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ON THE POLYNOMIAL RING OVER A MORI DOMAIN

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One of the open problems about Mori domains is the following: if A is a Mori domain (i.e. a domain with the ascending chain condition on divisorial ideals), is A[X] also a Mori domain ?

J.Querré has shown this is true if A is integrally closed (cf. [6, Section 3, Théorème 5]) and M.Roitman found the same result if A contains an uncountable field (cf.[8]).There are no example where A is Mori and A[X] is not.

I shall give a new class of Mori domains for which the polynomial ring is also a Mori domain. A domain of its may be not integrally closed or may not contain an uncountable field. So this class contains new examples with respect to the ones given by J.Querré or M.Roitman.

Suppose A is a Mori domain and A^* is the complete integral closure of A. Suppose further that $(A:A^*) \neq (0)$.

I recall the algorithm given in [2, Section 1] to get A^* in terms of prime ideals of A. Put $A_0 = A$ and

a) $A_{i+1} = (A_i:\mathfrak{R}_i)$, where $\mathfrak{R}_i$ is the intersection of the strong maximal divisorial ideals of A_i (if A_i has some strong maximal divisorial ideal);

b) $A_{i+1} = A_i$, if A_i does not have any strong maximal divisorial ideal.

We get in this way a sequence of Mori overrings of A, that

is stationary for some m and $A_m = A^*$ (cf.[2, Theorem 1.8]):

$$A = A_0 \subset A_1 \subset \ldots \subset A_m = A^* \qquad (*)$$

Consider now the particular case where for each i, i = 0,...,m-1, $\mathfrak{R}_i$ is a radical ideal of A_{i+1}. I call in this case A "seminormal in A*"; this definition is due to the following points:

i) if A is Noetherian, then $A^* = \bar{A}$ (integral closure of A) and our particular condition holds if and only if A is seminormal (in $\bar{A}$) in the usual sense;

ii) there are many similarities between Noetherian seminormal domains and Mori domains "seminormal in the complete integral closure": in both cases, A_i is obtained from A_{i+1} by a glueing of prime ideals (cf. for details [10, Theorem 2.1] and [2, Corollary 3.7]).

The property ii) indicates also how to construct examples of domains of this type. We get for example in this way the domains A = k + XYk[X,Y] or B = k[X] + Yk[X,Y,Z]. Notice that, on the contrary of the Noetherian case, in the Mori case, we do not have any finiteness type-restriction, so that two lines can be "glued" in a point as in Spec(A) or a plain can be "glued" in a line as in Spec(B) (cf. for other examples [2, Examples 3.12]).

THEOREM. Let A be a Mori domain such that $(A:A^*) \neq (0)$. If A is "seminormal in A*", then A[X] is also a Mori domain.

Proof. Let $A_i = B \subset A_{i+1} = C$ be the generic step of the sequence (*). Thus $C = (B:\mathfrak{R})$, where, if $P_1,\ldots,P_n$ are the strong maximal divisorial ideals of B (cf. [2, Proposition

1.5]), we have $\mathfrak{R} = P_1 \cap \ldots \cap P_n$.

By [2, Proposition 2.7 and Corollary 2.8], we know that $B = C \cap B_1 \cap \ldots \cap B_n$, where for each j, $j = 1, \ldots, n$, B_j is the pullback of the diagram

$$\begin{array}{ccc} & & k(P_j) = B_{P_j}/P_jB_{P_j} \\ & & \downarrow \\ S_j^{-1}C & \longrightarrow & S_j^{-1}C/S_j^{-1}P_j \end{array}$$

(where $S_j = B - P_j$).

Since A* is a Krull domain (cf.[1, Corollary 18]), it is well known that A*[X] also is a Mori (in fact Krull) domain. Thus to prove the Theorem it is enough to show (in the generic step of the sequence (*), $B \subset C$) that if C[X] is a Mori domain, then B[X] is also a Mori domain.

Indeed $B[X] = C[X] \cap B_1[X] \cap \ldots \cap B_n[X]$. So, by [7, Théorème 2], it is enough to show that $B_j[X]$ is a Mori domain (for $j = 1, \ldots, n$).

Let's fix an index j and let's denote, for simplicity, B_j by $\mathfrak{B}$, $S_j^{-1}C$ by $\mathfrak{C}$, $S_j^{-1}P_j$ by $\mathfrak{p}$ and $k(P_j)$ by k. From the previous pullback diagram, we get the following pullback diagram (cf.[4, Lemma 2]):

$$\begin{array}{ccc} \mathfrak{B}[X] & \longrightarrow & k[X] \\ & & \downarrow \\ \mathfrak{C}[X] & \longrightarrow & \mathfrak{C}/\mathfrak{I}\,[X] = \mathfrak{C}[X]/\mathfrak{I}[X] \end{array}$$

If L is the quotient field of k[X], we have k[X] = $\mathfrak{C}/\mathfrak{I}[X] \cap L$, where the intersection is made in the total quotient ring of $\mathfrak{C}/\mathfrak{I}[X]$. Thus, by a result of Roitman (cf.[9, Theorem 4.15]), $\mathfrak{B}[X]$ is a Mori domain if: i) $\mathfrak{C}[X]$ is a Mori domain, ii) $\mathfrak{I}[X]$ is a Mori ideal, iii) $\mathfrak{I}[X]$ is a prime ideal of $\mathfrak{B}[X]$. Actually $\mathfrak{C}[X] = S_j^{-1}C[X] = S_j^{-1}(C[X])$ is a Mori domain, because C[X] is Mori (cf.[5,Corollaire 3]). Moreover $\mathfrak{I}$ is a radical ideal of $\mathfrak{C}$ (cf.[2, Proposition 3.3,2)]) and is also a Mori ideal, because it is a prime (in fact maximal) ideal of the Mori domain $\mathfrak{B}$ (cf.[8, Theorem 6.2]). Thus, by [9, Proposition 4.9, (b)], $\mathfrak{I}$ is a finite intersection of prime ideals of $\mathfrak{C}$. We easily deduce that $\mathfrak{I}[X]$ is also a finite intersection of prime ideals of the Mori domain $\mathfrak{C}[X]$. Thus, by [8, Theorem 6.2], $\mathfrak{I}[X]$ is a Mori ideal. Moreover $\mathfrak{I}[X]$ is prime in $\mathfrak{B}[X]$ and so we conclude that $\mathfrak{B}[X]$ is a Mori domain.

It is not difficult to show that if A is an integrally closed Mori domain such that (A:A*) = (0), then A is "seminormal in A*" and so, by the Theorem, A[X] is Mori. Thus in this case the polynomial ring inherits the Mori property and we get as a consequence of the Theorem the mentioned result of J.Querré in a particular case:

COROLLARY. *Let* A *be a Mori domain such that* $(A:A^*) \neq (0)$. *If* A *is integrally closed, then* A[X] *is a Mori domain*.

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