Individual Choice Sequences in the Work of L.E.J. Brouwer

Joop Niekus
ILLC Amsterdam

Résumé : Par des suites de choix, nous comprenons des suites qui ne sont pas déterminées complètement par une loi arithmétique. Elles sont des objets caractéristiques de l'intuitionnisme de Brouwer. Nous prétendons qu'à partir de 1927, l'utilisation par Brouwer de suites de choix particulières n'est pas reconnu comme tel. Nous prétendons que l'utilisation de ces suites dans la méthode du sujet créatif, après la seconde guerre mondiale, n'a pas à être mis en relation avec l'utilisation de celles-ci dans les années vingt et qu'elles sont mal interprétées. Nous montrons où se trouvent ces suites de choix dans l'œuvre de Brouwer et comment elles doivent être traitées.

Abstract: Choice sequences are sequences not completely determined by an algorithmic law. We maintain that the introduction of particular choice sequences by Brouwer in the late twenties was not recognised as such. We claim that their later use in the method of the creative subject was not traced back to this original usage and has been misinterpreted. We show where these particular choice sequences appear in the work of Brouwer and we show how they should be handled.

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1 Introduction

Brouwer made his first steps in the foundations of mathematics in his thesis of 1907 ([Brouwer 1907]). In the decade after this his main concern was topology. He returned to the foundations in 1918 when he presented with [Brouwer 1918] a reconstruction of mathematics along the lines set out in his thesis. The reconstruction, called intuitionism, has two striking features. Firstly, it is independent on the law of the excluded middle. Secondly, real numbers are introduced by infinitely proceeding sequences with terms chosen more or less freely from objects already constructed: *choice sequences*. He kept publishing on the subject until 1930.

After a lapse of more than fifteen years he started to publish again on the subject in 1948. Characteristic for his papers in this second period of intuitionistic activity is a technique for deriving counterexamples against consequences of the law of the excluded middle. The technique has generally been considered to be radically new; in this period it is known as the method of the creative subject.

Extensive research has been done on choice sequences. The standard text on the subject is A.S. Troelstra's monograph [Troelstra 77]. This work is based on ideas originating with G. Kreisel (see [Kreisel 1963], [Kreisel 1968] and [Kreisel and Troelstra 1970]). It contains a considerable amount of technical work on formal systems of classes of choice sequences. For these systems Troelstra proves elimination theorems: a sentence with quantification over choice sequences can be translated into an equivalent sentence without choice sequences. So, what can be proved with choice sequences can also be proved without them.

The formal systems of [Troelstra 77] are not relevant for the method of the creative subject. In order to reconstruct this method, Kreisel and Troelstra developed the theory of the idealised mathematician. From seemingly straightforward assumptions concerning the properties of the idealised mathematician, Troelstra derived a paradox, which could not be resolved satisfactorily, see [Troelstra and van Dalen 1988, 842-846] and our analysis in [Niekus 1987, 434-436]. The method of the creative subject is controversial, also within intuitionism.

The subject of this paper is Brouwer's use of *individual choice sequences*, which are particular sequences of which the terms are not algorithmically determined. They have not been an object of study after Brouwer; the research was concentrated on global properties. They do not occur in [Troelstra 77], or in [Troelstra 82], a survey article on Brouwer's use of choice sequences. Our aim is twofold. Firstly we want to show where individual choice sequences appear in the work of Brouwer; secondly we want to show how they should be treated.

In Section 3 we give the first example known to us that Brouwer deliberately uses an individual choice sequence. It is from a lecture in 1927, but not published until 1991 ([Brouwer 91]). We also give the first in print, which is in [Brouwer 30]. Further we cite a fragment from [Brouwer 48], which we think shows without doubt, that Brouwer is exploiting individual choice sequences, as introduced in 1927, in the method of the creative subject.

All three examples of choice sequences deal with the same result: the non-equivalence of apartness and inequality. Only in [Br 48] does Brouwer give a proof, which we shall analyse in the Section 4. In our view the properties of a choice sequence are determined during the construction, which is in the future. So reasoning about a choice sequence should involve principles of the logic of time. We shall use one very obvious in our reconstruction. We do not use an idealised mathematician.

In Section 5 we shall compare our reconstruction with the reconstruction with the idealised mathematician, which is the standard one in intuitionism.

We start in Section 2 with the necessary definitions.

2 Definitions

In intuitionism, mathematics consists of mental constructions of the human individual only; there are no mathematical truths outside the human mind. The prime material for these constructions is the sequence of natural numbers $\mathbb{N}$. Of this sequence the first element is given and every next one is constructible from its predecessors. They have their origin in "our perception of the move of time" ([Brouwer 1981, 4]).

From the natural numbers the integers $\mathbb{Z}$ and rational numbers $\mathbb{Q}$ can be constructed in a standard way. The resulting mathematics up to this point is contained in classical mathematics. It is the introduction of real numbers by infinitely proceeding sequences that gives intuitionism its special character. We shall use for this purpose sequences of rational numbers (a_n) that are constructed in the following way:

Any $q \in \mathbb{Q}$ is admitted as a choice for the first element a_0; $q \in \mathbb{Q}$ is admitted as next element a_{n+1} in the already constructed sequence $a_0, a_1, \ldots, a_n$ iff $|a_n - q| < 2^{-n}$.

A thus constructed sequence is a *real number generator* (rng).

If the construction is completely determined by an algorithmic law from the first term onward, we call the rng a *lawlike sequence*. We reserve the name *choice sequence* for rng's having no complete algorithmic description. Brouwer called these sequences "unfertig", which is German for incomplete. Such a sequence reaches its completion during the construction. We shall see examples of these incomplete objects below.

Between two rng's (a_n) and (b_n) we define the relation R by

$$(a_n)R(b_n) \quad \text{iff} \quad \forall k\, \exists n\, \forall m > n \quad |a_m - b_m| < 2^{-k}.$$

R is an equivalence relation. The real numbers are the equivalence classes of this relation. They form the *full continuum*, also called the *continuum*. The *reduced continuum* is formed by the equivalence classes of R restricted to the lawlike rng's. We shall denote real numbers by $a, b, \ldots x, y, \ldots$ Arithmetical operations and relations can be defined on real numbers via the representatives, as in the following definitions.

$$\begin{aligned}
a < b \quad &\text{iff} \quad \text{for representatives } (a_n) \text{ and } (b_n) \\
&\qquad \exists k\, \exists n\, \forall m \quad (b_{n+m} - a_{n+m}) > 2^{-k}, \\
a > b \quad &\text{iff} \quad b < a, \\
a\#b \quad &\text{iff} \quad a < b \;\text{ or } b < a.
\end{aligned}$$

Since in intuitionism a proof of $\exists n A(n)$ requires a proof of $A(n_0)$ for some specific n_0, one should distinguish between the apartness relation $a\#b$ and the inequality relation $a \neq b$. The latter expression means $\neg(a = b)$, i.e. the supposition of $a = b$ is contradictory. Each of the three examples of choice sequences in section 3 deals with the non-equivalence of these two relations.

The way we introduced real numbers is by no means exclusive. We could have used another modulus of convergence in the definition of R, e.g. k^{-1} instead of 2^{-k}. We could have add a clause in our definition of rng's that q is of the form $a.2^{-n}, a \in \mathbb{Z}, n \in \mathbb{N}$, in which case rng's are infinite convergent sequences of *dual fractions*. Instead of sequences of rational numbers we could have used sequences of *intervals* of rational numbers. In one of the examples below Brouwer had introduced the real numbers by sequences the n-th term a $\lambda^{(n)}$-interval, which is encompassed by its predecessor. A $\lambda^{(n)}$-interval has the form $[a.2^{-n-1}, (a+2).2^{-n-1}]$, with $a \in \mathbb{Z}, n \in \mathbb{N}$. There is no essential difference between these definitions; they result in the same continuum.

3 Individual choice sequences — where they appear

Brouwer started to explore the distinction reduced versus full continuum in his Berlin lectures of 1927, the text not published until 1991 ([Brouwer 91]). In the following way he showed that the above defined relation $<$ is not a "full" order on the reduced continuum.[1]

Let k be the smallest number such that the k-th up to the $k+9$-th digit in the decimal expansion of π form the sequence 0123456789. We define (b_n) as follows:

$$\begin{aligned} b_n &= 0 \quad \text{iff} \quad n < k \\ b_n &= (-2)^{-k} \quad \text{iff} \quad n \geqslant k. \end{aligned}$$

The sequence (b_n) clearly generates a real number, say r. Given an algorithm to calculate the digits of the decimal expansion of π, (b_n) is completely algorithmic, so r is an element of the reduced continuum. For this real number Brouwer claims that neither $r < 0$, nor $r > 0$, nor $r = 0$ holds. He concludes: $<$ is not a full order on the reduced continuum.

[1] In the original German text the proof that $<$ is not a full order on the reduced continuum and not an order on the continuum were interwoven. The original text:

> Wir wollen zunächtst zeigen, daß das naive "vor" und nach weder im Kontinuum noch im reduzierten Kontinuum eine Ordnung, geschweige denn eine "vollständige" Ordnung der Punkkerne abgibt. Dazu betrachten wir eine mathematische Entität oder Species S, eine Eigenschaft E, und definieren wie folgt den Punkt s des Kontinuums: Das n-te λ-Intervall λ_n ist eine symmetrischum den Nullpunkte gelegenes $\lambda^{(n-1)}$-Intervall, so lange man die Gültigheit noch die Absurdität von E für S kennt, dagegen ist es ein symmetrisch um den Punkt 2^{-m}, bzw. um den Punkt -2^{-m} gelegenes $\lambda^{(n)}$-Intervall, wenn $n \geqslant m$ und zwischen der Wahl des $(m-1)$-ten und der Wahl des m-ten Intervalles ein Beweis der Gültigkeit bzw. der Absurdität von E für S gefunden worden ist. Weiter bezeichnen wir mit k_1 die kleinste natürliche Zahl n mit der Eigenschaft, daß die n-te bis $(n+9)$-te Ziffer der Dezimalbruchenentwicklung von π eine Sequenz 0123456789 bilden und dazu definieren wir wie folgt den Punkt r des reduzierten Kontinuums: Das n-te λ-Intervall λ_n ist ein symmetrisch um den Nullpunkt gelegenes $\lambda^{(n-1)}$-Intervall, solange $n < k_1$; für $n \geqslant k_1$ aber ist λ_n das symmetrisch um den Punkt $(-2)^{-k_1}$ gelegene $\lambda^{(n)}$. Alsdann ist der zu s gehörende Punktkern des Kontinuums $\neq 0$, aber solange man weder die Absurdität noch die Absurdität der Absurdität von E für S kennt, weder > 0 noch < 0. Bis zum stattfinden einer dieser beide Entdeckungen kannalso das Kontinuum nicht geordnet sein. Weiter ist der zu r gehörende Punktkern des reduziertes Kontinuums, solange die Existenz von k_1 weder bewiesen noch noch ad absurdum geführt ist, weder $= 0$, noch > 0, noch < 0. Bis zum stattfinden einer dieser beiden Entdeckungen ist also das reduzierte Kontinuum nicht vollständig geordnet. ([Brouwer 1991, 31-32]).

Note the role of time in this argument. Neither $r < 0$ nor $r > 0$ did hold for Brouwer because he could not give a specific natural number n_0 such that $k = n_0$. Recently it has been discovered that k exists and that it is even, so $r > 0$ holds *now*.[2]

"Neither $r < 0$ nor $r > 0$ hold" is an example of the *weak negation*, expressing just the absence of a proof. The *strong negation* of an assertion A means that from the assumption of A a contradiction can be derived. Although both negations appear in Brouwer's work, only the strong negation has been formalised in intuitionistic logic. It is with respect to the strong negation that Brouwer refutes the law of the excluded middle: $A \vee \neg A$ holds when either a proof of A is available or a proof showing its contradictority. As long as there are unsolved mathematical problems, it is of course not generally valid.

A property of the strong negation is that $\neg\neg A$ is not equivalent with A, it is weaker. Therefore $\neg\neg A \vee \neg A$, which we shall meet below, is weaker than $A \vee \neg A$.

The technique for showing $>$ is not a full order on the reduced continuum, was not new. Already in [Brouwer 1908] Brouwer used unknown properties of the decimal expansion of π to demonstrate the unreliability of the law of the excluded middle. But he proved a stronger statement for the full continuum. He constructed a real number a such that $a \neq 0$ holds, but neither $a > 0$ nor $a < 0$ can be proved. This argument was new, and it was special. It runs as follows:

Let A be a proposition which is not *tested*, i.e. neither $\neg A$ nor $\neg\neg A$ has been proved. In connection with this proposition A we define the sequence (a_n) as follows.

- $a_n = 0$ as long as, at the choice of a_n, neither A nor $\neg A$ has been proved;
- $a_{k+n} = 2^{-k}$, for all n, if a proof of A has been found between the choice of a_{k-1} and a_k;
- $a_{k+n} = -2^{-k}$, for all n, if a proof of $\neg A$ has been found between the choice of a_{k-1} and a_k.

This sequence generates a real number a, for which $a \neq 0$ holds, but as long as neither A nor ¬A has been proved, neither $a > 0$ nor $a < 0$ holds. Brouwer concludes: as long as neither one of these proofs has been found, the continuum is not ordered.

[2] According to [Borwein 1998], $k = 17,387,594,880$.

The sequence generating a above depends on whether or not some proof will be found, so it is not algorithmic. It is the first time that Brouwer deliberately gives an example of a choice sequence.[3] He does not give further argument here for the result. We delay our first comment till the next choice sequence, which is in [Brouwer 30], the text of his second Vienna lecture of 1928.

After the Berlin lectures Brouwer generalized his π-expansion technique by introducing the notion of a *fleeing property* f for natural numbers. It satisfies the following conditions: for each natural number it is decidable whether it possesses f or not, no natural number possessing f is known, the assumption of the existence of a number$f$ is not known to be contradictory. The critical number λ_f of a fleeing property f is the smallest natural number possessing f.

For Brouwer the standard example of a fleeing property for a natural number m was the m-th digit in the decimal expansion of π being the first of the row 0123456789, as we saw above. The real number r defined there is an example of a *dual checking number* of that fleeing property. Since for this property λ_f is now known, it is not a fleeing property anymore.

Brouwer applied the new notion in [Br 30], the text of the second of two lectures he held in Vienna in 1928. He examines the continuum with regard to seven properties, all classically valid. Each time he distinguishes between the reduced and the full continuum. When sufficient, he uses a lawlike real, as below:

> That the continuum (and also the reduced continuum) is **not discrete** follows from the fact that the number $1/2 + p_f$, with p_f the dual checking number of the fleeing property f, is neither equal to $1/2$, nor apart from $1/2$.[4]

[3]In his proof of the negative continuity theorem of [Brouwer 1927] Brouwer may seem to use an individual choice sequence. This precedes the famous continuity theorem. According to Brouwer the theorem is, contrary to the continuity theorem, an immediate consequence of basic intuitionistic principles, and its proof appears in his lectures from 1918 (contrary to the continuity theorem). However, the proof is not at all clear. There are, among others, reconstructions in [Posy 1976], [Martino 1985] and [Troelstra 1982], all different.

Brouwer continuously developed his ideas on choice sequences during his entire career. As it seems to us the notion of choice sequence was not sufficiently crystallized out when he used it in [Brouwer 1927]. At least he never used it again in the same manner. In [Brouwer 1981, 81] Brouwer proves the negative continuity theorem again. Whereas he was fully exploiting choice sequences as introduced in [Brouwer 1991] in his method of the creating subject, he now uses a lawlike sequence in his proof.

The second proof is not taken into consideration in the reconstructions mentioned.

[4]"Daß das Kontinuum (und ebenso das reduzierte Kontinuum) **nicht diskret ist**,

But if necessary he uses a choice sequence:

> That $<$ is not an order on the continuum is demonstrated by the real number p, generated by the sequence (c_n), its terms chosen such that $c_1 = 0$ and $c_v = c_{v+1}$ with only the following exception. Whenever I find the critical number λ_f of some particular fleeing property f, I choose the next c_v equal to -2^{-v}, and when I find a proof this critical number does not exist, I choose c_v equal to 2^{-v}. This number p is unequal to 0, but nevertheless it is not apart from zero.[5]

Note the difference between these two sequences. If for some natural number m it is proved that $m = \lambda_f$, then the number defined in the first fragment becomes $1/2 + 2^{-m}$. Such a relation does not follow from the definition in the second case.

Since the Berlin lecture notes were not published until 1991, the sequence used in the second fragment is the first choice sequence of Brouwer in print. It is a peculiar fact that this sequence has never been recognised as something special.[6] Whether the sequence is the same as in his Berlin lecture depends on whether one may conclude from the definition of a fleeing property that it is *non-tested*. An indication that Brouwer intended to give the same example is that in [Brouwer 48], which we treat below, he gave the fleeing property used in the definition of r in [Brouwer 91] as an example of a non-tested proposition. As in [Brouwer 91], there is no further proof of the above result, or of the results obtained with other choice sequences.[7]

As any infinite sequence in intuitionism, a choice sequence is given by a description of the construction of its terms. But the description does not determine the sequence algorithmically. In the examples above

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folgt z. B. daraus, daß die Zahl $1/2 + p_f$, wo p_f die duale pendelzahl der fliehenden Eigenschaft f vorstellt, weder gleich 1/2 noch von 1/2 verschieden ist." ([Brouwer 1975, 435]).

[5]"Daß das Kontinuum durch die der Anschauung entnommene Reihenfolge ihrer Elemente **nicht geordnet ist**, erweist sich am Elemente p, für dessen bestimmte konvergente Folge $c_1, c_2, \dots c_1$im Nullpunkt und jedes $c_{v+1} = c_v$ gewählt wird, mit der einzige Ausnahme, daß ich, sobald von einer bestimmten fliehenden Eigenschaft f mir eine Lösungszahl λ_f bekannt wird, das nächste c_v gleich -2^{-v-1}wähle, und daß ich, sobald mir eine Beweis der Absurdität dieser Lösungszahl bekannt wird, das nächste c_v gleich 2^{-v-1}wähle. Dieses Element p ist von Null verschieden, ist aber trotzdem weder kleiner als Null noch größer als Null." ([Brouwer 1975, 436]).

[6]The choice sequence of [Brouwer 1991] has for the first time been recognised as such and cited in [Dalen 1999]. No reconstruction is given. The fact that Brouwer uses choice sequences in [Brouwer 1930] is mentioned there; they are not shown.

[7]There are two other choice sequences in [Brouwer 30], on [Brouwer 1975, 437] and on [Brouwer 1975, 438]. They are also in [Niekus 2002, 11].

the values of the terms are made to depend on future mathematical experiences of the one who constructs the sequence. In the examples above Brouwer denotes with "I" or "we" the one who constructs it.

It is remarkable that Brouwer fell into inactivity after the introduction of these choice sequences. After [Brouwer 30] he hardly published anything for more then fifteen years.

After the Second World War he became active again. His papers of that period are characterised by what has been considered as a new method, which is known as *the method of the creative subject.* We treat the example from [Brouwer 48]. His proof of the non-equivalence of apartness and inequality starts with the following definition:

> Let α be a mathematical assertion that cannot be tested, i.e. for which no method is known to prove either its absurdity or the absurdity of its absurdity.
>
> Then the creating subject can, in connection with this assertion α, create an infinitely proceeding sequence $a_1, a_2, a_3, \ldots$ according to the following direction: As long as, in the course of choosing the a_n, the creating subject has experienced neither the truth, nor the absurdity of α, a_n is chosen equal to 0.
>
> However, as soon as between the choice of a_{r-1} and a_r the creating subject has obtained a proof of the truth of α, a_r as well as a_{r+v} for every natural number v is chosen equal to 2^{-r}. And as soon as between the choice of a_{s-1} and a_s the creating subject has experienced the absurdity of α, a_s, as well as a_{s+v} for every natural number v is chosen equal to -2^{-s}.
>
> This infinitely proceeding sequence $a_1, a_2, a_3, \ldots$ is positively convergent, so it defines a real number ρ. ([Brouwer 1975, 478]).

There has been discussion about what the expression *creating subject* could mean. Note that in Brouwer's view mathematics consists of mental constructions, created by the human individual. A definition, e.g. as above, cannot be but a description of a construction, to be carried out by that individual. As it seems to us, Brouwer denotes with the expression *creating subject* the individual who can carry out the construction, where he used "we" or "I" before. Interpreted this way, the definition in [Br 48] is the same as in the examples of choice sequences above.

Brouwer himself remarks in the introduction of [Brouwer 48] that he uses this example in his lectures from 1927, and there is no other candidate in these lectures.

We conclude that Brouwer applies individual choice sequences, as introduced in [Brouwer 91], in the method of the creative subject.

4 Individual choice sequences — how to treat them

Contrary to [Brouwer 91] and [Brouwer 30], Brouwer gives in [Brouwer 48] a detailed proof of the stated result, which we shall analyse in this section. We shall take the expression *the creating subject* to denote *any mathematician*, which could be ourselves. Therefore we shall use *we* in our reconstruction, as Brouwer did in his early use of choice sequences.

Thus the sequence defining ρ in the definition given in [Brouwer 48] above is a sequence we *can* construct. We reason about this sequence before the construction actually has started, we just use the definition. Since the terms of the sequence depend on our future mathematical experience, we need principles for reasoning about the future. We start with the formalisation of such principles.

Suppose the future to be divided into ω discrete stages. For a mathematical assertion φ and a natural number n

$$G_n\varphi$$

is defined as: on the n-th stage from now we are going to have a proof of φ. A proof of $G_n\varphi$ may depend on information coming free before stage n, but it may also be the case that $G_n\varphi$ holds because we already have a proof of φ now, since we suppose that a proof remains valid in the move of time. This is expressed by adopting for all natural numbers m and n, and for any mathematical assertion φ

$$G_n\varphi \Rightarrow G_{n+m}\varphi. \tag{1}$$

Let the present be stage 0, which is expressed by

$$\varphi \Leftrightarrow G_0\varphi. \tag{2}$$

Then obviously, for any mathematical assertion

$$\varphi \Rightarrow \exists n G_n\varphi \quad \text{is valid.} \tag{3}$$

From (3) immediately follows

$$\neg\exists n G_n\varphi \Rightarrow \neg\varphi. \tag{4}$$

The principles above are general principles for reasoning about future mathematical activity. They will be used in the reconstruction below; the basic step is (4).

We repeat the definition of ρ in [Br 48] with "we" instead of "creating subject"; A is a mathematical proposition which is not tested, i.e. we have neither a proof of $\neg A$ nor of $\neg\neg A$ *now*.

As long as, while choosing values for (a_n), we neither have attained a proof of A nor of $\neg A$, we take $a_n = 0$. If we find a proof of A between the choice of a_{n-1} and a_n, we take $a_{n+m} = 2^{-n}$ for all m. If we find a proof of $\neg A$ between the choice a_{n-1} and a_n we take $a_{n+m} = -2^{-n}$ for all m.

The sequence (a_n) is convergent, so it defines a real number, say ρ.

We connect the definition of (a_n) with the introduced stages by taking the division of stages such, that a_n is chosen at stage n. Given this division, if for some k $G_k A$ holds, then $\rho \geqslant 2^{-k}$ holds as well. So we have

$$\exists n G_n A \Rightarrow \rho > 0, \tag{5}$$

and analogously

$$\exists n G_n \neg A \Rightarrow \rho < 0. \tag{6}$$

Note that the newly introduced term $G_n A$ cannot be used in the definition of (a_n), e.g. by defining $a_n = 0$ iff neither $G_n A$ nor $G_n \neg A$ holds. The fact that $G_n A$ is not valid now, does not exclude that we will find a proof of A before stage n.

We are now ready for the proof. After the definition of ρ in [Brouwer 48] we cited in Section 3, Brouwer continues with:

> If for this real number ρ the relation $\rho > 0$ were to hold, then $\rho < 0$ would be impossible, so it would be certain α could never be proved to be absurd, so the absurdity of the absurdity of α would be known, so α would be tested, which it is not. Thus the relation $\rho > 0$ does not hold.
>
> Further, if for the real number ρ the relation $\rho < 0$ were to hold, then $\rho > 0$ would be impossible, so it would be certain that α could never be proved to be true, so the absurdity of α would be known, so again α would be tested, which it is not. Thus the relation $\rho < 0$ does not hold.
>
> Finally let us suppose that the relation $\rho = 0$ holds. In this case neither $\rho < 0$ nor $\rho > 0$ could ever be proved, so neither the absurdity nor the truth of α could ever be proved, so the absurdity as well as the absurdity of the absurdity of α would be known. This is a contradiction, so the relation $\rho = 0$ is absurd, in other words the real numbers ρ and 0 are different. ([Brouwer 1975, 478-479]).

We rewrite the first paragraph of this proof as follows:

If $\rho > 0$ holds, then $\neg(\rho < 0)$ holds, so $\neg\exists n G_n \neg A$ holds, so $\neg\neg A$ holds, and A would be tested. Since A is not tested, $\rho > 0$ does not hold.

Crucial in this rewriting is that "it would be certain that α could never be proved to be absurd" is expressed by $\neg\exists n G_n \neg A$. The reasoning in the rewritten paragraph is valid because of the following implications:

1. $\rho > 0 \Rightarrow \neg(\rho < 0)$
2. $\neg(\rho < 0) \Rightarrow \neg\exists n G_n \neg A$ because of (6)
3. $\neg\exists n G_n \neg A \Rightarrow \neg\neg A$ because of (4).

Analogously, if $\rho < 0$ were to hold, $\neg A$, and so the fact that A is tested, would follow from

1. $\rho < 0 \Rightarrow \neg(\rho > 0)$
2. $\neg(\rho > 0) \Rightarrow \neg\exists n G_n A$ because of (5)
3. $\neg\exists n G_n A \Rightarrow \neg A$ because of (4).

So $\rho < 0$ does not hold either.

However, if $\rho = 0$ were to hold, it would follow from

1. $\rho = 0 \Rightarrow \neg(\rho < 0)$
2. $\neg(\rho < 0) \Rightarrow \neg\exists n G_n \neg A$
3. $\neg\exists n G_n \neg A \Rightarrow \neg\neg A$

and from

4. $\rho = 0 \Rightarrow \neg(\rho > 0)$
5. $\neg(\rho > 0) \Rightarrow \neg\exists n G_n A$
6. $\neg\exists n G_n A \Rightarrow \neg A$

that $\neg A$ and $\neg\neg A$ were to hold, which is a contradiction, i.e. $\rho \neq 0$ does hold.

5 The standard reconstruction

In this last section we would like to point out the difference between our reconstruction of [Brouwer 1975] and the standard one in intuitionistic research ([Troelstra 1988, 842-846]). In the standard reconstruction the expression *creating subject* is changed in *creative* subject and interpreted as *the idealised mathematician* (for short IM).[8] The ω discrete stages cover all the mathematical activity of the IM, and do not, as in our version, extend from the present to the future before us. The basic notion of the resulting *theory of the creative subject* (for short TCS) is: the IM has a proof of φ at stage n. We express this by $I_n\varphi$. The TCS consists of the following axioms; n and m natural numbers, ϕ can be any mathematical assertion:

$$I_n\varphi \vee \neg I_n\varphi \quad \text{for all } n \tag{7}$$

$$I_n\varphi \Rightarrow I_{n+m}\varphi \quad \text{for all } n \text{ and } m \tag{8}$$

$$\varphi \Rightarrow \exists n I_n\varphi \tag{9}$$

$$\exists n I_n\varphi \Rightarrow \varphi \tag{10}$$

For a proposition A we can define (a_n) by

- $a_n = 0$ if $\neg I_n A \;\&\; \neg I_n \neg A$
- $a_n = 2^{-m}$ if $\exists m(m < n \;\&\; \neg I_m A \;\&\; I_{m+1} A)$
- $a_n = -2^{-m}$ if $\exists m(m < n \;\&\; \neg I_m \neg A \;\&\; I_{m+1} \neg A)$.

Brouwer's result is obtained as follows. Let ρ is the real number generated by (a_n).

A follows from $\rho > 0$ because of

1. $\rho > 0 \Rightarrow \exists n I_n A$
2. $\exists n I_n A \Rightarrow A$ because of (10).

Analogously ¬A follows from $\rho < 0$ because

1. $\rho < 0 \Rightarrow \exists n I_n \neg A$
2. $\exists n I_n \neg A \Rightarrow \neg A$ because of (10).

[8] In the recent [Atten 2004, 64-71] again *creating* subject is used, but the interpretation is as in [Troelstra 1988].

So for this reconstruction it is sufficient that A is an undecided proposition. It does not explain why Brouwer did resort to an untested proposition. In our reconstruction in section IV above the principle comparable with (10),

$$\exists n G_n \varphi \Rightarrow \varphi \tag{11}$$

is not valid; it would eliminate the distinction we want to make with G_n. Therefore, the untestedness is necessary in our reconstruction.

Finally we remark that the standard reconstruction has never been connected with the work of Brouwer in the late twenties.

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