PHILOSOPHIA SCIENTIÆ

LADISLAV KVASZ

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Philosophia Scientiæ, tome 3, nº 2 (1998-1999), p. 183-201
<http://www.numdam.org/item?id=PHSC_1998-1999__3_2_183_0>

The Epistemological Foundations of Geometry in 19 th Century

Ladislav Kvasz
Faculty of Mathematics and Physics
Comenius University - Bratislava

Abstract. The aim of the paper is to offer a theory of the development of the epistemic subject in the history of non-Euclidean geometry. Its basic idea is to use Wittgenstein's picture theory of meaning from the *Tractatus* as a tool for analysing the meaning of geometric pictures from the works of Lobachevsky, Beltrami, Cayley, and Klein.

Résumé. L'article prend pour but de présenter une théorie sur le développement du sujet épistemique dans l'histoire de la géometrie non-euclidienne. L'idée de base ici est l'application de la „*Picture theory of meaning*" empruntée du *Tractatus* de Wittgenstein pour arriver à une analyse du sens des images géometriques dans les oeuvres de Lobachevsky, Beltrami, Cayley et Klein.

Philosophia Scientiae, 3 (2), 1998-1999, 183-202

The present paper is a further development of the ideas presented in our article Henri Poincaré and the *Epistemological Interpretation of the Erlangen Program* [Kvasz 1994]. In that article we suggested to distinguish three kinds of epistemic subject used in geometry : the *subject* of projective geometry, the *meta-subject* of the Beltrami-Klein model and the *scattered subject* of the Erlangen Program. We showed that the differences between Kant's philosophy of geometry and Poincaré's philosophy of geometry are a consequence of the different kinds of epistemic subject, on which their theories are based. While Kant formulated his philosophy of geometry from the point of view of the meta-subject, Poincaré used the scattered subject.

In the present paper we would like to offer a theory of the development of the epistemic subject using the idea of the form of language from Wittgenstein's *Tractatus*. In the *Tractatus* Wittgenstein described the epistemic subject as „*the limit of the world*" and connected it with the form of language. Our basic idea is to use Wittgenstein's picture theory of meaning to understand the geometric pictures from the works of Lobachevsky, Beltrami, Cayley, and Klein. We will examine the pictures of geometry and try to find the development of their form (in the sense of Wittgenstein). But before turning to modern geometry, it is necessary to make a short detour to Renaissance painting.

1. The language of the perspectivist paintings

The Renaissance painters started to paint the world as they saw it, to paint it from a particular point of view, to paint it in perspective. They wanted to paint the objects in such a way that the picture would evoke in the spectator the same impression as if he were looking at the real object. So it had to evoke the illusion of depth. To reach this goal the painter had to follow three principles of perspective :

Perspective of size - the remote objects are to be painted smaller

Perspective of colours - the remote objects are to be painted with dimmer colours

Perspective of outlines - the remote objects are to be painted with softer outlines

By following these principles a special line appears on the painting – **the horizon**. In fact the painter is not allowed to create it by a stroke of his brush. It is not permitted to paint the horizon, which only shows itself when the picture is completed. According to proposition 2.172 of the *Tractatus* („*A picture cannot, however, depict its pictorial form : it displays it.*"), the horizon belongs to the form of the language. It corresponds to the boundary of the world pictured by the painting, and therefore, according to proposition 5.632 („*The subject does not belong to the world : rather, it is a limit of the world*"), the horizon belongs to the subject. So besides the signs of the language which express definite objects, there are expressions on the painting connected not with the objects, but with the subject.

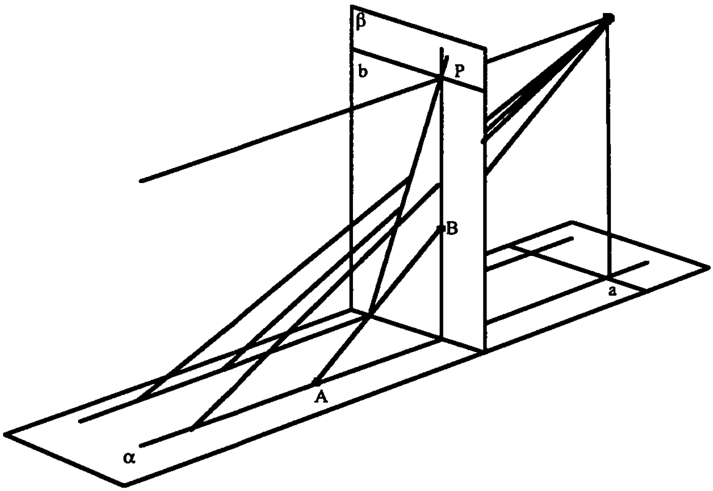
Albrecht Dürer showed us in one of his drawings [see Kvasz 1994] a method by which it is possible to create a perspectivist painting. We will describe Dürer's procedure, because it enables us to show what is common and what is different in perspectivist and projective picturing. Imagine that we want to paint some object so that its picture would evoke in the spectator exactly the same impression as if he were looking onto the original object. Let us take a perfectly transparent foil, fix it onto a frame and put it between our eye and the object we are intending to paint. We are going to dab paint onto the foil, point by point in the following way. We choose some point on the object (let it for instance be brown), mix paint of exactly the same colour and dab it on that point of the foil, where the ray of light coming from the brown point of the object into our eye, intersects the foil. If we have mixed the paint well, the dabbing of the paint onto the foil should not be visible. After some time spent by such dotting we create a picture of the object, which evokes exactly the same impression as the object itself.

By similar procedures the Renaissance painters discovered the principles of perspective. Among other things, they discovered that in order to **evoke the illusion of two parallel lines**, for instance two opposite sides of a ceiling, they had to draw two convergent lines. They discovered this but did not know why it was so. The answer to this, as well as many other questions, was given by projective geometry.

2. The language of projective geometry

Gérard Desargues, the founder of projective geometry came up with an excellent idea. He replaced the object with its picture. So while the painters formulated the problem of perspective as a relation between the picture and reality, Desargues formulated it as a problem of the relation between two pictures. Suppose that we already have a perfect perspective picture of an object, for instance of a jug, and let us imagine a painter who wants to paint the jug using Dürer's dotting procedure. At a moment when he is not paying attention, we can replace the jug by its picture. If the picture is good, the painter should not notice it, and instead of painting a picture of a jug he could start to paint a picture of a picture of the jug. Desargues did exactly this, and it was the starting point of projective geometry. The advantage brought by Desargues' idea is that, instead of the relation between a three-dimensional object and its two-dimensional picture we have to deal with a relation between two two-dimensional pictures. After this replacement of the object by its picture, it is easy to see that Dürer's dotting procedure becomes a central projection of one picture onto the other with its centre in our eye. Thus the **centre of projection represents the point of view from which the two pictures make the same impression.**

To make the central projection a mapping, Desargues had first of all to supplement both planes with infinitely remote points. After this the line *a* consists of those points of the plane α which are mapped onto the infinitely remote points of the second plane β. On the other hand, the line *b* consists of the images of the infinitely remote points of the plane α. So by supplementing each plane with the infinitely remote points, the central projection becomes a one-to-one mapping.



In the pictures of projective geometry there is a remarkable point – different from all other points – the centre of projection. As shown above, the centre of projection represents the eye of the painter from Dürer's drawing. Besides this point the pictures of projective geometry contain also a remarkable straight line. It is the line *a*. It is not difficult to see that the line *a* represents the horizon. But it is important to realise one basic difference between the horizon in a perspectivist painting and in a picture of projective geometry. In projective geometry the horizon is a straight line, it belongs to the language. It is not something that shows itself only when the picture is completed, as in the case of the paintings. Desargues drew the horizon, made from it an ordinary line, a sign of the language. Thus **the point of view is explicitly incorporated into language.** It is present in the form of the centre of projection and of the horizon, which belongs to this centre.

3. The language of non-Euclidean geometry

It is an interesting historical fact that even though Girolamo Saccheri and Johann Henrich Lambert discovered many propositions of non-Euclidean geometry, they persisted in believing that the only possible geometry is the Euclidean one. The break through in this question started only with Carl Friedrich Gauss, Janos Bolyai and Nikolaj Ivanovich Lobachevsky, who came to the conviction, that besides the Euclidean geometry another geometry is also possible. Gauss first called the new geometry anti-Euclidean, then astral, and later invented the name non-Euclidean, which is used currently. In order to determine the form of language and the kind of the epistemic subject of the non-Euclidean geometry, we will analyse the derivation of the cosine theorem by Lobachevsky.

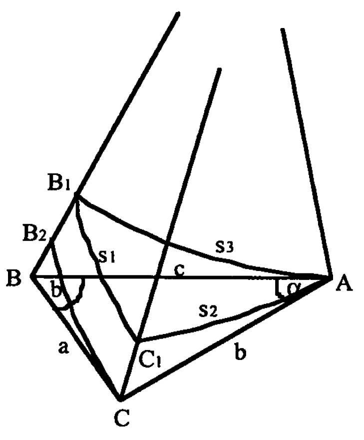
In non-Euclidean geometry there are two objects, which are similar to straight lines. First is the equidistant, which is the set of all points lying on one side of a straight line at a constant distance from it. The second is the limit line, which is the set of points towards which a circle passing through a fixed point approaches when its diameter grows behind any boundary. In the Euclidean geometry both the equidistant as well as the limit line, are straight lines. In the non-Euclidean geometry, on the other hand, they are not straight lines. Many of the incorrect proofs of Euclid's fifth postulate were based on the mistake that the geometer constructed an equidistant but regarded it as a straight line. But to say that the equidistant is a straight line means in other words to accept Euclid's fifth postulate. These propositions are equivalent. Therefore every proof of Euclid's fifth postulate based on the assertion that the equidistant is a straight line is a circular proof.

In a similar way one dimension higher we can form an equidistant surface to a plane and a limit surface (from a sphere passing through a fixed point, whose diameter grows beyond any boundary). On the limit surface there are limit lines lying in a similar way to straight lines lying on the plane. Lobachevsky discovered an interesting circumstance, namely, that on the limit surface there holds an analogy to Euclid's fifth postulate. If we chose a limit line l and a point P not lying on this line on the limit surface, then there is exactly one limit line, passing through the point P not intersecting the line l. That means that on the limit surface, for limit lines taken instead of the ordinary lines, the whole Euclidean geometry holds. So does among others the cosine theorem :

$$s_1{}^2 = s_2{}^2 + s_3{}^2 - 2s_2s_3.\cos a \qquad (1)$$

where s_1, s_2, s_3 are the lengths of the sections of the limit lines forming the sides of the triangle and α is the magnitude of the angle by the vertex A.

The formula (1) holds on the limit surface. But Lobachevsky wanted to derive the trigonometric formulas for triangles whose sides are sections of the straight lines of the non-Euclidean plane and not segments of limit lines. For this purpose he used the following picture. There ABC is a triangle on the non-Euclidean plane, which touches the limit surface in the point A. AB^1C^1 is the projection of the triangle ABC onto this limit surface. For the triangle AB^1C^1, as it is a triangle on the limit surface, Euclidean geometry holds.



Lobachevsky succeeded in finding the formulas that connect the lengths of the straight line segments of the plane with the lengths of their projections on the limit surface. These formulas a

$$s^3 = \sigma.\text{th}\,\frac{c}{k} \quad s^2 = \sigma.\text{th}\,\frac{b}{k} \quad s^1 = \sigma\,\frac{th\frac{a}{k}}{ch\frac{b}{k}} \qquad \text{(T)}$$

In these formulas there are two constants k and σ. The functions sh(x), ch(x), and th(x) are the so called hyperbolic sine, hyperbolic cosine and the hyperbolic tangent. With the help of these translatory formulas (T) it is possible to translate the cosine theorem (1) from the limit surface onto the non-Euclidean plane. We obtain :

$$ch\frac{a}{k} = ch\frac{b}{k}.ch\frac{c}{k} - sh\frac{b}{k}.sh\frac{c}{k}.\cos\alpha$$

The derivation of the cosine theorem is an important achievement, because this theorem plays a central role in trigonometry.

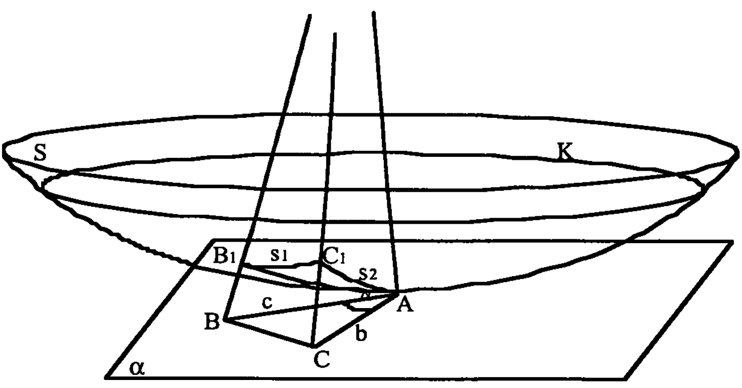
But how did Lobachevsky derive this formula ? First he **embedded** into the non-Euclidean space a fragment of Euclidean geometry (in the form of the limit surface), and then he **transmitted** the geometrical relations from this fragment onto the non-Euclidean plane. So this picture represents a junction of two languages. These two languages are separated from each other. One language, the Euclidean, is situated on the limit surface. The other, the non-Euclidean is situated on the plane. The formulas (T) establish the translation between these two languages. The formulas (T) are neither formulas of Euclidean geometry, nor formulas of non-Euclidean geometry. They belong to the metatheory, connecting these two geometries.

Nevertheless there is another, maybe even more striking change brought about by Lobachevsky. How was it possible to draw these pictures ? The line AC_1, the one, which is drawn on the picture, cannot be a segment of any limit line. It is drawn on ordinary Euclidean „paper“, and there does not exist anything like a limit line on this paper. In order to be able to understand this picture, we have to know that we must not take literally what we are looking at. Sure, we are looking onto an ordinary network of lines of the Euclidean plane, and on the Euclidean plane there is nothing like a limit line. But the situation in many respects resembles the Renaissance painting. The painting, strictly speaking, also does not have any depth, but nevertheless it is able to evoke depth. The ability to evoke depth is connected with the introduction of a new epistemic subject, the external point of view, which we have to take up, in order to see what we have to see. From this point of view we see looking at two straight lines which obviously converge and intersect, two parallel sides of a ceiling. We think that with Lobachevsky we are dealing with something similar, namely with the **external interpretative subject**. This does not have the form of a point of view. It is more an interpretative distance, which consists in the ability, for instance, to see a non-Euclidean triangle, which, strictly speaking, it is impossible to draw, beyond the triangle ABC, which is in the picture present in the form of an ordinary Euclidean triangle. The interpretative subject becomes a part of the language in the sense that, in the drawing of the picture it is taken into consideration. Anyone unable to take up this interpretative standpoint cannot understand the pictorial language. But it is external, because it is implicit, nobody is able to say exactly what Lobachevsky wants from us.

The pictures of Lobachevsky are thus based on two subjects. One of them, the internal, is an ordinary Desarguean centre of projection, from which the two triangles ABC and AB^1C^1 make exactly the same impression, so this is the subject that constitutes the correspondence of the object and its image. The other, the external, is the interpretative subject, which constitutes the interpretative distance. The basic problem of the pictures of Lobachevsky is that the internal subject is non-Euclidean, while the external subject is Euclidean. So in the pictures there is a conflict between forms of languages. The language that he uses is Euclidean, but what he wants to express is non-Euclidean.

4. The language of the Beltrami's model

Eugenio Beltrami, who constructed its first model, proved the consistency of non-Euclidean geometry in 1868. Felix Klein suggested a simplified version of this model in 1871, and we would like to discuss it briefly. But before we turn to the interpretation of Beltrami's model, we must return to the picture, which was used in the derivation of the trigonometric formulas by Lobachevsky. There we drew only a small sector of the limit surface in the form of the triangle AB_1C_1. Let us now draw a greater part of this surface as well as of the plane.

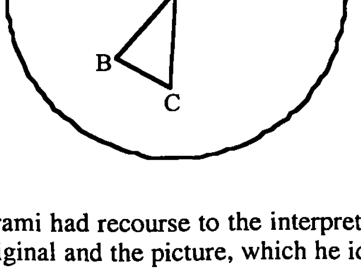


We can see that the plane is pictured not onto the whole limit surface, but only onto part of it in the form of a circle. The parameter s in the formulas

(T) is the diameter of this circle. The limit surface is in fact a sphere of non-Euclidean geometry, whose centre is infinitely remote. The projection from the limit surface onto the plane, used by Lobachevsky, happens from this infinitely remote centre. In the interpretation of Desargues we have mentioned that the centre of projection represents the point of view. So we can say that Lobachevsky used an improper internal perspectivist subject. In addition he also had an external interpretative subject.

Beltrami came with an original idea, which in many respects resembles Desargues; namely he **stuck the picture onto the original**. The central problem of Lobachevsky was that in his pictures he had a conflict of grammars. Beltrami removed this conflict when he identified the external interpretative subject with the internal projective subject. In other words, he looked at the above picture from the improper centre of projection. What did he see ? We know that the centre of projection is that point from which the image and the original make exactly the same impression. That means that from this point the limit surface (more exactly those parts of it onto which the whole plane is projected) and the non-Euclidean plane look exactly the same. The triangles ABC and AB_1C_1 blend.

It could seem that, by this identification of the picture and the original, the information gets completely lost. Lobachevsky's transition of the trigonometric formulas from the limit surface onto the plane was based exactly on the fact that these objects were different, and so they could correspond to one another.

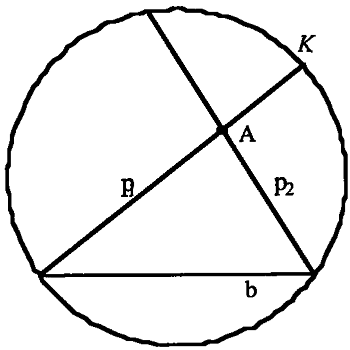


But here Beltrami had recourse to the interpretative subject. The difference between the original and the picture, which he identified physically, was transferred onto the shoulders of the interpretative subject. In a manner similar to Desargues when he turned the external perspectivist subject of the Renaissance paintings into the internal subject of the projective geometry having the form of the centre of projection, Beltrami internalised the interpretative subject of Lobachevsky. He turned the implicit requirements of Lobachevsky, which required one to see behind Euclidean lines non-Euclidean objects, into explicit rules. So the interpretation becomes expressible within the language. It gets the form of a dictionary (its first three lines refer to the picture above, and the last two lines to the next picture) :

SIGN	EXTERNAL LANGUAGE	INTERNAL LANGUAGE
K	*a circle on the Euclidean plane*	*the horizon of the non-Euclidean plane*
A, B, C	*points inside of the circle*	*points of the non-Euclidean plane*
D	*a point outside the circle*	*? ? ? ? ?*
b	*a chord of the circle*	*a straight line of the non-Euclidean plane*
p^1, p^2	*chords not intersecting the chord b*	*parallels to the straight line b*

But let us come back to Beltrami. When he looked onto the picture of Lobachevsky from the improper centre of projection, as already mentioned, the Euclidean objects on the limit surface blended with the non-Euclidean objects of the plane (for instance the triangles *ABC* and AB^1C^1 appear exactly the same). But that means, that **it is possible to draw them**! The conflict of grammars is overcome. We can draw everything „in a Euclidean way" and interpret it „in non-Euclidean terms". This is the advantage of the fact that the interpretative subject is directly present within the language. The interpretation, which for Lobachevsky was an implicit understanding of what the author wanted to express, has instead the character of explicit naming. We draw an Euclidean object and interpret it by definition as the non-Euclidean object with which it is identified in the projection. So this explicit incorporation of the interpretation in the language makes it possible to draw the non-Euclidean plane. We in fact draw the circle of the limit surface which, as Lobachevsky discovered, is Euclidean. The non-Euclidean plane is projected onto this circle, and thus from the centre of projection they make exactly the same impression. This makes it possible to name the geometrical objects inside of this circle after those non-Euclidean objects, with which these Euclidean, and so drawable, objects blend. That is why we call the interpretative subject as internal, because the naming, what is normally something implicit, based on showing, happens here explicitly, in the language.

The Beltrami-Klein model is based on a simple picture, in which a circle with some of its chords is represented.

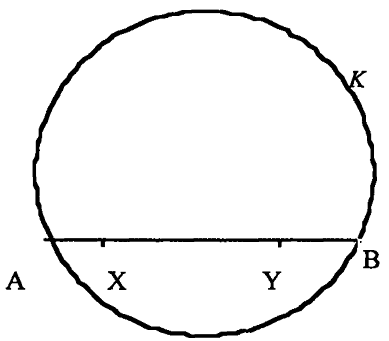


The points inside the circle (the circle excluding the circular line) represent the whole non-Euclidean plane, and the chords (excluding their endpoints) represent its straight lines. In this model all the axioms of Euclidean geometry are satisfied. Through any two different points of this plane there passes exactly one straight line, etc. - all axioms with the exception of the axiom of parallels. If we choose any straight line, for instance *b*, and any point which does not lie on this line, for instance A, through this point we can thus draw two straight lines (p^1 and p^2) which do not intersect the line *b*.

5. Cayley's relativisation of the metric

Although Beltrami's model definitely removed all doubts concerning the consistency of non-Euclidean geometry, nevertheless this is a model of non-Euclidean geometry within Euclidean geometry. This means that these geometries do not have equal status. First there must be given an Euclidean plane, in order to draw on it the circle *K* and in the inside of this circle to make a model of the non-Euclidean plane. Nevertheless it is not difficult to realise, that the inequality of the status of Euclidean and non-Euclidean geometries in the Beltramian model originates more in our physical constitution than in the nature of these geometries. So the primacy of the Euclidean geometry, which lies at the basis of the Beltrami's model, is more a weakness of this model than strength.

The English mathematician Arthur Cayley had the idea to look at Euclidean geometry from the position of the non-Euclidean, that is, to reverse the order in which they enter into the construction of Beltrami's model. He wanted to give up the view of Euclidean geometry as something given, as something binding, and try to see it also as a model. The basic idea of Cayley emerged when he examined the problem of introducing the concept of distance into the Beltrami's model. Let us imagine small beings, for which the interior of the circle *K* is their world. How would they measure distances ? Our usual metric is not suitable, because in it the distance from a point inside the circle *K* to its circumference is finite. Nevertheless for our beings this distance would be infinite, because the circumference of the circle is the horizon of their world, and as in each world the horizon is infinitely remote. So the usual concept of distance is unsatisfactory.



Cayley's suggestion was to look at this circle from the standpoint of projective geometry. The points A and B, in which the line XY intersects the circumference of the circle K, do not belong to the world of our small beings (for them these points are infinitely remote), but for us they are readily available. Here the interpretation leaves the object language of the model and resorts to the metalanguage. If we are looking for the concept of the distance between points X and Y, we have four points available (in the metalanguage), namely X, Y, A and B. And four points define an projective invariant magnitude, which was already discovered by Desargues, namely the double proportion :

$$(A, B; X, Y) = \frac{|AX|}{|BX|} : \frac{|AY|}{|BY|}$$

For the sake of brevity we omit the historical details and present only the resultant formula, which expresses the distance between points X and Y in the model as :

$$d(X,Y) = |\ln(A, B; X, Y)| \qquad (2)$$

It is not difficult to see that when the point X approaches the point A, the double proportion converges to zero, its logarithm drops to minus infinity, and so in absolute magnitude we get the value we need. As the point X approaches the horizon, its distance from the point Y grows beyond any bound. Of course, this formula is not in the language of our small beings. They do not understand what the point A is. But this is not important. The whole Beltramian model is formulated in the external language. What is important is that the distance is formulated with the use of a projective invariant, namely the double proportion.

The suggestion of Cayley was to **look at the whole of Beltrami's model from the standpoint of the projective geometry**. Let us forget that the Beltrami's model is constructed on the Euclidean plane and imagine that it is constructed on the projective plane. What is a projective plane ? It is what Desargues made from the Euclidean plane. So we have to forget the parallels, the distances and the angles. What remains is a snarl of lines, which intersect each other. Now onto this „clean" plane we draw the circle *K*. This circle intersects every straight line at two points, and with the help of these points we introduce the concept of the distance, using formula (2). So the circle introduces non-Euclidean distance. Besides this the circle divides the straight lines into three groups - those which intersect inside the circle, those which intersect onto its circumference, and those which intersect outside the circle. The lines of the first kind are secants, the second are the parallels, and the third is a special kind of lines, which Lobachevsky called divergents. So we see that besides the concept of distance, the circle can also be used to introduce the concept of parallels and the other kinds of relationships between straight lines. Also the concept of angles can be introduced using only the circle *K*. So the circle *K* is that object, which induces the non-Euclidean structure onto the projective plane.

Instead of Beltrami's construction

$$\mathbf{E} \longrightarrow \mathbf{L}$$

in which the model of the non-Euclidean geometry was based on the Euclidean plane, Cayley suggested a different scheme

$$\mathbf{E} \longrightarrow \mathbf{P} \longrightarrow \mathbf{L}.$$

The first arrow indicates the transition from the Euclidean plane to the projective plane and it consists in **disregarding** its Euclidean structure, that is, disregarding the Euclidean concepts of parallels, distances and magnitudes of angles. The second arrow indicates the passing from the projective plane to the non-Euclidean plane, and it consists in the **introduction** of the non-Euclidean structure, that is, introducing the non-Euclidean concepts of parallels, distances and magnitudes of angles. This second step is based on the circle K. Cayley has called this circle the absolute.

So Cayley comprehended the role that the circle K played in the Beltrami's model. It constitutes the non-Euclidean structure of the model's geometry. So he was able to formulate a fundamentally new question, namely the question : What constitutes the Euclidean-ness of the Euclidean plane ? In the framework of the Beltrami's model it was not possible to ask this question. If we take the Euclidean plane as something given, then the question of how can we introduce the Euclidean geometry onto this plane is meaningless. The transition

$$\mathbf{E} \longrightarrow \mathbf{E}$$

it is impossible to describe. What constitutes a language is inexpressible in this language. In Cayley's framework raises the same question very naturally, namely what constitutes the Euclidean-ness of the Euclidean plane. It asks by what geometrical object should we replace Beltrami's circle K in order to get the scheme

$$\mathbf{E} \longrightarrow \mathbf{P} \longrightarrow \mathbf{E}$$

that is, in order to get Euclidean geometry again in the projective plane. We have seen that Cayley brought a qualitatively deeper insight into the structure of the Euclidean geometry. He was able to find what constitutes its Euclidean structure. But what made this fundamentally new insight possible ? We have seen that it was a seemingly small shift, namely the transition from the Euclidean to the projective plane as the basis of the model. But how did Cayley accomplish this transition ? By an appeal ! He did not change anything in Beltrami's picture, rather he just asked us to forget that it is drawn on the Euclidean plane and instead to see the whole picture as drawn on the projective plane. That means, we must forget the parallels, the distances and the magnitudes of angles. But who can do this ? We think, nobody.

So here we are dealing with an appeal, similar to that, on which the whole of perspectivist painting is based, in that case, seeing two parallel sides of a ceiling behind the two intersecting lines of the painting. The appeal of Lobachevsky is analogous, asking us to see behind lines of the Euclidean plane the objects of non-Euclidean geometry. In a similar way to Dürer or Lobachevsky, Cayley is also asking us to abandon what we are looking at, namely a Euclidean plane, and instead to see the projective plane. Thus we can interpret this appeal just as we did in the other two cases, namely as the introduction of a new kind of external subject into the language. It is external, because the transition from the Euclidean plane to the projective plane is not explicitly described; Cayley does not tell us what we have to do in order to see the projective structure. He only requires an implicit understanding of what would happen if we were able to see the plane without the parallels, distances and magnitudes of angles. Of course, strictly speaking, we are unable to see in this way, but we understand what he wants to tell us. If we are willing to follow him, we are then able to understand the fundamental question of what constitutes the Euclidean structure of the Euclidean geometry.

The external subject introduced by Cayley we would like to call the integrative subject, because it makes it possible to integrate the Euclidean and the non-Euclidean geometry into one system. From this point of view we see the common foundation of the projective plane, from which both Euclidean and non-Euclidean geometries emerge in a uniform way, with the help of the curve called the absolute by Cayley. The circumstance that the Euclidean geometry had a prior position in our world is only a question of physics and cognitive science, but not of geometry. From Cayley on, the Euclidean and the non-Euclidean geometries are equivalent from the geometrical point of view.

Cayley's transition to the projective plane as the base of geometry makes it possible to go further to an even more radical question

$$\mathbf{E} \longrightarrow \mathbf{P} \longrightarrow \ ?$$

What geometries are possible ? This is a matter of taking different curves in the role of the absolute, and then finding out what kind of geometry is created on the projective plane. This is a basically new question, namely the question : What is geometry ? Cayley formulated this question, but the answer to it was given by Felix Klein.

6. The language of Klein's Erlangean program

The question „Which geometries are possible ?" was formulated by Cayley. In his framework the question meant : „Which curves taken as the absolute give rise to a geometry ?". Cayley understood that the role of the circle in the Beltrami's model constitutes the concept of the distance, and so he thought that the possible geometries could be found by examining different curves. But this approach is unsuitable to analyse even Euclidean geometry. The absolute is in this case degenerate and for this reason it does not create any metric. In addition, there are too many curves, many more than there could be geometries. Klein found a way out from this problem. The circle in the

Beltrami's model has an additional property, which Cayley did not notice. In the group of all projective transformations the circle defines a subgroup of those transformations, which transform the circle onto itself. By identifying the geometries with the subgroups of the projective group, Klein found a tool, which made it possible to give an answer onto Cayley's question. Not every curve is suitable for an absolute. The absolute can only be a curve such that it defines a subgroup within the projective group.

Thus Klein did the same thing with Cayley's integrative subject that Beltrami had done made with the interpretative subject of Lobachevsky, and Desargues with the perspectivist subject of the Renaissance painters : namely, he incorporated Cayley's subject into the language. The transformation group is exactly the tool that makes it possible to replace the appeal of Cayley to „forget the parallels, distances and angles", by an explicit instruction, „from the Euclidean group transit to the projective one". That is because the projective group is exactly that group which „destroys" the parallelness, the distances and the angles, leaving only the intersections and the double proportion. Thus Klein replaced Cayley's scheme

$$\mathbf{E} \longrightarrow \mathbf{P} \longrightarrow \; ? \qquad \text{by} \qquad \mathbf{G_E} \longrightarrow \mathbf{G_P} \longrightarrow \mathbf{G}_{?}.$$

Let us now come to a short epistemological analysis of Klein's Erlangen program. Our task is to explain why exactly the theory of groups was so successful in geometry. At first sight Klein's approach seems to be that he has taken a concept from algebra and brought it into geometry. So, why exactly this concept ? Could he have taken another ? Would it lead to different development of geometry ? What we need is an epistemological interpretation of the concept of group itself. The reconstruction we need can be found in *La Science et l'Hypothése* of Henri Poincaré. In his book Poincaré investigates the relation between the geometrical space and the spaces of our sensory perceptions. At first sight it could seem that these two spaces are identical. But this is not the case. Geometrical space is continuous, infinite, homogeneous, isotropic and three-dimensional. Poincaré shows that the space of our visual perceptions is neither homogeneous, nor isotropic nor three-dimensional. With the other sensory perceptions the situation is similar. That means that we can not derive the concept of the geometrical space from the space of any one isolated sensory organ. It is important to realise that, for the birth of the geometrical vision, visual impressions are not enough. The tactile and motor perceptions are necessary as well. So the concept of the geometrical space is from the very beginning connected with our body.

Poincaré asserts that we have derived the concept of geometrical space from the relations in which the changes of different kinds of perceptions (visual, tactile and motor) follow each other. The most important among these relations are the relations of **compensation**. We can compensate changes in the visual field by motion of our body or eyes. Poincaré showed that these compensations have the structure of a group, and that this group is the group of transformations of the Euclidean space. So the Euclidean space is neither the space of our sight, nor the space of our touch, nor the space in which are we moving. The Euclidean space is the structure in which these three sensorial spaces are integrated together. This analysis shows that the concept of group is something deeply concerned with us. The Euclidean group is the tool with the help of which each of us transcends the private world of his or her sensory perceptions. And in this way the concept of the group forms the ground on which the intersubjective language of spatial relations is based.

Given this background, it is not surprising that Felix Klein could very effectively use the concept of group in geometry, and that, with the help of the theory of groups, geometry reached a qualitatively higher level of abstraction. It is not surprising because the concept of group forms the foundation on which is constituted the concept of space. So Klein did not introduce an algebraic concept into geometry, rather he only made this concept explicit, which from the very beginnings lay implicit within the foundations of geometry. So we can say that Klein's Erlangen program was successful in bringing a unifying view to geometry, because the concept of group, on which this program is based, forms the epistemological foundation of the concept of space.

Now let us return to Poincaré's analysis of the concept of space. Poincaré speaks about compensatory relations between perceptions. But what does it mean to compensate ? To compensate means to attain sameness between the original and the new perceptions. But that is familiar to us, isn't it ? Let us recall Dürer. He also wanted two perceptions to be the same. The agreement is always an agreement from a specific point of view. We have characterised the point of view as that point from which the picture and the original make exactly the same impression. So also in the case of compensation we are dealing with some kinds of epistemic subject, subject for which the perceptions before and after the compensation are the same. The subject, which constitutes the group of compensatory relations and which forms the basis of Klein's approach to geometry, does not have the form of a point, as it did in the case of Desargues. For Desargues it was enough to introduce a single point into the geometrical language. Nor has it the form of two viewpoints, the external and the internal, as it did in the case of Beltrami. For Beltrami it was therefore enough to add a dictionary, which made it possible to translate theorems from the external into the internal language and vice versa. In the case of Klein viewpoints fill the whole space. The Euclidean space can be seen as the space of viewpoints. That is why we called this kind of epistemic subject as the scattered subject (Kvasz 1994). Now we prefer calling it the integrative subject, to lay more emphasis onto the concept of group, which is integrating all the possible viewpoints.

7. The development of the epistemic subject of the synthetic geometry

Following the development of the synthetic geometry we have seen a succession of changes leading from the Euclidean geometry to Klein's Erlangen program. If we would like to characterise the essence of these changes, we could say, that they were changes of the epistemic subject, which constitutes the language of geometry. We have found the following succession of languages :

*the language of the Renaissance painters based on the **external** perspectivist subject*

*the language of the projective geometry based on the **internal** perspectivist subject*

*the language of Lobachevsky based on the **external** interpretative subject*

*the language of the Beltrami's model based on the **internal** interpretative subject*

*the language of Cayley based on the **external** integrative subject*

*the language of Klein's Erlangen program based on the **internal** integrative subject*

If we compare this succession with the three stages described in [Kvasz 1994], we see, that beside some changes in terminology, each stage is now divided into two parts – one external and one internal. In this way the dynamic of the development of language becomes visible. The development consists in the incorporation of the external epistemic subject into the language. This incorporation is followed by the emergence of some new kind of epistemic subject. The succession of the different kinds of epistemic subject is not an accidental. It consists of changes, which link up one onto the other. Desargues, for instance, incorporated into the language of the projective geometry in the explicit form of the centre of projection the subject of the iconic language of the Renaissance painters. In this way he created the internal perspectivist subject. Lobachevsky, in his transition of the trigonometric formulas from the limit surface, already used this internal subject, but, as the whole picture was undrawable within Euclidean geometry, he was forced to introduce a second, external interpretative subject. Beltrami incorporated this external interpretative subject in the form of explicit rules of translation between the external and the internal languages. In this way he created the internal interpretative subject. Cayley disconnected the two languages, between which the translation took place and put the projective plane in between them. But to do this, he needed an additional structure of the language, based on a new kind of subject namely the integrative subject. But this subject was for Cayley only external, and Klein found the way to incorporate it into the language and to create the internal integrative subject.

We see that in this development always richer and richer zstructures of subjectivity are build into the language. First the subject in the form of the point of view, which is the basis of the subjectiveness of the personal view. Then the subject in the form of interpretation, which is the basis of the subjectiveness of meaning. And in the end the subject in the form of integration of all possible points of view, which is the basis of the subjectiveness of the possibilities of transcendence. We cannot deny that all these levels are parts of our own subjectivity. Everyone has his personal point of view, his own interpretation of reality and his unique potentiality of possibilities. So it is clear that we ourselves are the source from which geometry has taken the basic structures for its languages. In this development, deeper and deeper structures of our subjectivity were incorporated into the language.

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