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ON THE EXPLICIT SOLVABILITY OF CERTAIN TRANSCENDENTAL EQUATIONS[1]

by Maxwell ROSENLICHT

The subject matter of this paper was first treated by Liouville in the 1830's. Its classical exposition is due to Ritt [2]. The principal concept is that of differential field, that is a field with a specified derivation. For the purposes of this paper very little needs to be known about differential fields, except for two essential elementary facts. One is that the subset of constants of a given differential field, that is elements with derivative zero, is a subfield, the other is that any algebraic extension field of a given differential field has a unique derivation that extends the given one; for these and other matters related to the present paper we refer to [3]. A certain amount of the theory of algebraic functions of one variable is used, and on this matter we refer to [1].

Lemma 1. — Let k be a differential subfield of the differential field K, with K a finite extension of k of transcendence degree one. Then the derivation of K is continuous in the topology of any k-place of K.

Fix a k-place P of K. We have to show that if $u \in K$, $u \neq 0$, and $\mathrm{ord}_P\, u$ is large, then $\mathrm{ord}_P\, u'$ is also large. For this it suffices to show that the integer $\mathrm{ord}_P\, u' - \mathrm{ord}_P\, u = \mathrm{ord}_P\, u'/u$ is bounded from below. If t is a fixed element of K such that $\mathrm{ord}_P\, t = 1$ and we set $\mathrm{ord}_P\, u = \nu$, then $u = t^\nu v$, where $\mathrm{ord}_P\, v = 0$, and we have

$$\frac{u'}{u} = \nu\frac{t'}{t} + \frac{v'}{v}.$$

Thus we are reduced to proving that $\mathrm{ord}_P\, v'$ remains bounded from below as v ranges over the set of elements of K that are finite at P. The following proof of this fact has the advantage of being valid without any assumptions on field characteristic or separability. Since the subfield of K consisting of all elements that are separably algebraic over k is closed under differentiation, we may, if necessary, replace k by this subfield of K to ensure that k is separably algebraically closed in K. We then use the Riemann-Roch theorem (for K as a function field over the algebraic closure of k in K) to find an element $x \in K$ which has a zero at P and at no other k-place of K. For any k-place P' of K, not a pole of x and distinct from P, there is a polynomial $f_{P'}(x) \in k[x]$ with nonzero constant term such that $\mathrm{ord}_{P'} f_{P'}(x) > 0$. Hence for any $v \in K$ such that $\mathrm{ord}_P\, v \geq 0$ we can find an element $f(x) \in k[x]$ such that $f(0) \neq 0$ and $f(x)v$ has poles only at the poles

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of x. Then $f(x)v$ is in the integral closure of $k[x]$ in K, a finite $k[x]$-module, say with generators $y_1, \ldots, y_n$. Thus any $v \in K$ which is finite at P is of the form

$$v = \frac{f_1(x)}{f(x)} y_1 + \ldots + \frac{f_n(x)}{f(x)} y_n,$$

where $f_1, \ldots, f_n, f \in k[x]$ and $f(0) \neq 0$. Since $f, 1/f, f_1, \ldots, f_n, y_1, \ldots, y_n$ are all finite at P, a minor calculation shows that

$$\mathrm{ord}_P\, v' \geq \min \{\mathrm{ord}_P\, y_1', \ldots, \mathrm{ord}_P\, y_n', \mathrm{ord}_P\, x'\},$$

which completes the proof.

Lemma 2. — Let $k \subset K$ be differential fields of characteristic zero with the same field of constants, K being a finite algebraic extension of $k(t)$, where $t \in K$ is transcendental over k and either $t' \in k$ or $t'/t \in k$. Suppose that $c_1, \ldots, c_n \in k$ are linearly independent over the rational numbers $\mathbf{Q}$, that $u_1, \ldots, u_n$ are nonzero elements of K, and that $v \in K$. Let P be a k-place of K, e_P the ramification at P of K over $k(t)$. Then we have the following estimates for

$$\mathrm{ord}_P \left(\sum_{i=1}^{n} c_i \frac{u_i'}{u_i} + v' \right):$$

f P is not a pole of t (nor a zero of t if $t'/t \in k$), then this number is

$$\begin{cases} \mathrm{ord}_P\, v - e_P \\ -e_P \\ > -e_P \end{cases} \quad \textit{if} \quad \begin{cases} \mathrm{ord}_P\, v < 0 \\ \mathrm{ord}_P\, v \geq 0 \textit{ and } \mathrm{ord}_P\, u_i \neq 0 \textit{ for some } i = 1, \ldots, n \\ \mathrm{ord}_P\, v \geq 0 \textit{ and } \mathrm{ord}_P\, u_i = 0, \; i = 1, \ldots, n; \end{cases}$$

if $t' \in k$ and P is a pole of t then this number is

$$\begin{cases} \geq \mathrm{ord}_P\, v \\ \leq \mathrm{ord}_P\, v + e_P \\ \geq 0 \end{cases} \quad \textit{if} \quad \begin{cases} \mathrm{ord}_P\, v < 0 \\ \mathrm{ord}_P\, v < -e_P \\ \mathrm{ord}_P\, v \geq 0; \end{cases}$$

and, finally, if $t'/t \in k$ and P is a pole or zero of t, the number is

$$\begin{cases} \mathrm{ord}_P\, v \\ \geq 0 \end{cases} \quad \textit{if} \quad \begin{cases} \mathrm{ord}_P\, v < 0 \\ \mathrm{ord}_P\, v \geq 0. \end{cases}$$

There is a unique derivation on the algebraic closure $\overline{K}$ of K that extends the given derivation on K, and this induces a differential field structure on its subfields $\overline{k}$ and $\overline{k}(K)$. We claim that $\overline{k}$ and $\overline{k}(K)$ have the same field of constants. For suppose that $c \in \overline{k}(K)$ is constant. Write

$$c^N + a_1 c^{N-1} + \ldots + a_N = 0,$$

where $a_1, \ldots, a_N \in K$ and N is minimal. Differentiating gives

$$a_1' c^{N-1} + \ldots + a_N' = 0,$$

which contradicts the minimality of N unless $a_1' = \ldots = a_N' = 0$. Thus each a_i is a constant in K, hence in k, and thus $c \in \overline{k}$. Now it is known that there is at least one

$\bar{k}$-place P' of $\bar{k}(K)$ lying over P, that the function induced on K by $\mathrm{ord}_{P'}$ is just ord_P, and that $e_{P'}=e_P$. We may therefore, if necessary, replace k, K, P by $\bar{k}$, $\bar{k}(K)$, P' respectively to obtain the simplifying hypothesis $k=\bar{k}$. That is, we may, without loss of generality, assume that k is algebraically closed.

By Lemma 1, the derivation on K extends to a continuous derivation on the completion K_P of K with respect to P. For any $w \in K_P$ such that $\mathrm{ord}_P\, w > 0$ we can define $\exp w$ and $\log(1+w)$ by the usual series, and we get $\exp\log(1+w)=1+w$, $\log\exp w = w$, $(\exp w)'=(\exp w)w'$, $(\exp w_1)(\exp w_2)=\exp(w_1+w_2)$. In the work below we shall work with a specific $x\in K_P$ (the choice of x depending on the special case we are in) such that $\mathrm{ord}_P\, x=1$. We then have a natural identification of K_P with $k((x))$, the field of formal power series in x with coefficients in k, and can write

$$u_i = g_i x^{\mu_i} \exp\Big(\sum_{\nu>0} h_{i\nu} x^\nu\Big), \qquad i=1, \ldots, n$$
$$v = \sum_{\nu\geq\gamma} h_\nu x^\nu,$$

where each $g_i, h_{i\nu}, h_\nu \in k$, $g_i \neq 0$, and each $\mu_i \in \mathbf{Z}$. Thus

$$\frac{u_i'}{u_i} = \frac{g_i'}{g_i} + \mu_i \frac{x'}{x} + \sum_{\nu>0} (h_{i\nu}' x^\nu + \nu h_{i\nu} x^{\nu-1} x')$$

and
$$v' = \sum_{\nu\geq\gamma} (h_\nu' x^\nu + \nu h_\nu x^{\nu-1} x').$$

In the work below we write $e_P = e$ for simplicity.

Assume now that P is not a pole of t (nor a zero of t if $t'/t\in k$). For some $\alpha\in k$ we have $\mathrm{ord}_P(t-\alpha)=e$. In the case $t'\in k$ we choose $x \in K$ such that $t-\alpha = x^e$. This x will satisfy the demand $\mathrm{ord}_P\, x=1$ of the preceding paragraph. We also have $(x^e)'=t'-\alpha'$, an element of k that is nonzero since all the constants in K are in k, so that $\mathrm{ord}_P(x^e)'=0$. In the case $t'/t\in k$, P is not a zero of t, so that $\alpha\neq 0$, and here we choose $x\in K_P$ such that $t-\alpha=\alpha x^e$. We again have $\mathrm{ord}_P\, x=1$, and here $(t/\alpha)'/(t/\alpha)=t'/t-\alpha'/\alpha$, an element of k which is again nonzero, since all the constants of K are in k. Hence $0=\mathrm{ord}_P(t/\alpha)'=\mathrm{ord}_P(1+x^e)'=\mathrm{ord}_P(x^e)'$. Thus in each of the cases with which we are presently dealing (the first grouping of cases in the statement of the lemma) we have $\mathrm{ord}_P\, e x^{e-1}x'=0$, or $\mathrm{ord}_P\, x'=1-e$. The expression above then gives

$$\mathrm{ord}_P\left(\frac{u_i'}{u_i} - \mu_i\frac{x'}{x}\right) \geq 1-e, \qquad \text{so} \qquad \mathrm{ord}_P\left(\sum_{i=1}^n c_i \frac{u_i'}{u_i} - \left(\sum_{i=1}^n c_i\mu_i\right)\frac{x'}{x}\right) > -e.$$

Since $\mathrm{ord}_P\, x'/x = -e$, we get $\qquad \mathrm{ord}_P \sum_{i=1}^n c_i \frac{u_i'}{u_i} \geq -e,$

with the strict inequality holding if and only if $\sum_{i=1}^n c_i\mu_i = 0$, that is, $\mu_1=\ldots=\mu_n=0$. Finally, consider the expression given above for v'. If $h_\gamma\neq 0$ then $\mathrm{ord}_P\, v=\gamma$. If in addition we have $\gamma\neq 0$ then clearly $\mathrm{ord}_P\, v'=\gamma-e=\mathrm{ord}_P\, v-e$. Thus we arrive at all the estimates of the first case.

Now consider the case where $t' = a \in k$ and P is a pole of t. Here $\mathrm{ord}_P\, t = -e$ and we can choose our $x \in K_P$ of order one such that $t = x^{-e}$. Since $a = t' = -ex^{-e-1}x'$ and $a \neq 0$, we have $\mathrm{ord}_P\, x' = e + 1$. This time we deduce immediately that $\mathrm{ord}_P\, u_i'/u_i \geq 0$. As for v, again assume $h_\gamma \neq 0$, so that $\mathrm{ord}_P\, v = \gamma$. If in addition $h_\gamma' \neq 0$, we have $\mathrm{ord}_P\, v' = \gamma$. In general, the terms of order $\gamma + e$ in v' are

$$h'_{\gamma+e}x^{\gamma+e} + \gamma h_\gamma x^{\gamma-1}x' = \left(h'_{\gamma+e} - \frac{\gamma h_\gamma}{e}a\right)x^{\gamma+e},$$

and if h_γ is a constant then since a is not the derivative of an element of k this last expression can be zero only if $\gamma = 0$. Thus if $\mathrm{ord}_P\, v < 0$ we have $\mathrm{ord}_P\, v \leq \mathrm{ord}_P\, v' \leq \mathrm{ord}_P\, v + e$. This in effect ends this part of the proof.

For the last part, where $t'/t = a \in k$ and P is a pole or zero of t, we first note that if P is a zero of t then P is a pole of $1/t$ and that $(1/t)'/(1/t) = -a \in k$, so that we may restrict our attention to the case where P is a pole of t. As before, for a suitable $x \in K_P$ we have $t = x^{-e}$, and we take this x to be our specific element of order one. Here $t' = -ex^{-e-1}x' = at = ax^{-e}$, so that $x' = -ax/e$. Since $a \neq 0$, $\mathrm{ord}_P\, x' = 1$. Using the power series above we see immediately that $\mathrm{ord}_P\, u_i'/u_i \geq 0$, $i = 1, \ldots, n$, and that (provided $h_\gamma \neq 0$) $\mathrm{ord}_P\, v' \geq \gamma = \mathrm{ord}_P\, v$, with the term of order γ in v' being

$$h'_\gamma x^\gamma + \gamma h_\gamma x^{\gamma-1}x' = \left(h'_\gamma - \frac{\gamma h_\gamma}{e}a\right)x^\gamma.$$

This is not zero if $\gamma \neq 0$, for otherwise $(t^\gamma/h_\gamma^e)' = 0$, again contradicting the assumption that all constants of K are in k. This completes the proof.

Theorem. — *Let $k \subset K$ be differential fields of characteristic zero having the same field of constants, with k algebraically closed in K and K a finite algebraic extension of $k(t)$, $t \in K$ being transcendental over k and such that either $t' \in k$ or $t'/t \in k$. Suppose that $c_1, \ldots, c_n$ are constants of k that are linearly independent over the rational numbers $\mathbf{Q}$, that $u_1, \ldots, u_n$ are nonzero elements of K and that $v \in K$. Then if*

$$\sum_{i=1}^{n} c_i \frac{u_i'}{u_i} + v' \in k$$

we have $u_1'/u_1, \ldots, u_n'/u_n, v' \in k$; furthermore, in the case $t' \in k$ we have $u_1, \ldots, u_n \in k$ and $v = ct + d$ where c is a constant of k and $d \in k$, while in the case $t'/t \in k$ we have $v \in k$ and there exist integers $\nu_0, \nu_1, \ldots, \nu_n$, with ν_0 nonzero, such that $u_i^{\nu_0}/t^{\nu_i} \in k$, $i = 1, \ldots, n$.

Consider the differential fields $\overline{k} \subset \overline{k}(K)$, as in the first paragraph of the proof of Lemma 2. It is shown there that the constants of $\overline{k}(K)$ are those of $\overline{k}$, so that all of our assumptions hold for the pair of fields $\overline{k}$, $\overline{k}(K)$. If the theorem holds for this latter pair of fields then we deduce that $u_1'/u_1, \ldots, u_n'/u_n, v' \in \overline{k} \cap K = k$. In the case $t' \in k$ we also deduce that $u_1, \ldots, u_n \in \overline{k} \cap K = k$, and that there exists a constant $c \in \overline{k}$ and an element $d \in \overline{k}$ such that $v = ct + d$; since for any K-automorphism σ of the normal extension field $\overline{k}(K)$ of K we have $ct + d = v = v^\sigma = c^\sigma t + d^\sigma$, which implies, since t is transcendental over k, that $c^\sigma = c$, $d^\sigma = d$, we infer that indeed $c, d \in k$. In the case $t'/t \in k$ we get immediately that $v, u_1^{\nu_0}/t^{\nu_1}, \ldots, u_n^{\nu_0}/t^{\nu_n} \in \bar{k} \cap K = k$. Thus we get everything we wish to prove for k, K from the knowledge of the theorem for $\bar{k}$, $\bar{k}(K)$. Thus we may assume from now on that k is algebraically closed.

Let P be a k-place of K. First suppose that P is not a pole of t nor, if $t'/t \in k$, a zero of t. Since

$$\sum_{i=1}^n c_i \frac{u_i'}{u_i} + v' \in k \qquad \text{we have} \qquad \operatorname{ord}_P \left(\sum_{i=1}^n c_i \frac{u_i'}{u_i} + v' \right) = 0 \quad \text{or} \quad +\infty.$$

From Lemma 2 we deduce that $\operatorname{ord}_P v \geq 0$ and $\operatorname{ord}_P u_1 = \ldots = \operatorname{ord}_P u_n = 0$.

Now let P be a pole of t, and consider the case where $t' = a \in k$. As in the proof of Lemma 2, write $t = x^{-e}$, for a suitable x in the completion K_P, and use the power series expansions given there for $u_1, \ldots, u_n, v$. We get

$$\sum_{i=1}^n c_i \frac{u_i'}{u_i} + v' = \sum_{i=1}^n c_i \left(\frac{g_i'}{g_i} + \mu_i \frac{x'}{x} + \sum_{\nu > 0} (h_{i\nu}' x^\nu + \nu h_{i\nu} x^{\nu-1} x') \right) + \sum_{\nu \geq \gamma} (h_\nu' x^\nu + \nu h_\nu x^{\nu-1} x').$$

Since $t' = a = -e x^{-e-1} x'$, we have $x' = -(a/e) x^{e+1}$. The coefficient of x^e in the above expansion, which must be zero, since we have an element of k, works out to be

$$-\frac{a}{e} \left(\sum_{i=1}^n c_i \mu_i \right) + \sum_{i=1}^n c_i h_{ie}' + h_e'. \qquad \text{Hence} \qquad \frac{a}{e} \left(\sum_{i=1}^n c_i \mu_i \right) = \left(\sum_{i=1}^n c_i h_{ie} + h_e \right)'.$$

If $\sum_{i=1}^n c_i \mu_i \neq 0$, then a is the derivative of an element of k, so that t differs from an element of k by some constant, also an element of k, contrary to the assumption that t is transcendental over k. Thus $\sum_{i=1}^n c_i \mu_i = 0$. From the fact that $c_1, \ldots, c_n$ are linearly independent over $\mathbf{Q}$ we infer $\mu_1 = \ldots = \mu_n = 0$. As a consequence, the order of each u_i at *any* k-place of K is zero. Thus $u_1, \ldots, u_n \in k$. Hence also $u_i' \in k$, so $u_i'/u_i \in k$, therefore also $v' \in k$. Now resume the consideration of a pole P of t. The statement of Lemma 2 tells us that $\operatorname{ord}_P v \geq -e$, so we can write

$$v' = \sum_{\nu \geq -e} \left(h_\nu' x^\nu - \frac{\nu h_\nu a x^{\nu+e}}{e} \right) \in k.$$

It follows that $h_{-e}' = 0$, so h_{-e} is a constant. Furthermore, if we write

$$\alpha = \min \{ \nu \in \mathbf{Z} : h_\nu \neq 0 \text{ and } \nu \neq -e, 0 \},$$

the above expansion gives as the coefficients of x^α and $x^{\alpha+e}$ respectively the quantities h_α' and $h_{\alpha+e}' - \alpha h_\alpha a/e$, and both of these are necessarily zero. From this we deduce that h_α is a constant and a the derivative of an element of k, a contradiction. Thus α cannot exist, and therefore $v = h_{-e} x^{-e} + h_0 = h_{-e} t + h_0$, which ends this part of the proof.

Finally we consider the case $t'/t = a \in k$. Lemma 2 tells us that v cannot have a pole at any pole or zero of t. Thus v has no poles at all. Therefore $v \in k$. Since also $v' \in k$, we are reduced to

$$\sum_{i=1}^{n} c_i \frac{u_i'}{u_i} \in k.$$

Fix a pole P of t. As in the proof of Lemma 2, write $t = x^{-e}$, for a suitable x in the completion K_P, and use the power series expansions given there for $u_1, \ldots, u_n$. Our proof will be complete if we can show that all of the quantities $h_{i\nu}$ are zero. Since $x' = -ax/e$, the expressions for $u_1, \ldots, u_n$ give

$$\sum_{i=1}^{n} c_i \left(\frac{g_i'}{g_i} - \frac{\mu_i a}{e} + \sum_{\nu>0} \left(h_{i\nu}' - \frac{\nu h_{i\nu}}{e} a \right) x^{\nu} \right) \in k.$$

Therefore for each $\nu > 0$, $$(\sum_{i=1}^{n} c_i h_{i\nu})' = \frac{\nu a}{e} \sum_{i=1}^{n} c_i h_{i\nu}.$$

From this it follows that $$((\sum_{i=1}^{n} c_i h_{i\nu})^e / t^{\nu})' = 0.$$

Since each constant of K is in k and t is transcendental over k we deduce that

$$\sum_{i=1}^{n} c_i h_{i\nu} = 0.$$

Defining $\zeta_i = \exp e \sum_{\nu>0} h_{i\nu} x^{\nu} \in K_P$ for $i = 1, \ldots, n$, we have $\zeta_i = u_i^e t^{\mu_i} / g_i^e \in K$. Now for any $w \in K_P$ such that $\mathrm{ord}_P\, w > 0$ and any $c \in k$ we can define $(1+w)^c = \exp(c \log(1+w))$, and a number of more or less obvious identities will hold for such irrational powers. In particular, the above relations on the $h_{i\nu}$'s imply

$$\zeta_1^{c_1} \zeta_2^{c_2} \ldots \zeta_n^{c_n} = 1.$$

Our proof will be complete if we can show that each $\zeta_i = 1$. To do this, let D be a nonzero k-derivation of K, that is, a derivation of K that is zero on k. By Lemma 1, D extends to a continuous derivation of K_P. We thus get

$$\sum_{i=1}^{n} c_i \frac{D\zeta_i}{\zeta_i} = 0.$$

Hence we have an equality for differentials of the algebraic function field K over k

$$\sum_{i=1}^{n} c_i \frac{d\zeta_i}{\zeta_i} = 0.$$

In particular, the sum of the residues at any given k-place of K of $c_1 d\zeta_1/\zeta_1, \ldots, c_n d\zeta_n/\zeta_n$ is zero. But the residue at any k-place of K of $d\zeta_i/\zeta_i$ is the order at that place of ζ_i. Since $c_1, \ldots, c_n$ are linearly independent over $\mathbf{Q}$, the order at any k-place of K of each ζ_i is zero. Hence $\zeta_1, \ldots, \zeta_n \in k$. Since $\mathrm{ord}_P(\zeta_i - 1) > 0$ for $i = 1, \ldots, n$, we have $\zeta_1 = \ldots = \zeta_n = 1$, and this ends the proof.

The preceding theorem is a powerful tool for finding elementary solutions, if such exist, of certain types of transcendental equations, or for proving their nonexistence. As a simple illustration of its use, we give the following specific application.

Recall that a *liouvillian extension* of a differential field k is a differential field K of the type $K = k(t_1, t_2, \dots, t_\nu)$, where for each $i = 1, 2, \dots, \nu$, either $t_i' \in k(t_1, t_2, \dots, t_{i-1})$, or $t_i'/t_i \in k(t_1, t_2, \dots, t_{i-1})$, or t_i is algebraic over $k(t_1, t_2, \dots, t_{i-1})$. In other words, a liouvillian extension of a differential field is an extension field obtainable by a repeated adjunction of antiderivatives, exponentials, and algebraic elements.

Proposition. — Let k be a differential field of characteristic zero and let $y_1, y_2, \dots, y_n$, $z_1, z_2, \dots, z_n$ be elements of a liouvillian extension field of k having the some subfield of constants as k. Suppose that

$$\frac{y_i'}{y_i} = z_i', \qquad i = 1, \dots, n,$$

and that $k(y_1, \dots, y_n, z_1, \dots, z_n)$ is algebraic over each of its subfields $k(y_1, \dots, y_n)$ and $k(z_1, \dots, z_n)$. Then $y_1, \dots, y_n, z_1, \dots, z_n$ are all algebraic over k.

For suppose that $y_1, \dots, y_n, z_1, \dots, z_n$ are elements of the liouvillian extension field K of k, where K and k have the same field of constants. Then we have a chain of differential fields $k \subset K_0 \subset K_1 \subset \dots \subset K_\nu = K$ such that K_0 is algebraic over k and for each $i = 1, \dots, \nu$, the field K_i is either algebraic over K_{i-1} or algebraic over $K_{i-1}(t_i)$, for some $t_i \in K_i$ such that either t_i' or t_i'/t_i is in K_{i-1}. Replacing each K_i by its algebraic closure in K, we may assume that each field K_{i-1} is algebraically closed in K_i. Replacing k by K_0, if necessary, we may assume that $k = K_0$. Eliminating repetitions in the chain $K_0 \subset K_1 \subset \dots \subset K_\nu$, we may assume that each K_i is transcendental over K_{i-1}. Thus each field K_{i-1} is algebraically closed in K_i, which has the same subfield of constants as K_{i-1} and is a finite algebraic extension of $K_{i-1}(t_i)$ for some $t_i \in K_i$ such that t_i is transcendental over K_{i-1} and either t_i' or t_i'/t_i is in K_{i-1}. Using induction on ν, we are reduced to proving the proposition in the special case that the liouvillian extension field K of k which contains $y_1, \dots, y_n, z_1, \dots, z_n$ satisfies the conditions on the extension field K of the theorem, and in this case we have to show that $y_1, \dots, y_n, z_1, \dots, z_n \in k$. If in the statement of the theorem the hypothesis $t' \in k$ holds, then the theorem implies that $y_1, \dots, y_n \in k$, while if the alternate hypothesis $t'/t \in k$ holds, then we get $z_1, \dots, z_n \in k$. Since $k(y_1, \dots, y_n, z_1, \dots, z_n)$ is algebraic over both $k(y_1, \dots, y_n)$ and $k(z_1, \dots, z_n)$ and one of these latter fields is k, we get each y_i and each z_i algebraic over k, hence in k. Thus the proposition.

The condition that $k(y_1, \dots, y_n, z_1, \dots, z_n)$ be algebraic over each of its subfields $k(y_1, \dots, y_n)$ and $k(z_1, \dots, z_n)$ comes about as a matter of course in at least two special cases of practical moment. In one case, there is a y_0 in some extension field of $k(y_1, \dots, y_n, z_1, \dots, z_n)$ such that y_0 and z_1 are algebraically interdependent over k (that is, z_1 is algebraic over $k(y_0)$ and y_0 is algebraic over $k(z_1)$), y_1 and z_2 are algebraically interdependent over $k(y_0)$, y_2 and z_3 are algebraically interdependent over $k(y_0, y_1)$, $\dots, y_{n-1}$

and z_n are algebraically interdependent over $k(y_0, y_1, \ldots, y_{n-2})$, and finally y_n and y_0 are algebraically interdependent over $k(y_1, y_2, \ldots, y_{n-1})$. In the other case, there is a z_{n+1} in some extension field of $k(y_1, \ldots, y_n, z_1, \ldots, z_n)$ such that z_{n+1} and y_n are algebraically interdependent over k, z_n and y_{n-1} are algebraically interdependent over $k(z_{n+1})$, $\ldots$, z_2 and y_1 are algebraically interdependent over $k(z_{n+1}, z_n, \ldots, z_3)$, and finally z_1 and z_{n+1} are algebraically interdependent over $k(z_n, z_{n-1}, \ldots, z_2)$. Both these cases are illustrated below.

Of greatest interest are differential fields of meromorphic functions of a complex variable x, that is, fields of meromorphic functions on a region of the x-sphere which are closed with respect to the usual derivation d/dx and which contain $\mathbf{C}$. These fields all have $\mathbf{C}$ as their subfield of constants. Their most notable examples are the field $\mathbf{C}$ itself and the field $\mathbf{C}(x)$ of rational functions of the variable x. In applications one often uses the fact that if u, v are meromorphic functions on a region of the x-sphere that are algebraic over $\mathbf{C}(x)$ and $u = \exp v$, then necessarily $u, v \in \mathbf{C}$. (This fact comes from the case $k = \mathbf{C}$, $n = 1$ of the proposition. More directly, it is easily proved by looking at the residues of the exact differential $dv = du/u$ at the various places of the field $\mathbf{C}(x, u, v)$. For any place P, $0 = \text{res}_P\, dv = \text{res}_P\, du/u = \text{ord}_P\, u$. Therefore u has no zeros or poles, hence is constant. Hence also v is constant.)

Thus as a special consequence of the proposition there is a large class of transcendental equations connecting variables x and y that have no solution in any field of meromorphic functions of x that is a liouvillian extension of $\mathbf{C}(x)$. Specific examples that were considered by Liouville himself are the equations

$$y = e^{y/x}$$

and

$$y - a \sin y = x, \quad a \in \mathbf{C}, \quad a \neq 0$$

(Kepler's equation, [2, p. 56]). A more complicated example, worse than anything considered by Liouville, is

$$y(\log(x + \sqrt{y}))^2 + 5 \log((x-y)^{1/5} + (xy^3 + \log(x + \sqrt{y}))^{1/3}) = 17 x^5 y^2.$$

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