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DENNY GULICK

Domination in Analysis

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Domination in Analysis

by

Denny Gulick

1. Introduction

Let T be a completely regular Hausdorff space, and let

R^T = the set of all real-valued functions on T .

LSC(T) = the set of all lower semi-continuous functions in R^T .

C(T) = the set of all continuous functions in R^T .

Consider the following three conditions on an arbitrary such T .

Condition A. For each $h \in R^T$, there is an $f \in C(T)$ such that $f \geq h$.

Condition B. For each $g \in LSC(T)$, there is an $f \in C(T)$ such that $f \geq g$.

Condition C. For each $h \in R^T$, there is a $g \in LSC(T)$ such that $g \geq h$.

When does T satisfy each of the just-mentioned conditions? If we replace $\geq$ by $=$ in the conditions, then the responses are simple. Indeed, Condition A becomes $C(T) = R^T$, Condition B becomes $C(T) = LSC(T)$, and Condition C becomes $LSC(T) = R^T$, and each of these occurs precisely when T is discrete.

However, finding those T which satisfy the Conditions A, B, and C as actually posed does not involve such a trivial task, and we devote this article to an attempt at characterizing in terms of T when Conditions A, B, or C hold.

Of course Conditions A, B, and C hold whenever T is discrete. One's first impression might be that if T is not discrete then T would not satisfy any of the conditions. The situation which occurs when $T = [0,1]$ reinforces this impression. Indeed, let $(t_n)_{n=1}^{\infty}$ enumerate the rationals in $[0,1]$, and let

$$h_0(t) = \begin{cases} n, & t = t_n \text{ for some } n \in N \\ 0, & \text{all other } t \in [0,1] \end{cases}$$

$$g_0(t) = \begin{cases} 1/t, & 0 < t \leq 1 \\ 0, & \text{for } t = 0 \end{cases}$$

Then $h_0 \in R^T$ and is unbounded on every non-empty open subset of $[0,1]$, and $g_0 \in LSC(T)$ and is unbounded on $[0,1]$. Together these facts imply straightaway that $T = [0,1]$ satisfies none of the Conditions A, B, or C.

In Section 2 we show that it is consistent with the usual axioms of set theory to assume that no non-discrete T exists satisfying Condition A. We then tackle the problem of those T which satisfy Condition B, and show that if T does satisfy Condition B, then T must be a P-space. However, all those non-discrete P-spaces we know do not satisfy Condition B, and it remains unanswered whether or not any non-discrete ones exist. We end Section 2 by studying those T which satisfy Condition C. In particular we show that if T is denumerable, or if T is discrete except for one element, then it satisfies Condition C, while if T contains no isolated points and either is non-meager satisfying the second countability axiom or is locally compact, then T does not satisfy Condition C.

In Section 3 we discuss Condition C in a locally convex space setting. If $C_s(T)$ denotes the space of real-valued continuous functions on T, endowed with the simple convergence topology, then we show that T satisfies Condition C precisely when the bidual $C_s''(T)$ of $C_s(T)$ is identified with R^T as a locally convex space. We also delve into various properties of $C_s'(T)$ and $C_s''(T)$, including when they are barreled or when they are bornological.

We conclude the paper by collecting together some open questions.

The present paper has grown out of our recent article [5], in which we concentrated on the duality theory for $C_s(T)$. At the Conference in Bordeaux we discussed [5], and our attempt to render that talk in a somewhat more general context for these Proceedings resulted in the present paper. The overlap between the two articles resides primarily in the early part of Section 3 of the present paper, where we merely outline certain of the duality properties of $C_s(T)$. Nevertheless, it is our intention that the two papers can be read independently, and that they should serve as companions of one another.

Before we begin the paper proper we describe our notational conventions. Throughout the paper T stands for a completely regular Hausdorff space. If S and S' are subsets of T, then $S \setminus S' = \{t \in S : t \notin S'\}$, while χ_S denotes the characteristic function of S in T. We recall from [4] that T is extremally disconnected if every open set in T has an open closure in T, and T is a P-space provided that countable intersections of open sets in T are always open (and P-points are described accordingly). As in [4], βT is the Stone-Čech compactification of T, while νT is the repletion (or the Hewitt real-compactification) of T. We write $h > 0$ for the strictly positive $h \in R^T$, meaning that $h(t) > 0$ for all $t \in T$. If $n \in N$ (with N the set of positive integers), then the nth truncating function θ_n is defined on the reals by

$$\theta_n(r) = \begin{cases} r \ , & \text{if } |r| \le n \\ nr/|r| \ , & \text{if } |r| > n \end{cases}$$

For compact T the space $M(T)$ is the collection of all real-valued bounded Radon measures on T, and the point mass at $t \in T$ is δ_t.

If E is a locally convex space and $A \subseteq E$, then co bal A is shorthand for the convex balanced hull of A, and

$$\text{co bal } A = \left\{ \sum_{n=1}^{m} c_n f_n : f_n \in A, \ \sum_{n=1}^{m} |c_n| \le 1, \text{ and } m \in N \right\}.$$

The collection of all real-valued continuous linear forms on E is the dual E' of E, and we clothe E' in the dual topology, which is the topology of uniform convergence on the bounded subsets of E. The bidual of E is E'', and by definition E'' is the dual of E' with its dual topology. If $A \subseteq E$, then the polar A^{δ} in E' of A is given by the formula

$$A^{\delta} = \{\varphi \in E' : |\varphi(f)| \leq 1 \text{ , for all } f \in A\} .$$

If $B \subseteq E'$, then the polar B^{q} in E of B is given by the formula

$$B^{q} = \{f \in E : |\varphi(f)| \leq 1 \text{ , for all } \varphi \in B\} .$$

Finally, in our presentation we employ "iff" for the more cumbersome "if and only if".

2. Analysis of Conditions A, B, and C

In order to discuss Condition A, we first recall that a set T is measurable ("non-modéré" in French [2]) iff there exists a non-trivial countably additive measure μ on the power set of T which assumes only the values 0 and 1 and such that $\mu(\{t\}) = 0$ for all $t \in T$ (see [4]). A cardinal is called measurable iff there exists a measurable set T with the same cardinality. The following theorem presents several statements equivalent to the statement that the set T is measurable, and results in part from a discussion with M. Haddad, to whom we would like to express our appreciation. We note before stating Theorem 1 that certain of the statements in it are already known to be equivalent (see Chapter 12 of [4]).

THEOREM 1. Let T be a set, let T_d be the set T with the discrete topology, and let $T_0 = T \cup \{t_0\}$, with $t_0 \notin T$. Then the following statements are equivalent:

a. There exists a non-discrete topology $\mathcal{T}$ on T_0 such that T_0 satisfies Condition A.

b. There exists a non-discrete topology $\mathcal{T}$ on T_0 such that T is discrete for $\mathcal{T}$, and such that if $(U_n)_{n=1}^{\infty}$ is any partition of T, then $\bigcup_{n=1}^{m} U_n$ is a deleted neighborhood of t_0, for some m.

c. There exists a non-discrete topology $\mathcal{T}$ on T_0 such that T is discrete for $\mathcal{T}$, and such that if $(U_n)_{n=1}^{\infty}$ is any partition of T, then U_m is a deleted neighborhood of t_0, for some m.

d. There exists a non-discrete topology on T_0 which renders T_0 an extremally disconnected P-space.

e. A free ultrafilter closed under countable intersections exists on T.

f. T is measurable.

g. $\nu T_d \neq T_d$.

Proof. To prove that (a)$\Longrightarrow$(b), let $(U_n)_{n=1}^{\infty}$ be a partition of T, and equip T_0 with the topology of (a). Define h on T_0 by

$$h(t) = \begin{cases} n, & \text{all } t \in U_n, \text{ all } n \in N \\ 0, & t = t_0 \end{cases}$$

By (a) there exists an $f \in C(T_0)$ such that $f \geqslant h$. Since f is continuous at t_0, we know that $f \leq m$ on some neighborhood of t_0. By the definition of h, $\bigcup_{n=1}^{m} U_n$ is therefore a deleted neighborhood of t_0, so that the topology $\mathcal{T}$ of (a) is the topology we sought in order to prove (b). Now assume (b) and let $\mathcal{F}_0$ be the neighborhood filter of t_0 in $\mathcal{T}$. If $\mathcal{F} = \{F \setminus \{t_0\} : F \in \mathcal{F}_0\}$, then let $\mathcal{F}_1$ be an ultrafilter in T containing the filter $\mathcal{F}$. Let $\mathcal{T}_1$ denote the topology on T_0 derived from $\mathcal{T}$ by substituting the neighborhood filter $\mathcal{F}' = \{F \cup \{t_0\} : F \in \mathcal{F}_1\}$ of t_0 in $\mathcal{T}$ for $\mathcal{F}_0$. Let $(U_n)_{n=1}^{\infty}$ be a partition of T. By (b) there is an m such that $\bigcup_{n=1}^{m} U_n \in \mathcal{F} \subseteq \mathcal{F}_1$. Since $\mathcal{F}_1$ is an ultrafilter, one of these sets in $(U_n)_{n=1}^{m}$, say U_m, is in $\mathcal{F}_1$. Thus with respect to $\mathcal{T}_1$ we have (c). To prove that (c)$\Longrightarrow$(d), let U be open in the non-discrete topology on T_0 of (c). If $t_0 \in U$, then $\overline{U} = U$. If $t_0 \notin U$, then $\{U, T \setminus U\}$ is a partition of T, so that by (c) either U or $T \setminus U$ is a deleted neighborhood of t_0. Either way $\overline{U}$ is open in T_0, whereupon T_0 is extremally disconnected. To show that T_0 is a P-space, it suffices to show that t_0 is a P-point. To that end, let $(U_n)_{n=1}^{\infty}$ be a decreasing sequence of *deleted* neighborhoods of t_0, each minus the point t_0, and let $U_1 = T$. Then the collection $\left\{ \bigcap_{n=1}^{\infty} U_n, \{U_n \setminus U_{n+1}\}_{n=1}^{\infty} \right\}$ forms a partition of T, so must by (c) contain a deleted neighborhood of t_0. Evidently the only possibility is that $\bigcap_{n=1}^{\infty} U_n$ be this deleted neighborhood, proving that t_0 is a P-point. Next we prove that (d)$\Longrightarrow$(e). For that we let T_0 be that of (d), and let $\mathcal{F}_0$ be the neighborhood filter of t_0 in T_0 and let $\mathcal{F} = \{F \setminus \{t_0\} : F \in \mathcal{F}_0\}$. Then $\mathcal{F}$ is a filter on T, and since T_0 is a P-space, $\mathcal{F}$ is closed under countable intersections. Next we show that $\mathcal{F}$ is an ultrafilter. Let $S \subseteq T$. Then $T_0 = \overline{S} \cup \overline{T \setminus S}$, so assume that $t_0 \in \overline{S}$. Since T_0 is extremally disconnected, and since S is open in T_0 , we know that $\overline{S} = S \cup \{t_0\}$ is open in T_0 , which means that $S \in \mathcal{F}$. By construction $\mathcal{F}$ is free (see [4]). Thus (e) holds. We assume next the existence on T of the ultrafilter $\mathcal{F}$ of (e) and we define μ on the power set of T by

$$\mu(S) = \begin{cases} 1, \text{ if } & S \in \mathcal{F} \\ 0, \text{ if } & S \notin \mathcal{F} \end{cases}$$

By the properties of $\mathcal{F}$ it is immediate that μ is the desired measure giving us (f). The equivalence of (f) and (g) is precisely the content of Theorem 12.2 in [4]. A somewhat different proof that (f)$\Longrightarrow$(g) goes as follows. Let μ be the measure on T associated with the measurability of T. Let $h \in R^T$, let $p \in N$, and define $U_{np} = \{t \in T : (n-1)/2^p \leq h(t) < n/2^p\}$, for all $n \in Z$ (where Z denotes the collection of all integers). Then $(U_{np})_{n \in Z}$ partitions T, and by the definition of μ we see that for each $p \in N$ there exists an $n_p \in Z$ such that $\mu(U_{n_p p}) = 1$. Because μ is countably additive, $\lim_{p \to \infty} h(U_{n_p p})$ exists, so we define φ on R^T by

$$\varphi(h) = \lim_{p \to \infty} h(U_{n_p p}) \text{ , all } h \in R^T \text{ .}$$

Surely φ is well-defined, and after a little computation we see that φ is a positive linear form on R^T. By Hewitt's Theorem [7, Theorem 22], φ corresponds to a measure supported on a compact subset K of υT_d . Because the only compact subsets of T_d are finite, and because our original μ annihilates all finite subsets of T, we know that $K \nsubseteq T$. Thus $\upsilon T_d \neq T_d$, which proves (g). Finally we show that (g)$\Longrightarrow$(a). To do it, let $t_0 \in \upsilon T_d \setminus T_d$, and let $T_0 = T \cup \{t_0\}$ have the topology induced from υT_d. If h is defined on T_0 , then $g = h|_T$ is in $C(T_d)$, so g has a continuous extension $\tilde{g}$ defined on T_0. Letting $\tilde{f} = |\tilde{g}| + |h(t_0)|$, we observe that $\tilde{f} \in C(T_0)$ and $\tilde{f} \geqslant |h|$, completing the proof that (g)$\Longrightarrow$(a). ∎

We remark at this point that there are equivalent ways of saying that T is measurable other than those mentioned in Theorem 1; see for instance [2]. However the ones stated here are the ones which play a role in the present article. The following two corollaries tell us definitively when there exist T satisfying Condition A.

COROLLARY 2. There exists a non-discrete T satisfying Condition A iff there exists a measurable cardinal.

Proof. If T is non-discrete and satisfies Condition A, then with any stronger non-discrete topology it satisfies condition A. So take any stronger non-discrete topology on T for which all but one point of T is isolated. That space is the T_0 of statement (a) in Theorem 1. Since (a)$\Longrightarrow$(f), there exists a measurable cardinal. On the other hand, (f)$\Longrightarrow$(a) in Theorem 1 yields the converse. ∎

COROLLARY 3. It is consistent with the Zermelo-Fraenkel axioms to assume that no T exists satisfying Condition A.

Proof. Corollary 2, together with Theorem 12.5 and the remarks in 12.6 of [4], yields the result. ∎

In contrast to Corollaries 2 and 3, it is not known (at least to us) if it is consistent to assume that a T satisfying Condition A -- or equivalently a T of measurable cardinal -- does exist. If we assume that such a cardinal exists, it must be huge. Such a cardinal α must be extremely ultra-super-giant inaccessible, and indeed there must exist α inaccessible cardinals less than α, by a theorem of Tarski [12]. In addition, if we assume that such a cardinal exists, then curious phenomena occur in set theory [10].

On the other hand, more recently J. Silver proved that if the Zermelo-Fraenkel axioms, together with the axiom of choice and the axiom of the existence of a measurable cardinal, are consistent, then it is consistent to assume in addition that the generalized continuum hypothesis holds [11].

We turn our attention to Condition B, and begin with a preliminary result.

LEMMA 4. Let Λ denote the ordinals less than some infinite limit ordinal, and let $t_0 \in T$. Assume that there exists a collection $(U_\lambda)_{\lambda \in \Lambda}$ of open neighborhoods of t_0 such that

(i) $\overline{U_\mu} \subseteq U_\lambda$ for all $\mu, \lambda \in \Lambda$ with $\mu < \lambda$.

(ii) Whenever V is a neighborhood of t_0 there is a $\lambda_V \in \Lambda$ such that for each $\lambda \geqslant \lambda_V$ an $m \in N$ exists for which $V \cap U_\lambda \not\subseteq V \cap U_{\lambda+m}$.

Then T does not satisfy Condition B.

<u>Proof</u>. For any non-limit ordinal $\lambda \in \Lambda$, let $n_\lambda \in N$ such that $n_\lambda - n$ is a limit ordinal (or 0), and let $\Lambda_0 = \{\lambda \in \Lambda : n_\lambda = 1\}$. Define g by

$$g = \sum_{\lambda \in \Lambda_0} \sum_{n=1}^{\infty} \chi_{U_{\lambda+n} \setminus \bigcap_{m=1}^{\infty} U_{\lambda+m}} .$$

If $\lambda, \lambda' \in \Lambda_0$ with $\lambda < \lambda'$, then $U_{\lambda'+n} \subseteq \bigcap_{m=1}^{\infty} U_{\lambda+m}$, so we have easily that $g(t) = 0$, for all $t \in U_{\lambda'+n}$ and all $n \in N$. Thus g is real-valued. Because $\bigcap_{m=1}^{\infty} U_{\lambda+m}$ is always closed in T and $U_{\lambda+n}$ always is open, we know that $g \in LSC(T)$. Let V be a neighborhood of t_0, let $\lambda_0 \in \Lambda_0$ with $\lambda_0 \geqslant \lambda_V$, and let $n \in N$. By hypothesis there exists a $t \in (V \cap U_{\lambda_0+n}) \setminus (V \cap U_{\lambda_0+n+m})$ for some $m \in N$. Then $g(t) \geqslant n$, so that g is unbounded on every such neighborhood V of t_0, which is true of no $f \in C(T)$. Thus T does not satisfy Condition B. ∎

THEOREM 5. If T satisfies Condition B, then T is a P-space.

<u>Proof</u>. Assume that $t_0 \in T$ is not a P-point. Then there exists a

sequence $(U_n)_{n=1}^{\infty}$ of open neighborhoods of t_0 such that $\bigcap_{n=1}^{\infty} U_n$ is not a neighborhood of t_0, and we can as well assume that $\overline{U}_{n+1} \subseteq U_n$ for all n. Since $\bigcap_{n=1}^{\infty} U_n$ is not a neighborhood of t_0, it is clear that for any neighborhood V of t_0 and any n, there exists an $m_n > n$ such that $V \cap U_n \neq V \cap U_{m_n}$. Thus the hypotheses of Lemma 4 are satisfied, and we can derive the desired conclusion.∎

Unfortunately the converse to Theorem 5 is false. Indeed, let T be the non-limit ordinals less than the first uncountable ω_1, with ω_1 adjoined, and clothe T in the order topology. Then T is a P-space. It is easy to see that ω_1 fulfills the hypotheses on t_0 in Lemma 4, where $\Lambda = [1, \omega_1)$. Consequently T does not satisfy Condition B.

That Condition A implies Condition B means that if we assume the existence of measurable cardinals, then we know that there exist T's which satisfy Condition B. However, without the existence of measurable cardinals we do not know any T which satisfy Condition B, and evidently none exists which has but one non-isolated point (like the T_0 in Theorem 1). Conceivably no such T exists without the existence of measurable cardinals. This possibility becomes plausible when we note that every P-space is basically disconnected (Problem 4K of [4]), whereas a non-discrete extremally disconnected space exists only if measurable cardinals exist (Problem 12H of [4]) -- a slightly stronger result than (d)$\Longrightarrow$(f) of our Theorem 1. In other words, to find a non-discrete space which satisfies Condition B but which has non-measurable cardinality, one must locate a T of reasonable cardinality which is more disconnected than basically disconnected but which is not quite extremally disconnected. That job may not be so easy. Another way of looking at the situation is as follows.
If we assume that all cardinals are non-measurable, then a non-discrete space T satisfies Condition B only if every pointwise bounded family in C(T) is dominated by a continuous function but there ex-

ists a family $(f_\lambda)_{\lambda\in\Lambda} \subseteq C(T)$ such that $\sup_{\lambda\in\Lambda} |f_\lambda| \notin C(T)$ (see Problem 3N in [4]).

A non-discrete compact T cannot satisfy Condition B because a compact P-space is necessarily finite, by Problem 4K of [4]. But a non-discrete replete T satisfying Condition B is not ruled out a priori. What we know is the following result.

PROPOSITION 6. If T satisfies Condition B, then υT also satisfies Condition B.

<u>Proof</u>. Let $\tilde{g} \in LSC(\upsilon T)$. If $g = \tilde{g}|_T$, then $g \in LSC(T)$, so that by hypothesis there exists an $f \in C(T)$ with $f \geqslant g$. Now let $t_0 \in \upsilon T \setminus T$. For each $\varepsilon > 0$, there exists a neighborhood U_ε of t_0 on which $\tilde{g} \geqslant \tilde{g}(t_0) - \varepsilon$. The continuous extension $\tilde{f}$ of f on υT gives $\tilde{f}(t_0) \geqslant \tilde{g}(t_0)$. Thus $\tilde{f} \geqslant \tilde{g}$. ∎

From now on we concentrate on Condition C. Unlike the problems confronting us in finding non-discrete T satisfying either Condition A or Condition B, it is trivial to find non-discrete T satisfying Condition C, and we can describe with no effort two types of such T.

In the first place, if T is denumerable, with any completely regular Hausdorff topology whatever, then T satisfies Condition C. To see this, just let $T = (t_n)_{n=1}^{\infty}$ and let $h \in R^T$. Merely define g on T by $g(t_n) = \max\{|h(t_m)| : m = 1,\ldots,n\}$, for each $n \in N$. Then $g \in LSC(T)$ and $g \geqslant h$.

For the second type, we note that there are non-discrete T of arbitrarily large cardinality which satisfy Condition C. For if T is any space which contains but one non-isolated element t_0, then any $h \in R^T$ is lower semi-continuous except possibly at t_0, so such an h can be majorized by a suitable $g \in LSC(T)$.

On the other hand, if $T = [0,1]$, then T does not satisfy Condition C, as we saw at the outset of the paper. Because in some sense in any space satisfying Condition C there seem to be lower semi-continuous functions which oscillate arbitrarily wildly, let us make the following slightly pictorial definition (as in effect we did in [5]).

DEFINITION 7. The space T is <u>wildly oscillatory</u> iff T satisfies Condition C (i.e., if for each $h \in R^T$, there exists a $g \in LSC(T)$ such that $g \geqslant h$). If T is not wildly oscillatory, we say that T is <u>mildly oscillatory</u>.

In analyzing which T are wildly (or mildly) oscillatory, we will refer to what we call everywhere unbounded functions. We say that $h \in R^T$ is everywhere unbounded iff h is unbounded on each non-empty open subset of T. After a moment's reflection you will agree that T is mildly oscillatory if on the one hand LSC(T) contains no positive everywhere unbounded function while on the other hand R^T does contain an everywhere unbounded function. (Perhaps the converse is valid too, for those T which contain no isolated points.)

What type of T admit no positive everywhere unbounded function in LSC(T)? Recall that T is non-meager iff T is not the countable union of closed nowhere dense subsets [8, p. 213]. Then we can state and prove the following proposition.

PROPOSITION 8. T is non-meager iff LSC(T) admits no positive everywhere unbounded function.

<u>Proof</u>. If T is non-meager and if $g \in LSC(T)$ is positive, let

$$S_n = \{t \in T : g(t) \leq n\}, \text{ for each } n \in N.$$

Then each S_n is closed in T and $\bigcup_{n=1}^{\infty} S_n = T$. Because T is non-meager, there is an integer n such that S_n contains a non-empty open subset. For the converse, assume that LSC(T) admits no positive everywhere unbounded function. Let $(S_n)_{n=1}^{\infty}$ be an increasing sequence of closed subsets with union T, and for each $n \in N$ let $g(t) = n$ for all $t \in S_n \setminus S_{n-1}$ (where $S_0 = \emptyset$). Then $g \in LSC(T)$ and is positive, and consequently by hypothesis there exists an $m \in N$ and an open subset $U \subseteq T$ such that $g(t) \leq m$ for all $t \in U$. Thus $U \subseteq \bigcup_{n=1}^{m} S_n$. Now the regularity of T yields a non-empty open subset V of T such that $V \subseteq U$ and also a $p \leq m$ such that $V \subseteq S_p$. ∎

Since all Baire spaces, and in particular all locally compact spaces, are non-meager, Proposition 8 attests that for any such T, the space $LSC(T)$ contains no positive everywhere unbounded function.

If T contains no non-isolated points, then the non-existence of an everywhere unbounded function on T is closely related to Condition (b) of Theorem 1. In fact, T admits no everywhere unbounded function precisely when for each partition $(U_n)_{n=1}^{\infty}$ of T there exists a non-isolated $t \in T$ and an $m_t \in N$ such that $\bigcup_{n=1}^{m_t} U_n$ is a deleted neighborhood of t. As is evident, this latter condition is a modestly weakened version of condition (b), and is exactly condition (b) in the event that T has exactly one non-isolated element.

Remember that Theorem 1 and Corollary 3 tell us in effect that it is consistent with the usual axioms of set theory to assume that any T with one non-isolated point (and indeed any non-discrete T) does not satisfy condition (b). Therefore, with a little of the pollyanna in us we conjecture that it is consistent with the usual axioms of set theory to assume that every T which contains no isolated points admits an everywhere unbounded function. If our conjecture happens to be true, then at least for those T without isolated points a criterion for T to be mildly oscillatory (or to not satisfy Condition C) would evolve: such a space would be mildly oscillatory if T were non-meager.

Without a proof of the conjecture, we can in any case prove that many T admit everywhere unbounded functions. We begin with the following simple observation.

PROPOSITION 9. If T is separable, then T admits an everywhere unbounded function.

Proof. If $(t_n)_{n=1}^{\infty}$ is dense in T, then let $h(t_n) = n$, for all $n \in N$, and let $h(t) = 0$ for all other $t \in T$.∎

As a result of Proposition 9, if T is separable and non-meager--like the reals--then T is mildly oscillatory. Moreover, by using Hamel bases we can easily show that any topological vector space T over the reals admits an everywhere unbounded function, and moreover such a T is also mildly oscillatory.

Next we will show that each locally compact space without isolated points admits an everywhere unbounded function. We will utilize the notion of what might be called "homocardinality". We say that a space T is homocardinal iff $\text{card } U = \text{card } T$ for any non-empty open U in T (where "card" stands for "cardinality"). We are then ready to state and prove Proposition 10, the basic idea for whose proof is due to J. Saint-Raymond.

PROPOSITION 10. Let T have no isolated points, and let $S \subseteq T$ with $\overline{S} = T$. Let $(T_\lambda)_{\lambda \in \Lambda}$ be a pairwise disjoint collection of homocardinal open subsets of T such that

(a) $\overline{\bigcup_{\lambda \in \Lambda} T_\lambda} = T$

(b) For each $\lambda \in \Lambda$, there exists a base $\mathcal{U}_\lambda = (U_{\lambda\gamma})_{\gamma \in \Gamma_\lambda}$ of open subsets for T_λ such that $\text{card } \mathcal{U}_\lambda \le \text{card } (U_{\lambda\gamma} \cap S)$, for all $\gamma \in \Gamma_\lambda$, and such that if $\text{card } (U_{\lambda\gamma} \cap S) = \aleph_0$, then we have $\text{card } (U_{\lambda\gamma} \cap S) = \text{card } U_{\lambda\gamma}$.

Then S (and hence T) supports an everywhere unbounded function.

__Proof__. Fix $\lambda_0 \in \Lambda$ and let $\mathcal{U}_{\lambda_0} = (U_{\lambda_0 \gamma})_{\gamma \in \Gamma_{\lambda_0}}$ satisfy property (b). Designate by α the ordinal such that we may (and will) identify Γ_{λ_0} with the ordinals less than α. If card $(U_{\lambda_0 \gamma_0} \cap S) = \aleph_0$ for some $\gamma_0 \in \Gamma_{\lambda_0}$, then by assumption (b) we know that card $U_{\lambda_0 \gamma_0} = \aleph_0$. But T_{λ_0} is homocardinal, so this means that card $T_{\lambda_0} = \aleph_0$. Consequently we can define h_{λ_0} on T_{λ_0} as in Proposition 9. Henceforth we can therefore assume that card $(U_{\lambda_0 \gamma} \cap S) \geq \aleph_1$ for each $\gamma \in \Gamma_{\lambda_0}$, and for the moment we fix $\gamma_0 \in \Gamma_{\lambda_0}$. Then

$$\text{card } \gamma_0 < \alpha = \text{card } \mathcal{U}_{\lambda_0} \leq \text{card } (U_{\lambda_0 \gamma_0} \cap S).$$

Utilizing these inequalities, we can by transfinite induction find a distinct sequence $(t_{n\lambda_0\gamma_0})_{n=1}^{\infty} \subseteq U_{\lambda_0\gamma_0} \cap S$ such that

$$t_{n\lambda_0\gamma_0} \neq t_{m\lambda_0\gamma}\ , \text{ for all } \gamma \in \Gamma_{\lambda_0} \text{ such that } \gamma < \gamma_0, \text{ and all } m,n \in N.$$

Define h_{λ_0} on T_{λ_0} by

$$h_{\lambda_0}(t) = \begin{cases} n, \text{ if } t = t_{n\lambda_0\gamma_0}, \text{ for some } n \in N \text{ and } \gamma_0 \in \Gamma_{\lambda_0}. \\ 0, \text{ all other } t \in T_{\lambda_0}. \end{cases}$$

Since $(U_{\lambda_0 \gamma})_{\gamma \in \Gamma_{\lambda_0}}$ comprises a base for open sets in T_{λ_0}, this means that h_{λ_0} is evidently unbounded on all non-empty open subsets of $T_{\lambda_0} \cap S$ (and hence on all those non-empty open subsets of T_{λ_0}). Finally, utilizing the fact that the T_λ's are pairwise disjoint, we can let h be defined on T by

$$h(t) = \begin{cases} h_\lambda(t), \text{ all } t \in T_\lambda, \text{ all } \lambda \in \Lambda. \\ 0, \text{ all other } t \in T. \end{cases}$$

Then h inherits the required property from the h_λ's, since $\overline{\bigcup_{\lambda \in \Lambda} (T_\lambda \cap S)} = \overline{S} = T$. ∎

The S which appears in Proposition 10 is not needed in order to derive the following proposition, but it will appear in Example 12 and will play an essential role in Proposition 17 in Section 3. The next result is essentially due to J. Saint-Raymond.

PROPOSITION 11. If T is locally compact and contains no isolated points, then T supports an everywhere unbounded function--and thus T is mildly oscillatory.

Proof. First, by Zorn's Lemma let $(T_\lambda)_{\lambda\in\Lambda}$ be a maximal collection of pairwise disjoint open homocardinal subsets of T. If by chance $\overline{\bigcup_{\lambda\in\Lambda} T_\lambda} \neq T$, then there would exist an open $V \subseteq (T \setminus \overline{\bigcup_{\lambda\in\Lambda} T_\lambda})$ with minimal cardinality. This V would be homocardinal, contradicting the maximality of $(T_\lambda)_{\lambda\in\Lambda}$. Thus $\overline{\bigcup_{\lambda\in\Lambda} T_\lambda} = T$. Fix $\lambda\in\Lambda$, and for a pair $s,t \in T_\lambda$, let $U_{st} \subseteq T_\lambda$ be relatively compact and open, and such that $s \in U_{st}$ but $t \notin \overline{U}_{st}$. To show that $\mathcal{U}_\lambda = (U_{st})_{s,t\in T_\lambda}$ is a basis for open sets of T_λ, let U be any non-empty open subset of T_λ, let $s \in U$, and let $t_0 \in T_\lambda \setminus \{s\}$. Then $\overline{U}_{st_0} \setminus U$ is compact. Either $\overline{U}_{st_0} \subseteq U$ or else there exists an n such that

$$\bigcup_{n=1}^{m} (T \setminus U_{st_n}) \supseteq \bigcup_{n=1}^{m} (T \setminus \overline{U}_{st_n}) \supseteq (\overline{U}_{st_0} \setminus U) ,$$

which means that

$$U_{st_0} \cap \bigcap_{n=1}^{m} U_{st_n} = \bigcap_{n=0}^{m} U_{st_n} \subseteq U ,$$

confirming that $\mathcal{U}_\lambda$ is a base for open sets of T_λ. Moreover, card $\mathcal{U}_\lambda$ = card T_λ, so that by letting $S = T$ in Proposition 10, we demonstrate the existence of an everywhere unbounded function on T. That T is therefore mildly oscillatory now follows from Proposition 8. ∎

In preparation for Proposition 17, which appears later on, we give now an example of a T which admits an everywhere unbounded function on a special, non-trivial dense subset S.

EXAMPLE 12. Let $T = \beta N \setminus N$ and let S denote the collection of all P-points of T. If we assume the continuum hypothesis, then S is dense in T by Problem 6V of [4]. We will show that S admits an everywhere unbounded function. To that end, let U be non-empty and open in T, so that $U \cap S \neq \emptyset$. If $(U \cap S) \subseteq (s_n)_{n=1}^{\infty}$, then let $s_1 \in U \cap S$, and for all $n \geqslant 2$ let $V_n \subseteq U$ be a neighborhood of s_1 such that $s_n \notin V_n$. Then $V = \bigcap_{n=1}^{\infty} V_n$ is a non-empty G_δ containing $s_1 \in S$, so V is a neighborhood of s_1. But $V \cap S = \{s_1\}$ and s_1 is not isolated, which means that S cannot be dense in T, contradicting our previous assertion. Thus for every non-empty U in T, we know that card $(U \cap S) \geqslant \aleph_1$. Since N is locally compact, the space T is compact by 6.9d of [4], so that by the proof of Proposition 11 there exists a collection $(T_\lambda)_{\lambda \in \Lambda}$ of pairwise disjoint, homocardinal, open subsets of T such that $\overline{\bigcup_{\lambda \in \Lambda} T_\lambda} = T$. Problem 6S of [4] tells us that there is a basis of open sets in T (and hence in each T_λ) of cardinality $\aleph_1$. Then Proposition 10 finishes the proof that there exists an everywhere unbounded function on S.

In showing that a given T is wildly or mildly oscillatory, we have had to lean heavily on the hypothesis that T contain no isolated points. As yet we have found no method for circumventing this hypothesis, even when T is locally compact and very down to earth. For example, if T is the ordinals less than the first uncountable ω_1 (or less than any $\omega_\alpha > \omega_1$), then T is locally compact and is moreover eminently structured. Is T wildly oscillatory? It seems to us that whatever be the reply, its proof cannot be constructive. Could the answer conceivably entail a new axiom?

3. Wildly oscillatory spaces and locally convex spaces

In this section we determine which T are wildly (or mildly) oscillatory in the context of locally convex spaces. To begin with, we let $C_s(T)$ denote the space $C(T)$ with the topology of simple convergence on the elements of T. It is the analysis in [1] of various special locally convex space properties of $C_s(T)$ which originally drew our attention to $C_s(T)$.

Certain characteristics of $C_s(T)$ are evident. In the first place, $C_s(T)$ is always dense in R^T when R^T carries its product topology. Moreover, $C_s(T) = R^T$ iff T is discrete, as we mentioned at the outset of the article. Because the supremum of a set of continuous functions is always lower semi-continuous (though possibly infinite-valued), the collection $(B_g)_{g \in LSC(T),\ g>0}$ forms a base for the bounded sets of $C_s(T)$, where by definition $B_g = \{f \in C_s(T) : |f| \leq g\}$.

The duality theory for $C_s(T)$ is by and large known (see [5]); we mention a few pertinent properties of $C_s'(T)$ at this point. As a vector space the dual $C_s'(T)$ is identified with the set of all finite linear combinations of point masses δ_t, $t \in T$, and evidently $(B_g^{\circ})_{g \in LSC(T),\ g>0}$ forms a basis for neighborhoods of 0 in $C_s'(T)$ in the dual topology, where in the identification we obtain

$$B_g^{\circ} = \text{co bal}\left\{\delta_t/g(t) : t \in T\right\}$$

[5, Lemma 2]. As a result, the only subsets of $C_s'(T)$ which are bounded in the dual topology of $C_s'(T)$ are those contained in finite-dimensional subspaces [5, Corollary 4]. An immediate consequence of this is the fact that $C_s'(T)$ is always a semi-Montel space.

Primarily because the (weak) completion of $C_s(T)$ is R^T, the bidual $C_s''(T)$ of $C_s(T)$ admits the following identification:

$$C_s''(T) = \{h \in R^T: \quad g \geqslant |h| \quad \text{for some} \quad g \in LSC(T)\}.$$

The bidual topology on $C_s''(T)$ turns out to be the topology of simple convergence, since the bounded subsets of $C_s'(T)$ are contained in finite dimensional subspaces. Now the form that $C_s''(T)$ takes as a vector space may remind you of our original Condition C. The relationship will be spelled out directly in Theorem 16.

From the characterization of $C_s''(T)$, we observe that $C_s''(T)$ lies between $C_s(T)$ and R^T, so that $C_s''(T)$ is always dense in R^T. The fact that $C_s''(T)$ contains all functions bounded on T plays a decisive role in the structure of $C_s''(T)$, as we begin to see in the following lemma.

LEMMA 13. The collection $(D_h)_{h \in R^T,\ h>0}$ forms a base for the bounded subsets of $C_s''(T)$, where $D_h = \{k \in C_s''(T) : |k| \leq h\}$. Moreover, if $(t_n)_{n=1}^m \subseteq T$, then there exists a $k \in D_h$ such that $k(t_n) = h(t_n)$, for $n = 1,\ldots,m$.

Proof. That the collection forms a base for the bounded sets is trivial, since the topology on $C_s''(T)$ is the topology of simple convergence. If $h \in R^T$ with $h > 0$, and if $(t_n)_{n=1}^m \subseteq T$, then define $k \in R^T$ by

$$k(t) = \begin{cases} h(t_n), & n = 1,\ldots,m \\ 0, & \text{all other} \quad t \in T \end{cases}$$

Since $C_s''(T)$ contains all the bounded functions defined on T, we know that $k \in C_s''(T)$ and hence that $k \in D_h$. ∎

PROPOSITION 14 a. $C_s'''(T) = (R^T)'$ for each T.

b. $C_s'''(T)$ is a Montel space, for each T.

Proof. Part (b) follows from (a) since $(R^T)'$ is a Montel space for all T. Consequently we only need prove (a). However, as a vector space, $C_s'(T) = (R^T)'$, and in addition, $C_s'(T)$ is a semi-Montel space, which means that $C_s'''(T) = (R^T)'$ as a vector space.

Thus we only have to make certain that $C_s'''(T)$ has the right topology. It is easy to check that sets of the form co bal$\{\delta_t/h(t): t\in T\}$, where $h\in R^T$ and $h>0$, form a base of neighborhoods of 0 in $(R^T)'$. But via Lemma 13 the collection $(D_h^\delta)_{h\in R^T,\ h>0}$ forms a base of neighborhoods of 0 in $C_s'''(T)$. It therefore suffices to prove that for each $h\in R^T$ with $h>0$, we have

$$D_h^\delta = \text{co bal}\ \{\delta_t/h(t) : t\in T\}\ .$$

On the one hand, clearly $D_h^\delta \supseteq \text{co bal}\ \{\delta_t/h(t) : t\in T\}$. On the other hand, if $\mu\in C_s'''(T)$ but $\mu\notin \text{co bal}\ \{\delta_t/h(t) : t\in T\}$, then

$$\mu = \sum_{n=1}^{m} c_n\, \delta_{t_n}/h(t_n)\ ,$$

where the $(c_n)_{n=1}^m$ and the $(t_n)_{n=1}^m$ are appropriate, where $\sum_{n=1}^m |c_n|>1$, and where $|c_n|>0$, for $n=1,\ldots,m$. Now let $k\in C_s''(T)$ be defined by

$$k(t) = \begin{cases} (\,|c_n|/c_n\,)\ h(t_n)\ , & \text{for}\ t=t_n\ ,\ n=1,\ldots,m \\ 0, & \text{all other}\ t\in T \end{cases}$$

Then $k\in D_h$ while $\mu(k) = \sum_{n=1}^m |c_n| > 1$. Thus $\mu\notin B_h^\delta$. As a result $B_h^\delta \subseteq \text{co bal}\ \{\delta_t/h(t) : t\in T\}$, and the proof is complete.∎

As an immediate consequence of Proposition 14 we have

COROLLARY 15. $C_s''''(T) = R^T$ for each T.

It is apparent by virtue of Corollary 15 that although $C_s(T)$ is reflexive only when T is discrete, on the other hand, $C_s''''(T)$ is always reflexive.

For each T, the inclusions

$$C_s(T) \subseteq C_s''(T) \subseteq C_s''''(T) = R^T$$

hold. In addition, $C_s(T) = C_s''(T)$ iff T is discrete. It is therefore very natural to ask when $C_s''(T) = R^T$ (which it turns out is the same as asking when $C_s''(T)$ is reflexive!). The answer is contained in Theorem 16, which is closely related to Theorem 9 in [5]. Before we state Theorem 16 we recall that a locally convex space for which all weak* compact subsets in its dual are equicontinuous is called a strictly Mackey space (see [3]).

THEOREM 16. For an arbitrary T, the following statements are equivalent:

a. $C_s''(T) = R^T$.

b. T is wildly oscillatory (i.e., T satisfies Condition C).

c. $C_s'(T)$ is bornological.

d. $C_s'(T)$ is ultrabornological.

e. $C_s'(T)$ is infrabarreled.

f. $C_s'(T)$ is a Montel space.

g. $C_s'(T)$ is barreled.

h. $C_s'(T)$ is a strong Mackey space.

i. There exists a completely regular Hausdorff space U and a homeomorphism $\varphi : C_s''(T) \to C_s(U)$ which is onto and which preserves either the natural order or the natural pointwise multiplication of both spaces.

<u>Proof</u>. The fact that (a)$\Longleftrightarrow$(b) is obvious. Assume that (b) holds, and let $B \subseteq C_s'(T)$ be convex and balanced and absorb all bounded subsets of $C_s'(T)$. Then $B \supseteq \text{co bal}\,\{\delta_t/h(t) : t \in T\}$ for some $h \in R^T$ with $h > 0$. By (b) we know that there is a strictly positive $g \in LSC(T)$ such that $h \leq g$, so that

$$B \supseteq \text{co bal}\,\{\delta_t/g(t) : t \in T\} = B_g^{\delta} .$$

Consequently B is a neighborhood of 0 in $C_s'(T)$, verifying that $C_s'(T)$ is bornological. Thus (c) holds. Because the bounded subsets of $C_s'(T)$ are relatively compact in $C_s'(T)$, we have (c)$\Longleftrightarrow$(d) and also (e)$\Longleftrightarrow$(f)$\Longleftrightarrow$(g) (see [8, p. 231]). In any locally convex space, (d)$\Longrightarrow$(e) and (g)$\Longrightarrow$(h) . Now to show that $(h) \Longrightarrow (a)$, assume that (a) is false. Let $h \in R^T \setminus C_s''(T)$, and let $D = \{0\} \cup \{h(t)\chi_{\{t\}} : t \in T\}$. Note that D is compact (thus automatically weak* compact) in $C_s''(T)$ since any neighborhood of 0 in $C_s''(T)$ contains all but finitely many elements of D. However, if D were equicontinuous, then there would necessarily be a $g \in LSC(T)$ such that $g > 0$ and such that

$$D^{\circ} \supseteq B_g^{\circ} = \text{co bal}\{\delta_t/g(t) : t \in T\} .$$

But then $|h| \leq g$, which is false since $h \notin C_s''(T)$. Thus $C_s'(T)$ is not a strong Mackey space, which completes the proof that $(h) \Longleftrightarrow (a)$. That $(a) \Longrightarrow (i)$ comes from letting U be T with the discrete topology, so that $C_s(U) = R^U = R^T$. Then φ is merely the identity function. Finally, to prove that $(i) \Longrightarrow (a)$, we observe that if $\varphi(g) = 1$, then $\varphi(D_g) = B_1$. However, D_g is compact in $C_s''(T)$, while B_1 is compact in $C_s(U)$ iff U is discrete. Thus $C_s(U)$ is reflexive, and consequently $C_s''(T)$ is also reflexive, so by Corollary 15, we obtain $C_s''(T) = R^T$, which proves (a) . ∎

Several comments are in order with respect to Theorem 16. First of all, the equivalence of (a) and (b) fulfills our promise to characterize in locally convex space terms those T which are wildly oscillatory. Secondly, the φ of (i) need not be linear. However, if φ happens to be a linear homeomorphism onto $C_s(U)$, then we conjecture that φ need not preserve either the order or the multiplication of the spaces entailed in order for (i) to be equivalent to (a) .

Finally, the condition "$C_s'(T)$ is a Mackey space" is conspicuously absent from the array of statements in Theorem 16, and for a very good reason. We simply do not know for what T the space $C_s'(T)$ is a Mackey space. Conceivably $C_s'(T)$ is a Mackey space iff T is wildly oscillatory, but to prove it one would in all likelihood need a much more penetrating understanding of the convex compact subsets of $C_s''(T)$ than we now have. In fact, for us to find even one T for which $C_s'(T)$ is not a Mackey space has been no trivial task. We present the example as Proposition 17.

PROPOSITION 17. If $T = \beta N \setminus N$ and if we assume the continuum hypothesis, then $C_s'(T)$ is not a Mackey space.

<u>Proof</u>. Let S be the set of P-points in T. By Example 12 we know that S is dense in T, and that there exists a (positive) $h_0 \in R^T$ which is everywhere unbounded on S. Since T is compact, Proposition 8 tells us that no positive $g \in LSC(T)$ is everywhere unbounded. Therefore, if $h = h_0 \chi_S$ on T, then h is not majorized by any $g \in LSC(T)$. Now let

$$A = \{h(t)\chi_{\{t\}} : t \in T\},$$

so that $A \subseteq C_s''(T)$, and let $D = \overline{\text{co bal } A}^{R^T}$. Evidently D is compact in R^T. Using the fact that each point of S is a P-point in T, we will show that $D \subseteq C_s''(T)$. With an argument similar to one found in Example 11 of [6], we will first show that if $k \in D$, then $k(t) \neq 0$ for at most countably many $t \in T$ (which such t of course must be in S). In that direction, let $k \in R^T$ and assume that $k(s) \neq 0$ for uncountably many $s \in S$. For each $r > 0$, let

$$S_r = \{s \in S : h(s) \leq r\},$$

so that $\bigcup_{r>0} S_r = S$. Then there necessarily exists an $\epsilon > 0$ and $r' > 0$ such that $|k(t)| > \epsilon$ for uncountably many (and hence countably many) elements of $S_{r'}$. Next, let $p \in N$ such that $p\epsilon/2 > r'$ and let $(s_m)_{m=1}^p \subseteq S_{r'}$ such that $|k(s_m)| > \epsilon$, for all $m \in N$. Finally, let

$$V = \{j \in R^T : |j(s_m)| < \epsilon/2 , m = 1,\dots,p\},$$

so that V is a neighborhood of 0 in R^T. Now if $j \in V$, then

$|(k+j)(s_m)| > \varepsilon/2$, for all $m = 1,\ldots,p$, whereupon

$$\sum_{m=1}^{p} |(k+j)(s_m)| > p\varepsilon/2 > r' .$$

On the other hand, if $j' \in$ co bal A , then

$$j' = \sum_{n=1}^{q} c_n h(t_n) \chi_{\{t_n\}} ,$$

for appropriate $(t_n)_{n=1}^{q} \subseteq T$ and $(c_n)_{n=1}^{q}$, where $\sum_{n=1}^{q} |c_n| \leq 1$, and where without loss of generality we can assume that $(t_n)_{n=1}^{q} \supseteq (s_m)_{m=1}^{p}$. Since $h(s_m) \leq r'$ for all $m = 1,\ldots,p$, we have

$$\sum_{m=1}^{p} |j'(s_m)| \leq \sum_{n=1}^{q} |c_n| r' \leq r' .$$

Thus $(k+V) \cap$ co bal $A = \emptyset$, so that $k \notin D$. Consequently if $k \in D$, then $k(t) \neq 0$ for at most countably many $t \in T$. Now we can easily prove that $D \subseteq C_s''(T)$. For if $k \in D$, then $T \setminus k^{-1}(0)$ is contained in some countable set $(u_n)_{n=1}^{\infty} \subseteq S$, and since S consists entirely of elements which are P-points in T , this means that there exists a sequence $(U_n)_{n=1}^{\infty}$ of pairwise disjoint open subsets of T such that $u_n \in U_n$, for all $n \in N$. If we let

$$g = \sum_{n=1}^{\infty} |k(u_n)| \chi_{U_n} ,$$

then $g \in LSC(T)$, while $g \geqslant |k|$, so that $k \in C_s''(T)$. This proves that $D \subseteq C_s''(T)$. By the definition of the Mackey topology, the set D° is a neighborhood of 0 in the Mackey topology of $C_s'(T)$. On the other hand, if D° were a neighborhood of 0 in $C_s'(T)$ with its dual topology, then there would have to be a positive $g \in LSC(T)$ such that $B_g^{\circ} \subseteq D^{\circ}$. But then $B_g^{\circ\circ} \supseteq D^{\circ\circ} \supseteq D \supseteq A$, forcing $|h| \leq g$, which contradicts the assertion at the beginning of the proof that no such g can be everywhere unbounded on S . Thus D° is not a neighborhood of 0 in $C_s'(T)$, which means finally that $C_s'(T)$ is not a Mackey space. ▌

The presence of the dense set of P-points in the T of Proposition 17 was critical to our argument. If there had been no dense subset of P-points, then such a D might not be contained in $C''_s(T)$ and therefore could not be used to show that $C'_s(T)$ is not a Mackey space. Witness the following example, where $T = [0,1]$ and where $S = (s_n)_{n=1}^{\infty}$ denotes the rationals in $[0,1]$. Let

$$h_0(t) = \begin{cases} 2^{2n}, & \text{for } t = s_n, \text{ all } n \in N \\ 0, & \text{for all other } t \in [0,1] \end{cases}$$

Then h_0 is positive and unbounded on every open subset of S and of T. Let $h = h_0 \chi_S$, and let $A = \{h(t)\chi_{\{t\}} : t \in [0,1]\}$. Then A is relatively compact in $C''_s([0,1])$, while on the other hand

$$D = \overline{\text{co bal } A}^{\,R^{[0,1]}}$$

is not contained in $C''_s([0,1])$. The reason why is that if

$$h_m = \sum_{n=1}^{m} (1/2^n)\, h(s_n)\chi_{\{s_n\}}, \quad \text{for all } m \in N,$$

then $(h_m)_{m=1}^{\infty} \subseteq \text{co bal } A \subseteq D \subseteq C''_s([0,1])$. If $h' = \lim_{m\to\infty} h_m$, then $h'(s_n) > 2^n$ for each $n \in N$, so that consequently h' is unbounded everywhere on $[0,1]$ and thus by Proposition 8 we know that h' cannot be in $C''_s([0,1])$.

The example just cited proves nothing in regard to whether or not $C'_s([0,1])$ is a Mackey space. In fact, we do not know if $C'_s([0,1])$ is a Mackey space. Of course, if $C'_s(T)$ is a Mackey space precisely when T is wildly oscillatory, then obviously $C'_s([0,1])$ would have to be a non-Mackey space.

Once one has gone this far with $C_s(T)$, $C'_s(T)$, and $C''_s(T)$, he quite naturally casts a further glance toward the locally convex space structure of $C''_s(T)$. Without effort he finds that $C''_s(T)$ is complete iff T is discrete, and moreover $C''_s(T)$ is semi-complete iff T is discrete. In addition, he readily observes that $C''_s(T)$ is metrizable iff card $T \leq \aleph_0$, while $C''_s(T)$ is separable iff the cardinality of T is no greater than that of the continuum.

In a slightly different direction he discovers that $C''_s(T)$ is barreled for each T . We prove this result now. It will be used in the proofs of Proposition 19 and Corollary 20, which although concerning $C_s''(T)$, nevertheless hark back to Condition A and the ideas occurring at the beginning of this paper.

PROPOSITION 18. $C_s''(T)$ is barreled for every T .

Proof. Assume that $A \subseteq C_s'''(T)$ is contained in no finite dimensional subspace of $C_s'''(T)$. Then there exist $(t_n)_{n=1}^{\infty} \subseteq T$ and $(\mu_n)_{n=1}^{\infty} \subseteq A$ such that $t_n \in \operatorname{supp}\mu_n$ and $t_n \notin \operatorname{supp}\mu_m$ for all $n > m$. Via Lemma 1.3 of [1] we can find (by taking a suitable subsequence if necessary) a pairwise disjoint collection $(U_n)_{n=1}^{\infty}$ of open subsets of T associated with $(t_n)_{n=1}^{\infty}$ such that $t_n \in U_n$ for all $n \in N$. Inductively let $h_m = a_m \chi_{U_m}$, where $a_m > 0$ is chosen to be so large that $|\mu_m(\sum_{n=1}^{m} h_n)| > m$. Then $h = \sum_{n=1}^{\infty} h_n \in LSC(T)$ and hence $h \in C_s''(T)$. In addition

$$|\mu_m(h)| = |\mu_m(\sum_{n=1}^{m} h_n)| > m \text{ , for all } m \in N.$$

Consequently A is not weak* bounded in $C_s'''(T)$. The weak* bounded subsets hence are automatically equicontinuous. This just means that $C_s''(T)$ is barreled. ∎

The fallout from Proposition 18 includes the following facts. The space $C_s''(T)$ is infrabarreled, is a Mackey space, and is moreover a strongly Mackey space. You might think that under these circumstances $C_s''(T)$ should always be bornological. That this hunch is essentially--but not exactly--accurate is the content of the following final proposition.

PROPOSITION 19. $C_s''(T)$ is bornological iff T is non-measurable.

Proof. Assume that T is non-measurable. We will prove that $C_s''(T)$ is bornological. Since $C_s''(T)$ is barreled by Proposition 18, this means that $C_s''(T)$ is a Mackey space. Thus we need only show that any linear form φ on $C_s''(T)$ which is bounded on all bounded subsets of $C_s''(T)$ is continuous--which would mean that φ corresponds to a measure of the form $\mu = \sum_{n=1}^{m} a_n \delta_{t_n}$, $t_n \in T$. Now any linear form φ which is bounded on the bounded subsets of $C_s''(T)$ is bounded on $D_1 = \{h \in R^T : |h| \leq 1\}$, so that by the Riesz-Kakutani Theorem φ corresponds on the span of D_1 to a $\mu \in M(\beta T_d)$, where T_d denotes the set T with the discrete topology. The correspondence is given by

$$\varphi(h) = \int_{\beta T_d} \tilde{h} d\mu = \mu(\tilde{h}), \text{ for all } h \text{ bounded on } T,$$

where $\tilde{h}$ is the (unique) continuous extension of h to βT_d. Now assume that there exists an infinite sequence $(t_n)_{n=1}^{\infty} \subseteq \beta T_d$ in the support of μ. By Lemma 1.3 of [1], there exists a subsequence $(t_{n_p})_{p=1}^{\infty} \subseteq \beta T_d$, and also the sequence $(U_p)_{p=1}^{\infty}$ of pairwise disjoint open subsets of βT_d, such that $t_{n_p} \in U_p$ for all $p \in N$. Without loss of generality we will assume that $\mu(U_p) = b_p > 0$, for each p. Let $\tilde{h}_p = (p/b_p)\chi_{U_p}$, let $h_p = \tilde{h}_p|_T$, and let $h = \sum_{p=1}^{\infty} h_p$. Then $h \in R^T$ because the U_p's are pairwise disjoint. Also $h > 0$ and $h_p \in D_h$ for each p. But for each $p \in N$ we have

$$\varphi(h_p) = \mu(\tilde{h}_p) = p,$$

contradicting the fact that φ is bounded on each bounded subset, including D_h. Consequently $\mu = \sum_{n=1}^{m} a_n \delta_{t_n}$, with $(t_n)_{n=1}^{m} \subseteq \beta T_d$. Next we assume that $t_1 \in \beta T_d \setminus T_d$ and $a_1 \neq 0$. By assumption T is non-measurable, so that Theorem 12.2 of [4] tells us that $\nu T_d = T_d$, and thus by Theorem 8.4 of [4] there exists an $h \in R^T$ such that the continuous extension $\tilde{h}$ to νT_d gives $\tilde{h}(t_1) = \infty$ and $\tilde{h} = 0$ on $(t_n)_{n=2}^{m}$. But then $|\varphi(\theta_n \circ h)| \xrightarrow[n]{} \infty$, rendering φ unbounded on D_h. This contradiction proves that $t_n \in T$ for each n. Consequently on the span of D_1 the form φ corresponds to $\mu = \sum_{n=1}^{m} a_n \delta_{t_n}$, and each $t_n \in T$. Finally, let $h \in C_s''(T)$, and for each $n \in N$ let $h_n = n\,(h - (\theta_n \circ h)) \in C_s''(T)$. If $h' = \sup_{n \in N} |h_n|$, then $h' \in R^T$ and $(h_n)_{n=1}^{\infty} \subseteq D_{h'}$. Since φ is bounded on $D_{h'}$ we know that

$$\varphi(h - (\theta_n \circ h)) = \varphi(h_n / n) \xrightarrow[n]{} 0 ,$$

whereupon

$$\varphi(h) = \lim_{n \to \infty} \varphi(\theta_n \circ h) = \mu(\tilde{h}) .$$

Thus φ corresponds to μ on all of $C_s''(T)$, meaning that $\varphi \in C_s'''(T)$. This is exactly what we needed to prove in order for $C_s''(T)$ to be bornological. To prove the converse, let T be measurable. Then $\nu T_d \neq T_d$ by Theorem 12.2 of [4]. For $t_0 \in \nu T_d \setminus T_d$ the measure δ_{t_0} describes a linear form φ bounded on all bounded subsets of R^T (and hence on all bounded subsets of $C_s''(T)$). But φ is not continuous on $C_s''(T)$ because it does not correspond to a measure of the form $\sum_{n=1}^{m} a_n \delta_{t_n}$, with $t_n \in T$. Thus $C_s''(T)$ is not bornological. ∎

COROLLARY 20. There exists a T satisfying Condition A iff there exists a T such that $C_s''(T)$ is not bornological.

<u>Proof</u>. This follows directly from Theorem 1 and Proposition 19. ∎

We first observe that the cardinality of T determines more when $C_s''(T)$ is bornological than does the topology on T. This was also the case for metrisability and separability of $C_s''(T)$. In each of these cases the fact that $C_s''(T)$ contains all the bounded functions on T played a critical part.

Although the just-proved Proposition 7 of [9] effectively gives our Proposition 19, our proof--quite independent from that in [9]--might shed some light on when $C_s''(T)$ is ultrabornological. Nevertheless a proof of a criterion under which $C_s''(T)$ is ultrabornological would probably involve a deeper understanding of the convex compact subsets of $C_s''(T)$ than we now have.

4. Unresolved Questions

We collect here several open questions which arose during the course of this paper.

1. Is it consistent with the usual axioms of set theory to assume that no non-discrete T exists which satisfies Condition B ?
2. Is it consistent with the usual axioms of set theory to assume that if T contains no isolated points then there exists an everywhere unbounded function in R^T ?
3. If there exists an everywhere unbounded function in LSC(T), then is T wildly oscillatory?
4. Is there a manageable criterion in terms of T alone which tells precisely when T is wildly oscillatory?
5. If T consists of the ordinals less than the first uncountable ω_1 (or for that matter, those less than ω_α for any $\alpha > 0$), is T wildly oscillatory?
6. Can the φ occurring in Theorem 16 be replaced by a homeomorphism which is merely linear and onto?
7. Is $C_s'(T)$ a Mackey space iff T is wildly oscillatory? In particular, is $C_s'([0,1])$ a Mackey space?
8. Is $C_s''(T)$ ultrabornological for all non-measurable T ?

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University of Maryland
College Park, Maryland 20742
l'Université de Paris-VI
l'Université de Paris-XI (Orsay)