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ITERATED INTEGRALS AND HOMOTOPY PERIODS

by Bohumil CENKL

The cohomology of the loop space of a simply connected space is well understood, and an extensive work has been done in this area since the appearance of the fundamental paper of Adams [1] in 1956. Nevertheless it is an interesting observation of Chen [4] that, over R or C, this cohomology can be computed from certain complex (whose objects are called iterated integrals), constructed from the de Rham complex of the underlying space.

From that result it seemed obvious that one should relate the de Rham theory also to the study of the fundamental group in general. First results in this direction were obtained again by Chen [4], where he used the de Rham theory over C or R. A conceptual description of the situation over C,R or Q was independently given by Sullivan in the framework of minimal models. Some of the results were obtained, by still another method, by Stallings.

In this lecture, I relate the approach of Chen to that of Sullivan. The differential graded algebra of iterated integrals is redefined so that it is an algebra over C,R,Q or Z. The fundamental properties are proved by different method then in Chen's original work, which relayed heavily on the differential forms over

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* Part of the research was done while the author was a guest at the University of Lille.

C or R. Chen's theorem for the loop space of a simply connected manifold is proved also over Z. For a nonsimply connected space there is defined a map T from the minimal model $\mathcal{M}$ of the tower of Eilenberg-MacLane spaces associated with the fundamental group of the space to the algebra of iterated integrals related to the space. This map T is the analogue of the translation of a power series connection of Chen. The purpose of this map T is twofold. On one side it gives the isomorphism of the algebras of nilpotent quotients of the lower central series, tensored with C,R or Q with the duals of the subalgebras of the minimal model of the tower of Eilenberg-MacLane spaces, with the same coefficients. This isomorphism is called the de Rham theorem for the fundamental group. On the other hand the image in the algebra of iterated integrals consists of all homotopy periods (homotopy invariants of pointed maps of a circle into the space). It is shown, that this homotopy periods are completely determined by the matrix Massey products in the minimal model $\mathcal{M}$.

The iterated integrals, in conjunction with the minimal model, appear to be a usefull tool in the homotopy theory.

1. Iterated integrals and Adams theorem

1.1 Singular cubic cohomology

This is included in here just to establish the notation.

Let I be the segment [0,1], and let $I^n =$ $= I\times\ldots\times I$ (n-times) be the standard n-cube with the coordinates $(t_1,\ldots,t_n)$, $0 \leq t_i \leq 1$. We define now two sets of maps, called the face and degeneracy maps respectively:

$$\lambda_i^{\varepsilon} : I^{n-1} \to I^n \ , \ \mu_i : I^n \to I^{n-1}$$

$$\lambda_i^{\varepsilon}(t_1,\ldots,t_{n-1}) = (t_1,\ldots,t_i,\varepsilon,t_{i+1},\ldots,t_{n-1}),$$

$$\mu_i(t_1,\ldots,t_n) = (t_1,\ldots,\hat{t}_i,\ldots,t_n),$$

$i = 1,2,\ldots,n$, $\varepsilon = 0,1$. These maps satisfy the relations:

(i) $\lambda^{\eta}_j \lambda^{\varepsilon}_i = \lambda^{\varepsilon}_{i+1} \lambda^{\eta}_j$, $j \leq i$

(ii) $\mu_i \mu_j = \mu_j \mu_{i+1}$, $j \leq i$

(iii) $\mu_j \lambda^{\varepsilon}_i = \lambda^{\varepsilon}_{i-1} \mu_j$, $j < i$

$\mu_j \lambda^{\varepsilon}_i = \text{identity}$, $j = i$

$\mu_j \lambda^{\varepsilon}_i = \lambda^{\varepsilon}_i \mu_{j-1}$, $j > i$.

A singular n-cube in a topological space X is a continuous map $f : I^n \to X$. We say that the n-cube f " $I^n \to X$ is degenerate if there exists a singular (n-1)-cube $g : I^{n-1} \to X$ such that $f = g \cdot \mu_n$. The (n-1)-cube g is then uniquelly determined by f ; namely $g = f\lambda^0{}_n = f\lambda^1{}_n$. We denote by $Q_n(X)$ the free abelian group whose basis is the set of all singular n-cubes in X. On the free abelian group

$$\Omega(X) = \sum_{n=0}^{\infty} \Omega_n(X)$$

we define the oeprator $\partial_\Omega : \Omega(X) \to Q(X)$,

$$\partial_\Omega f = \sum_{i=1}^{n} (-1)^{i+1} (f \cdot \lambda^1{}_i - f \cdot \lambda^0{}_i) .$$

From the relation (i) we get $\partial_Q \partial_\Omega = 0$. On the other hand from (iii) if follows that the subgroup

$$D(X) = \sum_{n=0}^{\infty} D_n(X)$$

of $Q(X)$ generated by the degenerate cubes is closed under ∂_Ω, the particular $\partial_Q D_{n-1}(X) \subset D_{n-1}(X)$. Hence, if we define

$$K(X) = Q(X)/D(X),$$

$K(X) = \sum_{n=0}^{\infty} K_n(X)$, $K_n(X)$ naturally isomorphic with $\Omega_n(X)/D_n(X)$, we get a complex $\{K(X), \partial_K\}$,

∂_K being the differential induced on the quotient $K(X)$. The cohomology of this complex, with coefficients in the abelian group G is called the singular cubic homology of X with coefficients in G.

By $K^n(X,G)$ we denote the group $\mathrm{Hom}\,(K_n(X),G)$, that means the group of cubic n-dimensional cochains with coefficients in G. The coboundary operator will be denoted by δ_K. And the homology and cohomology group will be denoted by $H_*(X,G)$ and $H^*(X,G)$ respectively. The face and degeneracy operators λ^ε_i, μ_i induce the maps

$$\ell^\varepsilon_i : K_n(X) \to K_{n-1}(X),\ \ell^\varepsilon_i(f) = \lambda^\varepsilon_i \circ f ,$$

and

$$m_i : K_{n-1}(X) \to K_n(X),\ m_i(g) = \mu_i \circ g .$$

From now on we take $G = k$ to be C,R,Q or Z. Let $A(I^n) = \bigoplus_{p \geq 0} A^p(I^n)$ be the module of differential forms, considered as a k-module, on the n-cube I^n. Then we define

$$A^p(X) = \{w = (w_n)_{n \geq 0} ,\ w_n \in \mathrm{Hom}_k(K_n(X), A^p(I^n)) \;|\; (\lambda^\varepsilon_i)^* w_n = w_{n-1} \circ \ell^\varepsilon_i ,\ (\mu_i)^* w_{n-1} = w_n \circ m_i\} .$$

The abelian group $A(X) = \sum_{p\geq 0} A^p(X)$ has a structure of a commutative graded algebra with respect to the product

$$\wedge : A^p(X) \times A^q(X) \to A^{p+q}(X),$$

$$(w_n' \wedge w_n'')(f) = w_n'(f) \wedge w_n''(f),$$

$$w' \wedge w'' = \Sigma\, w_n' \wedge w_n'' , \; w' = \Sigma w_n', \; w'' = \Sigma w_n'' .$$

On $A(X)$ there is the differential

$$d : A^p(X) \to A^{p+1}(X),$$

$$(dw_n)(f) = d(w_n(f)),\; dw = \Sigma\, dw_n,\; w = \Sigma w_n .$$

Thus $A(X)$ becomes a differential graded algebra. We talk about the cubic singular de Rham complex $\{A(X), d\}$ of the topological space X. The cohomology of this complex $H^*_{DR}(X;k)$ is called the cubic singular de Rham cohomology of the topological space X. Then we have

Theorem 1. There is an isomorphism

$$H^*_{DR}(X;k) \cong H^*(X;k)$$

where on the right hand side is the singular cubic cohomology with coefficients k.

Proof. The statement follows from a more general theorem proved in [2] and [8] once a transition between the simplicial and cubic theories is made.

1.2 An algebraic construction.

Let M be an oriented differentiable manifold and let $\Omega(M)$ be the space of piecewise smooth loops at a base point x_0 on M. The simplest iterated integrals, as introduced originally by Chen [3], can be defined as follows: Let $\alpha : I^n \to \Omega(M)$ be a representative for $\alpha' \in K_n(\Omega(M))$. Then α defines $\hat{\alpha} : I^{n+1} \to M$, $\hat{\alpha}(x,t) = \alpha(x)(t)$, $(x,t) \in I^n \times I = I^{n+1}$. Then for any $a \in A^{q+1}(M) =$ = the module of differential (q+1)-forms on M

$$\int_0^1 \hat{\alpha}^* a \quad \text{is a q-form on } I^n, \text{ and } \int_0^t \hat{\alpha}^* a$$

is a q-form on I^{n+1}. Because the integration takes place in I^{n+1} (it is in fact an integration over the fibre I in the projection $I^n \times I \to I^n$) it is convenient to introduce this as an algebraic operation. Thus we look more closely at the algebraic version of the algebras $A^*(I)$, $A^*(S^1)$, $A^*(I^n)$, $A^*(I\times I^n)$, $A^*(S^1\times I^n)$. The corresponding algebraic objects will be denoted by K, L, V, A and $\tilde{A}$ respectively.

Let k be a commutative ring with a unit. Assume that we are given the following data :

(i) A DGA-algebra (The definition is section 6.)

$$K = \bigoplus_{r \geq 0} K^r \,, \quad K^s = 0 \,, \quad s > 1$$

with the differential and augmentation

$$\delta_K : K^r \to K^{r+1} \,, \quad \varepsilon_K : K \to k \,,$$

$\varepsilon_K(K^r) = 0$ for $r \geq 1$. The cohomology ring $H^*(K)$ is trivial, i.e. $H^0(K) \cong k$, $H^r(K) = 0$ for $r \geq 1$.

(ii) There is a DGA-algebra

$$L = \bigoplus_{r \geq 0} L^r \,, \quad L^s = 0 \,, \quad s > 1$$

with the differential and augmentation

$$\delta_L : L^r \to L^{r+1} \,, \quad \varepsilon_L : L \to k \,,$$

$\varepsilon_L(L^r) = 0$, $r \geq 1$. The cohomology ring has the property $H^0(L) \cong k$, $H^1(L) \cong k$, $H^s(L) = 0$, $s > 1$.

(iii) Let $V = \bigoplus_{r \geq 0} V^r$ be a DGA-algebra over k with the differential and augmentation

$$\delta_V : V^r \to V^{r+1} \,, \quad \varepsilon_V : V \to k \,.$$

(iv) There is a DGA-algebra morphism

$$\nu_0 : L \to K$$

which induces an isomorphism $\nu_0^* : H^0(L) \to H^0(K)$.

Now we define two new DGA-algebras

$A = \bigoplus_{r\geq 0} A^r$, $A = K \otimes_k V$, with the differential d_A,

and

$\tilde{A} = \bigoplus_{r\geq 0} \tilde{A}^r$, $\tilde{A} = L \otimes_k V$, with the differential $d_{\tilde{A}}$.

The product in A and $\tilde{A}$ is simply denoted by $\cdot$.

The morphism ν_0 induces the DGA-algebra morphism

$$\nu : \tilde{A} \to A \quad , \quad \nu = \nu_0 \otimes 1.$$

Note that both A and $\tilde{A}$ are bigraded, with the gradation given by

$A^{r+1} = A^{0,r+1} \oplus A^{1,r}$, $A^{0,r+1} = K^0 \otimes V^{r+1}$, $A^{1,r} = K^1 \otimes V^r$.

We denote by λ', λ'' the projections

$$\lambda' : A^{r+1} \to A^{1,r} \ , \ \lambda'' : A^{r+1} \to A^{0,r+1} \quad .$$

In fact any element $a \in A^{r+1}$ can be written uniquelly in the form $a = t \otimes a' + a''$, $t \otimes a' \in A^{1,r}$, $a'' \in A^{0,r}$. Thus $\lambda'(a) = t \otimes a'$, $\lambda''(a) = a''$.

Analogously we have

$\tilde{A}^{r+1} = \tilde{A}^{0,r+1} \oplus \tilde{A}^{1,r}$, $\tilde{A}^{0,r+1} = L^0 \otimes V^{r+1}$, $\tilde{A}^{1,r} = L^1 \otimes V^r$.

The augmentation ε_K induces the morphism

$\varepsilon_A : A^r \to A^{-1,r} = k \otimes_k V^r = V^r$, $\varepsilon_A(A^{s,r}) = 0$, $s > 0$.

And similarly ε_L induces the morphism

$$\varepsilon_{\tilde{A}} : \tilde{A}^r \to \tilde{A}^{-1,r} = k \otimes_k V^r = V^r, \quad \varepsilon_{\tilde{A}}(\tilde{A}^{s,r}) = 0, \; s > 0$$

The differential d_A in A ,

$$d_A = (d_K \otimes 1) + (-1)^s (1 \otimes d_V) : A^{s,r} \to A^{r+s+1}$$

has a natural splitting $d_A = d_A' + d_A''$, $d' : A^{s,r} \to A^{s,1+r}$, $d'' : A^{s,r} \to A^{s,r+1}$. And similarly in $\tilde{A}$, $d_{\tilde{A}} = d_{\tilde{A}}' + d_{\tilde{A}}''$. The unit $i : k \to K^0$ together with the module morphism $h_K : K^1 \to K^0$ and the augmentation $\varepsilon_K : K^0 \to k$, such that in the resolution

$$0 \to k \xrightarrow{i} K^0 \xrightarrow{d_K} K^1 \to 0$$

$$\varepsilon_K \qquad h_K$$

hold the relations:

$\varepsilon_K \cdot i = 1_k$, $h_K \cdot d_K + i \cdot \varepsilon_K = 1_{K^0}$, is the contracting homotopy for $\varepsilon_K : K \to k$. This gives the contracting homotopy for the cochain transformation $\varepsilon_A : A^r \to A^{-1,r}$, $r \geq 0$.

There are the module morphisms

$$h' : A^{1,r} \to A^{0,r} ,$$

$h' = h_K \otimes 1$, and

$$c : A^{-1,r} \to A^{0,r} ,$$

$= i \otimes 1$, such that

$\varepsilon_A \cdot \iota = 1_{A^{-1,r}}$, $h' \cdot d_A' + \iota \cdot \varepsilon_A = 1_{A^{0,r}}$, and $h' \cdot d_A'' + d_A'' \cdot h' = 0$. Thus we have an analogue of the above sequence (after tensoring with V^r)

$$0 \to A^{-1,r} \to A^{0,r} \overset{d_A'}{\to} A^{1,r} \to 0 \quad .$$

$$\varepsilon_A : A^{0,r} \to A^{-1,r} \qquad h' : A^{1,r} \to A^{0,r}$$

Now we define the morphism of modules

$$h = (-1)^r \, h'\lambda' \; : A^{r+1} \to A^{0,r}$$

and

$$\sigma : A^{r+1} \to \iota(A^{-1,r}) \; ,$$

$\sigma(A^{0,r+1}) = 0$. We require that σ satisfies the additional properties :

$$\sigma \, d_A' = 0 \text{ on } \nu(A^{r+1}) \; , \; d_A'' \, \sigma + \sigma \, d_A'' = 0 \; .$$

From now on we assume these axioms for the operations.

<u>Proposition 1.</u> We have the relations

(1) $d_A'' \, h - h \, d_A'' = 0$

(2) $h' \, d_A' = 1 - \varepsilon_A$

(3) $d_A' \, h' = 1$

(4) $\lambda' d_A' = d_A'$

(5) $d_A' \lambda' = 0$

(6) $d_A'' \lambda' - \lambda' d_A'' = 0$

(7) $d_A'\sigma = 0$

(8) $\sigma d_A' = 0$ on the image $\nu(\tilde{A}^{r+1})$

(9) $d_A''\sigma + \sigma d_A'' = 0$

Proof. All these properties follow immediately from the above definitions. We look more closely only at (1). First of all observe that it is enough to show that

(10) $$h'd_A'' + d_A''h' = 0$$

because from here we have , for any $a \in A^{r+1}$,
$d_A''h(a) = (-1)^r d_A''h'\lambda'(a) = (-1)^{r+1}h'd_A''\lambda'(a) =$
$= (-1)^{r+1}h'\lambda'd_A''(a) = h\, d_A''(a)$. But (10) follows from the fact that for $t \otimes a' \in K^1 \otimes V^r = A^{1,r}$,
$h'd_A''(t \otimes a') = -(h_K \otimes 1)(t \otimes d_A''a') = -h_K(t)\otimes d_A''a' =$
$= -d_A''(h_K(t)\otimes a') = -d_A''(h_K \otimes 1)(t \otimes a') = -d_A''h'(t \otimes a')$.

Let $\bar{B}(\mathbb{A})$ be the bar construction on a DGA-algebra $\mathbb{A}$ over k. Recall that $B(\mathbb{A})$ is the vector space generated by

$$[a_1 \mid \ldots \mid a_r],$$

where $a_i \in \mathbb{A}$ and the degree of a_i, denoted by $|a_i|$ is ≥ 1. As usually, for the bar construction on a DGA-algebra (graded positively)

$$\deg [a_1|\ldots|a_r] = |a_1|+\ldots+|a_r|-r .$$

There is defined the shuffle product

$[a_1|\dots|a_r]*[a_{r+1}|\dots|a_{r+s}] =$

$$= \sum_{\pi} (-1)^{\sigma(\pi)} [a_{\pi(1)}|\dots|a_{\pi(r+s)}] \quad ,$$

where the summation goes over all (r,s)-shuffles π and $\sigma(\pi) = \Sigma(|a_i|-1)(|a_{r+j}|-1)$ summed over all pairs $(i,r+j)$ such that $\pi(r+j)<\pi(i)$.
[For this look at S. MacLane "Homology" pp 312-133].
Now we apply the bar construction to the algebra A. This product makes $\bar{B}(A)$ a commutative algebra.
There is also the coproduct

$$\psi[a_1|\dots|a_r] = \sum_{i=0}^{r} [a_1|\dots|a_i] \otimes [a_{i+1}|\dots|a_r] \ ,$$

the differential $\partial = \partial' + \partial''$ defined by

$$\partial'[a_1|\dots|a_r] = \sum_{i=1}^{r-1} (-1)^{|a_1|+\dots+|a_i|-i-1} [a_1|\dots|a_i a_{i+1}|\dots|a_r$$

$$\partial''[a_1|\dots|a_r] = \sum_{i=1}^{r} (-1)^{|a_1|+\dots+|a_{i-1}|-1} [a_1|\dots|d_A a_i|\dots|a_r] \ .$$

<u>Proposition 2.</u> $\bar{B}(A)$ with the product $*$, coproduct ψ and the differential ∂ is a differential Hopf algebra.

<u>Proof.</u> It is enough to check that $*$, ψ commute with the differential.

Now we define the morphism of modules

$$H_0 : \bar{B}(A) \to A^{0,*} \qquad \text{by}$$

$$H_0([a_1|\dots|a_r]) = h(h(\dots h(a_1)a_2)\dots a_r) .$$

<u>Lemma 1.</u> $d_A'' H_0([a_1|\dots|a_r]) =$

$$= \sum_{i=1}^{r} (-1)^{|a_1|+\dots+|a_{i-1}|-i-1} H_0([a_1|\dots|da_i|\dots|a_r]) +$$

$$+ \sum_{i=1}^{r-1} (-1)^{|a_1|+\dots+|a_i|-i} H_0([a_1|\dots|a_i a_{i+1}|\dots|a_r]) +$$

$$+ (-1)^{|a_1|+\dots+|a_r|-r} H_0([a_1|\dots|a_{r-1}]).\ \lambda''(a_r) +$$

$$+ (-1)^{|a_1|} H_0([\iota\ \varepsilon\ \lambda''(a_1)\cdot a_2|a_3|\dots|a_r]) +$$

$$+ \sum_{i=2}^{r-1} (-1)^{|a_1|+\dots+|a_i|-i-1} H_0([\iota\varepsilon\lambda''(H_0([a_1|\dots|a_{i-1}])\cdot a_i)a_{i+1}|a_{i+2}|\dots|a_r])$$

$$+ (-1)^{|a_1|+\dots+|a_r|-r-1} \iota\varepsilon\ \lambda''(H_0([a_1|\dots|a_{r-1}])\cdot a_r) .$$

<u>Proof.</u> By induction with respect to r . For $r=1$ we get

$$d_A''\ H_0([a])=d''\ h(a)=h(d''a)=h(da)-h(d'a)=$$

$$= h(da)+(-1)^{|a|-1}\ h'\lambda'(d'a''+d'(t\otimes a'))=h(da)+$$

$$+ (-1)^{|a|-1}\ a''+(-1)^{|a|}\ \iota\ \varepsilon(a'')=H_0([da])+(-1)^{|a|-1}\ a''+$$

$$+ (-1)^{|a|}\ \iota\ \varepsilon\ \lambda''\ (a) .$$

Similar computation is done for $r=2$ and $r=3$ as the 4th and 5th term occure there for the first time. Suppose that the formula is true for r.

Then

$$d''_A H_0([a_1|\dots|a_{r+1}]) = d''_A h(H_0([a_1|\dots|a_r])\cdot a_{r+1} =$$

$$= h(d''_A H_0([a_1|\dots|a_r])\cdot a_{r+1} +$$

$$+ (-1)^{|a_1|+\dots+|a_r|-r} h(H_0([a_1|\dots|a_r])\cdot d''_A a_{r+1} =$$

$$= h(d''_A H_0([a_1|\dots|a_r]\cdot a_{r+1})$$

$$+ (-1)^{|a_1|+\dots+|a_r|-r} h(H_0([a_1|\dots|a_r])\cdot d_A a_{r+1} +$$

$$+ (-1)^{|a_1|+\dots+|a_r|-r-1} h(H_0([a_1|\dots|a_r])\cdot d'_A a_{r+1}).$$

Now we compute the last term. Namely

$$(-1)^{|a_1|+\dots+|a_r|-r-1} h(H_0([a_1|\dots|a_r])\cdot d'_A a_{r+1}) =$$

$$= (-1)^{|a_1|+\dots+|a_r|-r} H_0([a_1|\dots|a_{r-1}|a_r\cdot a''_{r+1}]) +$$

$$+ (-1)^{|a_1|+\dots+|a_{r+1}|-r-1} H_0([a_1|\dots|a_r])\cdot a''_{r+1} +$$

$$+ (-1)^{|a_1|+\dots+|a_{r+1}|-r} \iota\,\varepsilon\,\lambda''(H_0([a_1|\dots|a_r])\cdot a_{r+1}) .$$

And finally, using this and the assumption, we get the formula.

Q.E.D.

And for the composition

$$H = \sigma\cdot\hat{H}_0 : \bar{B}(A) \to \iota(A^{-1,*}) ,$$

$$\hat{H}_0([a_1|\dots|a_{r+1}]) = H_0([a_1|\dots|a_r])a_{r+1}$$

we get $H([a_1|\dots|a_{r+1}]) = \sigma\{H_0([a_1|\dots|a_r])\cdot a_{r+1}\}$

<u>Lemma 2.</u> $d_A H([a_1|\dots|a_r]) = H(\partial[a_1|\dots|a_r]) +$

$$+ (-1)^{|a_1|-1} H([\iota\,\varepsilon\,\lambda''(a_1)\cdot a_2|a_3|\dots|a_{r+1}]) +$$

$$+\sum_{i=2}^{r-1}(-1)^{|a_1|+\ldots+|a_i|-i} H([\iota\varepsilon\lambda''(H_0([a_1|\ldots|a_{i-1}])\cdot a_i)a_{i+1}|a_{i+2}|a_{i+3}|\ldots|a_{r+1}$$

$$+(-1)^{|a_1|+\ldots+|a_r|-r}\sigma(\iota\ \varepsilon\ \lambda''(H_0([a_1|\ldots|a_{r-1}])\cdot a_r)\cdot a_{r+1}) +$$

$$+\sigma\ d'_A(H_0([a_1|\ldots|a_r])\cdot a_{r+1}) .$$

<u>Proof.</u> $d_A\ H([a_1|\ldots|a_{r+1}])=d_A\ \sigma\{H_0([a_1|\ldots|a_r])\cdot a_{r+1}\} =$

$$= d''_A\ \sigma\{H_0([a_1\ \ldots\ a_r])\cdot a_{r+1}\} =$$

$$= -\sigma\{d''_A\ H_0([a_1|\ldots|a_r])\cdot a_{r+1}\}+(-1)^{|a_1|+\ldots+|a_r|-r-1}\sigma\{H_0([a_1|\ldots|a_r])\cdot d''_A\ a_{r+1}\}$$

$$= -\sigma\{d''_A\ H_0([a_1|\ldots|a_r])\cdot a_{r+1}\}+(-1)^{|a_1|+\ldots+|a_r|-r-1}\sigma\{H_0([a_1|\ldots|a_r])\cdot d_A\ a_{r+1}\}$$

$$+(-1)^{|a_1|+\ldots+|a_r|-r}\sigma\{H_0([a_1|\ldots|a_r])\cdot d'_A\ a_{r+1}\} .$$

Next we use the formula from lemma 1 and compute the last term

$$(-1)^{|a_1|+\ldots+|a_r|-r}\ \sigma\{H_0([a_1|\ldots|a_r])\cdot d'_A\ a_{r+1}\} =$$

$$= \sigma\ d'_A\{H_0([a_1|\ldots|a_r])\cdot a_{r+1}\} +$$

$$+(-1)^{|a_1|+\ldots+|a_r|-r-1}\ \sigma\{H_0([a_1|\ldots|a_{r-1}])\cdot a_r\cdot a''_{r+1}\} .$$

Both these formulas substituted give the desired form to $d_A\ H([a_1|\ldots|a_r])$.

<u>Remark</u>. The importance of lemma 2 is obviously that under certain conditions one expects the map H to commute with the differentials.

The map $H : \bar{B}(A) \to \iota(A^{-1,*})$ is not multiplicative as it stands. In order to make it multiplicative we must impose two additional axioms on the maps σ and h.

Suppose that for any a, b $\varepsilon\ A^r$, $r \geq 1$,

(11) $\sigma\{a\}\cdot\sigma\{b\}=\sigma\{h(a)\cdot b\}+(-1)^{(|a|-1)(|b|-1)}\sigma\{h(b)\cdot a\}$,

(12) $h(a)\cdot h(b)=h(h(a)\cdot b)+(-1)^{(|a|-1)(|b|-1)}h(h(b)\cdot a)$.

From now on we assume that h,σ always satisfy these two additional properties. This leads to

Lemma 3. The map H is a morphism of GA-algebras

Proof. The multiplicativity of H is proved by induction in the following steps. First, from (12) it follows that

$$H_0([a_{r+2}])\cdot H_0([a_1|\ldots|a_r])=h(a_{r+2})\cdot h(h(\ldots h(a_1)a_2\ldots a_r)=$$

$$=\sum_{i=1}^{r+1}(-1)^{(|a_1|+\ldots+|a_{i-1}|-i+1)\cdot(|a_{r+2}|-1)}H_0([a_1|\ldots|a_{i-1}|a_{r+2}|a_i|\ldots|a_r])$$

This gives

$$H([a_1|\ldots|a_{r+1}])\cdot H([a_{r+2}])=\sigma\{H_0([a_1|\ldots|a_r])\cdot a_{r+1}\}\cdot\sigma\{a_{r+2}\}=$$

$$=\sigma\{H([a_1|\ldots|a_{r+1}])\cdot a_{r+2}\}+$$

$$+(-1)^{(|a_1|+\ldots+|a_{r+1}|-r-1)(|a_{r+2}|-1)}\sigma\{H_0([a_{r+2}])\cdot H_0([a_1|\ldots|a_r])\cdot a_{r+1}\}$$

And this, by the above formula for H_0, is equal to

$$\sum_{i=1}^{r+2}(-1)^{(|a_i|+\ldots+|a_{r+1}|-r-i)(|a_{r+2}|-1)}H([a_1|\ldots|a_{i-1}|a_{r+2}|a_i|\ldots|a_{r+1}])=$$

$= H([a_1|\ldots|a_{r+1}] * [a_{r+2}])$.

All this was done by induction with respect to the number of elements in the first component of the *-product. Analogously, proceeding by induction with respect to the number of elements in the second component of the *-product we get

$$H([a_1|\ldots|a_r])\cdot H([a_{r+1}|\ldots|a_{r+s}]) =$$

$$= H([a_1|\ldots|a_r] * [a_{r+1}|\ldots|a_{r+s}]) .$$

Remark. The formulas (11) and (12) are the algebraic versions of the Fubini theorem.

1.3 The Adams Theorem.

In this part we give a new proof of Chen's de Rham theorem [4] and also we suggest a different proof for the theorem of Adams [1] ,over k .

Let M be a differentiable manifold and $A^*(M)$ the de Rham complex. We restrict ourselves to the manifold although the proof can be modified easily for the general case.

For any $\alpha' \in K_n(\Omega(M))$ represented by

$\alpha : I^n \to \Omega M$, $\hat{\alpha} : I^{n+1} \to M$, $\hat{\alpha}(x,t) = \alpha(x)(t)$,

$x \in I^n = t \in I$ the map $\hat{\alpha}^* : A^*(M) \to A^*(I^{n+1})$

induces the morphism of the differential Hopf algebras

$$\hat{\alpha}^* : \overline{B}(A^*(M)) \to \overline{B}(A^*(I^{n+1})) .$$

Now, going back to the definition of H with $A=A^*(I^{n+1})$, $A^{-1,*}=A^*(I^n)$, we get the composition

$$\bar{B}(A^*(M)) \overset{\hat{\alpha}^*}{\to} \bar{B}(A^*(I^{n+1})) \overset{H}{\to} A^*(I^n).$$

And now we define the map

(13) $\xi : \bar{B}(A^*(M)) \to A(\Omega(M))$.

$\xi([a_1|\dots|a_r])(\alpha) = H\cdot\hat{\alpha}^*([a_1|\dots|a_r])$.

Lemma 4. ξ is a morphism of DGA-algebras.

Proof. That ξ is a morphism of graded algebras follows immediately from the fact that $\hat{\alpha}^*$ is morphism of GA-algebras and from Lemma 3.

In order to establish that ξ commutes with the differentials we must show that for any $\alpha \in K_n(\Omega(M))$ and any form $w \in A^*(M)$ holds $\varepsilon_A \hat{\alpha}^* w=0$. In order to do this we return back to algebra.

Suppose that we have a decomposition $L^o=k \oplus L^o/k$ such that $i(s) = s+0$ and that $d_L : 0 \oplus L^o/k \to L^1$ is an isomorphism. Under these conditions we can prove the following

Fact. $\varepsilon_L h_L = 0$ on L^1.

Proof. For any $\ell \in L^1, h_L(\ell)=\ell_1+\ell_2 \in k \oplus L^o/k$. Thus $\varepsilon_L h_L(\ell) = \varepsilon_L(\ell_1)+\varepsilon_L(\ell_2)=\ell_1+\varepsilon_L(\ell_2)$, and

$i\, \varepsilon_L (\ell_2) = -h_L d_L(\ell_2) + \ell_2$, and

$\varepsilon_L\, i\, \varepsilon_L (\ell_2) = -\varepsilon_L h_L d_L (\ell_2) + \varepsilon_L (\ell_2)$.

Because $\varepsilon_L\, i$ = identity we get $\varepsilon_L(\ell_2) = -\varepsilon_L\, h_L\, d_L(\ell_2) + \varepsilon_L(\ell_2)$, therefore $\varepsilon_L\, h_L(d_L(\ell_2)) = 0$. And because d_L is an isomorphism, we get $\varepsilon_L\, h_L = 0$. Q.E.D.

Because in our case $L^o = A^o(S^1)$, S^1 with the base point s_0, $A^o(S^1)|s_0 = k$ and $L^o = k \oplus A^o(S^1)/k$. By our definition $\nu(\tilde{A}^{0,r}) = L^o \otimes V^r =$

$$= (k \otimes V^r) \oplus (A^o(S^1)/k \otimes V^r) =$$

$$= A_1^{0,r} \oplus A_2^{0,r} \text{ with the obvious notation.}$$

The augmentation $\varepsilon_{\tilde{A}} \equiv 0$ on $\tilde{A}_2^{0,r}$ by the above observation. Recall that $\tilde{A}^{0,r} \subset A^r(I^n \times S^1)$. Then $\tilde{A}_1^{0,r}$ = the r-forms in the x variables which are constant functions in the t variable, and $\tilde{A}_2^{0,r}$ = the r-forms in the x variables that vanish at the points (x, s_0), s_0 = the base point of S^1.

Now, for any $\alpha \in K_n(\Omega(M))$ the map $\hat{\alpha} : I^{n+1} \to M$ has the property $\hat{\alpha}(x,0) = \hat{\alpha}(x,1) = *$. Thus for any $w \in A^*(M)$, $\hat{\alpha}^* w \mid (x,0) = \hat{\alpha}^* w \mid (x,1) = 0$. Therefore $\hat{\alpha}^* w \in \tilde{A}_2^{0,r}$, and finally $\varepsilon_A \hat{\alpha}^* w = 0$. This implies that for $\alpha \in K_n(\Omega(M))$ all the terms involving $\iota \in \lambda''$ in the formula from lemma 2, where $a_j = \hat{\alpha}^* w_j$ for $w_j \in A^*(M)$, vanish. The last term $\sigma d'_A(H_0([\hat{\alpha}^* w_1 | \ldots | \hat{\alpha}^* w_r]) \cdot \hat{\alpha}^* w_{r+1}) = 0$ simply because $H_0([\hat{\alpha}^* w_1 | \ldots | \hat{\alpha}^* w_r]) \cdot \hat{\alpha}^* w_{r+1}$ belongs to $\nu(\tilde{A})$, by (8).

Remark. Because ξ is morphism of graded algebras which commutes with the differentials, the image $\xi(\bar{B}(A^*(M))$ in $A(\Omega(M))$ is also a DGA-algebra.

Let $[w_1|\ldots|w_r] \in \bar{B}(A^*(M))$, where $w_1,\ldots,w_r$ are differential forms on M of degree ≥ 1. Then $\xi([w_1|\ldots|w_r])$ is the element $\int w_1 \ldots w_r$ in Chen's notation [4].

Definition. The algebra $C = C(\Omega(M), A^*(M)) = \xi(\bar{B}(A^*(M))$ is the algebra of iterated integrals (of the first kind).

We define the filtration $\{F_rC\}$, $k = F_0C \subset F_1C \subset \ldots \subset F_rC = C$ of the Hopf algebra of iterated integrals C by

$F_rC = \{\xi([a_1|\ldots|a_s]) \mid a_i \in A^p(M),\ p \geq 1,\ 0 \leq s \leq r\}$,

$F_rC = 0$, $r < 0$. From the definition of the differential in C it is clear that $d(F_rC) \subset F_rC$.

The filtration of the bar construction, as it is defined in S. MacLane "Homotopy" p.309, $F_r = F_r\bar{B}(A^*(M))$ = the submodule spanned by the elements $[a_1|\ldots|a_s]$, $a_i \in A^p(M)$, $p \geq 1$, $0 \leq s \leq r$. $F_r = 0$ for $r < 0$. And from the formulas defining the differential ∂ in $\bar{B}(A^*(M))$ we get immediately $\partial(F_r) \subset F_r$.

It turns out that for the comparison of the spectral sequences of these two filtrations it is more suitable to replace the DGA-algebra $A^*(M)$ by somewhat smaller algebra with the same cohomology. Namely, let

$$\bar{A}^0(M) = A^1(M)/dA^0(M),\ \bar{A}^k(M) = A^{k+1}(M),\ k \geq 1 .$$

Then we define the filtration $\bar{F}_r = F_r\bar{B}(\bar{A}^*(M)), r \geq 0$. The surjection $\pi : A^*(M) \to \bar{A}^*(M)$ gives the surjective maps $\pi : F_r \to \bar{F}_r$, $r \geq 0$, and $\pi : F_r/F_{r-1} \to \bar{F}_r/\bar{F}_{r-1}$.

The DGA-algebra morphism ξ (13) induces the map

$$F\xi : F_r/F_{r-1} \to F_rC/F_{r-1}C .$$

In fact we get

<u>Proposition 3.</u> The map $F\xi$ factorizes through $\bar{F}_r/\bar{F}_{r-1}$ so that the following diagram is commutative

$$\begin{array}{ccc} F_r/F_{r-1} & & \\ & \overset{F\xi}{\searrow} & \\ \pi\downarrow & & F_rC/F_{r-1}C \\ & \overset{\bar{F}\xi}{\nearrow} & \\ \bar{F}_r/\bar{F}_{r-1} & & \end{array}$$

Proof. It is enough to show that $a_i = da$, $a \in A^0(M)$, $[a_1|\ldots|a_{i-1}|da|a_{i+1}|\ldots|a_r] \in F_r$ but $\notin F_{r-1}$ the image $\xi([a_1|\ldots|da|\ldots|a_r]) \in F_{r-1}C$.

For many $\alpha \in K_n(\Omega(M))$, $\hat{\alpha} : I^n \times I \to M$, $a \in A^*(M)$ denote $\hat{\alpha}^* a = c$. Then $\xi([a_1|\ldots|da|\ldots|a_r])(\alpha) =$ $H \cdot \hat{\alpha}^*([a_1|\ldots|d_A a|\ldots|a_r]) =$

$$= \begin{cases} \sigma\{H_0([c_1|\ldots|d_A c_i|\ldots|c_{r-1}])\cdot c_r\} & \text{for } 1 \leq i < r , \\ \sigma\{H_0([c_1 \ldots c_{r-1}]) \cdot d_A c_i\} & \text{for } i = r , \end{cases}$$

where $d_A = d_A' + d_A''$ is the differential in $A^*(I^{n+1})$. In the case $1 < i < r$ we have, where the signs are ommitted as they turn out to be irrelevant,

$$H_0([c_1|\ldots|d_A c|\ldots|c_{r-1}]) =$$
$$= (-1)^\bullet\ h\ d_A\ [h(\ldots h(c_1)\ldots c_{i-1})\cdot c] \ +$$
$$+ (-1)^\bullet\ h\ [d_A h(\ldots h(c_1)\ldots c_{i-1})\cdot c] \ =$$
$$= (-1)^\bullet\ h\ (\ldots h(c_1)\ldots c_{i-1})\cdot c + (-1)^\bullet d_A'' h\{h(\ldots h(c_1)\ldots c_{i-1})\cdot c\}$$
$$+ (-1)^\bullet\ \iota\ \varepsilon_A\ \{h(\ldots h(c_1)\ldots c_{i-1})\cdot c\} \ +$$
$$+ (-1)^\bullet\ h\{h(\ldots h(c_1)\ldots c_{i-2})\cdot c_{i-1}\cdot c\} \ +$$
$$+ (-1)^\bullet\ h[h\{d''_A\ (h(\ldots h(c_1)\ldots)c_{i-1}\}\cdot c] \ .$$

Because α maps I^n into the loop space the 3^{rd} term is zero, and because $a \in A^o(M)$ the 2^{nd} and the 5^{th} terms are zero. Therefore we get

$$\sigma\{H_0([c_1|\ldots|d_A c|\ldots|c_{r-1}])\cdot c_r\} =$$
$$= (-1)^\bullet \sigma\{H_0([c_1|\ldots|c_{i-1}|c\cdot c_{i+1}|c_{i+2}|\ldots|c_{r-1}])\cdot c_r\} \ +$$
$$+ (-1)^\bullet \sigma\{H_0([c_1|\ldots|c_{i-1}\cdot c|c_{i+1}|\ldots|c_{r-1}])\cdot c_r\} =$$

$= (-1)^{\bullet}\ \xi([a_1|\dots|a_{i-1}|a\cdot a_{i+1}|\dots|a_r])(\alpha)\ +$

$+\ (-1)^{\bullet}\ \xi([a_1|\dots|a_{i-1}\cdot a|a_{i+1}|\dots|a_r])(\alpha)$

which is equal to $\xi([a_1|\dots|da|\dots|a_r])(\alpha)$ for every $\alpha \in K_n(\Omega(M))$. Hence we have shown that for $1 < i < r$, $\xi([a_1|\dots|da|\dots|a_r]) \in F_{r-1}C$.

A similiar argument works for $i=1,r$.

Proposition 4. If M is path connected then

$$\overline{F}\xi\ :\ \overline{F}_r/\overline{F}_{r-1} \longrightarrow F_rC/F_{r-1}C$$

is bijective.

Proof. Let $\alpha_j' : I^{d_j} \to \Omega(M)$ be the map associated with the element $\alpha_j \in K_{d_j}(\Omega(M))$, $j = 1,2,\dots,n$.

For each element $[a_1|\dots|a_n]$ representing a class in $\overline{F}_n/\overline{F}_{n-1}$, $a_j \in \overline{A}^{d_j}$, we define an n-linear map $b_{a_1\dots a_n}$ on the singular cubic chains $\alpha_1,\dots,\alpha_n$ by

$$b_{a_1\dots a_n}(\alpha_1,\dots,\alpha_n)=\Big(\int_{I^{d_1+1}} \hat{\alpha}_1^{\,*}a_1\Big)\bullet\dots\bullet\Big(\int_{I^{d_n+1}} \hat{\alpha}_n^{\,*}a_n\Big)\ ,$$

where the integration is taken over the oriented cubes in the euclidean space and where $\bullet$ stands for the usual product in R.

It is easy to see that the maps $b_{a_1\dots a_n}$ span the space of all n-linear maps $L_n(K_*(\Omega(M))$ from $K_*(\Omega(M))$ into R.

For a q-cube $\alpha : I^q \to \Omega(M)$ we define the reduced cube

$$\tilde{\alpha} = \begin{cases} \alpha & \text{for } q > 0 \\ \alpha - \varepsilon & \text{for } q = 0 \end{cases},$$

where $\varepsilon : I^0 \to *$ is the null loop.

Thus we have a monomorphism

$$\gamma_n : \bar{F}_n/\bar{F}_{n-1} \to L_n(K_*(\Omega(M))) \quad .$$

On the other hand, any element in $F_nC/F_{n-1}C$ can be represented in the form $\xi([a_1|\dots|a_n])$ + terms of lower dimension; $a_j \in A^*(M)$. And the evaluation of the element $\xi([a_1|\dots|a_n])+\dots$ in $A^*(\Omega(M))$ on the sequence of reduced cubes $\alpha_1,\dots,\alpha_n$; $\alpha_j \in K_*(\Omega(M))$ via integration give a map

$$\eta_n : F_nC/F_{n-1}C \to L_n(K_*(\Omega(M)))$$

such that

$$\eta_n(\xi([a_1|\dots|a_n])+\dots) = b_{[a_1|\dots|a_n]} \quad .$$

From the construction we see that γ_n is a monomorphism. The map λ_n is the diagram

$$\begin{array}{ccc} \bar{F}_n/\bar{F}_{n-1} & \xrightarrow{\gamma_n} & L_n(K_*(\Omega(M))) \\ & \lambda_n \searrow \quad \nearrow \eta_n & \\ & F_nC/F_{n-1}C & \end{array}$$

is injective, as follows by induction. And finally we see that $\eta_n(F_nC/F_{n-1}C) = \gamma_n(\overline{F}_n/\overline{F}_{n-1})$ This proves that λ_n is bijective.

Lemma 5. The homomorphism $\xi : \overline{B}(A^*(M)) \to C$ of DGA-algebras induces an isomorphism of the cohomologies.

Proof. Because the cohomology of $\overline{B}(A^*(M))$ is isomorphic to that of $\overline{B}(\overline{A}^*(M))$ and the bijective map $\overline{F}\xi$ induces an isomorphism of the spectral sequences already on the E_1-level, we get the statement.

From the last lemma and from the theorem of Adams [1], using the pairing of the bar and cobar constructions, we get

Theorem 2 (Chen). Let M be a connected and simply connected differentiable manifold, and let $A^*(M)$ be the de Rham complex of M with a finite cohomology. Then there is an isomorphism

$$H^*(\Omega(M);k) \cong H^*(C;k)$$

of the singular cohomology of $\Omega(M)$ with the homology of the DGA-algebra of iterated integrals on M.

Remark. The statement of the theorem is not the most general possible. The formulation of this theorem for a topological space M, with the correct limitations can be found in [6]. Our proof works for that case as well. In fact our proof, being algebraic, goes through without change even for k = Q, while the original proof depends heavily on the differential forms over R or C.

A slight modification of our definitions makes the statement of the theorem correct even when k = Z. In that case $A^*(M)$ is the de Rham complex with integral coefficients [4].

On the other hand the algebra of the iterated integrals can be used to prove the theorem of Adams. From the recent work of Cartan [8] and Watkis [2] we know that the cohomology of $A(\Omega(M))$ with the coefficients k = R,Q,C (or even Z, with the correct interpretation of the functor A) is isomorphic to the cubic singular cohomology of $\Omega(M)$. On the other hand the algebra of the iterated integrals C is a DGA subalgebra of $A(\Omega(M))$. It turns out that for a connected and simply connected manifold M there is an inverse map $\eta : A(\Omega(M)) \to C$ which is a quasi-isomorphism of DGA algebras. Thus, by using the pairing of bar and cobar construction, or equivalently the pairing of C with the bar construction we have an independent proof of the theorem of Adams [1].

Because the cohomology of the commutative DGA-algebra $\overline{B}(A^*(M))$ (with respect to the *-product) is isomorphic with the cohomology of $\Omega(M)$ we can iterate the whole process. Namely, there is well defined DGA-algebra morphism

$$\xi_2 : \overline{B}(\overline{B}(A^*(M)) \to A(\Omega^2(M)).$$

And in general a DGA-algebra morphism

$$\xi_n : \overline{B}_n(A^*(M)) = \underbrace{\overline{B}(\ldots(\overline{B}}_{\text{n-times}}(A^*(M))\ldots) \to A(\Omega^n(M)).$$

The image $C_n = \xi_n(\overline{B}_n(A^*(M))$ in $A(\Omega^n(M))$ is the algebra of iterated integrals of the n-th kind.

2. De Rham theorem for the fundamental group and homotopy periods

We recall the construction of the minimal model [12] $\mathcal{M}$ for the fundamental group $G = \pi_1(M)$ of a CW-complex M. Then a map T from the differential graded algebra $\mathcal{M}$ to a commutative DG-algebra $A^*(M)$ over k = C,R or Q is constructed. This map is an extension of the parallel transport defined by Chen [4]. This map T is then used in the proof of the de Rham theorem for the group Γ. As a by-product we get all the homotopy invariants, called homotopy periods in [7] of the pointed map of S^1 into M.

2.1 Minimal model

The lower central series for the fundamental group $G_1 = G = \pi_1(M)$ of a CW complex M is the system of groups

$$G_1 = G \supset G_2 \supset G_3 \supset \dots , G_{n+1} = [G_1 G_n],$$

where G_n/G_{n+1} is the center of G/G_{n+1}, and $0 \to G_n/G_{n+1} \to G^{(n+1)} \to G^{(n)} \to 1$, $G^{(n)} = G/G_n$, is the central extension. To the lower central series corresponds the tower of nilpotent quotients

$$G^{(2)} \leftarrow G^{(3)} \leftarrow G^{(4)} \leftarrow \dots$$

and the Postnikov tower of Eilenberg-MacLane spaces

$$K(G^{(2)},1) \leftarrow K(G^{(3)},1) \leftarrow K(G^{(n)},1) \leftarrow \dots .$$

With each of these CW complexes we associate the k-vector space $A^*(K(G^{(n)},1))$ of differential forms. Then with the Postnikov tower there is associated the system of compatible DG-algebras

$$A^*(K(G^{(2)},1)) \to A^*K(G^{(3)},1)) \to \dots .$$

Let

$$\rho^{(n)} : {}_{(n)}M^* \to A^*(K(G^{(n+1)},1)), \ n \geq 1$$

be the minimal model for the DG-algebra $A^*(K(G^{(n+1)},1))$.

The maps $\rho^{(n)}$ are graded algebra morphisms commuting with the differentials.

The algebraic version of the Serre spectral sequence of the fibration

$$K(G_n/G_{n+1},1) \to K(G^{(n+1)},1) \to K(G^{(n)},1)$$

gives us a complete description of the minimal model

$$M^* = \lim_{(n)} M^*$$

of the limit

$$\lim A^*(K(G^{(n)},1).$$

${}_{(1)}M^*$ is a free, over k, algebra generated by the elements $\alpha_i^{(1)}$ of the basis $\alpha_1^{(1)},\ldots,\alpha_{k_1}^{(1)}$ for $H^1(K(G^{(2)},1);k) \cong H^1(A^*(K(G^{(2)},1)) \cong \mathrm{Hom}\ (G^{(2)},k)$. The differential d in ${}_{(1)}M^*$ is defined by $d\alpha_i^{(1)}=0$. Because $\alpha_i^{(1)}$ is one-dimensional, ${}_{(1)}M^*$ is the exterior algebra on the generators $\alpha_1^{(1)},\ldots,\alpha_{k_1}^{(1)}$.

Next recall, that there is an injective transgression map

$$T : H^1(K(G_n/G_{n+1},1);W) \to H^2(K(G^{(n)},1);W)$$

for any abelian group W. If $\iota \in H^1(K(G_n/G_{n+1},1);G_n/G_{n+1}) = \mathrm{Hom}(G_n/G_{n+1}, G_n/G_{n+1})$ is the identity element, then

$\tau(\iota) \in H^2(K(G^{(n)},1); G_n/G_{n+1})$ as a map $\tau(\iota): K(G^{(n)},1) \to$ $\to K(G_n/G_{n+1},2)$ defines $K(G^{(n+1)},1)$ by the full-back of the path space

$$PK(G_n/G_{n+1},2) \to K(G_n/G_{n+1},2) .$$

This allows us to define the next stage of the minimal model.

${}_{(2)}\mathcal{M}^*$ is simply the extension ${}_{(1)}\mathcal{M}^*[\alpha_1^{(2)},\ldots,\alpha_{k_2}^{(2)}]$ when we adjoin to ${}_{(1)}\mathcal{M}^*$ the generators $\alpha_i^{(2)}$, where $\alpha_1^{(2)},\ldots,\alpha_{k_2}^{(2)}$ is the basis for $H^1(K(G_2/G_3);k)$. This basis is determined by the one forms $a_1^{(2)},\ldots,a_{k_2}^{(2)}$ in $A^1(K(G^{(3)},1))$. These forms can be chosen in such a way that $da_i^{(2)} = \rho^{(1)}(\beta_i^{(1)}) \in A^2(K(G^{(2)},1))$, where $\beta_i^{(1)}$ are in ${}_{(1)}\mathcal{M}^2$ and $\rho^{(1)}(\beta_i^{(1)})$ represents $\tau(d\alpha_i^{(2)})$ in $H^2(K(G^{(1)},1);k)$. The elements β_1 are linear combinations of products of elements from ${}_{(1)}\mathcal{M}^*$. The differential in ${}_{(2)}\mathcal{M}^*$ is defined by $d\alpha_i^{(2)} = \beta_i^{(1)}$. And the map $\rho^{(2)}: {}_{(2)}\mathcal{M}^* \to A^*(K(G^{(3)},1))$ is the extension of $\rho^{(1)}$ defined by sending

$$\rho^{(2)} : \alpha_i^{(2)} \to a_i^{(2)} .$$

The construction of ${}_{(n)}\mathcal{M}^*$ proceeds by induction. Therefore we can summarise, and state

<u>Proposition 5.</u> The tower of nilpotent quotients gives, over k, the tower of minimal models

$$\begin{array}{ccccccc} {}_{(1)}\mathcal{M}^{*} & \to & {}_{(2)}\mathcal{M}^{*} & \to & {}_{(3)}\mathcal{M}^{*} & \to & \cdots \\ \rho^{(1)}\downarrow & & \rho^{(2)}\downarrow & & \rho^{(3)}\downarrow & & \\ A^{*}(K(G^{(2)},1)) & \to & A^{*}(K(G^{(3)},1)) & \to & A^{*}(K(G^{(4)},1)) & \to & \cdots \end{array}$$

where

$${}_{(n)}\mathcal{M}^{*} = \bigwedge(\alpha_1^{(1)},\ldots,\alpha_{k_1}^{(1)},\ldots,\alpha_1^{(n)},\ldots,\alpha_{k_n}^{(n)})$$

is the exterior algebra on the generators, with the differential

$$d\alpha_i^{(k)} = \beta_i^{(k-1)}$$

Let $L_n = [{}_{(n)}\mathcal{M}^{*}]^{*}$ be the dual of the differential graded algebra. ${}_{(n)}\mathcal{M}^{*}$ being minimal generated by elements in degree 1 implies that L_n is nilpotent Lie algebra, with the bracket defined as a dual of the differential. Thus the tower of minimal models determines the tower of nilpotent Lie algebras

$$0 \leftarrow L_1 \leftarrow L_2 \leftarrow L_3 \leftarrow \cdots \quad .$$

Let M be a CW complex with the fundamental group $G = \pi_1(M)$. To be specific, we take for example $M = K(G,1)$. Then we have the Postnikov tower

$$\begin{array}{ccccc} K(G,1) & & & & \\ p_2\downarrow & \searrow p_3 & & & \\ K(G^{(2)},1) & \leftarrow & K(G^{(3)},1) & \leftarrow & \cdots \end{array}$$

where the maps p_n are given by the projections $G \to G/G_n = G^{(n)}$. With this diagram is associated the system of commutative DG-algebras. In particular we get the maps

$$ {}_{(n)}\mathcal{M}^1 \overset{\rho^{(n)}}{\to} A^1(K(G^{(n+1)},1)) \overset{p_{n+1}^*}{\to} A^1(M) . $$

The composition ${}_{(n)}\tilde{\omega} = p_{n+1}^* \cdot \rho^{(n)}$ extends to the morphism of DG-algebras

$$ {}_{(n)}\tilde{\omega} : {}_{(n)}\mathcal{M}^* \to A^*(M), $$

and in the limit we get $\tilde{\omega} = \lim {}_{(n)}\tilde{\omega} : \mathcal{M}^* \to A^*(M)$. For each n the map ${}_{(n)}\tilde{\omega}$ defines a unique element

$$ {}_{(n)}\omega \in A^*(M) \otimes L_n , $$

and

$$ \omega = \lim {}_{(n)}\omega \in A^*(M) \otimes L , \quad L = \lim L_n . $$

Remark. The element ω viewed as an L-valued form on M is a special case of a power series connection studied by Chen [4]. In the terminology of [4] this is a flat connection because as a map it commutes with the differentials.

2.2 Homotopy periods

The differential of the generator $\alpha_i^{(k)}$ of ${}_{(k)}\mathcal{M}^1$ is the element $d\alpha_i^{(k)} = \beta_i^{(k-1)} \in {}_{(k-1)}\mathcal{M}^2$. But

${}_{(k-1)}M^*$ is a free algebra on the generators $(\alpha_1^{(1)},\ldots,\alpha_{k_{k-1}}^{(k-1)})$. Therefore $\beta_i^{(k-1)}$ is the sum of products of generators from ${}_{(k-1)}M^*$. Take those $\alpha^{(k-1)}$'s that occure in $\beta_i^{(k-1)}$, and take the differentials $d\alpha_j^{(k-1)} = \beta_j^{(k-2)}$ again. Then consider all the $\alpha_r^{(k-2)}$'s in $\beta_j^{(k-2)}$ etc. The system of all these $\alpha_j^{(k-1)}$, $\alpha_r^{(k-2)}$, etc is called the pyramid of the generator $\alpha_i^{(k)}$. Note that such a pyramid is uniquelly determined by each $\alpha_i^{(k)} \varepsilon {}_{(k)}M^1$. We denote the pyramid associated with $\alpha_i^{(k)}$ by $P(\alpha_i^{(k)})$.

In order to give a more formal definition of the pyramid $P(\alpha_i^{(k)})$, we first of all observe that $\alpha_i^{(k)}$ is an element in some defining system for a matrix Massey products.

Let $(W_1,\ldots,W_m)$ be a system of matrices with the following two properties:

(i) The entries of the matrix W_r are the elements of a subset of the generators $(\alpha_1^{(1)},\ldots,\alpha_{k_1}^{(1)})$ for ${}_{(1)}M^*$.

(ii) The products of the matrices W_iW_{i+1} is defined for $i = 1,2,\ldots,m-1$.

Let us denote by V_i the matrix of the cohomology classes determined by W_i. We assume that $V_iV_{i+1}=0$ for $i = 1,2,\ldots,m-1$. Then denote W_i by $A_{i-1,i}$ and define inductively the matrices

A_{ij} by the relations $0=A_{ij}+\sum_{k=i+1}^{j-1} A_{ik} A_{kj}$, $1<j-i<m$

and so that A_{ij}, $j-i=n$, has as its entries only the generators from the set

$$\alpha_1^{(n)},\ldots,\alpha_{k_n}^{(n)} \quad .$$

Then the system of all the matrices A_{ij}, $0\leq i<j\leq n$, $(i,j)\neq(0,m)$ is called the defining system for the matrix Massey product $\langle V_1,\ldots,V_m\rangle$. It can be put into the form of upper trianglular block matrix

$$A = \begin{pmatrix} 0 & A_{01} & A_{02} & \cdots & A_{0\,m-1} & * \\ & 0 & A_{12} & & A_{1\,m-1} & A_{1m} \\ & & 0 & & & \\ & & & & & A_{m-1\,m} \\ & & & & 0 & \\ & 0 & & & & 0 \end{pmatrix}$$

For more details we refer to [5].

<u>Definition.</u> A pyramid $P(A_{rs})$ of the matrix A_{rs},from the defining system A, is the system of matrices A_{ij}, $r\leq i<j\leq s$.

Observe that the pyramid $P(A_{rs})$ is closed with respect to the relations

$$\text{(14)} \qquad dA_{ij} + \sum_{k=i\pm 1}^{j-1} A_{ik} A_{kj} = 0,\ r \leq i < j \leq s.$$

In fact the system of matrices A_{ij}, $r < i < j < s$

is a defining system for the matrix Massey product $\langle V_r, V_{r+1}, \ldots, V_s \rangle$.

Definition. A pyramid $P(a_{rs})$ of the entry a_{rs} of the matrix A_{rs}, from the defining system A, is the set of elements from the matrices A_{ij}, $r \leq i < j \leq s$, which is closed under the relations (14). The element a_{rs} is called the vertex of $P(a_{rs})$.

If all the matrices A_{ij} in A are one by one matrices then we have the defining system for the "ordinary" Massey product $\langle V_1, \ldots, V_m \rangle$.

Summarising the above discussion in terms of the introduced terminology we have

Proposition 6. Each element $\alpha_i^{(k)} \in {}_{(k)}M^1$ is a vertex of a pyramid $P(\alpha_i^{(k)})$ associated with some defining system A for the matrix Massey products.

Let ME be the set of square matrices with entries in a graded k-module E. Then for any two matrices $A, B \in ME$, such that the product AB is defined, we define $A \otimes B$ to be a matrix with the entries from $E \otimes E$. The multiplication in $A \otimes B$ is given by the multiplication of matrices.

From the elements in the pyramid $P(A_{rs})$ we construct inductively the following objects, thought of as polynomials in the elements of $P(A_{rs})$.

$$A^{(2)}_{i\ i+2} = A_{i\ i+2} + A_{i\ i+1} \otimes A_{i+1\ i+2}$$

for $r \leq i < s-1$,

$$A^{(3)}_{i\ i+3} = A_{i\ i+3} + A_{i\ i+1} \otimes A_{i+1\ i+3} + A^{(2)}_{i\ i+2} \otimes A_{i+2\ i+3}$$

for $r \leq i < s-2$. And in general

$$A^{(n)}_{i\ i+n} = A_{i\ i+n} + A_{i\ i+1} \otimes A_{i+1\ i+n} + \sum_{k=2}^{n-2} A^{(k)}_{i\ k+i} \otimes A_{k+i\ i+n} + A^{(n-1)}_{i\ i+n-1} \otimes A_{i+n-1\ i+n}$$

for $r \leq i < s-n+1$ and for $2 \leq n \leq s-r$. The successive substitutions give an element

$$I(A_{rs}) = A^{(r-s)}_{rs}.$$

We call this element the polynomial of the pyramid $P(A_{rs})$ or simply the polynomial of A_{rs}. And if we trace an individual element we get the polynomial $I(a_{rs})$ of the element a_{rs} of the matrix A_{rs}, or the polynomial of the pyramid $P(a_{rs})$.

In particular it makes sense to talk about the polynomial $I(\alpha^{(n)}_i)$ associated with the element $\alpha^{(n)}_i \in {}_{(n)}\mathcal{M}^1$. Obviously $I(\alpha^{(n)}_i)$ belongs to the tensor algebra of $\mathcal{M}^1$, $T\mathcal{M}^1$.

The polynomial $I(A_{rs})$ is well defined for a pyramid constructed with entries from any DG-algebra. In particular, from the algebra $A^*(M)$. And if for $\omega_1, \omega_2 \in A^1(M)$ the tensor product $\omega_1 \otimes \omega_2$ is denoted by $[\omega_1|\omega_2]$ and the single element $\omega \in A^1(M)$ by $[\omega]$ with the grading from the bar construction, then we can define for each $\alpha_i^{(n)} \in {}_{(n)}\mathcal{M}^1$,

$$\omega_i^{(n)} = \rho^{(n)} \cdot p_n^* \ \alpha_i^{(n)}$$

as an element in $A^1(M)$; and $I(\omega_i^{(n)}) \in \overline{B}(A^*(M))$. The ξ-image of $I(\omega_i^{(n)})$,

$$\Omega(\alpha_i^{(n)}) = \xi(I(\omega_i^{(n)})) \in C^0.$$

Let $(X_1^{(1)}, \ldots, X_{k_1}^{(1)}, \ldots, X_1^{(n)}, \ldots, X_{k_n}^{(n)})$ be the generators for L_n, dual to $(\alpha_1^{(1)}, \ldots, \alpha_{k_n}^{(1)})$.

Proposition 7.

$$T = 1 + \Sigma\ \Omega\ (\alpha_i^{(n)}) \cdot X_i^{(n)} \ ,$$

where the summation goes over all the generators of $\mathcal{M}^*$, is an element of

$$C^0 \otimes L\ .$$

Example. Let $M = \bigvee_n S^1$ be the wedge of n circles. Then the fundamental group $G = \pi_1(M)$ is a free group, over k, on n 1-dimensional generators. The n-th stage of the minimal model ${}_{(n)}\mathcal{M}^*$ for $K(G^{(n)},1)$ is the free algebra $k(\alpha_1,\ldots,\alpha_n;\alpha_{ij};\ldots,\alpha_{i_1,\ldots,i_n})$, where the generators α satisfy

$$d\alpha_{ij} + \alpha_i\alpha_j = 0,\ d\alpha_{ijk} + \alpha_i\alpha_{jk} + \alpha_{ij}\alpha_k = 0,\ \text{etc.}$$

Let $(X_1,\ldots,X_n,X_{ij},\ldots,X_{i_1\ldots i_n})$ be the dual generators for L_n.

$$I(\alpha_{ij}) = \alpha_{ij} + \alpha_i \otimes \alpha_j,$$

$$I(\alpha_{ij\varepsilon}) = \alpha_{ijk} + \alpha_i \otimes \alpha_{jk} + \alpha_{ij} \otimes \alpha_k + \alpha_i \otimes \alpha_j \otimes \alpha_k,\ \text{etc.}$$

and

$$T = 1 + \Sigma\xi([\omega_i]) + \Sigma\xi([\omega_{ij}] + [\omega_i|\omega_j])X_{ij} + \Sigma\xi([\omega_{ijk}] + [\omega_i|\omega_{jk}] + [\omega_{ij}|\omega_k] + [\omega_i|\omega_j|\omega_k])X_{ijk} + \cdots,$$

where the ω's are from $A^1(M)$; $\omega_i = \rho^{(1)}p_2^*\alpha_i,\ldots$.

Example. Let M be a manifold such that $H^1(M;k)$ is determined by the closed 1-forms $a_1,a_2,c_{11},c_{12},c_{21},c_{22},b_1,b_2$. Furthermore assume that the matrix Massey product $\langle a,c,b\rangle$, where $a = (a_1,a_2)$, $c = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22}\end{pmatrix}$, $b = \begin{pmatrix} b_1 \\ b_2\end{pmatrix}$ is trivial. Such a manifold exists. For example a connected nilpotent Lie group with these properties can be easily found.

In this case the minimal model M^* for the lower central series of $G = \pi_1(M)$ is the free algebra with the generators $\alpha_1,\alpha_2,\omega_{11},\omega_{12},\omega_{21},\omega_{22},\beta_1,\beta_2$, where

$\rho^{(1)} p_2^* \alpha = a$, $\rho^{(1)} p_2^* \omega = c$ and $\rho^{(1)} p_2^* \beta = b$. The defining system for the matrix Massey product $\langle \alpha, \omega, \beta \rangle$ in M^* is given by the matrix

$$\begin{pmatrix} 0 & \alpha & \gamma & \sigma \\ & 0 & \omega & \varepsilon \\ 0 & & 0 & \beta \\ & & & 0 \end{pmatrix} \quad \alpha = (\alpha_1, \alpha_2), \gamma = (\gamma_1, \gamma_2), \quad \omega = \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix}, \ \varepsilon = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix}, \ \beta = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}.$$

In this case $M^* = {}_{(n)}M^*$, $n \geq 3$.

$$I(\gamma_i) = \gamma_i + \Sigma \alpha_j \otimes \omega_{ji}, \quad I(\varepsilon_i) = \varepsilon_i + \Sigma \omega_{ij} \otimes \beta_j,$$

$$I(\sigma) = \sigma + \Sigma \gamma_i \otimes \beta_i + \Sigma \alpha_i \otimes \varepsilon_i + \Sigma \alpha_i \otimes \omega_{ij} \otimes \beta_j .$$

If a,c,b,g,e,s stand for the image of $\alpha, \omega, \beta, \gamma, \varepsilon, \sigma$ in $A^1(M)$ and if A,C,B,G,E,S are the duals of $\alpha, \omega, \beta, \gamma, \varepsilon, \sigma$; with appropriate indices. Then T as an element of $C^0 \hat{\otimes} L$, $L_n = L$, $n \geq 3$, takes the form

$$\begin{aligned} T = 1 &+ \Sigma\xi[a_i]\, A_i + \Sigma\xi[c_{ij}]\, C_{ij} + \\ &+ \Sigma\xi[b_i]\, B_i + \Sigma\xi([g_i] + \Sigma[a_j|c_{ji}])\, G_i + \\ &+ \Sigma\xi([e_i] + \Sigma[c_{ij}|b_j])\, E_i + \\ &+ \xi(\Sigma[a_i|c_{ij}|b_j] + \Sigma[g_i|b_i] + \\ &+ \Sigma[a_i|e_i] + [s])\, S . \end{aligned}$$

Remark. These two examples are typical in the sense that they show what form the coefficients of T take with respect to the basis for L.

2.3 De Rham theorem

Theorem 3. The coefficients $\Omega(\alpha_i^{(n)})$ [and also $\Omega_0(\alpha_i^{(n)})$] are homotopy periods.

From the theorem we get immediately

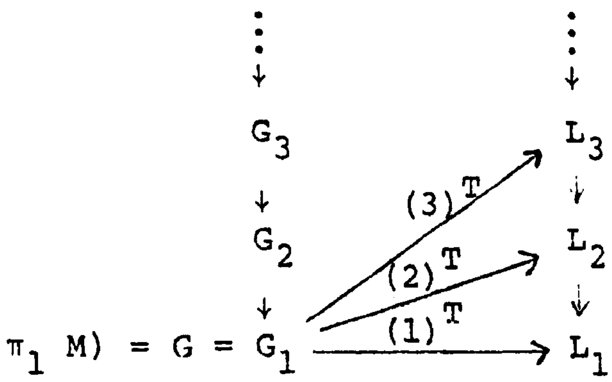
Corallary. The element $T \in C^0 \hat{\otimes} L$ determines a unique L-valued function on the group G.

Proof of the theorem. $T \in C^0 \otimes L$ means that for any loop $\lambda : I^0 \to \Omega(M)$, $T(\lambda) \in L$, i.e. T is a well defined function on the space of loops $\Omega(M)$. Observe that the coefficients of T are linear combinations of sums of iterated integrals of the form

$$\xi([\omega_1|\ldots|\omega_r]) \in C^0,\ \omega_i \in A^1(M),$$

where $\xi = \sigma \cdot \hat{H}_0$, $\hat{H}_0 : \bar{B}(A^*(I^{n+1})) \to A^{1,*}$, for any $n \geq 1$. For the polynomial $I(\omega_i^{(n)})$ associated with a pyramid $P(\alpha_i^{(n)})$ it can be shown by a direct computation, similar to that from the proof of Proposition 2, that $\hat{H}_0(I(a_i^{(n)}))$ is a closed 1-form on I^{m+1} for any $\alpha : I^m \to \Omega(M)$, $\hat{\alpha} : I^{m+1} \to M$; $a_i^{(n)} = \hat{\alpha}^*(\omega_i^{(n)})$, $m = 1$. Then from the Stokes formula it follows that $\Omega(\alpha_i^{(n)})$ is a homotopy invariant. Similarly for $\Omega_0(\alpha_i^{(n)})$.

Let $\lambda_n : L \to L_n$ be the projection. Then ${}_{(n)}T = \lambda_n \cdot T$ is an L_n-valued function on $G=G_1$. Thus we have the following picture



The coefficients of $_{(n)}T$ involve the iterated integrals of the form $\xi([\omega_1|\ldots|\omega_r])$ where r is at most equal to n, $r \leq n$.

<u>Proposition 8.</u> The function $_{(n)}T$ is zero on the subgroup G_{n+1}.

<u>Proof.</u> Let $\alpha : I^0 \to \Omega(M)$, $\beta : I^0 \to \Omega(M)$ be representations for $\bar{\alpha} \in G_1$, $\bar{\beta} \in G_s$. Then one can show directly from the definition of ξ that

$$\xi([\omega_1|\ldots|\omega_r])(\alpha\beta) = \xi([\omega_1|\ldots|\omega_r])(\alpha) + \xi([\omega_1|\ldots|\omega_r])(\beta),$$

where $\alpha\beta$ is the composition of loops. From here it follows that $\xi([\omega_1|\ldots|\omega_r])([\alpha\beta]) = 0$ whenever $r \leq s$. A more detailed computational proof of this proposition can be found in [3].

<u>Corallary.</u> $_{(n)}T$ induces the map

$$_{(n)}T : G_n/G_{n+1} \to L_n .$$

Let us denote $grG = \bigoplus_{n\geq 1} gr_n G$, $gr_n G = G_n/G_{n+1}$. grG has a structure of a Lie algebra defined by the commutator on G. The family of the maps ${}_{(n)}T$ defines a unique map

$$\Gamma : gr\, G \to L$$

as a limit of the sequence of maps Γ

Proposition 9. The maps Γ is a morphism of the Lie algebras.

Proof. From the proof of the last proposition if follows that only the "longest" iterated integrals in $\Omega(\alpha_i^{(n)})$ enter when this is applied to an element α representing the n-fold commutator $\alpha \in G_n$. Then the proof proceeds by induction.

By tracing the duals $X_i^{(n)}$ of the individual generators $\alpha_i^{(n)}$ in ${}_{(n)}M^*$ it can be shown that

Theorem 4. For each $n \geq 1$

$$\Gamma_n : gr_n G \otimes k \to L_n$$

is an isomorphism of Lie algebras.

Remark. The Proposition 9 and Theorem 4 are more precise statements then the Theorem 3.4.1 in [4], even over C and R.

The Theorem 4, over Q, was stated and a direct proof was sujested (without the map T) by Sullivan in his lectures in Paris in 1973.

As was already mentioned in the introduction, I have chosen an approach which ties together the work of Chen, May and Sullivan.

References

[1] J.F. Adams, On the cobar construction, Colloque Lauvain 1956, 81-87 (same in Proc. Nat. Acad. Sc. 42, 409-412 (1956)).

[2] H. Cartan, Théories cohomologiques, Inventiones Math. 35(1976), 261-272.

[3] K.T. Chen, Algebras of iterated path integrals and fundamental groups, T.A.M.S. 156 (1971), 359-379.

[4] K.T. Chen, Iterated integrals of differential forms and loop space homology, An. Math. 97 (1973), 217-246.

[5] J.P. May, Metric Massey products, J. Algebra 12 (1969), 533-568.

[6] D. Sullivan, Infinitesimal computations in topology, (preprint).

[7] D. Sullivan, Differential forms and topology of manifolds, Proceedings of the Conference on manifolds, Tokyo, 1973.

[8] C. Watkiss, Cochaines commutatives sur les ensembles simpliciaux, (preprint).