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SUR LES FONCTIONS SYMETRIQUES;

PAR M. WORONTZOFF.

Soit

$$U = F(x_1, x_2, \ldots, x_n) = \Phi(a_0, a_1, \ldots, a_n)$$

une fonction symétrique rationnelle et entière des racines de l'équation

$$f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n = 0.$$

On sait que

$$\sum_{k=1}^{k=n} \frac{du}{dx_k} = -\sum_{r=1}^{r=n} (n - r + 1) a_{r-1} \frac{du}{da_r}.$$

Nous nous proposons ici de généraliser ce théorème. Comme

$$\frac{dx_k}{da_r} = -\frac{x_k^{n-r}}{f'(x)_k},$$

où

$$f'(x_k) = a_0(x_k - x_1)(x_k - x_2)\dots(x_k - x_{k-1})(x_k - x_{k+1})\dots(x_k - x_n)$$
$$= na_0 x_k^{n-1} + (n-1)a_1 x_k^{n-2} + \dots + a_{n-2}x_k + a_{n-1}$$
$$= \left(\frac{-1}{x_k}\right)(a_1 x_k^{n-1} + 2a_2 x_k^{n-2}) + \dots + (n-1)a_{n-1}x_k + na_n,$$

. . . .

on a

$$(1)\quad \left\{\begin{aligned} & A_0 \frac{du}{da_0} + A_1 \frac{du}{da_1} + \dots + A_r \frac{du}{da_r} + \dots + A_n \frac{du}{da_n} \\ & = \sum_{k=1}^{k=n} - \frac{1}{f'(x_k)} (A_0 x_k^n + A_1 x_k^{n-1} + \dots + A_r x_k^{n-r} + \dots + A_n) \frac{du}{dx_k} \text{ [1]}. \end{aligned}\right.$$

Posons, pour abréger,

$$x_1^m + x_2^m + \dots + x_n^m = S_m,$$

$$-(n-r-m+1)a_{r+m-1} + \sum_{c=1}^{c=m} a_{r+m-c} S_{c-1}$$

$$= -\sum_{c=1}^{c=r} a_{r-c} S_{m+c-1} = \sum_{k=1}^{k=n} x_k^m \frac{da_r}{dx_k} = H_{(r)}^{(m)},$$

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(1) Soit, par exemple, $U = S_q$, alors de la formule (1) on déduit

$$\sum_{k=1}^{k=n} \frac{a_0 x_k^{n+q-1}}{f'(x_k)} = \sum_q x_1^{q_1} x_2^{q_2} \dots x_n^{q_n} = -\frac{a_0}{q} \frac{ds_q}{da_0} \left(= \dots = -\frac{a_0}{q+r} \frac{ds_{q+r}}{da_r}\right)$$
$$= \sum (-1)^i i! \frac{a_0^{-i} a_1^{\alpha} a_2^{\beta} \dots a_n^{\lambda}}{\alpha! \beta! \dots \lambda!}$$

(*Nouvelles Annales*, p. 382; 1888), où la somme $\sum_q x_1^{q_1} \dots x_n^{q_n}$ se rapporte à toutes les solutions entières positives de l'équation

$$q_1 + \dots + q_n = q.$$

d'où

$$H_0^{(m)} = 0, \qquad H_r^{(0)} = -(n-r+1)a_{r-1}, \qquad H_r^{(1)} = ra_r,$$
$$H_r^{(2)} = (r+1)a_{r+1} - \frac{a_r a_1}{a_0}, \qquad \ldots.$$

Maintenant, en prenant successivement dans l'égalité (1),

$$A_0 = a_0, \qquad A_1 = a, \qquad \ldots, \qquad A_r = a_r, \qquad \ldots, \qquad A_n = a_n,$$
$$A_0 = 0, \qquad A_1 = -na_0, \qquad \ldots,$$
$$A_r = -(n-r+1)a_{r-1}, \qquad \ldots, \qquad \ldots, \qquad A_n = -a_{n-1},$$
$$A_0 = 0, \qquad A_1 = a_1, \qquad A_r = ra_r, \qquad \ldots, \qquad A_n = na_n,$$

ou

$$A_0 = -na_0, \qquad A_1 = -(n-1)a_1, \qquad \ldots,$$
$$A_r = -(n-r)a_r. \qquad \ldots, \qquad A_n = 0,$$
$$A_0 = 0, \qquad A_1 = 2a_2 - \frac{a_1^2}{a_0}, \qquad \ldots,$$
$$A_r = (r-1)a_{r+1} - \frac{a_r a_1}{a_0}, \qquad \ldots,$$

ou

$$A_0 = a_1, \qquad A_1 = 2a_2, \qquad \ldots, \qquad A_r = (r+1)a_{r+1}, \qquad \ldots,$$
$$\ldots\ldots\ldots \qquad \ldots\ldots\ldots \qquad \ldots\ldots \qquad \ldots\ldots\ldots\ldots\ldots \qquad \ldots\ldots$$
$$A_0 = 0, \qquad A_1 = H_1^{(m)}, \qquad \ldots, \qquad A_r = H_r^{(m)}, \qquad \ldots,$$

ou

$$A_0 = -a_0 L_{m-1}, \qquad A_1 = H_1^{(m)} - a_1 S_{m-1}, \qquad \ldots,$$
$$A_r = H_r^{(m)} - a_r S_{m-1}, \qquad \ldots,$$

et en remarquant que

$$H_1^{(m)} x_k^{n-1} + H_2^{(m)} x_k^{n-2} + \ldots + H_{n-1}^{(m)} x_k + H_n^{(m)}$$
$$= a_1 x_k^{n+m-2} + 2a_2 x_k^{n+m-3} + \ldots + na_n x_k^{m-1},$$

on trouve respectivement les formules suivantes

$$a_0 \frac{du}{da_0} + a_1 \frac{du}{da_1} + \ldots + a_{n-1} \frac{du}{da_{n-1}} + a_n \frac{du}{da_n} = 0,$$

et

$$\sum_{k=1}^{k=n} \frac{du}{dx_k} = -\left[na_0 \frac{du}{da_1} + (n-1)a_1 \frac{du}{da_2} + \ldots \right.$$
$$\left. + 2a_{n-2} \frac{du}{da_{n-1}} + a_{n-1} \frac{du}{da_n}\right],$$

$$\sum_{k=1}^{k=n} x_k \frac{du}{dx_k} = \quad a_1 \frac{du}{da_1} + 2a_2 \frac{du}{da_2} + \ldots$$
$$+ (n-1)a_{n-1} \frac{du}{da_{n-1}} + na_n \frac{du}{da_n}$$
$$= -\left[na_0 \frac{du}{da_0} + (n-1)a_1 \frac{du}{da_1} + \ldots \right.$$
$$\left. + 2a_{n-2} \frac{du}{da_{n-2}} + a_{n-1} \frac{du}{da_{n-1}}\right],$$

$$\sum_{k=1}^{k=n} x_k^2 \frac{du}{dx_k} = \quad \left(2a_2 - \frac{a_1^2}{a_0}\right) \frac{du}{da_1} + \left(3a_3 - \frac{a_2 a_1}{a_0}\right) \frac{du}{da_2} + \ldots$$
$$+ \left(na_n - \frac{a_{n-1}a_1}{a_0}\right) \frac{du}{da_{n-1}} + \left(-\frac{a_n a_1}{a_0}\right) \frac{du}{da_n}$$
$$= a_1 \frac{du}{da_0} + 2a_2 \frac{du}{da_1} + \ldots$$
$$+ (n-1)a_{n-1} \frac{du}{da_{n-2}} + na_n \frac{du}{da_{n-1}},$$

. . . .

$$\sum_{k=1}^{k=n} x_k^m \frac{du}{dx_k} = \sum_{k=1}^{k=n} x_k^m \left(\frac{du}{da_1}\frac{da_1}{dx_k} + \frac{du}{da_2}\frac{da_2}{dx_k} + \ldots + \frac{du}{da_n}\frac{da_n}{dx_k}\right)$$
$$= \sum_{r=1}^{r=n} \mathrm{H}_r^{(m)} \frac{du}{da_r} = \sum_{r=0}^{r=n-1} (\mathrm{H}_r^{\prime(m)} - a_r \mathrm{S}_{m-1}) \frac{du}{da_r}.$$

Exemples. — 1° Si l'on pose $\mathrm{U} = \mathrm{S}_i$, $i > 0$, on a

$$a_1 \frac{ds_i}{da_0} + 2a_2 \frac{ds_i}{da_1} + \ldots$$
$$+ (n-1)a_{n-1} \frac{ds_i}{da_{n-2}} + na_n \frac{ds_i}{da_{n-1}} = i\mathrm{S}_{i+1};$$

2° Soit $u = \frac{a_q}{a_0}$, on obtient, pour $m > 1$,

$$\frac{1}{a_0}\sum_{r=1}^{r=n} H_r^{(m)} \frac{da_q}{da_r} + \frac{H_q^{(m)}}{a_0} = (-1^q)\sum x_1^m x_2 x_3 \ldots x_q,$$

$$H_q^{(m)} = (-1)^q a_0 \sum x_1^m x_2 \ldots x_q.$$