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ROBIN HARTSHORNE

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CURVES WITH HIGH SELF-INTERSECTION ON ALGEBRAIC SURFACES

by Robin HARTSHORNE, Harvard University

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§ 0. Introduction.

If X is a subvariety of an algebraic variety V, one can ask to what extent V is determined by its formal completion $\hat{V}$ along X. This question has been studied by Chow [1], Hironaka [7], Matsumura [8], and myself [6]. A typical result states that if X is suitably ample, for example if X is an ample divisor on V, or V is a projective space and $\dim X \geq 1$, then V is determined birationally by the formal scheme $\hat{V}$.

In this paper we ask, to what extent is V determined by X and its normal bundle $\mathscr{N}$ in V, if $\mathscr{N}$ is sufficiently ample? Given a variety X and a vector bundle $\mathscr{N}$ on X, there is always at least one embedding of X with normal bundle $\mathscr{N}$, namely the zero-section of the vector bundle itself. So we ask, how similar is the embedding of X in V to the zero-section of the vector bundle $\mathscr{N}$?

We will say that two embeddings $X \to V_1$ and $X \to V_2$ are *equivalent* if there is a birational map $f: V_1 \to V_2$ which is an isomorphism on an open neighborhood of X in V_1, and induces the identity on X.

It seems that the most interesting case is when X is a curve on a surface F. For if X is a curve, and $\dim V \geq 3$, one can say very little, whereas if $\dim X \geq 2$, one can use obstruction theory to obtain a strong uniqueness result (see § 4 below). Hence our main interest is to classify embeddings of curves on surfaces with sufficiently ample normal bundle, which in this case means with sufficiently high self-intersection. The embedding is never unique, because for any geometrically ruled surface $\pi: \mathbf{P}(\mathscr{E}) \to C$ (see § 1) one can find sections with arbitrarily high self-intersection. However, our main Theorem (4.1) asserts that these are the only possibilities. We show that if X is a non-singular curve of genus g on a non-singular surface F, with $X^2 > 4g+5$, then the embedding $X \to F$ is equivalent to a section of a geometrically ruled surface. The same statement is true if we assume only $X^2 \geq 4g+5$, with a single exception, namely the non-singular cubic curve in the projective plane.

For the proof, we first use Enriques' classification of surfaces, to show that F is a (birationally) ruled surface. Next we study irreducible (possibly singular) curves Y on a minimal model F′ of a ruled surface, to obtain bounds on Y^2 in terms of $p_a(Y)$. Finally, we examine what happens when we blow up points of F′ to recover the original surface F.

As a by-product of our method, we obtain a new proof (3.3) of the theorem of Noether-Nagata which gives a classification of the minimal models of rational surfaces.

§ 1. Preliminaries.

We fix an algebraically closed base field k of arbitrary characteristic. A *curve* or *surface* is an integral scheme proper over k, of dimension one or two. A *ruled surface* is a surface which is birational to a product $\mathbf{P}^1 \times C$, where C is a curve.

If C is a non-singular curve, and $\mathscr{E}$ is a locally free sheaf of rank two on C, we consider $\mathbf{P}(\mathscr{E})$, the associated $\mathbf{P}^1$-bundle of $\mathscr{E}$ [EGA, II.4.1]. We call the triple $(\mathbf{P}(\mathscr{E}), \pi, C)$ a *geometrically ruled surface*, where $\pi : \mathbf{P}(\mathscr{E}) \to C$ is the projection. Note that two locally free sheaves $\mathscr{E}_1$ and $\mathscr{E}_2$ give rise to isomorphic geometrically ruled surfaces if and only if $\mathscr{E}_1 \cong \mathscr{E}_2 \otimes \mathscr{L}$ for some invertible sheaf $\mathscr{L}$ on C. If C is the projective line, then every locally free sheaf is a direct sum of invertible sheaves [2], so we get a complete set of geometrically ruled surfaces by taking the surfaces $F_e = \mathbf{P}(\mathscr{O} \oplus \mathscr{O}(e))$, for $e = 0, 1, 2, \ldots$, where $\mathscr{O}(1)$ is an invertible sheaf of degree one.

If $\mathscr{L} \subseteq \mathscr{E}$ is a sub-line bundle, i.e., an invertible subsheaf of $\mathscr{E}$ such that $\mathscr{E}/\mathscr{L}$ is also invertible, then $\mathscr{L}$ gives rise to a sheaf of homogeneous ideals $\sum_{n \geq 1} \mathscr{L}^n$ of the sheaf of graded algebras $\sum_{n \geq 0} S^n(\mathscr{E})$, and hence a subscheme $D \subseteq \mathbf{P}(\mathscr{E})$, which is actually a section of the projection $\pi : \mathbf{P}(\mathscr{E}) \to C$. One finds that the normal bundle to D is $(\mathscr{E}/\mathscr{L}) \otimes \mathscr{L}^{\vee}$, and so $D^2 = \deg \mathscr{E} - 2 \deg \mathscr{L}$. There are arbitrarily negative sub-line bundles of $\mathscr{E}$, and hence there are sections D with D^2 arbitrarily large.

If D is a section of $\mathbf{P}(\mathscr{E})$, and if f denotes a fibre of the projection π, then the group of divisors modulo numerical equivalence on $\mathbf{P}(\mathscr{E})$ is the free abelian group generated by D and f. Thus any divisor Y is numerically equivalent to some combination

$$Y \equiv mD + nf, \qquad m, n \in \mathbf{Z}.$$

If we denote D^2 by d, then an easy calculation gives

$$Y^2 = m^2 d + 2mn$$
$$2p_a(Y) - 2 = m^2 d + 2mn - md - 2n + m(2\gamma - 2),$$

where $\gamma =$ genus C.

If $C = \mathbf{P}^1$, we will use D to denote the section of F_e corresponding to the sub-line bundle $\mathscr{O} \subseteq \mathscr{O} \oplus \mathscr{O}(e)$, so that $D^2 = e$. We denote by D_e the section of F_e, for $e \geq 1$, corresponding to the sub-line bundle $\mathcal{O}(e) \subseteq \mathcal{O} \oplus \mathcal{O}(e)$. Thus $D_e^2 = -e$. Note $D_e \equiv D - ef$. If Y is a curve on F_e, with $Y \equiv mD + nf$, then the formulae above become

$$Y^2 = m^2 e + 2mn$$
$$2p_a(Y) - 2 = m^2 e + 2mn - me - 2n - 2m.$$

Note that if Y is an effective curve on any geometrically ruled surface, then $m \geq 0$. If Y is an effective curve on a geometrically ruled rational surface F_e, which does not contain the section D_e as a component, then also $n \geq 0$, because

$$\begin{aligned} Y.D_e &= (mD + nf).(D - ef) \\ &= me + n - me = n, \end{aligned}$$

and $Y.D_e \geq 0$.

Now we define some rational maps which we will use later. If F is a non-singular surface, and $P \in F$ a point, we denote by $\mathrm{dil}_P : F \to F'$ the blowing-up, or dilatation, at P. If $Y \subseteq F$ is an exceptional curve of the first kind, we denote by $\mathrm{cont}_Y : F \to F'$ the blowing-down, or contraction, of Y. If $\pi : \mathbf{P}(\mathcal{E}) \to C$ is a geometrically ruled surface, and P is a point of $\mathbf{P}(\mathcal{E})$ we define elm_P to be the result of blowing up P, and then blowing down the fibre of π passing through P. Let $\mathcal{O}(1)$ be the tautological invertible sheaf on $\mathbf{P}(\mathcal{E})$. Then $\pi_*(\mathcal{I}_P(1)) = \mathcal{E}'$ is a locally free sheaf of rank two on C, which is a subsheaf of $\pi_*(\mathcal{O}(1)) = \mathcal{E}$. One sees easily that

$$\mathrm{elm}_P : \mathbf{P}(\mathcal{E}) \to \mathbf{P}(\mathcal{E}')$$

is the rational map induced from the inclusion of sheaves $\mathcal{E}' \subseteq \mathcal{E}$.

On the rational surfaces F_e, we define two more rational maps: $\mathrm{refl} : F_0 \to F_0$ is the map which interchanges the two families of lines on $F_0 = \mathbf{P}^1 \times \mathbf{P}^1$; and $\mathrm{int}_P : F_1 \to F_1$, for a point $P \notin D_1$, consists of blowing up P and blowing down D_1. The maps elm_P, refl, and int_P are called *elementary transformations*.

We call a non-singular surface F a *minimal model* if every birational morphism $F \to F'$ to a non-singular surface F' is an isomorphism. Using Castelnuovo's criterion for exceptional curves of the first kind, one sees easily that every geometrically ruled surface is a minimal model, with the exception of F_1. The surface F_1 has a unique exceptional curve of the first kind, D_1, and $\mathrm{cont}_{D_1}(F_1) = \mathbf{P}^2$. Later ((2.1) and (3.3)) we will prove the converse, which is the classification of minimal models of ruled and rational surfaces.

We will say that a rational map is a *morphism*, if it is regular at every point. For basic results on birational maps of surfaces, we refer to [14].

§ 2. Curves on non-rational ruled surfaces.

Theorem **(2.1)** (1). — *Let* C *be a non-singular curve of genus* $\gamma > 0$. *Then any minimal model of the function field* $K = k(C)(t)$ *is a geometrically ruled surface* $\pi : \mathbf{P}(\mathcal{E}) \to C$, *for a*

(1) This result seems to be well known, but the only published proof we could find ([13], chap. V, Th. 1 *a*, p. 86) is very complicated, so we include a proof here. It was also proved by Knapp ([9], chap. II).

suitable locally free sheaf of rank two on C. *Furthermore, any birational map between two such minimal models* $\mathbf{P}(\mathscr{E})$ *and* $\mathbf{P}(\mathscr{E}')$ *can be factored into a product of an automorphism of* $\mathbf{P}(\mathscr{E})$ *and a finite number of elementary transformations.*

To prove the theorem, it will be sufficient to prove the following

Lemma (**2.2**). — *Let* F *be any non-singular model of* K, *and let* $f' : \mathrm{F} \to \mathbf{P}(\mathscr{E}')$ *be any birational map. Then there is a finite product of elementary transformations* $g : \mathbf{P}(\mathscr{E}') \to \mathbf{P}(\mathscr{E})$, *such that* $f = g \circ f'$ *is a morphism.*

Proof. — Consider all possible birational maps $f = g \circ f'$, where g is a product of elementary transformations. For any such f, we let $\lambda(f)$ be the least number of times one must blow up a point of F in order to obtain a surface G which dominates $\mathbf{P}(\mathscr{E})$. Now choose g so that $\lambda(f)$ is minimum. We will prove that $\lambda = 0$, i.e., f is a morphism.

If $\lambda > 0$, then G contains an irreducible exceptional curve X of the first kind such that $\mathrm{cont}_{\mathrm{X}}(\mathrm{G})$ dominates F. Let Y be the image of X in $\mathbf{P}(\mathscr{E})$. If Y is a single point, then $\mathrm{cont}_{\mathrm{X}}(\mathrm{G})$ also dominates $\mathbf{P}(\mathscr{E})$, which contradicts the minimality of λ. Hence Y is a curve. Since Y is rational, and the base curve C is non-rational, Y must be a fibre of $\mathbf{P}(\mathscr{E})$. Now $\mathrm{X}^2 = -1$, and $\mathrm{Y}^2 = 0$, so there is a point $\mathrm{P} \in \mathrm{Y}$ such that G dominates $\mathrm{dil}_{\mathrm{P}}(\mathbf{P}(\mathscr{E}))$. Hence G also dominates $\mathrm{elm}_{\mathrm{P}}(\mathbf{P}(\mathscr{E}))$. But the image of X in $\mathrm{elm}_{\mathrm{P}}(\mathbf{P}(\mathscr{E}))$ is a point, so $\mathrm{cont}_{\mathrm{X}}(\mathrm{G})$ dominates $\mathrm{elm}_{\mathrm{P}}(\mathbf{P}(\mathscr{E}))$, which again contradicts the minimality of λ.

Before stating the next result, we observe that for any model F of the function field $\mathrm{K} = k(\mathrm{C})(t)$, the inclusion $k(\mathrm{C}) \to \mathrm{K}$ induces a rational map $\mathrm{F} \to \mathrm{C}$, which is necessarily a morphism. Furthermore, we can always find a morphism of F to a minimal model F′, compatible with the projections to C.

Theorem (**2.3**). — *Let* X *be an irreducible curve on a non-rational ruled surface* F. *Let* m *be the degree of the projection* $\pi : \mathrm{X} \to \mathrm{C}$, *and assume* $m > 1$. *Then*

$$\mathrm{X}^2 \leq \frac{2m}{m-1}(p_a(\mathrm{X}) - 1).$$

Proof. — We first consider the case when F is a minimal model. Then by the previous theorem, F is a geometrically ruled surface $\mathbf{P}(\mathscr{E})$, and from the discussion of the previous section, we have $\mathrm{X} \equiv m\mathrm{D} + nf$, where D is a section of F, and f is a fibre of the projection $\pi : \mathrm{F} \to \mathrm{C}$. Note that the integer m is the one defined above.

We will use the formulae of section one,

$$\mathrm{X}^2 = m^2 d + 2mn$$
$$2p_a(\mathrm{X}) - 2 = m^2 d + 2mn - md - 2n + m(2\gamma - 2),$$

where γ is the genus of C.

If $p_a(\mathrm{X}) \leq 1$, then from the existence of the finite map $\pi : \mathrm{X} \to \mathrm{C}$, we conclude $p_a(\mathrm{X}) = 1 = \gamma$. In that case

$$0 = m^2 d + 2mn - md - 2n$$
$$= (m-1)(md + 2n),$$

so that either $m=1$, which contradicts the hypothesis $m>1$, or $md+2n=0$, which implies $X^2=0$, and the inequality holds.

Now we assume $p_a(X)>1$. If $X^2\leq 0$, there is nothing to prove, so we may assume also $X^2>0$. Then a simple calculation shows that

$$\frac{X^2}{2p_a(X)-2}=\frac{m}{m-1+\alpha}$$

where

$$\alpha=\frac{m(2\gamma-2)}{md+2n}.$$

But $\gamma\geq 1$, so $\alpha\geq 0$, and we have

$$\frac{X^2}{2p_a(X)-2}\leq\frac{m}{m-1},$$

which is the desired result, in case F is a minimal model.

For the general non-rational ruled surface F, we use induction on the number of irreducible exceptional curves one must contract on F to obtain a minimal model. Let $F\to F'$ be a single contraction, so that $F=\mathrm{dil}_P(F')$ for a suitable point $P\in F'$. Let Y be the image of X in F'. Then by induction, we may assume the result true for Y, i.e.,

$$Y^2\leq\frac{2m}{m-1}(p_a(Y)-1).$$

If $r\geq 0$ is the multiplicity of the point P for Y, then we have

$$X^2=Y^2-r^2$$

$$p_a(X)=p_a(Y)-\frac{1}{2}r(r-1).$$

Thus we find

$$X^2\leq\frac{2m}{m-1}(p_a(X)-1)+\frac{m}{m-1}r(r-1)-r^2.$$

But since Y admits a finite morphism $\pi:Y\to C$ of degree m to a non-singular curve C, the multiplicity of any point on Y is bounded by m. So $r\leq m$, and

$$\frac{m}{m-1}r(r-1)-r^2=\frac{r}{m-1}(r-m)\leq 0,$$

which completes the proof.

Corollary (**2.4**). — *Let* X *be a non-singular curve of genus* $g>0$ *on a non-rational ruled surface* F. *Then either*

a) $X\cong C$, *and the embedding* $X\to F$ *is equivalent to a section of the geometrically ruled surface* $\pi:\mathbf{P}(\mathscr{E})\to C$, *or*

b) $X^2\leq 4g-4$.

Proof. — Let m be the degree of the projection $\pi : X \to C$. If $m=0$, then X must be a rational curve, which is impossible. On the other hand, if $m \geq 2$, then $\dfrac{m}{m-1} \leq 2$, so the theorem implies that $X^2 \leq 4g-4$.

If $m=1$, then $\pi : X \to C$ is an isomorphism, and we will show that the embedding $X \to F$ is equivalent to a section of a geometrically ruled surface. Consider morphisms $f : F \to \mathbf{P}(\mathscr{E})$ of F to geometrically ruled surfaces $\mathbf{P}(\mathscr{E})$, which exist by Theorem (2.1). For any such morphism f, $f(X)$ is a curve of degree one over C, hence a section of $\mathbf{P}(\mathscr{E})$ over C, and we have $X^2 \leq f(X)^2$. Choose f so that $f(X)^2$ is minimum. Then $X^2 = f(X)^2$. Indeed, if $X^2 < f(X)^2$, then there is a point $P \in f(X)$ which is fundamental for f. So $f' = \mathrm{elm}_P \circ f$ is also a morphism, and $f'(X)^2 = f(X)^2 - 1$, which contradicts the minimality of $f(X)^2$. It follows that f^{-1} is regular at every point of $f(X)$, and so f is an isomorphism in a neighborhood of X. So the embedding $X \to F$ is equivalent to the embedding $X \to f(X) \subseteq \mathbf{P}(\mathscr{E})$, and we are done.

Example. — Assume char. $k=0$, let C be a non-singular elliptic curve, and let $F = \mathbf{P}^1 \times C$. Let P_0 be a point of $\mathbf{P}^1$, and let $\mathfrak{a}$ be a divisor of degree $n \geq 2$ on C. Then by Bertini's theorem, the general member X of the complete linear system $|2P_0 \times C + \mathbf{P}^1 \times \mathfrak{a}|$ is irreducible and non-singular. One finds that the genus of X is $g = n+1$, and the self-intersection is $X^2 = 4g-4$. Thus the bounds of the Corollary are the best possible for curves on a ruled surface of genus $\gamma = 1$. Presumably one could get better bounds on ruled surfaces of higher genus.

§ 3. Curves on rational surfaces.

Proposition **(3.1)**. — *Let* F *be a rational surface which admits a morphism* $f_0 : F \to F_{e_0}$ *to one of the surfaces* F_{e_0} *(see* § 1*), and let* X *be a non-singular curve on* F. *Then there is a finite product* $g : F_{e_0} \to F_e$ *of elementary transformations, such that* $f = g \circ f_0$ *is a morphism, and such that* $Y = f(X)$ *is a curve with the following properties, where m and n are the integers defined by the relation* $Y \equiv mD + nf$:

a) *every multiple point of* Y *has multiplicity* $\leq \frac{1}{2}m$;
b) *if* $e=0$, *then* $m \leq n$;
c) *if* $e=1$, *then every multiple point of* Y *has multiplicity* $\leq n$;
d) *if* $m=1$, *then* $X^2 = Y^2$;
e) *if* $e=1$, $m=2$, *and* $n=0$, *then* $X^2 = Y^2$.

Proof. — We consider all products g of elementary transformations such that $g \circ f_0$ is a morphism. If $f_0(X)$ is a point P, we apply elm_P, and after a finite number of such steps, we may assume $g(f_0(X))$ is a curve. Among all such g, we choose one which first minimizes m, and secondly minimizes Y^2. This is possible, because for any morphism $f : F \to F_e$, we have $m(f(X)) \geq 0$, and $X^2 \leq f(X)^2$. Now we will show that the morphism $f = g \circ f_0$ has the required properties.

a) If Y has a multiple point P of multiplicity $r>\frac{1}{2}m$, we apply elm_P. Since X is non-singular, P is fundamental for f, so $f'=\mathrm{elm}_P\circ f$ is also a morphism. This blows up P, but also introduces a new multiple point P′ of multiplicity $r'=m-r$. Denoting the new situation with primes, we find $m'=m$, and

$$Y'^2=Y^2-r^2+r'^2<Y^2,$$

which contradicts the minimality of Y^2.

b) If $e=0$, we consider $\mathrm{refl}\circ f$. This interchanges the roles of m and n, so by minimality of m, we conclude $m\leq n$.

c) If $e=1$, let P be a multiple point of Y, of multiplicity r. If $P\in D_1$, then $r\leq Y.D_1=n$. If $P\notin D_1$, we consider $f'=\mathrm{int}_P\circ f$. Then $m'=Y.D-r=m+n-r$, so by the minimality of m, we have $r\leq n$.

d) If $m=1$ and $X^2<Y^2$, then there is a point $P\in Y$ which is fundamental for f. We apply elm_P, and find $m'=m=1$, and $Y'^2=Y^2-1$, which contradicts the minimality of Y^2.

e) Suppose $e=1$, $m=2$, $n=0$, and $X^2<Y^2$. Then again there is a point $P\in Y$ which is fundamental for f. P cannot be in D_1, since $Y.D_1=n=0$. So we apply int_P, and obtain $m'=m+n-1=1$, which contradicts the minimality of m.

Proposition **(3.2)**. — *Let* X *be an irreducible curve on a rational surface* F, *and suppose there is a morphism* $f: F\to F_e$ *having the properties* a), b), c), d), e) *of the previous proposition. Then*

(i) *if* $m=2$, *then* X *is non-singular, and* $X^2\leq 4g+4$;

(ii) *if* $m=3$, *then* X *is non-singular, and* $X^2\leq 3g+6$;

(iii) *if* $m\geq 4$ *and either* $e\neq 1$, *or* $e=1$ *and* $n\geq\frac{1}{2}m$, *then*

$$X^2\leq\frac{2m}{m-2}(p_a(X)-1);$$

(iv) *if* $m\geq 4$ *and* $e=1$ *and* $n<\frac{1}{2}m$, *then*

$$X^2<\frac{2k}{k-3}(p_a(X)-1),$$

where $k=m+n$.

Proof. — We consider first the case when $F=F_e$. Then we will apply the formulae of § 1

$$X^2=m^2e+2mn$$
$$2p_a(X)-2=m^2e+2mn-me-2m-2n.$$

(i) If $m=2$, X is non-singular by *a)*, and we have

$$X^2=4e+4n$$
$$2p_a(X)-2=2e+2n-4,$$

so

$$X^2=4p_a(X)+4.$$

(ii) If $m=3$, X is non-singular by *a)*, and we have

$$X^2=9e+6n$$
$$2p_a(X)-2=6e+4n-6,$$

so

$$X^2=3p_a(X)+6.$$

(iii) If $m\geq 4$, then

$$2p_a(X)=(m-1)(me+2n-2)$$
$$\geq 3(4e+2n-2).$$

If $e=0$, then $n\geq m\geq 4$, so this is at least 18. If $e>0$, it is at least 6. In either case, $p_a(X)>1$, so we can consider

$$\frac{X^2}{2p_a(X)-2}=\frac{m}{m-1-\alpha},$$

where

$$\alpha=\frac{2m}{me+2n}.$$

We will show $\alpha\leq 1$, which implies

$$\frac{m}{m-1-\alpha}\leq\frac{m}{m-2},$$

and hence the result. Indeed, if $e=0$, then $m\leq n$, so $\alpha\leq 1$. If $e=1$, then $n\geq\frac{1}{2}m$, so $\alpha\leq 1$. If $e>1$, then $n\geq 0$, so $\alpha\leq 1$.

(iv) If $m\geq 4$, and $e=1$, we have

$$2p_a(X)-2=m(m-3)+2n(m-1)\geq 4.$$

So we can consider the quotient $X^2/(2p_a(X)-2)$. Substituting $m=k-n$, we find

$$\frac{X^2}{2p_a(X)-2}=\frac{k}{k-3+\beta}$$

where

$$\beta=\frac{kn-3n^2}{k^2-n^2}.$$

Now we are assuming that $n<\frac{1}{2}m$, so $3n<k$, and $\beta>0$. Hence

$$X^2<\frac{2k}{k-3}(p_a(X)-1),$$

as required.

This completes the proof in case $F=F_e$. For the general case, we use induction on the number of dilatations one must apply to F_e in order to obtain F. So let us consider a morphism $F\to F'$ over F_e, such that $F=\mathrm{dil}_P(F')$ for a suitable point $P\in F'$. Let Y be the image of X in F'. By induction, we may assume the results true for Y.

(i) and (ii) Since Y is non-singular, X is also, the genus is the same, and $X^2\leq Y^2$.

(iii) Let $r \geq 0$ be the multiplicity of the point P on Y. Then

$$X^2 = Y^2 - r^2$$

$$p_a(X) = p_a(Y) - \frac{1}{2} r(r-1),$$

so we find
$$X^2 \leq \frac{2m}{m-2}(p_a(X)-1) + \frac{m}{m-2} r(r-1) - r^2.$$

But by condition *a)*, we have $r \leq \frac{1}{2} m$, so

$$\frac{m}{m-2} r(r-1) - r^2 = \frac{r}{m-2}(2r-m) \leq 0,$$

which gives the result.

(iv) Again let r be the multiplicity of P on Y. Then we find

$$X^2 \leq \frac{2k}{k-3}(p_a(X)-1) + \frac{k}{k-3} r(r-1) - r^2.$$

In this case, by condition *c)*, we have $r \leq n$, so $r \leq \frac{1}{3} k$, and

$$\frac{k}{k-3} r(r-1) - r^2 = \frac{r}{k-3}(3r-k) \leq 0,$$

which gives the result.

Theorem **(3.3)** (Nagata [11]) (1). — *Any minimal model of a rational function field* $K = k(t, u)$ *is isomorphic to* $\mathbf{P}^2_k$, *or to one of the surfaces* F_e, $e \neq 1$. *Furthermore, any birational map between two such minimal models can be factored into a product of an automorphism of the first, and a product of a finite number of elementary transformations and maps of the form* $\mathrm{dil}_P : \mathbf{P}^2 \to F_1$ *and* $\mathrm{cont}_{D_1} : F_1 \to \mathbf{P}^2$.

For the proof, it will be sufficient to prove the following

Lemma **(3.4)**. — *Let* F *be any rational surface, and let* $f' : F \to F_{e'}$ *be a birational map. Then there is a finite product of elementary transformations* $g : F_{e'} \to F_e$ *such that either* $f = g \circ f'$ *is a morphism, or* $F = \mathbf{P}^2$, $e = 1$, *and* $f = \mathrm{dil}_P \circ h$, *for a suitable automorphism* $h : F \to F$.

Proof. — Consider all possible birational maps $f = g \circ f'$, where $g : F_{e'} \to F_e$ is a product of elementary transformations. For any such f, let $\lambda(f)$ be the least number of dilatations one must apply to F to obtain a surface G dominating F_e. Choose g so that $\lambda(f)$ is minimum. Then we will show that either $\lambda(f) = 0$, in which case f is a morphism, or $\lambda(f) = 1$, $F = \mathbf{P}^2$, and $e = 1$.

Suppose $\lambda(f) > 0$. Then blow up $\lambda(f)$ points of F, to obtain a surface G dominating F_e. Let X be an irreducible exceptional curve of the first kind on G such that $\mathrm{cont}_X(G)$ dominates F. Applying Proposition (3.1), we may assume without loss of

(1) This theorem was also proved by Knapp ([9], chap. III).

generality that the morphism $\varphi: G \to F_e$ has the properties *a), b), c), d), e)* of the proposition.

Now consider $Y=\varphi(X)$ in F_e. We distinguish various cases, according to the value of m.

$m=0$. — Then Y is a ruling, $Y^2=0$, so there is a point $P \in Y$ which is fundamental for φ (since $X^2=-1$). Then $\mathrm{cont}_X(G)$ dominates $\mathrm{elm}_P(F_e)$, which contradicts the minimality of $\lambda(f)$.

$m=1$. — In this case $Y^2=-1$, by *d)*, so we must have $e=1$, $Y=D_1$. If φ is an isomorphism, then $F=\mathbf{P}^2$, $\lambda(f)=1$, and we are done. Otherwise, for some point $P \in F_1$, $P \notin D_1$, $\mathrm{cont}_X(G)$ dominates $\mathrm{int}_P(F_1)$ which contradicts the minimality of $\lambda(f)$.

$m=2$. — In this case Y is non-singular, so $p_a(Y)=0$, and $e+n=1$. Thus either $e=0$, $n=1$, which is impossible by *b)*, or $e=1$, $n=0$, which is impossible by *e)*, for in that case $Y^2=4$.

$m=3$. — Again Y is non-singular, so $p_a(Y)=0$, and we have $3e+2n=2$. Thus $e=0$, $n=1$, which is impossible by *b)*.

$m \geq 4$. — We apply (iii) or (iv) of Proposition (3.2), finding $X^2 \leq -\frac{2m}{m-2}$ or $X^2 \leq -\frac{2k}{k-3}$. But $X^2=-1$, so neither of these is possible.

Corollary **(3.4)** (Noether [12]). — *Any birational transformation of the projective plane into itself is a product of a projective collineation and a finite product of standard quadratic transformations, each consisting of blowing up three non-collinear points and blowing down the lines joining them.*

Proof. — This follows easily from the theorem (see Nagata [11, Theorem 6]).

Theorem **(3.5)** [1]. — *Let* X *be a non-singular curve of genus* $g \geq 0$ *on a rational surface* F. *Then either*

a) $X \cong \mathbf{P}^1$, *and the embedding* $X \to F$ *is equivalent to a section of a geometrically ruled rational surface* $F_e \to \mathbf{P}^1$, *or*

b) $g=1$, *and the embedding* $X \to F$ *is equivalent to a non-singular cubic curve in* $\mathbf{P}^2$, *in which case* $X^2=9$, *or*

c) $X^2 \leq 4g+4$.

Proof. — If $F=\mathbf{P}^2$, we calculate explicitly. Let m be the degree of X. Then

$$X^2=m^2$$

$$g=\frac{1}{2}(m-1)(m-2).$$

Thus
$$X^2=4g+4-(m-2)(m-4).$$

[1] The fact that $X^2 \leq 9$ for $g=1$ was proved by Nagata ([11, Th. 4]).

So if $m \leq 2$ or $m \geq 4$, we have $X^2 \leq 4g+4$. The unique exception is $m=3$, which is case *b)* above.

If $F \neq \mathbf{P}^2$, then by Theorem (3.3), F dominates some one of the surfaces F_e. By Proposition (3.1), we can choose a morphism $F \to F_e$ having the properties *a)*, *b)*, *c)*, *d)*, *e)*, and then draw the conclusions of Proposition (3.2). We distinguish various cases, according to the value of m.

$m=0$. — Then $Y^2=0$, $X^2 \leq 0$, so *c)* holds trivially.

$m=1$. — Then by *d)*, $X^2=Y^2$, so the morphism $F \to F_e$ is an isomorphism in a neighborhood of X. Y is a section of the projection $\pi : F_e \to \mathbf{P}^1$, so the embedding $X \to F$ is equivalent to this section.

$m=2$. — Then $X^2 \leq 4g+4$ by (i).

$m=3$. — Then by (ii), we have $X^2 \leq 3g+6$. If $g \geq 2$, this implies $X^2 \leq 4g+4$, so *c)* holds. We treat the cases $g=0, 1$ separately.

$g=0$. — Going back to our formulae, we have in this case $6e+4n=4$, so $e=0, n=1$, which is impossible by *b)*.

$g=1$. — In this case, $6e+4n=6$, so $e=1, n=0$, and $Y^2=9$. If $X^2<Y^2$, then $X^2 \leq 8$, and *c)* holds. If $X^2=Y^2$, then the embedding $X \to F$ is equivalent to the embedding $X \to Y \subseteq F_1$. Now $n=0$, so $Y.D_1=0$, so the embedding $X \to F$ is also equivalent to the embedding $X \to Y \subseteq \operatorname{cont}_{D_1} F_1 = \mathbf{P}^2$. But this is the non-singular cubic curve of case *b)*.

$m \geq 4$. — If case (iii) applies, then we note that $m \geq 4 \Rightarrow \dfrac{m}{m-2} \leq 2$, so

$$X^2 \leq 4g-4,$$

and *a fortiori*, *c)* holds.

If case (iv) applies, we have

$$X^2 \leq \frac{2k}{k-3}(g-1).$$

If $k \geq 6$, then $\dfrac{k}{k-3} \leq 2$, so $X^2 \leq 4g-4$, and *c)* applies. We treat the exceptional cases $k=4, 5$ separately. Note that Y must be non-singular in these cases, since every multiple point of Y has multiplicity $\leq n$, and $n=0, 1$.

$k=4$. — $m=4$, $n=0$, we have $g=3$, $Y^2=16$, so $X^2 \leq Y^2 = 4g+4$.

$k=5$. — Then either $m=4, n=1, g=6, Y^2=24$, or $m=5, n=0, g=6, Y^2=25$. In either case $X^2 \leq Y^2 < 4g+4$.

Example. — On $F_0 = \mathbf{P}^1 \times \mathbf{P}^1$, the general curve X of type $m=2$, $n \geq 2$ is irreducible and non-singular. Its genus is $g=n-1$, and we find $X^2=4g+4$. For $g=0$, consider the conic in $\mathbf{P}^2$, with self-intersection 4. Thus for all $g \geq 0$, the bounds of the theorem are the best possible.

§ 4. Conclusions and Generalizations.

We now return to the question posed in the introduction. If X is a non-singular variety embedded in a non-singular variety V, we define the *normal bundle* $\mathcal{N}_{X/V}$ to be the dual of the locally free sheaf $\mathcal{I}/\mathcal{I}^2$ on X, where $\mathcal{I}$ is the sheaf of ideals of X in V. We ask, if $\mathcal{N}$ is sufficiently ample, to what extent is V determined by X and $\mathcal{N}$? In particular, we would like to compare the embedding $X \to V$ to the standard embedding $j : X \to \mathbf{V}(\mathcal{N}^\vee)$, the zero-section of the geometric vector bundle

$$\mathbf{V}(\mathcal{N}^\vee) = \operatorname{Spec} \sum_{n \geq 0} S^n(\mathcal{N}^\vee)$$

[EGA, II.1.7)]. Of course the phrases " sufficiently ample " and " determined by " are vague, and we must make them precise in each context.

Let us first consider a curve X on a surface F. Then the normal bundle $\mathcal{N}$ is a line bundle on X, and we can use its degree, which is the self-intersection of X on F, as a measure of its ampleness. We thus come to the following theorem, which is the main result of this paper.

Theorem **(4.1)**. — *Let* X *be a non-singular curve of genus* $g \geq 0$ *on a non-singular surface* F, *and assume that* $X^2 > 4g+5$. *Then* F *is a ruled surface, and the embedding* $X \to F$ *is equivalent* (see § 0) *to a section of a geometrically ruled surface* $\pi : \mathbf{P}(\mathcal{E}) \to C$. *The same statement is true if we assume only* $X^2 \geq 4g+5$, *with a single exception, namely the non-singular cubic curve in the projective plane (or an equivalent embedding).*

Proof. — Let K be the canonical divisor class on F. Then $2g-2 = X.(X+K)$, so as soon as $X^2 > 2g-2$, we have $X.K < 0$. In this case, the proof of Enriques' classification of surfaces shows that F is a ruled surface [10, § 1].

If F is non-rational, then the result follows from Corollary (2.4) in case $g > 0$. If $g = 0$, then X is a component of a fibre of the projection $\pi : F \to C$, so that $X^2 \leq 0$, which is impossible.

If F is a rational surface, the result follows from Theorem (3.5).

For curves on surfaces, we may still ask whether some of these sections of geometrically ruled surfaces are equivalent to each other, or even formally equivalent, in the sense that the formal completions of the surfaces along the curves are isomorphic formal schemes. The answer to both questions is no, as we see in the following two propositions.

Proposition **(4.2)**. — *Let* C *be a non-singular curve, let* $X \cong C$, *and let* $X \to \mathbf{P}(\mathcal{E}_1)$ *and* $X \to \mathbf{P}(\mathcal{E}_2)$ *be two sections of geometrically ruled surfaces over* C. *Let* $f : \mathbf{P}(\mathcal{E}_1) \to \mathbf{P}(\mathcal{E}_2)$ *be a birational map, which induces an equivalence of the two embeddings. Assume either that* C *is not rational, or that* $X^2 \geq 2$. *Then* f *is an isomorphism.*

Proof. — We are assuming that f induces an isomorphism of an open neighborhood U_1 of X in $\mathbf{P}(\mathcal{E}_1)$ to an open neighborhood U_2 of X in $\mathbf{P}(\mathcal{E}_2)$. The complement of U_i cannot contain any fibre of the ruled surface. Thus in case C is not rational, the complement of U_i contains no rational curves, so f must be biregular everywhere. If C is rational and $X^2 \geq 2$, then the complement of U_i may contain a rational curve. But one checks easily that this occurs only when X is the standard section D on a surface F_e, $e \geq 2$, in which case U_i may contain the exceptional section D_e. But still the complement of U_i contains no exceptional curves of the first kind, so f is biregular everywhere.

Proposition **(4.3)**. — *Let* $X \rightarrow F_1$ *and* $X \rightarrow F_2$ *be two embeddings of a curve* X *in surfaces* F_1, F_2 *with* $X^2 > 0$ *in each case. Suppose there is an isomorphism* $g: \hat{F}_1 \xrightarrow{\approx} \hat{F}_2$ *of the formal completions of* F_1 *and* F_2 *along* X, *which induces the identity on* X. *Then there is a birational map* $f: F_1 \rightarrow F_2$, *which gives an equivalence of the two embeddings, and such that the induced map* $\hat{f}: \hat{F}_1 \rightarrow \hat{F}_2$ *of the completions is equal to* g.

Proof. — According to a theorem of Hironaka ([7], § 2, Theorem IV*), the natural map $K(F_i) \rightarrow K(\hat{F}_i)$, of the field of rational functions on F_i to the field of formal-rational functions on $\hat{F}_i$, is an isomorphism, for $i = 1, 2$. Thus g induces an isomorphism $K(F_1) \rightarrow K(F_2)$, and hence a birational map $f: F_1 \rightarrow F_2$. Now for any point $x \in X$, we have

$$\mathcal{O}_{x, F_i} = \mathcal{O}_{x, \hat{F}_i} \cap K(F_i).$$

Hence one sees easily that f is biregular in a neighborhood of X, that $\hat{f} = g$, and that f induces the identity on X.

Remark. — This proof shows that the same result is true (namely formal isomorphism implies equivalence) whenever one has two embeddings $X \rightarrow V_1$ and $X \rightarrow V_2$ of a variety X in varieties V_1, V_2 such that the natural maps $K(V_i) \rightarrow K(\hat{V}_i)$ are isomorphisms. This says that X is G3 in V_1 and V_2, in the terminology of Hironaka and Matsumura ([8], Definitions (2.9), p. 64).

Now let us consider an embedding of a curve X in a variety V of dimension ≥ 3. In this case we will show by example that there are embeddings of X in $\mathbf{P}^3$ with arbitrarily ample normal bundle. Thus we do not have an analogue of Theorem (4.1). One may hope, however, to prove something about embeddings of X in non-rational varieties V of dimension ≥ 3.

Example **(4.4)**. — *Let* X *be a non-singular curve. For any coherent sheaf* $\mathcal{F}$ *on* X, *there is an embedding* $X \rightarrow \mathbf{P}^3$, *whose normal bundle* $\mathcal{N}$ *has the following properties*:

1) $\mathcal{F} \otimes \mathcal{N}$ *is generated by global sections on* X.

2) $H^1(X, \mathcal{F} \otimes \mathcal{N}) = 0$.

In particular, $\mathcal{N}$ is always ample [AVB, § 8], and by taking $\mathcal{F} = \mathcal{O}_X(-n)$, we can make $\deg \mathcal{N}$ arbitrarily large.

To get this embedding, take an invertible sheaf $\mathcal{L}$ on X such that 1) $\mathcal{F} \otimes \mathcal{L}$ is generated by global sections; 2) $H^1(X, \mathcal{F} \otimes \mathcal{L}) = 0$, and 3) $\mathcal{L}$ is very ample on X. For example, $\mathcal{L} = \mathcal{O}(n)$ for sufficiently large n will do, if $\mathcal{O}(1)$ is any ample invertible sheaf on X. Then $\mathcal{L}$ gives an embedding of X into $\mathbf{P}^N$, where $N = \dim H^0(X, \mathcal{L}) - 1$. We may assume $N \geq 3$. Now the generic projection of X into $\mathbf{P}^3$ is non-singular, hence gives an embedding of X. And by construction, the normal bundle $\mathcal{N}$ to X in this embedding is a quotient of a direct sum of four copies of $\mathscr{L}$. It follows that 1) and 2) above hold for $\mathscr{N}$.

We turn finally to embeddings of varieties X of dimension ≥ 2. We show that if $\mathscr{N}$ is sufficiently ample in a cohomological sense, then any embedding of X with normal bundle $\mathscr{N}$ is formally isomorphic to the zero-section of the vector bundle itself.

Proposition **(4.5)**. — *Let* X *be a complete non-singular variety of dimension* r, *and let* $\mathscr{N}$ *be a locally free sheaf on* X *such that*

$$H^{r-1}(X, \omega_X \otimes \Omega^1_X \otimes \Gamma^n(\mathscr{N})) = 0 \quad \textit{for} \quad n \geq 1$$

and

$$H^{r-1}(X, \omega_X \otimes \mathscr{N}^{\vee} \otimes \Gamma^n(\mathscr{N})) = 0 \quad \textit{for} \quad n \geq 2.$$

(*Here we have written* ω_X *for* Ω^r_X, *and* $\Gamma^n(\mathscr{N})$ *for* $S^n(\mathscr{N}^{\vee})^{\vee}$.) *Then any smooth formal scheme* $\mathfrak{X}$, *with reduced scheme of definition* X, *and with normal bundle* $\mathscr{N}$, *is isomorphic to the formal completion of* $\mathbf{V}(\mathscr{N}^{\vee})$ *along its zero-section.*

Proof. — For any $n \geq 1$, let X_n be the closed subscheme of $\mathfrak{X}$ defined by $\mathscr{I}^n$ where $\mathscr{I}$ is the ideal of X. Then $\mathscr{O}_{X_{n+1}}$ is an extension of $\mathscr{O}_{X_n}$ by $\mathscr{I}^n/\mathscr{I}^{n+1} = S^n(\mathscr{N}^{\vee})$. We will use obstruction theory [SGA 1, exposé III, § 5] to show that all these extension, are trivial, which gives the result.

Since X is non-singular, the extensions are locally trivial. The automorphisms of the sheaf of rings $\mathscr{O}_{X_{n+1}}$ inducing the identity on $\mathscr{O}_{X_n}$ are given by the sheaf $\mathscr{A}_n = \mathit{Der}(\mathscr{O}_{X_n}, \mathscr{I}^n/\mathscr{I}^{n+1})$ of derivations of $\mathscr{O}_{X_n}$ into $\mathscr{I}^n/\mathscr{I}^{n+1}$. Hence all possible such extensions (if there is at least one) are classified by $H^1(X, \mathscr{A}_n)$.

Using duality on X, our hypotheses give

$$H^1(X, \check{\Omega}^1 \otimes S^n(\mathscr{I}/\mathscr{I}^2)) = 0 \quad \text{for} \quad n \geq 1$$

and

$$H^1(X, \mathscr{N} \otimes S^n(\mathscr{I}/\mathscr{I}^2)) = 0 \quad \text{for} \quad n \geq 2.$$

Now $\mathscr{A}_1 = \mathit{Der}(\mathscr{O}_X, \mathscr{I}/\mathscr{I}^2) = \mathit{Hom}(\Omega^1_X, \mathscr{I}/\mathscr{I}^2) = \check{\Omega}^1_X \otimes (\mathscr{I}/\mathscr{I}^2)$. So $H^1(X, \mathscr{A}_1) = 0$, and the first extension is trivial. It follows that for any $n \geq 2$, $\Omega^1_{X_n} \otimes \mathscr{O}_X \cong \Omega^1_X \oplus (\mathscr{I}/\mathscr{I}^2)$. Now for $n \geq 2$,

$$\mathscr{A}_n = \mathit{Hom}(\Omega^1_{X_n}, \mathscr{I}^n/\mathscr{I}^{n+1}) \cong (\check{\Omega}^1_X \oplus \mathscr{N}) \otimes (\mathscr{I}^n/\mathscr{I}^{n+1}).$$

So $H^1(X, \mathscr{A}_n) = 0$ by our hypotheses.

Examples. — The hypotheses on $\mathscr{N}$ will be satisfied if $r \geq 2$, $\mathscr{L}$ is an ample invertible sheaf on X, and $\mathscr{N} = \mathscr{L}(\nu)$ for ν sufficiently large; or if $r = 1$, and $\mathscr{N} = \mathscr{L}(\nu)$ for ν sufficiently negative. Thus for curves on surfaces, we have formal uniqueness of sufficiently negative embeddings.

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