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MEASURE THEORETIC ZERO SETS IN INFINITE DIMENSIONAL SPACES AND APPLICATIONS TO DIFFERENTIABILITY OF LIPSCHITZ MAPPINGS.

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In the first part of the present paper we discuss without proof some results which has appeared with full proof in [3].

In the second part we show how our concept of zero set can be applied to the differentiability theory of Lipschitz mappings between separable Frechet spaces.This work is independent and simultaneous to some closely related work by P. Mankiewicz (see [5]).Although our zero sets were not known by Mankiewicz he has so much information about the sets of differentiability that the overlapping between his work and ours is considerable.We feel that the zero set concept allow a more natural approach and in any case our results gives a very strong motivation for the measure theoretic zero sets.

In the definition of the measure theoretic zero sets and the proofs of some of their fundamental properties we only need that our separable Frechet space A is an abelian topological group which is separable and admit an invariant complete metric d.

For this reason we consider in the sequel an abelian Polish group $(G,+,\emptyset)$ .G is metrizable and therefore admit an invariant metric d which is automatically complete. Let $A \subseteq G$ be a universally measurable set.By our definition A is a Haar zero set if there exist a probability measure u on G such that $\chi_A * u = 0$ (this ,,testing measure,, is of course in no reasonable sense unique).The above equality is equivalent with the requirement that every translate of A is a zero set for the measure u.

It is easily seen using the Fubini theorem that in the special case where G is locally compact our definition gives exactly the zero sets for the Haar measure (see [3]). Therefore we call our zero sets Haar zero sets or simply zero sets if no confusion is likely to arise.An essential difference between the Haar zero sets and the topological zero sets (first category sets) is that the topological zero sets only depends on the topology while the Haar zero sets depends both on the topology and the group structure. Although there are some strong analogies between these two zero set concepts they are disjoint even in the locally compact case.

If u is a ,,testing measure,, for the Haar zero set A and v is a ,,testing measure,, for the zero set B then we see (using the Fubini theorem) that $u * v$ is a ,,testing measure,, for $A \cup B$ .This shows that a finite union of zero sets is a zero set.Using the completeness of the group G and a slight modification of this argument it can be shown that a countable union of zero sets is a zero set (see [3] ,Th. 1).

Let $A, B \subseteq G$ be universally measurable sets.We define

$$F(A,B) = \{ g \in G \mid (g+A) \cap B \text{ is not a Haar zero set} \}.$$

Then $F(A,B)$ is always open in G (possibly empty) (see [3] ,Th.2).

If A is a universally measurable set in G which is not a Haar zero set then $0 \in F(A,A)$ .Since $F(A,A) \subseteq A-A$ we conclude that $A-A$ is a neighbourhood.This shows that in the non locally compact case every compact set is a Haar zero set.In the non locally compact case therefore there does not exist any probability measure u such that every Haar zero set is a zero set for u or such that every zero set for u is a Haar zero set.

We now shift our attention to a separable Frechet space A .It is easy to prove that every analytic hyperplane H in A is closed.It does not seem to be known whether or not every universally measurable hyperplan H is closed.Of course it follows from the preceding results that a universally measurable hyperplane H is a Haar zero set (otherwise $H-H=H$ would be open).Therefore the problem is the measure theoretic analogue to the (also open) problem whether or not every first category hyperplane is closed.These problems seems to be of the same degree of difficulty.Of course it is not trivial that there exist hyperplanes which are not universally measurable.Roughly speaking the following result shows that ,,almost every,, hyperplane is not universally measurable (in a similar sense ,,almost every,, hyperplane is of the second category).The proof is an adaptation of an argument shown to the author by W.Roelcke .

Let a_i $(i \in I)$ be an algebraic basis for the separable Frechet space A and let b_i be the coefficient functionals. Then at most finitely many of the hyperplanes $H_i = b_i^{-1}(0)$ are universally measurable (see [3], Th.3).

A function $f: A \longrightarrow B$ where A and B are Frechet spaces is called Lipschitz if for every continuous seminorm q on B there exist a continuous seminorm p_q on A and a constant $C_q > 0$ such that

$$q(f(x)-f(y)) \leq C_q \, p_q(x-y)$$

for all $x, y \in A$.

We call f weakly locally Lipschitz if for every $a \in A$ and every continuous seminorm q on B there exist a neighbourhood $U_{q,a}$ of a and $C_{q,a} > 0$, $p_{q,a}$ continuous seminorm on A with

$$q(f(x)-f(y)) \leq C_{q,a} \, p_{q,a}(x-y)$$

for all $x, y \in U_{q,a}$.

The Frechet space A is a Radon Nikodym space if for all Lipschitz mappings $f: \mathbb{R} \longrightarrow A$ the mapping f is differentiable in almost every point with respect to Lebesgue measure. This class of spaces has been studied by S.D.Chatterjii in the Banach space case (see [4]).

Theorem 1: Let $f: \mathbb{R} \longrightarrow A$ be a weakly locally Lipschitz mapping from the euclidean space $\mathbb{R}^n$ into the real Radon Nikodym Frechet space A. Then f is differentiable in almost every point with respect to Lebesgue measure on $\mathbb{R}^n$.

Proof: We note that weakly locally Lipschitz is equivalent with locally Lipschitz (the neighbourhood $U_{q,a}$ may be choosen independently of q) since $\mathbb{R}^n$ is locally compact (that this holds in general is dubious). In the special case where $A = \mathbb{R}^m$ this result is due to Rademacher (see [6]). We give a new proof which is much shorter and easier then any known proof. For a fixed direction $h \in \mathbb{R}^n$ it is easily shown using the Fubini theorem (note that f is (locally) Lipschitz on any line) that the directional derivative

$$r_f(x,h)=\lim_{\lambda\to 0} \lambda^{-1}(f(x+\lambda h)-f(x))$$

exist for almost every $x \in \mathbb{R}^n$ with respect to Lebesgue measure.

Let $h_i \in \mathbb{R}^n$ be a dense sequence such that $\{h_i\}$ forms a subgroup H of $\mathbb{R}^n$. From the preceding remark it follows that we may choose a Borel set $S \subseteq \mathbb{R}^n$ such that $\mathbb{R}^n \setminus S$ is a zero set for the Lebesgue measure and such that $r_f(s,h_i)$ exist for all $s \in S$ and all $i \in \mathbb{N}$. A trivial estimation shows that in fact $r_f(s,h)$ exist for all $s \in S$ and $h \in \mathbb{R}^n$ and depends continuously on h (indeed is Lipschitz in h) for fixed $s \in S$. The delicate point of the proof is to show that $r_f(s,h)$ for almost every s is linear in h. It is enough to show that for $i,j \in \mathbb{N}$ we have

$$r_f(s,h_i+h_j)-r_f(s,h_i)-r_f(s,h_j)= \Phi_{ij}(s)=0$$

for almost every $s \in \mathbb{R}^n$.

Note that $\Phi_{ij}(s)$ is well defined for almost every $s \in \mathbb{R}^n$. To show that $\Phi_{ij}(s)=0$ for almost every $s \in \mathbb{R}^n$ it is enough to show that for all $s \in \mathbb{R}^n$

$$(\Phi_{ij} * \psi)(s) = \int \Phi_{ij}(s-t)\,\psi(t)\,dt = 0$$

for any C^∞ function ψ on $\mathbb{R}^n$ with compact support. We have

$$(\Phi_{ij} * \psi)(s) = \int \Phi_{ij}(s-t)\,\psi(t)\,dt =$$
$$\lim_{\lambda \to 0} \lambda^{-1}((f*\psi)(s+\lambda h_i+\lambda h_j)-(f*\psi)(s+\lambda h_i)-(f*\psi)(s+\lambda h_j))$$

since the Lebesgue theorem of dominated convergence allows us to shift the order of integration and going to the limit. Now $f*\psi$ is a C^∞ mapping from $\mathbb{R}^n$ into A, hence we have that the limit is zero. This finishes the proof of theorem 1.

The next result may turn out to be the principal motivation for our definition of the Haar zero sets.

Theorem 2: Let $f: A \longrightarrow B$ be a weakly locally Lipschitz mapping from the real separable Frechet space into the real Radon Nikodym Frechet space B. Then there exist a Borel set $D \subseteq A$ such that $(A \setminus D)$ is a Haar zero set and such that for all $x \in D$ the directional derivative

$$r_f(x,a) = \lim_{\lambda \to 0} \lambda^{-1}(f(x+\lambda a)-f(x))$$

exist for all $a \in A$ and uniformly for a in compact sets (for fixed $x \in D$). Moreover for fixed $x \in D$ $r_f(x,a)$ depends continuously and linearly on $a \in A$. If A is a Schwartz space (in particular if A is nuclear) then f is differentiable in the strong (Frechet) sense in every point $x \in D$.

Proof:Let $A_1 \subsetneq A_2 \subsetneq \dots \subsetneq A_n \subsetneq \dots$ be an increasing sequence of finite dimensional subspaces whose union is dense in A. For each n let $D_n \subsetneq A$ be the set of all $x \in A$ such that $r_f(x,a)$ exist for all $a \in A_n$ and depends linearly on $a \in A_n$ .It is easy to show that D_n is Borel measurable and an application of theorem 1 on each sideclass of A_n shows that $(A \setminus D_n)$ is a Haar zero set (the Lebesgue measure on A_n is testing measure).We put $D_f = \bigcap_n D_n$ and it is easily seen that D_f fulfills the conditions in the theorem.With a modification of the above argument and the proof of theorem 1 in [3] we could obtain a probability measure u on A which is a testing measure for all $(A \setminus D_f)$.

It is known that even for real valued Lipschitz functions on separable Hilbert spaces one cannot obtain Frechet differentiability in at least one point (see [7]).

The Frechet space A is a Schwartz space if for every continuous seminorm q on A there exist a continuous seminorm $p > q$ such that the canonical mapping from $\bar{A}_p$ into $\bar{A}_q$ is compact.This means that the unit ball for the seminorm p is compact in the topology induced by q .It is now easy to see that we can prove that the directional derivative exist uniformly on a whole neighbourhood, but this means that f is Frechet differentiable (for each fixed $x \in D$).

The problems we now intend to consider makes sense (suitable modified) for Frechet spaces.But since even the Banach space case is not very well understood we shall now consider only Banach spaces.

Let A and B be real Banach spaces and $f:A \longrightarrow B$ a Lipschitz equivalence (f is bijective and both f and f^{-1} fulfills a Lipschitz condition).It is an open problem whether or not the existence of such an f implies the existence of a linear homeomorphism.Trying to apply the preceding material one turns into the following

Problem: Does any Lipschitz equivalence between separable Banach spaces preserve the Haar zero sets ?

If this has an affirmative answer then Lipschitz equivalence implies existence of a linear topological equivalence at least for real separable Radon Nikodym Banach spaces.

Another problem which may be studied with the tools developed above is a varionational problem.Let $F \subseteq A$ be a norm closed subset of the real separable Banach space A. For $x \in A$ we want to find $y \in F$ with $\|x-y\| = \inf_{z \in F} \{\|x-z\|\}$. If A is not reflexive we can choose a closed hyperplane F such that no point $x \in (A \setminus F)$ has a nearest point $y \in F$.This follows from a well known theorem of James (see).Thus in order to obtain reasonable results we restrict to reflexive Banach spaces.The natural approach is to consider the Lipschitz function $f(x) = \inf_{y \in F} \{\|x-y\|\}$ and differentiate.We shall state a few conjectures (considering only separable Banach spaces).

Conjectures :

I) The differential $r_f(x,.)$ is for almost every $x \in (A \setminus F)$ an element in the dual space A' of norm one and for almost every $x \in F$ we have $r_f(x,.)=0$. It seems most likely that this is true also in the non reflexive case.

II) If A is reflexive then for almost every $x \in A$ there exist $y \in F$ with $\|x-y\| = f(x)$ and y is unique for almost every $x \in A$ if A is strictly convex.

We can prove the conjectures if F is weakly compact. It is known that the varionational problem has a solution for at least a dense set of $x \in A$ if A is uniformly convex.

We shall now apply the preceding material to function spaces. Let (X, ϱ) be a compact metric space and let $F \subseteq C(X)$ be a point separating space of continuous real valued functions on X. Suppose that F admit a separable Frechet space topology $\mathcal{F}$ finer then the topology of uniform convergence (special case : F is uniformly closed). Let $Y \subseteq X$ be a closed subset. Define $f_Y : F \longrightarrow \mathbb{R}$ by

$$f_Y(g) = \sup_{y \in Y} \{ g(y) \} .$$

It is easily shown that f_Y is a Lipschitz function on F. Suppose $g \in F$ is a point of differentiability for f_Y. We see easily that $r_{f_Y}(g,.)$ is a measure with mass 1 in a point $y_0 \in Y$ such that $g(y_0) = \sup_{y \in Y} g(y)$ and $g(y) < g(y_0)$ for $y \in Y$ with $y \neq y_0$.

Thus for each particular closed set $Y \subseteq X$ almost every $g \in F$ assumes its maximal value on Y in exactly one point $y_0 \in Y$ .In particular the set of $g \in F$ with unique maximum in Y is dense in F.A Borel set $S \subseteq Y$ is called thin (relative to F and Y) if the set of $g \in F$ with a unique maximum in Y assumed in a point $s \in S$ is a Haar zero set in F. It should be interesting to have some investigations of this concept of thinness in some special cases.

We shall now list without any specific ordering some problems some of them already mentioned in the text.

Problem I) Let $A,B \subsetneq G$ be universally measurable subsets of the abelian Polish group $(G,+,\mathcal{O})$.Suppose that both A and B are not Haar zero sets.Is then $F(A,B) \neq \emptyset$?

Problem II) Let A_i $(i \in I)$ be a family of universally measurable pairwise disjoint subsets of the abelian Polish group $(G,+,\mathcal{O})$ which are not Haar zero sets.Is then I at most countable ?

Problem III) Let $f:A \dashrightarrow B$ be a bijective mapping between the separable real Frechet spaces A and B.Suppose both f and f^{-1} are (locally) Lipschitz.Does then f preserve the Haar zero sets ?

This would be a very interesting result even for separable reflexive Banach spaces.

Problem IV) Under some smothness assumptions on the Banach space A it is known that every continuous convex function f on A is Frechet differentiable in a dense G_δ set (see [2]).Does the measure theoretic analogues of these results hold ?

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