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QUOTIENT RINGS OF RIGHT NOETHERIAN RINGS AT SEMI-PRIME IDEALS

by Gerhard O. MICHLER

In this talk of the ring theory meeting at the University of Lyon in July 1973 a survey was given on some properties of the ring R_N of right quotients of a right Noetherian ring R (with identity element) at a two-sided semi-prime ideal N. The results presented here are mainly due to J. Lambek and the author (10), W. Schelter and L. Small (13), and J. Nill (11) .

In the first part of the talk we showed that localizing at the injective hull E(R/N) of the (unitary) right R-module R/N is the same as localizing with respect to the multiplicative set

$$\mathcal{C}(N) = \{c\varepsilon R \mid cr \ \varepsilon \ N \text{ implies } r \ \varepsilon \ N\}$$

((10), Proposition 2.2).

We say we are localizing at N and call the localization $h : R \to R_N$ the *ring of right quotients of* R *at* N.

Extending a theorem of A.G. Heinicke (5) we then proved that the localization functor Q_N is right exact if and only if the N-closure $\tilde{N}$ of h(N) in R_N is such that $R_N/\tilde{N}$ is a finite direct sum of simple R_N-modules containing at least one representative of each isomorphism class of simple R_N-modules ((10), Proposition 2.5). In general $\tilde{N}$ is not a two-sided ideal of R_N ((9), Example 5.9). In fact Q_N is right exact and $\tilde{N}$ is a two-sided ideal of R_N if and only if R satisfies the right Öre condition with respect to $\mathcal{C}(N)$. Another equivalent statement asserts that $\tilde{N}$ is the Jacobson radical of R_N and $R_N/\tilde{N}$ is semisimple Artinian ((10), Proposition 2.5).

As an application of this theorem J. Lambek and the author obtained in theorem 3.3 of (10) the following generalisation of L.W. Small's theorem on right Noetherian right orders in right Artinian rings : If N is any semi-prime ideal of the right Noetherian ring R, then R_N is a right Artinian classical ring of right fractions of R with respect to $\mathcal{C}(N)$ if and only if some power of N is N-torsion and for every $r \in R$ with $cr = 0$ for some $c \in \mathcal{C}(N)$ there is a $c' \in \mathcal{C}(N)$ such that $rc' = 0$. These two conditions are trivially satisfied when $\mathcal{C}(N)$ consists of regular elements, and N is nilpotent.

The right Noetherian ring R with Jacobson radical J is a *classical semi-local ring,* if R/J is Artinian, and if every right ideal E of R is closed in the J-adic topology, i.e. $E = \bigcap_{n=1}^{\infty} (E+J^n)$. Generalising well known results on commutative Noetherian rings Theorem 5.3. of (10) asserts that the ring R_N of right quotients of the right Noetherian ring R at the two-sided semi-prime ideal N is a classical semi-local ring with Jacobson radical $\tilde{N}$ if and only if for every right ideal E of R there exists an integer $n > 0$ such that

$$E \cap N^{(n)} \leq cl_N(EN),$$

where $N^{(n)}$ denots the n-th right symbolic power of N, and where $cl_N(EN)$ is the N-closure of EN. If $\hat{R}_N$ denotes the $\tilde{N}$-adic completion of R_N, then (under these equivalent conditions) $\hat{R}_N$ is the bicommutator of $E_R(R/N)$. Furthermore, the Jacobson radical of $\hat{R}_N$ is a finitely generated right ideal of $\hat{R}_N$. Therefore these results generalise well known results in commutative algebra and of J. Kuzmanovich (6) on non-commutative Dedekind rings. However, also every semi-prime ideal N of an enveloping algebra of a finite-dimensional nilpotent Lie algebra, or of a group algebra of a finitely generated nilpotent group satisfies this right symbolic Artin-Rees condition.

The theorems on quotient rings mentioned so far all require or assert that the quotient functor Q of the torsion theory $\mathcal{T}$ is right exact, which for any $\mathcal{T}$ holds for every right Noetherian right hereditary ring R by Theorem 4.6 of O. Goldman (3) Even if R is a commutative Noetherian ring of global projective dimension $\mathrm{gl.dim}\, R = 2$ there are torsion theories whose quotient functors are not right exact, e.g. B. B. Stenström (14), p. 40. However we now show :

THEOREM 1. - *Let* R *be a semi-prime right and left Noetherian ring with* $\mathrm{gl.dim} R \leq 2$. *Then for any torsion theory* $\mathcal{T}$ *the ring of right quotients* $R_{\mathcal{T}}$ *of* R *with respect to* $\mathcal{T}$ *is right Noetherian.*

Before we give its proof we should like to mention that this theorem generalizes Theorem 6.2 of J. Nill (11) who obtained this result for the special case of (non commutative) domains. The proof is based on the following lemma which for domains is due to R. Hart (4).

LEMMA 1. - *Let* Y *be an essential right ideal of a semi-prime right Noetherian ring* R *with classical right quotient ring* Q.
Then $Y^* = \{q \in Q \mid qY \leq R\}$ *can be embedded into a finitely generated free left* R *-module* F *such that* F/Y^* *is a submodule of a finitely generated free left* R *-module.*

Proof. - Since Q is semisimple Artinian, every essential right ideal Y of R is generated by finitely many regular elements $y_i \in Y$, $i = 1,2\ldots,n$, by theorem 5.5 of J.C. Robson (12). Thus $Y^* = \bigcap_{i=1}^{n} Ry_i^{-1}$. Let $F = \sum_{i=1}^{n} \oplus Ry_i^{-1}$, and let $\alpha \in \mathrm{Hom}_R(Y^*,F)$ be defined by

$$\alpha(a) = (a,a,\ldots,a) \in F,\ \forall a \in Y^*.$$

Then α is a monomorphism. If $N = \alpha(Y^*)$, and if $\tau \in \mathrm{Hom}_R(F, \sum_{i=2}^{n} \oplus (Ry_1^{-1} + Ry_i^{-1}))$ is defined by

$$\tau(a_1,a_2,\ldots,a_n) = (a_1-a_2,\ a_1-a_3,\ldots,a_1-a_n)$$

for every n-tupel $\{a_j \in Ry_j^{-1} \mid j = 1,2\ldots,n\}$, then it easy to see that $N = \ker\tau$. Hence

$$0 \to \bigcap_{i=1}^{n} Ry_i^{-1} \to F \to \sum_{i=2}^{n} \oplus\, (Ry_1^{-1} + Ry_i^{-1}) \to 0$$

is a short exact sequence. Since R satisfies the right Ore condition with respect to the regular elements, there are regular elements $d_i, x_{i1}, t_i \in R$ such that

$$d_i = y_1 x_{i1} = y_i t_i,\ i = 2,3,\ldots,n.$$

Hence $Ry_i^{-1} + Ry_1^{-1} \leq Rd_i^{-1}$, which completes the proof of Lemma.

LEMMA 2. - *Let* R *be a semi-prime right and left Noetherian ring, and let* $\mathcal{T}$ *be any torsion theory on* Mod R. *Then the torsion ideal* T(R) *of* R *with respect to* $\mathcal{T}$ *is semi-prime.*

Proof. - Let $\mathcal{D}$ be the filter of $\mathcal{T}$-dense right ideals of R. Since T(R) is a two-sided ideal of R, and since R is left Noetherian, the right annihilator A of T(R) is a two-sided ideal of R belonging to $\mathcal{D}$. By A.W. Goldie (2) A is contained in only finitely many maximal annihilator ideals W_k , $k = 1,2,\ldots,s$, and each W_k is a two-sided prime ideal. Let $P = \bigcap_{k=1}^{s} W_k$. Then $P \in \mathcal{D}$, and $T(R) \leq A_{\ell} \leq P_{\ell} \leq T(R)$, where X_{ℓ} denotes the left annihilator of the ideal X. Thus

$$T(R) = P_{\ell} = \bigcap_{j=s+1}^{t} W_j \text{ , if } W_{s+1}\text{ , } W_{s+2}\text{ , }\ldots,\ W_t$$

are the remaining finitely many minimal prime ideals of R. In particular, T(R) is a semi-prime ideal of R.

By means of these two lemmas it is now easy to give the PROOF of THEOREM 1 : Since $R_{\mathcal{T}}$ is the quotient ring of R/T(R) with respect to the torsion theory $\mathcal{T}$, we may assume by lemma 2 that T(R) = 0. If $\mathcal{D}$ denotes the filter of $\mathcal{T}$-dense right ideals, and if Q is the classical right quotient ring of R, then

$R = \{q \in Q \mid q^{-1}R \in \mathcal{D}\}$, and every $X \in \mathcal{D}$ is an essential right ideal of R.

Let Y be a right ideal of $R_{\mathcal{F}}$, and let $0 \neq y \in Y$. Then $yD \leq R$ for some $D \in \mathcal{D}$ Hence $y \in D^* = \{q \in Q \mid qD \leq R\}$, and $y D^{**} \leq R$, where $D^{**} = \{v \in Q \mid D^*v \leq R\}$. As $D \leq D^{**}$, its follows that $D^{**} \in \mathcal{D}$. Since D is an essential right ideal of R lemma 1 asserts the existence of finitely generated free left R-modules G and F such that $D^* \leq G$ and $A = G/D^* \leq F$. Hence there is an exact sequence

$$0 \longleftarrow F/A \longleftarrow F \longleftarrow A \longleftarrow G \longleftarrow D^* \longleftarrow 0.$$

Because of $gl.\dim R \leq 2$, it follows that D^* is a projective left R-module, hence $V = D^{**} \cong \mathrm{Hom}_R(D^*, R)$ is a projective right R-module contained in R. Taking projective coordinates $v_i \in V$, $\phi_i = q_i \in \mathrm{Hom}_R(V,R) = V^* = D^*$, $i = 1,2,\ldots, m < \infty$, we obtain $v = \sum_{i=1}^{m} v_i\phi_i(v) = \sum_{i=1}^{m} v_i\, q_i\, v$ for every $v \in V$. Pick a regular element $c \in V$. Then $c = \sum_{i=1}^{m} v_i\, q_i\, c$, and

$$1 = \sum_{i=1}^{m} v_i\, q_i \in VR_{\mathcal{F}}$$

because $D^* \leq R_{\mathcal{F}}$. Therefore

$yVR_{\mathcal{F}} = yD^{**} R_{\mathcal{F}} = yR_{\mathcal{F}}$, and

$y \in yR_{\mathcal{F}} \leq yVR_{\mathcal{F}} \leq (Y \cap R)R_{\mathcal{F}}$

for every $y \in Y$. Hence $Y \leq (Y \cap R)R_{\mathcal{F}} \leq Y$, and Y is a finitely generated right ideal of $R_{\mathcal{F}}$.

REMARK. - Checking the proof one at once observes that we actually have proved the following more general result : Let R be a semiprime right Noetherian ring with left projective global dimension $l.\ gl \dim R \leq 2$, if $\mathcal{F}$ is any torsion theory on Mod(R) such that R is $\mathcal{F}$-torsionfree, then $R_{\mathcal{F}}$ is right Noetherian.

In order to contrast Theorem 1 we now should like to mention the recent example of a right Artinian ring R whose maximal ring of right quotients $Q_{max}(R)$ is not right Artinian, which is due to W. Schelter and L. Small (13). This is very surprising and remarkable, because H. Storrer (15) showed that $Q_{max}(R)$ is right Artinian if R is right *and* left Artinian.

Before we state the example, we mention two theorems proved by J. Nill (11) showing that it is actually only one example in a series of such examples.

Let M be a right D-module, where D is any ring. Let F be a subring of $End_R(M)$. Then M is an F - D - bimodule. Form $R = \begin{pmatrix} F & M \\ 0 & D \end{pmatrix}$. Then R is a ring as is well known. If $E_D(X)$ denotes the injective hull of the right D-module X, then Theorem 7.6 of J. Nill (11) asserts that

$$H = \begin{pmatrix} Hom_D(M,E_D(M) & E_D(M) \\ Hom_D(M,E_D(D) & E_D(D) \end{pmatrix}$$

is always an injective right R-module containing R. Furthermore, if H is the injecti- injective hull of R, then

$$Q_{max}(R) = \begin{pmatrix} End(M) & M_{\mathfrak{F}} \\ M^* & D_{\mathfrak{F}} \end{pmatrix} ,$$

Where $\mathcal{D}_{\mathfrak{F}}$ is the filter of all $E_D(M)$-dense and $E_D(D)$-dense right ideals of D, and where $M^* = Hom_D(M,D)$.

Using this result J. Nill then shows in Theorem 7.21 of (11) (with the same notation, and without the above condition on H) that the ring R has a smallest dense right ideal if and only if the filter $\mathcal{D}_{\mathfrak{F}}$ of $\mathfrak{F}$-dense right ideals contains a smallest element. Furthermore, he shows in Proposition 7.29 : Let D be a ring with a composition series $D > B > A > 0$, and let $M = D/A$. If B is not a dense right ideal of D, then D is the smallest element in $\mathcal{D}_{\mathfrak{F}}$, and so

$$Q_{max}(R) = \begin{pmatrix} \mathrm{End}(M) & \mathrm{Hom}_D(D,M) \\ M^* & \mathrm{End}_D(D) \end{pmatrix}$$

Example. (Samll-Schelter (13)) : Let D be the right Artinian ring of J.E. Björk ((1), Example 2.5), and let F be its ground field. Then D has a composition series meeting the conditions of the above proposition due to J. Nill. Furthermore, $\mathrm{End}_D(M)$ is not right Artinian, and not even right Noetherian as is shown by J.E. Björk (1). Hence $Q_{max}(R)$ is *not* right Artinian. In fact $Q_{max}(R)$ is the right quotient ring of R at the maximal two-sided ideal

$$P = \begin{pmatrix} F & M \\ 0 & B \end{pmatrix}$$

Finally we remark that working with Björk's ring directly W. Schelter and L. Small (13) found this particular example before J. Nill (11).

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