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Poincaré on Mathematical Intuition

A Phenomenological Approach to Poincaré's Philosophy of Arithmetic

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Abstract. My focus in this paper is Poincaré's notion of pure intuition. I intend to show that this notion does not belong, as many critics claim, to the psychology of mathematics, but rather to a much broader approach to the foundations of arithmetics, which can be put together with the transcendental philosophies of mathematics of Kant and Husserl. I also argue, against some of the critics of Poincaré, that his attacks on logicism are due to what Poincaré sees as the epistemological limitations of logicism, not to its anti-psychologistic tenor.

1. Introduction

Critics of Poincaré tend to dismiss his criticism of logicism, formalism and mathematical realism as the voice of reaction, as the last gasp of die-hard psychologism and put Poincaré's notions of intuition squarely under the label of mere subjective convictions with no epistemic force. I believe that this reading of Poincaré, although right in some points, is in general basically wrong. Although Poincaré himself didn't take some notions of intuition as epistemologically fundamental, there is in his thought a concept of intuition that does play a fundamental epistemic role and cannot be confined to the subjective realm of private experiences. I also believe that Poincaré's criticisms of the new trends in the foundations of mathematics of his days do cut some ice and deserve, together with his notion of a form of apodictic evidence, some sympathetic attention. It will be evident in the sequel that I read Poincaré with an eye on Husserl and that I'll try to bring both Poincaré's notion of the intuition of numbers and Poincaré's criticism of formalists, logicists and realists under the scope of a Husserlian-type radical epistemology. So, Poincaré's concept of pure intuition is my primary concern here.

2. Poincaré's Intuitionism

Like most who approach these matters there is in Poincaré some degree of sloppiness in the use of the terms *intuition* and *evidence*. For Poincaré they are more or less synonymous. However I want to draw a distinction between them. I will take the word *intuition* (or *intuition of*) as denoting a type of mental experience that provides one with objectivities, i.e. objects both real and ideal, states-of-affairs, structures, and whatever one can classify as an object in the most generic sense. By *evidence* (or *intuition that*) I want to understand another type of mental experience where certain objectivities are experienced as fulfilling what is thought of them. In other words, evidences give us the truth understood as the adequation between what is thought or said and that of which something is said or thought. The evidence is the experiencing of this adequation. For the moment I do not want to discuss the nature of these experiences. From a psychologist's point of view they can be viewed as natural phenomena to be explained by natural science, but there is a non-naturalist approach which gives them a much more prominent rôle in the grounding of knowledge.

Poincaré admits many forms of intuition and evidence. The first is perception and imagination which, according to him, have a rôle to play not only in natural sciences but also in mathematics. The second form is inductive reasoning of the type used in the natural sciences. The third is reasoning by analogy, probably one of the most fecund methods of mathematical discovery. Poincaré must have had this method in high esteem for he at some point 'defines' mathematics as the art of calling different things by the same name. A fourth type of intuition is a sort of directedness which guides our steps throughout a logical proof.

Of course all these types of intuition and evidenciation are not independent of each other, and if we were careful we would probably find some more, but I don't want to dwell on them for too long. The reason is that Poincaré does not accord them anything more than a heuristic role. What they give us cannot be taken independently of logical reasoning.

An altogether different thing happens with what Poincaré called the intuition of pure number. Here I'll be careful to keep in sight the distinction mentioned earlier between intuition and evidence since, in fact, Poincaré is dealing with two different experiences. On the one hand, the intuition of a structure in which a certain formal sequence is given and, on the other, the evidence of the principle of complete induction as a form of reasoning valid for whatever matter fills that formal sequence. This evidence is obviously based on intuition in the sense that it is the intuitive experience that provides the objectivity as having precisely the property which is thought in the principle of complete induction. To put it in other words, by the means of intuition, thinks Poincaré, we gain access to a *structure* which underlies the sequence of natural numbers or any other notion recursively defined. With this structure in our mind's eye, so to speak, we can *see* — and it is this seeing that constitutes our experience of evidence — that with respect to that recursively defined notion we have the right, guaranteed by its very essence, to reason inductively.

That Poincaré recognizes this distinction is made clear in his letter to *Mind* answering Russell's criticisms (Letter to Mind (1906), apud J. Vuillemin: preface to [Poincaré 1902]).

This distinction is relevant to understanding Poincaré's criticism of the logicists' efforts, as Poincaré sees it, to derive the principle of complete induction *analytically* from a *definition* of the natural numbers. That the validity of the principle depends of an experience of evidenciation is enough to rule out a merely analytic, that is purely logical, derivation. Therefore, the principle of induction is not true in virtue of what its terms mean, it is not a mere logical consequence of a conventional definition, and it is not true in the manner of the principles of natural sciences. The principle of complete induction is a *necessary* truth evidentiated on the basis of an intuitive experience, ideally reproducible, by the means of which we constitute, and can always constitute, an objective structure *adequately given* and whose essence can be, consequently, completely and exhaustively surveyed. This is what gives the evidence its apodictic character. The principle of complete induction is, in Poincaré's words, the only true synthetic a priori truth of mathematics. Of course, we still have to justify the use of so many strong notions and clarify matters a bit more. So, let's try and do so.

Let's examine the details of the intuition of pure numbers. Given any possible act of certain type the evidence is forced upon us that an act of the *same* type is ideally reproducible. We see that what has been carried out once can be carried out again, and we see this by turning our attention inward and focusing on the mental experience itself and not on its content. This is a typical example of a second-order mental experience called *reflexion*. So we can say that upon reflexion we can see with evidence that any act-type is ideally reproducible. It is an immediate consequence of this that, given the necessary temporal structure of consciousness, an indefinite sequence of repetitions is set before us, that is a sequence of the same act-type which by the action of abstraction is reduced to its abstract form, a sequence of moments or, if you like, a sequence of points with a beginning and no end.

Again we can see that whatever holds for the first term of this series, and whenever it holds for a given term also holds for the next, must necessarily hold for any member of the series, and this is the now evident principle of complete induction.

Poincaré's argument for the evidence of the principle goes like this:

> il n'est que l'affirmation de la puissance de l'esprit qui se sait capable de concevoir la répétition indéfinie d'un même acte dès que cet acte est une fois possible.
> L'esprit a de cette puissance une intuition directe et l'expérience ne peut être pour lui qu'une occasion de s'en servir et par là d'en prendre conscience [...]. L'induction mathématique, c'est-à-dire la démonstration

> par récurrence, s'impose [...] nécessairement, parce qu'elle n'est que l'affirmation d'une propriété de l'esprit lui-même. [Poincaré 1968, 41-42]

So, we see that for Poincaré necessity is rooted in the structure of consciousness. Of course those for whom nothing but empty tautologies can be necessary will not be impressed by Poincaré's point of view and will dismiss it as sheer 'psychologism'. I'll come back to these matters later on, but for the time being let me note that once we refuse to see consciousness exclusively within the natural perspective, as the locus of psychological or psycho-physical phenomena, and we see it as *pure* conciousness where everything that exists, in the most general sense of the word, must find its *meaning* and its *genesis* as being precisely what it is, then Poincaré's words gain a new light. We can now see them as describing a constitutive experience where the *sense* 'natural numbers' is constituted, and also as describing the genesis of a necessary truth on the basis of apodictic evidence. Of course, by taking this standpoint, we are already in the domain of phenomenology.

Poincaré himself, although at times close to it, never crossed the division line between pure descriptive psychology and phenomenology, but it is not difficult to attribute to him the same tendency towards genetic analyses that phenomenologists display.

Logicists claim that Poincaré misunderstood their task, this may be so, but they also misunderstood Poincaré's criticism. A true critique of mathematical knowledge in the style of Kant was his aim, one which, according to him, logicists never reached because they didn't understand that reducing mathematics to logic only shifted the problem, and in place of the task of explaining mathematical knowledge we have now the task of explaining logical knowledge. With the extra burden of explaining the far from obvious assumption of why the latter is more fundamental than the former.

Let's produce some textual evidence for the claim that Poincaré indeed saw the origin of all being in consciousness and that the naive natural attitude of realist metaphysics should be overcome by a more thoroughly philosophical perspective. In *Dernières Pensées* we read that Poincaré assumes

> qu'un objet n'existe que quand il est pensé, et qu'on ne saurait concevoir un objet pensé indépendamment d'un sujet pensant. [Poincaré 1913, 94]

and that for the realists

> [le monde] existerait encore même s'il n'y avait [...] aucun sujet pensant. Cela, c'est le point de vue du sens commum, et ce n'est que par réflexion qu'on peut être amené à l'abandonner. [*ibid.*]

It seems to me obvious that Poincaré's is not advocating some bizarre form of idealism where the substance of the world *comes into existence* by means of a divine creative power of the human mind. What he is telling us is that whatever exists do so in a certain mode, with a certain sense, and that it is precisely the many senses of being that find their origin in consciousness. In particular the mathematical way of being, we must find in the life of consciousness its specific sense or otherwise take the absurd standpoint of mathematical realism.

But, of course, Poincaré himself falls short of complete immersion in constitutive phenomenology, he still believes that psychology could play this rôle, but the fact that he undeniably saw the need for the rôle to be played and sketched its main lines is enough to secure him a place in the pre-history of phenomenology, a place not unlike the one occupied by Husserl's *Philosophy of Arithmetic* in the context of his philosophy.

But let's go back to Poincaré's pure intuition and raise a question which any of his opponents is quick to bring up. How can what is given to *me* in *my* conciousness have *objective* validity? This is the main problem with any form of psychologism, they say, it cannot go beyond the limits of *subjective* experiences with no bearing on an objective world. Needless to point out that many criticisms of same tenor were directed at Husserl's idealism. The answer is always the same, if the notion of an *objective* realm has a sense it is in pure consciousness that it got it. It is in my consciousness that the *other* and an *objective reality* which is common to me and the other are constituted, i.e. are given a sense. Even without a clear-cut concept of pure consciousness, Poincaré's answer goes along the same lines. He says:

> Mais ce que nous appelons la réalité objective c'est, en dernière analyse, ce qui est commun à plusieurs êtres pensants, et pourrait être commun à tous. [Poincaré 1970, 23]

But Poincaré's notion of objective reality has a peculiarity related to his avowed structuralism. Poincaré believes that nothing is objective, i.e. intersubjective, that is not capable of being conveyed by language. Objective existence finds its dwelling in language. And again we notice a point of contact with Husserl who, in *The Origin of Geometry*, says that it is in language that objective idealities find their 'bodies'. But language, Poincaré believes, cannot grasp the substance of the world, sensations in particular. These remain private objects. Only *relations* among sensations can be captured by language, only they can be communalized and thus only relations are objective.

Therefore it is precisely these relations that are scientifically or mathematically relevant. Mathematics is then concerned, strictly speaking, not with objects but with *structures*, and this explains why what is given in the intuition of pure numbers are not numbers properly said but the *structure* of the numerical sequence. This structure can be objectified in the most adequate way, for, ideally, it can be given as *the same* in any individual consciousness. Consequently, the principle of complete induction is not only a necessary but also an *objective* truth.

Let's try and summarize what we've got so far. Among Poincaré's many notions of intuition one stands out that, unlike the others, plays not a heuristic but an epistemic rôle. Only the intuition of pure number can give us access to an objectively existing mathematical entity, the structure of the non-negative integers which underlies any recursively defined notion. On the basis of this intuition another objectifying experience, that of evidence, gives us, and *only* it can give us, the truth of the principle of complete induction which says that with respect to a recursively defined notion we are justified in reasoning by recursion.

My main point is that, for Poincaré, these experiences cannot be confined, as logicists like Frege would claim, to a private subjective realm from where they cannot have any bearing on an objective reality. It is Poincaré's belief that, on the contrary, only these experiences can ultimately justify the *objective* validity of the most basic of all mathematical truths (as well as, one might add, those *logical* principles and rules which logicists take for granted). Poincaré, I claim, is striving for a radical grounding of mathematical knowledge, true to his Kantian allegiance, that foreshadows phenomenological themes and strategies.

Poincaré believes that the principle of complete induction is a synthetic *a priori* truth. By this he understands that the principle cannot be analytically derived from purely logical principles, that it's not a definition of the natural numbers, and that it cannot be an analytic consequence of such a definition. Let's see what his arguments are for each of these attemps at reducing the principle to an analytic truth or to a mere conventional definition. The first one Poincaré doesn't take too seriously. According to him the reduction of arithmetic to logic is very close to being a fraud. What logicists did, he says, was basically to transfer arithmetic to books conveniently labelled as books of logic. The second he chooses as his favorite target. One way of *defining* the notion of natural number is to call by this name whatever satisfies the principle of induction, that is,

$$\text{Number } (x) \Leftrightarrow \forall F(F(o) \wedge \forall y(F(y) \rightarrow F(Sy)) \rightarrow F(x))$$

or, in words, a natural number is that which satisfies any hereditary property.

Poincaré thinks that a notion can only have the right of citizenship in mathematics if we can *prove* that is does not lead to contradiction. To exist, he says, has only one sense in mathematics and that is to be free from contradiction.

We have two ways of showing that the purported definition of natural numbers is an acceptable one. We can give it substance by exhibiting the objects to which the definition applies, i.e. the natural numbers themselves. But this, says Poincaré, will bring us back to the *intuition* of the natural numbers. In other words, if we take the principle of induction as a definition we need the intuition which, according to Poincaré, grounds the principle to give the definition an object, which leaves us right on the same place.

Another way of securing consistency for the definition is to show that the formal context where it occurs is consistent in the following way. The consequences of formal systems (in the logical sense) are in general infinite in number, so we cannot check one by one in the search for the absurdity that if found would prove inconsistency. All we can do, Poincaré thinks, is to check that the rules of inference are reliable, that is, do not generate a contradiction from non-contradictory premises and then to conclude *by induction*, since proofs are of finite length, that contradictions will never appear, given that the most basic premises do not contradict each other. The *petitio principii* is now evident, the principle of induction would be used in the proof that the principle of induction consistently defines the notion of natural number. This, thinks Poincaré, is the kiss of death for the logicists' attempt to define the notion of natural number. Of course, logicists never claimed that absolute proofs of consistency were possible. Hilbert, for instance, who for Poincaré was also a logicist, took some minimal amount of arithmetic, including a weaker form of the principle of induction, for granted. But Poincaré was not sensitive to these niceties, the entire idea of a *logical* rendition of the notion seemed for him completely misleading.

But the purported definition of natural numbers has another problem. The notion as defined is a hereditary one, that is, it holds for zero and whenever it holds for a given object it also holds for its successor, but it is defined by a sentence containing a quantifier ranging over the class of *all* hereditary properties. So, a notion is defined in terms of a class which contains it. This constitutes a vicious-circle and consequently the definition like all that fall into this trap is called *impredicative*. And, says Poincaré, an impredicative definition doesn't define anything, basically because it originates an infinite regress in which nothing is really defined.

Let's see now why the principle of induction is not a *logical* consequence of the recursive definition of number. We certainly know why, the principle is not included in the recursive definition, on the contrary it works as its closure condition. But what Poincaré actually says is that, as already mentioned, the principle requires an experience of evidence for its justification. So, it can't be a purely logical consequence of the recursively defined notion of natural number. What the principle says is that we have the right to reason by recursion with respect to the notion so defined.

There are other problems with the logicists' attempt at defining the notion of natural numbers. Besides the *petitio* it involves and its impredicativity it is circular in still another sense. Attempts at defining particular numbers such as zero and one, on top of being, thinks Poincaré, ludicrously elaborate are circular because they mention numerical terms. Goldfarb [1987] charges Poincaré of commiting an elementary logical mistake for not noticing that these terms are easily eliminable by appropriate paraphrases that logical language allows. Needless to say, Poincaré wouldn't be in the least impressed by this power of logistics to circumvent the mention of numerical terms. The reason is that although they are not *mentioned* numerical ideas are still *used* here. The adequate paraphrasing logicists rely on must necessarily make use of differentiated variables and in order to write down the definition the logicist must already have the notion he is defining.

Of course this doesn't constitute a problem for logicism because it doesn't give itself the scope of Poincaré's approach to the foundations of arithmetic. This criticism of Poincaré is nonetheless worth mentioning because it shows clearly what his aim was, to trace back the *origin* of the notion of number, a task Poincaré accomplished by the means of descriptive psychology. All of Poincaré's charges against the logicists rely on the presupposition that logicism wants to deny that there is such a thing as the origin of the notion of number and to give it a purely conventional rendering.

The formal definition must always rest on a previous 'understanding' that gives the definition its 'grasp', and it's precisely the nature of this 'understanding' that Poincaré thinks philosophically revelant to analyse.

3. Conclusions

Poincaré's opposition to logicists and formalists, on the one hand, and to mathematical realists on the other, shows that his philosophy of mathematics assume two basic premises. The first is that mathematical knowledge is made up of something more than empty tautologies and is concerned with proper mathematical entities. Mathematics, he thinks, has specific non-logical evident principles of reasoning, such as the principle of complete induction, applicable not only in number theory but in the whole of mathematics. Mathematics is concerned with structures which permeate all of its domains, such as the structure of the sequence of natural numbers, which cannot be ultimately presented by explicit or implicit definitions but must be given in intuition. The second of the basic premisses is that mathematical entities do not exist independently of being thought by a community of thinkers who agree on what they are thinking about. The objectivity of mathematical entities is constituted precisely in harmonious intersubjectivity.

My view is that these premises cannot be easily dismissed as misguided 'psychologism' or epistemological naturalism and must be granted their legitimate rôle as guidelines for a true philosophical analysis of the *ultimate* grounds of truth and existence in mathematics. A task later taken over by phenomenology.

Usually Poincaré's adversaries and critics are quick to charge him of defending that sort of psychologism criticized by Frege who, they believe, said the last word on the matter. As a matter of fact, I believe that the realism or, if you like, the objectivism present in Frege's notion of number is only the other face of the coin where psychologism is printed, and that both must be overcome by a transcendental analysis of the proper mode of being of mathematical entities and its rooting in consciousness. This analysis is already present in Poincaré, although still in a sketchy and incomplete form, which can be dismissed only by those who either refuse to be thoroughly philosophical or are still in the grip of naive metaphysics.

Realism and psychologism are both aspects of naturalism, a view which can conceive existence only in the mode of existence of the natural (physical or psychological) object. Consciousness in the naturalistic perspective is nothing but a bundle of a certain type of natural phenomena. For this reason psychologism is incompetent to deal with ideal objects like mathematical objects, it can think of them only as *real* objects, i.e. as *ideas* in the psychological (Fregean) sense. Critics of psychologism, like Frege, are right in pointing to this false reduction but they cannot do so much better. In their hands idealities in general, judgments or *thoughts* in the Fregean sense, ideal objects such as numbers or objective truths leave the interior of naturalized conciousness to wander like ghosts out of space and time but sometimes coming in contact with us in some supernatural way. They cannot do any better than that because they share the psychologists' naturalist prejudice of conceiving existence exclusively under the mode of 'natural being' which is either psychological or psycho-physical or else "*that which is out there existing in itself independently of me*", i.e. the mode of being of physical objects. Since ideal objects are obviously not real in the sense of not being objects of the natural world, there's no other way but to create a bizarre parallel reality in which they reside *in the manner* of physical objects in this world.

Of course, it is not an easy thing to explain how we gain access to this supernatural reality. Gödel, for instance, cannot do any better than postule some sort of sensibility which he calls, *faute de mieux*, intuition. Unlike Poincaré who gives us the actual mental experiences which constitute intuition, Gödel's brand of intuition is nothing more than a happy incident in which we can ascend to the Platonic world of realists but can't tell how.

Psychologists, in their turn, can only conceive intuition as a psychological or psychophysical phenomenon, a natural occurrence within that small corner of the natural world which is our mind, it can only provide subjective conviction with no bearing on objective truth or objective existence. Needless to say the bad reputation that the notion of intuition got for itself comes as no suprise.

The solution to the dilemma is of course to give up naturalism, that is the false dichotomy between a private realm of being which depends on me for its existence and a public realm of being existing independently of me. We must find ways of conceiving different modes of being in which transcendence and objectivity are *not* inconsistent with dependence on consciousness. We must learn how to distinguish objectivities from the mental experiences in which they are conceived. All this we can, of course, find in Husserl who taught us that there's no being or truth that is not given by consciouness. This view differs from psychologism because Husserl de-naturalizes conciousness giving it a new dimension, that of transcendental consciouness where not only the ideal world but *also the natural world* must find their sense. Naturalism is only another word for philosophical *naïveté,* it presupposes too much. Husserl tells us that even the world of the natural sciences must be constituted, i.e. given a sense in consciousness. The same is also valid for mathematical realms, considered in the realist *as well as* in the constructivist mode of being.

There have been some recent attempts at finding a phenomenological vindication of mathematical realism. This project is bound to failure simply because realism is a perspective held within the naive naturalist viewpoint surpassed by phenomenology. The sense "*transcendent object existing by and in itself independently of my mental life*" is a possible sense of an intentional object. But the quotation mark around the expression cannot be removed, that mode of being cannot be attached to the object meant as if it *really* were so. The word 'really' would bring back the naturalist perspective we've given up.

To be transcendent is to be always open to new experiences, to be objective is to be the same for everyone, to explain how an objective domain of transcendent beings can be constituted is a task for phenomenology, to think that this is the realm realists have in mind is to misunderstand the meaning of 'constitution' in phenomenological context. To remember that for Husserl the *ideal* object is that which presents itself *as product* is enough to make realism an impossible view within phenomenology.

Something along these lines can also be said with respect to attempts at siding Husserl with intuitionistic logic. No logic, understood as a 'corpus' of principles of reasoning can be given once and for all, independently of the realm it is supposed to rule. The logic that is going to rule over judgments about a realm of objectivities depends on the sense of being attached to these objectivities. Any judgment can be conceived as having an *intrinsic* truth-value once we *assume* that it is *capable* of being filled out by an adequate intuition. Classical logic rests on certain presuppositions we make, these presuppositions constitute an idealization that may go along with certain modes of being.

Coming back to Poincaré, and concluding, I suggest that although not a phenomenologist Poincaré carried out, with respect to his notion of the intuition of pure number, a description of mental experiences which falls within the domains of pure psychology, a discipline which runs parallel to phenomenology proper. His project was thoroughly foundational in the sense of being a search for the ultimate grounds of mathematical knowledge. From this perspective he criticizes logicism, formalism and, in the end, mathematical realism where he senses philosophical *naïveté* for ignoring precisely the constitutive rôle played by consciousness. For Poincaré there can't be any true understanding of mathematical knowledge that falls short of telling us how and why we ended up focusing our attention on certain specific mathematical entities and reasoning about them in the way we do. We certainly cannot see this enterprise as a mere description of a psychological nature of how we get to know mathematics but, on the contrary, as an analysis, transcendental in scope, ontological and epistemological in nature, of the genesis of objective entities and principles of reasoning which they must obey. In other words, Poincaré's task is to provide a *transcendental* rather than a *psychological* justification of arithmetic. Poincaré's allegiance to Kant has nothing to do with how both understand the notion of intuition (which incidentally has very different meanings in each of them) but instead with how both understand the task of explaining mathematical knowledge. Goldfarb says that

> the constrast with Frege shows how Poincaré is — despite his disclaimer — construing the project of the foundations of mathematics as being concerned with matters of the psychology of mathematics, and faulting logicism for getting it wrong. [Goldfarb 1987]

According to Goldfarb Poincaré is concerned with how we *get to know* what we know in mathematics, when Poincaré is actually concerned with the *ultimate* foundation of mathematical knowledge, its primary rational basis, and faulting logicism for not getting it at all.

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