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GRADED BRAUER GROUPS AND K-THEORY WITH LOCAL COEFFICIENTS

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The Bott periodicity theorem shows that the real K-theory, KO, and the complex K-theory, KU, are generalised cohomology theories graded by $\mathbf{Z}_8$ and $\mathbf{Z}_2$ respectively. Our aim is to define a " K-theory with local coefficients " $K^\alpha(X)$ (K denotes either KO or KU) which shall generalize the usual groups $K^n(X)$, $n\in\mathbf{Z}_8$ or $n\in\mathbf{Z}_2$.

The ordinary cohomology with local coefficients $H^n(X, \alpha)$ is defined for $(n, \alpha)\in\mathbf{Z}\times H^1(X, \mathbf{Z}_2)$. At least when X is a connected finite CW-complex, $KO^\alpha(X)$ is defined for $\alpha\in\mathbf{Z}_8\times H^1(X, \mathbf{Z}_2)\times H^2(X, \mathbf{Z}_2)$ and $KU^\alpha(X)$ is defined for

$$\alpha\in\mathbf{Z}_2\times H^1(X, \mathbf{Z}_2)\times \mathrm{Tors}(H^3(X, \mathbf{Z})),$$

and these may be called K-theory with local coefficients. The sets which index K-theory appear here naturally as " graded Brauer groups " associated with the space X; these groups were essentially studied by Serre [7] and Wall [17]. These graded Brauer groups, together with another less important set (§ 2), seem to index all reasonable " K-theories with local coefficients ".

One motivation for this work is that it gives a complete satisfactory " Thom isomorphism " theorem in K-theory. More precisely, if V is a real vector bundle on X, the KO-theory of its Thom space is isomorphic to $KO^\alpha(X)$, $\alpha^{-1}=(d(V), w_1(V), w_2(V))$, where $d(V)\equiv\dim(V) \bmod 8$ and where the $w_i(V)$ are the first two Stiefel-Whitney classes of V. However not all α are of this form.

GBrO(X), the real graded Brauer group of X, is in fact the direct sum of $\mathbf{Z}_8$ with an extension of $H^1(X, \mathbf{Z}_2)$ by $H^2(X, \mathbf{Z}_2)$ if X is a connected finite CW-complex (2). (The extension splits set-theoretically only.) If $\alpha, \alpha'\in$ GBrO(X), there is a product $KO^\alpha(X)\otimes KO^{\alpha'}(X)\to KO^{\alpha\alpha'}(X)$; this is constructed by means of Fredholm operators in Hilbert space. So $\bigoplus_\alpha KO^\alpha(X)$ is a graded ring; GBrU(X), the complex graded Brauer group of X, has analogous properties.

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More generally, if GBr(X) denotes either GBrO(X) or GBrU(X) (as appropriate), the tensor product of bundles of algebras induces a composition:

$$\hat{\otimes}: \quad \mathrm{GBr}(X)\times\mathrm{GBr}(X') \to \mathrm{GBr}(X\times X').$$

Then one can define a " cup product ":

$$K^{\alpha}(X)\times K^{\alpha'}(X') \to K^{\alpha\hat{\otimes}\alpha'}(X\times X').$$

The same techniques enable us to define " Adams operations ":

$$\psi^n: \quad KU^{\alpha}(X) \to KU^{\alpha^n}(X)\otimes\Omega_n$$

for odd n, with $\Omega_n=\mathbf{Z}[\omega]/(\Phi_n(\omega))$, where Φ_n is the n-th cyclotomic polynomial. For $n\equiv 1 \pmod 4$, there are also operations:

$$\psi^n: \quad KO^{\alpha}(X) \to KO^{\alpha^n}(X)\otimes\Omega_n.$$

If $\alpha=0$, these operations are essentially the ordinary ψ^n operations.

1. Graded algebras

k will denote either of the fields $\mathbf{R}$ or $\mathbf{C}$. By a *graded k-algebra* we shall mean a $\mathbf{Z}_2$-graded associative k-algebra $A=A_0\oplus A_1$ with unit and of finite dimension as a vector space.

The *graded radical* $\mathfrak{r}(A)$ of A is defined to be the intersection of its maximal left graded ideals. It is a graded ideal. The graded radical of $A/\mathfrak{r}(A)$ is 0 and so $A/\mathfrak{r}(A)$ is said to be *semisimple*. The same argument as that used for ungraded algebras in Chapter 4 of [2] shows that:

Lemma 1. — *A semisimple graded k-algebra is a product of simple graded k-algebras.*

The simple graded k-algebras A may be classified as follows. If $A_1=0$ and $k=\mathbf{R}$, A must be isomorphic to $\mathbf{M}_n(\mathbf{R})$, $\mathbf{M}_n(\mathbf{H})$ or $\mathbf{M}_n(\mathbf{C})$ for some integer n. If $A_1=0$ and $k=\mathbf{C}$, A must be isomorphic to $\mathbf{M}_n(\mathbf{C})$ for some integer n.

Otherwise, according to Lemma 3 of [17], either A is simple (as an ungraded algebra) or A_0 is simple and there exists an element $u\in Z(A)\cap A_1$ such that $A_1=A_0.u$ and $u^2=1$ (Z(A) denotes the centre of A). In either event, if $k=\mathbf{R}$, either $Z(A)\cap A_0=\mathbf{R}$ and A is *central* in the sense of [17], or $Z(A)\cap A_0=\mathbf{C}$ and A is a simple central graded $\mathbf{C}$-algebra. If $k=\mathbf{C}$, A is necessarily central.

In [17] Wall has classified the central simple k-algebras; a list of their isomorphism classes is given below. Note that if u is an element of an algebra A such that $u^2=\pm 1$, we write $Z(u)=\{a\in A\,|\,a.u=u.a\}$ and $Z^*(u)=\{a\in A\,|\,a.u=-u.a\}$; $\mathbf{H}$ denotes the quaternion division $\mathbf{R}$-algebra and $1, i, j, k$ is its usual basis; $1\mapsto 1$ and $i\mapsto i$ will specify an embedding $\mathbf{C}\to\mathbf{H}$; I_n will denote the $n\times n$ unit matrix.

Simple central graded $\mathbf{R}$-algebras are classified by their *type* (an element of $\mathbf{Z}_8$)

and their *size* (either a positive integer n, or an unordered pair of positive integers (p, q)). The eight types are as follows:

[1; n] $A=\mathbf{M}_n(\mathbf{C})$; $A_0=\mathbf{M}_n(\mathbf{R})$; $A_1=i.\mathbf{M}_n(\mathbf{R})$.

[2; n] $A=\mathbf{M}_n(\mathbf{H})$; $A_0=\mathbf{M}_n(\mathbf{C})=Z(u)$; $A_1=Z^*(u)$; $u=i.I_n$.

[3; n] $A=\mathbf{M}_n(\mathbf{H})\oplus\mathbf{M}_n(\mathbf{H})$; $A_0=\mathbf{M}_n(\mathbf{H})$; $A_1=u.\mathbf{M}_n(\mathbf{H})$.
(Here $u\in Z(A)\cap A_1$ is such that $u^2=1$.)

[4; p, q] $A=\mathbf{M}_{p+q}(\mathbf{H})$; $A_0=Z(u)$; $A_1=Z^*(u)$.
(Here u is the diagonal matrix whose first p diagonal entries are 1 and whose last q diagonal entries are -1). Let $[4; n]=[4; n, n]$. Let $[4; n, 0]=\mathbf{M}_n(\mathbf{H})$.

[5; n] $A=\mathbf{M}_{2n}(\mathbf{C})$; $A_0=\mathbf{M}_n(\mathbf{H})$; $A_1=u.\mathbf{M}_n(\mathbf{H})$.
(The embedding $A_0\to A$ is specified thus: for each $M\in\mathbf{M}_n(\mathbf{R})\subset\mathbf{M}_n(\mathbf{H})$:

$$M\mapsto\begin{pmatrix}M & 0\\ 0 & M\end{pmatrix};\quad i.M\mapsto\begin{pmatrix}iM & 0\\ 0 & -iM\end{pmatrix};$$

$$j.M\mapsto\begin{pmatrix}0 & M\\ -M & 0\end{pmatrix};\quad k.M\mapsto\begin{pmatrix}0 & iM\\ iM & 0\end{pmatrix}$$

$u=i.I_{2n}$.)

[6; n] $A=\mathbf{M}_{2n}(\mathbf{R})$; $A_0=Z(u)$; $A_1=Z^*(u)$.
(Here u is the matrix $\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}$ where $I=I_n$.)

[7; n] $A=\mathbf{M}_n(\mathbf{R})\oplus\mathbf{M}_n(\mathbf{R})$; $A_0=\mathbf{M}_n(\mathbf{R})$; $A_1=u.\mathbf{M}_n(\mathbf{R})$.
(Here $u\in Z(A)\cap A_1$ is such that $u^2=1$.)

[8; p, q] $A=\mathbf{M}_{p+q}(\mathbf{R})$; $A_0=Z(u)$; $A_1=Z^*(u)$.
(Here u is the diagonal matrix whose first p diagonal entries are 1 and whose last q diagonal entries are -1). Let $[8; n]=[8; n, n]$. Let $[8; n, 0]=\mathbf{M}_n(\mathbf{R})$.

Simple central graded $\mathbf{C}$-algebras are classified by their *type* (an element of $\mathbf{Z}_2$) and their *size* (either a positive integer n, or an unordered pair of positive integers (p, q)). The two types are as follows:

(1; n) $A=\mathbf{M}_n(\mathbf{C})\oplus\mathbf{M}_n(\mathbf{C})$; $A_0=\mathbf{M}_n(\mathbf{C})$; $A_1=u.\mathbf{M}_n(\mathbf{C})$.
(Here $u\in Z(A)\cap A_1$ is such that $u^2=1$.)

(2; p, q) $A=\mathbf{M}_{p+q}(\mathbf{C})$; $A_0=Z(u)$; $A_1=Z^*(u)$.
(Here u is the diagonal matrix whose first p diagonal entries are 1 and whose last q diagonal entries are -1). Let $(2; n)=(2; n, n)$. Let $(2; n, 0)=\mathbf{M}_n(\mathbf{C})$.

If A and B are two graded k-algebras, their *graded tensor product* $C = A\hat{\otimes}B$ has underlying graded vector space $A\otimes_k B$ and has its product subject to the rule:

$$(a\hat{\otimes}b_j).(a_i\hat{\otimes}b) = (-1)^{ij}(a.a_i)\hat{\otimes}(b_j.b) \quad \text{for } a_i \in A_i \text{ and } b_j \in B_j.$$

For example, $[t; n]\hat{\otimes}[t'; n'] = [t+t'; a(t,t').n.n']$, where $a(t,t')$ is 1, 2, 4 or 8. The following lemma is Theorem 2 of [17]:

Lemma **2.** — *The graded tensor product of two simple central graded k-algebras is simple and central.*

It may be verified that the type of such a tensor product is the sum of the types of the factors. Both the lemma and this additivity property are valid also for simple central algebras A with $A_1 = 0$ provided that $\mathbf{M}_n(\mathbf{H})$ is assigned type $4 \in \mathbf{Z}_8$, $\mathbf{M}_n(\mathbf{R})$ is assigned type $0 \in \mathbf{Z}_8$ and $\mathbf{M}_n(\mathbf{C})$ is assigned type $0 \in \mathbf{Z}_2$.

Let $V = \mathbf{R}^{p+q}$ equipped with the quadratic form:

$$Q(x_1, \ldots, x_{p+q}) = -x_1^2 - \ldots - x_p^2 + x_{p+1}^2 + \ldots + x_{p+q}^2.$$

The *Clifford algebra* $C(Q) = C^{p,q}$ is defined to be the quotient of the tensor algebra $T(V)$ by the ideal generated by all elements of the form $x\otimes x - Q(x)$. It is naturally $\mathbf{Z}_2$-graded. It has dimension 2^{p+q}; $C^{p,q}\hat{\otimes}C^{r,s} = C^{p+r,q+s}$. See [9] for more details. As $C^{1,0} = \mathbf{C}$ and $C^{0,1} = \mathbf{R}\oplus\mathbf{R}$, $C^{p,q}$ is a simple central graded $\mathbf{R}$-algebra of type $p-q$. Complex Clifford algebras can also be constructed.

2. Bundles of simple central R-algebras

Let X be a paracompact connected space. Let A be a graded $\mathbf{R}$-algebra. Let Aut(A) be the Lie group of $\mathbf{R}$-automorphisms of A. A *bundle of* A's on X is a fibre bundle with base X, fibre A and group Aut(A). The isomorphism classes of these bundles form the set $H^1(X, \underline{\mathrm{Aut}}(A))$, where the underline denotes " sheaf of continuous functions ". If A is one of the algebras mentioned explicitly in the last section, it is possible to find a compact closed subgroup $\mathrm{Aut}_0(A)$ of Aut(A) such that the induced map $H^1(X, \underline{\mathrm{Aut}_0}(A)) \to H^1(X, \underline{\mathrm{Aut}}(A))$ is bijective for all X. (This follows from the theory of p. 51 of [8] and some explicit checking; the details will not be needed.)

Consider first bundles of $\mathbf{M}_n(\mathbf{R})$'s and $\mathbf{M}_n(\mathbf{H})$'s over X (for all n). The automorphism groups are $\mathbf{PGL}_n(\mathbf{R})$ and $\mathbf{GL}_n(\mathbf{H})/\mathbf{R}^*$ respectively (by the Skolem-Noether theorem; see p. 66 of [2]). The exact sequence:

$$1 \to \mathbf{R}^* \to \mathbf{GL}_n(\mathbf{R}) \to \mathbf{PGL}_n(\mathbf{R}) \to 1$$

defines a coboundary map $w_2 : H^1(X, \underline{\mathbf{PGL}_n}(\mathbf{R})) \to H^2(X, \underline{\mathbf{R}}^*) = H^2(X, \mathbf{Z}_2)$; w_2 is similarly defined in the quaternionic case. If $\mathscr{A}$ and $\mathscr{B}$ are two such bundles, so is $\mathscr{A}\otimes\mathscr{B}$, and an argument similar to the proof of Lemma 4 below shows that

$w_2(\mathscr{A}\otimes\mathscr{B})=w_2(\mathscr{A})+w_2(\mathscr{B})$. A bundle $\mathscr{A}$ of $\mathbf{M}_n(\mathbf{R})$'s is said to be *negligible* if it is of the form END(E) for some real vector bundle E or equivalently if $w_2(\mathscr{A})=0$.

The isomorphism classes of such bundles form a commutative monoid under $\otimes$; the classes of negligible bundles form a submonoid. The *orthogonal Brauer group* of X, BrO(X), is defined to be the quotient. There is an injective homomorphism, natural in the obvious sense, $\Phi:\mathrm{BrO}(\mathrm{X})\to\mathbf{Z}_2\oplus\mathrm{H}^2(\mathrm{X},\mathbf{Z}_2)$. (This shows that BrO(X) is in fact a group. Φ is defined as the product of w_2 with the map which assigns $0\in\mathbf{Z}_2$ to bundles of $\mathbf{M}_n(\mathbf{R})$'s and $1\in\mathbf{Z}_2$ to bundles of $\mathbf{M}_n(\mathbf{H})$'s.)

This remains valid if X is paracompact and has finitely many components provided that $\mathbf{Z}_2$ is replaced by $\mathrm{H}^0(\mathrm{X},\mathbf{Z}_2)$ where appropriate; analogous situations later will not be commented on. The following theorem will be proved in § 4:

Theorem **3.** — *Let* X *be a finite* CW-*complex. Then*:

$$\Phi:\mathrm{BrO}(\mathrm{X})\to\mathrm{H}^0(\mathrm{X},\mathbf{Z}_2)\oplus\mathrm{H}^2(\mathrm{X},\mathbf{Z}_2)$$

is an isomorphism.

The automorphism group of $\mathrm{A}=[8;n]=\mathbf{M}_{2n}(\mathbf{R})$ is that subgroup of

$$\mathrm{Aut}(\mathbf{M}_{2n}(\mathbf{R}))=\mathbf{GL}_{2n}(\mathbf{R})/\mathbf{R}^*$$

which leaves invariant the subspaces A_0 and A_1. Call it E_n. It may be verified that $\mathrm{E}_n=\mathrm{E}'_n/\mathbf{R}^*$, where $\mathrm{E}'_n=\mathrm{F}_n\cup\mathrm{F}'_n\subset\mathbf{GL}_{2n}(\mathbf{R})$, and where $\mathrm{F}_n=\left\{\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}\middle|\, a,b\in\mathbf{GL}_n(\mathbf{R})\right\}$, and $\mathrm{F}'_n=\left\{\begin{pmatrix}0 & a\\ b & 0\end{pmatrix}\middle|\, a,b\in\mathbf{GL}_n(\mathbf{R})\right\}$. Finally, G_n is defined by the exactness and commutativity of the diagram:

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbf{R}^* & \longrightarrow & \mathrm{F}_n & \longrightarrow & \mathrm{G}_n & \longrightarrow & 1\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbf{R}^* & \longrightarrow & \mathrm{E}'_n & \longrightarrow & \mathrm{E}_n & \longrightarrow & 1\\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathbf{Z}_2 & = & \mathbf{Z}_2 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}$$

Hence, if $\mathscr{A}$ is a bundle of $[8;n]$'s on X, this diagram defines its *characteristic classes* $w_1(\mathscr{A})\in\mathrm{H}^1(\mathrm{X},\mathbf{Z}_2)$ and $w_2(\mathscr{A})\in\mathrm{H}^2(\mathrm{X},\mathbf{R}^*)=\mathrm{H}^2(\mathrm{X},\mathbf{Z}_2)$.

A bundle $\mathscr{A}$ of [8; n]'s on X is said to be *negligible* if

$$\mathscr{A}_0 = \mathrm{HOM}(\mathrm{V}, \mathrm{V}) \oplus \mathrm{HOM}(\mathrm{W}, \mathrm{W}) \quad \text{and} \quad \mathscr{A}_1 = \mathrm{HOM}(\mathrm{V}, \mathrm{W}) \oplus \mathrm{HOM}(\mathrm{W}, \mathrm{V})$$

for some real vector bundles V and W. The exact cohomology sequences obtained from the diagram show that $\mathscr{A}$ is negligible if and only if both $w_1(\mathscr{A}) = 0$ and $w_2(\mathscr{A}) = 0$.

Let HO(X) be the set $\mathrm{H}^1(\mathrm{X}, \mathbf{Z}_2) \times \mathrm{H}^2(\mathrm{X}, \mathbf{Z}_2)$. The rule:

$$(a, b) . (a', b') = (a + a', b + b' + a . a')$$

gives it the structure of a 4-torsion abelian group; it is an extension of H^1 by H^2. Then, for $\mathscr{A}$ as above, set $w(\mathscr{A}) = (w_1(\mathscr{A}), w_2(\mathscr{A}))$.

Lemma **4.** — *If* $\mathscr{A}$ *and* $\mathscr{B}$ *are bundles of* [8; n]'s *and* [8; n']'s *respectively on* X, *then* $w(\mathscr{A} \hat{\otimes} \mathscr{B}) = w(\mathscr{A}) . w(\mathscr{B})$.

Proof. — The maps $\mathrm{E}'_n \to \mathbf{Z}_2$ and $\mathrm{E}'_{n'} \to \mathbf{Z}_2$ will both be denoted by $a \mapsto \bar{a}$. Choose an open cover $\mathfrak{U} = \{\mathrm{U}_i\}$ of X such that the restrictions of $\mathscr{A}$ and $\mathscr{B}$ to each U_i are product bundles. Set $\mathrm{U}_{ij} = \mathrm{U}_i \cap \mathrm{U}_j$ and $\mathrm{U}_{ijk} = \mathrm{U}_i \cap \mathrm{U}_j \cap \mathrm{U}_k$. It is convenient to use the same symbol for a function as for its restriction to a smaller domain of definition. Hence there exist functions $\alpha_{ij} : \mathrm{U}_{ij} \to \mathrm{E}_n$ such that: 1) $\alpha_{ii} = 1$; 2) $\alpha_{ij} . \alpha_{ji} = 1$; and 3) $\alpha_{ij} . \alpha_{jk} . \alpha_{ki} = 1$ on U_{ijk}. These functions determine $\mathscr{A}$ up to isomorphism in the usual way. Now choose functions $a_{ij} : \mathrm{U}_{ij} \to \mathrm{E}'_n$ such that: 4) $a_{ii} = 1$; 5) $a_{ij} . a_{ji} = 1$; and 6) $a_{ij} \mapsto \alpha_{ij}$ under the morphism $\mathrm{E}'_n \to \mathrm{E}_n$. Then $a_{ij} . a_{jk} . a_{ki} = \mathrm{A}_{ijk} : \mathrm{U}_{ijk} \to \mathbf{R}^*$ and the set $\{\mathrm{A}_{ijk}\}$ forms a Čech 2-cocycle. Note that $w_1(\mathscr{A})$ is specified by the Čech 1-cocycle $\{\bar{a}_{ij}\}$ and that $w_2(\mathscr{A})$ is specified by the Čech 2-cocycle $\{\mathrm{A}_{ijk}\}$. Let b_{ij} and B_{ijk} be the corresponding objects for $\mathscr{B}$. If $\mathscr{C} = \mathscr{A} \hat{\otimes} \mathscr{B}$ has corresponding objects c_{ij} and C_{ijk}, $c_{ij} = (a_{ij} \hat{\otimes} 1) . (1 \hat{\otimes} b_{ij})$. Hence $\bar{c}_{ij} = \bar{a}_{ij} + \bar{b}_{ij}$, which shows that $w_1(\mathscr{C}) = w_1(\mathscr{A}) + w_1(\mathscr{B})$. Further:

$$\begin{aligned}\mathrm{C}_{ijk} &= (a_{ij} \hat{\otimes} 1) . (1 \hat{\otimes} b_{ij}) . (a_{jk} \hat{\otimes} 1) . (1 \hat{\otimes} b_{jk}) . (a_{ki} \hat{\otimes} 1) . (1 \hat{\otimes} b_{ki}) \\ &= (\mathrm{A}_{ijk} \hat{\otimes} 1) . (1 \hat{\otimes} \mathrm{B}_{ijk}) . (\bar{b}_{ij} . \bar{a}_{jk} + \bar{b}_{jk} . \bar{a}_{ki} + \bar{b}_{ij} . \bar{a}_{ki}),\end{aligned}$$

where the field $\mathbf{Z}_2$ is considered as a subgroup of $\mathbf{R}^*$. The result that:

$$w_2(\mathscr{C}) = w_2(\mathscr{A}) + w_2(\mathscr{B}) + w_1(\mathscr{A}) . w_1(\mathscr{B})$$

follows from the fact that the third bracketed term represents the cup product.

If $\mathscr{A}$ is a bundle of [t; n]'s on X, and $\mathscr{B}$ is a product bundle of [$8-t$; n']'s on X, the above lemma shows that $w(\mathscr{A} \hat{\otimes} \mathscr{B})$ is independent of n'. This is defined to be $w(\mathscr{A}) = (w_1(\mathscr{A}), w_2(\mathscr{A}))$. Likewise, if $\mathscr{A}$ is a bundle of [4; p, q]'s or [8; p, q]'s on X, $w(\mathscr{A})$ is defined; in this case $w_1(\mathscr{A}) = 0$ if $p \neq q$. Likewise, if $\mathscr{A}$ is a bundle of $\mathbf{M}_n(\mathbf{H})$'s or $\mathbf{M}_n(\mathbf{R})$'s on X, $w(\mathscr{A})$ is defined to be $(0, w_2(\mathscr{A}))$. Now Lemma 4 implies:

Lemma **5.** — *If* $\mathscr{A}$ *and* $\mathscr{B}$ *are bundles of simple central graded* **R**-*algebras on the paracompact space* X, $w(\mathscr{A} \hat{\otimes} \mathscr{B}) = w(\mathscr{A}) . w(\mathscr{B})$.

The set of isomorphism classes of such graded bundles is a commutative monoid under $\hat{\otimes}$; the set of classes of negligible bundles is a submonoid; the *orthogonal graded Brauer group* of X, GBrO(X), is defined to be the quotient. (Note that the same group is obtained by considering only bundles of $[t; n]$'s for all t, n.) If X is connected, let $\tau : \mathrm{GBrO(X)} \to \mathbf{Z}_8$ be the homomorphism " type "; define Ψ to be $\tau \times w$. Note that Ψ is injective. Let $i : \mathrm{BrO(X)} \to \mathrm{GBrO(X)}$ be the obvious injection. Then the following diagram, in which the right hand vertical arrow is the product of the standard injections, commutes:

$$\begin{array}{ccc} \mathrm{BrO(X)} & \xrightarrow{\Phi} & \mathbf{Z}_2 \oplus \mathrm{H}^2(\mathrm{X}, \mathbf{Z}_2) \\ \downarrow{\scriptstyle i} & & \downarrow \\ \mathrm{GBrO(X)} & \xrightarrow{\Psi} & \mathbf{Z}_8 \oplus \mathrm{HO(X)} \end{array}$$

The definition of Ψ and the commutativity of the diagram immediately extend to the case when X is paracompact and has finitely many components.

Theorem **6.** — *Let* X *be a finite* CW-*complex. Then*:

$$\Psi : \mathrm{GBrO(X)} \to \mathrm{H}^0(\mathrm{X}, \mathbf{Z}_8) \oplus \mathrm{HO(X)}$$

is an isomorphism.

Proof. — It is already known that Ψ is injective, and it suffices to assume that X is connected. For $a \in \mathrm{H}^1(\mathrm{X}, \mathbf{Z}_2)$ and $b \in \mathrm{H}^2(\mathrm{X}, \mathbf{Z}_2)$ it is required to construct a bundle $\mathscr{A}$ of $[8; n]$'s on X such that $w(\mathscr{A}) = (a, b)$. By Theorem 3 there exists a bundle $\mathscr{B}$ of $\mathbf{M}_n(\mathbf{R})$'s such that $v(\mathscr{B}) = b$; the diagram shows that $w(\mathscr{B} \hat{\otimes} [8; 1]) = (0, b)$, where $[8; 1]$ is the product bundle of $[8; 1]$'s. Let V be a real line bundle on X such that $w_1(\mathrm{V}) = a$ and let W be the Whitney sum of V with $\mathrm{X} \times \mathbf{R}^7$. Then Lemma 7 below shows that $w(\mathrm{C(W)}) = (a, 0)$. Now Lemma 4 shows that $w(\mathrm{C(W)} \otimes \mathscr{B}) = (a, b)$, proving the theorem.

If V is a real vector bundle on the compact space X, its *Clifford bundle* C(V) ([9], § 1.1) is a bundle of central simple graded **R**-algebras. This construction defines a homomorphism $c : \mathrm{KO(X)} \to \mathrm{GBrO(X)}$; c need not be surjective. In fact, if c is surjective, the classical fibration

$$\mathrm{B\,Spin}(n) \to \mathrm{BSO}(n) \to \mathrm{K}(\mathbf{Z}_2, 2)$$

has a cross-section for every n. This is clearly impossible (use Steenrod squares for instance).

Lemma **7.** — *Let* V *be a real vector bundle on the paracompact space* X, *provided by a negative definite quadratic form. Then* $w_i(\mathrm{V}) = w_i(\mathrm{C(V)})$ *for* $i = 1, 2$.

Proof. — Clearly either $w_1(\mathrm{C(V)}) = 0$ for all X and V or $w_1(\mathrm{C(V)}) = w_1(\mathrm{V})$. To show that the first alternative is false, consider the case when $\mathrm{X} = \mathbf{P}_1(\mathbf{R})$ and V is the

Hopf line bundle. Then $C(V)=1\oplus V$ and $\mathscr{A}=C(V)\hat{\otimes}[7;1]$ is such that $\mathscr{A}_0=1\oplus V$ and $w_2(\mathscr{A})=0$. (An easy direct check shows that $w_2(C(L))=w_2(C(L)\hat{\otimes}[7;1])=0$ for any line bundle L on any X.) As V is not stably trivial, $\mathscr{A}_0$ cannot be isomorphic to HOM(W, W)⊕HOM(W′, W′) for any vector bundles W, W′. Hence $w_1(C(V))\neq 0$. Similarly, there exist universal constants $a, b \in \mathbf{Z}_2$ such that $w_2(C(V))=a.w_1(V)^2+b.w_2(V)$. Consider the case when $X=K(\mathbf{Z}_2\times\mathbf{Z}_2, 1)$ (or its 3-skeleton); let L_1 and L_2 be line bundles on X such that $\pi_1=w_1(L_1)$ and $\pi_2=w_1(L_2)$ generate $H^1(X,\mathbf{Z}_2)$. Then, if $V=L_1\oplus L_2$,

$$\begin{aligned} a.(\pi_1+\pi_2)^2+b.\pi_1.\pi_2 &= w_2(C(V)) = w_2(C(L_1)\otimes C(L_2)) \\ &= w_2(C(L_1))+w_2(C(L_2))+w_1(C(L_1)).w_1(C(L_2)) = \pi_1.\pi_2. \end{aligned}$$

Hence $a=0$ and $b=1$, as required.

Remark. — We will not deal with bundles of non-central simple graded **R**-algebras, i.e. with bundles of $(1;n)$'s or $(2;p,q)$'s *considered as* **R***-algebras.* It is possible to define equivalence classes of these and obtain a " graded Brauer set " on which the graded orthogonal Brauer group acts. This set has no group structure.

3. Bundles of simple central C-algebras

Again, X will denote a paracompact connected space. If A is a graded **C**-algebra, a bundle of A's on X is defined analogously to the real case.

Consider first bundles of $\mathbf{M}_n(\mathbf{C})$'s on X (for all n); $\mathbf{M}_n(\mathbf{C})$ has **C**-automorphism group $\mathbf{PGL}_n(\mathbf{C})$. The lower exact row of the commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbf{Z}_n & \longrightarrow & \mathbf{SL}_n(\mathbf{C}) & \longrightarrow & \mathbf{PGL}_n(\mathbf{C}) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \mathbf{C}^* & \longrightarrow & \mathbf{GL}_n(\mathbf{C}) & \longrightarrow & \mathbf{PGL}_n(\mathbf{C}) & \longrightarrow & 1 \end{array}$$

defines a coboundary map $v: H^1(X, \underline{\mathbf{PGL}_n(\mathbf{C})}) \to H^2(X, \underline{\mathbf{C}^*}) = H^3(X,\mathbf{Z})$. The upper row shows that the image of v is n-torsion. Once again it may be proved that $v(\mathscr{A}\otimes\mathscr{B})=v(\mathscr{A})+v(\mathscr{B})$. A bundle $\mathscr{A}$ of $\mathbf{M}_n(\mathbf{C})$'s is said to be *negligible* if it is isomorphic to END(E) for some complex vector bundle E or equivalently if $v(\mathscr{A})=0$.

The isomorphism classes of such bundles form a commutative monoid under ⊗; the classes of negligible bundles form a submonoid; the *unitary Brauer group* of X, BrU(X), is defined to be the quotient and is indeed a group. The following theorem will be proved in § 4:

Theorem **8** (Serre). — *Let* X *be a finite* CW*-complex. Then* $v: \mathrm{BrU}(X) \to H^3(X,\mathbf{Z})$ *is injective. Its image is the torsion subgroup.*

The automorphism group of $A=(2;n)$ is that subgroup of

$$\mathrm{Aut}(\mathbf{M}_{2n}(\mathbf{C}))=\mathbf{GL}_{2n}(\mathbf{C})/\mathbf{C}^*$$

which leaves invariant the subspaces A_0 and A_1. Call it E_n (no confusion will arise with E_n of § 2) and construct E'_n, F_n and G_n as before. Then a diagram like the first diagram of § 2 but with $\mathbf{C}^*$ replacing $\mathbf{R}^*$ may be used to define maps $u_1 : H^1(X, \underline{\mathbf{PGL}_n}(\mathbf{C})) \to H^1(X, \mathbf{Z}_2)$ and $u_2 : H^1(X, \underline{\mathbf{PGL}_n}(\mathbf{C})) \to H^2(X, \underline{\mathbf{C}}^*)=H^3(X, \mathbf{Z})$. Hence if $\mathscr{A}$ is a bundle of $(2;n)$'s on X, its *characteristic classes* $u_1(\mathscr{A})\in H^1(X, \mathbf{Z}_2)$ and $u_2(\mathscr{A})\in H^3(X, \mathbf{Z})$ are defined. As $u_2(\mathscr{A})=v(\mathscr{A}^*)$, where $\mathscr{A}^*$ is the bundle of $\mathbf{M}_{2n}(\mathbf{C})$'s underlying $\mathscr{A}$, $2n.u_2(\mathscr{A})=0$.

A bundle $\mathscr{A}$ of $(2;n)$'s on X is said to be *negligible* if

$$\mathscr{A}_0=\mathrm{HOM}(V, V)\oplus\mathrm{HOM}(W, W) \quad \text{and} \quad \mathscr{A}_1=\mathrm{HOM}(V, W)\oplus\mathrm{HOM}(W, V)$$

for some complex vector bundles V and W. Once again, $\mathscr{A}$ is negligible if and only if both $u_1(\mathscr{A})=0$ and $u_2(\mathscr{A})=0$. Let HU(X) be the set $H^1(X, \mathbf{Z}_2)\times \mathrm{Tors}(H^3(X, \mathbf{Z}))$. If $\beta : H^2(X, \mathbf{Z}_2) \to H^3(X, \mathbf{Z})$ is the Bockstein, the rule $(a, b).(a', b')=(a+a', b+b'+\beta(a.a'))$ gives it the structure of a torsion abelian group; it is an extension of H^1 by $\mathrm{Tors}(H^3)$. Then, for $\mathscr{A}$ as above, set $u(\mathscr{A})=(u_1(\mathscr{A}), u_2(\mathscr{A}))$. The following lemma is proved in the same way as Lemma 4:

Lemma **9.** — *If* $\mathscr{A}$ *and* $\mathscr{B}$ *are bundles of* $(2;n)$'s *and* $(2;n')$'s *respectively on* X, *then* $u(\mathscr{A}\otimes\mathscr{B})=u(\mathscr{A}).u(\mathscr{B})$.

As before, this lemma enables us to define $u(\mathscr{A})$ when $\mathscr{A}$ is a bundle of $(1;n)$'s, $(2;p,q)$'s or $\mathbf{M}_n(\mathbf{C})$'s. Now Lemma 9 implies:

Lemma **10.** — *If* $\mathscr{A}$ *and* $\mathscr{B}$ *are bundles of simple central graded* **C***-algebras on the compact space* X, $u(\mathscr{A}\hat{\otimes}\mathscr{B})=u(\mathscr{A}).u(\mathscr{B})$.

As before, the *unitary graded Brauer group* of X, GBrU(X) may be defined. If X is connected, a homomorphism $\Psi : \mathrm{GBrU}(X) \to \mathbf{Z}_2\oplus \mathrm{HU}(X)$ and an injection $i : \mathrm{BrU}(X) \to \mathrm{GBrU}(X)$ may be defined. Once again, the following diagram commutes:

$$\begin{array}{ccc} \mathrm{BrU}(X) & \xrightarrow{v} & \mathrm{Tors}(H^3(X, \mathbf{Z})) \\ \downarrow{\scriptstyle i} & & \downarrow \\ \mathrm{GBrU}(X) & \xrightarrow{\Psi} & \mathbf{Z}_2\oplus\mathrm{HU}(X) \end{array}$$

Theorem **11.** — *Let* X *be a finite* CW*-complex. Then:*

$$\Psi : \quad \mathrm{GBrU}(X) \to H^0(X, \mathbf{Z}_2)\oplus\mathrm{HU}(X)$$

is an isomorphism.

Proof. — The proof is analogous to that of Theorem 6 and so will be omitted.

4. The existence theorems

This section is devoted to the proofs of Theorems 3 and 8. Throughout it X will denote a finite connected CW-complex. Note that the injectivity of Φ and v has already been proved.

Lemma **12.** — *If* V *is a k-vector bundle on* X, *there exists a vector bundle* W *on* X *such that* $V\otimes W$ *is trivial.*

Proof. — Let X have dimension d and V have rank r. Then $([V]-r)\in K(X)$ is nilpotent (see p. 127 of [3]). Let N be an integer such that $([V]-r)^N=0$. If

$$x=r^{N-1}-r^{N-2}.([V]-r)+\ldots+(-1)^{N-1}.([V]-r)^{N-1},$$

$M.x.[V]=M.r^N\in K(X)$, where the integer M is chosen such that $M.x$ has rank $\rho=M.r^{N-1}>d$. So there exists a vector bundle W′ such that $[W']-n=M.x$ for some positive integer n. Now the injection $\mathbf{O}(\rho)\to\mathbf{O}(\rho+n)$ induces isomorphisms $\pi_i(\mathbf{O}(\rho))\to\pi_i(\mathbf{O}(\rho+n))$ for $i\leq d$, and hence induces a bijection $[X, B_{\mathbf{O}(\rho)}]\to[X, B_{\mathbf{O}(\rho+n)}]$. Hence there exists a vector bundle W such that $[W]=M.x$. As $[V\otimes W]=M.r^N>d$, $V\otimes W$ is trivial. (If $k=\mathbf{C}$, $\mathbf{O}$ must be replaced by $\mathbf{U}$ in this proof.)

Now, for definiteness, assume that $k=\mathbf{R}$. Let K′PO(X) be the quotient of the commutative monoid of isomorphism classes of $\mathbf{R}$-vector bundles on X with composition induced by the tensor product by the submonoid consisting of the classes of trivial bundles. By Lemma 12 it is a commutative group. Further, it is divisible. For if m is an integer and with the notation of the above proof, the formal binomial expansion of $r^{N-1}.m^{N-1}.(N-1)!.(1+([V]-r)/r)^{1/m}$ may be used as above to construct a vector bundle V_m such that $V_m^{\otimes m}$ is a multiple of V. Further, it is torsion-free, and hence is a $\mathbf{Q}$-vector space. For if V is a vector bundle such that $V^{\otimes m}$ is trivial of rank r^m, Lemma 12 shows that there is a vector bundle W such that $T=W\otimes(V^{\otimes(m-1)}\oplus r.V^{\otimes(m-2)}\oplus\ldots\oplus r^{m-1})$ is trivial and $[V\otimes T]=[r.T]\in KO(X)$. So again, for a suitable positive integer M, $M.V\otimes T\approx M.r.T$, and so V has image 0 in K′PO(X).

Let KPO(X) be the quotient of the commutative monoid of isomorphism classes of bundles of $\mathbf{M}_n(\mathbf{R})$'s on X (for all n) with composition induced by the tensor product by the submonoid consisting of the classes of trivial bundles. As the tensor product of such a bundle with its opposed bundle is negligible, Lemma 12 shows that KPO(X) is a group. The endomorphism bundle construction induces a homomorphism $i: K'PO(X)\to KPO(X)$. The characteristic class w_2 defines a homomorphism $w_2: KPO(X)\to H^2(X, \mathbf{Z}_2)$. The sequence:

$$0\to K'PO(X)\xrightarrow{i} KPO(X)\xrightarrow{w_2} H^2(X, \mathbf{Z}_2)$$

is clearly exact. Now the group $\mathbf{PO}_n=\mathbf{O}(n)/\mathbf{Z}_2$ acts on $\mathbf{R}^n\otimes\mathbf{R}^n$ in the obvious way; hence a bundle $\mathscr{A}$ of $\mathbf{M}_n(\mathbf{R})$'s on X induces a vector bundle of rank n^2; this construction induces a homomorphism $2j: KPO(X)\to K'PO(X)$. As $j.i=1$ the sequence splits.

If $X = \mathbf{S}^p$, the p-sphere, w_2 is surjective. This statement is trivial unless $p=2$. If $\mathscr{A} = C(V)\hat{\otimes}[6; 1]$ (considered as a bundle of $\mathbf{M}_4(\mathbf{R})$'s), where V is the Hopf complex line bundle (considered as a real 2-plane bundle) on $X = \mathbf{S}^2 = \mathbf{P}^1(\mathbf{C})$, Lemma 7 shows that $w_2(\mathscr{A})$ is the non-zero element of $H^2(X, \mathbf{Z}_2)$. Now $K'PO(\ )$, $KPO(\ )$ and $H^2(\ , \mathbf{Z}_2)$ are all half exact homotopy functors; a theorem of Brown (see p. 7.1 of [5]) shows that $j \oplus w_2 : KPO(X) \to K'PO(X) \oplus H^2(X, \mathbf{Z}_2)$ is an isomorphism for all X.

It suffices to prove Theorem 3 in the case when X is connected. The surjectivity of w_2 implies the surjectivity of Φ, which therefore is an isomorphism.

Remark. — Brown's theorem and the " Pontryagin character " may now be used to show that $K'PO(X) \approx \prod_{i>0} H^{4i}(X, \mathbf{Q})$. This yields the periodicity of the homotopy of the direct limit $\mathbf{PO}$ of the groups $\mathbf{PO}_n$.

As Theorem 8 is proved in [7] we will omit the modifications needed to make the above argument work in the case $k = \mathbf{C}$.

5. K-Theory with local coefficients

Let $\mathscr{A}$ be a bundle of A's on the compact space X, where A is a graded k-algebra (§ 1). Denote by $\mathscr{E}^{\mathscr{A}}(X)$ the category of graded k-vector bundles which are projective $\mathscr{A}$-modules in the obvious sense, with morphisms of degree 0. $\bar{\mathscr{E}}^{\mathscr{A}}(X)$ is the category whose objects are those of X but whose morphisms are not necessarily of degree 0. Both $\mathscr{E}^{\mathscr{A}}(X)$ and $\bar{\mathscr{E}}^{\mathscr{A}}(X)$ are "*prebanach categories*" (see [11]) and the forgetful functor $\varphi : \mathscr{E}^{\mathscr{A}}(X) \to \bar{\mathscr{E}}^{\mathscr{A}}(X)$ is a Banach functor. The *Grothendieck group* $K^{\mathscr{A}}(X)$ is the K-group $K(\varphi)$ of the Banach functor φ. For example, if $\mathscr{A} = k$ (the product bundle), $K^{\mathscr{A}}(X)$ is isomorphic to the well-known group $K(X)$. More generally, if $\mathscr{A} = C^{p,q}$, $K^{\mathscr{A}}(X)$ is the group $K^{p,q}(X) = K^{p-q}(X)$ introduced in [11].

If $\alpha \in GBr(X)$ (which means $GBrO(X)$ if $k=\mathbf{R}$, $GBrU(X)$ if $k = \mathbf{C}$), and if $\mathscr{A}$ is a bundle of central simple graded k-algebras of class α, it will be shown later that $K^{\mathscr{A}}(X)$ depends only on α. It is defined to be $K^{\alpha}(X)$.

If $\mathfrak{r}_x$ denotes the graded radical of $\mathscr{A}_x$ where $x \in X$, $\mathscr{R} = \bigcup \mathfrak{r}_x$ is a sub-bundle of graded ideals of $\mathscr{A}$ and $\mathscr{A}/\mathscr{R}$ is a bundle of $A/\mathfrak{r}(A)$'s, i.e. of semisimple graded k-algebras. Define a functor $\theta : \mathscr{E}^{\mathscr{A}}(X) \to \mathscr{E}^{\mathscr{A}/\mathscr{R}}(X)$ by the formula $\theta(E) = (\mathscr{A}/\mathscr{R}) \otimes_{\mathscr{A}} E$ for $E \in \mathscr{E}^{\mathscr{A}}(X)$.

Proposition **13**. — *The functor* θ *induces an isomorphism* $\theta_* : K^{\mathscr{A}}(X) \to K^{\mathscr{A}/\mathscr{R}}(X)$.

Proof. — Following § 2.1 of [9] we give another description of $K^{\mathscr{B}}(X)$ for every $\mathscr{B}$. Consider triples $(E, \varepsilon_1, \varepsilon_2)$ where E is a $\mathscr{B}$-module and where ε_1 and ε_2 are *gradings* of E. This means that ε_1 and ε_2 are two involutions of E (regarded as an ordinary bundle) such that $\varepsilon_i b = b\varepsilon_i$ for $b \in \mathscr{B}_0$ and $\varepsilon_i b = -b\varepsilon_i$ for $b \in \mathscr{B}_1$. Moreover (E, ε_i) is assumed to be a *graded* projective module over $\mathscr{B}$. A triple $(E, \varepsilon_1, \varepsilon_2)$ is called *elementary* if ε_1 is homotopic to ε_2 among the gradings of E. The group $K^{\mathscr{B}}(X)$ is then the quotient of the monoid constructed with such triples by the equivalence relation generated by the addition of elementary triples (see § 2.1 of [9] for a proof of an analogous statement). The *cosuspension* $\Sigma\mathscr{B}$ of $\mathscr{B}$ (not the suspension of [13]) is defined to be $C^{0,1}\hat{\otimes}\mathscr{B}$. A graded projective module over $\mathscr{B}$ may be thought of as an ordinary projective module over $\Sigma\mathscr{B}$. Hence every element of $K^{\mathscr{B}}(X)$ can be written $(E, \varepsilon_1, \varepsilon_2)$ where $E=(\Sigma\mathscr{B})^n$ for a certain n and where ε_1 is the grading induced by the canonical generator of $C^{0,1}$. In this context, θ_* is simply defined by the formula $\theta_*((\Sigma\mathscr{A})^n, \varepsilon_1, \varepsilon_2)=((\Sigma(\mathscr{A}/\mathscr{R}))^n, \bar{\varepsilon}_1, \bar{\varepsilon}_2)$, where $\bar{\varepsilon}_i$ is the image of ε. Let $\eta(t)$ be a homotopy between $\eta(0)=\bar{\varepsilon}_1$ and $\eta(1)=\bar{\varepsilon}_2$. The argument of Lemma 1.3 of [14] shows the existence of a continuous family $\zeta(t)$ of gradings of $(\Sigma\mathscr{A})^n$ such that $\overline{\zeta(t)}=\eta(t)$ and $\zeta(0)=\varepsilon_1$. If we put $\lambda(t)=1+t\zeta(1)\varepsilon_2$, $\lambda(t)$ is invertible by Nakayama's lemma. Then the homotopy $\xi(t)$ given by $\xi(t)=\zeta(2t)$ for $0\leq t\leq\frac{1}{2}$ and $\xi(t)=(\lambda(2t-1))^{-1}\zeta(1)\lambda(2t-1)$ for $\frac{1}{2}\leq t\leq 1$ connects ε_1 and ε_2 in the set of gradings of E. Hence θ_* is injective. Now, for every bundle $\mathscr{B}$ of graded algebras, let $F(n, \mathscr{B})$ be the Banach space of endomorphisms $\varepsilon: (\Sigma\mathscr{B})^n \to (\Sigma\mathscr{B})^n$ such that $\varepsilon b=(-1)^i b\varepsilon$ for every $b\in\mathscr{B}_i\subset\Sigma\mathscr{B}$. Let $\mathrm{Grad}(n, \mathscr{B})\subset F(n, \mathscr{B})$ be the subset of endomorphisms ε such that $\varepsilon^2=1$. There are obvious maps $F(n, \mathscr{A}) \to F(n, \mathscr{A}/\mathscr{R})$ and $\mathrm{Grad}(n, \mathscr{A}) \to \mathrm{Grad}(n, \mathscr{A}/\mathscr{R})$; ε_1 is a canonical base point for $\mathrm{Grad}(n, \mathscr{A})$ whilst $\bar{\varepsilon}_1$ is one for $\mathrm{Grad}(n, \mathscr{A}/\mathscr{R})$. If $\eta\in F(n, \mathscr{A}/\mathscr{R})$ and if $\gamma=\eta\bar{\varepsilon}_1$, $\gamma b=b\gamma$ for every $b\in\mathscr{A}/\mathscr{R}$. Hence $F(n, \mathscr{A}) \to F(n, \mathscr{A}/\mathscr{R})$ is surjective. Let $((\Sigma(\mathscr{A}/\mathscr{R}))^n, \bar{\varepsilon}_1, \eta)$ be a triple which specifies an element of $K^{\mathscr{A}/\mathscr{R}}(X)$. Now there exists an $\varepsilon\in F(n, \mathscr{A})$ such that $\bar{\varepsilon}=\eta$ and $\varepsilon^2=1+k$, where $k\in\mathbf{M}_n(\mathscr{R})$ is nilpotent. So the square root $\sqrt{(1+k)}=1+\frac{1}{2}k-\ldots$ exists. Put $\varepsilon_2=\varepsilon\sqrt{(1+k)}$; then $\theta_*((\Sigma\mathscr{A})^n, \varepsilon_1, \varepsilon_2)=((\Sigma(\mathscr{A}/\mathscr{R}))^n, \bar{\varepsilon}_1, \eta)$. Hence θ_* is surjective and the proof is complete.

Remark. — In order to calculate $K^{\mathscr{A}}(X)$ it is now sufficient to assume that $\mathscr{A}$ is semisimple. As $K^{\mathscr{A}\times\mathscr{B}}(X)$ is isomorphic to $K^{\mathscr{A}}(X)\oplus K^{\mathscr{B}}(X)$, it is further sufficient to assume that all the simple factors of the fibre of $\mathscr{A}$ are isomorphic. It is false, unfortunately, that all bundles of semisimple graded k-algebras are products of bundles of simple graded k-algebras.

The definition of $K^{\mathscr{A}}(X)$ may be generalized in many directions in the usual way. First of all, we introduced *relative groups* $K^{\mathscr{A}}(X, Y)$ when Y is a closed subspace of X (consider triples $(E, \varepsilon_1, \varepsilon_2)$ such that $\varepsilon_{1|Y}=\varepsilon_{2|Y}$ as in § 2.1 of [9]); for Y empty we recover the definition of $K^{\mathscr{A}}(X)$. If $\mathscr{C}$ is a Banach category, $\mathscr{C}$-vector bundles may be considered instead of ordinary vector bundles (cf. [9]). Denote by $K^{\mathscr{A}}(X; \mathscr{C})$ ($K^{\mathscr{A}}(X, Y; \mathscr{C})$ in the relative case) the group so obtained. An interesting example (see below) is the category $\mathscr{C}=\check{\mathscr{H}}$ of [10]. Finally, if G is a compact Lie group acting continuously on X and $\mathscr{A}$, we may consider " G-$\mathscr{A}$-vector bundles " (i.e. there is the relation $g.(a.e)=(g.a).(g.e)$ for $g\in G$, $a\in\mathscr{A}_x$, $e\in E_x$, $x\in X$). In this way a group $K_G^{\mathscr{A}}(X)$ is obtained. A slight generalisation may be obtained by considering augmented groups as in [15].

$K^{\mathscr{A}}(X)$ is a homotopy invariant in the following sense. Let $\mathscr{A}\times I$ be a bundle of algebras over $X\times I$, where I is the unit interval. The inclusions $i_0: X\to X\times\{0\}\subset X\times I$ and $i_1: X\to X\times\{1\}\subset X\times I$ are homotopic. If a is an element of $K^{\mathscr{A}\times I}(X\times I)$, $i_0^*(a)=i_1^*(a)$.

6. The Thom isomorphism

By " abus de langage " let us denote by $\mathscr{A}$, E, ... the inverse images $f^*\mathscr{A}, f^*E$, ... for any map $f: Y\to X$. As in [9], there is a homomorphism:

$$t:\quad K^{\Sigma\mathscr{A}}(X;\mathscr{C})\to K^{\mathscr{A}}(X\times\mathbf{D}^1, X\times\mathbf{S}^0;\mathscr{C})$$

defined by the formula $t(E,\varepsilon_1,\varepsilon_2)=(E',\varepsilon_1(\theta),\varepsilon_2(\theta))$. Here E′ is E regarded as an $\mathscr{A}$-module and $\varepsilon_i(\theta)$ is the grading of E′ defined over the point $\theta\in\mathbf{D}^1=[0,\pi]$ by $\varepsilon_i(\theta)=\varepsilon\cos\theta+\varepsilon_i\sin\theta$ where $\varepsilon\in C^{0,1}\subset C^{0,1}\hat{\otimes}\mathscr{A}$ is the canonical generator of $C^{0,1}$.

Theorem **14.** — *For every Banach category* $\mathscr{C}$, *t is an isomorphism.*

Proof. — The proof is analogous to that of Theorem (2.2.2) of [9].

Remark. — If ΣX denotes the pair $(X\times\mathbf{D}^1, X\times\mathbf{S}^0)$, the theorem takes the striking form $K^{\Sigma\mathscr{A}}(X)\approx K^{\mathscr{A}}(\Sigma X)$. No analogue of this theorem in algebraic K-theory is known. Theorem 14 is, of course, still true for all the generalizations mentioned in § 5. Also, if V is a vector bundle on X with a positive quadratic form Q, the methods of [9] define a homomorphism $t: K^{\mathscr{A}\hat{\otimes}C(V)}(X;\mathscr{C})\to K^{\mathscr{A}}(B(V), S(V);\mathscr{C})$.

Theorem **15.** — *The generalized homomorphism t is an isomorphism.*

Proof. — This may be proved as in [9] by using Mayer-Vietoris arguments (cf. [18]).

To show that $K^{\mathscr{A}}(X)$, where $\mathscr{A}$ is a bundle of simple central graded k-algebras on X, depends only on the class of $\mathscr{A}$ in GBr(X), it is convenient to interpret $K^{\mathscr{A}}(X)$ as the *graded Grothendieck group* of the category $\mathscr{E}^{\mathscr{A}}(X)$ (cf. § 2.1 of [9]).

Theorem **16.** — *Let* $E=E_0\oplus E_1$ *be a* $\mathbf{Z}_2$*-graded k-vector bundle on* X *with graded endomorphism bundle* END(E). *If* $\mathscr{A}$ *is a bundle of graded k-algebras on* X, *the additive functor* $\varphi:\mathscr{E}^{\mathscr{A}}(X)\to\mathscr{E}^{\mathscr{A}\hat{\otimes}END(E)}(X)$ *defined by* $\varphi(F)=F\hat{\otimes}E$ *is an equivalence of graded Banach categories. In particular,* φ *induces an isomorphism* $\varphi_*: K^{\mathscr{A}}(X)\to K^{\mathscr{A}\hat{\otimes}END(E)}(X)$.

Proof. — As the question is local there is no loss of generality in assuming that $E_0=X\times k^p$ and $E_1=X\times k^q$. So $\mathscr{A}\hat{\otimes}END(E)=\mathbf{M}_{p+q}(\mathscr{A})$ with a certain grading. If μ' is a homomorphism from $\varphi(F)$ to $\varphi(F')$, linear algebra shows that μ' is of the form $\varphi(\mu)$. Hence φ is fully faithful. It remains to prove that φ is essentially surjective; it suffices to prove that $\mathscr{A}\otimes END(E)$ is isomorphic to some $\varphi(F)$. For $F=\mathscr{A}\otimes E$, this is satisfied.

Definition **17.** — *For* $\alpha\in GBr(X)$, $K^{\alpha}(X)$ *is defined (up to canonical isomorphism) as the group* $K^{\mathscr{A}}(X)$ *for any* $\mathscr{A}$ *with class* α. (This is justified by Theorem 16.)

Let $\varphi : \mathscr{C} \to \mathscr{C}'$ be a quasi-surjective Serre functor (in the sense of [9]) between two Banach categories. As in [9] and [14] we can define a connecting homomorphism $\partial : K^{\mathscr{A}}(X\times\mathbf{D}^1, X\times\mathbf{S}^0; \mathscr{C}') \to K^{\mathscr{A}}(X; \varphi)$. This yields an exact sequence:

$$K^{\mathscr{A}}(X\times\mathbf{D}^1, X\times\mathbf{S}^0; \mathscr{C}) \to K^{\mathscr{A}}(X\times\mathbf{D}^1, X\times\mathbf{S}^0; \mathscr{C}') \to$$
$$\to K^{\mathscr{A}}(X; \varphi) \to K^{\mathscr{A}}(X; \mathscr{C}) \to K^{\mathscr{A}}(X; \mathscr{C}').$$

The homomorphism t of Theorem 14 may be used to obtain the theorem:

Theorem **18.** — *There exists an exact sequence*:

$$K^{\Sigma\mathscr{A}}(X; \mathscr{C}) \to K^{\Sigma\mathscr{A}}(X; \mathscr{C}') \to K^{\mathscr{A}}(X; \varphi) \to K^{\mathscr{A}}(X; \mathscr{C}) \to K^{\mathscr{A}}(X; \mathscr{C}').$$

The following theorem is proved in the same way:

Theorem **19.** — *If $\mathscr{A}$ is a bundle of graded k-algebras on the compact space* X *and* Y *is a closed subspace of* X, *there exists an exact sequence*:

$$K^{\Sigma\mathscr{A}}(X; \mathscr{C}) \to K^{\Sigma\mathscr{A}}(Y; \mathscr{C}) \to K^{\mathscr{A}}(X, Y; \mathscr{C}) \to K^{\mathscr{A}}(X; \mathscr{C}) \to K^{\mathscr{A}}(Y; \mathscr{C}).$$

7. The multiplicative structure

As in [10], let $\mathscr{H}$ be the Banach category of k-Hilbert spaces and let $\check{\mathscr{H}}$ be the Banach category with the same objects as $\mathscr{H}$ but with $\check{\mathscr{H}}(H_1, H_2) = \mathscr{H}(H_1, H_2)/\mathscr{K}(H_1, H_2)$ where $\mathscr{K}(H_1, H_2)$ is the space of all completely continuous maps $H_1 \to H_2$; $\varphi : \mathscr{H} \to \check{\mathscr{H}}$ will denote the canonical functor and $\mathscr{E}$ is the Banach category of finite dimensional k-vector spaces; X again denotes a compact space.

Let $\kappa : K^{\mathscr{A}}(X; \mathscr{E}) \to K^{\mathscr{A}}(X; \varphi)$ be defined (as in [10]) by $\kappa(E, \varepsilon_1, \varepsilon_2) = (E, \varepsilon_1, \varepsilon_2)$. Then the following is proved in the same way as Proposition 5 of [10]:

Proposition **20.** — κ *is an isomorphism. Also*

$$\kappa^{-1}.\partial : \quad K^{\Sigma\mathscr{A}}(X; \check{\mathscr{H}}) \to K^{\mathscr{A}}(X; \mathscr{E}) \approx K^{\mathscr{A}}(X)$$

is an isomorphism.

More generally, if we have an exact sequence of prebanach categories (cf. [13]) $0 \to \mathscr{C}' \to \mathscr{C} \to \mathscr{C}'' \to 0$, there is an exact sequence:

$$K^{\Sigma\mathscr{A}}(X; \mathscr{C}') \to K^{\Sigma\mathscr{A}}(X; \mathscr{C}) \to K^{\Sigma\mathscr{A}}(X; \mathscr{C}'') \to K^{\mathscr{A}}(X; \mathscr{C}') \to K^{\mathscr{A}}(X; \mathscr{C}) \to \dots$$

In particular, if $\mathscr{C}$ is *flabby* (" flasque ", see Definition 3 of [13]),

$$K^{\Sigma\mathscr{A}}(X; \mathscr{C}'') \approx K^{\mathscr{A}}(X; \mathscr{C}').$$

Hence there is an isomorphism $K^{\mathscr{A}}(X; \mathscr{C}) \approx K^{\Sigma\mathscr{A}}(X; S\mathscr{C})$ for every Banach category $\mathscr{C}$.

As in [10] we can define a group $\overline{K}^{\mathscr{A}}(X)$ by considering self-adjoint Fredholm operators. More precisely, consider couples (E, D), where E is a Hilbert bundle on X which is a graded $\mathscr{A}$-module (i.e. a $\Sigma\mathscr{A}$-module) and where $D : E \to E$ is a *quasi-graduation* of E. This means that D is a continuous family of self-adjoint Fredholm operators commuting with elements of $(\Sigma\mathscr{A})_0$ and anti-commuting with elements of $(\Sigma\mathscr{A})_1$. Then $\overline{K}^{\mathscr{A}}(X)$ is the Grothendieck group associated to the commutative monoid of homotopy classes of such couples. Using spectral theory as in [10], we can define an isomorphism $u : \overline{K}^{\mathscr{A}}(X) \to K^{\Sigma\mathscr{A}}(X; \check{\mathscr{H}})$. Eventually the following analogue of Theorem 6 of [10] is obtained:

Theorem **21.** — $j = \kappa^{-1}.\partial.u$: $\overline{K}^{\mathscr{A}}(X) \to K^{\mathscr{A}}(X)$ *is an isomorphism.*

The methods of [12] give an explicit inverse to this isomorphism. This fact is not needed here.

The groups $\overline{K}^{\mathscr{A}}(X)$ enable a *cup product* to be defined:

$$K^{\mathscr{A}}(X) \otimes K^{\mathscr{A}'}(X') \to K^{\mathscr{A} \hat{\otimes} \mathscr{A}'}(X \times X').$$

The formula $(E, D) \cup (E', D') = (E \hat{\otimes} E', D \hat{\otimes} 1 + 1 \hat{\otimes} D')$ is used, where $\hat{\otimes}$ means the *graded completed* tensor product. The multiplication is associative and distributive with respect to the addition. It is commutative in the following sense: define $T : \mathscr{A} \hat{\otimes} \mathscr{A}' \to \mathscr{A}' \hat{\otimes} \mathscr{A}$, covering the canonical isomorphism $X \times X' \to X' \times X$, by $T(x \otimes x') = (-1)^{\delta(x)\delta(x')}.x' \otimes x$, where δ denotes the degree. Define $T' : E \hat{\otimes} E' \to E' \hat{\otimes} E$ similarly. Then the pair (T', T) define an isomorphism.

In particular, if $X = X'$ and $\mathscr{A}$ and $\mathscr{A}'$ are bundles of central simple graded k-algebras, this product composed with the restriction to the diagonal defines a product $K^{\alpha}(X) \otimes K^{\alpha'}(X) \to K^{\alpha\alpha'}(X)$, where α, α' are respectively the images of $\mathscr{A}$, $\mathscr{A}'$ in GBr(X).

Remark. — In the case when $\mathscr{A}$ and $\mathscr{A}'$ are Clifford algebras of vector bundles, the cup product is usually defined by using the Thom isomorphism (Theorem 15). The methods of [10] and [14] show that all reasonable compatibilities between the two definitions hold.

8. Adams operations

Let $\mathscr{A}$ be a bundle of graded k-algebras on the compact space X. An action of the symmetric group $\mathfrak{S}_n$ on $\mathscr{A}^{\hat{\otimes} n}$ is defined by the formula (in which the elements a_i are supposed to be homogeneous):

$$\sigma.(a_1 \hat{\otimes} a_2 \hat{\otimes} \ldots \hat{\otimes} a_n) = (-1)^N . a_{\sigma(1)} \hat{\otimes} a_{\sigma(2)} \hat{\otimes} \ldots \hat{\otimes} a_{\sigma(n)},$$

where $N = N(\sigma; a_1, a_2, \ldots, a_n)$ is the number of inversions induced by σ of two a_i's of odd degree. Note that:

$$N(\tau; a_1, a_2, \ldots, a_n) + N(\sigma; a_{\tau(1)}, a_{\tau(2)}, \ldots, a_{\tau(n)}) \equiv N(\sigma\tau; a_1, a_2, \ldots, a_n) \ (2);$$

this states that an $\mathfrak{S}_n$-action is indeed defined by this rule and may be verified by examining the case when $\mathscr{A}=C^{1,0}$ In this case the action of $\mathfrak{S}_n$ on $\mathscr{A}^{\hat{\otimes} n}=C^{n,0}$ is simply that induced by the action of $\mathfrak{S}_n$ on $\mathbf{R}^n$.

Definition **22.** — *The symmetric tensor power* $\varphi_n : K^{\mathscr{A}}(X) \to K_{\mathfrak{S}_n}^{\mathscr{A}^{\hat{\otimes} n}}(X)$ *is defined by the formula:*

$$\varphi_n(E, D)=(E^{\hat{\otimes} n}, D\hat{\otimes} 1\hat{\otimes} \ldots \hat{\otimes} 1+\ldots+1\hat{\otimes} \ldots \hat{\otimes} 1\hat{\otimes} D).$$

Remarks. — The action of $\mathscr{A}^{\hat{\otimes} n}$ on $E^{\hat{\otimes} n}$ is given by the formula (in which the elements are homogeneous) $(a_1\hat{\otimes} \ldots \hat{\otimes} a_n).(e_1\hat{\otimes} \ldots \hat{\otimes} e_n)=(-1)^M.a_1e_1\otimes \ldots \otimes a_ne_n$ where $M=M(a_1, \ldots, a_n; e_1, \ldots, e_n)$ is the number of inversions of odd degree elements in the permutation $(a_1, \ldots, a_n, e_1, \ldots, e_n) \mapsto (a_1, e_1, a_2, e_2, \ldots, a_n, e_n)$. (The example of $C^{1,0}$ considered as a module over itself shows that this is well-defined, and that, if $\sigma\in\mathfrak{S}_n$, $\lambda\in\mathscr{A}^{\hat{\otimes} n}$, $e\in E^{\hat{\otimes} n}$, $\sigma.(\lambda.e)=(\sigma.\lambda).(\sigma.e)$. Hence φ_n is well-defined.) Note that if $\mathscr{A}$ is a Clifford bundle $C(V)$, φ_n is the algebraic translation of the map $K(V)\to K_{\mathfrak{S}_n}(V^n)$ of [4] (cf. [15]).

It is now convenient to make the following definition. Let $f: B\to C$ be an isomorphism of algebras. Let P be a B-module and Q be a C-module. A group (iso)morphism $\bar{f}: P\to Q$ will be called an (iso)morphism of modules if $\bar{f}(\lambda.x)=f(\lambda).\bar{f}(x)$ for $\lambda\in B$ and $x\in P$. The example that we are interested in is when $B=(\mathscr{A}\hat{\otimes}\mathscr{A}')\hat{\otimes} \ldots \hat{\otimes}(\mathscr{A}\hat{\otimes}\mathscr{A}')$ (n copies); $C=(\mathscr{A}\hat{\otimes} \ldots \hat{\otimes}\mathscr{A})\hat{\otimes}(\mathscr{A}'\hat{\otimes} \ldots \hat{\otimes}\mathscr{A}')$ (n copies in each bracket); $P=(E\hat{\otimes}E')^{\hat{\otimes} n}$ and $Q=E^{\hat{\otimes} n}\hat{\otimes}E'^{\hat{\otimes} n}$. Here E is an $\mathscr{A}$-module, E′ an $\mathscr{A}'$-module and $\mathscr{A}$ and $\mathscr{A}'$ are bundles of graded k-algebras on X and X′ respectively. $f: B\to C$ is defined by the formula $f(a_1\hat{\otimes} a'_1\hat{\otimes} \ldots \hat{\otimes} a_n\hat{\otimes} a'_n)=(-1)^M(a_1\hat{\otimes} \ldots \hat{\otimes} a_n)\hat{\otimes}(a'_1\hat{\otimes} \ldots \hat{\otimes} a'_n)$ with M as above, and the isomorphism $\bar{f}$ is defined by an analogous formula. Hence there is induced a canonical isomorphism $T : K_{\mathfrak{S}_n}^{(\mathscr{A}\hat{\otimes}\mathscr{A}')^{\hat{\otimes} n}}(X\times X') \to K_{\mathfrak{S}_n}^{\mathscr{A}^{\hat{\otimes} n}\hat{\otimes}\mathscr{A}'^{\hat{\otimes} n}}(X\times X')$. Then the operations φ_n are seen to have the multiplicative property expressed by the commutativity of:

$$\begin{array}{ccc}
K^{\mathscr{A}}(X)\times X^{\mathscr{A}'}(X') & \longrightarrow & K^{\mathscr{A}\hat{\otimes}\mathscr{A}'}(X\times X') \\
\Big\downarrow{\scriptstyle \varphi_n^{\mathscr{A}}\times\varphi_n^{\mathscr{A}'}} & & \Big\downarrow{\scriptstyle \varphi_n} \\
 & & K_{\mathfrak{S}_n}^{(\mathscr{A}\hat{\otimes}\mathscr{A}')^{\hat{\otimes} n}}(X\times X') \\
 & & \Big\downarrow{\scriptstyle T} \\
K_{\mathfrak{S}_n}^{\mathscr{A}^{\hat{\otimes} n}}(X)\times K_{\mathfrak{S}_n}^{\mathscr{A}'^{\hat{\otimes} n}}(X') & \longrightarrow & K_{\mathfrak{S}_n}^{\mathscr{A}^{\hat{\otimes} n}\hat{\otimes}\mathscr{A}'^{\hat{\otimes} n}}(X\times X')
\end{array}$$

The groups $K_{\mathfrak{S}_n}^{\mathscr{A}^{\hat{\otimes} n}}(X)$ seem to be very hard to compute in general. For example, if X is a point and $\mathscr{A}=C^{1,0}$, the explicit determination of this group is related to the explicit determination of the representation ring of a certain group $\mathfrak{F}'_n$, which double covers $\mathfrak{S}_n$. Part of this task has been done by Schur [16].

In order to define more acceptable invariants, the following basic idea is useful. Let G be a subgroup of $\mathfrak{S}_n$ such that there exists a homomorphism $\mu : G \to U(\mathscr{A}^{\widehat{\otimes} n})$ (U denotes the group of invertible elements of degree 0) such that $\sigma . a = \mu(\sigma) . a . \mu(\sigma^{-1})$ for $\sigma \in G$ and $a \in \mathscr{A}^{\widehat{\otimes} n}$. Then we can "twist" the action of G on every G-$\mathscr{A}^{\widehat{\otimes} n}$-module P in the following way: set $\sigma * m = \mu(\sigma^{-1})\sigma . m$. Note that for all $m \in P$, $\sigma \in G$, $\tau \in G$ and $a \in \mathscr{A}^{\widehat{\otimes} n}$, $\sigma * (\tau * m) = (\sigma\tau) * m$ and $\sigma * (am) = a(\sigma * m)$.

Lemma **23**. — *If* $\mathscr{E}^{\mathscr{A}^{\widehat{\otimes} n}}$ *is the category of graded* G-$\mathscr{A}^{\widehat{\otimes} n}$*-modules with the trivial action of* G *on* $\mathscr{A}^{\widehat{\otimes} n}$ *and if* $K_{(G)}^{\mathscr{A}^{\widehat{\otimes} n}}$ *is the associated graded* Grothendieck *group* (G *is as above*), *the graded categories* $\mathscr{E}_{(G)}^{\mathscr{A}^{\widehat{\otimes} n}}$ *and* $\mathscr{E}_{G}^{\mathscr{A}^{\widehat{\otimes} n}}$ *are isomorphic. In particular,* $K_{(G)}^{\mathscr{A}^{\widehat{\otimes} n}}(X) \approx K_{G}^{\mathscr{A}^{\widehat{\otimes} n}}$.

Proof. — Define $\varphi : \mathscr{E}_{G}^{\mathscr{A}^{\widehat{\otimes} n}}(X) \to \mathscr{E}_{(G)}^{\mathscr{A}^{\widehat{\otimes} n}}$ by $\varphi(P) = P'$ and $\varphi(f) = f'$, where P′ is P as an $\mathscr{A}^{\widehat{\otimes} n}$-module with the action $*$ of G and where f' coincides with f on the underlying $\mathscr{A}^{\widehat{\otimes} n}$-modules. Since $\mu(\sigma)$ is of degree 0, the new action of G is of degree 0 and the functor is well-defined. Likewise, an inverse φ' to φ may be constructed.

Two cases are of special interest. Firstly, that when $G = \mathbf{Z}_n$, the cyclic group generated by the permutation $g = (1\ 2\ \ldots\ n)$; n will always be taken to be *odd*; if further $n = 4p+1$, the second case is when $G = \mathfrak{D}_n$, the dihedral group generated by g and the permutation $t = (1\ 4p)(2\ 4p-1)\ldots(2p\ 2p+1)$. $\mathbf{Z}_n$ is normal in $\mathfrak{D}_n$; the quotient group is $\mathbf{Z}_2$ and it acts on $\mathbf{Z}_n$ by inversion. $\mathfrak{D}_n$ is generated by g and t with the relations $g^n = t^2 = 1$ and $tg = g^{-1}t$.

Lemma **24**. — *Let* $\mathscr{A}$ *be a bundle of central simple graded k-algebras on* X. *If n is odd, there exists a homomorphism* $\mu : \mathbf{Z}_n \to U(\mathscr{A}^{\widehat{\otimes} n})$ *such that* $\sigma . a = \mu(\sigma) a \mu(\sigma^{-1})$ *for* $\sigma \in \mathbf{Z}_n$ *and* $a \in \mathscr{A}^{\widehat{\otimes} n}$. *If* $n = 4p+1$, *there also exists a homomorphism* $\mu' : \mathfrak{D}_n \to U(\mathscr{A}^{\widehat{\otimes} n})$ *extending* μ *such that* $\sigma . a = \mu'(\sigma) a \mu'(\sigma^{-1})$ *for* $\sigma \in \mathfrak{D}_n$ *and* $a \in \mathscr{A}^{\widehat{\otimes} n}$.

Proof. — The Schur multiplicators $H^2(\mathbf{Z}_n, \mathbf{R}^*)$, $H^2(\mathbf{Z}_n, \mathbf{C}^*)$ and $H^2(\mathfrak{D}_n, \mathbf{C}^*)$ are all zero; [16] and a little calculation show that $H^2(\mathfrak{S}_n, \mathbf{R}^*) = \mathbf{Z}_2 \oplus \mathbf{Z}_2$ and that the restriction to $H^2(\mathfrak{D}_n, \mathbf{R}^*) = \mathbf{Z}_2$ maps the element corresponding to $\mathfrak{F}'_n$ to 0. The Skolem-Noether theorem and some explicit checking show that if A is any of the algebras listed in § 1, all automorphisms σ of A are induced by conjugation by a unit $x \in A_0 \cup A_1$, determined up to a central homogeneous unit. As $g^n = 1$, its action must be induced by a unit $x \in A_0$. As t is the product of an even number of conjugate elements of $\mathfrak{S}_n$, its action must also be induced by a unit $x \in A_0$. The result now follows from the theory of projective representations.

Denote by $v : K_{G}^{\mathscr{A}^{\widehat{\otimes} n}}(X) \to K_{(G)}^{\mathscr{A}^{\widehat{\otimes} n}}(X)$ the isomorphism of Lemma 23 for $G = \mathbf{Z}_n$

or $\mathfrak{D}_n$ and n as above. Then it is easily seen that the following diagram commutes:

$$\begin{array}{ccc} K_G^{\mathscr{A}^{\hat{\otimes} n}}(X)\times K_G^{\mathscr{A}'^{\hat{\otimes} n}}(X') & \longrightarrow & K_G^{\mathscr{A}^{\hat{\otimes} n}\hat{\otimes}\mathscr{A}'^{\hat{\otimes} n}}(X\times X') \approx K_G^{(\mathscr{A}\hat{\otimes}\mathscr{A}')^{\hat{\otimes} n}}(X\times X') \\ \Big\downarrow{\scriptstyle v\times v} & & \Big\downarrow{\scriptstyle v} \\ K_{(G)}^{\mathscr{A}^{\hat{\otimes} n}}(X)\times K_{(G)}^{\mathscr{A}'^{\hat{\otimes} n}}(X') & \longrightarrow & K_{(G)}^{\mathscr{A}^{\hat{\otimes} n}\hat{\otimes}\mathscr{A}'^{\hat{\otimes} n}}(X\times X') \approx K_{(G)}^{(\mathscr{A}\hat{\otimes}\mathscr{A}')^{\hat{\otimes} n}}(X\times X') \end{array}$$

We now treat in detail the case when $k=\mathbf{C}$ and $G=\mathbf{Z}_n$. The case when $k=\mathbf{R}$ and $G=\mathfrak{D}_n$ will be dealt with later.

The following lemma (which may be generalised to the case when $\mathbf{Z}_n$ is replaced by any other finite group) is proved in the same way as Proposition (2.1) of [4], i.e. by use of the canonical projection operators in the group algebra.

Lemma **25.** — *Let* F *be a complex Hilbert bundle on* X *with an action of the cyclic group* $\mathbf{Z}_n$. *Then the bundle* F *splits into the sum* $F_0\oplus F_1\oplus\ldots\oplus F_{n-1}$. *The generator* g *of* $\mathbf{Z}_n$ *acts on* F_r *by the multiplication by* ω^r, *where* $\omega=\exp(2\pi i/n)$.

If $D: F\to F'$ is a morphism of $\mathbf{Z}_n$-bundles, we shall write $D_r: F_r\to F'_r$ for the restriction of D to F_r and F'_r. Let Ω_n be the subring of $\mathbf{C}$ generated over $\mathbf{Z}$ by the above ω. In other words, $\Omega_n=\mathbf{Z}[\omega]/(\Phi_n(\omega))$, where Φ_n is the n-th cyclotomic polynomial.

Definition **26.** — *If* (F, D) *is an element of* $KU_{(\mathbf{Z}_n)}^{\mathscr{A}^{\hat{\otimes} n}}(X)$, Tr(F, D), *its trace, is the element* $\sum_{r=0}^{n-1}(F_r, D_r)\otimes\omega^r$ *of* $KU^{\mathscr{A}^{\hat{\otimes} n}}(X)\otimes\Omega_n$.

Proposition **27.** — *The "trace function"* Tr *is multiplicative. In other words, if* $(F, D)\in KU_{(\mathbf{Z}_n)}^{\mathscr{A}^{\hat{\otimes} n}}(X)$ *and* $(F', D')\in KU_{(\mathbf{Z}_n)}^{\mathscr{A}'^{\hat{\otimes} n}}(X')$, *the elements* $\mathrm{Tr}(F, D)\cup\mathrm{Tr}(F', D')$ *and* $\mathrm{Tr}(F\hat{\otimes}F', D\hat{\otimes}1+1\hat{\otimes}D')$ *are equal when we identify* $KU^{\mathscr{A}^{\hat{\otimes} n}\otimes\mathscr{A}'^{\hat{\otimes} n}}(X\times X')$ *and* $KU^{(\mathscr{A}\otimes\mathscr{A}')^{\hat{\otimes} n}}(X\times X')$ *by the canonical isomorphism.*

Proof. — The proposition is a direct consequence of the definitions and of the fact that $\omega^n\omega^m=\omega^{n+m}$.

Lemma **28.** — *Let* $d>1$ *be a divisor of* n. *Let the couples* $(G_0, D_0), \ldots, (G_{d-1}, D_{d-1})$ *be as in* § 7 *and suppose that the generator* g *of* $\mathbf{Z}_n$ *acts on* $(F, D)=(G_0\oplus\ldots\oplus G_{d-1}, D_0\oplus\ldots\oplus D_{d-1})$ *by the matrix* $\gamma=(\gamma_{ij})$, *where the only non-zero* γ_{ij} *are* $\gamma_{10}=\gamma_0$, $\gamma_{21}=\gamma_1$, $\ldots$, $\gamma_{d-1,d-2}=\gamma_{d-2}$ *and* $\gamma_{0,d-1}=\gamma_{d-1}$. *Then* $\mathrm{Tr}(F, D)=0$.

Proof. — As G_0 and G_i can be identified by the isomorphism $\gamma_{i-1}\ldots\gamma_0$, there is no loss of generality in assuming that $G_0=G_i$, $D_0=D_1=\ldots=D_{d-1}$ and that each γ_i is the identity. Then $\mathrm{Ker}(g-\omega^{rn/d})\approx G_0$. As $\sum_r\omega^{rn/d}=0$, $\mathrm{Tr}(F, D)=0$.

Definition **29.** — *The " Adams operation ":*

$$\psi^n: \quad \mathrm{KU}^{\mathscr{A}}(\mathrm{X}) \to \mathrm{KU}^{\mathscr{A}^{\hat{\otimes} n}}(\mathrm{X}) \otimes \Omega_n$$

is the composite:

$$\mathrm{KU}^{\mathscr{A}}(\mathrm{X}) \xrightarrow{\varphi_n^{\mathscr{A}}} \mathrm{KU}_{\mathfrak{S}_n}^{\mathscr{A}^{\hat{\otimes} n}}(\mathrm{X}) \longrightarrow \mathrm{KU}_{\mathbf{Z}_n}^{\mathscr{A}^{\hat{\otimes} n}}(\mathrm{X}) \xrightarrow{v} \mathrm{KU}_{(\mathbf{Z}_n)}^{\mathscr{A}^{\hat{\otimes} n}}(\mathrm{X}) \xrightarrow{\mathrm{Tr}} \mathrm{KU}^{\mathscr{A}^{\hat{\otimes} n}}(\mathrm{X}) \otimes \Omega_n.$$

Theorem **30.** — *The Adams operation* ψ^n *is additive,* i.e. $\psi^n(x+y) = \psi^n(x) + \psi^n(y)$. *Moreover, it is multiplicative in the sense that there is a commutative diagram:*

$$\begin{array}{ccccc}
\mathrm{KU}^{\mathscr{A}}(\mathrm{X}) \otimes \mathrm{KU}^{\mathscr{A}'}(\mathrm{X}') & & \xrightarrow{\quad \cup \quad} & & \mathrm{KU}^{\mathscr{A} \hat{\otimes} \mathscr{A}'}(\mathrm{X} \times \mathrm{X}') \\
\downarrow{\scriptstyle \psi^n \otimes \psi^n} & & & & \downarrow{\scriptstyle \psi^n} \\
\mathrm{KU}^{\mathscr{A}^{\hat{\otimes} n}}(\mathrm{X}) \otimes \mathrm{KU}^{\mathscr{A}'^{\hat{\otimes} n}}(\mathrm{X}') & \xrightarrow{\cup} & \mathrm{KU}^{\mathscr{A}^{\hat{\otimes} n} \hat{\otimes} \mathscr{A}'^{\hat{\otimes} n}}(\mathrm{X} \times \mathrm{X}') \otimes \Omega_n & \xleftarrow{\mathrm{T}} & \mathrm{KU}^{(\mathscr{A} \hat{\otimes} \mathscr{A}')^{\hat{\otimes} n}}(\mathrm{X} \times \mathrm{X}') \otimes \Omega_n
\end{array}$$

Proof. — The second assertion is a direct consequence of the diagrams following Definition 22 and Lemma 24, and of Proposition 27. Write $x = (\mathrm{E}_0, \mathrm{D}_0)$ and $y = (\mathrm{E}_1, \mathrm{D}_1)$. Then $(\mathrm{E}_0 \oplus \mathrm{E}_1)^{\hat{\otimes} n} = \bigoplus (\mathrm{E}_{i_1} \hat{\otimes} \mathrm{E}_{i_2} \hat{\otimes} \ldots \hat{\otimes} \mathrm{E}_{i_n})$, where the direct sum is taken over all i_r. This expression is the sum of $\mathrm{E}_0^{\hat{\otimes} n}$, $\mathrm{E}_1^{\hat{\otimes} n}$ and of bundles of the form $\mathrm{G} = \mathrm{G}_0 \oplus \ldots \oplus \mathrm{G}_{d-1}$ $(d > 1)$, where d divides n. Here the action of $\mathbf{Z}_n$ on G is as in Lemma 28. So $\mathrm{Tr}(\mathrm{F}, \mathrm{D}) = 0$. Hence $\psi^n(x+y) = \psi^n(x) + \psi^n(y)$, as required.

Proposition **31.** — *Let* $\mathrm{E} = \mathrm{E}_0 \oplus \mathrm{E}_1$ *be a* $\mathbf{Z}_2$*-graded* **C**-*vector bundle on* X *with graded endomorphism bundle* $\mathscr{A} = \mathrm{END}(\mathrm{E})$. *Let* λ *be the isomorphism* φ_* *of Theorem* 16. *Then the following diagram is commutative:*

$$\begin{array}{ccccc}
\mathrm{KU}(\mathrm{X}) & \xrightarrow{\psi^n} & \mathrm{KU}(\mathrm{X}) & \xrightarrow{i} & \mathrm{KU}(\mathrm{X}) \otimes \Omega_n \\
\downarrow{\scriptstyle \lambda} & & & & \downarrow{\scriptstyle \lambda \otimes 1} \\
\mathrm{KU}^{\mathscr{A}}(\mathrm{X}) & & \xrightarrow{\qquad \psi^n \qquad} & & \mathrm{KU}^{\mathscr{A}^{\hat{\otimes} n}}(\mathrm{X}) \otimes \Omega_n
\end{array}$$

where the upper ψ^n *is the ordinary Adams operation* ([1], [4]) *and i is the inclusion.*

Proof. — If $\mathscr{A} = \mathbf{C}$, the argument of [4] shows that our ψ^n is the usual ψ^n operation. For other $\mathscr{A}$, λ is defined by the cup product with E or $\mathrm{E}^{\hat{\otimes} n}$ as appropriate. The action of $\mathfrak{S}_n$ on $\mathscr{A}^{\hat{\otimes} n}$ is induced by the natural action of $\mathfrak{S}_n$ on $\mathrm{E}^{\hat{\otimes} n}$ and is thus given by inner automorphisms. This implies that the action of $\mathbf{Z}_n$ on $(v\varphi_n^{\mathscr{A}})(\mathrm{E}, \mathrm{D})$ is trivial. The result follows.

Remark. — If $\mathscr{B}$ is another bundle of simple central graded $\mathbf{C}$-algebras, exactly the same proof shows the commutativity of the diagram:

$$\begin{array}{ccc} \mathrm{KU}^{\mathscr{B}}(\mathrm{X}) & \xrightarrow{\qquad\qquad\psi^n\qquad\qquad} & \mathrm{KU}^{\mathscr{B}^{\hat{\otimes} n}}(\mathrm{X})\otimes\Omega_n \\ \Big\downarrow{\scriptstyle\lambda} & & \Big\downarrow{\scriptstyle\lambda} \\ \mathrm{KU}^{\mathscr{B}\hat{\otimes}\mathscr{A}}(\mathrm{X}) & \xrightarrow{\psi^n} \mathrm{KU}^{(\mathscr{B}\hat{\otimes}\mathscr{A})^{\hat{\otimes} n}}(\mathrm{X})\otimes\Omega_n \approx & \mathrm{KU}^{\mathscr{B}^{\hat{\otimes} n}\hat{\otimes}\mathscr{A}^{\hat{\otimes} n}}(\mathrm{X})\otimes\Omega_n \end{array}$$

This shows that ψ^n is essentially an operation from $\mathrm{KU}^{\alpha}(\mathrm{X})$ to $\mathrm{KU}^{\alpha^n}(\mathrm{X})\otimes\Omega_n$ for $\alpha\in\mathrm{GBrU}(\mathrm{X})$.

Now consider the case when $k=\mathbf{R}$ and $n=4p+1$. Following End [6], we will define operations ψ^n in this case. In view of the preceding discussion, we need define only:

$$\mathrm{Tr}:\ \mathrm{KO}^{\mathscr{A}^{\hat{\otimes} n}}_{(\mathfrak{D}_n)}(\mathrm{X}) \to \mathrm{KO}^{\mathscr{A}^{\hat{\otimes} n}}(\mathrm{X})\otimes\Omega_n.$$

If F is a (real) Hilbert bundle on X with an action of $\mathfrak{D}_n$, set $\mathrm{F}'=\mathrm{F}\otimes\mathbf{C}$ and $\mathrm{F}'_s=\mathrm{Ker}(g-\omega^s)$. Let c denote the complex conjugation on F'. Then its restriction $c_s:\mathrm{F}'_s\to\mathrm{F}'_{-s}$ is $\mathbf{C}$-anti-linear whilst the restriction of t, $t_{-s}:\mathrm{F}'_{-s}\to\mathrm{F}'_s$ is $\mathbf{C}$-linear. So F'_s is naturally isomorphic to the complexification of the *real* Hilbert bundle $\mathrm{F}_s=\mathrm{Ker}(tc-1)\cap\mathrm{Ker}(g-\omega^s)$. If $\mathrm{D}:\mathrm{F}\to\mathrm{G}$ is a morphism of real Hilbert $\mathfrak{D}_n$-bundles, write $\mathrm{D}_s:\mathrm{F}_s\to\mathrm{G}_s$ for the restriction. If (F, D) are as in § 8, define:

$$\mathrm{Tr}(\mathrm{F},\mathrm{D})=\sum_s(\mathrm{F}_s,\mathrm{D}_s)\otimes\omega^s.$$

As in Definition 29, the ψ^n operation is defined to be the composite:

$$\mathrm{KO}^{\mathscr{A}}(\mathrm{X}) \xrightarrow{\varphi_n^{\mathscr{A}}} \mathrm{KO}^{\mathscr{A}^{\hat{\otimes} n}}_{\mathfrak{S}_n}(\mathrm{X}) \longrightarrow \mathrm{KO}^{\mathscr{A}^{\hat{\otimes} n}}_{\mathfrak{D}_n}(\mathrm{X}) \xrightarrow{v} \mathrm{KO}^{\mathscr{A}^{\hat{\otimes} n}}_{(\mathfrak{D}_n)}(\mathrm{X}) \xrightarrow{\mathrm{Tr}} \mathrm{KO}^{\mathscr{A}^{\hat{\otimes} n}}(\mathrm{X})\otimes\Omega_n.$$

It is easy to extend all the propositions proved above to the real case. Moreover, the following diagram, in which c is the complexification, commutes:

$$\begin{array}{ccc} \mathrm{KO}^{\mathscr{A}}(\mathrm{X}) & \xrightarrow{\qquad\qquad\psi^n\qquad\qquad} & \mathrm{KO}^{\mathscr{A}^{\hat{\otimes} n}}(\mathrm{X})\otimes\Omega_n \\ \Big\downarrow{\scriptstyle c} & & \Big\downarrow{\scriptstyle c} \\ \mathrm{KU}^{\mathscr{A}\otimes\mathbf{C}}(\mathrm{X}) & \xrightarrow{\psi^n} \mathrm{KU}^{(\mathscr{A}\otimes\mathbf{C})^{\hat{\otimes} n}}(\mathrm{X})\otimes\Omega_n \approx & \mathrm{KU}^{\mathscr{A}^{\hat{\otimes} n}\otimes\mathbf{C}}(\mathrm{X})\otimes\Omega_n \end{array}$$

The following diagram, in which r is the realification, also commutes (cf. [6]):

$$\begin{array}{ccc}
\mathrm{KU}^{\mathscr{A}\otimes\mathrm{C}}(\mathrm{X}) & \xrightarrow{\psi^n} \mathrm{KU}^{(\mathscr{A}\otimes\mathrm{C})^{\hat{\otimes} n}}(\mathrm{X})\otimes\Omega_n \approx & \mathrm{KU}^{\mathscr{A}^{\hat{\otimes} n}\otimes\mathrm{C}}(\mathrm{X})\otimes\Omega_n \\
\downarrow{\scriptstyle r} & & \downarrow{\scriptstyle r} \\
\mathrm{KO}^{\mathscr{A}}(\mathrm{X}) & \xrightarrow{\qquad\qquad\psi^n\qquad\qquad} & \mathrm{KO}^{\mathscr{A}^{\otimes n}}(\mathrm{X})\otimes\Omega_n
\end{array}$$

Remark. — Simple examples show that Ω_n is necessary to define ψ^n in both cases.

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