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The Early Reception in France of the Work of Poincaré and Lyapunov in the Qualitative Theory of Differential Equations

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Résumé. Ce travail débute par une brève analyse de l'influence des contributions de Poincaré sur les travaux de Lyapunov en théorie qualitative des équations différentielles. Il se poursuit par un examen de l'impact des travaux de Poincaré et Lyapunov sur les contributions de Picard, Painlevé et Hadamard en théorie des oscillations et de la stabilité. L'article se termine par une analyse des exposés des travaux de Poincaré et Lyapunov dans les grands traités français d'analyse du XXe siècle.

Abstract. This work starts with a short analysis of the influence of Poincaré on the works of Lyapunov in the qualitative theory of differential equations. Then follows a description of the impact of Poincaré and Lyapunov's works on the contributions of Picard, Painlevé and Hadamard in oscillation and stability theory. The paper ends with an analysis of the description of the works of Poincaré and Lyapunov in the most important French treatises on analysis during the XXth century.

1. Introduction

The beginning of the qualitative theory of nonlinear differential equations is associated with the names of Henri Poincaré [see Darboux 1913; Hadamard 1921; 1922; 1933; Picard 1913] and of Alexandr M. Lyapunov [see Grigorian 1980; Smirnov 1992].

Specially influential have been Poincaré's three volumes *Les Méthodes Nouvelles de la Mécanique Céleste* which appeared between 1892 and 1899, and the original Russian version of Lyapunov's *Problème général de la Stabilité du Mouvement*, published in 1892. The reader can consult [Mawhin 1994] for an analysis of their contents.

It may be of some interest to describe briefly some aspects of the early reception of this work in France, specially by the light of a recently discovered letter of Lyapunov to Picard, published in [Mawhin 1994].

2. The Influence of Poincaré's Work on Lyapunov's Memoir

When Lyapunov publishes his fundamental memoir [Lyapunov 1892], Poincaré has worked in the qualitative theory of differential equations and in celestial mechanics since more than ten years. His 1879 Thesis *Sur les propriétés des fonctions définies par des équations aux différences partielles* [Poincaré 1879], although mostly devoted to partial differential equations, contains techniques akin with various problems in ordinary differential equations. A note in the *Comptes Rendus de l'Académie des Sciences de Paris* in 1880 is Poincaré's starting point of a long series of papers on the *Courbes définies par une équation différentielle* [Poincaré 1881-1886], ended in 1886, and whose contents has been analyzed in [Gilain 1991]. In celestial mechanics, one must mention, besides a large number of notes in the *Comptes Rendus,* the pioneering paper of 1883 on periodic solutions of the three body problem [Poincaré 1883] and the famous memoir of 1890 crowned by the King Oscar Prize [Poincaré 1890]. See Goroff's introduction in the English translation of [Poincaré 1892-1899, I, 1-93].

In the preface of his memoir, Lyapunov generously acknowledges Poincaré's influence. Speaking of the mathematical justification of the linearization method for the study of the stability of an equilibrium, Lyapunov observes that:

> l'essai unique, autant que je sache, de solution rigoureuse de la question appartient à M. Poincaré, qui, dans le mémoire remarquable sous bien des rapports «Sur les courbes définies par une équation différentielle» (*Journal de Mathématiques*, 3e série, t. VII et VIII ; 4e série, t. I et II), et en particulier dans ses deux dernières parties, considère des questions de stabilité relatives au cas des systèmes d'équations différentielles du second ordre et s'arrête aussi à quelques questions voisines, se rapportant aux systèmes du troisième ordre. Bien que M. Poincaré se borne à des cas très particuliers, les méthodes dont il se sert permettent des applications beaucoup plus générales et peuvent encore conduire à beaucoup de nouveaux résultats. C'est ce qu'on verra par ce qui va suivre, car, dans une grande partie de mes recherches, je me suis guidé par les idées développées dans le Mémoire cité. [Lyapunov 1892, 205]

More explicit details about the relation with Poincaré's results are given in Chapter II of [Lyapunov 1892]: *Etude des mouvements permanents* [291, 294, 300-301, 314, 353, 392]. In particular, Lyapunov mentions the pioneering contributions of Poincaré to what is called to-day the problem of determining the conditions under which an equilibrium of a planar differential system is a *center,* i.e. is surrounded by a one-parameter family of closed orbits. Some results of [Poincaré 1881-1886] will also be mentioned in Chapter III of [Lyapunov 1892]: *Etude des mouvements périodiques* [415, 455].

Describing later his method of development of solutions of ordinary differential in power series, Lyapunov mentions in a footnote of the *Préface* that:

> les séries dont il est question ont été considérées, dans des hypothèses plus particulières, dans mon Mémoire «Sur les mouvements hélicoïdaux permanents d'un corps solide dans un liquide» (*Communications de la Société mathématique de Kharkow*, 2e série, t. I, 1888). J'ai appris dans la suite que M. Poincaré avait considéré ces séries, dans les mêmes hypothèses, dans sa Thèse *Sur les propriétés des fonctions définies par les équations aux différences partielles* (1879). [Lyapunov1892, 205-206]

This connection is explicited in the Chapter II of [Lyapunov 1892]: *Etude des mouvements permanents.*

Finally, an addendum to the *Préface* mentions that:

> Pendant l'impression de cet ouvrage, laquelle s'étendit à plus de deux années, ont paru deux ouvrages très intéressants de M. Poincaré, qui traitent de questions se rappprochant beaucoup de celles que j'ai considérées. Je parle de son Mémoire «Sur le problème des trois corps et les équations de la Dynamique», paru dans les *Acta Mathematica*, t. XIII, peu de temps après que j'eus commencé à faire imprimer mon travail, ainsi que du premier Volume, qui vient de paraître, de son Traité intitulé: *Les méthodes nouvelles de la Mécanique céleste*, Paris, 1892. Dans le premier se trouvent certains resultats analogues à ceux que j'ai obtenus, ce que j'indique aux endroits convenables de mon ouvrage. Quant au second, je n'ai pas encore eu le temps de l'étudier en détail ; mais, pour ce qui concerne les questions que j'ai considérées, il ne renferme pas, à ce qu'il paraît, de compléments essentiels au Mémoire des *Acta Mathematica*. [Lyapunov 1892, 207-208]

In a footnote at the beginning of Chapter III of [Lyapunov 1892]: *Etude des mouvements périodiques,* Lyapunov observes that:

> La question des solutions périodiques des équations différentielles non linéaires est considérée aussi, quoique à un autre point de vue, dans le dernier Mémoire de Poincaré: «Sur le problème des trois corps et les équations de la Dynamique» (*Acta Mathematica*, t. XIII). [*ibid.*, chap. III, 392]

In the same chapter, when he states his theorem that a linear canonical system with periodic coefficients has a reciprocal characteristic equation, Lyapunov mentions that:

> ce théorème est indiqué aussi par M. Poincaré dans son Mémoire «Sur le problème des trois corps et les équations de la Dynamique» (*Acta Mathematica*, t. XIII, p. 99-100), où l'auteur le base aussi sur les relations de la forme (29). Mais je le connaissais avant la publication de ce Mémoire et, en février 1900, je l'ai communiqué, sous la forme précédente, à la Société Mathématique de Kharkow, avec d'autres propositions se rapportant à l'équation caractéristique (*Communications de la Société mathématique de Kharkow*, 2e série, t. II : extrait des procès-verbaux des séances). [*ibid.*, 415]

Summarizing, we see that Lyapunov's work has been influenced by Poincaré's early one and regularly overlaps with some of his further contributions. In several occasions, and specially in dealing with the question of stability, Lyapunov transforms some remarks made by Poincaré in special situations into a powerful general method. On the other hand, Poincaré never mentioned Lyapunov in his subsequent papers or books dealing with the qualitative theory of ordinary differential equations.

3. The Influence of Poincaré's and Lyapunov's Work on Picard, Painlevé and Hadamard

We want to analyze shortly in this section the early reception in France (before 1900) of some aspects of Poincaré and Lyapunov's work on nonlinear differential equations. The reader can consult [Dell'Aglio and Israel 1987 and 1989] for an interesting analysis of the early reception of this work in Italy (in particular its influence upon some papers of Tullio Levi-Civita), the paper by Diner in [Dahan-Dalmedico, Chabert and Chemla 1992] for the Soviet contributions between the two World Wars and [Dahan-Dalmedico 1994] for the development in the USA after the second World War.

In 1896 (and even in 1895 for some Chapters which first appeared separately), Emile Picard published the third volume of his famous *Traité d'analyse* [Picard 1891-1896], based upon his lectures at the Faculty of Science of Paris during the three preceding years on *some of the questions which are of particular interest today to analysts and whose study can be fruitfully continued.* Chapter VIII of this volume, entitled *On periodic solutions and on asymptotic solutions of some differential equations*, is essentially devoted to Poincaré's theory of periodic solutions given in the first volume of the *Méthodes nouvelles.* The essential difference is that Picard uses his favorite method of successive approximations, instead of Cauchy's method of majorants, to prove the fundamental theorem on the existence and continuous dependence of solutions of ordinary differential equations depending upon one parameter.

Another feature is a somewhat more extended discussion of the existence of periodic solutions of an autonomous differential system around an equilibrium. Picard considers in some detail the case of two differential equations

$$(1) \qquad dx/dt = f(x, y, \mu), \; dy/dt = g(x, y, \mu),$$

where $f(0, 0, \mu) = g(0, 0, \mu) = 0$ for all μ and the linearized equation around (0, 0, 0) has all its solutions periodic, say of period T. He searches for small T-periodic (or more generally (T + T')-periodic solutions of (1), T' small) and shows by a very sketchy analysis that the Poincaré's equations giving the initial conditions of possible T-periodic solutions have, for small values of μ, a closed curve of solutions near the origin, tending to it when μ goes to zero. Such a conclusion is uncorrect without further assumptions, as shown by elementary counterexamples [see Mawhin 1994, 28].

This was noticed and mentioned to Picard by Lyapunov in a letter written on January 20 1895, discovered by Albert Libchaber in a copy of Lyapunov's memoir [1892], and published with his kind permission as Appendix III of [Mawhin 1994, 34-43]. After reading Volume III of Picard *Traité d'analyse* [1891-1896] and specially Chapter VIII on periodic and asymptotic solutions, Lyapunov wrote this letter to inform Picard about his own results contained in [Lyapunov 1892] and to raise questions about the validity of Picard's claim mentioned above. Lyapunov's letter starts as follows:

> En lisant le dernier fascicule paru de votre admirable Traité d'Analyse (tome III, fasc. 1), j'y ai rencontré plusieurs questions intéressantes, parmi lesquelles il y en a deux qui ont pour moi un intérêt tout particulier, puisque de ces questions-là je me suis occupé moi-même. J'ai en vue la question des solutions périodiques de certaines équations différentielles sous les conditions que vous indiquez au n. 20 du chapitre VIII (p. 184) et aussi celle de ces solutions asymptotiques dont l'existence vous établissez aux n.n. 26-28 (p. 192-197). Ces questions j'avais aussi traité dans mon ouvrage sur la stabilité du mouvement, et quant à l'existence de solutions asymptotiques elle s'y trouve établie sous les conditions plus générales encore.

The main part of the letter is essentially a very readable French summary of the Russian monograph *Problème général de la stabilité du mouvement* [Lyapunov 1892]. In particular, the last part of the letter describes Lyapunov's famous existence theorem for the periodic solutions, near an equilibrium, of general Hamiltonian systems when all the characteristic roots of the linearized part of the right-hand member are imaginary and satisfy a non-resonance condition. A similar result also holds if the system is not Hamiltonian but admits a suitable first integral. Lyapunov then observes that the reasoning of Picard about the existence of periodic solutions near an equilibrium mentioned above is not conclusive, when such a first integral does not exist.

Interestingly enough, in 1897, Picard presented to the *Comptes Rendus* a paper of Paul Painlevé with title *Sur les petits mouvements périodiques des systèmes* [1897a]. In this note, after recalling the short discussion of Poincaré [1892-1899] about the periodic solutions near an equilibrium and the version given by Picard [1891-1896], Painlevé observes that this discussion of [Picard 1891-1896] is uncorrect and proposes to overcome the difficulties. After exhibiting a counterexample to Picard's conclusion, Painlevé claims to be able to prove that if the characteristic equation of the linearized system admits a couple of simple imaginary roots and if the other roots are not integer multiples of them, then Poincaré's equations giving the initial conditions of periodic solutions have a curve of solutions for μ small, providing a one parameter family of closed orbits for (1). Stated in this way, the result is of course again uncorrect.

However, in another note published the same year [Painlevé, 1897b], Painlevé quotes only his previous 'theorem' for a mechanical system admitting a potential U, which is a then a special case of Lyapunov's theorem (whose name or work are not mentioned). Later, in an analysis of his scientific work [Painlevé 1900, 106-107], Painlevé reports about the first note as proving the existence of an infinity of small periodic motions in the vicinity of an equilibrium where the potential is not maximum, *the method failing only in exceptional cases.* As Painlevé never published the proofs and details of the results anounced in the two notes, it is difficult to have a precise idea of which result Painlevé had really in mind. The second note [Painlevé 1897b] sketches a method to prove the existence of a family of small periodic solutions whose period tends to infinity when the amplitude tends to zero, in the situation (not covered by Lyapunov) of a potential minimum at zero and beginning with terms of degree larger than two.

In any case, Picard, who presented the notes of Painlevé to the French Academy of Sciences, does not seem to have noticed their link with the results of [Lyapunov 1892] communicated to him by Lyapunov's letter. Were it the case, he would surely have asked Painlevé to compare them with the conclusions of his notes. Moreover, despite Lyapunov's letter and Painlevé's note, the sections devoted to the periodic solutions near an equilibrium in the subsequent editions of Picard's *Traité d'analyse* have remained unaltered. The only possible influence of Lyapunov's letter is the addition, in the second edition (1908) of Volume III of the *Traité* , of a section to Chapter VIII with the title *De la stabilité et de l'instabilité des intégrales de certaines équations différentielles; théorème de M. Liapounoff sur l'instabilité de l'équilibre.* But this section may as well have been inspired by Lyapunov's paper published in 1897 in the *Journal de Liouville* [Lyapunov 1897], summarizing some of the concepts and results of [Lyapunov 1892] on stability and giving new instability conditions based upon the use of an auxiliary function (Lyapunov's second method).

The French translation of [Lyapunov 1892] was published, the same year as the second edition of Volume III of [Picard 1891-1896], in the *Annales de la Faculté des Sciences de Toulouse.* It would be interesting to find out who was instrumental in publishing this translation.

Another connoisseur and continuator of Poincaré in the qualitative theory of differential equations is of course Jacques Hadamard. In a work presented to the French Academy of Science, crowned by the Prix Bordin in 1896 [Hadamard 1897], and published the next year in the same issue of the *Journal de Liouville* as Lyapunov's paper [Lyapunov 1897], Hadamard studies the stability and asymptotic behavior of the trajectories of dynamical systems through auxiliary functions quite similar to the ones introduced by Lyapunov's second method in [Lyapunov 1892].

Indeed, Hadamard mentions, at the end of Section 36 of his paper, that the instability of the equilibrium of a conservative mechanical system having a potential whose quadratic part contains at least a positive coefficient, had already been proved by Lyapunov in 1892.

> dans un Mémoire malheureusement écrit en langue russe, mais dont un extrait a été inséré au journal de M. Jordan en 1897, et dont j'ignorais l'existence lorsque j'ai communiqué à l'Académie des Sciences les remarques qui précèdent et la démonstration qui s'y rattache étroitement. [Lyapunov 1892]

Again, it does not seem that Lyapunov's letter had left a strong impression in Picard's mind when Hadamard's memoir was discussed at the Academy for the attribution of the Prix Bordin!

Painlevé also contributed the same year to the converse of the Lagrange-Dirichlet stability theorem in a note to the *Comptes Rendus* [Painlevé 1897c], again presented by Picard. This paper mentions the contributions of Lyapunov and Hadamard (without references) and discusses briefly, in the case of two degrees of freedom, some conditions for instability when the potential does not necessarily contains terms of the second order. Once more, Painlevé did not publish later the detailed proofs of his results, and he returned for the last time to those questions in a note of 1904 [Painlevé 1904], which gives a now classical counterexample to the converse of the Lagrange-Dirichlet stability theorem.

Another important contribution of Hadamard to the theory of dynamical systems is his famous memoir on the geodesics of surfaces with negative curvature. Among his conclusions, Hadamard writes:

> Si l'on ne suit les trajectoires que pendant un temps déterminé, quelconque d'ailleurs, on peut imaginer que les erreurs sur les données initiales aient été rendues assez minimes pour ne pas altérer sensiblement la forme de ces trajectoires pendant le susdit intervalle de temps. Ce qui précède nous montre qu'il n'est en aucune façon légitime d'en tirer une conclusion analogue relativement à l'allure finale de ces mêmes courbes. Celle-ci peut fort bien dépendre (et

dépend, en effet, dans les problèmes relativement simples auxquels est consacré le présent Mémoire) de propriétés discontinues, arithmétiques des constantes d'intégration. [Hadamard 1898, 71-72]

This is the phenomenon met by Poincaré in [Poincaré 1890] and in the third volume of [Poincaré 1892-1899]. Its fundamental philosophical significance was already noticed in 1906 by Pierre Duhem [1906, 206-215], in his sharp discussion of the physical implications of Hadamard's results. One can consult [Dahan-Dalmedico, Chabert and Chemla 1992] (and in particular the papers of Chabert and Dahan Dalmedico/Chabert), [Ruelle 1991] and Goroff's introduction in the English translation of [Poincaré 1892-1899, I, 1-93] for recent interesting discussions.

4. The Presence of Poincaré's and Lyapunov's Work in French Treatises on Analysis during the XXth Century

The short analysis of the previous section shows that the contributions of Poincaré and Lyapunov were already influential in France at the end of the century in the work of mathematicians like Picard, Painlevé and Hadamard. Their theories were important enough to be included in several large treatises on analysis written in French. The third volume of the *Cours d'analyse mathématique* of Edouard Goursat [1910-1915, 1-45] contains, from its second edition, a chapter entitled *Intégrales infiniment voisines,* which is an introduction to Poincaré's theory of periodic solutions and to Lyapunov's stability theory. The second volume of the *Cours d'analyse mathématique* of Georges Valiron [1942, 353-360] has a chapter devoted to Poincaré's discussion of singular points and limit cycles of planar autonomous systems.

The most recent treatise keeping this tradition seems to be the *Cours d'analyse de l'Ecole Polytechnique* of Jean Favard [1960, 131-171], whose third volume contains the foundations of Lyapunov's stability theory and Poincaré's qualitative theory of planar autonomous systems, together with more recent developments. The *Formulaire de Mathématiques à l'usage des physiciens et des ingénieurs,* published under the direction of Maurice Fréchet, contains a volume *Equations différentielles* written by Robert Campbell and Georges Reeb [1964]. It provides a lot of information about singular points and periodic orbits of ordinary differential equations.

Possibly influenced by Bourbaki [1961], the authors of more recent French treatises on analysis like [Dieudonné 1962-1982] or [Schwartz 1967] have restricted the chapters on ordinary differential equations to the fundamental theory of Cauchy's problem and the study of linear systems. Stability theory and periodic solutions appear only in the few books on ordinary differential equations published in French in this period by Maurice Roseau [1966; 1970; 1976], and Nicolas Rouche and Jean Mawhin [1973].

The qualitative theory of ordinary differential equations is also present in several other specialized French monographs. For example, the *Mémorial des Sciences Mathématiques* series contains, between the two World Wars, the volumes of Emile Cotton [1928] with contributions to the Poincaré-Lyapunov theory of asymptotic solutions, of Michel Petrovitch [1931] on the qualitative integration of differential equations, of Edouard Husson [1932] on the trajectories of Dynamics and of Henri Dulac [1934] on the singular points of differential equations. We can quote also Georges Bouligand's book *Lignes de niveau, lignes intégrales* of 1937 [Bouligand 1937], which collects lectures given in 1930-1931 on qualitative integration. His *Mécanique rationnelle* [Bouligand 1959], which has seen many editions, contains also an introduction to the work of Poincaré, Lyapunov and Hadamard.

Poincaré's new methods of celestial mechanics were also instrumental in the mathematical development of the old quantum theory, as clearly shown by Gustave Juvet, who writes in 1926:

> Cela prouve la prodigieuse vigueur des méthodes de Poincaré d'être immédiatement aptes à s'appliquer aux problèmes que pose aujourd'hui la théorie des quanta. [Juvet 1926, 75]

But, at the same moment, the old theory of quanta was superseded by quantum mechanics and Poincaré's work lost its interest among physicists until the recent rebirth linked to chaos theory, with a few exceptions like Yves Rocard, Jules Haag and Théodore Vogel [1965b; 1973] who published monographs of a more applied nature dealing with oscillations and stability [Haag 1952-1955; Haag and Chaléat 1960; Rocard 1941; 1949; 1954]. One has to mention also the proceedings of two international conferences respectively held in Porquerolles in 1951 [Porquerolles 1953] and in Marseille in 1964 [Vogel 1965a]. It is interesting to quote an excerpt of the *Introduction* to this last volume, written by Théodore Vogel:

> Une tradition, dont l'origine se trouve dans les motifs qui incitèrent H. Poincaré à en entreprendre l'étude systématique, classe dans la Mécanique les problèmes relatifs aux équations différentielles non linéaires. Cette tradition n'est pas sans justification, dans la mesure où la Mécanique est, des branches anciennes de la Physique, celle où l'approximation linéaire des lois d'évolution laisse sans réponse satisfaisante le plus grand nombre de questions, celle aussi dont les applications techniques font le plus intervenir des dispositifs aussi essentiellement non linéaires que le régulateur par «tout ou rien».

> Tradition qui a pourtant ses inconvénients; car elle fait participer les études qui nous occupent ici de la situation inconfortable faite, certainement en France, mais apparemment aussi dans d'autres pays, à la Mécanique elle-même.

Finally, in the first half of this century, the contributions of French mathematicians and scientists like E. and H. Cartan, J. Chazy, P. le Corbeiller, E. Cotton, A. Denjoy, H. Dulac, E. Esclangon, P. Fatou, J. Favard, R. Gosse, T. Got, J. Haag, A. Liénard, E. Maillet, G. Reeb, Y. Rocard, Th. Vogel and others have kept alive, although away from the main stream of mathematical research, the qualitative theory of nonlinear differential equations initiated by Poincaré and Lyapunov. But this is another story...

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