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ON A PROBLEM OF NIEMINEN

By William F. DONOGHUE, Jr.

In a recent publication [3] T. Nieminen raises the following question: If T is a bounded operator on a Hilbert space, the spectrum of which is a subset of $|z|=1$ and whose resolvent satisfies the inequality $\|R_z\| \leq \big||z|-1\big|^{-1}$ on the resolvent set, does it follow that T is unitary? We show that it does.

It is convenient to recall a definition: if T is a bounded operator on a Hilbert space, the set W(T) of complex numbers of the form (Tx, x) where $\|x\|=1$ is called the numerical range of T and is a convex set contained in a circle of radius $\|T\|$ [4, 1]. Our Theorem is then the following.

Theorem. — The following three classes of bounded operators on the Hilbert space $\mathscr{H}$ are identical:

(I) The unitary operators.

(II) The operators T for which

(*a*) T^{-1} exists and is everywhere defined and $\|Tx\| \geq \|x\|$ for all x in $\mathscr{H}$ and

(*b*) W(T) is a subset of the unit disc.

(III) The operators T for which

(*a'*) 0 is in the resolvent set and $\|R_0\| \leq 1$ and

(*b'*) for an unbounded sequence of numbers $t_n > 1$, $\|R_z\| \leq (t_n - 1)^{-1}$ if $|z| = t_n$.

Proof. — It is easy to see that the unitary operators are in class III. We shall show first that operators in the class III are in class II, and then that the operators in II are unitary.

It is obvious that conditions (*a*) and (*a'*) are equivalent. If, now, T satisfies (*a'*) and (*b'*) and $\lambda = t_n e^{i\theta}$, we will have for any y in $\mathscr{H}$, $\|y\| \leq (t_n-1)^{-1}\|(T-\lambda)y\|$ whence

$$(t_n-1)^2\|y\|^2 \leq \|Ty\|^2 + t_n^2\|y\|^2 - t_n[e^{i\theta}\overline{(Ty, y)} + e^{-i\theta}(Ty, y)].$$

Now if $\|y\|=1$ and $(Ty, y) = \rho e^{i\varphi}$ we obtain

$$2t_n[\rho\cos(\theta-\varphi)-1] \leq \|Ty\|^2 - 1.$$

Since the sequence t_n is unbounded and θ arbitrary we infer that $\rho \leq 1$, i.e. (*b*).

It T satisfies (*a*), then, by a known theorem [2], $T = VA$ where V is unitary and A is positive and also satisfies (*a*). It is clear that the spectrum of the self-adjoint A lies in the interval $1 \leq \lambda \leq \|A\|$, hence we may write $A = I + H$ where I is the identity operator and H is positive. Accordingly, $T = V + VH$ and it is obvious that any operator of this form, where V is unitary and H is positive and bounded, satisfies (*a*).

Let x be a normalized element of $\mathcal{H}$ for which $|(\mathrm{V}x, x)| = 1 - \varepsilon$ for some small ε which may be 0. We will have

$$(\mathrm{T}x, x) = (\mathrm{V}x, x) + (\mathrm{VH}x, x) = (\mathrm{V}x, x) + (\mathrm{H}x, \mathrm{V}^{-1}x).$$

Choose a normalized element y such that $(x, y) = 0$ and $\mathrm{V}^{-1}x = \alpha x + \beta y$; evidently $\alpha = \overline{(\mathrm{V}x, x)}$ and $|\alpha|^2 + |\beta|^2 = 1$. Thus, from (*b*)

$$|(\mathrm{T}x, x)| = |(\mathrm{V}x, x)[1 + (\mathrm{H}x, x)] + \bar{\beta}(\mathrm{H}x, y)| \leq 1.$$

Let $h = (\mathrm{H}x, x)$; from the Schwarz inequality we obtain

$$|(\mathrm{H}x, y)|^2 \leq (\mathrm{H}x, x)(\mathrm{H}y, y) \leq ||\mathrm{H}||\, h$$

and we know that $|\beta| \leq \sqrt{2\varepsilon}$; hence $(1 - \varepsilon)(1 + h) \leq 1 + \sqrt{2||\mathrm{H}||\varepsilon h}$. It follows that h vanishes if ε does and that the ratio h/ε is bounded as ε approaches 0. We infer that there exists an M such that

$$(\mathrm{H}x, x) \leq \mathrm{M}[1 - |(\mathrm{V}x, x)|]$$

for all normalized x.

Let $\mathrm{V} = \int_0^1 e^{2\pi i\lambda} d\mathrm{E}_\lambda$ be the spectral representation of V; for an integer N let $\mathrm{P}_k = \mathrm{E}_{k/\mathrm{N}} - \mathrm{E}_{(k-1)/\mathrm{N}}$, $k = 1, 2, \ldots, \mathrm{N}$. If x is a normalized element in the range of P_k, then the number $(\mathrm{V}x, x) = \int_0^1 e^{2\pi i\lambda} d(\mathrm{E}_\lambda x, x)$ is evidently in the convex hull of the complex numbers on an arc of length $2\pi/\mathrm{N}$ of the unit circle, whence $1 - |(\mathrm{V}x, x)| \leq \pi^2/\mathrm{N}^2$. It will follow that

$$||\mathrm{H}x||^2 = (\mathrm{H}^2 x, x) \leq ||\mathrm{H}||\, \mathrm{M}\pi^2/\mathrm{N}^2$$

and therefore

$$||\mathrm{H}x|| \leq \mathrm{C}\, ||x||/\mathrm{N}$$

for all x in the range of P_k, the constant C being independent of N.

If, now, z is an arbitrary vector in the space and $y = \mathrm{H}z$, then, for such x,

$$|(y, x)| = |(\mathrm{H}z, x)| = |(z, \mathrm{H}x)| \leq ||z|| \cdot ||\mathrm{H}x|| \leq ||z|| \cdot ||x||\, \mathrm{C}/\mathrm{N}$$

and for $x = \mathrm{P}_k y$

$$||\mathrm{P}_k y||^2 = (y, \mathrm{P}_k y) \leq ||z|| \cdot ||\mathrm{P}_k y||\, \mathrm{C}/\mathrm{N}$$

or

$$||\mathrm{P}_k y|| \leq ||z||\, \mathrm{C}/\mathrm{N}.$$

Accordingly

$$||y||^2 = \sum_{k=1}^{\mathrm{N}} ||\mathrm{P}_k y||^2 \leq \mathrm{N}\, ||z||^2 \mathrm{C}^2/\mathrm{N}^2 = ||z||^2 \mathrm{C}^2/\mathrm{N}.$$

Since N is arbitrary, $y = 0$, and since z is arbitrary, $\mathrm{H} = 0$ and $\mathrm{T} = \mathrm{V}$, whence T is unitary.

If we recall that a contraction on a Hilbert space is any linear transformation of bound at most 1, we can state the essential part of our result in the following convenient form:

Corollary. — If S is a contraction on a Hilbert space which has a bounded inverse for which $W(S^{-1})$ is a subset of the unit disc, then S is unitary.

One might be led to the conjecture that any bounded operator T for which the resolvent satisfies the inequality $||R_z|| \leq 1/d(z)$ where $d(z)$ is the distance from z to the spectrum of T must be normal. That this is not the case is shown by the following simple counterexample. Let A be an operator of bound ≤ 1 which is not normal; W(A) will be a subset of the unit disc and, by a known theorem, for $|z|>1$ the resolvent of A will be bounded by the reciprocal of the distance from z to W(A), *a fortiori* by $(|z|-1)^{-1}$. Let B be an operator which is normal and whose spectrum is exactly the unit disc. If $\mathscr{H}$ is the direct sum of two Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ and T is defined as the operator A on $\mathscr{H}_1$ and B on $\mathscr{H}_2$, then it is easy to see that the spectrum of T is the unit disc, that T is not normal, and that $||R_z|| \leq (|z|-1)^{-1}$ on the resolvent set, where R_z is the resolvent of T. We remark that the conjecture would be correct if the Hilbert space were finite dimensional.

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