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Are Statistical Laws Genuine Laws ?
A Concern of Poincaré and Boltzmann.

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Introduction

Usually a law is understood as a rule which governs all the things of a certain area to which the law is applicable. Since statistical laws govern most of the things of such an area but allow exceptions for some of them – even if those exceptions are very unprobable – there is the question whether statistical laws are genuine laws at all.

The prototype of law which governs all the things of a certain area has always been the dynamical law as it occurs in Newton's Theory. At the time of Poincaré and Boltzmann however a considerable number of phenomena (like those of heat, friction, diffusion, radiation etc.) were known which couldn't be explained by dynamical laws but quite well by statistical laws. Both Poincaré and Boltzmann discovered important new properties of these kinds of laws. These discoveries led to a series of new important questions with a wide range of application from microcosmos to macrocosmos, from quantum theory to modern cosmology. Some of these questions are :

(1) Are all laws (ultimately) dynamical laws ?

(2) Are all laws statistical laws ? Could it be that all laws which govern the microstructure of the universe (of nature) are in fact statistical and the deterministic appearence is only on the surface of certain macroscopic phenomena ?

(3) A new discovery, dynamical chaos, shows that even within the area which is governed by dynamical laws the behavior of the system can change radically just by changing slightly some initial conditions. Could it therefore be that all the basic laws are dynamical laws and asymmetric initial conditions cause random behavior only explainable by statistical laws or chaotic behavior hardly explainable by any law ?

(4) Is the reversibility with respect to time (time symmetry) of the basic laws of Quantum Mechanics and General Relativity compatible with the irreversibility (with respect to time) of the statistical laws which govern life, order, and self similarity ? Is the irreversibility with respect to time an essential property of laws such that the basic laws of today's physics (the laws of QM and GR) are incomplete ?

Although Poincaré and Boltzmann didn't answer these questions which are still open today they had ingenious ideas about them and proved important theorems which have been used as basic steps towards a solution of these problems.

1. Can all laws be reduced to dynamical laws ?

Dynamical laws like Newton's laws of motion characterize a physical system by three important conditions :

(1) The state of the system at any given time t_i is a definite function of its state at an ealier time t_{i-1}. A unique earlier state (corresponding to a unique solution of the equation) leads under the time evolution to a unique final state (again corresponding to a unique solution of the equation).

This property is a more precise description of Laplace's idea. It is usually taken as the defining condition for determinism.

(2) The system is periodic, i.e. the state of the system repeats itself after a finite period of time and continues to do so in the absence of external disturbing forces.

(3) The system has a certain type of stability. Assume we make very small changes in the initial states, say within a neighbourhood distance of e. Then the distance of the state h(e) is proportionally small (no more than a linearly increasing function of time). This kind of stability with respect to small perturbations is called "perturbative stability" which holds in all linear systems.

According to (1) dynamical laws are deterministic in the following sense : Given an initial state (or any state of the system at a certain time t_0) any other state in the future at t_{0+i} can be predicted and any other state in the past at t_{0-j} can be retrodicted by applying the law ; i.e. by calculating a solution of the differential equation.

Assume a film is made of the world, i.e. of the events happening in the whole universe. After the film is developed we cut it into pieces corresponding to single film-pictures. Now we put the single pictures successively in time (in the order of time) into a long card index box like the cards of a library catalogue. Then one special state of the universe at a certain time t corresponds to one such card (film picture) of the catalogue. One can follow one trajectory across the (perpendicular to the) catalogue-cards.

Interpreted with the help of this illustration Laplace's determinism means that it suffices to know the law(s) of nature and one single catalogue card (film picture) corresponding to one state (of the universe) at a certain time t in order to construct all other cards of the catalogue, i.e. to predict and to retrodict all the other states of the universe.

The mechanistic world view underlying Laplace's determinism was based on the belief that all physical systems are – if analyzed in its inmost structure –

ultimately mechanical systems. Since a clock was understood as a paradigm example of a mechanical system the main thesis of the mechanistic world view could be expressed by saying that all complex systems (things) of the world – even most complicated ones like gases, swarms of moscquitos or clouds – are ultimately (i.e. if we would have enough knowledge of the detailed interaction of the particles) – clocks. Or to put it in Popper's words : "All clouds are clocks" [Popper 1965, 210].

That Laplace's idea is not satisfied in certain areas was discovered a long time ago. Thermodynamics is one example, fiction, diffusion, radiation are others. Such discoveries led to another global question :

2. Are all laws (ultimately) statistical laws ?

After the discovery of statistical laws in thermodynamics and later in other areas there was a general doubt with respect to the mechanistic and deterministic interpretation of the world.

That there are physical truths which are statistical in character was clear for Boltzmann and for Poincaré who both underline the importance of Maxwell's, Clausius', Gibbs' and Carnot's discoveries :

"Schon Clausius, Maxwell u.a. haben wiederholt darauf hingewiesen, daß die Lehrsätze der Gastherorie den Charakter statistischer Wahrheiten haben. Ich habe besonders oft und so deutlich als mir möglich war betont, daß das Maxwellsche Gesetz der Geschwindigkeitsverteilung unter Gasmolekülen keineswegs wie ein Lehrsatz der gewöhnlichen Mechanik aus den Bewegungsgleichungen allein bewiesen werden kann, daß man vielmehr nur beweisen kann, daß dasselbe weitaus die größte Wahrscheinlichkeit hat und bei einer großen Anzahl von Molekülen alle übrigen Zustände damit verglichen so unwahrscheinlich sind, daß sie praktisch nicht in Betracht kommen" [Boltzmann 1896, 567].

Poincaré after commenting on Carnot's principle and discussing irreversible processes which cannot be explained with the help of classical mechanics gives an example :

"A drop of wine falls into a glass of water ; whatever may be the law of the internal motion of the liquid, we shall soon see it colored of a uniform rosy tint, and however much from this moment one may shake it afterwards, the wine and the water do not seem capable of again separating. Here we have the type of the irreversible physical phenomenon : to hide a grain of barely in a heap of wheat, this is easy ; afterwards to find it again and get it out, this is practically impossible. All this Maxwell and Boltzmann have explained ; but the one who has seen it most clearly, in a book too little read because it is a little difficult to read, is Gibbs, in his 'Elementary Principles of Statistical Mechanics'" [Poincaré 1958, p. 97].

One of the first philosophers who noticed that a certain imperfection in all "clocks" allows to enter chance and randomness was Charles Sanders Peirce :

"But it may be asked whether if there were an element of real chance in the universe it must not occasionally be productive of signal effects such as could not pass unobserved. In answer to this question, without stopping to point out that there is an abundance of great events which one might to be tempted to suppose were of that nature, it will be simplest to remark that physicists hold that the particles of gases are moving about irregularly, substantially as if by real chance, and that by the principles of probabilities there must occasionally happen to be concentrations of heat in the gases contrary to the second law of thermodynamics, and these concentrations occurring in explosive mixtures, must sometimes have tremendous effects" [Peirce 1935, 1960, 6. 47].

The question was now : Could it not be the case that all laws are statistical and the deterministic outlook is only on the surface of macroscopic phenomena ? That is all complex systems (things) of the world are in fact – in its inmost structure, i.e. on the atomic level – like gases or swarms of mosquitos or clouds. This led to another extreme picture "All clocks are clouds" [Popper 1965, 210].

The question whether all physical laws can be reduced or based on statistical laws was however not a serious topic at the time of Poincaré and Boltzmann. One reason for that was that Quantum Theory was not yet available. Rather there were two important questions :

(1) Are physical laws which are statistical like the second law of thermodynamics (the law of entropy) compatible with the basic dynamical laws (of classical mechanics) ?

(2) Are the statistical laws, like the law of entropy explainable with the help of (or reducible to) dynamical laws ?

Zermelo thought to have proved that the answer to (1) is negative. But Boltzmann explains the misunderstandings of Zermelo and shows that there is no incompatibility.[1] Planck hopes that (2) is true and stresses that he does not go as far as Zermelo, who was Planck's assistant at this time.

"Zermelo, however, goes farther [than I], and I think that incorrect. He believes that the second law, considered as a law of nature, is incompatible with any

1 I'll come back to this point in chapter 4 when time-irreversibility is discussed.

mechanical view of nature. The problem becomes essentially different, however, if one considers continuous matter instead of discrete mass-points like the molecules of gas theory. I believe and hope that a strict mechanical significance can be found for the second law along this path, but the problem is obviously extremely difficult and requires time."[2]

"The principle of energy conservation requires that all natural occurrences be analyzable ultimately into so-called conservative effects like, for example, those which take place in the motion of a system of mutually attracting or repelling material points, or also in completely elastic media, or with electromagnetic waves in insulators. [...] On the other hand, the principle of the increase of entropy teaches that all changes in nature proceed in one direction. [...] From this opposition arises the fundamental task of theoretical physics, the reduction of unidirectional change to conservative effects.[3]

At the turn of the century and in the first half of it many physicists accepted a view which can be roughly stated as follows :

In respect to some areas (mainly macroscopic) deterministic laws with good predictibility for single events give an adequate description and explanation.

In respect to other areas (thermodynamics, friction, diffusion, radiation and microscopic areas) statistical laws with no good predictability for the single event but with predictibility for the whole aggregate give an adequate description and explanation.

Understood in this way it was compatible that for example the pendulum interpreted as a macroscopic dynamical system obeys Newton's laws and allows strict prediction whereas interpreted as a microscopic system, i.e. in respect to its atomic structure behaves in some of its features like a cloud and can then be adequately described by statistical laws without strict predictions for single particles.[4]

Quantum mechanics, already from its beginning put again a different complexion on the question of the status of statistical laws and their interpretation. It is clear that the laws mainly provided by this theory are statistical laws. But it is not clear whether the theory refers to individual quantum systems or only to big ensembles of prepared systems. The main difficulty here is that for every individual system, say an individual photon, the value of the observable before the measurement is objectively undetermined whereas a sufficiently large

2 Planck in a letter to his friend Leo Graetz. Cited in Kuhn (1978), p.27.

3 A paper read to the Russian Academy of Science in 1897. Cited in Kuhn (1978), p.28.

4 The underlying deeper question here is of course that of the completeness or incompleteness of physical laws (laws of nature). Cf. Weingartner (1997).

number of photons satisfy a statistical law telling relative frequencies in the experiment. Thus although there is indeterminacy in a good objective sense for every individual system there is a strict law if the ensemble is large enough such that we can speak of an objective and definite (i.e. Yes/No) property of the whole system.

Concerning such an emergence of law out of lawless behaviour of the individual system Wheeler spoke of "law without law".[5] However, at the time when Wheeler wrote this article one didn't have the kind of experiments to show unambigously this emergence of statistical laws as they are known today : Split-beam experiments with photons (and other particles). Soon after its discovery by Heisenberg and Schrödinger quantum mechanics was interpreted as a probabilistic theory (by Born) and understood by many physicists that way since.

In this context many philosophical interpretations ranging from more subjective and epistemic interpretations of probability like lack of information to more objective ones like potentiality and propensity have been proposed.[6]

Concerning such interpretations there are two questions : (A) Are there enough reasons to assume one of these interpretations if there are no experimental situations which show some property of the single system which could serve as a basis for such an interpretation ? (B) Do these interpretations distract from a more important feature which was expressed by the above considerations : that quantum mechanics is not so much a probabilistic theory but answers Yes/No questions with the help of statistical laws. And this gives an answer to the main question of this article : that at least from the point of view of this important area statistical laws are genuine laws.

Boltzmann and Poincaré couldn't be aware of these developments. But they understood one of the most important problems here : How can the "law" of entropy emerge from random behaviour of individual systems ? Can such a "rule" be a law at all ? What is its relation to that what (at that time) is unquestionable a law, i.e. the dynamical law ? And these questions have been problems expecially in quantum mechanics since its existence but have received new answers in the light of new experiments.

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5 Cf. Wheeler (1983). For a detailled theoretical interpretation of such experiments see Mittelstaedt (1997a).

6 For a discussion of such interpretations in the light of the question whether quantum mechanics is a probabilistic theory see Mittelstaedt (1997b).

3. The Discovery of Chaotic Motion

3.1 Do all laws obey the principle : Similar causes lead to similar effects ?

If the answer is Yes there shouldn't be cases with very small differences in causes which lead to very big differences in their effects. But there are such cases. A first warning concerning such a principle we find in Aristotle :

"The least initial deviation from the truth is multiplied later a thousandfold» [Aristotle (Heav) 271b8].

Attentive people seem to have been always aware of counterexamples to the above principle from situations in every day life. Thus experienced highlanders in mountainous countries know very well that extremely small initial events can lead to a bursting of an avalanche whereas another time the same small event leads to nothing (no comparable effect). A specific counterexample is due to Maxwell who discusses explicitely the question of the validity of the above principle :

"There is another maxime which must not be confounded with that quoted at the beginning of this article[7], which asserts 'That like causes produce like effects'. This is only true when small variations in the initial circumstances produce only small variations in the final state of the system. In a great many physical phenomena this condition is satisfied ; but there are other cases in which a small initial variation may produce a very great change in the final state of the system, as when the displacement of the "points" causes a railway train to run into another instead of keeping its proper course" [Maxwell (MaM), p. 13].

What these examples describe is a strong sensitivity with respect to initial conditions. And this is one of the most important necessary conditions of chaotic motion. Observe however that it is not sufficient. In Maxwell's example the running of the train in a completely different direction is not a chaotic phenomenon, though some aspects of the crash may have chaotic properties.

Hadamard and Poincaré were the first to mathematically attack the problem of the sensitivity with respect to initial conditions. Jacques Hadamard made calculations of a billard game with negative (concave) curvature ; i.e. a billard table which is not plane but has dents. Of such a billiard game he could prove the sensitive dependence upon initial conditions.[8] In connection with that

7 The one to which Maxwell refers is "The same causes will always produce the same effects" which he discusses earlier.

Hadamard understood that the behavior of a system becomes completely unpredictable if a small error is committed in the initial data. This result and also the so called Henón-attractor can be viewed as modern mathematical interpretations of Aristotle's passage above.

Calculations of Chirikov and Berry showed how drastic the sensitive dependence on initial conditions could be. Imagine the gravitational force of an electron (somewhere in the universe, say 10^{10} lightyears away) is taken away for a moment. Could there be a difference in the interaction (with respect to collision) of air-molecules on the earth ? After how many collisions with other molecules the extremely small gravitational difference far away could have the effect of failing instead of meeting (collision) another molecule ? The answer is very astonishing indeed, it is between 56 and 60 only. If we take a man being in 1m distance of a billiard table and his gravitational force on the billiard balls the respective answer is 9. Already Bernoulli in the 18th century made similar calculations with respect to games (known as "Bernoulli-shift").

The sensitive dependence with respect to initial conditions is measured by a positive Liapunov exponent. This exponent measures in fact two things : (1) First it measures the exponential separation of adjacent conjugate (with respect to the starting point) points. (2) Second it measures the loss of information about the position of a point (in an interval) after one iteration. This loss of information is proportional to the socalled Kolmogorov entropy.[9]

3.2 The Question of Integrability and the Prize-Question

Poincaré worked extensively on these problems – even earlier than Hadamard - in his Mécanique Céleste. In this work he answered the prize question of the King Oscar II (of Sweden) in the negative. The prize-question (announced in 1885) was this :

> For an arbitrary system of mass points which attract each other according to Newton's laws, assuming that no two points ever collide, give the coordinates of the individual points for all time as the sum of a uniformly convergent series whose terms are made up of known functions.

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8 The respective proof for a billiard table with positive (convex) curvature (i.e. with hillocks or also with round corners) is more difficult and has been given only in the seventies by Sinai. Cf. Cornfeld–Fomin–Sinai (1982). For a lucid exposition of some of the main-ideas of Hadamard, Poincaré and Boltzmann see Ruelle (1991). Ruelle has done extensive research on the theory of socalled "Strange Attractors".

9 For more details and references see Weingartner (1996) chapter 2.

The price was given to Poincaré. However he did not really solve the problem but gave strong reasons that such series do not exist, i.e. that contrary to the expectation these series of perturbation theory in fact diverge.[10]

The prize-question was partially answered by Kolmogorov in 1954 and solved by his pupil Arnol'd in 1963. A special case of it was answered by Moser. Hence the name KAM-theorem. It gives an answer to the question whether an integrable system (with an arbitrary number of degrees of freedom) survives weak perturbation. The theorem says that the answer is positive and that the invariance with respect to small perturbation or the stability is proportional to the degree of irrationality of the rotation number r of the curve of the trajectory. This has led to a new (weakened) concept of stability which holds for the majority of the orbits ; i.e. the majority of solutions (for the respective differential equations) are quasiperiodic. The system is then partially integrable ("KAM-integrable").

In his mechanique celeste Poincaré was far ahead of his time. Until 1885 it was generally assumed that all dynamical systems are integrable. This is more or less true for the two-body system (for example earth and sun) but not any more if a third body interfers. Poincaré showed that a perturbed physical system cannot be represented by an integrable Hamiltonian (which would consist of the sum of a free Hamiltonian plus an interaction-potential with a coupling constant for the perturbation). This was an insight which was fully understood only about 80 years later when Kolmogorov and his pupil Arnol'd took up the matter and when other scientists investigated more and more the many-body problem. Today we know that the solar system is partially chaotic where Mercury shows chaotic motion to a higher degree than Mars and Mars to a higher degree than Venus and Earth [Laskar 1994]. In fact Poincaré conjectured that the real movement of the planets is more complicated than their description by Kepler's and Newton's laws would suggest :

> And Newton's law itself ? Its simplicity, so long undetected, is perhaps only apparent. Who knows if it be not due to some complicated mechanism, to the impact of some subtle matter animated by irregular movements, and if it has not become simple merely through the play of averages and large numbers ? In any case, it is difficult not to suppose that the true law contains complementary terms which may become sensible at small distances. [Poincaré 1952, 148]

10 Weiserstrass though in the price-committee (together with Mittag-Leffler and Hermite) was rather astonished about Poincaré's answer. Since he arrived at a proof to the contrary for very special frequencies. Today it is known that there are exceptions for very special frequencies.

> Kepler remarks that the positions of a planet observed by Tycho are all on the same ellipse. Not for one moment does he think that, by a singular freak of chance, Tycho had never looked at the heavens except at the very moment when the path of the planet happened to cut that ellipse. What does it matter then if the simplicity be real or if it hide a complex truth ? [*ibid.*, 149]

A very unexpected experiment, performed only since 1984, is that with a spherical pendulum under some very special initial conditions. Students of physics know that the spherical pendulum was always one of the best examples to demonstrate motion governed by dynamical laws. But the new discovery with this and similar experiments show that even inside the area of the applicability of dynamical laws a special change of initial conditions can make the motion chaotic.

The socalled spherical pendulum consisting of a small weight attached to the lower end of a string (of length l) has a period $T_0 = 2\pi\ (l/g)^{1/2}$ of sinusodial oscillations (provided the oscillations are small). The spherical pendulum (under normal conditions with small amplitudes) shows a regular behavior with the three important characteristics given in chapter 1.

The important new discovery is now that this simple physical system becomes chaotic if the top end is forced to move back and forth (maximal displacement D) with a slightly different period T greater than T_0 , provided that D is about 1/64 of l and not more than about a tenth of the energy of motion is dissipated by damping (air resistance etc.). In 1984 Miles showed experimentally that the system is chaotic for values of $T = 1{,}00234T_0$. It has to be emphasized however that this does not just mean that the system becomes unstable in the sense of simple bifurcation. Unstability in the sense of simple bifurcation has been known for a long time. In this case the pendulum weight makes a back and forth oszillation in the same plane and by forcing the upper end this movement begins to be unstable. Such a simple bifurcation where the plane is not changed occurs when $T = 0{,}989T_0$ and slightly above. But for $T = 1{,}00234T_0$ the pendulum is breaking out of the plane, the number of further bifurcations are arbitrarily increasing, the dependence on initial conditions is completely random such that there is no predictability (or only for very short times). [Lighthill 1986]

3.3 Chaotic motion and statistical description

Chaotic motion[11] has a number of other properties besides the two mentioned above (sensitive dependence on initial conditions and non-integrability). Elsewhere I have described eight characteristics of chaotic behavior.[12] Here a further point has to be mentioned which throwed some new light on statistical laws. The development through time of a chaotic system cannot be described by single trajectories. Or in other words : Such a description would be an untenable idealization. And this is so although the equations are dynamical (and in this sense deterministic) laws. The reason is that one usually assumes that a trajectory has a certain robustness ; i.e. it wouldn't vary too much if the initial conditions are changed slightly. But for chaotic systems this robustness is not available because of the sensitive dependence with respect to initial conditions. Two trajectories which differ extremely little with respect to their initial conditions (their starts) diverge exponentially in the course of time (after a finite, relatively small number of iterations). This situation leads to new attempts to apply a statistical description. Instead of single trajectories one has to take into account ensembles of trajectories which have new properties and can only be described by statistical laws.

4. Boltzmann versus Poincaré ?

Is Boltzmann's statistical mechanics as an explanation for time-irreversible processes incompatible with Poincaré's recurrence theorem ?

4.1 Poincaré's recurrence theorem says that for a Hamiltonian system a trajectory returns to a given neighborhood of a point an infinite number of times (if the time is infinite).

To give an illustration : skiing in fresh powder snow is a great pleasure. But if the slope is small and one is skiing down frequently the slope will be filled with traces and after some time no new space (powder snow) is left and thus one has to use one's own traces again (recurrence). This illustration tells us already some important conditions : The motion has to be area-preserving (the skier is not supposed to leave the slope) and in a finite region.

Another example : Place one beetle on the first square of a chess board with borders (so that it cannot leave the chess board). After some time of running around the beetle will again come to the first square (recurrence). In these cases it is easy to see that Poincaré's recurrence theorem holds : After a sufficiently

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11 In the sense of dynamical chaos. So called Quantum-chaos has different properties.

12 Weingartner (1996) chapter 1.3. Moreover it should be mentioned that there are different levels of chaotic behavior which correspond to levels of increasing disorder. They begin with integrability and quasiperiodic motion. On a higher level there is KAM integrability and weak perturbation (the non-integrable part being a restricted chaotic layer). On a still higher level of disorder there is chaotic motion. Cf. Chirikov (1991).

long time a certain state of the system recurs again, or the system comes arbitrarily close to the respective earlier state again.

In 1893 Poincaré introduced a method to check a trajectory, the socalled Poincaré map. This is a plane which cuts the trajectory line. Plotting the points in which a trajectory cuts the plane (usually meant in one direction) gives a sectional view of the trajectory. If there is recurrence in the same point (on the Poincaré map) the motion is cyclic and periodic (normal undisturbed spherical pendulum). Other possibilities are : The motion may approach a fixed point (like a spiral) ; it may be a motion around a torus (in this case the points in the map are arranged on an inner or outer circle and the motion is quasiperiodic). If the motion is chaotic the Poincaré map has space filling points (is continuous).

4.2 Irreversible processes

At the time of Poincaré and Boltzmann numerous examples of irreversible processes were known. One simple example of Poincaré has been cited already in chapter 2. A crucial physical experiment which shows irreversibility is Carnot's process which is frequently discussed by Poincaré :

> Let us commence with the principle of Carnot. This is the only one which does not present itself as an immediate consequence of the hypothesis of central forces ; more than that, it seems, if not to directly contradict that hypothesis, at least not to be reconciled with it without a certain effort. If physical phenomena were due exclusively to the movements of atoms whose mutual attraction depended only on the distance, it seems that all these phenomena should be reversible. [...] On this account, if a physical phenomenon is possible, the inverse phenomenon should be equally so, and one should be able to reascend the course of time. Now, it is not so in nature, and this is precisely what the principle of Carnot teaches us ; heat can pass from the warm body to the cold body ; it is impossible afterwards to make it take the inverse route and to reestablish differences of temperature which have been effaced. Motion can be wholly dissipated and transformed into heat by friction ; the contrary transformation can never be made exceptpartially. [Poincaré 1958, 96]

We are surrounded with other irreversable processes : Flow of water, flow of glacier, flow of heat, diffusion, friction, phenomena of electric transport, absorption and dispersion, transmission and relaxation phenomena, radiation, floods, avalanches, lightenings, growing, aging, propagation etc.

For a number of such processes, especially those occurring in gases Boltzmann has given a theoretical explanation with his statistical mechanics. His theory is based on earlier investigations especially of Maxwell and Clausius. This is not the place to go into details of Boltzmann's H-Theorem or the second law of thermodynamics (i.e. the law of entropy). But some simple points are in order. Entropy is a measure for the reversibility of a process. For reversible processes the entropy cannot increase, for irreversible processes it increases. The law of entropy says that in a closed system the entropy can remain the same or must increase, it cannot decrease. Boltzmann connected the entropy with the atomic hypotheses, i.e., roughly with the assumption that matter is composed of a huge number of little balls dancing around wildly. At that time the atomic structure of matter was not generally accepted. Mach's scepticism about it is wellknown. To some objections Boltzmann said once :

> Auf Einwände, welche gegen diese Theorie von Poincaré in sehr feiner und scharfsinniger, von Bertrand in minder höflicher und auch minder scharfsinniger Weise gemacht wurden und auch in Deutschland Widerhall fanden, will ich hier nicht eingehen. Diese Sache bildet noch immer den Gegenstand von Kontroversen, doch glaube ich auch von den Molekülen beruhigt sagen zu können :
>
> Und dennoch bewegen sie sich ! [Boltzmann 1897c, 608]

Assume a litre of air consisting of about 2,7 . 10^{22} molecules. Then it will be understandable that this system of 2,7 . 10^{22} molecules can be in a huge number of different (micro-)states, the number is about $10^{5 \cdot 1022}$ (i.e. 10 to the power of a 5 with 22 zeros) so as to realize the macrostate "litre of air". Thus the same (for our eyes or lungs the same) macrostate can be realized by a huge number of different microstates. Boltzmann's discovery was that the probability of such a macrostate can be defined as the number of microstates which can realize the macrostate and that this number (more accurately the logarithm of it) is the entropy. Thus the law of entropy has a new interpretation : The probability of a macrostate of a closed system can only increase or stay the same, it cannot decrease. That means that the time development of closed systems goes in the direction from less probable states to more probable states. This means of course that these processes are time irreversible. For example the Big Bang Theory of modern cosmology says that the whole universe developes into more and more probable states according to Boltzmann's idea.

4.3 But how are the irreversible processes and especially the law of entropy compatible with Poincaré's recurrence theorem ? And more generally, how are they compatible with the dynamical laws (recall chapter 2) ?

With respect to the first Zermelo thought he had proved that the two are incompatible. And since recurrence and reversibility with respect to time holds for the dynamical laws he thought he had proved also that Boltzmann's Statistical Mechanics cannot be a correct theory of the irreversible processes.[Zermelo 1896a,b]

However, as the replies of Boltzmann [Boltzmann 1896, 1897a] show Zermelo partially neglected and partially misunderstood important conditions in connection with Poincaré's recurrence theorem.

4.31 The first thing which is made clear by Boltzmann is that Poincaré's recurrence theorem is not applicable under conditions where the number of molecules is infinite and time is increasing and can be very long but finite. On the other hand if the conditions are such that time is infinite and the number of molecules is very large (but finite and in a finite space) then Poincaré's theorem is applicable.

4.32 The second thing which Boltzmann realized very clearly is that the probability of recurrence depends very much on the complexity of the system. This can be easily explained and will be understood intuitively. Remember the examples of chapter 4.1. Imagine we place now 10 beetles on each square of the chess board and let them run around. What is the probability now that the state of the whole system will recur, i.e. that each of the 640 beetles will be in the same start position again (assume we have marked each individual) ? It is easy to understand that this probability will be much lower than the one that a single beetle will come back to its starting point.

Thus recurrence is not impossible but has very low probability if the system is more complex. Imagine there are thousands of skiers on the slopes of a big ski resort (the cable cars and lifts of a big ski region in Austria can take up about 60.000 people per hour). The probability that at some later time t_0 all skiers will be again at a position in which all the skiers were at the time t_i earlier such that the whole state of this system would recur has much lower probability (even if we assume that they go on skiing days and nights) than the recurrence of a single skier to an earlier position.

Imagine now the molecules of a litre of gas (of oxygen or nitrogen at normal temperature and pressure). As mentioned above there are about $2{,}7 \cdot 10^{22}$ molecules moving around with high speed having about $10^5 \cdot 10^{22}$ different (possible) states. It will be understood from extrapolating the above examples with the ants and skiers that a recurrence of one state of the whole system will have very low probability.

Bearing in mind that the probability of recurrence depends highly on the complexity of the system one of the most decisive points of Boltzmann is now the following : Recurrence of a state very close to the initial state after a very (infinitly) long time is compatible with a time-irreversible development of a system in the sense that the recurrence of a state of lower entropy (compared to the one already realized) has extremely low probability (although it is not impossible). A contradiction with Boltzmann's Statistical Mechanics would only arise if it would follow from Poincaré's Recurrence Theorem that the time of recurrence is an observable length of time. Such an observation of recurrence would refute Boltzmann's theory, though not on logical grounds (there is no impossibility but only extremely low probability) but on empirical grounds. But as Boltzmann states such a consequence does of course not follow from Poincaré's Recurrence Theorem and therefore Zermelo could not prove that :

> Die Konsequenz des Poincaré'schen Satzes, daß abgesehen von wenigen singulären Zustandsverteilungen ein dem Anfangszustande sehr naher Zustand nach einer, wenn auch sehr langen Zeit immer wiederkehren muß, steht daher in vollstem Einklange mit meinen Lehrsätzen.
>
> Nur der Schluß, daß an den mechanischen Grundanschauungen irgend etwas geändert oder diese gar aufgegeben werden müßten, darf daraus nicht gezogen werden. Dieser Schluß wäre nur berechtigt, wenn sich aus den mechanischen Grundanschauungen irgend eine mit der Erfahrung in Widerspruch stehende Konsequenz ergäbe. Dies wäre aber nur der Fall, wenn Hr. Zermelo beweisen könnte, daß die Zeitdauer dieser Periode, innerhalb welcher der alte Zustand des Gases nach dem Poincaréschen Satze eintreten muß, eine beobachtbare Länge hat. Es dürfte nun zwar schon a priori evident sein, daß, wenn etwa eine Trillion winziger Kugeln, jede mit einer großen Geschwindigkeit begabt, zu Anfang der Zeit in einer Ecke eines Gefäßes mit absolut elastischen Wänden beisammen waren, sich dieselben in kurzer Zeit ziemlich gleichmäßig im Gefäße verteilen werden, und daß die Zeit, wo sich alle ihre Stöße so kompensiert haben, daß sie alle wieder in derselben Ecke zusammenkommen, so groß sein muß, daß sie niemand zu erleben imstande ist. Zum Überfluß ergibt die im Anhange beigefügte Rechnung für diese Zeit einen Betrag, dessen enorme Größe wahrhaft beruhigend ist. [Boltzmann 1896, 571]

It does not seem that Poincaré would have not understood this. On the contrary he frequently discusses irreversible phenomena. He didn't interpret his own theorem in the way Zermelo interpreted it. But it seems that also Zermelo finally – after Boltzmann's refuting criticism of his two essays – gave up his position and resigned with respect to physics since he became engaged in Set

Theory. Boltzmann wrote also a special paper on Poincaré's Recurrence Theorem which shows how seriously he took the matter [Boltzmann 1897b]. There he explains again his standpoint more mathematically also mentioning some points of Zermelo.

4.4 Time-reversible basic laws and the Law of Entropy

Against Boltzmann's theory Loschmidt (in 1876) raised an objection which used the possible reversibility of the velocities of the molecules. Assume all molecules in a gas move "backwards". This is permitted if thermodynamics is explained by statistical mechanics, in general it is permitted because the equations of mechanical motion are time-reversible. The time-reversibility would allow H to increase (contrary to Boltzmann's H-theorem) and E (entropy) to decrease.

But also here there is no contradiction because the reversal of velocitics has extremely low probability even if it is not impossible. Independently of that computer-simulations show [Prigogine-Strengers 1993, 210ff] that the velocity distribution developes into the equilibrium of the Maxwell-Boltzmann distribution as Boltzmann has predicted. Moreover one can simulate reversal of velocities with such experiments. In the simulation experiments after a certain number of collisions (50 or 100) the velocity is reversed all of a sudden. It can be observed then that the H-function increases temporarily. But on the whole the development behaves according to Boltzmann's theory. If the number of collisions increase, H does not reach its initial value again and its tops will be lower and lower. It can also be observed that if there is already a high number of collisions before the beginning of the reversal of velocities the difficulty to reach again an initial value of H increases.

All this leads again to the two questions raised in chapter 2 : How are the basic laws of physics compatible with the law of entropy ? And are the statistical laws like the law of entropy explainable with the help of dynamical laws ?

To the last question there is no ultimate answer and the main problem which existed for Boltzmann is still not solved : I mean the problem that the basic laws of Quantum Mechanics and of General Relativity are time symmetric (time-reversibility) whereas the law of entropy defines an arrow of time (time-irreversibility). So how can the arrow of time be explained by the basic laws ?

There are different views about it : Some, like Chirikov [Chirikov 1996, ch. 2.2], think that further research on chaos could show in more detail how chaotic phenomena – though they are non-integrable systems and not describable by usual trajectories (having a kind of robustness, i.e. non-sensitivity) – still can have dynamical laws as their basis.

Chirikov's main point seems to be that non-recurrency and irreversibility with respect to time have to be distinguished. They are not equivalent notions. Thus one may keep reversibility with respect to time (time-symmetry) of the basic laws and of the processes described by them. On the other hand the arrow of time implied by statistical laws like the law of entropy can be interpreted as an internal arrow of the processes in which entropy increases. This amounts to a more modest interpretation of the arrow of time in statistical laws or in statistical processes : The internal arrow of the process means non-recurrency but it leaves the reversibility with respect to time untouched.

Others, like Prigogine [Prigogine-Stengers 1993, ch. 6, 8, 12], think that the basic laws of nature are irreducibly probabilistic, i.e. statistical laws which cannot be reduced to other (more basic) laws. The statistical laws used in physics so far were understood as (in principle) reducible laws. A sign for this understanding was the attempt to describe physical processes with the help of trajectories and wave functions. Processes of chaos and order seem to show however that ensembles of trajectories (corresponding to large Poincaré-systems) are needed for a more realistic description. Irreversibility with respect to time and the probabilistic character are then objective basic properties of the system. Consequently the arrow of time becomes an essential property of nature.

I think today that it is more reasonable to defend the more modest view of Chirikov : what statistical laws in the sense of the law of entropy show is non-recurrence. But strong non-recurrence suffices for strong experience of an arrow of time, first of all in experience of everyday life. But it seems that non-recurrence is also sufficient to explain even from a scientific point of view the appearing "arrow of time" in macroscopic processes where life is involved and in the cosmological evolution.

As far as I can see Boltzmann never claimed more than non-recurrence or better the extremely low probability of recurrence. He clearly points out that for the universe both directions of time are indistinguishable [Boltzmann 1897a, 579]. And although he uses the term irreversibility (in German : Irreversibilität) he speaks of the "Schein der Irreversibilität" (the appearence of irreversibility) where he explains the compatibility in the sense of question (1) of chapter 2 [Boltzmann 1898, Vol. II, 257].

Independently of these questions concerning the "arrow of time" it has been mentioned already at the end of chapter 2 that it is problematic to think of the laws of quantum mechanics as irreducibly probabilistic in character and probably too rash to think so of the basic laws of nature. In any case the question in the title of this article : "Are statistical laws genuine laws ?" is easily answered by what has been said so far : They are certainly genuine laws although they might not be as basic as a few laws of Quantum Mechanics and General Relativity.

Concerning the first question above, i.e. that of compatibility of the two types of laws a nice Gedankenexperiment of Lee [Lee 1998, ch. Symetries and Asymetries] shows that there is no difficulty that the underlying basic laws are time symmetric and the laws on a higher level are statistical laws in Boltzmann's sense which are not time symmetric and where recurrence has very low probability :

Asume a number of airports with flight connections in such a way that between any two of these airports the number of flights going both ways along any route is the same. This property will stand for microscopic reversibility. Some of the airports may have more than one air connection (they are connected with more than one other airport) whereas other airports have a connection only to one airport (let's call such airports dead end airports). A passenger starting from a dead end airport (or starting from any other airport) can reach any other airport and can also get back to his starting airport with the same ease. This property stands for macroscopic reversibility. In this case we have both microscopic and macroscopic reversibility.

But suppose now we were to remove in every airport all the signs and flight informations, while maintaining exactly the same number of flights. A passenger starting from a dead end airport A will certainly reach the next airport B since that is the only airport connected with A. But then – especially when assuming that B has many flight connections – it will be very difficult to get further to his final destination, in fact it will be a matter of chance. Moreover his chance to find back to his dead end airport A will be very small indeed. Thus in this case we have microscopic reversibility maintained but macroscopic irreversibility and both are not in conflict.

It will be easily understood that also here the probability of a recurrence decreases drastically if the system is more complex. In the above example this would mean to replace the single passenger by thousands of passengers flying around. The probability that all these passengers will recur at the same time to their starting position will be extremely low. This is so even with flight information but without the probability will be still much lower. Thus we have an example of a kind of an internal arrow of time on the higher level, while time symmetry on the lower level.

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