrak{K}_1)$. Let $\varphi_\varepsilon(t) = \max(0, t-\varepsilon)$. We claim $\varphi_\varepsilon(A) \in \mathfrak{K}_1$. For, writing $E_\varepsilon = \chi_{(\varepsilon,\infty)}(A)$, we have $E_\varepsilon \leq A/\varepsilon$, so that $E_\varepsilon \in \mathfrak{K}_{\mathscr{N}}$ and hence $E_\varepsilon \in \mathfrak{K}_1$. Since $\varphi_\varepsilon(A) \leq ||A|| E_\varepsilon$, we have $\varphi_\varepsilon(A) \in \mathfrak{K}_1$ and since φ_ε is continuous, $\varphi_\varepsilon(A) \in \mathscr{A} \cap \mathfrak{K}_1$. The proof of the lemma is completed by noting that as $\varepsilon \to 0$, $\varphi_\varepsilon(A) \to A$ uniformly.

To show that $\rho^{-1}(\mathfrak{K}_{\mathscr{N}}) = \mathfrak{J}$, it suffices by lemma (2.3) to prove that $\rho^{-1}(\mathfrak{K}_1) \subset \mathfrak{J}$. We remind the reader that for $S \in \mathscr{N}$, the trace norm of S is defined by

(1) See Remark (3.7).

$||S||_1 = \sup\{|\mathrm{tr}(SR)| : R \in \mathfrak{K}_1, ||R|| = 1\}$, and $\mathfrak{K}_1 = \{S \in \mathcal{N}; ||S||_1 < \infty\}$, $||SR||_1 \leq ||S||_1 ||R||$. Suppose now that $A = W_\varphi + J \in \rho^{-1}(\mathfrak{K}_1)$; we derive a contradiction from the assumption that φ is not zero. It follows, then, that $A \in \mathfrak{J}$ as was to be proved.

If φ is not zero, some Fourier coefficient of φ is not zero, so $\varphi = ae^{i\lambda(\cdot)} + \varphi_1$ with $a \neq 0$ and φ_1 does not contain the term $e^{i\lambda(\cdot)}$. Thus

$$M_{\chi(0,\alpha)} T_{-\lambda}(W_\varphi + J) = aM_{\chi(0,\alpha)} + B + J'$$

where $J' \in \mathfrak{J}$ and $B = M_{\chi(0,\alpha)} T_{-\lambda} W_{\varphi_1}$ satisfies $\mathrm{tr}\, \rho(B) = 0$. By lemma (2.3), we can write $J = J_1 + J_2$ with $J_1 \in \rho^{-1}(\mathfrak{K}_1)$ and $||J_2|| \leq \frac{|a|}{2}$. Hence

$$\begin{aligned} ||\rho(W_\varphi + J)||_1 &\geq |\mathrm{tr}(\rho(M_{\chi(0,\alpha)} T_{-\lambda}(W_\varphi + J)))| \\ &\geq |a|\alpha - ||J_1||_1 - \frac{|a|}{2} ||\rho(M_{\chi(0,\alpha)} T_{-\lambda})||_1 \geq \frac{|a|}{2}\alpha - ||J_1||. \end{aligned}$$

As $\alpha \to \infty$, we conclude that $||\rho(W_\varphi + J)||_1 = \infty$, contradicting the assumption that $W_\varphi + J \in \rho^{-1}(\mathfrak{K}_1)$.

To prove the equality of a-ind and t-ind, note that both satisfy (2.1) (i)-(iv). Since they both provide homomorphisms from $(\mathcal{A}/\mathfrak{J})^d \to \mathbf{R}$, it suffices to check equality on $\{W_\lambda; \lambda \geq 0\}$ which are representatives for generators of $(\mathcal{A}/\mathfrak{J})^d$. Since $\sigma(W_\lambda) = e^{i\lambda(\cdot)}$, we have $t\text{-ind}(W_\lambda) = -\lambda$. On the other hand, $\rho(W_\lambda)$ is an isometry so that $\ker \rho(W_\lambda) = 0$, while $\ker \rho(W_\lambda^*) = L^2(0, \lambda) \otimes \ell^2(\mathbf{R}_d)$. Thus the projection onto $\ker \rho_\lambda(W^*)$ is $\widetilde{M}_{\chi(0,\lambda)}$, so that $a\text{-ind}(W_\lambda) = -\mathrm{tr}(\widetilde{M}_{\chi(0,\lambda)}) = -\int_0^\lambda dt = -\lambda$. This completes the proof of the theorem.

§ 3. Other groups.

In this section we extend the result of § 2 to locally compact groups G of the form H×V where

(i) H is a locally compact abelian group with $\hat{\mathrm{H}}$ torsion free;

(ii) $\hat{\mathrm{V}} = \mathbf{R}$ or $\hat{\mathrm{V}}$ is a subgroup of $\mathbf{R}_d$.

Let $\mathrm{H}^2(\mathrm{G})$ be the subspace of $\mathrm{L}^2(\mathrm{G})$ consisting of functions whose Fourier transform is supported in $\hat{\mathrm{H}} \times \hat{\mathrm{V}}^+$ where $\mathrm{V}^+ = \hat{\mathrm{V}} \cap \mathbf{R}^+$, and let P be the orthogonal projection onto this subspace. As before, write $W_\varphi = PM_\varphi | \mathrm{H}^2(\mathrm{G})$; let $\mathcal{A}(\mathrm{G})$ denote the C*-algebra on $\mathrm{H}^2(\mathrm{G})$ generated by $\{W_\varphi, \varphi \in \mathrm{AP}(\mathrm{G})\}$, and $\mathfrak{J}(\mathrm{G})$ the closed ideal generated by commutators. Note that § 2 is the case $\mathcal{A} = \mathcal{A}(\mathbf{R})$ and note that AP(H) lies in the center of $\mathcal{A}(\mathrm{G})$, for translation by elements of $\hat{\mathrm{H}}$ commutes with P.

As in the real case, $\mathcal{A}(\mathrm{G})/\mathfrak{J}(\mathrm{G}) \simeq \mathrm{AP}(\mathrm{G}) \simeq \mathrm{C}(\mathrm{G}^\mathrm{B})$ (1) [10]. The symbol map $\sigma : \mathcal{A}(\mathrm{G}) \to \mathrm{AP}(\mathrm{G})$ is again $\sigma(W_\varphi + J) = \varphi$. According to Proposition (3.1) below, the set of components of Fred($\mathcal{A}$(G), $\mathfrak{J}$(G)), i.e., $(\mathrm{AP}(\mathrm{G}))^d$ is isomorphic *via* a generalized

(1) See Remark (3.7).

mean motion with $\hat{G}=\hat{H}\times\hat{V}$. One is therefore tempted to define a topological index on Fred($\mathscr{A}$(G), $\mathfrak{I}$(G)) with values in $\hat{G}$. However, we want the equality of the t-index with an a-index; and if $\eta\in\hat{H}$, the operator W_η has inverse $W_{-\eta}$ so that a-ind(W_η) will have to be zero. Therefore in defining the t-index we project onto $\hat{V}$, the second summand in $\hat{G}$: If A is invertible mod $\mathfrak{I}$(G), and the mean motion of $\sigma(A)=(\eta,\lambda)\in\hat{H}\times\hat{V}$, then t-ind$(A)=-\lambda$. Thus the t-ind is defined using the following proposition originally proved by Bohr [3] for AP(**R**) and Van Kampen [13] for G with $\hat{G}$ torsion free:

Proposition **(3.1)**. — $(\mathrm{AP}(G))^d\simeq(\hat{G})_d$.

To obtain the analytic index, we need to exhibit the representation $\rho_G:\mathscr{A}(G)\to\mathscr{N}$, which we obtain by reducing the general case to the case $G=\mathbf{R}$ of § 2. We have observed that $\mathrm{AP}(H)\simeq C(H^B)$ lies in the center of $\mathscr{A}$(G). We leave to the reader that $\mathscr{A}$(G) is naturally isomorphic to $C(H^B,\mathscr{A}(V))$, the continuous functions from H^B to $\mathscr{A}$(V). Let $e:\mathscr{A}(G)\simeq C(H^B,\mathscr{A}(V))\to\mathscr{A}(V)$ be the homomorphism which is evaluation at the identity of H^B. The map e is the extension of the map $W_{\varphi.\psi}\mapsto\varphi(e)W_\psi$ where $\varphi\in\mathrm{AP}(H)$ and $\psi\in\mathrm{AP}(V)$. When $V=\mathbf{R}$, then $\rho\circ e$ gives the desired representation $\rho_G:\mathscr{A}(\mathbf{R})\to\mathscr{N}$ as in section 2. When $\hat{V}=\mathbf{R}_d$, we use Proposition (3.3) below which establishes a natural isomorphism $\mu:\mathscr{A}(\mathbf{R})\overset{\rightarrow}{\simeq}\mathscr{A}(\hat{\mathbf{R}}_d)$. Now our desired representation is $\rho\circ\mu^{-1}\circ e$. When $\hat{V}\hookrightarrow\mathbf{R}_d$, it is not difficult to show that this injection induces the natural isometric injection $i:\mathscr{A}(V)\to\mathscr{A}(\hat{\mathbf{R}}_d)$ and our desired representation becomes $\rho\circ\mu^{-1}\circ i\circ e$.

Since the ideals generated by commutators are preserved by the above homomorphisms, the representation ρ_G of $\mathscr{A}(G)\to\mathscr{N}$ sends $\mathfrak{I}(G)\to\mathfrak{K}_{\mathscr{N}}$, and hence sends Fred($\mathscr{A}$(G), $\mathfrak{I}$(G)) into Fredholm operators in $\mathscr{N}$. The analytic index is then defined as in section 2 and the index theorem carries through as before: the formal properties hold for both the topological and analytic index and one checks both indices on the representatives $\{W_g, g\in\hat{G}\}$ of the group $(\mathscr{A}(G)/\mathfrak{I}(G))^d$. We have then

Theorem **(3.2)**. — *If* $A\in\mathscr{A}(G)$, $\rho(A)$ *is Fredholm in* $\mathscr{N}$ *if* σ_A *is invertible in* AP(G) *in which case* a-ind$(A)=t$-ind(A).

It remains to prove the

Proposition **(3.3)**. — *The map* $T_\lambda\mapsto T_\lambda$, $\lambda\in\mathbf{R}^+$, *extends to a* $*$*-isomorphism* $\mu:\mathscr{A}(\mathbf{R})\overset{\rightarrow}{\simeq}\mathscr{A}(\mathbf{R}_d)$.

Proof. — Since we will be representing operators on $L_2(\mathbf{R}_d)$ and $L_2(\mathbf{R})$, to prevent confusion we will let T_λ, T'_λ be translation by λ on $L_2(\mathbf{R}_d)$, $L_2(\mathbf{R})$ respectively. Similarly, if ψ is a step function on **R** with a finite number of discontinuities and continuous from the right, we let M_ψ, M'_ψ be multiplication by ψ on $L_2(\mathbf{R}_d)$, $L_2(\mathbf{R})$ respectively. Let $\mathscr{M}$, $\mathscr{M}'$ denote the C*-algebras generated by $\{M_\psi\}$, $\{M'_\psi\}$ and let $\mathscr{B}$, $\mathscr{B}'$ denote the C*-algebras generated by $\{\mathscr{M}, T_\lambda\}_{\lambda\in\mathbf{R}}$ and $\{\mathscr{M}', T'_\lambda\}_{\lambda\in\mathbf{R}}$ respectively. Since $||M_\psi||=\sup_{\lambda\in\mathbf{R}}|\psi(\lambda)|=||M'_\psi||$, the map $M'_\psi\mapsto M_\psi$ extends to a $*$-isomorphism $\tau_0:\mathscr{M}'\overset{\rightarrow}{\simeq}\mathscr{M}$.

Lemma **(3.4)**. — *The map* $\sum_j M'_{\psi_j} T'_{\lambda_j} \mapsto \sum_j M_{\psi_j} T_{\lambda_j}$ *extends to a continuous homomorphism* $\tau : \mathscr{B}' \to \mathscr{B}$.

Proof. — $\|\sum_j M_{\psi_j} T_{\lambda_j}\| \leq |\langle \sum_j M_{\psi_j} T_{\lambda_j} f, g\rangle| + \varepsilon$ for some $f, g \in L_2(\mathbf{R}_d)$ which are finite linear combinations of δ functions, with $\|f\| = \|g\| = 1$. If

$$f(t) = \begin{cases} \alpha_i, & t = t_i, \quad i = 1, \dots, N \\ 0 & t \neq t_i \end{cases},$$

let

$$f'_k(t) = \begin{cases} \sqrt{k}\,\alpha_i & t \in \left[t_i, t_i + \frac{1}{k}\right] \\ 0 & \text{otherwise} \end{cases}.$$

Note that $f'_k \in L_2(\mathbf{R})$, $\lim_{k\to\infty} \|f'_k\| = \|f\|$, and $\lim_{k\to\infty} \langle \sum_j M'_{\psi_j} T'_{\lambda_j} f'_k, g'_k\rangle = \langle \sum_j M_{\psi_j} T_{\lambda_j} f, g\rangle$. Hence $\|\sum_j M_{\psi_j} T_{\lambda_j}\| \leq \|\sum_j M'_{\psi_j} T'_{\lambda_j}\|$ and the lemma is proved.

To show that the kernel of the above map τ is zero, we introduce a one parameter group of automorphisms φ'_t, φ_t of $\mathscr{B}'$ and $\mathscr{B}$ respectively which is conjugation by $e^{it\lambda}$. Clearly $\varphi_t \circ \tau = \tau \circ \varphi'_t$.

Lemma **(3.5)**. — (i) *For each* $B \in \mathscr{B}$, *the function* $t \mapsto \varphi_t(B)$ *lies in* $AP(\mathbf{R}, \mathscr{B})$, i.e., *it is continuous almost periodic in norm.*

(ii) *The mean* $p(B) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \varphi_t(B) \in \mathscr{M}$. *If* $B \geq 0$, *then* $p(B) \geq 0$ *and* $B \geq 0$, $p(B) = 0$ *implies* $B = 0$.

(iii) *The corresponding primed statements hold and* $\tau \circ \rho' = \rho \circ \tau$.

Proof. — (i) If $B = \sum_j M_{\psi_j} T_{\lambda_j}$, then $\varphi_t(B) = \sum_j e^{-i\lambda_j t} M_{\psi_j} T_{\lambda_j}$ so that $\varphi_t(B) \in AP(\mathbf{R}, \mathscr{B})$; now take uniform limits.

(ii) If $B = M_\psi T_\lambda$, then $p(M_\psi T_\lambda) = \begin{cases} 0, & \lambda \neq 0 \\ M_\psi, & \lambda = 0 \end{cases}$, which lies in $\mathscr{M}$. Since ρ is continuous (it is norm decreasing), $p(\mathscr{B}) \subset \mathscr{M}$. The second sentence of (ii) is proved by looking at the AP functions $\langle \varphi_t(B) f, f\rangle$ for all $f \in L_2(\mathbf{R}_d)$.

(iii) The proofs of the primed statements are the same. The equation $\tau \circ \rho' = \rho \circ \tau$ holds for $B' = M'_\psi T'_\lambda$ and hence on all of $\mathscr{B}'$.

We apply this lemma to show that τ is a $*$-isomorphism of $\mathscr{B}'$ with $\mathscr{B}$. For if $\tau(B') = 0$, then $\tau(C') = 0$ with $C' = (B')^* B' \geq 0$. Hence $(\tau \circ \rho')(C') = 0$. Since τ is injective, $\rho'(C') = 0$ so that by (3.5) (ii) we have $C' = 0$ and $B' = 0$. Thus τ is injective and hence an isometry. Since $\tau(B')$ contains the dense set $\{\sum M_\psi T_\lambda\}$, τ is surjective.

To complete the proof of the proposition, note that, for $\psi = \chi_{\mathbf{R}^+}$, $M_\psi \in \mathscr{M}$ and $M'_\psi \in \mathscr{M}'$, and that $\mathscr{A}(\mathbf{R})$ is naturally isomorphic to $M'_\psi \mathscr{B}' M'_\psi$, $\mathscr{A}(\mathbf{R}_d)$ to $M_\psi \mathscr{B} M_\psi$. Hence the desired isomorphism is just τ restricted to $M'_\psi \mathscr{B}' M'_\psi$.

The following corollary to Theorem (3.2) is of some use.

Corollary **(3.6)**. — *If* $A = W_\varphi + J \in \mathscr{A}(G)$ *is invertible, then the mean motion of* φ *lies in* $\hat{H}$.

We end § 3 with two remarks related to the proof of proposition (3.3).

Remark **(3.7).** — In the proof of that proposition we had a continuous homomorphism $\tau : \mathscr{B}' \to \mathscr{B}$ and the main problem was to prove $\tau^{-1}(0) = (0)$. This was done by introducing one parameter groups of automorphisms φ'_t, φ_t commuting with τ. Since $\varphi_t(\mathscr{B})$ is almost periodic, averaging allows us to conclude that $\tau^{-1}(0) \neq (0) \Rightarrow \tau^{-1}(0) \cap (\{B' \in \mathscr{B}' ; \varphi'_t(B') = B'\} = \mathscr{M}')$ is not empty. Since $\tau|\mathscr{M}'$ is injective, the theorem follows. This idea works in many different situations. The first example is the known fact that $\mathscr{A}/\mathfrak{J} \simeq \mathrm{AP}(\mathbf{R})$. The map $\varphi \mapsto \{W_\varphi + \mathfrak{J}\}$ from $\mathrm{AP}(\mathbf{R}) \to \mathscr{A}/\mathfrak{J}$ is norm decreasing, so continuous. There is no kernel, for φ_t operates on both algebras and the invariant elements in $\mathrm{AP}(\mathbf{R})$ consists only of constants. Another example gives an alternate proof to the fact that $\rho^{-1}(\mathfrak{K}_{\mathscr{N}}) = \mathfrak{J}$ needed in Theorem (2.2). Rephrased, we want to show that the map $\mathscr{A}/\mathfrak{J} \to \mathscr{N}/\mathfrak{K}_{\mathscr{N}}$ induced by ρ has no kernel. Again φ operates on both sides and the invariant elements of $\mathscr{A}/\mathfrak{J} = \mathrm{AP}(\mathbf{R})$ are the constants.

Remark **(3.8).** — The use of the almost periodic mean in the proof of Proposition (3.3) gives a different method of constructing $\mathscr{N}$ and the representation ρ. Let $\mathscr{Q} = \{A \in \widetilde{\mathscr{A}} : p(A)$ has compact support$\}$. It is an ideal in $\widetilde{\mathscr{A}}$. Define a trace on $\mathscr{Q}$ by $\mathrm{tr}(A) = \int_{-\infty}^{\infty} p(A)(x)dx$ and an inner product by $\langle A_1, A_2 \rangle = \mathrm{tr}(A_1 A_2^*)$. The left regular representation of $\widetilde{\mathscr{A}}$ acts on the Hilbert space completion of $\mathscr{Q}$ to give the desired representation ρ.

§ 4. Open Problems.

First we observe that the ordinary integer valued analytic index is independent of the (irreducible) representation of the C*-algebra used to define the index. To be precise, suppose $\{\mathscr{A}, \mathfrak{J}\}$ is a pair consisting of a C*-algebra $\mathscr{A}$ with identity and closed two sided ideal $\mathfrak{J}$. Let $\{\mathscr{B}, \mathfrak{K}\}$ denote the pair where $\mathscr{B}$ is all bounded operators on a separable Hilbert space and $\mathfrak{K}$ is the ideal of compact operators. As usual, $\mathrm{Fred}(\mathscr{A}, \mathfrak{J})$ denotes the elements of $\mathscr{A}$ invertible mod $\mathfrak{J}$.

Theorem **(4.1).** — *Suppose* ρ_j, $j = 1, 2$ *are two faithful irreducible $*$-representations of* $\{\mathscr{A}, \mathfrak{J}\} \to \{\mathscr{B}, \mathfrak{K}\}$. *Then* $A \in \mathrm{Fred}(\mathscr{A}, \mathfrak{J})$ *implies* $\rho_j(A) \in \mathrm{Fred}(\mathscr{B}, \mathfrak{K})$ *and* $\mathrm{ind} \circ \rho_1 = \mathrm{ind} \circ \rho_2$ *on* $\mathrm{Fred}(\mathscr{A}, \mathfrak{J})$.

Proof. — If $\mathfrak{J} = 0$, $\mathrm{Fred}(\mathscr{A}, \mathfrak{J}) = \mathscr{A}^0$ and $\mathrm{ind} \circ \rho_j = 0$. If $\mathfrak{J} \neq 0$, then $0 \neq \rho_j(\mathfrak{J}) \subset \mathfrak{K}$ and the irreducibility of ρ_j implies that $\rho_j(\mathfrak{J}) = \mathfrak{K}$. Since ρ_j is faithful, $\rho_j : \mathfrak{J} \to \mathfrak{K}$ is an isometry. Since separable irreducible representations of $\mathfrak{K}$ are equivalent, there exists a representation ρ'_1 of $\mathscr{A}$ equivalent to ρ_1 and $\rho'_1 = \rho_2$ on $\mathfrak{J}$. But then $\rho'_1 = \rho_2$ on $\mathscr{A}$, for $A \in \mathscr{A}$ and $J \in \mathfrak{J}$ implies $\rho'_1(A)\rho'_1(J) = \rho'_1(AJ) = \rho_2(AJ) = \rho_2(A)\rho_2(J) = \rho_2(A)\rho'_1(J)$. Hence $\rho'_1(A) = \rho_2(A)$ for all $A \in \mathscr{A}$ and ρ_1 and ρ_2 are equivalent on $\mathscr{A}$ so that $\mathrm{ind} \circ \rho_1 = \mathrm{ind} \circ \rho_2$.

If we weaken the hypothesis of the above theorem and assume that ρ_j are not irreducible but only type I_∞ factor representations, then in order for $\rho_j(\mathfrak{J})$ to be compact operators (so that $\rho_j(\mathrm{Fred}(\mathscr{A}, \mathfrak{J}))$ are Fredholm), the commutant of the I_∞ factor must be a finite matrix algebra. We conclude the

Corollary. — *Suppose* ρ_j, $j=1, 2$ *are two type* I_∞ *factor isometric* $*$*-representations of* $\{\mathscr{A}, \mathfrak{J}\} \to \{\mathscr{M}, \mathfrak{K}_{\mathscr{M}})$ *where* $\mathscr{M}$ *is the type* I_∞ *factor with nontrivial ideal* $\mathfrak{K}_{\mathscr{M}}$, *and the commutant of* $\mathscr{M}$ *is a finite matrix algebra. Then* $\rho_j(\mathrm{Fred}(\mathscr{A}, \mathfrak{J}))$ *are Fredholm operators and* $\mathrm{ind}\circ\rho_j$ *are rationally related,* i.e., *there exists positive integers* m_1, m_2 *such that* $m_1 \,\mathrm{ind}\circ\rho_1 = m_2 \,\mathrm{ind}\circ\rho_2$ (1).

We have made the observations above for the discrete case in order to emphasize the following question in the II_∞ case for the real valued index.

Problem 1. — *Suppose* ρ_j, $j=1, 2$ *are two type* II_∞ *factor faithful* $*$*-representations of* $\{\mathscr{A}, \mathfrak{J}\} \to \{\mathscr{N}_j, \mathfrak{K}_{\mathscr{N}_j}\}$, *so that* $\rho_j(\mathrm{Fred}(\mathscr{A}, \mathfrak{J})) \subset \mathrm{Fred}(\mathscr{N}_j, \mathfrak{K}_{\mathscr{N}_j})$. *Is there a positive real number* λ *such that* $\lambda \,\mathrm{ind}\circ\rho_1 = \mathrm{ind}\circ\rho_2$? *Does it help to assume* $\mathscr{N}_1 \simeq \mathscr{N}_2$?

When $\mathscr{A} = \mathscr{A}(\mathbf{R})$, the main case of this paper, it is easy to check that this question has an affirmative answer. We have seen that $(\mathscr{A}/\mathfrak{J})^d \cong \mathbf{R}$ so that $\mathrm{ind}\circ\rho : \mathbf{R} \to \mathbf{R}$. But if one uses the translations T_λ to represent elements of $(\mathscr{A}/\mathfrak{J})^d$, it is easy to see that $\mathrm{ind}\circ\rho$ must be order preserving since the projections on the kernels of T_λ are increasing with λ. Hence $\mathrm{ind}\circ\rho$ is order preserving and must be multiplication by a positive scalar.

In this paper, we started with a topological index for $\{\mathscr{A}(\mathbf{R}), \mathfrak{J}(\mathbf{R})\}$ and constructed an appropriate representation in $\{\mathscr{N}, \mathfrak{K}_{\mathscr{N}}\}$ whose analytic index gave the topological one. The remarks in the previous paragraph show that any II_∞ representation would do. If, however, problem 1 has a negative solution in general, it is natural to ask which " topological indices " can occur as analytic indices of some II_∞ factor representation. To be specific, let $\{\mathscr{A}, \mathfrak{J}\}$ be a C*-algebra with identity and ideal $\mathfrak{J}$. By a topological index we mean an $h \in \mathrm{Hom}((\mathscr{A}/\mathfrak{J})^0, \mathbf{R})$ satisfying:

$A \in \mathrm{Fred}(\mathscr{A}, \mathfrak{J})$ with $BA = I \Rightarrow h(A+\mathfrak{J}) \leq 0$.

Note that if h is a topological index, then

$A \in \mathrm{Fred}(\mathscr{A}, \mathfrak{J})$ with $AC = I \Rightarrow h(A+\mathfrak{J}) \geq 0$.

In particular, if A is invertible, then $h(A+\mathfrak{J}) = 0$, so that h induces an element of $\mathrm{Hom}((\mathscr{A}/\mathfrak{J})^d, \mathbf{R})$. By an analytic index, we mean $\mathrm{ind}_{\mathscr{N}}\circ\rho$ for some II_∞ factor representation $\rho : \{\mathscr{A}, \mathfrak{J}) \to \{\mathscr{N}, \mathfrak{K}_{\mathscr{N}}\}$. Note that every analytic index is a topological index.

Problem 2 (2). — *Given* $\{\mathscr{A}, \mathfrak{J}\}$, *which topological indices occur as analytic indices? If a topological index occurs as an analytic one, find all the corresponding quasi-equivalence classes of representations* ρ.

Though problems 1 and 2 are very general, there are some concrete examples. Consider the matrix generalization of $\{\mathscr{A}(\mathbf{R}), \mathfrak{J}(\mathbf{R})\}$ in section 2. That is the C*-algebra generated by $N \times N$ matrix valued almost periodic functions acting on $H^2 \otimes \mathbf{C}^N$. Denote

(1) Another way of saying the uniqueness is that if $\{\mathscr{A}, \mathfrak{J}\}$ is represented in a factor of type I_∞, the index is unique up to normalization of the dimension function.

(2) For the set of analytical indices to form a subgroup, one should allow anti-representations. In fact, for certain C*-algebras (including all those discussed in this paper) Jordan representations ρ should be allowed in the sense that then $\mathrm{ind}\circ\rho$ is a topological index.

this C*-algebra by $\mathscr{A}_N$ and let $\mathfrak{J}_N$ be the ideal such that $\mathscr{A}_N/\mathfrak{J}_N \cong AP(\mathbf{R}) \otimes \mathscr{M}_N$ where $\mathscr{M}_N$ is the $N \times N$ matrix algebra. Then $(\mathscr{A}_N/\mathfrak{J}_N)^0$ is the set of all continuous AP functions with values in $\mathbf{GL}(N)$. A topological index will be a real valued homomorphism of the group of homotopy classes of such functions. The II_∞ representation given in section 2 extends and gives a representation of $\{\mathscr{A}_N, \mathfrak{J}_N\} \to \{\mathscr{N} \otimes \mathscr{M}_N, \mathfrak{K}_{\mathscr{N}} \otimes \mathscr{M}_N\}$ and hence an analytic index. However we have not been able to find the corresponding topological index. One might expect that the topological index is the determinant followed by the mean motion. However, there are continuous almost periodic maps of $\mathbf{R}$ into $\mathbf{SL}(N)$ which are not homotopically trivial. We have been unable to compute their analytic index. So

Problem 3. — Find an index theorem in the $N \times N$ *matrix case.*

The generalization of section 2 to the case $G = H \times V$ with semigroup $\Sigma = H \times V^+$ in section 3 is very special.

Problem 4. — Find an index theorem for $\{\mathscr{A}(G, \Sigma), \mathfrak{J}(G, \Sigma)\}$ *where* (G, Σ) *is a locally compact abelian group and* Σ *a semigroup* (1).

It is unlikely that either the discrete or continuous case alone, i.e. type I_∞ or II_∞ representation, would do for the analytic index in general, but one will need a representation of mixed type. For example, let $\mathscr{A}$ be the C*-algebra on H^2 generated by W_φ, $\varphi \in AP(\mathbf{R}) + C_0(\mathbf{R})$, where $C_0(\mathbf{R})$ is the algebra of continuous functions vanishing at ∞. Let $\mathfrak{J}$ be the ideal generated by commutators. In this case $\mathfrak{J}$ contains the compact operators and it can be shown that $\mathscr{A}/\mathfrak{J} \simeq AP + C_0 \simeq C(X)$ where X can be described as follows: $X = (\bigcup_{r \in \mathbf{R}} (r, r)) \cup (B^{\mathbf{R}} \times \infty)$ in $B^{\mathbf{R}} \times (\mathbf{R} \cup \infty)$. Then $C(X)^0 \simeq C(\mathbf{S}^1)^0 \times AP(\mathbf{R})^0$ and $C(X)^d \cong \mathbf{Z} \times \mathbf{R}$.

Problem 5. — Problem 2 for this special case.

The problems above involving abstract index theorems were formulated in terms of a pair $(\mathscr{A}, \mathfrak{J})$. They can also be formulated for a single C*-algebra $\mathscr{A}_1$ (which above would be $\mathscr{A}/\mathfrak{J}$) and lead to some interesting questions even in the discrete case. We now call a topological index an element in $\mathrm{Hom}(\mathscr{A}_1^d, \mathbf{Z})$. An analytic index is an element $\mathrm{ind} \circ \tau$ where τ is a representation of $\mathscr{A}_1 \to \mathscr{B}/\mathfrak{K}$.

Problem 6. — Which topological indices occur as analytic indices?

Following Busby [7], representations of $\mathscr{A}_1$ in $\mathscr{B}/\mathfrak{K}$ are in 1-1 correspondence with extensions of $\mathfrak{K}$ by $\mathscr{A}_1$. So problem 6 can be rephrased: Find the map from the set of extensions of $\mathfrak{K}$ by $\mathscr{A}_1$ into $\mathrm{Hom}(\mathscr{A}_1^d, \mathbf{Z})$. The cases $\mathscr{A}_1 = C(\mathbf{S}^1)$ or $\mathscr{A}_1 = \mathscr{B}/\mathfrak{K}$ are of some interest.

In the case of $\mathscr{A}_1 = C(\mathbf{S}^1)$, one can show (Douglas) the following:

If $\mathscr{A}$ is an extension of $\mathfrak{K}$ by $C(\mathbf{S}^1)$, then the integer

$$N = -\,\mathrm{index}\ T = \dim\ker T^* - \dim\ker T$$

(1) See [12] for the quarter plane discrete case.

can be attached to $\mathscr{A}$, where T is any element in the coset corresponding to z. For each $N \neq 0$ the extension is obtained from the Toeplitz operators on the circle. For $N > 1$ the exact sequence $0 \rightarrow \mathfrak{K} \otimes \mathscr{M}_N \rightarrow \mathscr{C}_N \rightarrow C(\mathbf{S}^1) \rightarrow 0$ gives the extensions, where $\mathscr{C}_N$ is the C^*-algebra generated by $\mathscr{C} \otimes I_N$ and $I \otimes \mathscr{M}_N$. For $N = -1$ we take the analogue of Toeplitz operators $T_\varphi = (I-P) M_\varphi | (H^2)^\perp$ defined for $\varphi \in C(\mathbf{S}^1)$ and produce the extensions for $N < -1$ as before. For $N = 0$ there is an extension corresponding to each closed subset K [1] of $\mathbf{S}^1$ where $\mathscr{A}$ is generated by multiplication by continuous functions on some $L^2(\mu)$, where μ has support $\mathbf{S}^1$ and $\mu | K$ has support K, and the compact operators on $L^2(\mu | K) \subset L^2(\mu)$.

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[1] If K is perfect; a slight variation is necessary otherwise.